Differential Geometry: Midterm

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Problem 1.

(a) Prove that $V_k(\mathbb{R}^n)$ has the structure of a manifold and calculate its dimension.

(b) Note that $V_1(\mathbb{R}^3)$ is equal to the two sphere. Prove that $V_2(\mathbb{R}^3)$ is diffeomorphic to the collection of unit tangent vectors in S^2, that is the subset

$$UT(S^2) = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2,\ v \in T_x S^2, \text{ and } ||v|| = 1\}$$

Proof.

(a) Let f be the map $f\colon Mat_{n\times k} \to Sym(k)$ which sends $A \mapsto A^T A$. If it can be shown that

(i) $Mat_{n\times k}$ is a manifold with dimension nk,

(ii) $Sym(k)$ is a manifold with dimension $k(k+1)/2$, and

(iii) f is a submersion for all $p \in f^{-1}(I_k) = V_k(\mathbb{R}^n) \subset Mat_{n\times k}$,

then the corollary of the Implicit Function Theorem for submersions gives that $V_k(\mathbb{R}^n)$ has the structure of a manifold of dimension $nk - k(k+1)/2$.

(i) $Mat_{n\times k}$ is a manifold of dimension nk with atlas $\{(Mat_{n\times k}, \phi)\}$ containing one chart, where $\phi : Mat_{n\times k} \to \mathbb{R}^{nk}$ is the map

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \mapsto (x_{11}, x_{12}, \dots, x_{1k}, x_{21}, x_{22}, \dots, x_{nk}).$$

(ii) $Sym(k)$ is a manifold of dimension $k(k+1)/2$ with atlas $\{Sym(k), \psi)\}$ where $\psi\colon Sym(k) \to \mathbb{R}^{k(k+1)/2}$ sends

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix} \mapsto (a_{11}, a_{12}, \dots, a_{1k}, a_{22}, a_{23}, \dots, a_{kk})$$

which is a smooth map with smooth inverse. Thus $Sym(k)$ is a manifold of dimension $k(k+1)/2$.

(iii) f is smooth because the map

$$\phi^{-1} \circ f \circ \psi\colon \mathbb{R}^{nk} \xrightarrow{\phi^{-1}} Mat_{n\times k} \xrightarrow{f} Sym(k) \xrightarrow{\psi} \mathbb{R}^{k(k+1)/2}$$

is smooth in each component. In particular, matrix multiplication is defined to be the sum of the product of entries, the product of coordinates of $\mathbb{R}^M$ is smooth, and the sum of smooth functions is smooth.

f is a submersion at all points $p \in f^{-1}(I_k)$ because the Jacobian matrix has full rank. Notice that $f \circ \phi^{-1}(x_{11}, x_{12}, \ldots, x_{nk})$ has entries a_{ij} given by

$$a_{ij} = \sum_{m=1}^{n} x_{mi} x_{mj}$$

and so the entries of the Jacobian matrix are given by

$$\frac{\partial x}{\partial x_{ij}} \underbrace{\left(\sum_{m=1}^{n} x_{m\ell} x_{mh} \right)}_{a_{\ell h}} = \begin{cases} 2x_{ij} & j = \ell = h \\ x_{ih} & j = \ell \neq h \\ x_{i\ell} & j = h \neq \ell \\ 0 & j \neq \ell \text{ and } j \neq h \end{cases}$$

In order to show that f is a submersion, it is enough to show that f'_A is surjective onto $Sym(k)$ for all $A \in f^{-1}(I_k)$. Computing f'_A yields:

$$\begin{aligned} f'_A(B) &= \lim_{h \to 0} \frac{f(A + hB) - f(A)}{h} \\ &= \lim_{h \to 0} \frac{(A + hB)^T (A + hB) - A^T A}{h} \\ &= \lim_{h \to 0} \frac{(A^T + hB^T)(A + hB) - A^T A}{h} \\ &= \lim_{h \to 0} \frac{A^T A + hA^T B + hB^T A + h^2 B^T B - A^T A}{h} \\ &= \lim_{h \to 0} A^T B + B^T A + hB^T B \\ &= A^T B + B^T A \end{aligned}$$

In order to show that f is a submersion it is enough to show that all symmetric matrices can be written as $A^T B + B^T A$ where $A, B \in Mat_{n \times k}$ with A fixed. Note that if a_{ij} and b_{ij} are the entries of A and B respectively, then the entries of $A^T B + B^T A$ can be written

$$c_{ij} = \sum_{m=1}^{n} a_{mi} b_{mj} + b_{mi} a_{mj}.$$

A is full rank (because $A^T A = I_k$), so $A^T B + B^T A$ is full rank too for some choice of B. Thus B can be chosen so that $A^T B + B^T A$ is any arbitrary symmetric matrix.
Since every symmetric matrix can be represented as $A^T B + B^T A$, then f is a submersion for all $p \in f^{-1}(I_k)$, and the Implicit Function Theorem (submersion version) implies that $V_k(\mathbb{R}^n)$ is a manifold of dimension $nk - k(k+1)/2$.

