Differential Geometry: Homework 4

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Problem 1. Let $M \subset \mathbb{R}^N$ be a submanifold of $\mathbb{R}^N$, and $p \in M$ a point. Verify that the two extrinsic definitions of tangent space to M at p (the second one requiring us to further assume that M is $f^{-1}(y)$ for some smooth function $f\colon \mathbb{R}^N \to \mathbb{R}^{N-m}$ and some regular value $y \in \mathbb{R}^{N-m}$) and the first intrinsic definition of tangent space given in class are all naturally isomorphic.

Proof.
The proof will follow in three parts: starting with a curve $\alpha\colon (-\epsilon, \epsilon) \to M \in \mathbb{R}^m$ centered at $p \in M$, we will call the derivative at 0 $\alpha'(0) = v \in T_p^{(1)}M$; we will then show that this we will show that this exact vector satisfies the conditions of definition 2, so $v \in T_p^{(2)}M$; we will then construct a equivalence class $[\alpha]$ and use this equivalence class to recover v.

(1) $\implies$ (2) Define the map $\phi_{1\Rightarrow 2} : T_p^{(1)}M \to T_p^{(2)}M$ as the identity map, which is an obvious isomorphism. It is only necessary to check that an arbitrary vector $v \in T_p^{(1)}M \subset \mathbb{R}^m$ is indeed in $T_p^{(2)}M \subset \mathbb{R}^m$. It is enough to check that $df_p(v) = df_p(\alpha'(0)) = 0$.

(2) $\implies$ (3) Because v is, by construction, $\alpha'(0)$ for some curve α, let $\phi_{2\Rightarrow 3} : T_p^{(2)}M \to T_p^{(3)}M$ be defined by the map $\alpha'(0) \mapsto [\alpha]$.
Note that this is well-defined: if $v = \beta'(0)$ for some curve β then (in particular) for the chart $(M, \phi = \mathrm{id})$,

$$(\phi \circ \alpha)'(0) = \alpha'(0) = v = \beta'(0) = (\phi \circ \beta)'(0),$$

so $\alpha \sim \beta$.

(3) $\implies$ (1) Given an equivalence class $[\alpha] \in T_p^{(3)}M$, we must now recover $v \in T_p^{(1)}M$. Simply define $\phi_{3\Rightarrow 1} : T_p^{(3)}M \to T_p^{(1)}M$ using the identity chart (M, id) and taking the derivative:

$$\phi_{3\Rightarrow 1}([\alpha]) = (\mathrm{id} \circ \alpha)'(0) = \alpha'(0) = v.$$

This does not depend on choice of representative, because all elements in the equivalence class agree on the derivative at 0. $\square$

Problem 2. Let $M = \{\, (x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2} \,\}$.

(a) Show that $M - 0$ is a 2-dimensional submanifold of $\mathbb{R}^3 - 0$

(b) Let $\alpha \colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a smooth curve with image contained in M, such that $\alpha(0) = (0, 0, 0)$. Show that $\alpha'(0) = (0, 0, 0)$.

(c) Use part (b) to show that M is no a submanifold of $\mathbb{R}^3$.

Proof. (a) Let $f \colon \mathbb{R}^2 - 0 \to \mathbb{R}^3 - 0$ be given by $f(x, y) = (x, y, \sqrt{x^2 + y^2})$. Then f is clearly an injection, because the projection onto the first two coordinates π_{sub} returns the original coordinates: $\pi_{\text{sub}} \circ f = \text{id}$. Also, f is an injection because the Jacobian matrix

$$df = \begin{bmatrix} \frac{\partial}{\partial x} x & \frac{\partial}{\partial y} x \\ \\ \frac{\partial}{\partial x} y & \frac{\partial}{\partial y} y \\ \\ \frac{\partial}{\partial x}\sqrt{x^2 + y^2} & \frac{\partial}{\partial y}\sqrt{x^2 + y^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \\ 0 & 1 \\ \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

is of rank 2 for all points in the domain.
The domain $\mathbb{R}^2 - 0$ is a manifold because it is an open subset of $\mathbb{R}^2$, and f is an injective immersion, therefore $\text{im}(f) = M - 0$ is a submanifold of $\mathbb{R}^3 - 0$.

(b) Per the hint, denote $\alpha(t)$ as $\alpha(t) = (x(t), y(t), z(t))$, so that $z \colon (-\epsilon, \epsilon) \to \mathbb{R}$. Because α has image in M, $z(t) \geq 0$ for all t. Thus $z(0) = 0$ is a local minimum, and $z'(0) = 0$.

