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EVALUATION OF A NEW QUADRILATERAL THIN PLATE BENDING ELEMENT

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SUMMARY

A review of 4-node, 12 degrees-of-freedom quadrilateral elements for thin plates is presented. A new element called DKQ is discussed. The formulation is based on a generalization of the efficient and reliable triangular element DKT presented in References 1 and 2 and on the rectangular element QC presented in Reference 3. These elements are derived using the so-called discrete Krichhoff technique. A detailed numerical evaluation of the behaviour of the DKQ element for the computation of displacements and stresses for thin plate bending problems is presented and discussed. The DKQ element appears to be a simple and reliable 12 degrees-of-freedom thin plate bending element.

INTRODUCTION

Quadrilateral elements are attractive for the discretization of plates of arbitrary shapes, of folded plate structures and of some particular shells like cylindrical shells. The availability of simple, efficient and reliable elements for thin plates and shells represents one of the main characteristics of all finite element computer program libraries for structural analysis purposes. The elements having as degrees-of-freedom (DOF) the so-called engineering degrees-of-freedom at the corner nodes only are particularly attractive. For thin plate bending these DOF are the normal displacement w and the rotations, i.e. θ_x and θ_y around the in-plane local orthogonal axes x and y, respectively. A general quadrilateral plate bending element is shown in Figure 1.

In two recent papers[1,2] a detailed formulation and numerical evaluation of a 9 DOF triangular element called DKT, based on the discrete Krichhoff technique, was presented and discussed. The DKT element appears to be one of the best 9 DOF elements taking into account the following aspects:

1. Simplicity and clarity of the formulation (availability of the interpolation functions and of an explicit formulation without numerical integration).
2. Efficiency in the evaluation of displacements as well as stresses for static problems and of frequencies for dynamic problems.
3. Reliability of the results with respect to element aspect ratio without any aritficial or adjustable parameter for thin and very thin plates.

The purpose of this paper is to present the formulation and the evaluation of a quadrilateral element based on a generalization of the DKT element. The new element is called DKQ

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(discrete Kirchhoff quadrilateral). The evaluation includes the recent assessment criteria suggested by Robinson.[4,5]

The DKQ element belongs to a family of discrete Kirchhoff elements for the linear and nonlinear static and dynamic analysis of thin plate and shell structures which is developed at Université Laval, Canada and Université de Compiègne, France under the guidance of Professor Dhatt and the senior author. References 6 and 7 report some of the research results obtained so far on the subject.

We first review the available rectangular and quadrilateral bending elements having four nodes and the three classical DOF at each node only. Then follows the detailed formulation of the DKQ element (interpolation functions for the normal rotations, stiffness matrix, evaluation of bending moments). The validity and the performance of the element is demonstrated through a series of standard tests involving one or more elements (patch test, behaviour of an element in differential bending, convergence tests on square and rectangular thin plates, influence of distortion). The results for typical displacements and stresses are compared with those obtained using other 12 DOF elements. A practical problem where comparison with experimental results is possible is also considered. More details on the results are included in a report.[53]

REVIEW OF 12 DOF QUADRILATERAL PLATE BENDING ELEMENTS

Several textbooks and research papers published so far contain a more or less exhaustive review of elements for plate bending analysis (see, e.g., References 8, 9 and 10). The present review is restricted to the rectangular and quadrilateral elements having 3 DOF at the four corner nodes only (Figure 1). The review is also limited to the thin plate models where the numerical results

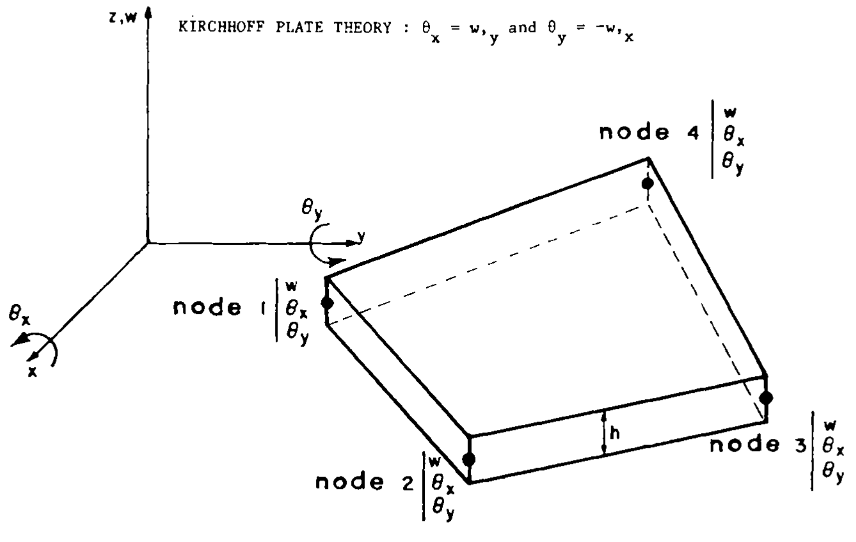


Figure 1. 12 DOF quadrilateral bending element

are associated with the so-called Kirchhoff–Poisson plate theory (12 DOF isoparametric quadrilateral elements for thick plates based on the so-called Mindlin plate theory are available and their formulation is straightforward—see References 11 and 12 among others). Since the final DOF are conventional displacement DOF the various available models are pure displacement models or hybrid (displacement or stress) type models.

Rectangular elements

The rectangular elements were the first elements for plate bending analysis. One of the first element was that of Melosh.[13] The formulation used the analogy between a plate and a system of crossing beams. A more important element using a complete cubic interpolation function plus two quartic terms (of the type x^3y and y^3x), often known as the ACM element,[14–18] is still implanted in several computer codes. It is an incompatible element but it successfully passes the patch test (Gallagher,[8] ref. 12.13, and Olson[19]). Other polynomial expressions have been proposed by Dawe (Gallagher,[8] ref. 12.9). A compatible but incomplete element was presented in References 20 and 18. A compatible rectangular element involving particular polynomial expressions on four subregions has been presented by Deak and Pian.[21]

Various 12 DOF rectangular elements have been proposed by Kikuchi and Ando[22] based on their simplified hybrid displacement functional. For the problems considered in Reference 22, a better convergence compared with the ACM element is reported. The paper by Mang and Gallagher,[23] however, questions the reliability of the numerical results obtained with that simplified functional.

Several rectangular elements based on the hybrid stress functional have been presented by Pian,[24,25] Severn and Taylor[26] and Neale *et al.*[27] They differ by the number of internal stress parameters considered.

A rectangular element called QC was presented by Dhatt.[3] It is based on the introduction of the Kirchhoff assumptions on a discrete manner. The present DKQ element is a generalization of the QC element (with a different presentation). A rectangular element using discrete Kirchhoff assumptions was also studied by Fried,[28] where the transverse shear strain energy was not neglected (in QC it is neglected). The behaviour of the element greatly depends on the value of an adjustable parameter related to the thickness to element length ratio.

Quadrilateral elements

A 12 DOF quadrilateral displacement element using a single polynomial expansion with a C^1 continuity for w does not exist. Quadrilateral elements can be defined through the assemblage of several triangular elements. If more than two subregions are considered, static condensation is used to define the 12 DOF element. Element Q15 in the SAP4[29] computer program results in the assemblage of four HCT,[18] 9 DOF triangular elements. Element Q19[30] is the result of the assemblage of four LCCT-11 triangular elements. Q15 and Q19 are compatible elements, but their formulation is not simple. Their performances appear in a later section of this paper.

Quadrilateral elements based on hybrid stress models have been presented by Allwood and Cornes,[31] and Torbe and Church.[32] Horrigmoe[33,34] uses the element with a linear internal moment distribution for the nonlinear analysis of shells. A great variety of 12 DOF hybrid stress elements has been also presented and discussed by Cook.[35–39] The elements are often obtained through the assemblage of triangular elements and static condensation. The arguments presented in the various papers may question the reliability of the elements for thin arbitrary plate situations.

The difficulty in deriving satisfactory 12 DOF quadrilateral element for thin plates is mainly due to the Kirchhoff plate theory where a C^1 continuity is required for the independent displacement w. One may consider the Mindlin plate theory where a C^0 continuity for the displacement w and the rotations θ_x and θ_y is only required if the convergence for thin plate situations is guaranteed. A 12 DOF quadrilateral element based on the Mindlin plate theory has been presented and tested by several authors like Hughes *et al.*,[11,40] and Hinton *et al.*[12,41,42] Bilinear interpolation functions are considered for w, θ_x and θ_y. Difficulties to obtain accurate results using this low order element for thin plate analysis are due to the fact that the terms of the stiffness matrix associated with the transverse shear energy is of order $0((l/h)^2)$ compared to the terms associated with the bending energy (where l is a characteristic length of the element and h is the thickness). For the problems considered in References 11, 12 and 41–43, reduced integration of the transverse shear strain energy leads to a good behaviour of the element for thickness to length ratios up to 10^4. The formulation of the element is very simple; however, we note that

1. A modification to the standard formulation is required to ensure correct convergence for very thin plate.[11]
2. Two spurious zero energy modes per element exist.
3. The results of some assessment tests presented in References 4 and 5 are not satisfactory.
4. The element is very sensitive to element aspect ratio (distortion).[12]

The QUAD4 element by McNeal[44,45] is an isoparametric 12 DOF quadrilateral element including transverse shear. The standard element is then modified using appropriate reduced and selective integration, and addition of special terms to improve the behaviour in bending. The formulation does not appear to be very simple and the results are dependent upon the value of an adjustable parameter. Good convergence behaviour is reported for square and rectangular plates.

