Week 11 Quiz Solutions.

1. (*3 points*) Find the tangent plane to the level surface $xyz + z^2 \cos x = -4$ at the point $(\pi, 0, 1)$.

   Solution. Let $f(x, y, z) = xyz + z^2 \cos(x)$. Since the gradient ∇f always points perpendicular to the level surface, the plane can be described as

$$f_x(\pi, 0, 1)(x - \pi) + f_y(\pi, 0, 1)(y - 0) + f_z(\pi, 0, 1)(z - 1) = 0$$

   where

$$\begin{aligned} f_x(x, y, z) &= yz - z^2 \sin(x) \qquad & f_x(\pi, 0, 1) &= -\sin(\pi) = 0 \\ f_y(x, y, z) &= xz & f_y(\pi, 0, 1) &= \pi \\ f_z(x, y, z) &= xy + 2z \cos(x) & f_z(\pi, 0, 1) &= 2\cos(\pi) = -2. \end{aligned}$$

   Therefore the plane is given by the equation

$$\pi y - 2(z - 1) = 0.$$

2. (*4 points*) Evaluate

$$\iint\limits_R xye^{y^2+x^2} \, \mathrm{d}A$$

   where $R = [0, 1] \times [0, 1]$.

   Solution.
   The integral can be written with the bounds

$$\int_{y=0}^{1} \int_{x=0}^{1} xye^{y^2+x^2} \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{1} (xe^{x^2})(ye^{y^2}) \, dx \, dy,$$

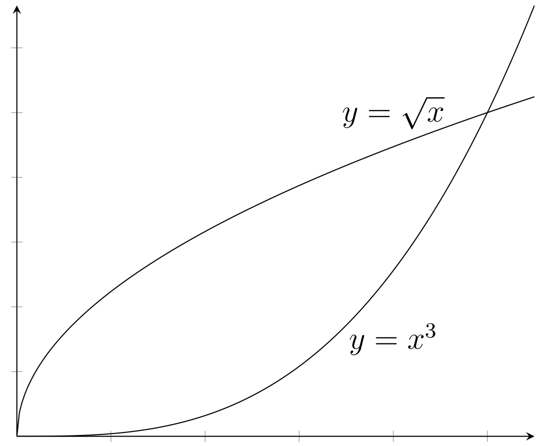
   so in particular it can be separated

$$\left(\int_{x=0}^{1} xe^{x^2} \, dx \right) \left(\int_{y=0}^{1} ye^{y^2} \, dy \right)$$

   By the substituion $u = x^2$ and $du = 2xdx$, this can be rewritten

$$\left(\int_{u=0}^{1} \frac{1}{2} e^u \, du \right)^2 = \frac{(e - 1)^2}{4}.$$

3. (*3 points*) Set up the double integral of $f(x, y) = e^x \cos(y^{15})$ over region R bounded below by $y = x^3$ and above by $y = \sqrt{x}$ in two different ways. Do not evaluate this double integral.



Solution.
The two curves intersect where $\sqrt{x} = x^3$, namely $(0, 0)$ and $(1, 1)$. Writing the bounds of y as a function of x

$$x^3 \leq y \leq \sqrt{x}$$
$$0 \leq x \leq 1$$

gives the integral

$$\int_{x=0}^{1} \int_{y=x^3}^{\sqrt{x}} e^x \cos(y^{15})\, dy\, dx.$$

Writing the bounds of x as a function of y

$$y^2 \leq x \leq \sqrt[3]{y}$$
$$0 \leq y \leq 1$$

gives the integral

$$\int_{y=0}^{1} \int_{x=y^2}^{\sqrt[3]{y}} e^x \cos(y^{15})\, dx\, dy.$$