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Improving stability and accuracy of Reissner–Mindlin plate finite elements via algebraic subgrid scale stabilization

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Abstract

Stabilized finite element methods for the solution of Reissner/Mindlin-type plate problems are presented. The formulations are based on previously described mixed formulations, like the assumed natural strain (ANS or MITC) concept or the discrete shear gap (DSG) method. In particular, the algebraic subgrid scale (ASGS) formulation is used for the stabilization term. The essential idea is to obtain stable elements and improve coarse mesh accuracy at the same time. It is shown how this can be achieved by a proper choice of stabilization parameters on the basis of physical insight into the mechanical behavior of shear deformable plates. In this context there is a strong relationship to concepts that have been developed long before stabilization techniques appeared in finite element technology, particularly the 'residual bending flexibility' or 'deflection matching' technique.

Keywords: Finite elements; Reissner/Mindlin plate theory; Algebraic subgrid scale stabilization; Discrete shear gap method

1. Introduction

This paper deals with the problem of transverse shear locking in shear deformable (Reissner/Mindlin type) plate and shell elements. There is a multitude of different concepts described in the literature to formulate 'locking-free' elements and it is interesting to note that the most popular of them appear to be almost or exactly identical in the case of structured meshes and rectangular elements. The quality—i.e. coarse mesh accuracy and thus efficiency—of such formulations thus mainly differs in the element behavior in the case of distorted meshes. In fact, well-established elements, like the MITC4 [9] (see also [12,14,18,20]) show significant locking and strong oscillations of transverse shear forces for certain mesh configurations, especially for tapered element shapes [4,5]. From a mathematical point of view, such elements are not stable.

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An effective means to obtain stable elements which are rather insensitive to mesh distortions are stabilization methods. Stabilized finite elements for plates have been described by Codina [8], Lyly et al. [17], among others. Reduced integrated elements with hourglass control, as pioneered by Flanagan and Belytschko [10] represent a special case of stabilized elements. Many papers on stabilization methods do not directly address the Reissner/Mindlin plate problem, but deal with the corresponding type of differential equations, see for instance [7,13]. It is impossible to review the entire literature here; the given references are those which have a certain relationship to the methods discussed in this study.

In the authors' view, these are among the best plate elements available. Other formulations which are both stable and insensitive to mesh distortions, like Auricchio and Taylor [1], Tessler and Hughes [21], perform well without stabilization. However, there is a strong relationship to stabilized methods: In [1] bubble functions are used (which are closely related to stabilization techniques, as pointed out by Hughes [13]) and the relationship of Tessler and Hughes [21] to stabilized elements will be discussed later in this paper.

An essential ingredient of stabilization methods is the stabilization parameter which typically introduces some arbitrariness into the formulation. In the case of plate elements its principle format—i.e. the dependence on plate thickness and element size—can be derived from mathematical arguments. But there is still a constant which can be chosen more or less arbitrarily. It will be demonstrated in this paper, how this factor can be used to improve coarse mesh accuracy and still retaining stability. This is done with the help of mechanical considerations, previously discussed in the context of 'residual bending flexibility' [19] or the 'deflection matching technique' [21].

2. Plate finite elements

The finite elements used as a basis for the stabilized method presented in this paper are formulated according to the discrete shear gap (DSG) method, originally proposed by Bletzinger et al. [6]. It can be classified as a 'B-bar' method because it results in a modification of the discretized differential operator (the strain displacement matrix or 'B-operator'). It can also be viewed as a collocation method with the nodes as sampling points for 'discrete shear gaps'. The most attractive features of this method are

- it is a uniform method for both triangular and quadrilateral elements and
- it can be applied—without any additional consideration—to elements of arbitrary polynomial order.

In the context of this paper the technical details of the method are irrelevant. In fact, the presented stabilization technique can be applied as well to other methods, like the ANS method. In the numerical examples we make reference to the MITC4 element, because it is much more widespread than the DSG family. But in fact, the four-node DSG element and the MITC4 appear to be identical, as far as the element stiffness matrix is concerned.

