Complex Analysis: Homework 9

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Problem 1. (page 166)
If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

Proof.
By Theorem 20,

$$\frac{1}{2\pi} \int\limits_{|z|=r} u \, d\theta = \alpha \log(r) + \beta$$

but since u is bounded the arithmetic mean is bounded for any r—in particular, the modulus is finite in the limit:

$$\lim_{r \to 0^+} \left| \frac{1}{2\pi} \int\limits_{|z|=r} u \, d\theta \right| = \lim_{r \to 0^+} |\alpha \log(r) + \beta| < \infty.$$

Thus $\alpha = 0$, and $u(0) = \beta$.

In general,

$$\alpha = - \int\limits_{|z|=r} r \, \frac{\partial u}{\partial r} \, d\theta$$

so this means that

$$\int\limits_{|z|=r} r \, \frac{\partial u}{\partial r} \, d\theta = 0$$

and therefore u is harmonic. $\square$

Problem 1. (page 171)
Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)\, d\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity (Poisson's integral for the half plane).

Proof.
Start by defining a Möbius transformation T and its inverse T^{-1}, where T maps the interior of the unit disk to the upper half plane and the boundary of the unit disk to the real axis.

$$T(z) = \frac{1}{2} \cdot \frac{1 - iz}{z - i}$$
$$T^{-1}(z) = \frac{1 + 2iz}{2z + i}$$

Then using equation (63) with $R = 1$ yields

$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, U(\theta)\, d\theta$$

(?) $\square$

Problem 3. (page 171)
In Exercise 1, assume that U has a jump at 0, for instance $U(+0) = 0$, $U(-0) = 1$. Show that $P_U(z) - \dfrac{1}{\pi} \arg z$ tends to 0 as $z \to 0$. Generalize to arbitrary jumps and to the case of the circle.

Proof.

$$P_U(z) - \frac{1}{\pi} \arg z = \frac{1}{\pi}\left(\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)\, d\xi - \arg z\right)$$

tends to 0 as $z \to 0$ if and only if

$$\lim_{z \to 0} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)\, d\xi - \arg z = 0.$$

$\square$

Problem 5. (page 171)
Show that the mean-value formula (62) remains valid for $u = \log|1+z|$, $z_0 = 0$, $r = 1$, and use this fact to compute

$$\int_0^\pi \log \sin\theta \, d\theta$$

Proof.
The mean value formula states that

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

so here we want to show that this formula works for the case

$$\log|1+0| = \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta}|\, d\theta.$$

□