(c) Because all curves with image in M centered at $(0, 0, 0)$ have derivative $\alpha'(0) = (0, 0, 0)$, by the first (extrinstic) defintion of a tangent space, the tangent space at $(0, 0, 0)$ has dimension 0, which is not the dimension of the manifold $M - 0$.

$\square$

Problem 3. Given a manifold M and a point p, as defined in class, let C_p^M denote the collection of all parametrized curves in M passing through p at 0:

$$C_p^M := \{\, (I, \alpha) \mid I \text{ any interval containing } 0,\ \alpha\colon I \to M \text{ smooth with } \alpha(0) = p \,\}.$$

As in class, we defined an equivalence relation $\sim$ on C_p^M as follows: pick any chart (U, φ) in M's atlas containing p, we say that $(I, \alpha) \sim (J, \beta)$ if $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$.

(a) Verify that $\sim$ is moreover independent of choice of chart (U, ϕ) in Ms maximal atlas containing p.

Once this is done, we can define the tangent space to p at M by $T_pM := C_p/\sim$.

(b) If W is an open subset of $\mathbb{R}^m$, and $q \in W$ any point, verify that there is an isomorphism of sets

$$C_q^W \xrightarrow{\sim} \mathbb{R}^m$$

which sends an equivalence $[(I, \alpha)]$ to $\alpha'(0)$ for any chosen representative (I, α) in the equivalence class (why is this well-defined?)

(c) Prove that there exists a unique vector space structure on T_pM such that for each chart (U, ϕ) containing p, the map

$$\Phi\colon T_pM \to C_{\phi(p)}^{\phi(U)} \xrightarrow{\sim} \mathbb{R}^m$$

is a linear isomorphism.

Proof.

(a) Suppose that for two curves α and β, $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$. Then $\psi \circ \alpha = \psi \circ \phi^{-1} \circ \phi \circ \alpha$ and $\psi \circ \beta = \psi \circ \phi^{-1} \circ \phi \circ \beta$. So by the chain rule in a sufficiently restricted neighborhood of p,

$$\begin{aligned}
(\psi \circ \alpha)'(0) &= (\psi \circ \phi^{-1} \circ \phi \circ \alpha)'(0) \\
&= (\psi \circ \phi^{-1})'(\phi \circ \alpha(0)) \cdot (\phi \circ \alpha)'(0) \\
&= (\psi \circ \phi^{-1})'(\phi(p)) \cdot (\phi \circ \alpha)'(0) \\
&= (\psi \circ \phi^{-1})'(\phi(p)) \cdot (\phi \circ \beta)'(0) \\
&= (\psi \circ \phi^{-1})'(\phi \circ \beta(0)) \cdot (\phi \circ \beta)'(0) \\
&= (\psi \circ \phi^{-1} \circ \phi \circ \beta)'(0) \\
&= (\psi \circ \beta)'(0)
\end{aligned}$$

Thus $\sim$ is independent of choice of chart.

(b) Let f be this function that maps $[(I, \alpha)] \mapsto \alpha'(0)$. This is well-defined because if $[(I, \alpha)] = [(J, \beta)]$, then $\alpha'(0) = \beta'(0)$ by definition of the equivalence relation.
To show that f is surjective, take some vector $\vec{v} \in \mathbb{R}^m$, and then the curve $\alpha(t) = \vec{v}t + p$ for an interval $(-\epsilon, \epsilon)$ small enough that the image of α, $\text{im} \subset W \subset \mathbb{R}^m$.
To see that f is injective, notice that if $f([(I, \alpha)]) = f([(J, \beta)])$, then $\alpha'(0) = \beta'(0)$, so by definition of $\sim$, $[(I, \alpha)] = [(J, \beta)]$.

(c) Let $[\alpha], [\beta] \in T_pM$ be two equivalence classes of curves centered at $p \in M$. Given some chart (U, ϕ) containing p, define vector addition to be

$$[\alpha] + [\beta] = [\phi^{-1}((\phi \circ \alpha) + (\phi \circ \beta))]$$

using ordinary pointwise addition in $\mathbb{R}^m$, and define scalar multiplication to be

$$c[\alpha(t)] = [\alpha(ct)].$$

Notice that $C^{\phi(M)}_{\phi(p)}/\sim$ is a set of equivalence classes of curves in $\mathbb{R}^m$, so if $[\alpha],[\beta] \in C^{\phi(M)}_{\phi(p)}/\sim$ then a natural defintion for addition is $[\alpha]+[\beta] = [\alpha+\beta]$, and a natural defintion for multiplication is $c \cdot [\alpha] = [c \cdot \alpha]$.
Then the map $\Phi : C^M_p/\sim \to C^{\phi(M)}_{\phi(p)}/\sim$ which (given a chart (U,ϕ) around p) maps $[\alpha] \mapsto [\phi \circ \alpha]$ is a linear isomorphism:

(i) Addition is linear:

$$\begin{aligned}
\Phi([\alpha]+[\beta]) &= \Phi([\phi^{-1}((\phi\circ\alpha)+(\phi\circ\beta))]) \\
&= [\phi\circ\phi^{-1}((\phi\circ\alpha)+(\phi\circ\beta))] \\
&= [(\phi\circ\alpha)+(\phi\circ\beta)] \\
&= [\phi\circ\alpha]+[\phi\circ\beta] \\
&= \Phi([\alpha])+\Phi([\beta])
\end{aligned}$$

(ii) Multiplication is linear:

$$\begin{aligned}
c\cdot\Phi([\alpha]) &= c\cdot[\phi\circ\alpha] && (1)\\
&= [c\cdot(\phi\circ\alpha)] && (2)\\
&= [\phi\circ\alpha(ct)] && (3)\\
&= \Phi([\alpha(ct)]) && (4)\\
&= \Phi(c\cdot[\alpha]) && (5)
\end{aligned}$$

where step (3) is justified by the chain rule, as the following derivatives are equal at $t=0$

$$\begin{aligned}
\frac{d}{dt}(\phi\circ\alpha)(ct) &= c\cdot(\phi\circ\alpha)'(ct) \\
\frac{d}{dt}(c\cdot(\phi\circ\alpha)) &= c\cdot(\phi\circ\alpha)'(t).
\end{aligned}$$

(iii) Φ is a bijection of sets. Let Φ^{-1} map $[\alpha]\mapsto[\phi^{-1}\circ\alpha]$. Then

$$\begin{aligned}
\Phi^{-1}\circ\Phi([\alpha]) &= \Phi^{-1}([\phi\circ\alpha]) = [\phi^{-1}\circ\phi\circ\alpha] = [\alpha], \text{ and}\\
\Phi\circ\Phi^{-1}([\alpha]) &= \Phi([\phi^{-1}\circ\alpha]) = [\phi\circ\phi^{-1}\circ\alpha] = [\alpha],
\end{aligned}$$

so Φ^{-1} is a two-sided inverse.

□

Problem 4. Give a detailed proof of the equivalence between the three definitions of T_pM given in class. Then, prove that the construction of the derivative

$$df_p \colon T_pM \to T_{f(p)}N$$

is the same for the three definitions.

Proof.
(1) $\implies$ (2) Let $\alpha : (-\epsilon, \epsilon) \to M$ be a curve centered at p, so that $[\alpha] \in T_p^{(1)}M = C^p/\sim$.
Next define $g_{p,M}^{(12)} \colon T_p^{(1)}M \to T_p^{(2)}M$ to be

$$g_{p,M}^{(12)}([\alpha]) = X_\alpha.$$

where $X_\alpha \colon C^\infty(p) \to \mathbb{R}$ is the (linear) map that sends $h \mapsto (h \circ \alpha)'(0)$.
Now it must be shown that (i) $g_{p,M}^{(12)}$ is well defined and (ii) X_α satisfies the Leibniz rule.

(i) Suppose that $[\alpha] = [\beta]$. Then for some chart ϕ around p,

$$\begin{aligned}
X_\alpha(h) &= (h \circ \alpha)'(0) \\
&= (h \circ \phi \circ \phi^{-1}\alpha)'(0) \\
&= (h \circ \phi^{-1})'(\phi \circ \alpha(0)) \cdot (\phi \circ \alpha)'(0) && \text{chain rule} \\
&= (h \circ \phi^{-1})'(\phi \circ \beta(0)) \cdot (\phi \circ \beta)'(0) && \alpha(0) = \beta(0) \text{ and } \alpha \sim \beta \\
&= X_\beta(h) && \text{via the above equalities with } \beta \text{ in place of } \alpha.
\end{aligned}$$

(ii) By the product rule

$$\begin{aligned}
X_\alpha(h \cdot g) &= ((h \cdot g) \circ \alpha)'(0) \\
&= \frac{d}{dt}\left[h(\alpha(t)) \cdot g(\alpha(t))\right]_{t=0} \\
&= (h \circ \alpha)'(0) \cdot g(\alpha(0)) + h(\alpha(0)) \cdot (g \circ \alpha)'(0) \\
&= X_\alpha(h) \cdot g(p) + h(p) \cdot X_\alpha(g).
\end{aligned}$$

