Counting structures on the $n \times k$ grid graph

Peter Kagey

University of Southern California

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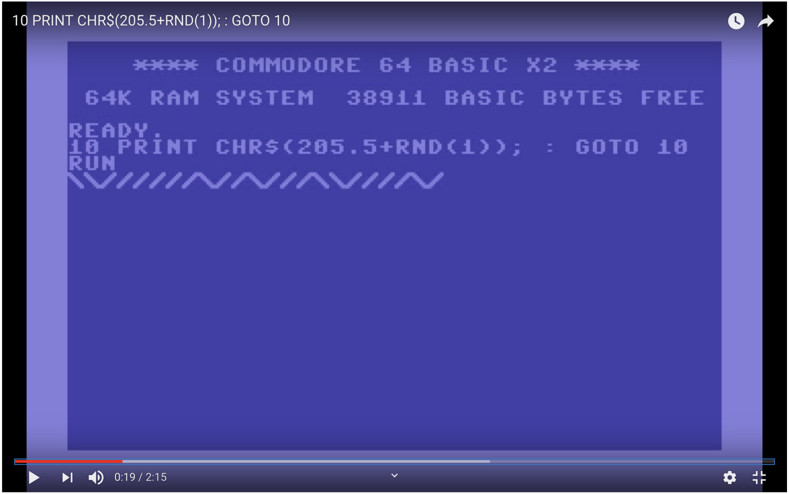
Commodore 64



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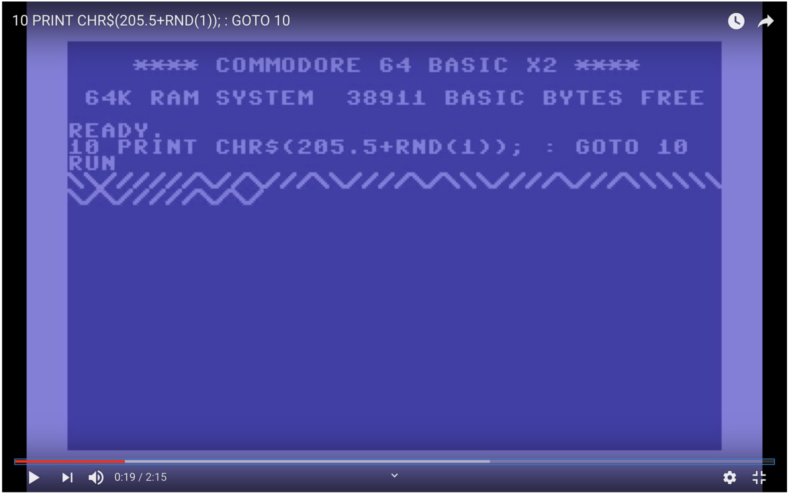
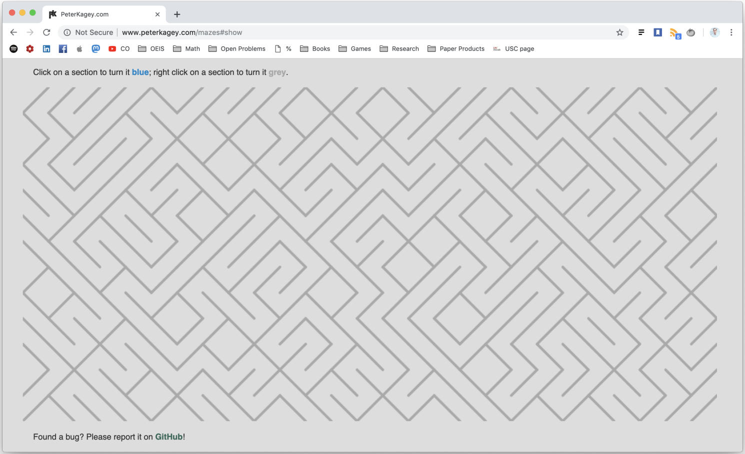