Formally, such methods can be traced back to a modified form of the deformation energy of the plate,

$$\Pi(w,\psi) = \tfrac{1}{2}t^3 a(\psi,\psi) + \tfrac{1}{2}\kappa G t(\mathbf{R}\psi - \nabla w, \mathbf{R}\psi - \nabla w) - g(w). \qquad (1)$$

The bilinear form $a(\psi,\psi)$ represents the bending energy, depending exclusively on the rotations ψ. The second term is the modified shear energy, with the shear correction factor κ and shear modulus G. $\mathbf{R}$ is a reduction operator, acting on the part of the transverse shear strains that depends on the rotations. It is the formal representation of the DSG method or the ANS method or any method that can be classified within in the group of 'locking-free' elements that use a modification of the strain displacement matrix. Finally, the linear form $g(w)$ expresses the work done by the external forces.

Eq. (1) also illustrates the reason for shear locking: while the bending stiffness tends to zero with t^3 as the plate gets thinner, the transverse shear stiffness is linear in t. Thus, even small errors in the transverse shear energy dominate the strain energy and can make the results completely useless. While the 'classical' methods, like ANS, reduce the error inside the bilinear form, stabilization methods modify the corresponding coefficient, making it approximately proportional to t^3 in the case of coarse meshes.

3. Stabilization methods

3.1. Stenberg's method

The first stabilization method that will be discussed in this paper has been presented by Lyly et al. [17] (it is called 'Stenberg's method' here, because it has been developed in his group). The corresponding functional reads

$$\overline{\Pi}(w,\psi) = \frac{1}{2}t^3 a(\psi,\psi) + \sum_K \frac{1}{2}\underbrace{\frac{\kappa G t^3}{t^2 + \alpha h_K^2}}_{\kappa G t^*}(\mathbf{R}\psi - \nabla w, \mathbf{R}\psi - \nabla w)_K - g(w), \tag{2}$$

where the modified transverse shear energy is split into a sum of the individual contributions of all elements K. The 'correction factor' for the shear energy

$$\tau = \frac{t^2}{t^2 + \alpha h_K^2} = \frac{1}{1 + \alpha \frac{h_K^2}{t^2}}, \qquad \kappa G t^* = \tau \kappa G t \tag{3}$$

depends on a 'typical element size' h_K and a weighting factor α. h_K can be the length of the longest edge of the element, the square root of its area or any other value that appropriately reflects the size of the element. α is any number larger than zero; in [17] $\alpha = 0.1$ is recommended. In the case of bilinear elements, the method is stable for any $\alpha > 0$, however, the elements tend to be too flexible, and thus inefficient, if α is chosen too large.

3.2. ASGS stabilization

An alternative possibility to derive stabilized plate elements is to use the algebraic subgrid scale (ASGS) technique. Stabilized plate elements based on algebraic subgrid scale stabilization have, for instance, been presented by Codina [7,8] see also [13].

The generic form of a stabilized method can be written as

$$\overline{\Pi} = \Pi + \Pi_{\text{stab}}. \tag{4}$$

When using ASGS we have

$$\Pi_{\text{stab}} = \sum_K \lambda_K b(-L^*(u), L(u))_K. \tag{5}$$

Here, $L(u)$ represents the differential operator associated with the problem at hand and $L^*(u)$ is its adjoint operator. In the context of problems in structural mechanics, like in this paper, Π is the deformation energy of the 'original' finite element method (in our context the DSG method), thus Π_{stab} has to be an energy expression as well. This can be ensured by a proper design of the bilinear form $b(\cdot,\cdot)$, typically involving the elasticity tensor or its inverse, respectively.

In this paper, stabilization terms are derived from the equilibrium equation for the bending moments,

$$\nabla \mathbf{m} - \mathbf{q} = \mathbf{0}, \tag{6}$$

neglecting distributed moment loading.

The corresponding ASGS stabilization term reads

$$\Pi_{\text{stab}}(w, \psi) = \sum_K \lambda_K (-\mathbf{C}_q^{-1} \cdot (-\nabla \mathbf{m} - \mathbf{q}), \nabla \mathbf{m} - \mathbf{q})_K, \tag{7}$$

where $\mathbf{m}$ and $\mathbf{q}$ are both functions of the displacement parameters w, ψ. $\mathbf{C}_q$ is the part of the elasticity tensor containing the transverse shear stiffness (in Cartesian coordinates and matrix notation $\mathbf{C}_q = \begin{bmatrix} \kappa G t & 0 \\ 0 & \kappa G t \end{bmatrix}$).

