Differential Geometry: Homework 5

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Problem 1. Let $M = f^{-1}(y)$ be the primage of a regular value $y \in \mathbb{R}^{N-m}$ of a (smooth) submersion $f\colon \mathbb{R}^N \to \mathbb{R}^{N-m}$.

(a) Let $\widetilde{TM} = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N : x \in M, v \in \ker df_x\}$. Show that as defined $\widetilde{TM}$ is a smooth submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2m$.

(b) Prove that there is a diffeomorphism between $\widetilde{TM}$ and the *tangent bundle of* M as defined in class:

$$\widetilde{TM} \cong TM$$

in a manner compatible with projection to M.

Proof.

(a) Because M is the preimage of a smooth submersion, by the implicit function theorem, M is an m-dimensional manifold, with maximal atlas $\mathcal{A}_M = \{\, (U_i^M, \phi_i^M : U_i^M \to \mathbb{R}^m)\,\}_{i\in I}$.

Let π be the projection onto the first N coordinates, then

$$\phi_i \circ \pi\colon \underbrace{\pi^{-1}(U_i)}_{\subset\mathbb{R}^{2m}} \to \mathbb{R}^m$$

is the composition of smooth maps, so is smooth.

Similarly, because f is a submersion, $df_x\colon \mathbb{R}^N \to \mathbb{R}^{N-m}$ (which is a linear map) is a surjection, and so is of full rank. Therefore $\ker df_x \cong \mathbb{R}^m$ via some isomorphism ψ_x. Thus we can construct an atlas for $\widetilde{TM}$ where each chart consists of the set $\widetilde{U}_i = U_i \times \mathbb{R}^m$ together with the function

$$\widetilde{\phi}_i\colon \widetilde{U}_i \to \mathbb{R}^{2m} \text{ which sends } \underbrace{(x, v)}_{\in U_i\times\mathbb{R}^m} \mapsto \underbrace{(\phi(x), \psi_x(v))}_{\in\mathbb{R}^{2m}}$$

and so $\widetilde{TM}$ is a submanifold of dimension $2m$.

(b) (I'm not sure I understand this problem.)
As defined in class,

$$TM = \bigsqcup_{p\in M} T_pM = \{\, (p, \vec{v}) : \vec{v} \in T_pM \,\}.$$

By using the second extrinsic defintion of a tangent space (given in lecture), $T_pM = \ker df_p$, so the identity map is a perfectly good diffeomorphism from $\widetilde{TM}$ to TM. (In order to use a different definition, one must only construct a diffeomorphism between the desired definition and the second extrinsic definition.)

□

Problem 2. Let M^m be a manifold of dimension m and $p \in M$ a point. Recall that $\mathcal{F}_p \subset C^\infty(p)$ is the ideal of germs of functions on M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \dots f_k$, where $f_i \in \mathcal{F}_p$.

(a) Prove that, in every set of local coordinates $(x_1, \dots, x_k)$ around the point p, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes to order k. You may assume a version of Taylors approximation theorem stated in class.

(b) Compute the dimension of $\mathcal{F}_p^k/\mathcal{F}_p^{k+1}$.

(c) Construct a smooth manifold along with a map to M, $E \xrightarrow{\pi} M$ whose "fiber" $E_p = \pi^{-1}(p)$ at the point $p \in M$ is $\mathcal{F}_p^1/\mathcal{F}_p^3$.

Proof.

(a) Because each germ $f_\ell \in \mathcal{F}_p$ consists of representatives that agree on a small neighborhood around p, all representatives have identical Taylor expansions

$$f_\ell(x) = \underbrace{f_\ell(p)}_{0} + \sum_i a_{\ell,i} x_i + \sum_{i,j} g_{\ell,ij}(x) x_i x_j$$

where $a_i^\ell, a_{ij}^\ell \in \mathbb{R}$, $g_{ij}^\ell \in C^\infty(p)$, and $f_\ell(p) = 0$ by defintion of $\mathcal{F}_p$. Then an element of $\mathcal{F}_p^k$ is generated by

$$f_1 f_2 \cdots f_k = \left(\sum_i a_{1,i} x_i + \sum_{i,j} g_{1,ij}(x) x_i x_j \right) \cdots \left(\sum_i a_{k,i} x_i + \sum_{i,j} g_{k,ij}(x) x_i x_j \right)$$

and therefore each term of each element in the generating set vanishes to order k, so the Taylor expansion of any element in $\mathcal{F}_p^k$ vanishes to order k.