Commodore 64

Javascript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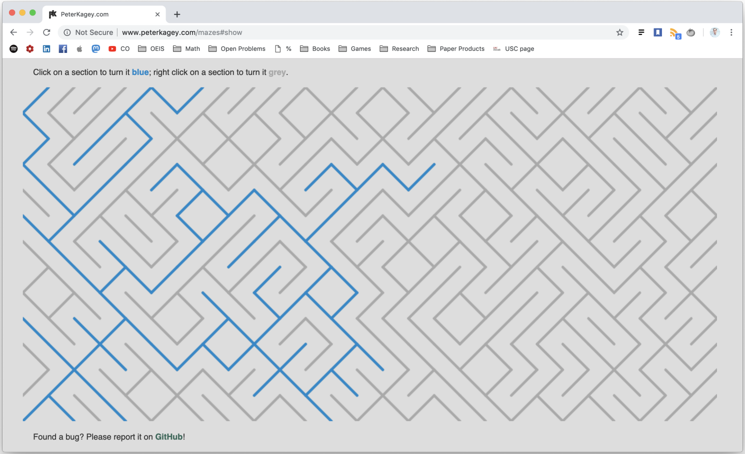
Javascript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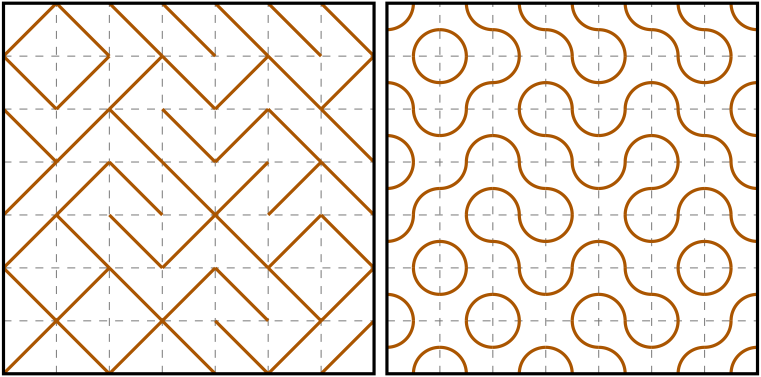
Counting grids



A295229: Number of tilings of the $n \times n$ grid, using diagonal lines to connect the grid points.

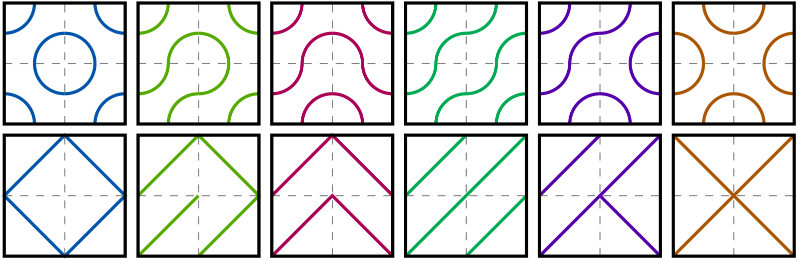
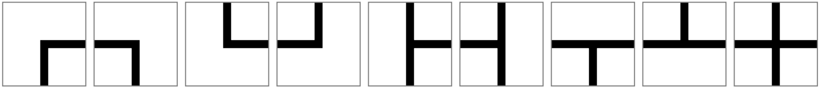
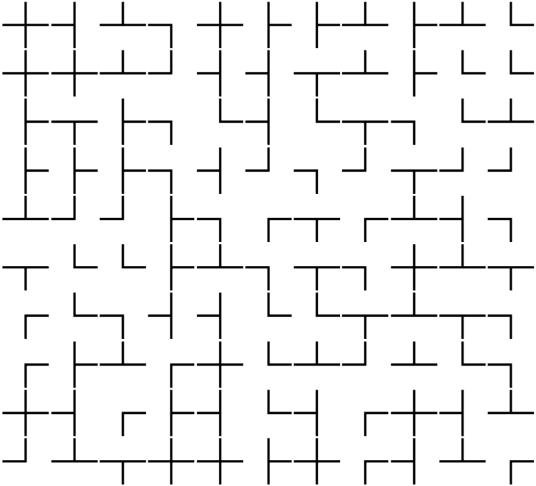


Figure 1: An example of the $a(2) = 6$ different ways to fill the 2×2 grid with diagonal tiles up to dihedral action of the square.