The LORA element presented by Robinson and Haggenmacher,[4,5] is a 9 stress parameters element (four for M_x and M_y and one for M_{xy}). The detailed derivation of the stiffness matrix is not given in Reference 4. The formulation is followed by the presentation of various results dealing with assessment criteria and standard tests. In general, a good behaviour of the element is reported. These results are compared with those obtained using DKQ in a later section.

The elements proposed by Cook,[35–39] Hughes *et al.*,[11,12,40] McNeal[44,45] and Robinson and Haggenmacher[4,5] should work for thick plates since they account for transverse shear deformation. However, the element proposed in this paper will not give the correct answer for thick plates since the transverse shear energy is neglected.

FORMULATION OF THE DKQ ELEMENT

The discrete Kirchhoff technique which was applied successfully to derive the triangular 9 DOF element DKT is considered to formulate the stiffness matrix of the new quadrilateral 12 DOF element (called DKQ). The basic arguments and formulative aspects pertaining to DKQ are given in the next section which obviously present similarities with the presentation given in Reference 1.

Stiffness matrix of the DKQ element

The formulation of the DKT and DKQ elements is based on the discretization of the strain energy where the transverse shear strain energy is neglected, i.e.

$$U = \sum_e U_b^e \quad \text{with} \quad U_b^e = \tfrac{1}{2}\int_{A^e} \langle\chi\rangle[D_b]\{\chi\}\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

U_b^e is the element strain energy due to bending. A^e is the element area. $\{\chi\}$ and $[D_b]$ (for a homogeneous isotropic plate) are given, respectively, by

$$\{\chi\} = \begin{Bmatrix} \partial\beta_x/\partial x \\ \partial\beta_y/\partial y \\ \partial\beta_x/\partial y + \partial\beta_y/\partial x \end{Bmatrix} \qquad [D_b] = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{2a, b}$$

E, ν and h are the Young's modulus, Poisson's ratio and thickness, respectively. β_x and β_y are the rotations of the normal to the undeformed middle surface in the x–z and y–z planes, respectively. U_b^e depends only upon β_x and β_y with a C^0 continuity required.

It is necessary to relate the rotations β_x and β_y to the transverse displacement w in such a way that the final element has the characteristics of a Kirchhoff type element, i.e.

(a) the nodal variables must be the displacement w and its derivatives $\theta_x = w,_y$ and $\theta_y = -w,_x$ with respect to x and y at the four corner nodes (with $w,_x$ standing for $\partial w/\partial x$);
(b) the Kirchhoff assumptions must be verified along the boundaries of the element in order to satisfy the Kirchhoff boundary conditions.

The formulation of the DKQ element is thus based on the following considerations:

1. β_x and β_y are defined by incomplete cubic polynomial expressions:

$$\beta_x = \sum_{i=1}^{8} N_i \beta_{x_i} \qquad \beta_y = \sum_{i=1}^{8} N_i \beta_{y_i} \tag{3a, b}$$

The shape functions $N_i(\xi, \eta)$, $i = 1, 8$ where ξ and η are parametric co-ordinates[9,46] are those of the 8-node Serendipity element.[9] β_{x_i} and β_{y_i} are transitory nodal variables affected at the corner and mid-nodes of the quadrilateral element (with straight sides) (Figure 2).

2. The Kirchhoff assumptions are introduced

(a) at the corner nodes:

$$\begin{Bmatrix} \beta_{x_i} + w,_{x_i} \\ \beta_{y_i} + w,_{y_i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad i = 1, 2, 3, 4 \tag{4}$$

(b) at the mid-nodes:

$$\beta_{s_k} + w,_{s_k} = 0 \qquad k = 5, 6, 7, 8 \tag{5}$$

where s represents the co-ordinate along the element boundary.

3. $w,_{s_k}$ is the derivative of the transverse displacement w with respect to s at the mid-node k, where w is defined by a cubic expression along each element side, i.e.

$$w,_{s_k} = \frac{-3}{2l_{ij}}(w_i - w_j) - \tfrac{1}{4}(w,_{s_i} + w,_{s_j}) \tag{6}$$

where $k = 5, 6, 7, 8$ is the mid-node of the sides $ij = 12, 23, 34, 41$, respectively. l_{ij} represents the length of the side ij (Figure 2).

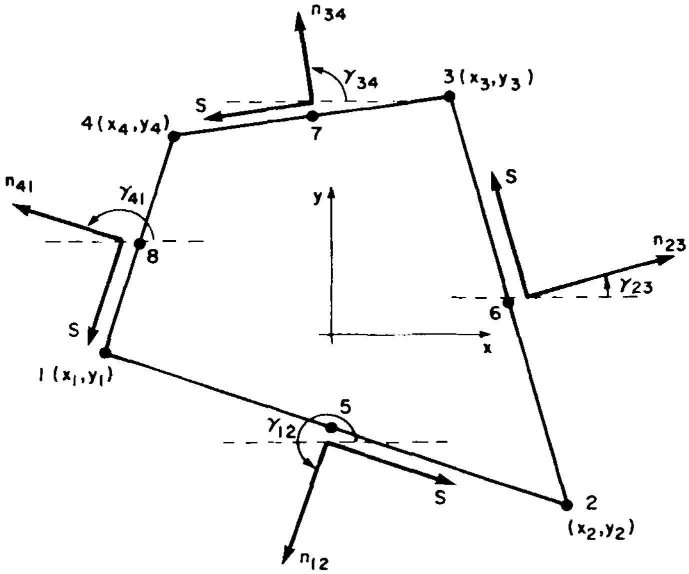
4. β_n varies linearly along the sides, i.e.

$$\beta_{n_k} = \tfrac{1}{2}(\beta_{n_i} + \beta_{n_j}) = -\tfrac{1}{2}(w,_{n_i} + w,_{n_j}) \tag{7}$$

where $k = 5, 6, 7, 8$ corresponds to the mid-node of the sides 12, 23, 34, 41.

We note that

(a) w is not defined in the interior of the element. w varies independently along the element sides. The nodal variable w at the four corner nodes appears through equation (6).



$x_{ij} = x_i - x_j$ $\quad i = 1,2,3,4$ et $j = 2,3,4,1$

$y_{ij} = y_i - y_j$

$\ell_{ij} = (x_{ij}^2 + y_{ij}^2)^{1/2}$ $\quad ij = 12,23,34,41$

$\gamma_{ij} = (x, n_{ij})$

$C = \cos \gamma_{ij} = -y_{ij}/\ell_{ij}$

$S = \sin \gamma_{ij} = x_{ij}/\ell_{ij}$

Figure 2. Geometry of the DKQ element

(b) The Kirchhoff assumptions are satisfied along the *entire* boundary of the element since $w_{,s}$ and β_s are both quadratic expressions along the element sides.
(c) Convergence towards the theory of thin plates is obtained for any element length to thickness ratio since the transverse shear energy is neglected (thus the DKQ element is not appropriate for thick plates but other simple elements exist for such situations).
(d) The classical beam bending element with two nodes and 4 DOF is obtained if nodes 4 and 3 are eliminated.
(e) The theoretical studies on error estimates and convergence properties, as established by Kikuchi[47] and Fried,[28,48] are valid for the DKQ element. The error in the evaluation of U_b^e is of order $0(l^2)$.
(f) The 12 DOF DKQ elements are such that w, $w_{,s}$, β_x, β_y and $w_{,n}$ are *compatible* along the element sides.

The *explicit* expression of the rotations β_x and β_y of a general quadrilateral in terms of the final DKQ nodal variables:

$$\langle U_n \rangle = \langle w_1 \ \theta_{x_1} \ \theta_{y_1} \ w_2 \ \theta_{x_2} \ \theta_{y_2} \ w_3 \ \theta_{x_3} \ \theta_{y_3} \ w_4 \ \theta_{x_4} \ \theta_{y_4} \rangle \tag{8}$$

with $\theta_{x_i} = w_{,y_i}$ and $\theta_{y_i} = -w_{,x_i}$ $i = 1, 2, 3, 4$ is obtained using equations (3)–(8):

$$\beta_x = \langle H^x(\xi, \eta) \rangle \{U_n\} \tag{9a}$$

$$\beta_y = \langle H^y(\xi, \eta) \rangle \{U_n\} \tag{9b}$$

with

$$
\begin{aligned}
\langle H^x \rangle &= \langle H_1^x \ldots H_{12}^x \rangle \\
\langle H^y \rangle &= \langle H_1^y \ldots H_{12}^y \rangle \\
H_1^x &= \tfrac{3}{2}(a_5 N_5 - a_8 N_8) \\
H_2^x &= b_5 N_b + b_8 N_8 \qquad (10) \\
H_3^x &= N_1 - c_5 N_5 - C_8 N_8 \\
H_1^y &= \tfrac{3}{2}(d_5 N_5 - d_8 N_8) \\
H_2^y &= -N_1 + e_5 N_5 + e_8 N_8 \qquad (11) \\
H_3^y &= -b_5 N_5 - b_8 N_8 = -H_2^x
\end{aligned}
$$

The functions $H_4^x, H_5^x, H_6^x, H_4^y, H_5^y, H_6^y$ are obtained from the above expressions by replacing N_1 by N_2 and indices 8 and 5 by 5 and 6, respectively. The functions H_7^x, H_8^x, H_9^x, H_7^y, H_8^y, H_9^y are obtained from the above expressions by replacing N_1 by N_3 and indices 8 and 5 by 6 and 7, respectively. The functions H_{10}^x, H_{11}^x, H_{12}^x, H_{10}^y, H_{11}^y, H_{12}^y are obtained from the above expressions by replacing N_1 by N_4 and indices 8 and 5 by 7 and 8, respectively.

$$
\begin{aligned}
a_k &= -\frac{x_{ij}}{l_{ij}^2} \qquad b_k = \tfrac{3}{4} x_{ij} y_{ij} / l_{ij}^2 \\
c_k &= (\tfrac{1}{4} x_{ij}^2 - \tfrac{1}{2} y_{ij}^2)/l_{ij}^2 \qquad d_k = -y_{ij}/l_{ij}^2 \qquad (12) \\
e_k &= (-\tfrac{1}{2} x_{ij}^2 + \tfrac{1}{4} y_{ij}^2)/l_{ij}^2
\end{aligned}
$$

with $k = 5, 6, 7, 8$ for the sides $ij = 12, 23, 34, 41$, where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ and $l_{ij}^2 = x_{ij}^2 + y_{ij}^2$. Details in obtaining $\langle H^x \rangle$ are presented in Appendix I.

