Topology: Homework 2

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Problem 1.
Let

$$\begin{aligned} p\colon \quad \mathbb{R} &\to S^1 \text{ by } x \mapsto (\cos x, \sin x) \\ \alpha\colon [0,1] &\to S^1 \text{ by } \ t \mapsto p(2\pi t), \end{aligned}$$

let f be some map with the property that $f(-x,-y) = -f(x,y)$ And let $\widetilde{\beta}\colon [0,1] \to \mathbb{R}$ be a lift of the path $\beta = f \circ \alpha$.

a. Show that there exists an integer $n_0 \in \mathbb{Z}$ such that $\widetilde{\beta}(\frac{1}{2}) = \widetilde{\beta}(0) + 2n_0\pi + \pi$.

Proof.
We know that $p \circ \widetilde{\beta} = f \circ \alpha$, which is to say,

$$(\cos(\widetilde{\beta}(t)), \sin(\widetilde{\beta}(t))) = f(\cos(2\pi t), \sin(2\pi t)).$$

And let's further decompose f as $f(x,y) = (f_x(x,y), f_y(x,y))$.
Thus, when $t = 0$, $\cos(\widetilde{\beta}(0)) = f_x(1,0)$. Similarly, when $t = \frac{1}{2}$, $\cos(\widetilde{\beta}(\frac{1}{2})) = f_x(-1,0) = -f_x(1,0)$. Therefore

$$\cos(\widetilde{\beta}(0)) + \cos(\widetilde{\beta}(\tfrac{1}{2})) = 0$$

so $\widetilde{\beta}(0)$ and $\widetilde{\beta}(\frac{1}{2})$ differ by π up to some multiple of 2π:

$$\widetilde{\beta}(\tfrac{1}{2}) = \widetilde{\beta}(0) + 2n_0\pi + \pi.$$

□

b. Show that, for the integer n_0 of part **a**, $\widetilde{\beta}(t) = \widetilde{\beta}(t - \frac{1}{2}) + 2n_0\pi + \pi$ for every $t \in \left[\frac{1}{2}, 1\right]$.

Proof.
Writing things explicitly gives

$$\begin{aligned} (\cos(\widetilde{\beta}(t)), \sin(\widetilde{\beta}(t))) &= f(\cos(2\pi t), \sin(2\pi t)) \\ (\cos(\widetilde{\beta}(t - \tfrac{1}{2})), \sin(\widetilde{\beta}(t - \tfrac{1}{2}))) &= f((\cos(2\pi(t - \tfrac{1}{2})), \sin(2\pi(t - \tfrac{1}{2})))) \\ &= f((\cos(2\pi t - \pi), \sin(2\pi t - \pi))) \\ &= f((-\cos(2\pi t), -\sin(2\pi t))) \\ &= -f((\cos(2\pi t), \sin(2\pi t))) \end{aligned}$$

So similarly to part **a**,

$$\cos(\widetilde{\beta}(t)) + (\cos(\widetilde{\beta}(t - \tfrac{1}{2})) = f_x(\cos(2\pi t), \sin(2\pi t)) - f_x(\cos(2\pi t), \sin(2\pi t)) = 0$$

the arguments $\widetilde{\beta}(t)$ and $\widetilde{\beta}(t - \frac{1}{2})$ must differ by π up to a multiple of 2π. By continuity, this multiple of 2π must be the same as in part **a**. Thus

$$\widetilde{\beta}(t) = \widetilde{\beta}(t - \tfrac{1}{2}) + 2n_0\pi + \pi.$$

Thus

$$\begin{aligned}
\widetilde{\beta}(1) &= \widetilde{\beta}(\tfrac{1}{2}) + 2n_0\pi + \pi, \\
\widetilde{\beta}(\tfrac{1}{2}) &= \widetilde{\beta}(0) + 2n_0\pi + \pi, \text{ and so} \\
\widetilde{\beta}(1) &= \widetilde{\beta}(0) + 4n_0\pi + 2\pi,
\end{aligned}$$

and because $4n_0 + 2 \neq 0$ for all $n_0 \in \mathbb{Z}$, $\widetilde{\beta}(1) \neq \widetilde{\beta}(0)$. □

c. Show that, if $x_0 = (1, 0)$ and $y_0 = f(1, 0)$, the induced homomorphism $f_*\colon \pi_1(S^1; x_0) \to \pi_1(S^1; y_0)$ is non-trivial.

Proof.
Consider the generating element $[\alpha] \in \pi(S^1; x_0)$, which maps to $[f \circ \alpha] = [p \circ \widetilde{\beta}]$ under f_*. Since part **b** shows that $\widetilde{\beta}(1) \neq \widetilde{\beta}(0)$, we know that $p \circ \widetilde{\beta}$ must describe a non-trivial loop around S^1, and thus the homomorphism is non-trivial. □

d. Consider the 2-dimensional sphere S^2 and identify the unit circle S^1 to its equator.

(i) Show that for every map $F\colon S^2 \to S^1$, the restriction $f\colon S^1 \to S^1$ which sends $(x, y) \mapsto F(x, y, 0)$ is homotopic to a constant map.