In order to increase flexibility in the choice of stabilization parameters (see Section 4.2) Eq. (7) is now slightly modified. In particular, it is desirable to treat the local ξ- and η-direction separately within each element. Expressing Eq. (6) in local element coordinates

$$\mathbf{m} = m^{\xi\xi} \mathbf{g}_\xi \otimes \mathbf{g}_\xi + m^{\xi\eta} \mathbf{g}_\xi \otimes \mathbf{g}_\eta + m^{\eta\xi} \mathbf{g}_\eta \otimes \mathbf{g}_\xi + m^{\eta\eta} \mathbf{g}_\eta \otimes \mathbf{g}_\eta,$$

$$\mathbf{q} = q^\xi \mathbf{g}_\xi + q^\eta \mathbf{g}_\eta, \qquad \nabla = \frac{\partial}{\partial \theta^\xi} \mathbf{g}^\xi + \frac{\partial}{\partial \theta^\eta} \mathbf{g}^\eta$$

leads to a set of two equations

$$\nabla \mathbf{m} - \mathbf{q} = m^{\xi\xi}_{,\xi} \mathbf{g}_\xi + m^{\xi\eta}_{,\eta} \mathbf{g}_\xi + m^{\eta\xi}_{,\xi} \mathbf{g}_\eta + m^{\eta\eta}_{,\eta} \mathbf{g}_\eta - q^\xi \mathbf{g}_\xi - q^\eta \mathbf{g}_\eta = 0 \tag{8}$$

$$\Rightarrow \quad m^{\xi\xi}_{,\xi} + m^{\xi\eta}_{,\eta} - q^\xi = 0, \tag{9}$$

$$m^{\eta\xi}_{,\xi} + m^{\eta\eta}_{,\eta} - q^\eta = 0. \tag{10}$$

For the sake of simplicity, the covariant base vectors $\mathbf{g}_\xi = \frac{\partial \mathbf{x}}{\partial \xi}$ and $\mathbf{g}_\xi = \frac{\partial \mathbf{x}}{\partial \eta}$ are assumed to be constant within one element. In the case of bilinear elements the influence of this assumption on the numerical behavior is negligible.

The corresponding elasticity matrix is

$$\mathbf{C}_q = \mathbf{J} \cdot \begin{bmatrix} \kappa G t & 0 \\ 0 & \kappa G t \end{bmatrix} \cdot \mathbf{J}^{\mathrm{T}}, \tag{11}$$

where $\mathbf{J}$ is the element Jacobian.

Rewriting Eq. (7) as

$$\Pi_{\text{stab}}(w, \psi) = \sum_K (-\Lambda \cdot \mathbf{C}_q^{-1} \cdot (-\nabla \mathbf{m} - \mathbf{q}), \nabla \mathbf{m} - \mathbf{q})_K, \tag{12}$$

where $\Lambda = \begin{bmatrix} \lambda_\xi & 0 \\ 0 & \lambda_\eta \end{bmatrix}$ is a diagonal matrix, renders the possibility to choose individual stabilization parameters for the ξ- and η-direction, i.e. for the residuals of Eqs. (9) and (10), respectively, in each element. In the case of rectangular elements, $m^{\xi\xi}_{,\xi}$ and $m^{\eta\eta}_{,\eta}$ vanish identically, leaving only contributions from transverse shear and twist in the stabilization terms. This will be of some importance later, when the mechanical significance of the stabilization terms is discussed (see Section 4.2).

3.3. Interrelation of both concepts

In the case of linear beam elements both methods are identical, as will be shown in the sequel (see also [15]). This similarity does not hold exactly in the case of plates but it is helpful for the interpretation of numerical results and the proper choice of stabilization parameters.