Equations (2) and (9) give

$$\{\chi\} = [B]\{U_n\}$$

with

$$
[B] = \begin{bmatrix} \langle H^x{}_{,x} \rangle \\ \langle H^y{}_{,y} \rangle \\ \langle H^x{}_{,y} + H^y{}_{,x} \rangle \end{bmatrix} = \begin{bmatrix} j_{11}\langle H^x{}_{,\xi}\rangle + j_{12}\langle H^x{}_{,\eta}\rangle \\ j_{21}\langle H^y{}_{,\xi}\rangle + j_{22}\langle H^y{}_{,\eta}\rangle \\ j_{11}\langle H^y{}_{,\xi}\rangle + j_{12}\langle H^y{}_{,\eta}\rangle + j_{21}\langle H^x{}_{,\xi}\rangle + j_{22}\langle H^x{}_{,\eta}\rangle \end{bmatrix} \qquad (13)
$$

where j_{11}, j_{12}, j_{21}, j_{22} are the components of the inverse of the Jacobian matrix $[J]$ of the transformation between the parent and the actual element.[46]

$$
[J] = \frac{1}{4}\begin{bmatrix} x_{21} + x_{34} + \eta(x_{12} + x_{34}) & y_{21} + y_{34} + \eta(y_{12} + y_{34}) \\ x_{32} + x_{41} + \xi(x_{12} + x_{34}) & y_{32} + y_{41} + \xi(y_{12} + y_{34}) \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad (14)
$$

We then have

$$
\begin{aligned}
j_{11} &= \frac{1}{\det[J]} J_{22} \qquad j_{12} = \frac{-1}{\det[J]} J_{12} \\
j_{21} &= \frac{-1}{\det[J]} J_{21} \qquad j_{22} = \frac{1}{\det[J]} J_{22}
\end{aligned} \qquad (15)
$$

$$
\det[J] = \frac{1}{8}(y_{42}x_{31} - y_{31}x_{42}) + \frac{\xi}{8}(y_{34}x_{21} - y_{21}x_{34}) + \frac{\eta}{8}(y_{41}x_{32} - y_{32}x_{41})
$$

The components of $\langle H^x m_\xi\rangle$, $\langle H^x,_\eta\rangle$, $\langle H^y,_\xi\rangle$ and $\langle H^y,_\eta\rangle$ can easily be expressed in terms of $N_{i,\xi}$ and $N_{i,\eta}$.

The stiffness matrix of the DKQ element is defined in a standard manner for displacement models as

$$[k^e]=\int_{A^e}[B]^{\mathrm{T}}[D_b][B]\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{-1}^{+1}\int_{-1}^{+1}[B]^{\mathrm{T}}[D_b][B]\det[J]\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{16}$$

A standard numerical integration scheme using 2×2 Gauss integration points have been found sufficient for the integration. It has been checked that only *three* zero energy modes per element are obtained for square and distorted elements and for length l to thickness h ratios ranging from 50 to 10^6 and using the 2×2 integration points. A 3×3 points scheme is theoretically necessary to integrate exactly on a rectangular element. The 2×2 scheme should not be understood here as a reduced integration scheme as proposed in several formulations to improve the element behaviours—see References 12, 42 and 43 among others.

Evaluation of bending moments and load vector for uniform loading

Once the 12 nodal displacements $\{U_n\}$ are known, the bending moments $\{M\}$ at any point x, y in the element are obtained: using equations (5a) and (18):

$$\{M(x,y)\}=[D_b][B(x,y)]\{U_n\} \tag{17}$$

with

$$\{M\}=\begin{Bmatrix} M_x \\ M_y \\ M_{xy}\end{Bmatrix}$$

In general, different values of bending moments are obtained along the element sides.

The external potential energy for a uniform loading of intensity $\bar{p}_z$ over an element is defined as

$$W^e=\bar{p}_z\int_{A^e} w\,\mathrm{d}x\,\mathrm{d}y=\langle U_n\rangle\{f^e\} \tag{18}$$

No interpolation function for w has been introduced for the derivation of the stiffness matrix. One can define a *simple* load vector $\{f_s^e\}$ by considering a linear interpolation for w over the element. For rectangular elements this gives

$$\langle f_s^e\rangle=\bar{p}_z\left\langle \frac{A^e}{4}\ 0\ 0\ \frac{A^e}{4}\ 0\ 0\ \frac{A^e}{4}\ 0\ 0\ \frac{A^e}{4}\ 0\ 0\right\rangle \tag{19}$$

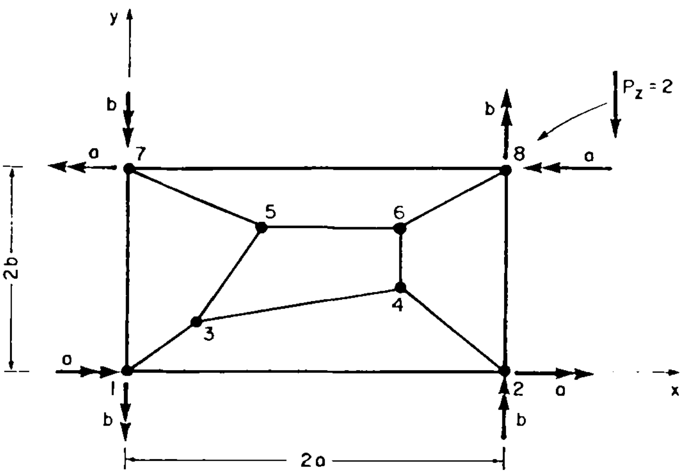
where A^e is the area of the element. In the case of a linear interpolation function, the error in the evaluation of potential energy and strain energy will be of the same order, i.e. $0(l^2)$. Once can also consider a cubic interpolation function for w to define a more *complete* load vector $\{f_c^e\}$. A cubic interpolation for w is more consistent with the derivation of the stiffness matrix since a cubic variation of w along the sides has been considered. The cubic polynomial will be, in fact, a complete cubic expansion plus the terms $\xi^3\eta$ and $\eta^3\xi$ in parametric co-ordinates. For a general quadrilateral a numerical integration is performed to obtain either $\{f_s^e\}$ or $\{f_c^e\}$.

ASSESSMENT TESTS AND COMPARATIVE RESULTS

In this section a presentation of the results of several classical and practical tests is proposed. The results are compared with most of the results the authors were able to extract from the literature and dealing *exclusively* with 12 DOF plate bending elements. Additional results are given in Reference 53. The subroutines associated with the DKQ element were made compatible with the use of the MEF[46] computer program and the computations were performed in double precision on a DEC-VAX 11/780 machine.

Patch test

Following the suggestions of Robinson[4,5] for the evaluation of low order plate bending elements, the patch test problem as defined in Figure 3 is considered. Five DKQ elements are assembled and subjected to a set of concentrated load and boundary conditions that lead theoretically to a uniform constant state of stresses on the rectangular plate. For various values of Poisson's ratio and positions of nodes 3, 4, 5, 6, M_x, M_y and M_{xy} are found equal to unity everywhere on the plate. This confirms the compatibility of the DKQ element. The problem considered in Reference 1, § 4.2.2, figure 16, is a special patch test problem and is also solved with success for various meshes involving the DKQ elements.[53]



Boundary conditions: $w = 0$ at nodes 1, 2, 7

Loading: $M_x = b$ at nodes 2, 8
$M_x = -b$ at nodes 1, 7
$M_y = a$ at nodes 1, 2
$M_u = -a$ at nodes 7, 8
$P_z = -2$ at nodes 8

Results: $M_x = M_y = M_{xy} = 1$ everywhere
(for $a = 20$, $b = 10$, $E = 1000$, $h = 1$, $w_8 = 12{\cdot}48$ with $\nu = 0.3$ and $w_8 = 9{\cdot}6$ for $\nu = 0$)