$$a(n) = \begin{cases} \frac{1}{8}\left(2^{n^2} + 2 \cdot 2^{n(n+1)/2} + 3 \cdot 2^{n^2/2} + 2 \cdot 2^{n^2/4}\right) & n \text{ even} \\ \frac{1}{8}\left(2^{n^2} + 2 \cdot 2^{n(n+1)/2} + 2^{(n^2+1)/2}\right) & n \text{ odd} \end{cases}$$

Other tiles





Baby's first corollary

Theorem (Corollary of Burnside's Lemma)

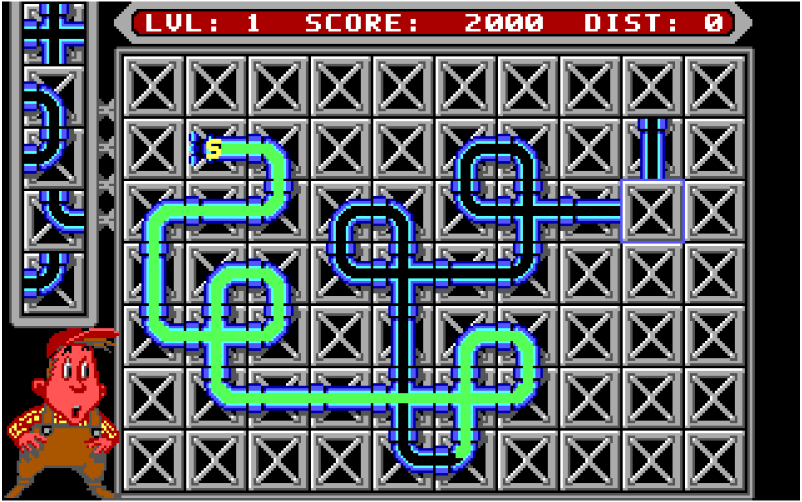
Let

- t *be the number of tiles,*
- q *be the number of tiles symmetric under a* 90° *rotation,*
- h *be the number of tiles symmetric under a* 180° *rotation,*
- d *be the number of tiles symmetric under a diagonal reflection, and*
- v *be the number of tiles symmetric under a vertical reflection.*

Then the number of tilings up to symmetries of the square is given by

$$a(n) = \begin{cases} \frac{1}{8}(t^{n^2} + 2qt^{(n^2-1)/4} + ht^{(n^2-1)/2} + (v^n + d^n)t^{(n^2-n)/2}) & n \text{ odd} \\ \frac{1}{8}(t^{n^2} + 3t^{n^2/2} + 2t^{n^2/4} + 2d^n t^{(n^2-n)/2}) & n \text{ even} \end{cases}$$

Pipe Mania



Leaf Free Grids

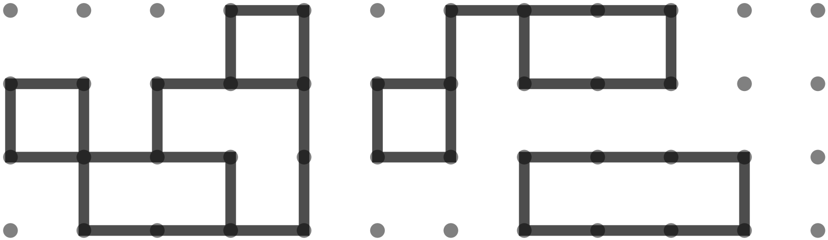
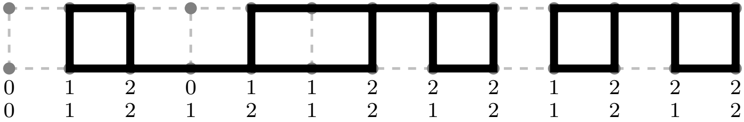


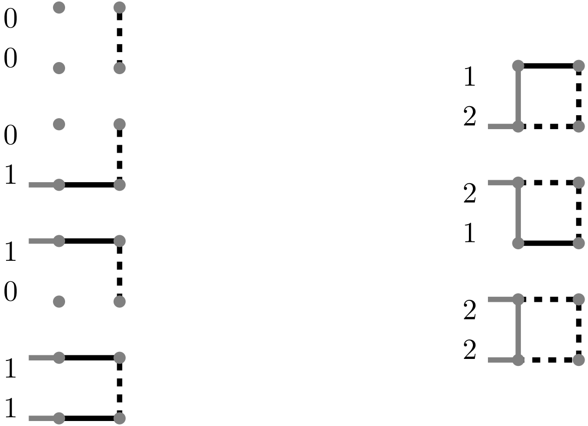
Figure 2: One of the $a_4(12) = 42650154782713601$ grids on the 12×4 grid.