(2) $\implies$ (1) Now we will (i) construct a map $g_{p,M}^{(21)} \colon T_p^{(2)}M \to T_p^{(1)}M$ that maps $X_\alpha \mapsto [\alpha]$ and (ii) show that this map is compatible with an arbitrary derivation.
Choose some chart ϕ centered at p. Let $\pi_k : \mathbb{R}^m \to \mathbb{R}$ be the projection map to the kth coordinate. Then $\pi_k \circ \phi : M \to \mathbb{R}$ can be viewed as the germ of a function at p. So let

$$\hat{\alpha}(t) = \phi^{-1}(X_\alpha(\pi_1 \circ \phi) \cdot t, \ldots, X_\alpha(\pi_m \circ \phi) \cdot t).$$

which is constructed so that $\hat{\alpha} \sim \alpha$

$$\begin{aligned}
(\phi \circ \hat{\alpha})'(0) &= (\phi(\phi^{-1}(X_\alpha(\pi_1 \circ \phi) \cdot t, \ldots, X_\alpha(\pi_m \circ \phi) \cdot t)))'(0) \\
&= (X_\alpha(\pi_1 \circ \phi) \cdot t, \ldots, X_\alpha(\pi_m \circ \phi) \cdot t)'(0) \\
&= (X_\alpha(\pi_1 \circ \phi), \ldots, X_\alpha(\pi_m \circ \phi)) \\
&= ((\pi_1 \circ \phi \circ \alpha)'(0), \ldots, (\pi_m \circ \phi \circ \alpha)'(0)) \\
&= (\phi \circ \alpha)'(0).
\end{aligned}$$

Therefore, let

$$g_{p,M}^{(21)}(X_\alpha) = [\hat{\alpha}(t)].$$

(This is a valid defintion for any arbitrary derivation, not just those in the image of $g_{p,M}^{(12)}$.)

(2) $\iff$ (1) — Derivative maps. We want to show that if $f \colon M \to N$, then

$$\begin{aligned}
df_p^{(1)}([\alpha]) &= g_{p,N}^{(21)} \circ df_p^{(2)} \circ g_{p,M}^{(12)}([\alpha]) \text{ and} \\
df_p^{(2)}(X_\alpha) &= g_{p,N}^{(12)} \circ df_p^{(1)} \circ g_{p,M}^{(21)}(X_\alpha).
\end{aligned}$$

(i) The first identity:

$$\begin{aligned}
g^{(21)}_{p,N} \circ df^{(2)}_p \circ g^{(12)}_{p,M}([\alpha]) &= g^{(21)}_{p,M} \circ df^{(2)}_p(X_\alpha) \\
&= g^{(21)}_{p,M}(X_\alpha \circ f^*) \\
&= [\phi^{-1}(X_\alpha \circ f^*(\pi_1 \circ \phi) \cdot t, \ldots, X_\alpha \circ f^*(\pi_m \circ \phi) \cdot t)] \\
&= [\phi^{-1}(X_\alpha(\pi_1 \circ \phi \circ f) \cdot t, \ldots, X_\alpha(\pi_m \circ \phi \circ f) \cdot t)] \\
&= [f \circ \alpha] \\
&= df^{(1)}_p[\alpha]
\end{aligned}$$

where the penultimate equality follows by the above argument that $\hat{\alpha} \sim \alpha$, instead with f pre-composed.

(ii) The second identity:

$$\begin{aligned}
g^{(12)}_{p,N} \circ df^{(1)}_p \circ g^{(21)}_{p,M}(X_\alpha) &= g^{(12)}_{p,M} \circ df^{(1)}_p[\alpha] \\
&= g^{(12)}_{p,M}([f \circ \alpha]) \\
&= X_{f \circ \alpha} \\
&= X_\alpha \circ f^* \\
&= df^{(2)}_p(X_\alpha)
\end{aligned}$$

where $X_{f\circ\alpha} = X_\alpha \circ f^*$ because for all g,

$$X_{f\circ\alpha}(g) = (g \circ f \circ \alpha)'(0) = X_\alpha(g \circ f) = X_\alpha \circ f^*(g).$$