(b) Each equivalence class of germs in $\mathcal{F}_p^k/\mathcal{F}_p^{k+1}$ consists of functions whose Taylor expansions vanish to order k, and that are equivalent if their order k terms are identical. Thus the dimension of $\mathcal{F}_p^k/\mathcal{F}_p^{k+1}$ is m^k, because there are m^k ways of choosing k elements from $\{x_1, \dots, x_m\}$ with replacement, and so there are m^k possible coefficients $a_{i_1 \dots i_k}$ for the order k term.

(c) Let

$$E = \bigsqcup_{p \in M} E_p = \{ (p, \mathcal{F}_p/\mathcal{F}_p^3) : p \in M \}.$$

and let $\pi : E \to M$ map $E_p \mapsto p$.

Denote the atlas of M by $\mathcal{A}_M = \{(U_i, \phi_i)\}_{i \in I}$, and let $\widetilde{U}_i = \pi^{-1}(U_i)$ (that is, the germs of all functions that vanish at a point in U_i modulo terms of order 3 and greater), and let $\widetilde{\phi}_p : \widetilde{U}_p \to \mathbb{R}^{2m+m^2}$ map (the germ of) a function to the point that it vanishes (p) at and the coefficients (of order 1 and 2) in its Taylor expansion centered at p (in local coordinates with respect to ϕ):

$$[f] = \left[\sum_{i=1}^m a_i x_i + \sum_{i=1}^m \sum_{j=1}^m a_{ij} x_i x_j \right] \mapsto (\underbrace{p_1, \dots, p_m}_{\in \mathbb{R}^m}, \underbrace{a_1, \dots, a_m, a_{11}, a_{12}, \dots a_{1m}, a_{21}, \dots a_{mm}}_{\in \mathbb{R}^{m+m^2}}).$$

$\square$

Problem 3. Let $f\colon M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{tikzcd} \Omega^0(N) \arrow[r, "f_0^*"] \arrow[d, "d_N"] & \Omega^0(M) \arrow[d, "d_M"] \\ \Omega^1(N) \arrow[r, "f_1^*"] & \Omega^1(M) \end{tikzcd}$$

Proof.
It is sufficient to show that $d_M(f_0^*(g)) = f_1^*(d_N(g))$ for all functions $g \in \Omega^0(N) = C^\infty(N)$.

Thus taking an arbitrary function $g \in C^\infty(N)$ and arbitrary point $p \in M$, the "upper right" path of the diagram yields a contangent vector in T^*M:

$$\begin{aligned} d_M(f_0^*(g))(p) &= d_M(g \circ f)(p) \\ &= (p, d(g \circ f)_p) \\ &= (p, [g \circ f - g \circ f(p)]) \in T^*M. \end{aligned}$$

Similarly, evaluating the "lower right" path of the diagram with the same function and point yields the same cotangent vector:

$$\begin{aligned} f_1^*(d_N(g))(p) &= (p, f^*(d_N(g)_{f(p)})) \\ &= (p, f^*[g - g(f(p))]) \\ &= (p, [g \circ f - g \circ f(p)]) \in T^*M. \end{aligned}$$

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Therefore $d_M(f_0^*(g)) = f_1^*(d_N(g))$, and the diagram commutes. $\square$

Problem 4. Give a detailed proof that the cotangent bundle T^*M is a smooth manifold and that the projection map $\pi\colon T^*M \to M$ is a smooth map.

Proof.
Use the defintion of T^*M from class:

$$T^*M = \bigsqcup_{p\in M} T_p^*M = \{\, (p, v^*) \mid p \in M, v^* \in T_p^*M \,\}$$

with a topology given by

$$\mathcal{T}_{T^*M} = \{\, W \subset T^*M \mid \widetilde{\phi}_i(W \cap \widetilde{U}_i) \text{ is open in } \mathbb{R}^{2n} \text{ for all } i \in I \,\}$$

where

(a) $\mathcal{A}_M = \{(U_i, \phi_i)\}_{i\in I}$ is M's maximal atlas.