The equilibrium equation for a straight, plane beam reads

$$\frac{\mathrm{d}}{\mathrm{d}x}M - Q = 0, \tag{13}$$

and the stabilization term according to Eq. (7) is

$$\Pi_{\mathrm{stab}}(w,\beta) = \sum_K \lambda_K \left(-\frac{1}{\kappa G t} \cdot \left(-\frac{\mathrm{d}}{\mathrm{d}x}M - Q \right), \frac{\mathrm{d}}{\mathrm{d}x}M - Q \right)_K. \tag{14}$$

Expressing the bending moment M and the transverse shear force Q in terms of transverse displacements w and rotations β

$$M = EI \cdot \frac{\mathrm{d}}{\mathrm{d}x}\beta, \qquad Q = \kappa GA \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta \right), \tag{15}$$

assuming a constant bending stiffness $EI =$ constant and making use of the fact that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\beta = 0$ in a linear element we arrive at

$$\Pi_{\mathrm{stab}}(w,\beta) = \sum_K \lambda_K \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta, -\kappa GA \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta \right) \right)_K. \tag{16}$$

The part of the deformation energy of a standard beam element which is related to transverse shear reads

$$\Pi_q(w,\beta) = \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta, \kappa GA \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta \right) \right) = \sum_K E_{qK}. \tag{17}$$

Obviously, the format of (17) resembles that of the stabilization term and we find

$$\Pi_q + \Pi_{\mathrm{stab}} = \sum_K (1 - \lambda_K)\kappa GA \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}w + \beta, \frac{\mathrm{d}}{\mathrm{d}x}w + \beta \right)_K, \tag{18}$$

which is the exact equivalent to the second term in Eq. (2), with $1 - \lambda_K = \frac{t^2}{t^2 + \alpha h_K^2} = \tau$.

In the case of bilinear plate elements the bending moments do not vanish completely and the equivalence is lost. This topic will be discussed again in Section 4.2.

4. Re-interpretation of stabilization terms—improving stability and accuracy

The stabilization methods described in the previous section have certain similarities to a couple of concepts described in earlier literature (e.g. [11,19,21]). Basically, all these concepts share the fact that the transverse shear energy is scaled with a factor depending on the slenderness of the element. It is interesting to observe that the principal design of this factor is always the same, although it is obtained with rather different methodologies and out of different motivations. Most of the methods have not been introduced to 'stabilize' finite elements but to make them more flexible in order to improve accuracy of beam and plate bending elements or simply to reduce ill-conditioning in the thin limit. Mathematically, the interrelations have been pointed out by Lyly [16] for a couple of different element types, see also [8].

In some sense, the idea of using stabilization parameters to 'tune' element behavior is related to the so-called free formulation by Bergan and co-workers (see for example [2,3]). Here, after ensuring certain basic qualities like exact representation of rigid body modes and satisfaction of the patch test, free parameters associated with higher deformation modes allow an 'arbitrary' element design.

4.1. Deflection matching technique

The nature of a couple of these methods is explained along the lines of the 'deflection matching technique' by Tessler and Hughes [21]. The basic idea is practically the same as in MacNeal's 'residual bending flexibility' [19].

The deflection matching technique can be reduced to one simple idea: take the *exact solution* for a certain deformation mode and compare it to the *finite element solution*. Then modify the transverse shear stiffness in the finite element method in order to match the exact result. Unfortunately, only the application to beam elements is straightforward. In the case of plates and shells additional considerations have to be made, somehow compromising the simplicity of the concept.

To get started, let's consider the case of a Timoshenko beam element. The exact solution for the displacement of a cantilever beam (length ℓ) under a tip load P is

$$w_{\text{exact}} = \frac{P\ell^3}{3EI} + \frac{P\ell}{\kappa GA}, \tag{19}$$

whereas one reduced integrated linear element yields

$$w_{\text{fe}} = \frac{P\ell^3}{4EI} + \frac{P\ell}{\kappa GA}. \tag{20}$$

Note that applying the ANS or DSG method instead of reduced integration leads to the same result in this case. Replacing the shear stiffness in the finite element formulation by a—yet unknown—modified shear stiffness κGA^* and solving the condition $w_{\text{fe}} = w_{\text{exact}}$ for κGA^* provides

$$\frac{P\ell^3}{4EI} + \frac{P\ell}{\kappa GA^*} = \frac{P\ell^3}{3EI} + \frac{P\ell}{\kappa GA} \Rightarrow \kappa GA^* = \frac{\kappa GA}{1 + \dfrac{\ell^2 \kappa GA}{12EI}}. \tag{21}$$

Applying the same idea to other systems and load cases always leads to the same result for κGA^*. In fact, it turns out that using this modified shear stiffness within the finite element formulation yields the exact solution according to Timoshenko beam theory for all cubic displacement fields, regardless of the thickness of the beam. For a two-node beam element this means that the exact stiffness matrix is obtained.