Figure 3. Patch test problem

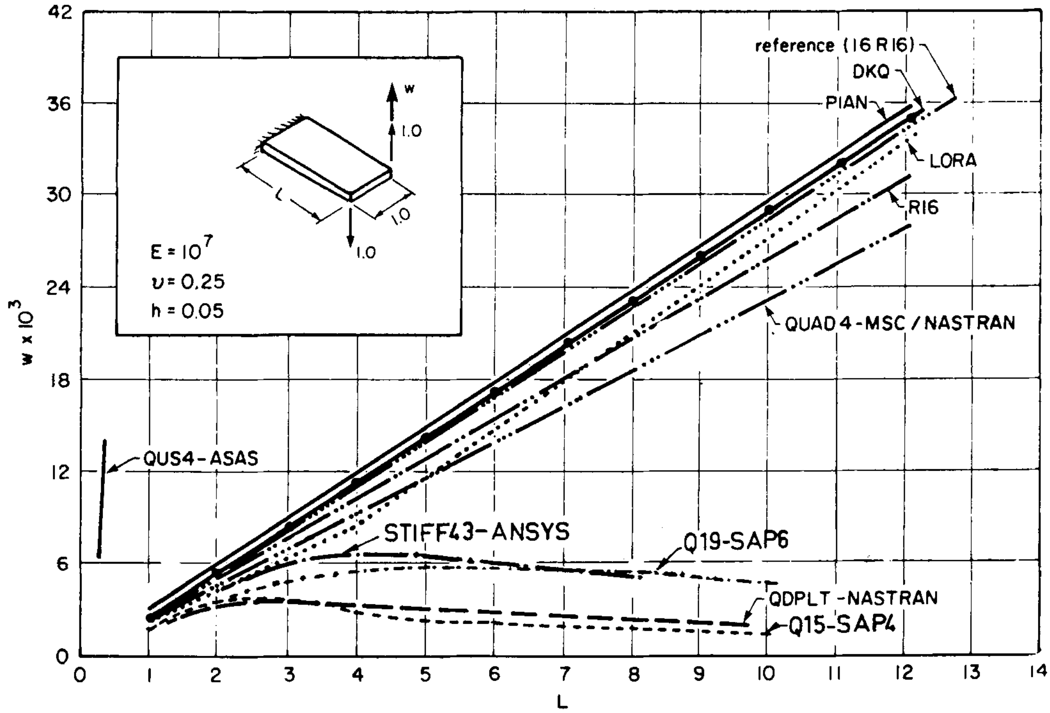


Figure 4. Performances of various elements for test A

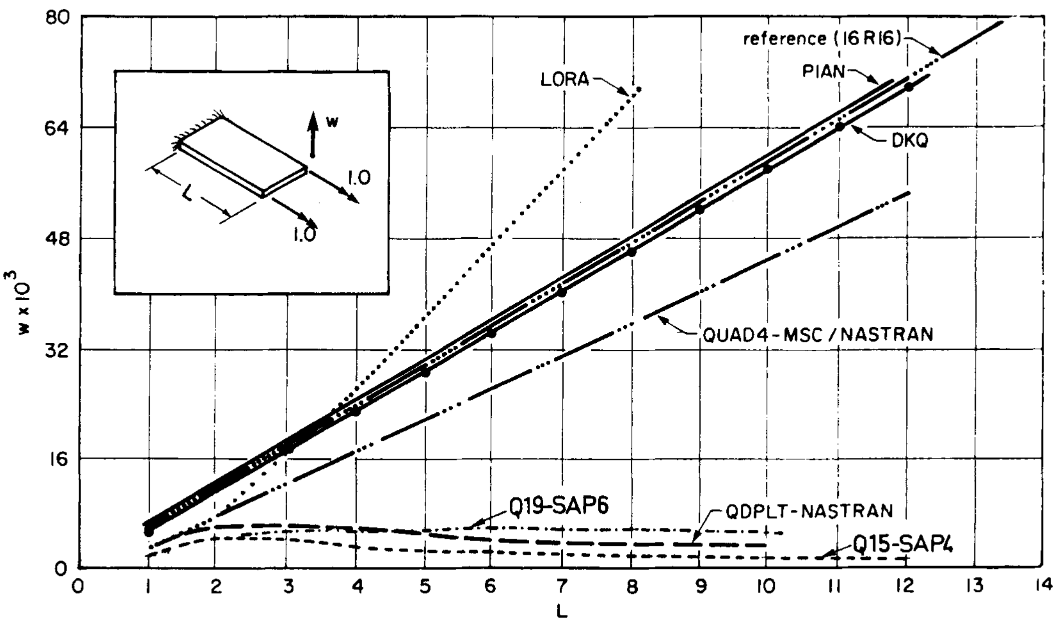
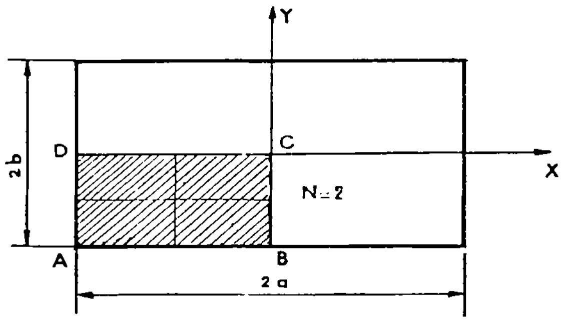


Figure 5. Performances of various elements for test B



ELEMENT	REFERENCES (Formulation or results)	SIGN	SHAPE	MODEL
LORA	4, 5	○	Quad.	equilibrium (hybrid)
QUAD4	44, 45, 4, 5	⊓	Quad.	displacement
QUS4	11, 12, 4, 5	●	Quad.	displacement
QDPLT	4, 5	▼	Quad. (4 triangles)	displacement*
DKT	this study	■	Quad.	displacement
QC	3	■	Rect.	displacement
DP	21		Rect. (4 triangles)	displacement
PIAN	24, 25, 4, 5	▲	Rect.	hybrid stress
H5, H9, (HTC)	37, 35, 36	◑	Quad. (4 triangles)	hybrid stress
ACM	14, 18, 51	△	Rect.	displacement*
M	15, 30, 51	▽	Rect.	displacement*
KQO	52		Quad.	displacement
KA (R2)	22	✱	Rect.	hybrid displacement
Q19	30, 51	◧	Quad. (4 triangles)	displacement
HCR	27		Rect.	hybrid stress
AC9, AC27	31		Quad.	hybrid stress

* non compatible

Figure 6. 12 DOF rectangular and quadrilateral elements

Cantilever rectangular plate under twisting loads

Two simple problems involving a single element are proposed by Robinson[4,5] to assess the plate bending elements in a situation of differential bending. A single element is clamped along one side (Figures 4 and 5). The data are: width = 1, $h = 0{\cdot}05$, $E = 10^7$, $\nu = 0{\cdot}25$. Two sets of concentrated loads are considered. In test A, two concentrated loads in opposite directions along z are applied at the two free nodes. The problems consist in evaluating the value of w at a free node in terms of the increasing length L of the element. In test B, two twisting couples ($M_y = 1$) are applied at the free corner nodes.

The results obtained using DKQ are reported in Figures 4 and 5. The results using Q15 (four HCT elements[18] in SAP 4[29]) and Q19 (four LCCT-11 elements[30] in SAP 6[49]) were also obtained and are reported. The reference values associated with the Kirchhoff thin plate theory is obtained using 16 compatible rectangular elements with 16 DOF[20] and are taken from

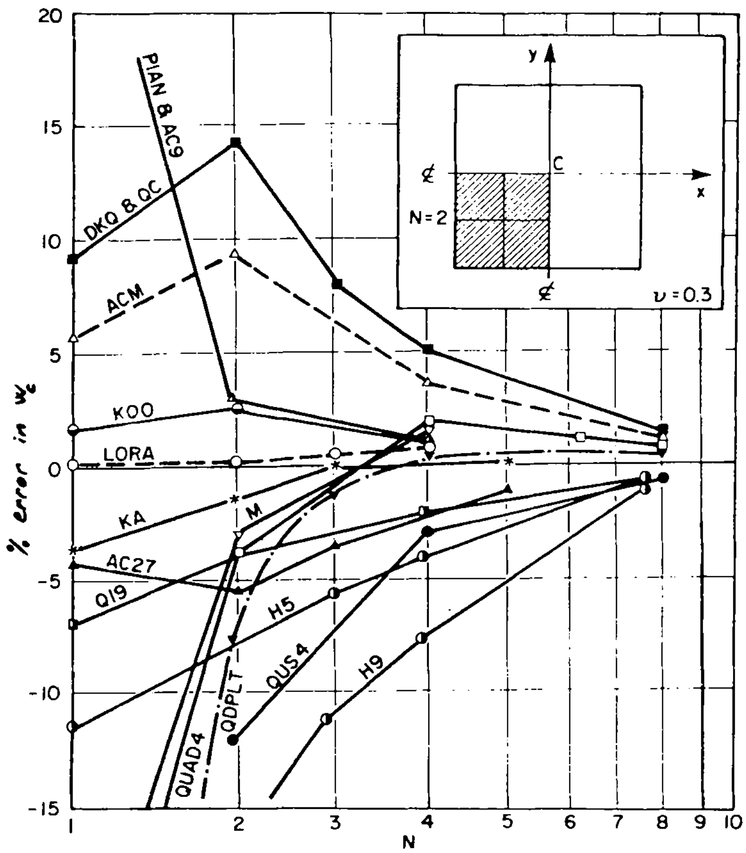


Figure 7. Clamped plate ($b/a = 1$). Concentrated load; central displacement

References 4 and 5. The results reported in Figures 4 and 5 dealing with the rectangular elements PIAN,[24,25] ACM (STIFF43-ANSYS)[14,16,18] and R16[20] and the quadrilateral elements LORA,[5] QUAD4-MSC/NASTRAN,[44] QUS4-ASAS,[11] and QDPLT-NASTRAN are taken directly from References 4 and 5.

It can be observed that the DKQ element like the DKT element[2] has an expremely good behaviour for the element aspect ratios considered. It performs better than the recent elements QUS4, LORA and QUAD4.

