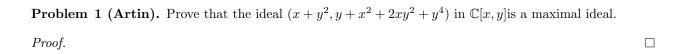
Math 510b: Homework 2

Peter Kagey

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Problem 2 (Artin). Let I be the principal ideal of $\mathbb{C}[x,y]$ generated by the polynomial $y^2 + x^3 - 17$. Which of the following sets generate maximal ideals in the quotient ring C[x,y]/I?

- (a) (x-1, y-4)
- (b) (x+1,y+4)
- (c) $(x^3 17, y^2)$

Problem 6 (Artin). Prove that the kernel of the homomorphism $\mathbb{Z} \to \mathbb{R}$ sending $x \mapsto 1 + \sqrt{2}$ ideal, and find a generator for this ideal.	$\bar{2}$ is a principal
Proof.	

Problem 7 (Artin). Let f be an irreducible polynomial in $\mathbb{C}[x,y]$, and let g be another polynomial. Prove that if the variety of zeros of g in \mathbb{C}^2 contains the variety of zeros of f, then f divides g.

Problem 8 (Artin). Determine the points of intersection of the two complex plane curves in each of the following

(a)
$$y^2 - x^3 + x^2 = 1, x + y = 1$$

(b)
$$x^2 + xy + y^2 = 1, x^2 + 2y^2 = 1$$

(c)
$$y^2 = x^3, xy = 1$$

(d)
$$x + y + y^2 = 0, x - y + y^2 = 0$$

Problem 9 (Artin). Prove that two quadratic polynomials f, g in two variables have at most four common zeros unless they have a non-constant factor in common.	
Proof.	

Problem 10 (Artin). An algebraic curve \mathcal{C} in \mathbb{C}^2 is called irreducible if it is the locus of zeros of an
irreducible polynomial $f(x,y)$ —one which cannot be factored as a product of nonconstant polynomials. A
point $p \in \mathcal{C}$ is called a singular point of the curve if $\partial f/\partial x = \partial f/\partial y = 0$ at p . Otherwise p is a nonsingular
point. Prove that an irreducible curve has only finitely many singular points.

Proof. \Box

Extra Problem. Let $R=\mathbb{Z}(\sqrt{-5})=\{a+b\sqrt{-5}:a,b\in\mathbb{Z}\}\subset\mathbb{C}.$ Define $N\colon R\to\mathbb{Z}_{\geq 0}$ by sending $a+b\sqrt{-5}\mapsto a^2+b^2.$

Show:

- (a) N(xy) = N(x)N(y) for all $x, y \in R$.
- (b) If x is a unit in R then N(x) = 1. Thus the only units in R are ± 1 .
- (c) There does not exist $x \in R$ with N(x) = 3.
- (d) If N(x) = 9 then x is irreducible in R.
- (e) Note that $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 \sqrt{-5})$, and conclude that 3 is irreducible in R but not prime.
- (f) Factorization into irreducible elements in R is not unique.
- (g) Comparing this example to $\mathbb{Z}[i]$, what goes wrong here that works for $\mathbb{Z}[i]$?
- (h) Find an ideal in R which is not principal.