Problem 5 solutions.

Problem 1. Evaluate the integral
$$\int (3x^2 - 1)\ln(x)\,dx.$$

Solution.
We'll use integration by parts. Let

$$\begin{aligned} u &= \ln(x) \qquad & du &= \frac{1}{x}\,dx \\ v &= x^3 - x \qquad & dv &= 3x^2 - 1\,dx. \end{aligned}$$

Then

$$\begin{aligned} \int (3x^2 - 1)\ln(x)\,dx &= (x^3 - x)\ln(x) - \int \frac{x^3 - x}{x}\,dx \\ &= (x^3 - x)\ln(x) - \int x^2 - 1\,dx \\ &= (x^3 - x)\ln(x) - \frac{x^3}{3} - x + c \end{aligned}$$

Problem 2. Evaluate the integral
$$\int_1^2 \frac{1}{4 + x^2}\,dx$$

Solution.
We'll use the identity
$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}.$$

Begin by dividing the denominator by 4,

$$\int_1^2 \frac{1}{4 + x^2}\,dx = \frac{1}{4}\int_1^2 \frac{1}{1 + x^2/4}\,dx$$

Then substituting $u = x/2$ and $du = dx/2$ gives

$$\int_1^2 \frac{1}{4 + x^2}\,dx = \frac{2}{4}\int_{u=1/2}^1 \frac{1}{1 + u^2}\,du = \left[\frac{1}{2}\arctan(u)\right]_{1/2}^1 = \frac{\pi}{8} - \frac{\arctan(\frac{1}{2})}{2}$$

Problem 3. Evaluate the integral
$$\int \arcsin(x)\,dx$$

Solution.
We'll use integration by parts. Let

$$u = \arcsin(x) \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$$
$$v = x \qquad dv = dx.$$

Then

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Substituting $u = 1 - x^2$ and $du = -2x\,dx$ gives

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + c = -\sqrt{1-x^2} + c$$

Thus the original integral becomes

$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + c$$

Problem 4. Evaluate the integral
$$\int \sin(\sqrt{y})\,dy$$

Solution.
This is a tricky substitution. Let

$$u = \sqrt{y} \quad \text{and} \quad du = \frac{1}{2\sqrt{y}}\,dy.$$

Now solving for dy gives
$$2\underbrace{\sqrt{y}}_{u}\,du = dy = 2u\,du.$$

Plugging this in gives

$$\int \sin(\sqrt{y})\,dy = 2\int u\sin(u)\,du.$$

And now using integration by parts with

$$w = u \qquad dw = du$$
$$v = -\cos(u) \qquad dv = \sin(u)\,du$$

gives

$$\begin{aligned}
2\int u\sin(u)\,du &= 2\left(-u\cos(u) + \int \cos(u)\,du\right) \\
&= 2\sin(u) - 2u\cos(u) + c \\
&= 2\sin(\sqrt{y}) - 2\sqrt{y}\cos(\sqrt{y}) + c
\end{aligned}$$

Problem 5. Evaluate the integral

$$\int e^{2x}\sin(x)\,dx.$$

Solution.
Here we do repeated integration by parts. First let

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x}\,dx \\ v &= -\cos(x) & dv &= \sin(x)\,dx. \end{aligned}$$

Then

$$\int e^{2x}\sin(x)\,dx = -\cos(x)e^{2x} + 2\int \cos(x)e^{2x}\,dx.$$

Next use

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x}\,dx \\ v &= \sin(x) & dv &= \cos(x)\,dx. \end{aligned}$$

This gives

$$\begin{aligned} \int e^{2x}\sin(x)\,dx &= -\cos(x)e^{2x} + 2\left(\sin(x)e^{2x} - 2\int \sin(x)e^{2x}\,dx\right) \\ &= -\cos(x)e^{2x} + 2\sin(x)e^{2x} - 4\int e^{2x}\sin(x)\,dx. \end{aligned}$$

Solving for our original integral gives

$$\int e^{2x}\sin(x)\,dx = \frac{1}{5}\left(2\sin(x)e^{2x} - \cos(x)e^{2x}\right).$$

Problem 6. Evaluate the integral

$$\int_0^5 2x + \sqrt{25 - x^2}\,dx.$$

Solution.
This is best solved geometrically. The fuction $\sqrt{25-x^2}$ describes the semicircle above the x-axis of radius 5, so integrating from 0 to 5 gives the area of quarter circle of radius 5, $25\pi/4$.
The other integral is quick:

$$\int_0^5 2x\,dx = [x^2]_0^5 = 25.$$

Thus

$$\int_0^5 2x + \sqrt{25 - x^2}\,dx = \frac{25\pi}{4} + 25.$$

Problem 7. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{2+x^2}}\,dx.$$

Solution.
This requires a substitution similar in spirit to problem 5.4. Let $u = 2+x^2$ and $du = 2x\,dx$. Then $x^2 = u-2$, so we can rewrite the integral as

$$\begin{aligned}
\frac{1}{2}\int_{u=2}^{3} \frac{u-2}{\sqrt{u}}\,du &= \frac{1}{2}\int_2^3 \sqrt{u} - \frac{2}{\sqrt{u}}\,du \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_2^3 \\
&= \sqrt{3} - 2\sqrt{3} - \frac{2}{3}\sqrt{2} + 2\sqrt{2} \\
&= \frac{4}{3}\sqrt{2} - \sqrt{3}
\end{aligned}$$

Problem 8. Evaluate the integral

$$\int_{-1}^1 x^3 e^{x^2+1}\,dx.$$

Solution.
We can use the same trick as before. Let $u = x^2+1$ and $du = 2x\,dx$. This means that $x^2 = u-1$. So

$$\frac{1}{2}\int_{u=2}^{2} (u-1)e^u\,du = 0.$$

based on the bounds of integration.
(It is enough to check that the original integrand is odd.)