Evaluation of displacements and stresses for rectangular plates

Square and rectangular clamped and simply-supported plates with aspect ratios of 1, 2 and 3 and subjected to a concentrated central load and a uniform loading are analysed using the DKQ elements. Regular meshes on a quarter-plate with $N = 1, 2, 3, 4$ and 8 elements per mid-side are considered (Figure 6). The results (displacement w at the centre and stresses at various

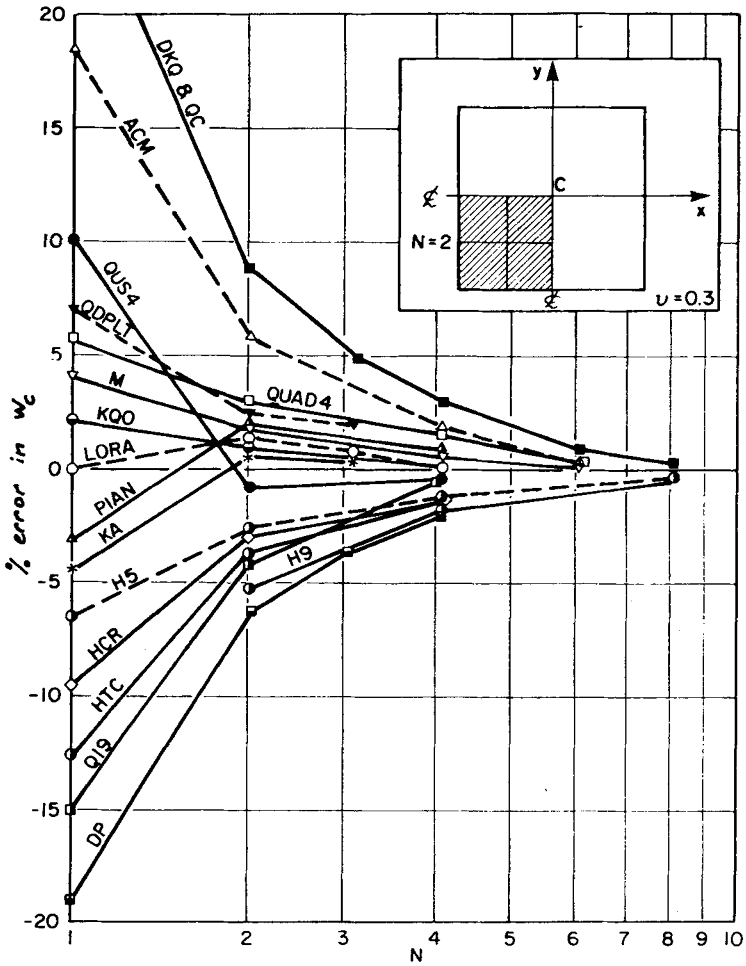


Figure 8. Simply-supported plate ($b/a = 1$). Concentrated load; central displacement

locations) are compared with different results taken from a great number of papers and involving rectangular and quadrilateral elements with 12 DOF only (Figure 6). The reference solutions are taken from Reference 50.

Figures 7–12 deal with the percentage of error on the central displacement versus the number of elements per mid-side for the various geometries, loading and boundary conditions. Figures

13–16 deal with the percentage of error on some significant bending M_x, M_y or twisting moments M_{xy} (corner reaction).

The convergence rates obtained with DKQ for the central displacement are, in general, less good than those reported for QUAD4 or LORA and, in general, no monotonic convergence is observed for these three elements. We note, however, that the convergence rate does not deteriorate with increasing aspect ratio (Figures 11 and 12).

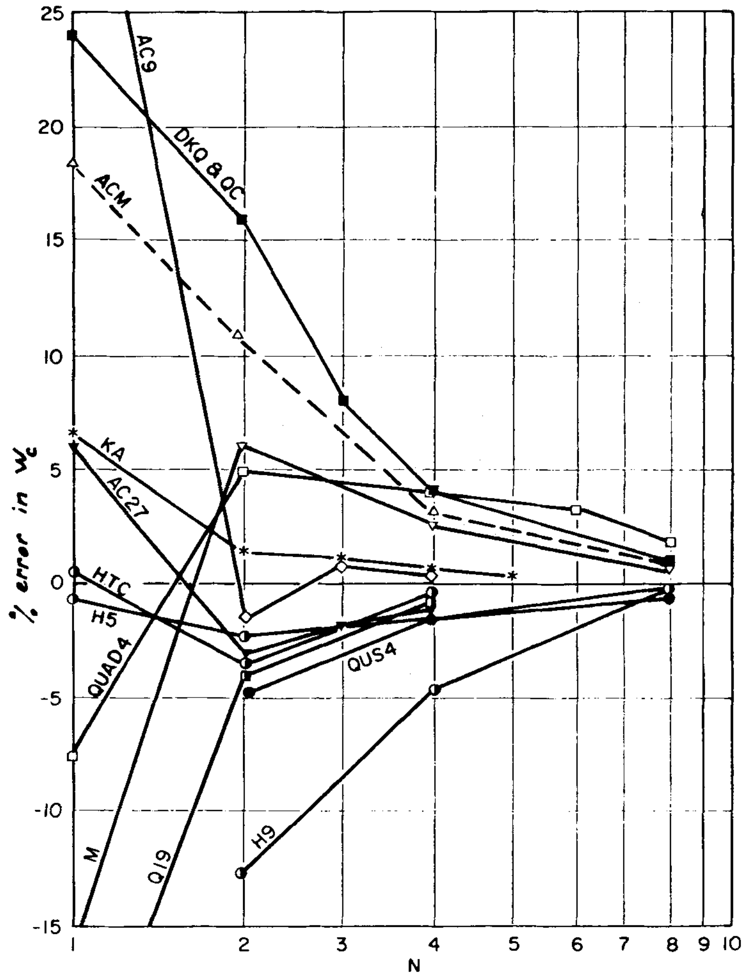


Figure 9. Clamped square plate. Uniform loading; central displacement

Not many papers include results dealing with the evaluation of stresses, even if they present equilibrium or hybrid stress elements. The stress results obtained with DKQ and reported in Figures 14–17 and in Reference 53 are found satisfactory. They have been computed directly at the corner nodes.

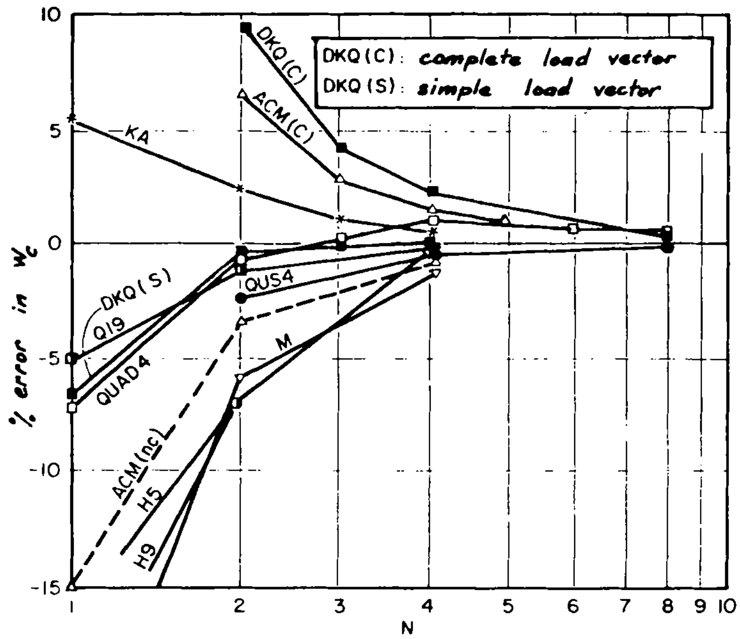


Figure 10. Simply-supported square plate. Uniform loading; central displacement

Analysis of a curved slab

The last example presented here deals with the analysis of a circular slab simply supported along two straight sides and free along the two curved sides. The data are given in Figure 18. This problem was considered by Coull and Das,[54] who presented analytical and experimental results, and Allwood and Cornes,[31] who analysed the plate with 12 DOF hybrid stress quadrilateral elements. The plate is subjected to a concentrated load on the symmetry plane but at three different locations (Figure 18).

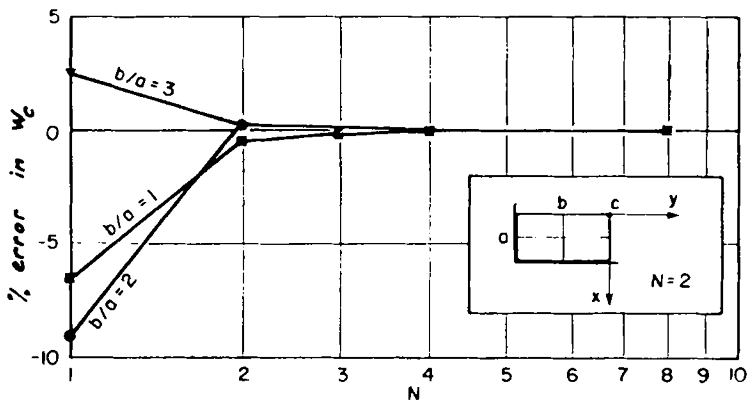


Figure 11. Simply-supported plate with uniform loading. Influence of aspect ratio b/a