Of course, exact stiffness matrices for beam elements are neither new nor very spectacular. The interesting point is the format of Eq. (21) compared to (3). With $I = \frac{bt^3}{12}$, $A = bt$, $G = \frac{E}{2(1+\nu)}$ and $\ell = h_K$ we have

$$\frac{\kappa GA}{1 + \dfrac{\ell^2 \kappa GA}{12EI}} = \frac{1}{1 + \dfrac{\kappa h_K^2}{2(1+\nu)t^2}} \kappa GA, \tag{22}$$

reproducing the modified stiffness in Eq. (3) with $\alpha = \frac{\kappa}{2(1+\nu)}$. Of course, this equivalence also applies to the ASGS method described in Section 3.3.

As the stabilization method is relatively insensitive with respect to the value of α (see the numerical examples in [4]) the intuitive idea is to make a choice which is optimal according to Eq. (22). Thus, a combination of the stabilizing effect described in the previous section and the improvement of accuracy shown in this section can be obtained.

4.2. Stabilization parameters for ASGS stabilization of plate elements

As mentioned earlier, the situation in the case of plates and shells is not as simple. First of all, transverse shear forces in x- and y-direction (or ξ- and η-direction, respectively) have to be treated separately. Mac-

Neal [19] states that, in addition, it has to be distinguished between transverse shear evolving from bending and twisting and that therefore the scalar value α emerges to a 2×2 matrix. Tessler and Hughes [21] describe 'element appropriate shear correction factors', depending on the element geometry. They carefully select these factors on the basis of numerical experiments.

Numerical experiments documented in [5] show that a value of $\alpha = \frac{\kappa}{2(1+\nu)}$ works very well in cases where bending is dominant in one specific direction, as could have been expected. As soon as two-dimensional bending and twist come into the picture the elements behave too flexible. One possibility is to simply reduce this value. However, the fact that we know in which cases $\alpha = \frac{\kappa}{2(1+\nu)}$ is too high, namely in the presence of twist, motivates the development of alternative, more 'intelligent' stabilization methods which take into account this information (it will turn out, however, that it is not advisable to enforce the *exact* representation of one-dimensional bending by all means, because the corresponding elements will always behave too flexible).

One such possibility is the ASGS stabilization described in Section 3.2. In contrast to the method by Lyly et al. [17], the stabilization terms contain also a contribution from twisting moments. In addition, individual stabilization parameters can be chosen for local ξ- and η-directions. This fact can be used to find 'optimal' stabilization parameters which improve accuracy not only in the case of one-dimensional bending, but in more general situations.

With

$$\Lambda = \begin{bmatrix} \lambda_\xi & 0 \\ 0 & \lambda_\eta \end{bmatrix}, \quad \lambda_\xi = \frac{1}{1+\dfrac{t^2}{\alpha_\xi l_\xi^2}}, \quad \lambda_\eta = \frac{1}{1+\dfrac{t^2}{\alpha_\eta l_\eta^2}}, \tag{23}$$

we propose

$$\alpha_\xi = 0.17\frac{l_\eta}{l_\xi}\frac{1}{2(1+\nu)}, \qquad \alpha_\eta = 0.17\frac{l_\xi}{l_\eta}\frac{1}{2(1+\nu)}. \tag{24}$$

Note, that $\lim_{l_\xi\to 0}\lambda_\xi = 0$ and $\lim_{l_\eta\to 0}\lambda_\eta = 0$, i.e. the stabilization terms vanish with mesh refinement. l_ξ and l_η are typical element lengths in the local directions, e.g. $l_\xi = 2\cdot|\mathbf{g}_\xi|$, $l_\eta = 2\cdot|\mathbf{g}_\eta|$. The parameter 0.17 is a result of numerical experiments. The factors $\frac{l_\eta}{l_\xi}$ and $\frac{l_\xi}{l_\eta}$ take into account different element behavior depending on the aspect ratio of the element. In fact, they help to avoid that the elements behave too stiff in the case of high aspect ratios. Finally, the factor $\frac{1}{2(1+\nu)}$ is directly transferred from the deflection matching technique for beams.

