Complex Analysis: Homework 12

Peter Kagey

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Problem 2. (page 193)
Prove that for $|z| < 1$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots = \frac{1}{1-z}$$

Proof.
By induction, notice that the partial products of $(1 + z^{2^k})$ have the form

$$\prod_{n=0}^{N}\left(1+z^{2^n}\right) = 1 + z + z^2 + \ldots + z^{2^{N+1}-1}.$$

The base case is clear, when $N = 0$, we have that the product is equal to $1 + z$. Then, the inductive step shows that

$$\begin{aligned}
\prod_{n=0}^{N}\left(1+z^{2^n}\right) &= \left(1+z^{2^N}\right)\prod_{n=0}^{N-1}\left(1+z^{2^n}\right) \\
&= \left(1+z^{2^N}\right)\left(1+z+z^2+\ldots+z^{2^N-1}\right) \\
&= \left(1+z+z^2+\ldots+z^{2^N-1}\right) + z^{2^N}\left(1+z+z^2+\ldots+z^{2^N-1}\right) \\
&= 1+z+z^2+\ldots+z^{2^N-1}+z^{2^N}+z^{2^N+1}+\ldots+z^{2^{N+1}-1} \\
&= \sum_{k=0}^{2^{N+1}-1} z^k
\end{aligned}$$

As we have seen before, the sum $1 + z + z^2 + z^3 + \ldots$ converges to $(1 - z)^{-1}$ for $|z| < 1$. Thus we get the original identity in the limit. $\square$

Problem 5. (page 193)
Suppose $|h| < 1$. Show that the function

$$\theta(z) = \prod_{n=1}^{\infty} (1 + h^{2n-1}e^z)(1 + h^{2n-1}e^{-z})$$

is analytic in the whole plane and satisfies the functional equation

$$\theta(z + 2\log h) = \frac{\theta(z)}{he^z}.$$

Proof.
Without doing anything clever

$$\begin{aligned}
\theta(z + 2\log h) &= \prod_{n=1}^{\infty} (1 + h^{2n-1}e^{z+2\log h})(1 + h^{2n-1}e^{-(z+2\log h)}) \\
&= \prod_{n=1}^{\infty} (1 + h^{2n+1}e^z)(1 + h^{2n-3}e^{-z}) \\
&= \frac{1}{he^z} \prod_{n=1}^{\infty} (1 + h^{2n+1}e^z)(e^z h + h^{2n-2}).
\end{aligned}$$

Now it is sufficient to show that

$$\prod_{n=1}^{\infty} (1 + h^{2n+1}e^z)(e^z h + h^{2n-2}) = \theta(z).$$

Convergence in the whole plane follows from splitting the product

$$\theta(z) = \prod_{n=1}^{\infty} (1 + h^{2n-1}e^z) \cdot \prod_{n=1}^{\infty} (1 + h^{2n-1}e^{-z})$$

which, by Theorem 6, converges if

$$\sum_{n=1}^{\infty} h^{2n-1}e^z < \infty \text{ and } \sum_{n=1}^{\infty} h^{2n-1}e^{-z} < \infty$$

but since $|h^2| < |h| < 1$, these can be evaluated directly

$$\begin{aligned}
\sum_{n=1}^{\infty} h^{2n-1}e^z &= \frac{e^z}{h} \sum_{n=1}^{\infty} (h^2)^n < \infty \text{ and} \\
\sum_{n=1}^{\infty} h^{2n-1}e^{-z} &= \frac{1}{he^z} \sum_{n=1}^{\infty} (h^2)^n < \infty \text{ for all } z \in \mathbb{C}.
\end{aligned}$$

Since the convergent product of analytic functions is analytic, $\theta(z)$ is analytic in the whole plane. $\square$

Problem 1. (page 197)
Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

Proof.
We know by Theorem 7 that we can construct a function g with simple zeros at each a_n. Then let

$$f(z) = \sum_{n=1}^{\infty} \underbrace{\frac{g(z)}{g'(a_n)(z-a_n)}}_{\to 1 \text{ as } z \to a_n} e^{\gamma_n (z-a_n)} \cdot A_n.$$

It is clear that $f(a_n) = A_n$ as long as f converges. □

Problem 3. (page 197)
What is the genus of $\cos\sqrt{z}$?

Proof.
The roots of $\cos z$ are all real: $\pi(1+2k)$ so the roots of $\cos\sqrt{z}$ are $\pi^2(1+2k)^2$. Thus the genus of $\cos\sqrt{z}$ is the least h such that the following sum converges

$$\sum_{k=0}^{\infty}\frac{1}{|\pi^2(1+2k)^2|^{h+1}}+\sum_{k=1}^{\infty}\frac{1}{|\pi^2(1-2k)^2|^{h+1}}.$$

Both sums converge when $h=0$ by the limit comparison test with $1/k^2$, so the genus of $\cos z$ is 0.

$\square$

Problem 5. (page 197)
Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real. (Hint: Consider $\operatorname{Im} f'(z)/f(z)$.)

Proof.
In the both the upper half plane and the lower half plane, f is nonzero, so f'/f is analytic, and $\operatorname{Im}(f'/f)$ is harmonic on both regions. By the maximum modulus principle, $|f'/f|$ must attain its minimum only on its boundary—thus $|f'/f|$ must have zeros only on the real axis. Therefore all zeros of f' are real. $\square$

Problem 3. (page 200)
What are the residues of $\Gamma(z)$ at the poles $z = -n$?

Proof.
Using $\Gamma(z) = \Gamma(z+1)/z$, and induction with base case $\Gamma(1) = 1$.

$$\lim_{z \to 0} z\Gamma(z) = \lim_{z \to 0} \Gamma(z+1) = \Gamma(1) = 1$$

Then

$$\begin{aligned}
&\lim_{z \to -n} z\Gamma(z) \\
&\qquad = \lim_{z \to -n} \Gamma(z+1) \\
&\qquad = \lim_{z \to -n} \frac{\Gamma(z+2)}{z+1} \\
&\qquad = \dots \\
&\qquad = \lim_{z \to -n} \frac{\Gamma(z + -z - 1)}{(z+1)(z+2)\dots(z+n+2)} \\
&\qquad = (-1)^n \frac{\Gamma(0)}{(1-n)(2-n) \cdot \ldots \cdot 2 \cdot 1}
\end{aligned}$$

Thus the residue is $(-1)^n/n!$. $\square$