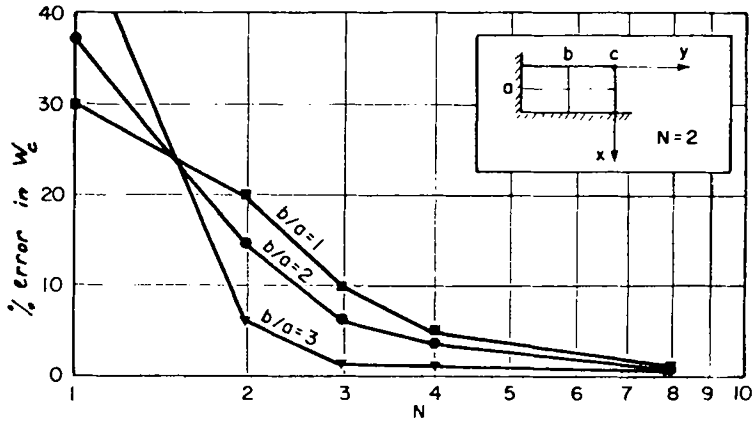


Figure 12. Clamped plate with uniform loading. Influence of aspect ratio b/a

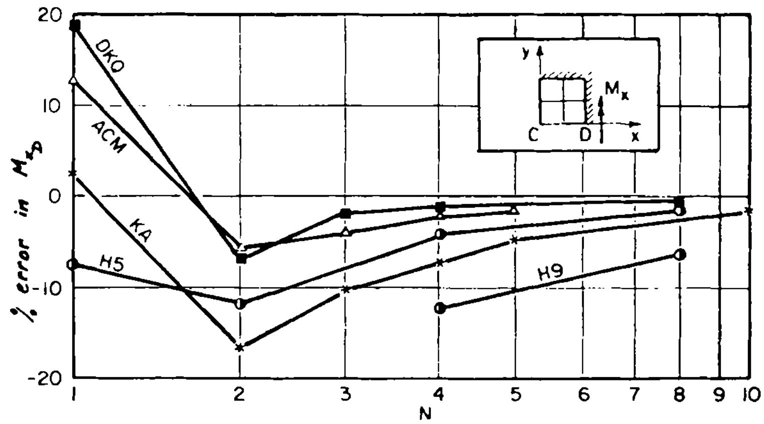


Figure 13. Clamped plate. Concentrated load; error in M_x at mid-side

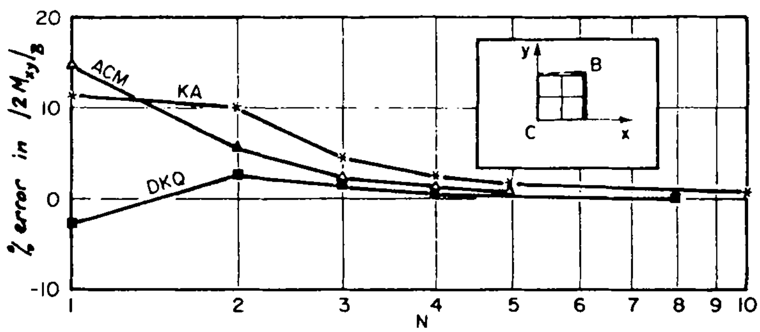


Figure 14. Simply-supported plate. Concentrated load; error in corner reaction

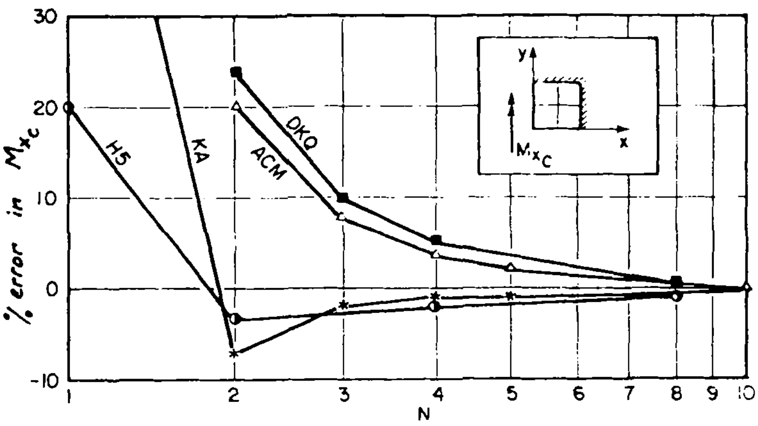


Figure 15. Clamped square plate. Uniform loading; error in bending moment at centre

Half of the plate was discretized, as in Reference 31, in 30 regular elements. Figures 19 and 20 deal with the variation of normal displacement and bending moments M_t and M_r between the ABC line, respectively, for the three positions of the load. The results obtained using the DKQ elements are in good agreement with the experimental results for the three loading cases with greater differences for the stresses than for the displacements. Results using DKQ and hybrid stress elements[31] are almost identical.

CONCLUSION

In this paper, the details of the formulation of a new quadrilateral 12 DOF element for the bending analysis of thin plates are presented. The element is obtained by generalizing the method used to derive a 9 DOF triangular element called DKT in References 1 and 2. Both

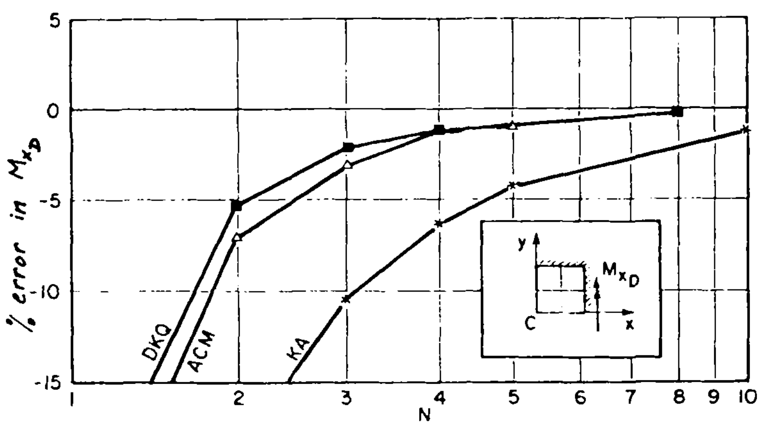
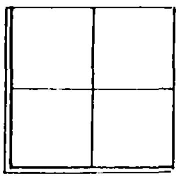
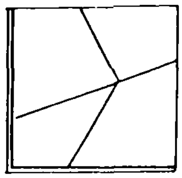
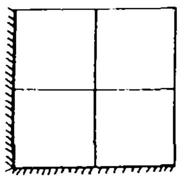
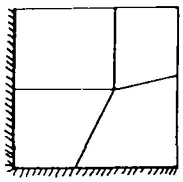
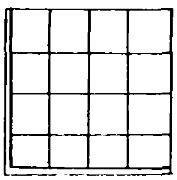
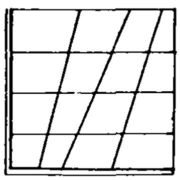


Figure 16. Clamped plate. Uniform loading; error in bending moment at mid-side

Mesh		Difference in w_c for irregular mesh compared to regular mesh
regular	irregular	
C 		SIMPLY SUPPORTED PLATE - uniform loading DKQ* : 1.05 % QUS4 [12] : 32 %
		CLAMPED PLATE - concentrated central load DKQ : 0.17 % AC9 [31] : 0.5 % AC27 [31] : 0.75 %
		SIMPLY SUPPORTED PLATE - concentrated central load DKQ : -0.5 % QUAD4 [44] : 4.6 % - uniform loading DKQ* : 0.45 % QUAD4 [44] : 6.6 %

ig a complete load vector $\{f_c^e\}$

Figure 17. Influence of mesh distortions on various 12 DOF elements

elements are based on the introduction of discrete Kirchhoff assumptions in a particular manner along the elements sides. An *explicit* expression of the rotations of the normal in terms of the 12 DOF is given. A detailed numerical evaluation is presenteed where the results (displacements and stresses) obtained using DKQ and many others 12 DOF elements are compared.

It is concluded that

1. The DKQ element is a compatible element without any spurious zero energy mode
2. Solutions in agreement with the Kirchhoff thin plate theory are obtained for any length to thickness ratio but without monotonic convergence.
3. Excellent results are observed for the tests proposed by Robinson[4,5] and involving a single element.
4. The DKQ element is not very sensitive to the elements distortions.
5. Convergence rates in displacements and stresses for the square and rectangular plates are satisfactory but are in general not as good as those reported for LORA[4] and QUAD4[44] for the displacements.

