Combinatorics: Homework 8

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Problem 34. [2]
Find all nonisomorphic posets P such that

$$F(J(P), x) = (1 + x)(1 + x^2)(1 + x + x^2)$$

Solution.

Problem 46 a. [2]
Let $f(n)$ be the number of sublattices of rank n of the boolean algebra B_n. Show that $f(n)$ is also the number of partial orders P on $[n]$.

Solution.
Let ϕ be a map which sends a poset P (with underlying set $[n]$) to $J(P) \subset B_n$. Since B_n is a finite distributive lattice, and every sublattice of a finite distributive lattice is a finite distributive lattice, we can take the set of join irreducibles of any rank n sublattice, and this will be isomorphic to a poset on $[n]$.

Problem 53. [2]
Let P be a finite n-element poset. Simplify the two sums

$$f(P) = \sum_{I \subset J(P)} e(I)e(\overline{I}),$$

$$g(P) = \sum_{I \subset J(P)} \binom{n}{\#I} e(I)e(\overline{I}),$$

where $\overline{I}$ denotes the complement $P - I$ of the order ideal I.

Proof. $\square$

Problem 57.

a. [2] Let P be an n-element poset. If $t \in P$, then set $\lambda_t = \#\{s \in P : s \leq t\}$. Show that

$$e(P) \geq \frac{n!}{\prod_{t\in P} \lambda_t}.$$

b. [2+] Show that the equality holds if and only if every component of P is a rooted tree.

Proof. a.

b. By induction, the base case is clear. When $n = 1$, there is only one poset (which is a rooted tree) with one linear extension.

$$e([1]) = \frac{1!}{\lambda_1} = 1$$

Thus given some rooted tree P, we can take the subposet $P - \hat{1}$, which is a disjoint union of rooted trees $P_1 + P_2 + \ldots + P_k$ with $n_1, n_2, \ldots, n_k$ elements respectively. Since P has a unique maximum, it must be labeled with n, then we can then choose which letters go in each sub-tree (using a multinomial coefficient), and then there are $e(P_i)$ ways to order the n_i labels for each P_i. Therefore

$$\begin{aligned} e(P) &= \binom{n-1}{n_1, n_2, \ldots, n_k} e(P_1)e(P_2)\ldots e(P_k) \\ &= \binom{n-1}{n_1, n_2, \ldots, n_k} \frac{n_1!}{\prod_{t\in P_1} \lambda_t} \frac{n_2!}{\prod_{t\in P_2} \lambda_t} \cdots \frac{n_k!}{\prod_{t\in P_k} \lambda_t} \\ &= \left(\frac{(n-1)!}{n_1!n_2!\ldots n_k!}\right) \frac{n_1!n_2!\ldots n_k!}{\prod_{t\in P-\hat{1}} \lambda_t} \\ &= \frac{(n-1)!}{\prod_{t\in P-\hat{1}} \lambda_t} \end{aligned}$$

Since all n elements of P are less than or equal to $\hat{1}$, $\lambda_{\hat{1}} = n$,

$$n \prod_{t\in P-\hat{1}} \lambda_t = \prod_{t\in P} \lambda_t$$

and thus

$$e(P) = \frac{n(n-1)!}{n\prod_{t\in P-\hat{1}} \lambda_t} = \frac{n!}{\prod_{t\in P} \lambda_t}$$

as desired. $\square$