5. Numerical experiments

The performance of the proposed four-node plate element is tested with the help of a couple of numerical experiments. Comparison is made to a four-node ANS element without stabilization and the stabilized MITC element proposed by Lyly et al. [17]. In particular, we use the following terminology:

- *MITC4:* The four-node shell element presented by Dvorkin and Bathe [9] (see also [12,14,18,20]), based on the ANS (or MITC) technique.
- *Stenberg:* The stabilized version of the MITC4, as proposed in the paper of Lyly et al. [17], the name 'Stenberg' is used because the group around this author published a couple of important papers on this subject.
- *Present:* The four-node element presented in this paper, based upon the DSG method along with an ASGS stabilization, using the stabilization parameters given in Eqs. (23) and (24).

5.1. Convergence (patch test)

All elements satisfy the patch test for constant bending and twist. The MITC4 element also satisfies the patch test for constant transverse shear forces (the same is true for the DSG element without stabilization), whereas the stabilized elements are much more flexible in this case. It should be mentioned, however, that a patch test for transverse shear is usually not performed, probably because the corresponding load case is somehow artificial (distributed bending moment). Moreover, convergence to the correct solution is ensured anyway, because the stabilization techniques are consistent, and the basic methods (ANS and DSG) do satisfy all patch tests and are known to be convergent.

5.2. Stability

We try to judge stability properties of the element numerically with the help of an academic example originally proposed by Bischoff [4]. Of course this does not replace a mathematical stability analysis. But the mesh configuration sketched in Fig. 1 is well-suited to reveal sensitivity of plate elements to mesh distortions which is, in turn, a typical symptom for an unstable method.

A quadratic plate with clamped edges, length/width $\ell = 4.0$ and thickness $t = 0.01$ is subject to a uniform load of $q = 1.0 \times 10^{-4}$. Material data are: Young's modulus $E = 30{,}000$, Poisson's ratio $\nu = 0.3$. One quarter of the plate is discretized, using symmetry, with a structured mesh of 4×4 finite elements. On the right hand side of the diagram it is indicated how mesh distortion is controlled with the parameter d. It is varied from $d = 0$ to 0.25. The resulting tapered element shape is particularly well suited to reveal distortion sensitivity of finite elements (in contrast to often performed distortion tests where the nodes are moved diagonally within the mesh).

From the numerical results compiled in Fig. 2 it can be seen that both stabilized elements are relatively insensitive to mesh distortions, while the MITC4 shows strong locking in the case of distorted meshes.

The left diagram shows the behavior of the elements for a certain mesh configuration ($d = 0.1$) as the plate slenderness increases. While all elements perform well in the case of thick plates, MITC4 without stabilization severely underestimates the transverse deflection in the thin limit.

In the second numerical test, the control parameter for mesh distortion is varied from $d = 0$ to 0.25, keeping the thickness constant ($t = 0.01$). The right diagram in Fig. 2 shows that all elements behave stiffer as the mesh distortion increases. However, the influence of mesh distortion on the stabilized elements is significantly smaller than in the case of MITC4.

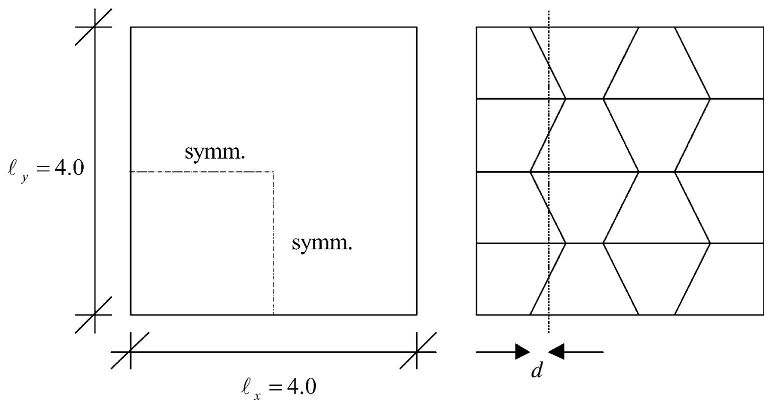

Fig. 1. Sensitivity to mesh distortion, problem setup.