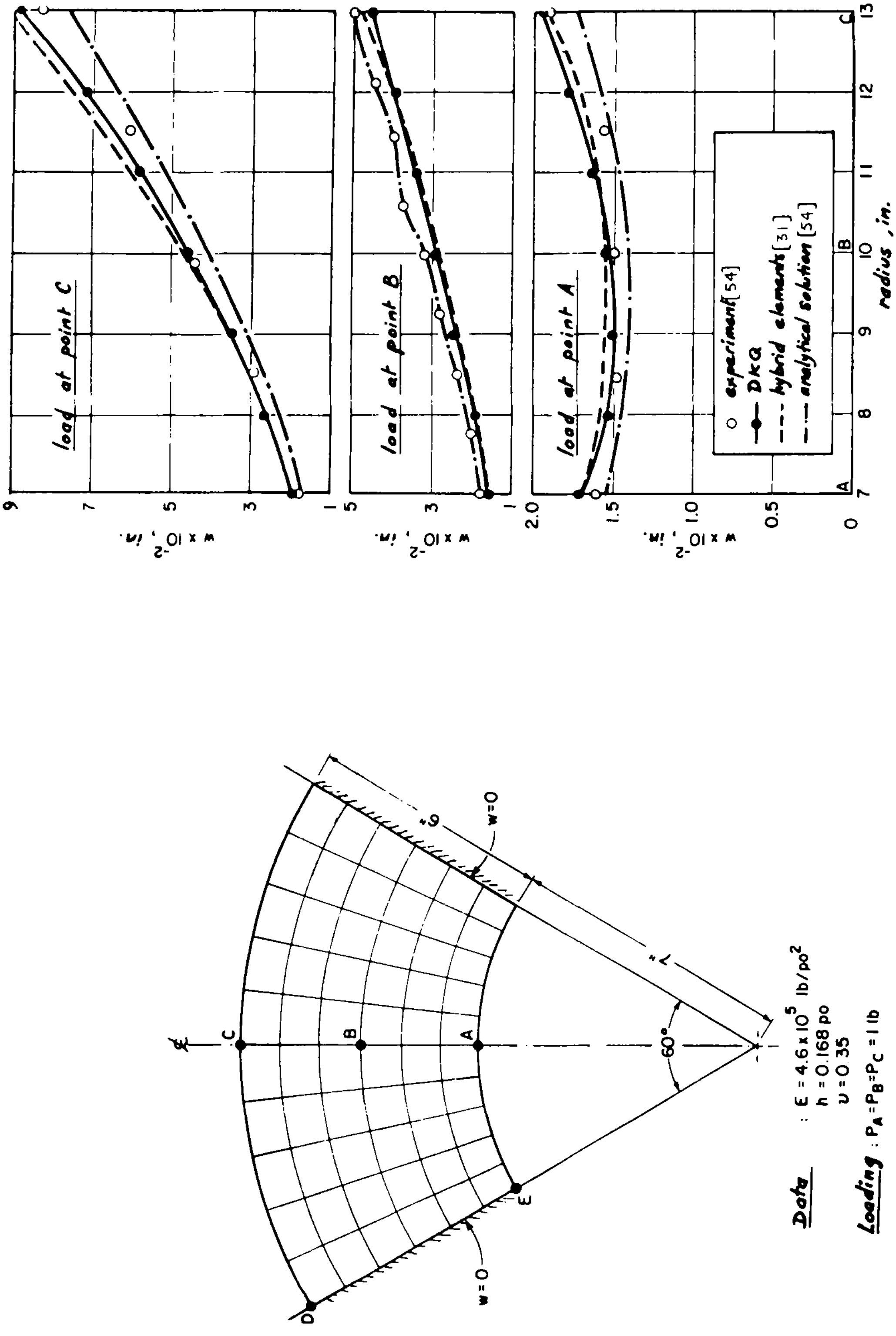


Figure 18. Curved slab. Data and mesh

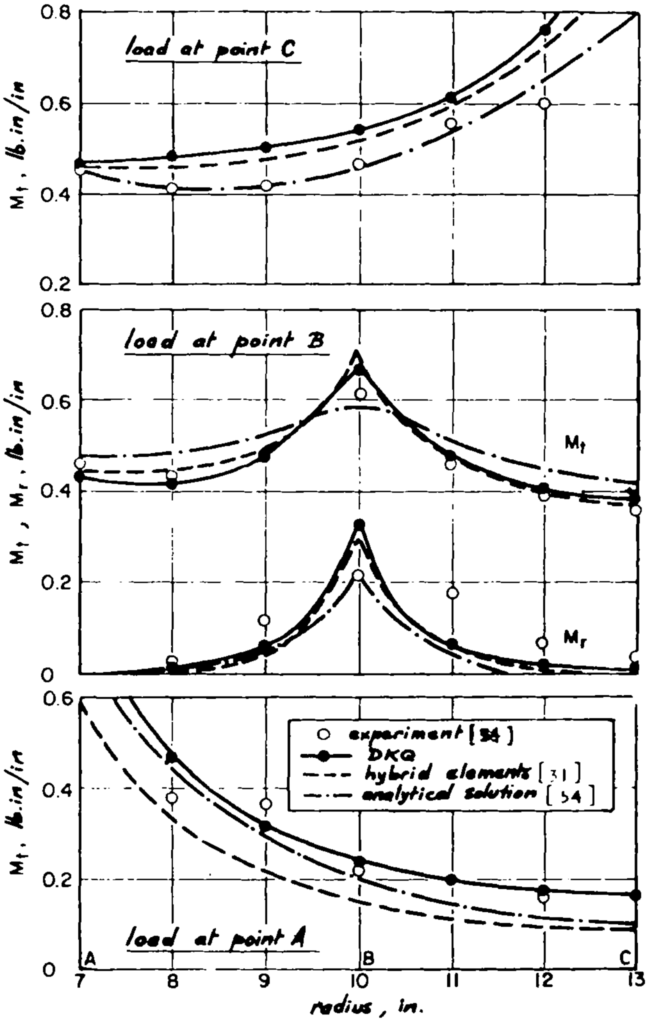


Figure 20. Curved slab. Bending moments along ABC

We emphasize that the construction of the stiffness matrix is direct and simple and follows the standard procedure for isoparametric membrane displacement elements.

From our experience it is found that the DKQ element is a simple and reliable engineering element for the analysis of thin plates of arbitrary shape.

ACKNOWLEDGEMENTS

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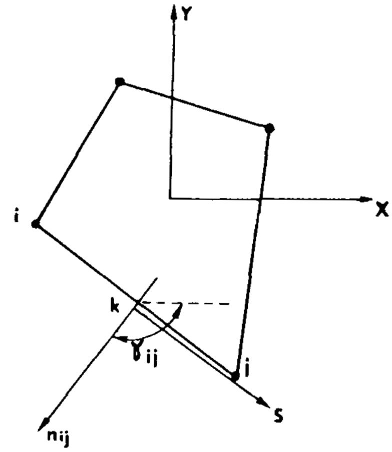
APPENDIX I: EXPLICIT INTERPOLATION FUNCTION FOR β_x

Let $\gamma_{ij} = \gamma_k = (\mathbf{x}, \mathbf{n}_{ij})$ and $\cos\gamma_k = C_k$ and $\sin\gamma_k = S_k$. Then

$$\begin{Bmatrix}\beta_x\\\beta_y\end{Bmatrix} = \begin{bmatrix}\cos\gamma_{ij} & -\sin\gamma_{ij}\\ \sin\gamma_{ij} & \cos\gamma_{ij}\end{bmatrix}\begin{Bmatrix}\beta_n\\\beta_k\end{Bmatrix} = \begin{bmatrix}C_k & -S_k\\ S_k & C_k\end{bmatrix}\begin{Bmatrix}\beta_n\\\beta_s\end{Bmatrix}_k \tag{20}$$



The quadratic variation of β_x and equation (4) give

$$\beta_x = \sum_{i=1}^{8} N_i B_{x_i} = -\sum_{i=1}^{4} N_i w,_{x_i} + \sum_{k=5}^{8} N_k \beta_{x_k}$$

Equations (20) and (5) give

$$\begin{aligned}\beta_x &= -\sum_{i=1}^{4} N_i w,_{x_i} + \sum_{k=5}^{8} N_k C_k \beta_{n_k} - \sum_{k=5}^{8} N_k S_k \beta_{s_k}\\ &= -\sum_{i=1}^{4} N_i w,_{x_i} + \sum_{k=5}^{8} N_k C_k \beta_{n_k} + \sum_{k=5}^{8} N_k S_k w,_{s_k}\end{aligned}$$

Equations (6) and (7) give

$$\begin{aligned}\beta_x = &-\sum_{i=1}^{4} N_i w,_{x_i} - \tfrac{1}{2}\sum_{k=5}^{8} N_k C_k (w,_{n_i} + w,_{n_j})\\ &-\sum_{k=5}^{8} N_k S_k\left(\frac{3}{2l_{ij}}(w_i - w_j) + \tfrac{1}{4}(w,_{s_i} + w,_{s_j})\right)\end{aligned}$$

Using equation (20) and $(w,_x)_i = -\theta_{y_i}$ and $(w,_y)_i = \theta_{x_i}$:

$$\begin{aligned}\beta_x = &\sum_{i=1}^{4} N_i \theta_{y_i} - \tfrac{1}{2}\sum_{k=5}^{8} N_k C_k (S_k\theta_{x_i} - C_k\theta_{y_i} + S_k\theta_{x_j} - C_k\theta_{y_j})\\ &-\sum_{k=5}^{8} N_k S_k \frac{3}{2l_{ij}}(w_i - w_j)\\ &-\sum_{k=5}^{8} N_k S_k \tfrac{1}{4}(C_k\theta_{x_i} + S_k\theta_{y_i} + C_k\theta_{x_j} + S_k\theta_{y_j})\end{aligned}$$

The above can be expressed in matrix form as follows:

$$\beta_x = \langle H_1^x \dots H_{12}^x \rangle \{U_n\} = \langle H^x \rangle \{U_n\}$$

with U_n defined by equation (8).

By defining

$$a_k = -S_k / l_{ij} \qquad b_k = -\tfrac{3}{4} C_k S_k \qquad c_k = -\tfrac{1}{2} C_k^2 - \tfrac{1}{4} S_k^2$$

with $ij = 12, 23, 34, 41$ for $k = 5, 6, 7, 8$ we obtain equation (10).

$$H_1^x = \tfrac{2}{3} N_5 a_5 - N_8 a_8$$

$$H_2^x = N_5 b_5 + N_8 b_8$$

$$H_3^x = N_1 - N_5 c_5 - N_8 c_8$$

For H_4^x, H_5^x, H_6^x replace 1 by 2, 5 by 6 and 8 by 5.
For H_7^x, H_8^x, H_9^x replace 1 by 3, 5 by 7 and 8 by 6.
For H_{10}^x, H_{11}^x, H_{12}^x replace 1 by 4, 5 by 8 and 8 by 7.

