Combinatorics: Homework 12

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Problem 1. Show that $R_2(3,4) = 9$.

Proof.
$R_2(3,4) > 8$ because the coloring of K_8 in Figure 1 has no subgraphs K_3 which are colored blue (solid), because the only blue edges are of neighbors around the border or opposite vertices. The coloring has no subgraphs of K_4 colored red (dashed) because no edges around the border are colored red, so a choice of K_4 must include "every other" vertex around the border, and the two choices are constructed so that either choice has two blue edges.

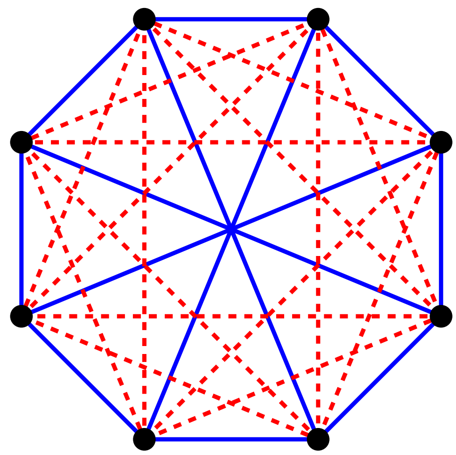

Figure 1: A 2-coloring of K_8 which has no red K_4 subgraphs or blue K_3 subgraphs.

Therefore it is sufficient to prove that $R_2(3,4) \leq 9$. We saw in class that $R_2(3,3) = 6$, and $R_2(2,4) = 4$. Therefore take any vertex v in K_9. Since v has degree 8, there are three cases to consider.

Case 1. Assume there are 5 red edges and 3 blue edges. Not all vertices can be in this case, because the sum of degrees must be even. If all nine vertices have three blue edges, then the sum of degrees in the blue subgraph is 27, a contradiction. Thus, it is always possible to choose a vertex in one of the following two cases.

Case 2. Assume there are 6 or more edges which are colored red. In this case, take six vertices which are connected to v via a red edge, and look at the K_6 subgraph on these vertices. Because $R_2(3,3) = 6$ either K_6 has a blue K_3 (in which case K_9 does too, and we're done) or K_6 has a red K_3, in which case $K_3 \cup v$ is a complete graph on 4 red vertices.

Case 3. Assume there are 4 or more edges which are colored blue. In this case, take four vertices which are connected to v via a blue edge, and look at the K_4 subgraph on these vertices. Because $R_2(4,2) = 4$ either K_4 is all red (and thus has a red K_4 subgraph, and we're done) or K_4 has a blue K_2 subgraph, in which case $K_2 \cup v$ is a complete graph on 3 blue vertices. □

Problem 2. Define M_n recursively by

$$\begin{aligned} M_1 &= 3 \\ M_m &= mM_{m-1} - m + 2. \end{aligned}$$

Show that

(a) $M_m = \lfloor m!\, e \rfloor + 1$

(b) If each edge on the complete graph K_{M_m} is colored in one of m colors, then there exists a monochromatic triangle.

Solution.

(a) By induction, the base case $m = 1$ is clear since $M_1 = \lfloor 1!\, e \rfloor + 1 = 2 + 1 = 3$. Thus the identity

$$\begin{aligned} \lfloor m!\, e \rfloor &= \left\lfloor \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!} + \frac{m!}{(m+1)!} + \ldots \right\rfloor \\ &= \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!}. \end{aligned}$$

holds for all $m \geq 1$ since

$$\frac{m!}{(m+1)!} + \frac{m!}{(m+2)!} + \ldots = \frac{1}{(m+1)} + \frac{1}{(m+1)(m+2)} + \ldots \leq e - 2 < 1$$

Therefore,

$$\begin{aligned} M_{m+1} &= (m+1)M_m - (m+1) + 2 \\ &= (m+1)(\lfloor m!\, e \rfloor + 1) - (m+1) + 2 \\ &= (m+1)\lfloor m!\, e \rfloor + 2 \\ &= (m+1)\left(\frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!}\right) + 2 \\ &= \frac{(m+1)!}{0!} + \frac{(m+1)!}{1!} + \ldots + \frac{(m+1)!}{m!} + 2 \\ &= \underbrace{\frac{(m+1)!}{0!} + \frac{(m+1)!}{1!} + \ldots + \frac{(m+1)!}{m!} + \frac{(m+1)!}{(m+1)!}}_{\lfloor (m+1)!\, e \rfloor} \underbrace{- \frac{(m+1)!}{(m+1)!}}_{-1} + 2 \\ &= \lfloor (m+1)!\, e \rfloor + 1. \end{aligned}$$

(b) By induction, where the base case $m = 1$ is clear from a 1-coloring of K_3 being a monochromatic triangle.

A vertex v in K_{M_m} has degree

$$\deg(v) = M_m - 1 = \lfloor m!\, e \rfloor = \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!}$$

, so by the pigeonhole principle, there are at least

$$\begin{aligned} \left\lceil \frac{\deg(v)}{m} \right\rceil &= \left\lceil \underbrace{\frac{(m-1)!}{0!} + \frac{(m-1)!}{1!} + \ldots + \frac{(m-1)!}{(m-1)!}}_{\lfloor (m-1)!\, e \rfloor} + \frac{(m-1)!}{m!} \right\rceil \\ &= \lfloor (m-1)!\, e \rfloor + 1 \\ &= M_{m-1} \end{aligned}$$

vertices with the same color, say, blue. Therefore, by the induction hypothesis, the subgraph on these M_{m-1} vertices either have a monochromatic triangle in the colors up to blue, or it has a blue edge, and this constructs a blue triangle.

Problem 3. Let $r \in \mathbb{P}$. Prove that any 2-coloring of the complete graph on $(r-1)^2+1$ vertices has either every red tree or every green tree in r vertices.

Proof.
As we showed in class, for a particular tree T on g vertices,

$$K_{(r-1)(g-1)+1}$$

either has a red K_r or a green T.

If $K_{(r-1)(g-1)+1}$ does not have a red K_r, it follows that it must have *every* green T on g vertices, since T was arbitrary to begin with.

If $K_{(r-1)(g-1)+1}$ does have a red K_r, then any tree on r vertices can be realized as a spanning tree on K_r, so K_r (and thus $K_{(r-1)(b-1)+1}$) has every red tree on r vertices. $\square$