

Complex Analysis: Homework 2

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Problem 5. (page 37)

Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x .

Proof.

□

Problem 3. (page 41)

Find the radius of convergence of the following power series:

(a) $\sum n^p z^n$

(b) $\sum \frac{z^n}{n!}$

(c) $\sum n! z^n$

(d) $\sum q^{n^2} z^n$ where $|q| < 1$

(e) $\sum z^{n!}$

Proof. (a) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |n^p|^{1/n} = \limsup_{n \rightarrow \infty} |n^p|^{1/n} = \limsup_{n \rightarrow \infty} n^{p/n} = 1$$

So the radius of convergence is $R = 1$.

(b) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \left(\frac{1}{n!} \right)^{1/n} < \limsup_{n \rightarrow \infty} \left(\frac{1}{N^n} \right)^{1/n} = \frac{1}{N}.$$

Because $R > N$ for all N , the radius of convergence is ∞ .

(c) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} (n!)^{1/n} > \limsup_{n \rightarrow \infty} (N^n)^{1/n} = N.$$

Because $R < 1/N$ for all N , the radius of convergence is 0.

(d) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} (q^{n^2})^{1/n} = \limsup_{n \rightarrow \infty} q^n = 0 \text{ for } |q| < 1.$$

Thus the radius of convergence is ∞ .

(e) Notice that $|z^{n!}| \geq |z^n|$ for $|z| \geq 1$, and $|z^{n!}| \leq |z^n|$ for $|z| < 1$.

$$\begin{aligned} \left| \sum z^{n!} \right| &\leq \left| \sum z^n \right| < \infty \text{ for } |z| < 1 \\ \left| \sum z^{n!} \right| &\geq \left| \sum z^n \right| = \infty \text{ for } |z| \geq 1 \end{aligned}$$

Thus the radius of convergence is 1.

□

Problem 8. (page 41)

For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent?

Proof. The sum $\sum_{n=0}^{\infty} w^n$ is convergent for $|w| < 1$. The sum in the problem is convergent when

$$\left| \frac{z}{1+z} \right| < 1 \implies |z| < |1+z|.$$

Letting $z = a + bi$, and comparing the squares of the absolute values:

$$a^2 + b^2 < (1+a)^2 + b^2$$

$$a^2 < 1 + 2a + a^2$$

$$-2a < 1$$

$$a > -1/2$$

Thus the sum converges when $\operatorname{Re}(z) > -1/2$.

□

Problem 3. (page 44)

Use the addition formulas to separate $\cos(x + iy)$ and $\sin(x + iy)$ in real and imaginary parts.

Proof. First note the identities of \sin and \cos with purely imaginary inputs:

$$\begin{aligned}\cos(iy) &= 1 - \frac{i^2 y^2}{2!} + \frac{i^4 y^4}{4!} - \cdots = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots = \cosh(y) \\ \sin(iy) &= iy - \frac{i^3 y^3}{3!} + \frac{i^5 y^5}{5!} - \cdots = i\left(y + \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right) = i \sinh(y)\end{aligned}$$

Then using the addition formulas

$$\begin{aligned}\cos(a + b) &= \cos(a) \cos(b) - \sin(a) \sin(b) \text{ and} \\ \sin(a + b) &= \cos(a) \sin(b) + \sin(a) \cos(b),\end{aligned}$$

it is clear that

$$\begin{aligned}\cos(x + iy) &= \cos(x) \cosh(y) + i \sin(x) \sinh(y) \text{ and} \\ \sin(x + iy) &= \sin(x) \cosh(y) + i \cos(x) \sinh(y).\end{aligned}$$

□

Problem 4. (page 44)

Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x)$$

Proof. Starting with the proof of \cos . From the above problem

$$\cos(z) = \cos(x) \cosh(y) + i \sin(x) \sinh(y)$$

so the square of the absolute value of $\cos(z)$ is

$$\begin{aligned} |\cos z|^2 &= (\cos(x) \cosh(y))^2 + (\sin(x) \sinh(y))^2 \\ &= (\cos(x) \cosh(y))^2 + (\sin(x) \sinh(y))^2 + (\cos(x) \sinh(y))^2 - (\cos(x) \sinh(y))^2 \\ &= \sinh^2(y)(\sin^2(x) + \cos^2(x)) + \cos^2(x)(\cosh^2(y) - \sinh^2(y)) \\ &= \sinh^2(y) + \cos^2(x) \end{aligned}$$

$$\begin{aligned} |\cos z|^2 &= (\cos(x) \cosh(y))^2 + (\sin(x) \sinh(y))^2 \\ &= (\cos(x) \cosh(y))^2 + (\sin(x) \sinh(y))^2 + (\sin(x) \cosh(y))^2 - (\sin(x) \cosh(y))^2 \\ &= \cosh^2(y)(\sin^2(x) + \cos^2(x)) + \sin^2(x)(\sinh^2(y) - \cosh^2(y)) \\ &= \cosh^2(y) - \sin^2(x) \end{aligned}$$

Thus adding the two different values together yields

$$\begin{aligned} 2|\cos z|^2 &= \sinh^2(y) + \cos^2(x) + \cosh^2(y) - \sin^2(x) \\ &= (\sinh^2(y) + \cosh^2(y)) + (\cos^2(x) - \sin^2(x)) \\ &= \cosh(2y) + \cos(2x) \end{aligned}$$

Similarly for \sin , from the above problem

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

so the square of the absolute value of $\cos(z)$ is

$$\begin{aligned} |\sin z|^2 &= (\sin(x) \cosh(y))^2 + (\cos(x) \sinh(y))^2 \\ &= (\sin(x) \cosh(y))^2 + (\cos(x) \sinh(y))^2 + (\sin(x) \sinh(y))^2 - (\sin(x) \sinh(y))^2 \\ &= \sinh^2(y)(\sin^2(x) + \cos^2(x)) + \sin^2(x)(\cosh^2(y) - \sinh^2(y)) \\ &= \sinh^2(y) + \sin^2(x) \end{aligned}$$

$$\begin{aligned} |\sin z|^2 &= (\sin(x) \cosh(y))^2 + (\cos(x) \sinh(y))^2 \\ &= (\sin(x) \cosh(y))^2 + (\cos(x) \sinh(y))^2 + (\cos(x) \cosh(y))^2 - (\cos(x) \cosh(y))^2 \\ &= \cosh^2(y)(\sin^2(x) + \cos^2(x)) + \cos^2(x)(\sinh^2(y) - \cosh^2(y)) \\ &= \cosh^2(y) - \cos^2(x) \end{aligned}$$

Thus adding the two different values together yields

$$\begin{aligned} 2|\sin z|^2 &= \sinh^2(y) + \sin^2(x) + \cosh^2(y) - \cos^2(x) \\ &= (\sinh^2(y) + \cosh^2(y)) - (\cos^2(x) - \sin^2(x)) \\ &= \cosh(2y) - \cos(2x) \end{aligned}$$

□

Problem 6. (page 47)
Determine all values of

(a) 2^i ,

(b) i^i ,

(c) $(-1)^{2i}$.

Proof.

□

Problem 7. (page 47)

Determine the real and imaginary parts of z^z .

Proof.

□

Problem ?. (page ?)

Proof.

□

Problem ?. (page ?)

Proof.

□

Problem ?. (page ?)

Proof.

□