(b) Consider the identity map from $V_2(\mathbb{R}^3)$ to the collection of unit tangent vectors of the unit sphere S^2 (called $UT(S^2)$ and defined above), which is smooth and has smooth inverse

$$\underbrace{(\vec{v}_1, \vec{v}_2)}_{\in V_2(\mathbb{R}^3)} \mapsto \underbrace{(\vec{v}_1, \vec{v}_2)}_{\in UT(S^2)}.$$

It is plain enough to see that $||\vec{v}_1|| = ||\vec{v}_2|| = 1$, because this is an explicit condition for both $V_2(\mathbb{R}^3)$ and $UT(S^2)$. Thus it is enough to check that the "orthogonality" condition in $V_2(\mathbb{R}^3)$ is compatible with the "tangent space" condition for $UT(S^2)$; that is, that f is a surjection.

Using the second extrinsic defintion of tangent space for $S^2 = f^{-1}(1)$ where $f(\vec{x}) = ||\vec{x}||$, we can compute the tangent space T_pS^2 to be vectors satisfying $df_p(\vec{v}) = 0$, where

$$\begin{aligned}
df_p(\vec{v}) &= v_1 \left(\frac{\partial f}{\partial x}\right)_p + v_2 \left(\frac{\partial f}{\partial y}\right)_p + v_3 \left(\frac{\partial f}{\partial z}\right)_p \\
&= v_1 \frac{p_1}{f(p)} + v_2 \frac{p_2}{f(p)} + v_3 \frac{p_3}{f(p)} \\
&= v_1 p_1 + v_2 p_2 + v_3 p_3 \\
&= \vec{v} \cdot p \\
&= 0.
\end{aligned}$$

Therefore the orthogonality condition is consistent with the tangent space condition.

$\square$

Problem 2.

(a) In $\mathbb{R}^2$, consider the vector fields X and Y defined by

$$X = e^{x^2+y^2} \frac{\partial}{\partial x} + \sin(xy) \frac{\partial}{\partial y}$$
$$Y = (x^2 + 3xy) \frac{\partial}{\partial x} + (x+y) \frac{\partial}{\partial y},$$

and compute the Lie bracket $[X, Y]$.

(b) Let $\mathcal{D} = \ker(dz + (x\,dy - y\,dx)) \subset T\mathbb{R}^3$ be the two-dimensional distribution considered in class. Verify that $\mathcal{D}$ is not integrable.

Proof.

(a) We can see how $[X, Y]$ behaves as a vector field by seeing where it maps the germ $f \in C^\infty(p)$ (given some point $p \in M$).

We defined $[X, Y]$ by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

So

$$\begin{aligned}
[X, Y]_p(f) &= X\left((x^2+3xy)\frac{\partial f}{\partial x} + (x+y)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad - Y\left(e^{x^2+y^2}\frac{\partial f}{\partial x} + \sin(xy)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&= e^{x^2+y^2}\frac{\partial}{\partial x}\left((x^2+3xy)\frac{\partial f}{\partial x} + (x+y)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad + \sin(xy)\frac{\partial}{\partial y}\left((x^2+3xy)\frac{\partial f}{\partial x} + (x+y)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad - (x^2+3xy)\frac{\partial}{\partial x}\left(e^{x^2+y^2}\frac{\partial f}{\partial x} + \sin(xy)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad - (x+y)\frac{\partial}{\partial y}\left(e^{x^2+y^2}\frac{\partial f}{\partial x} + \sin(xy)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&= e^{x^2+y^2}\left((2x+3y)\frac{\partial f}{\partial x} + (x^2+3xy)\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} + (x+y)\frac{\partial^2 f}{\partial x \partial y}\right)_{(x,y)=p} \\
&\quad + \sin(xy)\left(3y\frac{\partial f}{\partial x} + (x^2+3xy)\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} + (x+y)\frac{\partial^2 f}{\partial y^2}\right)_{(x,y)=p} \\
&\quad - (x^2+3xy)\left(2xe^{x^2+y^2}\frac{\partial f}{\partial x} + e^{x^2+y^2}\frac{\partial^2 f}{\partial x^2} + y\cos(xy)\frac{\partial f}{\partial y} + \sin(xy)\frac{\partial^2 f}{\partial y \partial x}\right)_{(x,y)=p} \\
&\quad - (x+y)\left(2ye^{x^2+y^2}\frac{\partial f}{\partial x} + e^{x^2+y^2}\frac{\partial^2 f}{\partial x \partial y} + x\cos(xy)\frac{\partial f}{\partial y} + \sin(xy)\frac{\partial^2 f}{\partial y^2}\right)_{(x,y)=p}
\end{aligned}$$

