Algebraic Combinatorics: Homework 1

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Problem 1. For $n \geq 0$, define a sequence $f(n)$ by the recurrence

$$f(n) = f(n-1) + f(n-2)$$

for $n \geq 2$ with initial conditions $f(0) = f(1) = 1$.

(a) Define $F(x) = \sum_{n\geq 0} f(n)x^n$ to be the ordinary generating function of $f(n)$. Use the recurrence relation for $f(n)$ to prove $F(x) = 1 + xF(x) + x^2F(x)$

(b) Solve (a) for $F(x)$ and use partial fraction decomposition to write $F(x)$ as

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}.$$

(c) Extract the coefficients of $F(x)$ from b to give an explicit formula for $f(n)$.

Solution.

(a)

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} f(n)x^n \\
&= 1 + x + \sum_{n\geq 2} f(n)x^n \\
&= 1 + x + \sum_{n\geq 2} f(n-1) + f(n-2)x^n \\
&= 1 + x + \sum_{n\geq 2} f(n-1)x^n + \sum_{n\geq 2} f(n-2)x^n \\
&= 1 + x + \sum_{n\geq 1} f(n)x^{n+1} + \sum_{n\geq 0} f(n)x^{n+2} \\
&= 1 + \underbrace{x + x\sum_{n\geq 1} f(n)x^n}_{xF(x)} + x^2 \underbrace{\sum_{n\geq 0} f(n)x^n}_{F(x)} \\
&= 1 + xF(x) + x^2F(x)
\end{aligned}$$

(b) Using the identity $F(x) = 1 + xF(x) + x^2F(x)$ and solving for $F(x)$ yields $F(x) = 1/(1-x-x^2)$. Then using the quadratic formula to find the roots of $1 - x - x^2$ (equivalently $x^2 + x - 1 = 0$) gives

$$\underbrace{\frac{-1-\sqrt{5}}{2}}_{-\varphi} \qquad \underbrace{\frac{-1+\sqrt{5}}{2}}_{-\overline{\varphi}}.$$

where $-(x+\varphi)(x+\overline{\varphi}) = 1 - x - x^2$. Then partial fraction decomposition on

$$\frac{1}{1-x-x^2} = \frac{A}{x+\varphi} + \frac{B}{x+\overline{\varphi}}$$

where A and B satisfy the system of equations

$$\begin{aligned} A + B &= 0 \\ \overline{\varphi}A + \varphi B &= 1 \end{aligned}$$

which gives $A = -B = 1/\sqrt{5}$ resulting in

$$\begin{aligned} F(x) &= \frac{1}{1-x-x^2} \\ &= \frac{1}{(x+\varphi)\sqrt{5}} - \frac{1}{(x+\overline{\varphi})\sqrt{5}} \\ &= \frac{1}{\varphi\sqrt{5}} \cdot \frac{1}{(x/\varphi+1)} - \frac{1}{\overline{\varphi}\sqrt{5}} \cdot \frac{1}{(x/\overline{\varphi}+1)} \end{aligned}$$

(c) We can write

$$\begin{aligned} F(x) &= \frac{1}{\varphi\sqrt{5}} \sum_{i\geq 0} (x/\varphi)^n - \frac{1}{\overline{\varphi}\sqrt{5}} \sum_{i\geq 0} (x/\overline{\varphi})^n \\ &= \frac{1}{\sqrt{5}} \left(\sum_{i\geq 0} \frac{x^n}{\varphi^{n+1}} - \sum_{i\geq 0} \frac{x^n}{\overline{\varphi}^{n+1}} \right) \\ &= \frac{1}{\sqrt{5}} \sum_{i\geq 0} \left(\frac{1}{\varphi^{n+1}} - \frac{1}{\overline{\varphi}^{n+1}} \right) x^n \\ &= \frac{1}{\sqrt{5}} \sum_{i\geq 0} \left(\frac{\overline{\varphi}^{n+1} - \varphi^{n+1}}{(\overline{\varphi}\varphi)^{n+1}} \right) x^n \\ &= \frac{1}{\sqrt{5}} \sum_{i\geq 0} \left(\varphi^{n+1} - \overline{\varphi}^{n+1} \right) x^n. \end{aligned}$$

