Complex Analysis: Homework 11

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Problem 1. (page 251)

If Ω is the punctured disk 0 < |z| < 1 and if f is given by $f(\zeta) = 0$ for $|\zeta| = 1$, f(0) = 1 show that all functions $v \in \mathfrak{B}(f)$ are ≤ 0 in Ω .

Proof.

Problem 2. (page 178)

Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for Re z > 1 and represent its derivative in series form.

Proof.

Assume that z = z + iy for x > 1. Then

$$|\zeta(z)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^z} \right| \tag{1}$$

$$\leq \sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right| \tag{2}$$

$$=\sum_{n=1}^{\infty} \left| \frac{1}{n^{z+iy}} \right| \tag{3}$$

$$=\sum_{n=1}^{\infty} \frac{1}{|n^x||n^{iy}|} \tag{4}$$

$$=\sum_{n=1}^{\infty} \frac{1}{n^x |e^{iy \log(n)}|} \tag{5}$$

$$=\sum_{n=1}^{\infty} \frac{1}{n^x} \tag{6}$$

which is known to converge by the integral test because x > 1. The derivative of $\zeta(z)$ can be computed term by term

$$\frac{d}{dz} \left[\zeta(z) \right] = \frac{d}{dz} \left[\sum_{n=1}^{\infty} n^{-z} \right]$$

$$= \sum_{n=1}^{\infty} \frac{d}{dz} \left[n^{-z} \right]$$

$$= \sum_{n=1}^{\infty} \frac{d}{dz} \left[e^{-z \log(n)} \right]$$

$$= \sum_{n=1}^{\infty} -e^{-z \log(n)} \log(n)$$

$$= \sum_{n=1}^{\infty} -n^{-z} \log(n)$$

Problem 4. (page 178)

As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeros in Ω then f(z) is either identically zero or has at most m zeros.

Proof.

Suppose for the sake of contradiction that f(z) has m+1 zeros. Then we can form a

Problem 5. (page 184)

The Fibonacci numbers are defined by $c_0 = 0, c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}.$$

Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n .

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

$$= \sum_{n=1}^{\infty} c_n z^n$$

$$= z + \sum_{n=2}^{\infty} c_n z^n$$

$$= z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n$$

$$= z + \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n$$

$$= z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2}$$

$$= z + z \sum_{n=1}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n$$

$$= z + z f(z) + z^2 f(z).$$

Then solving for f(z) yields

$$f(z) = \frac{z}{1 - z - z^2},$$

thus f is a rational function. Now we must find a power series representation of f, and its coefficients will give us the terms of the Fibonacci sequence. Since f has roots $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$, partial fraction decomposition yields

$$f(z) = -\frac{z}{(z+\varphi)(z+\psi)}$$
$$= \frac{A}{z+\varphi} + \frac{B}{z+\psi}$$

where A and B satisfy

$$-z = A(z + \psi) + B(z + \varphi)$$

and thus the system of equations

$$A + B = -1$$
$$A\psi + B\varphi = 0.$$

Solving for A and B gives

$$f(z) = \frac{1}{\sqrt{5}} \left(\frac{-\varphi}{z + \varphi} + \frac{\psi}{z + \psi} \right)$$
$$= \frac{1}{\sqrt{5}} \left(-\frac{1}{z/\varphi + 1} + \frac{1}{z/\psi + 1} \right).$$

So the identity

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

together with $-1/\psi = \varphi$ and $-1/\varphi = \psi$ gives

$$f(z) = \sum_{n=0}^{\infty} \frac{-1}{\sqrt{5}} (-z/\varphi)^n + \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (-z/\psi)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} ((-z/\psi)^n - (-z/\varphi)^n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\varphi^n z^n - \psi^n z^n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\varphi^n - \psi^n) z^n$$

Therefore

$$c_n = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

Problem 4. (page 186)

Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}.$$

Calculate B_1 , B_2 , and B_3 .

Proof.

(An idea from https://math.stackexchange.com/q/1006615/121988.)

We know that $f(z) = (e^z - 1)^{-1}$ has a pole at 0, so consider f(z) = 1/(zg(z)) where

$$g(z) = \frac{e^z - 1}{z} = \frac{1}{z} \left(-1 + \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

which is analytic everywhere. When $z \neq 0$, f(z) = h(z)/z where h(z) = 1/g(z). (?) Another idea:

$$f(z) = \frac{-1}{1 - e^z} = -\sum_{n=0}^{\infty} (e^z)^n = -\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)^n$$

Problem 5. (page 186) Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers.

Proof.

Problem 1. (page 190)

Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}, \text{ and } \qquad \sum_{n=1}^{\infty} \frac{1}{n^6}$$

with a complete justifaction of the steps that are needed.

Proof.

By equation (10), we know that

$$\pi \cot \pi z = \lim_{m \to \infty} \sum_{n = -m}^{m} \frac{1}{z - n} = \frac{1}{z} + \sum_{n = 1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Problem 2. (page 190) Express

$$\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form.

Proof.