$$
\begin{aligned}
&= e^{x^2+y^2}\left((2x+3y)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad + \sin(xy)\left(3y\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad - (x^2+3xy)\left(2xe^{x^2+y^2}\frac{\partial f}{\partial x}+y\cos(xy)\frac{\partial f}{\partial y}\right)_{(x,y)=p} \\
&\quad - (x+y)\left(2ye^{x^2+y^2}\frac{\partial f}{\partial x}+x\cos(xy)\frac{\partial f}{\partial y}\right)_{(x,y)=p}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\left((2x+3y-2x(x^2+3xy)-2y(x+y))e^{x^2+y^2}+3y\sin(xy)\right)\frac{\partial f}{\partial x}\right]_{(x,y)=p} \\
&\quad + \left[\left(e^{x^2+y^2}+\sin(xy)-(x^2y+3xy^2+x^2+xy)\cos(xy)\right)\frac{\partial f}{\partial y}\right]_{(x,y)=p}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
[X,Y] = &\left((2x+3y-2x(x^2+3xy)-2y(x+y))e^{x^2+y^2}+3y\sin(xy)\right)\frac{\partial}{\partial x} \\
&+ \left(e^{x^2+y^2}+\sin(xy)-(x^2y+3xy^2+x^2+xy)\cos(xy)\right)\frac{\partial}{\partial y}
\end{aligned}
$$

(b) $\mathcal{D} = \ker(dz + (x\,dy - y\,dx))$ means that $(dz + (x\,dy - y\,dx))_p \in T_p^*\mathbb{R}^3$. By Frobenius' Theorem we can prove that $\mathcal{D}$ is not integrable by verifying that it is not involutive. In other words, we just need to show there exists $X, Y \in \mathcal{D}$ such that $[X,Y] \notin \mathcal{D}$.

Let

$$
\begin{aligned}
X &= x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} \\
Y &= \frac{\partial}{\partial z}-\frac{1}{x}\frac{\partial}{\partial y}
\end{aligned}
$$

So that

$$
\begin{aligned}
(dz+(xdy-ydx))(X) &= dz(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})+(xdy)(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})-(ydx)(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}) \\
&= 0+xy-yx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(dz+(xdy-ydx))(X) &= dz(\frac{\partial}{\partial z}-\frac{1}{x}\frac{\partial}{\partial y})+(xdy)(\frac{\partial}{\partial z}-\frac{1}{x}\frac{\partial}{\partial y})-(ydx)(\frac{\partial}{\partial z}-\frac{1}{x}\frac{\partial}{\partial y}) \\
&= 1-\frac{x}{x}+0 \\
&= 0.
\end{aligned}
$$

Thus $X, Y \in \mathcal{D}$. Now to verify that $[X,Y] \notin \mathcal{D}$:

$$
\begin{aligned}
X\left(\frac{\partial}{\partial z} - \frac{1}{x}\frac{\partial}{\partial y}\right) - Y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) &= x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z} - \frac{1}{x}\frac{\partial}{\partial y}\right) \\
&\quad + y\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} - \frac{1}{x}\frac{\partial}{\partial y}\right) \\
&\quad - \frac{\partial}{\partial z}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\
&\quad + \frac{1}{x}\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\
\\
&= x\frac{\partial^2}{\partial x\partial z} + \frac{x}{x^2}\frac{\partial}{\partial y} - \frac{x}{x}\frac{\partial^2}{\partial x\partial y} \\
&\quad + y\frac{\partial^2}{\partial y\partial z} - \frac{y}{x}\frac{\partial^2}{\partial y^2} \\
&\quad - x\frac{\partial^2}{\partial z\partial x} - y\frac{\partial^2}{\partial z\partial y} \\
&\quad + \frac{\partial^2}{\partial y\partial x} + \frac{y}{x}\frac{\partial^2}{\partial y^2} + \frac{1}{x}\frac{\partial}{\partial y} \\
\\
&= \frac{2}{x}\frac{\partial}{\partial y}.
\end{aligned}
$$

And

$$
(dz + (xdy - ydx))([X,Y]) = (x\,dy)\left(\frac{2}{x}\frac{\partial}{\partial y}\right) = x\frac{2}{x} = 2 \neq 0,
$$

so $[X,Y] \notin \mathcal{D}$. Therefore $\mathcal{D}$ cannot be integrable because it is not involutive.

$\square$

Problem 3. Let $M^m \subset \mathbb{R}^N$ be a submanifold of $\mathbb{R}^n$. Given any $z_0 \in \mathbb{R}^{N-p}$, prove that for any open neighborhood U around z_0, there exists a $z \in U$ such that the horizontal slice $M \cap (\mathbb{R}^p \times \{z\})$ is an $(m - N + p)$-dimensional submanifold of M.

