Math 533: Homework 7

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Problem 1. Write the character table for S_4.

Proof. First we have cycle types corresponding to the partitions

$$\begin{aligned}
&1 \text{ permutation with cycle type } && (1,1,1,1)\\
&6 \text{ permutations with cycle type } && (2,1,1)\\
&3 \text{ permutations with cycle type } && (1,1,1,1)\\
&8 \text{ permutations with cycle type } && (3,1)\\
&6 \text{ permutations with cycle type } && (4).
\end{aligned}$$

We begin the character table with the trivial and the signed representations. Next, since there is only one way to write $4! = 24$ as a sum of five squares, namely $1 + 1 + 2^2 + 3^2 + 3^2$, this is the first column.

	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
χ^{triv}	1	1	1	1	1
χ^{sgn}	1	-1	1	1	-1
χ'	2	?	?	?	?
χ''	3	?	?	?	?
χ'''	3	?	?	?	?

We know that the defining/standard representation has

$$\begin{aligned}
\chi^{\text{std}}((1)(2)(3)(4)) &= 4\\
\chi^{\text{std}}((12)(3)(4)) &= 2\\
\chi^{\text{std}}((12)(34)) &= 0\\
\chi^{\text{std}}((123)(4)) &= 1\\
\chi^{\text{std}}((1234)) &= 0
\end{aligned}$$

And by taking the inner products with the two known characters gives

$$\begin{aligned}
\frac{1}{4!}\langle \chi^{\text{def}}, \chi^{(1,1,1,1)}\rangle &= \frac{1\cdot 4 + 6\cdot 2 + 3\cdot 0 + 8\cdot 1 + 6\cdot 0}{4!} = 1\\
\frac{1}{4!}\langle \chi^{\text{def}}, \chi^{(4)}\rangle &= \frac{1\cdot 4 - 6\cdot 2 + 3\cdot 0 + 8\cdot 1 - 6\cdot 0}{4!} = 0
\end{aligned}$$

Subtracting off the trivial representation gives

	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
χ^3	3	1	-1	0	-1

which is indeed irreducible

$$\langle \chi^3, \chi^3\rangle = \frac{1\cdot 3^2 + 6\cdot 1^2 + 3\cdot(-1)^2 + 8\cdot 0 + 6\cdot(-1)^2}{4!} = 1.$$

Then the tensor product of χ^{sgn} and χ^3 gives χ^4

	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ^4	3	−1	−1	0	1

which is also irreducible

$$\langle \chi^4, \chi^4 \rangle = \frac{1 \cdot 3^2 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2 + 8 \cdot 0 + 6 \cdot 1^2}{4!} = 1.$$

This leaves just one more row which can be deduced from orthogonality.

$$\langle \chi^{\text{triv}}, \chi^5 \rangle = 1 \cdot 2 + 6a + 3b + 8c + 6d = 0 \tag{1}$$
$$\langle \chi^{\text{sgn}}, \chi^5 \rangle = 1 \cdot 2 - 6a + 3b + 8c - 6d = 0 \tag{2}$$
$$\langle \chi^3, \chi^5 \rangle = 1 \cdot 6 + 6a - 3b + 8 \cdot 0 - 6d = 0 \tag{3}$$
$$\langle \chi^4, \chi^5 \rangle = 1 \cdot 6 - 6a - 3b + 8 \cdot 0 + 6d = 0 \tag{4}$$

Adding equations (1) and (2) gives $4 + 6b + 16c = 0$, and adding equations (3) and (4) gives $12 - 6b = 0$, so $b = 2$ and $c = -1$.

This gives

	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ^5	2	?	2	−1	?

but

$$\langle \chi^4, \chi^4 \rangle = \frac{1 \cdot 2^2 + 6a^2 + 3 \cdot 2^2 + 8 \cdot (-1)^2 + 6d^2}{4!} = \frac{4 + 6a^2 + 12 + 8 + 6d^2}{4!} = 1$$

so $a = d = 0$, and the full table is

	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ^{triv}	1	1	1	1	1
χ^{sgn}	1	−1	1	1	−1
χ^3	3	1	−1	0	−1
χ^4	3	−1	−1	0	1
χ^5	2	0	2	−1	0

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