Math 574: Homework 2

Peter Kagey

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Problem 1.

(a) Show that $A, B \in M_3(K)$ are similar if and only if they have the same minimal and characteristic polynomials.

Proof. A and B are similar if and only if they have the same Jordan normal form, so it is sufficient to compare the minimal and characteristic polynomials of matrices in Jordan normal form.

(a) A 3×3 matrix can have one of three Jordan normal forms

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \; A_2 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \text{or } A_3 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$

Notice that a matrix A

(i) similar to A_1 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly once,

(ii) similar to A_2 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly twice, and

(iii) similar to A_3 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly three times.

(b) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Both A and B have the same minimal and characteristic polynomials

$$\begin{aligned} m_A(x) &= m_B(x) = (1-x)^2 \\ p_A(x) &= p_B(x) = (1-x)^4, \end{aligned}$$

but A and B are not similar because they have different Jordan canonical forms.

□

Problem 2. Fix $A \in M_n(K)$ and let $C(A) = \{B : BA = AB\}$.

Proof.

(a) Firstly, suppose $f(x) \in K[x]$. Then

$$\begin{aligned} Af(A) &= A(c_0 I + c_1 A + \cdots + c_n A^n) \\ &= c_0 A + c_1 A^2 + \cdots + c_n A^{n+1} \\ &= (c_0 I + c_1 A + \cdots + c_n A^n)A \end{aligned}$$

so $f(A) \in C(A)$. Conversely, suppose v is a cyclic vector for A, that is $\{v, Av, A^2v, \cdots, A^{n-1}v\}$ is a basis for $M_n(K)$, meaning we can write

$$Bv = c_0 v + c_1 Av + \cdots + c_n A^n v = f(A)v.$$

Moreover, we can write $B(A^k v) = f(A)(A^k v)$, and since ${A^k}_{k=0}^{n-1}$ is a basis, $B = f(A)$ because it B and $f(A)$ send every basis vector to the same place.
Lastly, $\dim C(A) = n$ because the map

$$(c_0, c_1, \ldots, c_{n-1}) \mapsto c_0 I + c_1 A + \cdots + c_{n-1} A^{n-1} = f(A) = B$$

is surjective since the minimal polynomial is degree n.

(b) Suppose that A is not necessarily cyclic. Notice that the set $\{f(A) : f(x) \in k[x]\}$ is a subspace of $C(A)$ since it is closed under addition and scalar multiplication. However, $\dim \{f(A) : f(x) \in k[x]\}$ is the degree of the minimal polynomial.
Let J_A denote the Jordan normal form of A, $J_A = U^{-1}AU$. Next let $B' = U^{-1}BU$ (with the same U as above) and notice that $\dim(C(A)) = \dim(C(J_A))$:

$$AB = BA \Leftrightarrow A \underbrace{UU^{-1}}_{I} B = B \underbrace{UU^{-1}}_{I} A \Leftrightarrow \underbrace{U^{-1}AU}_{J_A} \underbrace{U^{-1}BU}_{B'} = \underbrace{U^{-1}BU}_{B'} \underbrace{U^{-1}AU}_{J_A}.$$

So it is enough to consider the Jordan normal form of A, thus apply (a) blockwise, since each block is cyclic.

□

Problem 3.

(a) Show that if N is nilpotent then $N^n = 0$.

(b) Show that $\dim \ker N$ is the number of Jordan blocks in its Jordan canonical form.

(c) How many similarity classes of 5×5 nilpotent matrices are there?

Proof.

(a) If $N^d = 0$, then there exists some $d' \leq n$ such that $N^{d'} = 0$ because the minimal polynomial $m_N(x) | x^d$, so the minimal polynomial is of the form $m_N(x) = x^{d'}$ with $d' <= n$, since the minimal polynomial has degree less than or equal to n.

