Matrix Analysis: Main ideas

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1 Definitions

1.1 Companion matrix

A **companion matrix** to a monic polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$$

is the $n \times n$ square matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

which has the property that its characteristic and minimal polynomials are $p(x)$.

1.2 Cyclic matrix

An $n \times n$ matrix over a field $\mathbb{F}$ is called a **cyclic matrix** if there exists a vector $\vec{v}$ such that $\{\vec{v}, A\vec{v}, \dots, A^{n-1}\vec{v}\}$ is a basis for $\mathbb{F}^n$.

1.3 Normal matrix

A matrix A is called **normal** if it commutes with its conjugate transpose, that is $A^*A = AA^*$.

1.4 Tensor product

...

1.5 Unitary matrix

A matrix U is called **unitary** if its conjugate transpose U^* is also its inverse, that is $U^*U = I$.

2 Decompositions

2.1 Schur decomposition

Each $A \in M_n(\mathbb{C})$ can be written

$$A = QUQ^{-1}$$

where Q is unitary and U is upper triangular.

2.2 Singular Value Decomposition

Let M be an $m \times n$ complex matrix. Then M can be written

$$M = U\Sigma V^*$$

where U and V are unitary matrices.

3 Theorems

3.1 Special cases of normal matrices

All unitary, Hermitian, and skew-Hermitian complex matrices are normal. All orthogonal, symmetric, and skew-symmetric real matrices are normal.

3.2 Spectral theorem

A matrix A is normal if and only if it is unitarily similar to a diagonal matrix, that is

$$A = U^* D U$$

for some unitary matrix U.

3.3 Hermitian matrices have real eigenvalues

3.4 Perron theorem

Any positive $n \times n$ matrix has a simple eigenvalue $\lambda_1 = r$ such that all other eigenvalues are strictly smaller than r. Moreover, there exists a corresponding eigenvector v such that all components are positive.