Week 6 Quiz Solutions.

1. (*7 points*) For $\vec{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$, find $\overrightarrow{T}, \overrightarrow{N}$ and $\overrightarrow{B}$ at the point $(3, 0, 0)$. Find an equation of the normal plane at this point.

   Solution.
   First, the curve $r(t) = (3, 0, 0)$ when $t = 0$, by the third coordinate.
   To compute the unit tangent vector, first compute $r'(t) = \langle -3\sin t, 3\cos t, 1\rangle$. Then the magnitude and the unit tangent vector are

$$|r'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{9(\sin^2 t + \cos^2) + 1} = \sqrt{10}$$
$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{10}}\langle -3\sin t, 3\cos t, 1\rangle$$

   Then the unit normal vector is

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{3}\langle -3\cos t, -3\sin t, 0\rangle = \langle -\cos t, -\sin t, 0\rangle.$$

   Next, the binormal vector is

$$\begin{aligned} B(t) &= T(t) \times N(t) \\ &= \frac{1}{\sqrt{10}}\langle -3\sin t, 3\cos t, 1\rangle \times \langle -\cos t, -\sin t, 0\rangle \\ &= \frac{1}{\sqrt{10}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin t & 3\cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} \\ &= \frac{1}{\sqrt{10}}\langle \sin t, -\cos t, 3\rangle. \end{aligned}$$

   Then evaulating at $t = 0$ gives

$$T(0) = \frac{1}{\sqrt{10}}\langle 0, 3, 1\rangle \qquad N(0) = \langle -1, 0, 0\rangle \qquad B(0) = \frac{1}{\sqrt{10}}\langle 0, -1, 3\rangle.$$

   Lastly, the normal plane is given by

$$3y + z + d = 0$$

   where the coordinates are the direction of $T(0)$. Since $(3, 0, 0)$ is a point on the plane, evaulating at this point yields $d = 0$. Therefore the equation of the normal plane is

$$3y + z = 0$$

2. (*3 points*) Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position:

$$\vec{a}(t) = \langle 1, 0, 3 \rangle, \qquad \vec{v}(0) = \langle 1, 0, 0 \rangle, \qquad \vec{r}(0) = \langle 0, 1, 0 \rangle.$$

Solution.
First integrate $\vec{a}(t)$ to get $\vec{v}(t)$

$$\vec{v}(t) = \left\langle \int 1\, dt, \int 0\, dt, \int 3\, dt \right\rangle = \langle t + c_1, c_2, 3t + c_3 \rangle.$$

Using the initial condition gives the velocity of the particle

$$\begin{aligned} \vec{v}(0) &= \langle 1, 0, 0 \rangle = \langle 0 + c_1, c_2, 3(0) + c_3 \rangle, \\ \vec{v}(t) &= \langle t + 1, 0, 3t \rangle. \end{aligned}$$

We repeat the process to find the position vector.

$$\vec{r}(t) = \left\langle \int t + 1\, dt, \int 0\, dt, \int 3t\, dt \right\rangle = \left\langle \frac{t^2}{2} + t + c_4, c_5, \frac{3t^2}{2} + c_6 \right\rangle.$$

Using the initial condition gives

$$\vec{r}(0) = \langle 0, 1, 0 \rangle = \left\langle \frac{0^2}{2} + 0 + c_4, c_5, \frac{0^2}{2} + c_6 \right\rangle = \langle c_4, c_5, c_6 \rangle,$$

so the position vector is given by

$$\vec{r}(t) = \left\langle \frac{1}{2}t^2 + t, 1, \frac{3}{2}t^2 \right\rangle.$$