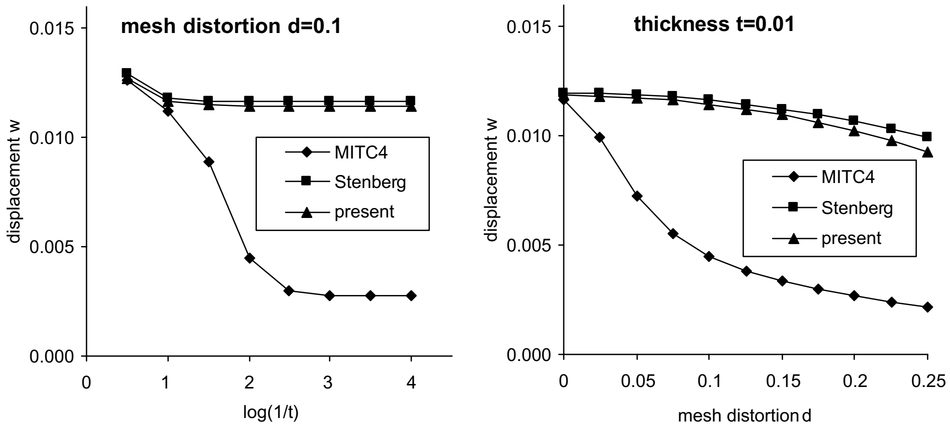

Fig. 2. Sensitivity to mesh distortion, results.

5.3. Accuracy

Finally, a number of numerical results is presented in order to demonstrate the fact that the proposed stabilization method, along with the 'mechanically motivated' stabilization parameters not only leads to stable elements but can also increase coarse mesh accuracy and thus efficiency of the elements. Basis for the numerical results is the quadratic plate introduced in the previous example. We then successively modify thickness, material data, boundary conditions and geometry of the system in order to test the presented elements in different situations.

As a certain choice of numerical experiments is always arbitrary one might argue that there is the possibility to construct cases in which performance of the compared elements is completely different. We therefore do not claim that these examples prove anything. However, they confirm that there is a potential to improve accuracy with stabilization methods and that it is sensible to include any available mechanical and geometric information a priori into the stabilization parameters.

We start with the problem data given in the previous section, changing Poisson's ratio to $\nu = 0.0$. The results are given in Fig. 3. The results for the same configuration with $\nu = 0.4$ are given in Fig. 4.

First of all, all elements perform well in these two cases. Stenberg's stabilized MITC element confirms the aforementioned critique on stabilized elements to be too flexible, especially in the second case. The DSG

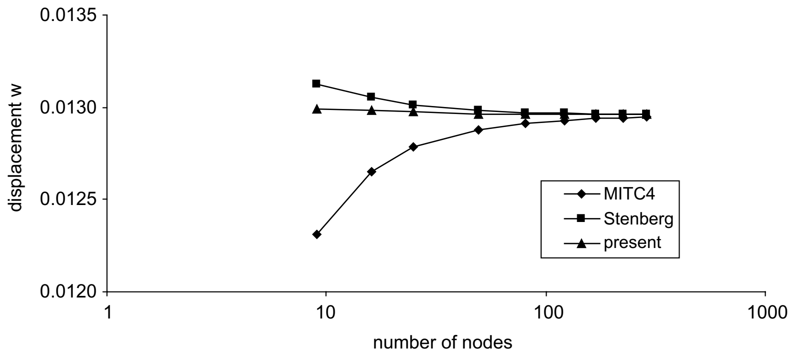

Fig. 3. Clamped quadratic plate, $\nu = 0.0$.

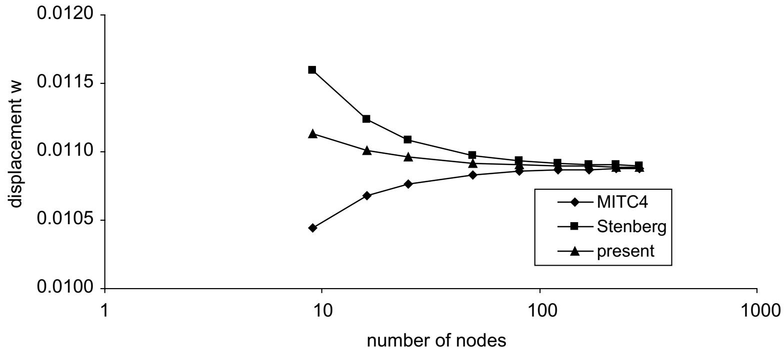

Fig. 4. Clamped quadratic plate, $\nu = 0.4$.