Proof.
Here's the construction, let $f \colon M^m \subset \mathbb{R}^N \to \mathbb{R}^p$ be the projection onto the last $N - p$ coordinates:

$$\underbrace{(x_1, x_2, \ldots, x_N)}_{\in M^m \subset \mathbb{R}^N} \overset{f}{\mapsto} (x_{p+1}, x_{p+2}, \ldots, x_N).$$

Because projection maps are smooth, (indeed, this projection is very nearly the model submersion) f is a C^∞ map. As such, we can use the corollary of Sard's theorem which states that the set of regular values of f are dense in $\mathbb{R}^{N-p}$. Thus, for any neighborhood U around z_0 there exists $z \in U \subset \mathbb{R}^{N-p}$ such that $f^{-1}(z)$ is a regular value.

By the corollary of the Implicit Function Theorem for submersion, $f^{-1}(z)$ can be given the structure of a manifold of dimension $\dim(M) - \dim(\mathbb{R}^{N-p}) = m - N + p$ since z is a regular value of f, meaning f is submersive at every point in $f^{-1}(z)$. □

Problem 4. Let $(q, \xi) \in N = T^*M$, and let (U, ϕ) be a chart around q, over which N is trivial.
Let $\lambda \colon N \to T^*N$ be defined as the 1-form that sends $(q, \xi) \mapsto \xi \circ d\pi_{(q,\xi)}$.

(a) Write an expression for $(\tilde{\phi}^{-1})^*(\lambda)$, and verify that λ is a smooth section.

(b) Let $\alpha \in \Omega^1(M)$ be a 1-form on M. Compute the pullback $\alpha^*(\lambda) \in \Omega^1(M)$ as a function of α.

Proof.

(a) Given $\tilde{\phi}^{-1}(q, \xi) = (q_1, \ldots, q_n, \xi_1, \ldots, \xi_n)$

$$\begin{aligned}
(\tilde{\phi}^{-1})^*(\lambda)(q_1, \ldots, q_n, \xi_1, \ldots, \xi_n) &= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), (\tilde{\phi}^{-1})^*(\xi \circ d\pi_{(q,\xi)})) && (1)\\
&= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), (\tilde{\phi}^{-1})^*[\xi \circ \pi - \xi(\pi(q, \xi))]) && (2)\\
&= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), (\tilde{\phi}^{-1})^*[\xi \circ \pi - \xi(q)]) && (3)\\
&= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), (\tilde{\phi}^{-1})^*[\xi \circ \pi]) && (4)\\
&= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), [\xi \circ \pi \circ \tilde{\phi}^{-1}]) && (5)\\
&= ((q_1, \ldots, q_n, \xi_1, \ldots, \xi_n), d(\xi \circ \pi \circ \tilde{\phi}^{-1})_{(q_1, \ldots, q_n, \xi_1, \ldots, \xi_n)}). && (6)
\end{aligned}$$

Step (1) follows from the definition of $(\tilde{\phi}^{-1})^*$, step (2) uses the third intrinsic defintion of the derivative map, step (3) calculates projection π, (4) uses that $\xi \in T_q^*M$, so $\xi(q)$ vanishes, (5) applies $(\tilde{\phi}^{-1})$, and (6) again uses the defintion of the derivative map.

In order to verify that $\lambda \colon N \to T^*N$ is a smooth section, it is enough to note that (i) the projection map composed with λ is the identity, and (ii) λ is smooth. The former comment follows from the above computation, and the latter follows by noting that $\xi \in T_p^*M$ is smooth (by defintiion), the projection map π is smooth, the chart map $\tilde{\phi}^{-1}$ is smooth, the composition of smooth maps is smooth, and the derivative map of a smooth map is smooth. Thus λ is a smooth section.

(b) Let $\alpha \in \Omega^1(M)$ be a 1-form on M. The computation of $\alpha^*(\lambda)(q)$ follows similarly to the one above:

$$\begin{aligned}
\alpha^*(\lambda)(q) &= (q, \alpha^*(\xi \circ d\pi_{\alpha(q)}))\\
&= (q, \alpha^*(\xi \circ [\pi - \pi(\alpha(q))])\\
&= (q, \alpha^*(\xi \circ [\pi - q])\\
&= (q, \alpha^*[\xi \circ \pi - \xi(q)])\\
&= (q, [(\xi \circ \pi - \xi \circ \pi \circ \alpha(q)) \circ \alpha])\\
&= (q, [\xi \circ \pi \circ \alpha - \xi \circ \pi \circ \alpha(q)])\\
&= (q, d(\xi \circ \pi \circ \alpha)_q)\\
&\in T^*M.
\end{aligned}$$

$\square$