$$a_2(1) = 1, a_2(2) = 2$$
$$a_2(n) = 5a(n-1) - 5a(n-2)$$

$$a_3(1) = 1, a_3(2) = 5, a_3(3) = 43, a_3(4) = 463$$
$$a_3(n) = 12a(n-1) - 6a(n-2) - 20a(n-3) - 5a(n-4)$$

Leaf Free Grids: The System of Recurrences





Example: A System of Recurrences

The 1×2 grid has initial conditions

$$a_{00}(1) = a_{11}(1) = 1$$
$$a_{10}(1) = a_{01}(1) = a_{12}(1) = a_{21}(1) = a_{22}(1) = 0,$$

and satisfies the system of first order homogeneous difference relations

$$\begin{aligned}
a_{00}(n+1) &= a_{00}(n) + a_{22}(n)\\
a_{01}(n+1) &= a_{01}(n) + a_{21}(n) + a_{22}(n)\\
a_{10}(n+1) &= a_{10}(n) + a_{12}(n) + a_{22}(n)\\
a_{11}(n+1) &= a_{00}(n) + a_{11}(n) + a_{12}(n) + a_{21}(n) + 2a_{22}(n)\\
a_{12}(n+1) &= a_{01}(n) + a_{21}(n) + a_{22}(n)\\
a_{21}(n+1) &= a_{10}(n) + a_{12}(n) + a_{22}(n)\\
a_{22}(n+1) &= a_{11}(n) + a_{12}(n) + a_{21}(n) + a_{22}(n).
\end{aligned}$$

System of Recurrences: Getting one long recurrence

Theorem (Corollary of Cayley–Hamilton theorem)

Given a system of k first order homogeneous linear recurrences,

$$\begin{aligned} a^{(1)}(n+1) &= \alpha_{11}a^{(1)}(n) + \ldots + \alpha_{1k}a^{(k)}(n) \\ \vdots &= \vdots \\ a^{(k)}(n+1) &= \alpha_{k1}a^{(1)}(n) + \ldots + \alpha_{kk}a^{(k)}(n) \end{aligned}$$

then each linear recurrence (and thus linear combination of recurrences) satisfies

$$a^{(i)}(n) = -\beta_{k-1}a^{(i)}(n-1) - \ldots - \beta_1 a^{(i)}(n-k-1) - \beta_0 a^{(i)}(n-k)$$

for $n > k$ where $A = \{\alpha_{ij}\}_{i,j=1}^{k}$ is the coefficient matrix, and

$$\chi_A(x) = \det(xI_k - A) = x^k + \beta_{k-1}x^{k-1} + \ldots + \beta_1 x + \beta_0$$

is the characteristic polynomial of A.

System of Recurrences: Getting one long recurrence

$$\begin{bmatrix} a_{00}(n+1) \\ a_{01}(n+1) \\ a_{10}(n+1) \\ a_{11}(n+1) \\ a_{12}(n+1) \\ a_{21}(n+1) \\ a_{22}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\chi_A(x) = \det(xI_k - A) = x^k + \beta_{k-1}x^{k-1} + \ldots + \beta_1 x + \beta_0$ be the characteristic polynomial of A.

Some conjectural recurrences

$$a_3(n) = 12*a_3(n-1) - 6*a_3(n-2) - 20*a_3(n-3) - 5*a_3(n-4)$$

$$a_4(n) = 36*a_4(n-1) - 7*a_4(n-2) - 201*a_4(n-3) + 49*a_4(n-4) + 20*a_4(n$$

$$a_5(n) = 103*a_5(n-1) + 1063*a_5(n-2) - 1873*a_5(n-3) - 20274*a_5(n-4)$$

Mazes and Spanning Trees

Systems of linear equations

Generalizations