REFERENCES

1. J. L. Batoz, K. J. Bathe and L. W. Ho, 'A study of three-node triangular plate bending elements', *Int. J. num. Meth. Engng*, **15**, 1771–1812 (1980).
2. J. L. Batoz, 'An explicit formulation for an efficient triangular plate bending element', *Int. J. Num. meth. Engng* (to appear).
3. G. Dhatt and S. Venkatasubbu, 'Finite element analysis of containment vessels', Proc. First Int. Conf. on Struct. Mech. in Reactor Tech., Berlin, Germany, vol. 5, paper J. 3/6 (Sept. 1971).
4. J. Robinson and G. Haggenmacher, 'Lora—an accurate four nodes stress plate bending element', *Int. J. num. Meth. Engng*, **14** (2), 296–306 (1979).
5. J. Robinson, 'Element evaluation. A set of assessment points and standard tests', Proc. F.E.M. in the Commercial Environment, vol. 1, pp. 217–248 (Oct. 1978).
6. J. L. Batoz and G. S. Dhatt, 'An evaluation of two simple and effective triangular and quadrilateral plate bending elements', Third World Congress on Finite Elements in Commercial Environment, Los Angeles, pp. K1–K17 (Oct. 1981).
7. J. L. Batoz and G. S. Dhatt, 'Elements triangulaires simples avec hypothèse de Kirchhoff-Love sous forme discréte pour l'analyse linéaire et non linéaire des plaques, coques surbaissées et coques profondes, *Méthodes Numériques pour les Sciences de l'Ingénieur*, GAMNI1, Dunod, Paris, 1979, pp. 197–206.
8. R. Gallagher, *Finite Element Analysis: Fundamentals*, Prentice-Hall, 1975.
9. O. C. Zienkiewicz, *The Finite Element Method*, 3rd edn, McGraw-Hill, 1977.
10. J. F. Abel, and C. S. Desai, 'Comparison of finite elements for plate bending', *ASCE*, **ST9**, 2143–2148 (Sept. 1972).
11. T. Hughes, R. Taylor and W. Kanoknukulachai, 'A simple and efficient finite element for plate bending', *Int. J. num. Meth. Engng*, **11**, 1529–1543 (1977).
12. E. Hinton and E. D. L. Pugh, 'Some quadrilateral isoparameteric finite elements based on Mindlin plate theory, Proc. Symp. on Applications of Computer Methods in Engineering, Los Angeles (Aug. 1977).
13. R. J. Melosh, 'A stiffness matrix for the analysis of thin plates in bending', *J. Aeronaut. Sci.*, **28** (34), (1961).
14. O. C. Zienkiewicz and Y. K. Cheung, 'The finite element method for analysis of elastic isotropic and orthotropic slabs', *Proc. Inst. Civ. Eng.*, **28**, 471–488 (1964).
15. R. J. Melosh, 'Basis of derivation of matrices for the direct stiffness analysis', *A.I.A.A. J.*, **1**, 1631–1637 (1963).
16. J. M. Argyris, 'Continua and discontinua, Proc. Conf. Matrix Methods in Struct. Mech., WPAFB, Ohio (1965).
17. J. L. Tocher and K. K. Kapur, 'Comment on basis of derivation of matrices for direct stiffness method', *A.I.A.A. J.*, **3**, 1215–1216 (1965).
18. R. W. Clough and J. L. Tocher, 'Finite element stiffness matrices for analysis of plate bending', Proc. Conf. Matrix Methods in Struct. Mech., WPAFB, Ohio (1965).
19. M. D. Olson, 'Compatibility of finite elements in structural mechanics', World Congress on F.E.M. in Structural Mechanics (October 1975).
20. F. K. Bogner, R. L. Fox and L. A. Schmit, 'The generation of inter-element compatible stiffness and mass matrices by the use of interpolation formulae', Proc. Conf. Matrix Methods in Struct. Mech., WPAFB, Ohio (1965).
21. A. L. Deak and T. H. Pian, 'Application of the smooth-surface interpolation to the finite element analysis', *A.I.A.A. J.*, **5**(1), 187–189 (1967).

22. F. Kikuchi and Y. Ando, 'Some finite element solutions for plate bending problems by simplified hybrid displacement method', *Nucl. Eng. Design*, **23**, 155–178 (1972).
23. H. Mang and R. Gallagher, 'A critical assessment of the simplified hybrid displacement method', *Int. J. num. Meth. Engng*, **11**, 145–167 (1977).
24. T. Pian, 'Element stiffness matrices for prescribed boundary stresses', Proc. Conf. Matrix Methods in Struct. Mech., WPAFB, Ohio, pp. 457–477 (1965).
25. Th. Pian and P. Tong, 'Rationalization in deriving element stiffness matrix by assumed stress approach', Proc. Conf. Matrix Methods in Struc. Mech., WPAFB, Ohio (1968).
26. R. T. Severn and P. R. Taylor, 'The finite element method for flexure of slabs when stress concentrations are assumed', *Proc. Instn Civ. Eng.*, **34**, 153–170 (1966).
27. B. K. Neale, R. D. Henshell and G. Edwards, 'Hybrid plate bending elements', *J. Sound Vib.*, **23**(1), 101–112 (July 1972).
28. I. Fried, 'Residual energy balancing technique in the generation of plate bending finite elements', *Comp Struct.*, **4**(4), 771–778 (1974).
29. K. J. Bathe, E. L. Wilson and F. Peterson, '*SAP IV*—A structural analysis program for static and dynamic response of linear systems', College of Engineering, University of California, Berkeley, Ca., U.S.A. (April 1974).
30. R. W. Clough and C. A. Felippa, 'A refined quadrilateral element for analysis of plate bending, Proc. Conf. on Matrix Methods in Struct. Mech., WPAFB, Ohio, pp. 399–440 (1968).
31. R. J. Allwood and G. M. Cornes, 'A polygonal finite element for plate bending problems using the assumed stress approach', *Int. J. num. Meth. Engng*, **1**, 135–149 (1969).
32. I. Torbe and K. Church, 'A general quadrilateral plate element', *Int. J. num. Meth. Engng*, **9**, 856–868 (1975).
33. G. Horrigmoe, 'Hybrid stress finite element model for nonlinear shell problems', Proc. 6th Canadian Conf. on Applied Mech., U.B.C. Vancouver, Canada (May 1977).
34. G. Horrigmoe, 'Hybrid stress finite element model for nonlinear shell problems', *Int. J. num. Meth. Engng*, **12**(12), 1819–1839 (1978).
35. R. Cook, 'Two hybrid elements for analysis of thick, thin and sandwich plates', *Int. J. num. Meth. Engng*, **5**, 277–288 (1972).
36. R. Cook, 'Some elements for analysis of plate bending', *ASCE*, **EM6**, 1453–1470 (1972).
37. R. Cook and S. Ladkany, 'Observations regarding assumed-stress hybrid plate elements', *Int. J. num. Meth. Engng*, **8**(3), 513–519 (1974).
38. R. Cook, 'Artificial adjustment of element stiffness with a particular reference to plate bending', *Int. J. num. Meth. Engng*, **10**, 945–976 (1976).
39. R. Cook, 'Further improvement of an effective plate bending element', *Comput. Struct.* **6**, 93–97 (1976).
40. W. Kanoknukulchai, 'A simple and efficient finite element for general shell analysis', *Int. J. num. Meth. Engng*, **14**(2), 179–200 (1979).
41. E. Hinton and N. Bicanic, 'A comparison of Lagrangian and Serendipity Mindlin plate elements for free vibrations analysis', *Comput. Struct.*, **10**, 483–493 (1979).
42. E. D. Pugh, E. Hinton and O. C. Zienkiewicz, 'A study of quadrilateral plate bending elements with reduced integration', *Int. J. num. Meth. Engng*, **12**(7) 1059–1078 (1978).
43. S. W. Lee and T. Pian, 'Improvement of plate and shell finite elements by mixed formulations', *A.I.A.A. J.* **16**(1), 29–34 (1978).
44. R. McNeal, 'A simple quadrilateral shell element', *Comput. Struct.* **8**, 175–183 (1978).
45. R. H. McNeal, 'Higher order versus lower order elements; economics and accuracy', Proc. World Congress on F.E.M. in the Commercial Environment, **1**, 201–215 (1978).
46. G. Dhatt and G. Touzot, *Une présentation de la méthode des éléments finis* (Ed. Maloine), Compiègne, 1981.
47. F. Kikuchi, 'On a finite element scheme based on the discrete Kirchhoff assumption', *Num. Math.*, **24**, 211–231 (1975).
48. I. Fried and S. K. Yang, 'Triangular, nine-degrees of freedom, C^0 plate bending element of quadratic accuracy', *Q. Appl. Math.*, **31**(3), 303–312 (1973).
49. *SAP6*, 'A structural analysis program for static and dynamic analysis', Civil Eng. Dept., University of Southern California, Los Angeles, Ca., U.S.A. (Feb. 1979).
50. S. Timoshenko and S. Woinowsky-Krieger, *Théorie des Plaques et Coques*, Dunod.
51. B. F. De Veubeke and G. Sander, 'An equilibrium model for plate bending', *Int. J. Solids Struct.*, **4**, 4447–468 (1968).
52. J. L. Batoz, 'Développements de nouveaux éléments simples courbés pour le calcul des coques minces surbaissées', *Thèse M. Sc.*, Université Laval, Québec (1971).
53. J. L. Batoz and M. Ben Tahar, 'Formulation et Evaluation d'un nouvel élément quadrilatéral à 12 D.L. pour la flexion des plaques minces', *Rapport Calcul des Structures*, Université de Technologie de Compiègne, France, 1981, 82 pp.
54. C. A. Coull and P. C. Das, 'Analysis of curved bridge decks', *Proc. Inst. Civ. Engrs*, **37**, 75–85 (1967).