(b) $\pi\colon T^*M \to M$ is a map that sends $(p, \vec{v}) \mapsto p$.

(c) $\widetilde{U}_i = \pi^{-1}(U_i) \subset T^*M$

(d) $\widetilde{\phi}_i\colon \widetilde{U}_i \to \phi_i(U_i) \times \mathbb{R}^m$ is a map that sends $(p, \vec{v}) \mapsto (\phi_i(p), d(\phi_i)_p(\vec{v}))$.

Then the atlas on T^*M is given by

$$\mathcal{A}_{T^*M} = \{(\widetilde{U}_i, \widetilde{\phi}_i)\}_{i\in I}.$$

It is sufficient to show that (i) $\{\widetilde{U}_i\}_{i\in I}$ is an open cover of T^*M, (ii) $\mathcal{A}_{T^*M}$ has smooth transition maps, and (iii) $\pi\colon T^*M \to M$ is a C^∞ map.

(i) Let $(p, v*) \in T_p^*M$. Then there exists some U_i such that $p \in U_i$ because the atlas $\mathcal{A}_M$ covers M. Take this U_i, and $\widetilde{(U)}_i = \pi^{-1}(U_i)$ is an open set which contains $(p, v*)$. Thus every point is in an open set, and $\{\widetilde{U}\}_{i\in I}$ is an open cover of T^*M.

(ii) Suppose that $(\widetilde{U}, \widetilde{\phi})$ and $(\widetilde{V}, \widetilde{\psi})$ are charts from open subsets of T^*M to $\mathbb{R}^{2n}$ with nonempty intersection. Then $\widetilde{\phi} \circ \widetilde{\psi}^{-1}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ maps

$$(x, y) \xmapsto{\widetilde{\psi}^{-1}} (\psi^{-1}(x), d\psi_x^{-1}(y)) \xmapsto{\widetilde{\phi}} (\phi \circ \psi^{-1}(x), d\phi_{\psi^{-1}(x)}(d\psi_x^{-1}(y))).$$

$\phi{\circ}\psi^{-1}$ is a C^∞ map because this is inherited from the C^∞ transition maps on M, and $d\phi_{\psi^{-1}(x)}(d\psi_x^{-1}(y))$ is smooth because it is the derivative of the C^∞ map

$$d\phi_{\psi^{-1}(x)}(d\psi_x^{-1}(y)) = d(\phi \circ \psi^{-1})_x(y)$$

(iii) By definition, π is a C^∞ map if for each point $(p, v*) \in T^*M$, there exists a chart (U_i, ϕ_i) around p such that $\phi_i \circ \pi \circ \widetilde{\phi}_i^{-1}\colon \mathbb{R}^{2m} \to \mathbb{R}^m$ is smooth. However, $\phi_i \circ \pi \circ \widetilde{\phi}_i^{-1}\colon$ is simply the projection

$$(x, y) \xmapsto{\widetilde{\phi}_i^{-1}} (\phi_i^{-1}(x), d(\phi_i^{-1})_x(y)) \xmapsto{\pi} \phi_i^{-1}(x) \xmapsto{\phi_i} x,$$

and projections are smooth.

∎

Problem 5. Let $f, g\colon M \to \mathbb{R}$ be smooth real-valued functions on a manifold M. Prove that

$$d(f \cdot g) = f \cdot dg + g \cdot df.$$

Proof.
Define the map $d\colon C^\infty(M) \to (M \to TM)$ is by the "third" intrinisc defintion of a tangent space:

$$d(h) = p \mapsto [h - h(p)].$$

Then

$$d(f \cdot g) = p \mapsto [f \cdot g - f(p)g(p)]$$

and

$$\begin{aligned}(f \cdot dg + g \cdot df) &= p \mapsto f(p)[g - g(p)] + g(p)[f - f(p)] \\ &= p \mapsto [f(p)g - f(p)g(p) + g(p)f - g(p)f(p)]\end{aligned}$$