(2) $\Longleftrightarrow$ **(3)** Now given some germ $[f] \in C^\infty(p)$, any representative can be written (via Taylor's Theorem) as

$$f(x) = f(p) + \sum_i a_i x_i + \sum_{i,j} a_{ij}(x) x_i x_j$$

so $X(f) = X(f - f(p))$ where $x \mapsto f(x) - f(p)$ vanishes at p. See also that X annihilates the generators of $\mathcal{F}_p^2$ (and so it annihilates $\mathcal{F}_p^2$): let $f = \sum_{i,j} g_{ij}\phi_i\phi_j$ where $\phi_i, \phi_j \in \mathcal{F}_p$ then by the Leibniz rule:

$$\begin{aligned}
X(g_{ij}\phi_i\phi_j) &= X(g_{ij}) \cdot \phi_i(p)\phi_j(p) + g_{ij}(p)X(\phi_i)\phi_j(p) + g_{ij}(p)\phi_i(p)X(\phi_j) \\
&= X(g_{ij}) \cdot 0 \cdot 0 + g_{ij}(p)X(\phi_i) \cdot 0 + g_{ij}(p) \cdot 0 \cdot X(\phi_j) \\
&= 0.
\end{aligned}$$

As shown in the previous homework, $\mathrm{Ann}(\mathcal{F}_p^2) \cong (\mathcal{F}_p/\mathcal{F}_p^2)^* = T^{(3)}_p M$ via some isomorphism $g^{(23)}_{p,M} = \Phi$, therefore the map $X_\alpha \mapsto \Phi(X_\alpha)$ will serve as an isomorphism from $T^{(2)}_p M$ to $T^{(3)}_p M$ with Φ^{-1} as its inverse.

(1) $\Longleftrightarrow$ **(3)** Once we have the above implications, it is easy enough to construct $g^{(13)}_{p,M}$ and $g^{(31)}_{p,M}$ as the compositions of the existing isomorphisms:

$$\begin{aligned}
g^{(13)}_{p,M} &= g^{(23)}_{p,M} \circ g^{(12)}_{p,M} \text{ and} \\
g^{(31)}_{p,M} &= g^{(21)}_{p,M} \circ g^{(32)}_{p,M}.
\end{aligned}$$

By the above work, we know (without explicitly checking) that these isomorphism respect the derivative maps. □

Problem 5. Let Γ be a group and M a smooth manifold. A (C^∞) action of Γ on M is a group homorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms on M. If $\gamma \in \Gamma$ and $x \in M$, we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

(a) Prove that if Γ acts freely and discontinously on M, then the quotient M/Γ naturally has the structure of a smooth manifold.

(b) Let $\mathbb{Z}_2$ act on $S_n \subset \mathbb{R}^{n+1}$ by sending $x \mapsto -x$. Using the standard manifold structure on S^n, prove that $S^n/\mathbb{Z}_2$ has the structure of a manifold, which is diffeomorphic to $\mathbb{R}P^n$, equipped with the smooth manifold structure defined on last week's homework.

Proof.

(a) Suppose that Γ acts freely and discontinuously on M. The maximal atlas $\mathcal{A}_M$ on M can be used to construct an atlas $\mathcal{A}_{M/\Gamma}$ on M/Γ. Denote $\mathcal{A}_M$ by

$$\mathcal{A}_M = \{(U_i,\, \phi_i \colon U_i \to \mathbb{R}^m)\}_{i \in I}.$$

Because Γ acts freely and discontinuously, around each point $p \in M$, there exists an open set $U_p \subset M$ such that $U_p \cap \gamma U_p = \emptyset$ for all $\gamma \neq \mathrm{id} \in \Gamma$, and the collection of charts $\{(U_p, \phi_p)\}_{p \in M} \subset \mathcal{A}_M$ is an atlas. Let the surjective map $q \colon M \to M/\sim$ be given by $x \mapsto [x] = \{\gamma x : \gamma \in \Gamma\}$, and let

$$\mathcal{A}_{M/\Gamma} = \{(V_p = q(U_p), \psi_p \colon V_p \to \mathbb{R}^m)\}_{p \in M}$$

where

$$\psi_p([x]) = \phi_p(\gamma x) \text{ for } \gamma \in \Gamma \text{ such that } \gamma x \in U_p,$$

where γ exists and is unique by construction.

(b) Denote S^n by $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Then $S^n/\mathbb{Z}_2 = \{\{x, -x\} : x \in S^n\}$ which has atlas

$$\mathcal{A}_{S^n/\mathbb{Z}_2} = \{(U_k, \phi_k)\}_{k=1}^n$$

where $U_k = \{x_k \neq 0\}$, and

$$\phi_k([(x_1, \ldots, x_k, \ldots, x_{n+1})]) = \left(\frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{n+1}}{x_k}\right) \in \mathbb{R}^n$$

which is the same atlas as $\mathbb{R}P^n$, only with maps restricted to $S^n \subset \mathbb{R}^{n+1}$.

$\square$