Complex Analysis: Homework 6

Peter Kagey

February 28, 2018

Problem 1. (page 136)
Show by use of (36), or directly, that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Proof.
Starting from (36) with $M = 1$ and $R = 1$, we have

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|$$

so multiplying by the denominator of the left hand side and dividing by the numerator of the right hand side yields

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \left| \frac{1 - \overline{f(z_0)} f(z)}{1 - \overline{z_0} z} \right|.$$

In particular, this holds in the limit:

$$\begin{aligned} \lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| &\leq \lim_{z \to z_0} \left| \frac{1 - \overline{f(z_0)} f(z)}{1 - \overline{z_0} z} \right| \\ |f'(z)| &\leq \left| \frac{1 - |f(z)|^2}{1 - |z|^2} \right| \\ &= \frac{1 - |f(z)|^2}{1 - |z|^2} \end{aligned}$$

where the final equality holds because $1 - |f(z)|^2 \geq 0$ and $1 - |z|^2 \geq 0$.

Thus dividing both sides by $1 - |f(z)|^2 \geq 0$ gives the desired inequality

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

$\square$

Problem 2. (page 136)
If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \leq \left|\frac{z-z_0}{z-\overline{z_0}}\right|$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y}$$

Proof.
For the first inequality, let

$$\hat{f}(z) = \frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}} \quad \text{and} \quad g(z) = \frac{z-z_0}{z-\overline{z_0}}$$

then $\hat{f}$ and g are both 0 at z_0, and are both analytic on the upper half plane punctured at $\overline{f(z_0)}$ and $\overline{z_0}$ respectively. By the hypothesis, we know that $|\hat{f}(z)| \leq 1$ and $|g(z)| \leq 1$. For z_0 on the boundary of $\operatorname{Im} z > 0$ (which is the real axis), $g(z) = 1$, so by Theorem $12'$ the function

$$\left|\frac{\hat{f}(z)}{g(z)}\right| = |\hat{f}(z)| \leq 1$$

on the boundary, so this must be the maximum in the upper half plane. Therefore

$$|\hat{f}(z)| \leq |g(z)|$$

.

The second inequality follows from the first. Multiplying by the denominator on the left hand side and dividing by the numerator on the right hand side

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq \left|\frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}}\right|$$

and so in the limit (Noting that for $w \in \mathbb{C},\ w - \overline{w} = 2\operatorname{Im} w$)

$$\lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq \lim_{z\to z_0}\left|\frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}}\right|$$
$$|f'(z)| \leq \left|\frac{2\operatorname{Im} f(z)}{2\operatorname{Im} z}\right|.$$

Thus dividing both sides by $|\operatorname{Im} f(z)|$ yields

$$\frac{|f'(z)|}{|\operatorname{Im} f(z)|} \leq \frac{1}{|\operatorname{Im} z|}$$
$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{\operatorname{Im} z},$$

where the final inequality holds by the hypothesis that $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$. $\square$

Problem 3. (page 136)
In Ex. 1 and 2, prove that equality implies that $f(z)$ is a linear transformation.

Proof.
(?) $\square$

Problem 1. (page 154)
How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

Proof.
Let $f(z) = 6z^3$, and let $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$ be the function described above. Then by the triangle inequality,

$$|f - g| = |z^7 - 2z^5 - z + 1| \leq |z|^7 + 2|z|^5 + |z| + 1,$$

so $|f - g| \leq 5$ on the boundary of the unit circle. Similarly,

$$|f| = 6|z|^3 = 6$$

on the boundary of the unit circle. Since $|f - g| < |f|$ on this curve, by Rouché's Theorem $g(z)$ has three roots because $f(z) = 6z^3$ has three roots.

$\square$

Problem 2. (page 154)
How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

Proof.
It is sufficient to count the number of roots in the disk of radius 2 and subtract the number of roots in the unit disk.
By Rouché's Theorem with $f(z) = z^4$ and $g(z) = z^4 - 6z + 3$, on the circle of radius 2, $|z| = 2$,

$$|z^4 - (z^4 - 6z + 3)| = |6z - 3| \leq 15 < 16 = 2^4 = |z|^4 = |z^4|$$

$|z^4 - (z^4 - 6z + 3)|$ has all four of its roots inside the circle of radius 2.
By another application of Rouché's Theorem, this time with $f(z) = -6z + 3$ and $g(z) = z^4 - 6z + 3$ on the unit circle, $|z| = 1$,

$$\begin{aligned}
|f - g| &= |-6z + 3 - (z^4 - 6z + 3)| = |-z^4| = |z|^4 = 1, \text{ and} \\
|f| &= |-6z + 3| \geq 6|z| - 3 = 3, \text{ so} \\
|f - g| &< |f| \text{ for } |z| = 1.
\end{aligned}$$

Since f has one root inside the unit circle (at $z = 1/2$), g also has one root inside the unit circle.
Therefore $z^4 - 6z + 3$ has three roots with modulus between 1 and 2. $\square$

Problem 3. (page 154)
How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? (Hint: Sketch the image of the imaginary axis and apply the argument principle to a large half disk.)

Proof.
The plan is to apply the argument theorem to the curve with one segment along the imaginary axis from Ri to $-Ri$ and the other segment along Re^{it} for $t \in [-\pi/2, \pi/2]$, and then making R big enough to catch all roots.
First consider the image of the imaginary axis:

$$f(it) = t^4 + -8it^3 - 3t^2 + 8it + 3 = (t^4 - 3t^2 + 3) + i(-8t^3 + 8t)$$

which has a strictly positive real part, because $t^4 - 3t^2 + 3$ has discriminant -3. Thus the image of the imaginary axis is entirely in the right half plane, and in particular does not wind around the origin.
Thus we're concerned with how many times the semicircle winds around the origin. The image of the semicircle

$$\begin{aligned} f(Re^{it}) &= R^4e^{4it} + 8R^3e^{3it} - 3R^2e^{2it} + 8Re^{it} + 3 \\ &= R^4e^{4it}\left(1 + \frac{8}{R}e^{-it} - \frac{3}{R^2}e^{-2it} + \frac{8}{R^3}e^{-3it} + \frac{3}{R^4}e^{-4it}\right) \end{aligned}$$

can be approximated by R^4e^{4it}, which winds around the origin twice on the interval $t \in [-\pi/2, \pi/2]$.
Therefore f has two roots in the right half plane. $\square$