Last Week Quiz Solutions.

1. (*5 points*) Show that $\vec{F}(x,y) = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$ is conservative and then find a potential function f (so that $\vec{F} = \nabla f$.) Use this f to calculate $\int_C \vec{F} \cdot d\vec{r}$ where C is any path from $(1,0)$ to $(2,1)$.

   Solution.
   To check that $\vec{F}(x,y)$ is conservative it is enough to check that

   $$\frac{\partial}{\partial y}[2xe^{-y}] = \frac{\partial}{\partial x}[2y - x^2e^{-y}].$$

   These are both equal to $-2xe^{-y}$, so the vector field is indeed conservative.

   Thus there exists some function f such that $F = \nabla f$, namely,

   $$\begin{aligned} f(x,y) &= \int 2xe^{-y}\,dx & &= \int 2y - x^2e^{-y}\,dy \\ &= x^2e^{-y} + f_1(y) & &= y^2 + x^2e^{-y} + f_2(x) \\ &= y^2 + x^2e^{-y} + c. \end{aligned}$$

   Then by the fundamental theorem of calculus for conservative vector fields

   $$\int_C \nabla f \cdot d\vec{r} = f(2,1) - f(1,0) = 1 + \frac{4}{e} + c - (1+c) = \frac{4}{e}.$$

2. (*5 points*) Set up but do not evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy, yz, zx \rangle$ and S is the portion of the paraboloid $z = 4 - x^2 - y^2$ which lies above the square $0 \le x \le 1,\ 0 \le y \le 1$, and has upward orientation.

Solution.
This can be parameterized by

$$\vec{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle.$$

so that

$$\begin{aligned} \vec{r}_x(x, y) &= \langle 1, 0, -2x \rangle \\ \vec{r}_y(x, y) &= \langle 0, 1, -2y \rangle, \end{aligned}$$

with cross product

$$\vec{r}_x(x, y) \times \vec{r}_y(x, y) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

Then integrated over the square region by the upward orientation and the right hand rule, $0 \le x \le 1$ and $0 \le y \le 1$.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \int_0^1 \int_0^1 \vec{F}(\vec{r}(x, y)) \cdot \underbrace{(r_x(x, y) \times r_y(x, y))}_{\vec{n}\,|r_x(x,y) \times r_y(x,y)|} \, dA \\ &= \int_0^1 \int_0^1 \langle xy, y(4 - x^2 - y^2), x(4 - x^2 - y^2) \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx \, dy \\ &= \int_0^1 \int_0^1 2x^2 y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2) \, dx \, dy. \end{aligned}$$