(b) Because the minimal polynomial is of the form $m_N(x) = x^{d'}$, the Jordan canonical form of N must have all zeros on the diagonal. Now it is enough to go block by block, and show that each block has nullity of exactly 1, namely

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(m-1)1} \\ a_{m1} \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \\ 0 \end{bmatrix}$$

so $a_1 1$ is a free variable and the dimension of the kernel of any block is 1. By induction on the number of Jordan blocks, the dimension of the kernel is equal to the number of blocks.

(c) By (a), the characteristic polynomial of a 5×5 nilpotent matrix is $p(x) = x^5$, so in Jordan canonical form, $a_{ii} = 0$. Thus the Jordan canonical form of A has zeros along the diagonal, with possibly some ones on the superdiagonal:

$$\begin{bmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus similarity classes of nilpotent matrices are in bijection with the seven partitions of 5 via the size of the Jordan blocks:

$$\begin{aligned}&5\\&4+1\\&3+2\\&3+1+1\\&2+2+1\\&2+1+1+1\\&1+1+1+1+1\end{aligned}$$

□

Problem 4. Let $T\colon V \to V$ be a linear transformation, and fix $v \in V$.

(a) Show that there is a unique polynomial $g(x)$ so that $h(T)v = 0$ if and only if $g|h$ and $g|m_T$.

(b) Show that the degree of G is the dimension of the span of $\{v, Tv, T^2v, \dots\}$.

Proof. First suppose that $\dim V = n$.

(a) First notice that $\{v, Tv, T^2v, \dots, T^nv\}$ cannot be linearly independent, because it consists of $n+1$ vectors. Now take the largest k such that $\{v, Tv, T^2v, \dots, T^kv\}$ *is* then, in particular,

$$h(T)v = c_{k+1}T^{k+1}v + c_kT^kv + \cdots + c_1Tv + c_0v = 0.$$

This forces c_{k+1} to be nonzero, so let $g(x) = h(x)/c_n$, so that $g(x)$ is monic.
Notice that there cannot be a different polynomial f of equal or lower degree than g, otherwise the set $\{v, Tv, T^2v, \dots, T^kv\}$ would be linearly dependent. Thus every polynomial $h(x)$ such that $h(T)v = 0$ is a multiple of $g(x)$.

(b) By construction, $\{v, Tv, T^2v, \dots, T^kv\}$ is the maximal linearly independent set. Moreover T^mv can be written as a linear combination of this linearly independent set for all $m > k$, so applying this procedure inductively, the aforementioned set is a basis for $\{v, Tv, T^2v, \dots\}$.

□

Problem 5.

Proof.

(a) Let $A = U^{-1}J_AU$, where J_A is the Jordan canonical form of A, which can be written $J_A = D_A + N_A$, with D_A the diagonal entries of J_A, and N_A the superdiagonal entries of J_A.

$$\underbrace{\begin{bmatrix}\lambda_1 & * & & & \\ & \lambda_2 & * & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & * \\ & & & & \lambda_n\end{bmatrix}}_{J_A} = \underbrace{\begin{bmatrix}\lambda_1 & 0 & & & \\ & \lambda_2 & 0 & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & 0 \\ & & & & \lambda_n\end{bmatrix}}_{D_A} = \underbrace{\begin{bmatrix}0 & * & & & \\ & 0 & * & & \\ & & 0 & \ddots & \\ & & & \ddots & * \\ & & & & 0\end{bmatrix}}_{N_A}$$

Notice that $p_{N_A}(x) = x^n$, so by Cayley-Hamilton, $N_A^n = 0$ and N_A. is nilpotent. Next, notice that

$$\begin{aligned}A &= U^{-1}J_AU\\ &= U^{-1}(D_A + N_A)U\\ &= U^{-1}D_AU + U^{-1}N_AU\end{aligned}$$

where $U^{-1}D_AU$ is clearly diagonalizable, and where $U^{-1}N_AU$ is nilpotent because

$$\begin{aligned}(U^{-1}N_AU)^n &= \underbrace{(U^{-1}N_AU)(U^{-1}N_AU)\dots(U^{-1}N_AU)}_{n} \\ &= U^{-1}\underbrace{N_A^n}_{0}U \\ &= 0.\end{aligned}$$

(b)

$\square$