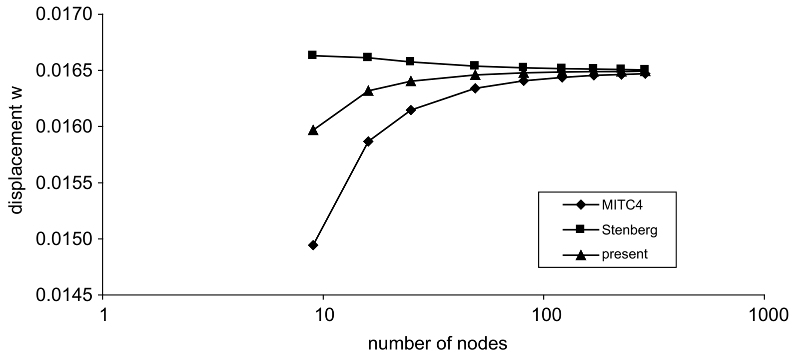

Fig. 5. Quadratic plate, $\nu = 0.4$, 'mixed' boundary conditions.

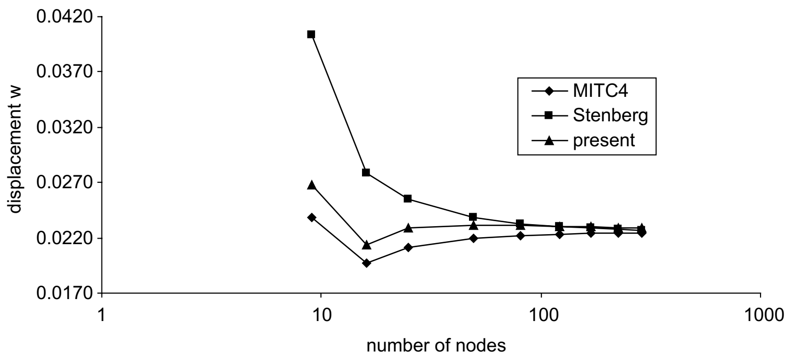

Fig. 6. Clamped quadratic plate, $\nu = 0.4$, aspect ratio = 3.

element with algebraic subgrid scale stabilization, presented in this paper, shows the best coarse mesh accuracy in these examples.

Next, the boundary conditions are changed from 'clamped' to 'soft simple support' (both rotations are free) on two opposite edges. The other edges remain clamped. This modification changes the relationship between bending and twist within the element, which has some implications on the stabilization, as discussed in the previous sections.

Here, both stabilized elements are significantly superior to MITC4. Stenberg's method leads to the best results in this case (see Fig. 5).

Finally, we return to a fully clamped plate and change the aspect ratio of the plate to 3 by setting $\ell_x = 12.0$, $\ell_y = 4.0$. The discretization is not changed, thus each individual element has an aspect ratio of 3 as well. The results are shown in Fig. 6.

The superior performance of the present stabilized element in this example is mainly due to the factors $\frac{l_\xi}{l_\eta}$ and $\frac{l_\eta}{l_\xi}$ in Eq. (24). Without those factors, the elements behave much stiffer.

6. Conclusions

It has been shown, how finite plate elements which perform well in the case of undistorted meshes, like the MITC4 element or the bilinear element based on the DSG method, can be improved with the help of stabilization techniques. In particular, an algebraic subgrid scale stabilization of the four-node DSG element has been presented.

The stabilization is formulated in a way that allows the choice of two different 'free parameters' α_ξ and α_η. These parameters can be used to improve coarse mesh accuracy of the elements, additional to enhancing their stability properties. Their choice is based on material and geometric properties of the individual elements.

It should be mentioned at this point that the values proposed in Eq. (24) are just one of numerous possibilities. Future work in this direction could address more rigorous methods to 'tune' the stabilization parameters on the basis of generic one-element tests, or analytical arguments.

The basic messages of the paper are

- Taking into account mechanical considerations and information (like the element aspect ratio and Poisson's ratio) help to find stabilization parameters that lead to more efficient elements with an improved coarse mesh accuracy.
- The specific format of the stabilization term, with individual stabilization in local ξ- and η-directions, allows for a more flexible design of the stabilization method in order to improve accuracy.

Another interesting fact is that using an ASGS stabilization naturally introduces the possibility to take into account the influence of twist within the stabilization term. This is not possible when stabilization techniques are used that only modify the transverse shear energy.

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