Math 510b: 2015 Final Exam

Peter Kagey

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Problem 1.

(i) Let R be a PID and let I be a non-zero ideal of R. Show that R/I is Artinian. Is the conclusion still true if R is only a UFD?

(ii) Give an example of a Dedekind domain which is not a UFD.

Proof.

(i) Since R is a PID, $I = \langle a \rangle$ for some $a \in R$. Since an Artinian ring is a ring that satisfies the descending chain condition on ideals, it is sufficient to show that all chains

$$I_1/\langle a \rangle \supset I_2/\langle a \rangle \supset \ldots = \langle a_1 \rangle/\langle a \rangle \supset \langle a_2 \rangle/\langle a \rangle \supset \ldots$$

are eventually constant, where (by the correspondence theorem) $I_n = \langle a_n \rangle \subset \langle a \rangle$. Now we can use the fact that R is a PID and thus a UFD: since a has a finite number of prime factors. In order for $\langle a_i \rangle/\langle a \rangle \supsetneq \langle a_{i+1} \rangle/\langle a \rangle$, $a_{i+1} \mid a_i$ and $a_{i+1} \neq ua_i$ (where u is a unit). This means that a_{i+1} must have fewer prime factors than a_i, thus the descending chain can only have a finite number of proper inclusions, and so satisfies the descending chain condition.

If R is a UFD, this is not true. For example, let $R = \mathbb{R}[x, y]$, then the ideals

$$\langle y \rangle/\langle x \rangle \supsetneq \langle y^2 \rangle/\langle x \rangle \supsetneq \langle y^3 \rangle/\langle x \rangle \supsetneq \ldots$$

do not satisfy the descending chain condition.

(ii) A Dedekind domain is

$\square$

Problem 2. Let R be a commutative k-algebra and let $S = M_n(R)$

Proof. $\square$

Problem 3. Let $R = \mathbb{C}[x, y]$ and consider the two ideals $I = (2x + y)$ and $J = (x^2 - y)$.

(a) Justify: I and J are both prime ideals of R, and each of them is the intersection of all the maximal ideals containing it.

(b) Give an explicit description of the maximal ideals containing each ideal, and then give a geometric interpretation of your answer using varieties in $\mathbb{C}^2$.

(c) Consider the ideal $I + J$. Determine whether or not it is a prime ideal. What is $\sqrt{I + J}$?

(d) Answer (c) for $I \cap J$

(e) Give a geometric interpretation of (c) and (d).

Proof.

(a) To see that I is a prime ideal, it is enough to see that $\mathbb{C}[x, y]$ is a domain, $2x+y$ is a degree 1 polynomial, and $x^2 - y$ is prime when viewed as $\mathbb{C}[y][x]$ via Eisenstein's criteria with prime y.

(b)

(c) Notice that

$$2x + y + x^2 - y = x^2 + 2x = x(x + 2) \in I + J,$$

but $x \notin I + J$ and $x + 2 \notin I + J$, so $I + J$ is not prime.

By Nullstellensatz, $\mathbb{I}(\mathbb{V}(I + J)) = \sqrt{I + J}$.

$$I + J = \{\alpha i + \beta j : \alpha, \beta \in \mathbb{C}[x, y], i \in I, j \in J\} = \langle 2x + y, x^2 - y\rangle$$

By definition,

$$\mathbb{V}(I + J) = \left\{(z, w) \in \mathbb{C}^2 : 2z + w = 0 = z^2 - w\right\} = \{(0, 0), (-2, 2)\},$$

because the system of equations

$$\begin{aligned} 2z + w &= 0 \\ z^2 - w &= 0 \end{aligned}$$

has solutions of $z = 0$ or $z = -2$ (via adding the two equations to form $z^2 + 2z = 0$) and corresponding values of $w = 0$ and $w = 2$. Then taking the ideal of polynomials vanishing on this variety gives

$$\begin{aligned} \mathbb{I}(\{(0, 0), (-2, 2)\}) &= \{f \in \mathbb{C}[x, y] : f(0, 0) = 0 = f(-2, 2)\} \\ &= \{f \in \mathbb{C}[x, y] : f(0, 0) = 0\} \cap \{f \in \mathbb{C}[x, y] : f(-2, 2) = 0\} \\ &= \langle x, y\rangle \cap \langle x + 2, y - 2\rangle. \end{aligned}$$

so $\sqrt{I + J} = \langle x, y\rangle \cap \langle x + 2, y - 2\rangle$

(d) Similarly, we will compute $\sqrt{I \cap J} = \mathbb{I}(\mathbb{V}(I \cap J))$.

$\square$

Problem 4.

Proof. $\square$

Problem 5.

Proof. $\square$