Therefore the coefficients are given by

$$f(n) = \frac{1}{\sqrt{5}} (\varphi^{n+1} - \overline{\varphi}^{n+1}) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Problem 2. Prove the product formula for exponential generating functions:

$$\left(\sum_{n\geq 0} f_n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} g_n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\right)\frac{x^n}{n!}$$

Solution.
Using the definition of multiplication of (ordinary) generating functions gives

$$\begin{aligned}
\left(\sum_{n\geq 0} f_n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} g_n \frac{x^n}{n!}\right) &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{f_k}{k!}\frac{g_{n-k}}{(n-k)!}\right)x^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{f_k}{k!}\frac{g_{n-k}}{(n-k)!}\frac{n!}{n!}\right)x^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\frac{1}{n!}\right)x^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\right)\frac{x^n}{n!}
\end{aligned}$$

as desired.

Problem 3. Assume that $1+\sum_{n\geq 1} f(n)x^n = \exp \sum_{n\geq 1} h(n)\frac{x^n}{n!}$, and show that the following are equivalent for $N \geq 1$ fixed.

1. $f(n) \in \mathbb{Z}$ for all $n \in [N]$.
2. $h(n) \in \mathbb{Z}$ and $\sum_{d|n} h(d)\mu(n/d) \cong 0 \pmod{n}$ for all $n \in [N]$.
3. $h(n) \in \mathbb{Z}$ and $h(n) \cong h(n/p) \pmod{p^r}$ whenever $n \in [N]$ and p is a prime that divides n at least once and at most r times.
4. there exists a polynomial $P(t) = \prod_{i=1}^{N}(t-\alpha_i) \in \mathbb{Z}[t]$ with complex roots such that $h(n) = \sum_{i=1}^{N} \alpha_i^n$ for all $n \in [N]$.

Solution.

Problem 4. Let $f(n) = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $g(n) = 2^n n!$.

(a) Show that $G(x) = F(x)^2$.

(b) Give a combinatorial proof.

Solution.

(a) Firstly, the generating function G is given by

$$G(x) = \sum_{n \geq 0} g(n) \frac{x^n}{n!} = \sum_{n \geq 0} 2^n n! \frac{x^n}{n!} = \sum_{n \geq 0} (2x)^n = \frac{1}{1-2x}.$$

Next the generating function F is given by

$$\begin{aligned}
F(x) &= \sum_{n \geq 0} f(n) \frac{x^n}{n!} \\
&= \sum_{n \geq 0} 1 \cdot 3 \cdot 5 \cdots (2n-1) \frac{x^n}{n!} \\
&= \sum_{n \geq 0} \left(\frac{1}{-2}\right)\left(\frac{3}{-2}\right)\left(\frac{5}{-2}\right) \cdots \left(\frac{2n-1}{-2}\right) (-2)^n \frac{x^n}{n!} \\
&= \sum_{n \geq 0} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-n+1)}{n!} (-2)^n x^n \\
&= \sum_{n \geq 0} \binom{-1/2}{n} (-2x)^n \\
&= (1 + (-2x))^{-1/2} \\
&= \frac{1}{\sqrt{1-2x}}
\end{aligned}$$

So it is clear that $F(x)^2 = G(x)$.

(b) Using the product formula for exponential generating functions gives

$$F(x) \cdot F(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} f(k) f(n-k) \right) \frac{x^n}{n!},$$

so it is sufficient to show that

$$g(n) = 2^n n! = \sum_{k=0}^{n} \binom{n}{k} f(k) f(n-k).$$

$g(n)$ gives the number of two-colorings of all permutations of $[n]$.