Proof.
I'll use Gin Park's trick and construct an explicit homotopy to the constant map $x \mapsto F(0, 0, 1) \in S^1$. Namely, let $H\colon S^1 \times [0, 1] \to S^1$ be given by

$$H((x, y), t) = F\left(x\sqrt{1 - t^2}, y\sqrt{1 - t^2}, \sqrt{t}\right).$$

Clearly H is continuous by composition, and

$$\begin{aligned}
H((x, y), 0) &= F(x, y, 0) = f(x, y) \text{ and} \\
H((x, y), 1) &= F(0, 0, 1),
\end{aligned}$$

so it is enough to check that $(x\sqrt{1 - t^2}, y\sqrt{1 - t^2}, \sqrt{t}) \in S^2$. But this follows easily since the norm squared is 1:

$$\begin{aligned}
(x\sqrt{1 - t^2})^2 + (x\sqrt{1 - t^2})^2 + (\sqrt{t})^2 &= x^2(1 - t^2) + y^2(1 - t^2) + t^2 \\
&= \underbrace{(x^2 + y^2)}_{=1}(1 - t^2) + t^2 \\
&= (1 - t^2) + t^2 \\
&= 1.
\end{aligned}$$

□

(ii) Show that there is no map $F\colon S^2 \to S^1$ such that $F(-x, -y, -z) = -F(x, y, z)$ for every $(x, y, z) \in S^2$.

Proof. This follows from parts **c** and **d** (i). Suppose that there were such a map. Then its restriction to the equator would satisfy the conditions for f above, and thus by part **c**, it would not be nullhomotopic. However, part **d** (i) showed that such a map *must* be nullhomotopic, a contradiction. Thus no map may exist. □

e. Let $f\colon S^2 \to \mathbb{R}^2$ be continuous. Show that there exists at least one pair of antipodal points that have the same image under g.

Proof. As per the hint, consider the map $F\colon S^2 \to S^1$ by

$$F(x,y,z) = \frac{g(x,y,z) - g(-x,-y,-z)}{||g(x,y,z) - g(-x,-y,-z)||}.$$

This function meets the criteria in part **d** (ii), namely

$$-F(x,y,z) = \frac{g(-x,-y,-z) - g(x,y,z)}{||g(x,y,z) - g(-x,-y,-z)||} = \frac{g(-x,-y,-z) - g(x,y,z)}{||g(-x,-y,-z) - g(x,y,z)||} = F(-x,-y,-z).$$

Therefore no such continuous function exists on all of S^2, and so there exists some (x,y,z) such that $||g(-x,-y,-z) - g(x,y,z)|| = 0$, and thus there exists some (x,y,z) such that $g(-x,-y,-z) = g(x,y,z)$. $\square$

f. Let A and B be two bounded domains in the xy-plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

(i) For each unit vector $\vec{u} \in S^2 \subset \mathbb{R}^3$, let $P_{\vec{u}}$ be the plane in $\mathbb{R}^3$ passing through the point $(0,0,1)$ and orthogonal to $\vec{u}$, and let $H_{\vec{u}}$ be the half-space delimited by $P_{\vec{u}}$ such that $\vec{u}$ points toward the interior of $H_{\vec{u}}$.
Show that there exists $\vec{u} \in S^2$ such that $\text{area}(A \cap H_{\vec{u}}) = \frac{1}{2}\,\text{area}(A)$ and $\text{area}(B \cap H_{\vec{u}}) = \frac{1}{2}\,\text{area}(B)$.

Proof. As suggested by the hint, let $g\colon S^2 \to \mathbb{R}^2$ be defined by

$$g(\vec{u}) = (\text{area}(A \cap H_{\vec{u}}), \text{area}(B \cap H_{\vec{u}})).$$

Notice that $P_{-\vec{u}} = P_{\vec{u}}^c$, so

$$g(-\vec{u}) = (\text{area}(A) - \text{area}(A \cap H_{\vec{u}}), \text{area}(B) - \text{area}(B \cap H_{\vec{u}}))$$

and moreover, $g(\vec{u}) = g(-\vec{u})$ precisely when

$$\begin{aligned}\text{area}(A \cap H_{\vec{u}}) &= \frac{1}{2}\,\text{area}(A)\\ \text{area}(B \cap H_{\vec{u}}) &= \frac{1}{2}\,\text{area}(B).\end{aligned}$$

It takes some measure-theoretic argument to show that g is continuous, but taking that for granted, part **e** proves that there exists a pair of antipodal points that have the same image under g. Thus the desired vector $\vec{u}$ is the one that satisfies the antipodal equality property. $\square$

(ii) Show that there exists a line in $\mathbb{R}^2$ that divides each of A and B into halves of equal area.

Proof. Simply choose any vector $\vec{u}$ that satisfies part **f** (i), and take the line which is the intersection of $P_{\vec{u}}$ and the xy-plane. If $\vec{u} = (0,0,-1)$, then $P_{\vec{u}}$ does not intersect the xy-plane. In this case, the areas of A and B are zero, so the any line will divide A and B into halves of equal area. $\square$