Now, in order to show that

$$[f \cdot g - f(p)g(p)] = [f(p)g + g(p)f - 2f(p)g(p)] \in \mathcal{F}_p / \mathcal{F}_p^2$$

are both representatives of the same equivalence class, it is enough to show that their Taylor expansions in local coordinates (around $\phi(p)$) agree up to their first order terms.

$$\begin{aligned}((f \cdot g) - f(p)g(p)) \circ \phi^{-1} &= (f \cdot g) \circ \phi^{-1} - f(p)g(p) \\ &= (f \circ \phi^{-1}) \cdot (g \circ \phi^{-1}) - f(p)g(p) \\ &= 0 + d((f \circ \phi^{-1}) \cdot (g \circ \phi^{-1}) - f(p)g(p))_{\phi(p)}(x) + \underbrace{\sum_{i,j} a_{ij}(x) x_i x_j}_{R(x)} \\ &= (d(f \circ \phi^{-1})_{\phi(p)} \cdot (g \circ \phi^{-1})(\phi(p)) \\ &\qquad + (f \circ \phi^{-1})(\phi(p) \cdot d(g \circ \phi^{-1})_{\phi(p)}))(x) + R(x) \qquad (1) \\ &= \left(d(f \circ \phi^{-1})_{\phi(p)} \cdot g(p)) + (f(p) \cdot d(g \circ \phi^{-1})_{\phi(p)})\right)(x) + R(x) \qquad (2)\end{aligned}$$

and

$$\begin{aligned}(f(p)g + g(p)f - 2f(p)g(p)) \circ \phi^{-1} &= f(p)(g \circ \phi^{-1}) + g(p)(f \circ \phi^{-1}) - 2f(p)g(p) \\ &= f(p) \cdot d(g \circ \phi^{-1})_{\phi(p)}(x) + g(p) \cdot d(f \circ \phi^{-1})_{\phi(p)}(x) + R(x) \qquad (3)\end{aligned}$$

So after a half-page of alphabet soup (where (1) follows from the product rule on functions from $\mathbb{R}^n$ to $\mathbb{R}$), it can be seen that the expressions in (2) and (3) are equal up to a second-order remainder term—meaning that the two germs are in the same equivalence class, and

$$d(f \cdot g) = f \cdot dg + g \cdot df.$$

□

Problem 6. Let $i\colon S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$ be the map $\theta \mapsto (\cos(\theta), \sin(\theta))$. Compute

$$i^*((x^2 + y)dx + (3 + xy^2)dy).$$

Proof.
By the footnote,

$$(x^2 + y)dx + (3 + xy^2)dy = (x, y) \mapsto ((x, y), (x^2 + y)dx + (3 + xy^2)dy),$$

so applying the 1-form to the function i^* yields the 1-form

$$\begin{aligned} i^*((x^2 + y)dx + (3 + xy^2)dy) &= \theta \mapsto (\theta, (\cos^2(\theta) + \sin(\theta))d(\cos) + (3 + \cos(\theta)\sin^2(\theta))d(\sin)) \\ &= \theta \mapsto (\theta, -(\cos^2(\theta) + \sin(\theta))\sin(\theta)d\theta + (3 + \cos(\theta)\sin^2(\theta))\cos(\theta)d\theta) \\ &= \theta \mapsto (\theta, (-\cos^2(\theta)\sin(\theta) + \sin^2(\theta) + 3\sin(\theta) + \cos(\theta)\sin^3(\theta))d\theta). \end{aligned}$$

Where $d(\cos) = -\sin(\theta)d\theta$ by the Taylor series expansion of $\cos - \cos(\theta)$,

$$\begin{aligned} d(\cos) &= \theta \mapsto d(\cos)_\theta \\ &= \theta \mapsto [\cos - \cos(\theta)] \\ &= \theta \mapsto [\varphi \mapsto \cos(\theta) - \varphi\sin(\theta) + \varphi^2 a(\varphi) - \cos(p)] \\ &= \theta \mapsto [\varphi \mapsto -\varphi\sin(\theta)] \\ &= \theta \mapsto -\sin(\theta)[\mathrm{id}] \\ &= -\sin(\theta)d\theta, \end{aligned}$$

and $d(\sin) = \cos(\theta)d\theta$ follows similarly. $\square$