Math 533: Homework 6

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Problem 1.

Proof.

(a) Let φ send $\lambda \in \mathrm{Comp}(n)$ to its partial sums, but leaving off n. Then φ^{-1} is just adding n to the set and taking first differences. For example:

$$\begin{aligned}
(4) &\mapsto \{\} \\
(1,3) &\mapsto \{1\} \\
(2,2) &\mapsto \{2\} \\
(1,1,2) &\mapsto \{1,2\} \\
(3,1) &\mapsto \{3\} \\
(1,2,1) &\mapsto \{1,3\} \\
(2,1,1) &\mapsto \{2,3\} \\
(1,1,1,1) &\mapsto \{1,2,3\}.
\end{aligned}$$

The refinement condition says that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq (\lambda'_{1,1}, \ldots \lambda'_{1,k_1}, \ldots, \lambda'_{n,1}, \ldots, \lambda'_{n,k_n}) = \lambda'$ where $\lambda'_{i,1} + \ldots + \lambda'_{i,k_i} = \lambda_i$ for each i. Then it is clear that

$$\begin{aligned}
\phi(\lambda) &= \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \ldots + \lambda_{n-1}\} \\
&\subseteq \{\lambda'_{1,1}, \ldots, \underbrace{\lambda'_{1,1} + \ldots + \lambda'_{1,k_1}}_{\lambda_1}, \ldots, \underbrace{\lambda'_{1,1} + \ldots + \lambda'_{n-1,k_{n-1}}}_{\lambda_1+\ldots+\lambda_{n-1}}, \ldots, \lambda'_{1,1} + \ldots + \lambda'_{n,k_n-1}\}.
\end{aligned}$$

(b) Gessel's fundamental basis is defined as

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta = \sum_{\beta \preceq \alpha} \sum_{i_1 < \cdots < i_k} x_{i_1}^{\beta_1} \cdots x_{i_k}^{\beta_k}$$

so it follows by the refinement condition

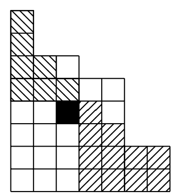
$$\begin{aligned}
F_\alpha &= \sum_{\varphi(\beta) \supseteq \varphi(\alpha)} \sum_{i_1 < \cdots < i_k} x_{i_1}^{\beta_1} \cdots x_{i_k}^{\beta_k} \\
&= \sum_{\varphi(\beta) \supseteq \varphi(\alpha)} \sum_{\substack{i_{1,1} = \cdots = i_{1,m_1} < \\ i_{2,1} = \cdots = i_{2,m_2} < \\ \cdots \\ i_{k,1} = \cdots = i_{k,m_k}}} x_{i_{1,1}} \cdots x_{i_{1,n_1}} x_{i_{2,1}} \cdots x_{i_{k,n_k}} \\
&= \sum_{\substack{i_1 \leq \ldots i_k \\ i_j < i_{j+1} \text{ if } j \in \varphi(\alpha)}} x_{i_1} \ldots x_{i_k}.
\end{aligned}$$

(c) Let f be a symmetric function written in the Schur basis. We'll check ρ and ψ on s_λ and extend by linearity. For example $\lambda = (2,1,1)$ has SYT corresponding to the compositions $(2,1,1), (1,2,1)$, and

$(1,1,2)$, which we'll index by the sets $\{2,3\}, \{1,3\}$, and $\{1,2\}$ respectively.

$$\begin{aligned}
s_{(2,1,1)} &= f_{(2,1,1)} + f_{(1,2,1)} + f_{(1,1,2)} \\
&= f_{\{2,3\}} + f_{\{1,3\}} + f_{\{1,2\}} \\
\rho(s_{(2,1,1)}) &= \rho(f_{(2,1,1)}) + \rho(f_{(1,2,1)}) + \rho(f_{(1,1,2)}) \\
&= f_{(1,1,2)} + f_{(1,2,1)} + f_{(2,1,1)} \\
&= s_{(2,1,1)} \\
\psi(s_{(2,1,1)}) &= \psi(f_{(2,1,1)}) + \psi(f_{(1,2,1)}) + \psi(f_{(1,1,2)}) \\
&= f_{\varphi(\{2,3\}^c)} + f_{\varphi(\{1,3\}^c)} + f_{\varphi(\{1,2\}^c)} \\
&= f_{\varphi(\{1\})} + f_{\varphi(\{2\})} + f_{\varphi(\{3\})} \\
&= f_{(1,3)} + f_{(2,2)} + f_{(3,1)} \\
w(s_{(2,1,1)}) &= s_{(2,1,1)'} \\
&= s_{(3,1)} \\
&= f_{(1,3)} + f_{(2,2)} + f_{(3,1)}
\end{aligned}$$

So in this particular example, ρ is the identity and ψ agrees with w. This is because under conjugation, inversions become non-inversions and vice versa, as illustrated here:



If the black box is filled with i, the upper section consists of possible positions for $i+1$ which create inversions, the lower section for non-inversions. These places switch under conjugation. (And conjugation sends SYT to SYT.)

$\square$

Problem 2.

Proof.

(a) The eight classes of standard shifted tableau are:

$$\begin{array}{cccc}
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4' \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 4 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 4' \\ \hline \end{array} \\
(2,2) & (2,1,1) & (1,3) & (1,2,1)
\end{array}$$

$$\begin{array}{cccc}
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3' \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 3 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2' & 3' \\ \hline \end{array} \\
(3,1) & (2,2) & (1,2,1) & (1,1,2)
\end{array}$$

And each shows up with multiplicity four since we can prime the diagonals independently.

(b) It is clear from the example that Q_γ is the sum over the standard shifted tableau of shape γ, and the multiplicity comes from the independence of priming the $\ell(\gamma)$ entries on the diagonal.

(c) Problem 1(c) worked out that

$$\begin{aligned} s_{(2,1,1)} &= f_{(2,1,1)} + f_{(1,2,1)} + f_{(1,1,2)} \\ s_{(3,1)} &= f_{(1,3)} + f_{(2,2)} + f_{(3,1)}. \end{aligned}$$

Also, there are two standard young tableaux of shape $(2,2)$:

$$\begin{array}{cc}
\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \\
(2,2) & (1,2,1)
\end{array}$$

with corresponding expansion in Gessel's fundamental basis

$$s_{(2,2)} = f_{(2,2)} + f_{(1,2,1)}.$$

Thus

$$Q_{(3,1)} = 4(s_{(2,1,1)} + s_{(3,1)} + s_{(2,2)}).$$

□

Problem 4.

Proof.

(a) If $\kappa\colon V \to \mathbb{N}_{>0}$ is proper, then acting on κ by any permutation of the integers σ also gives a proper coloring. This means that the sum is closed under permuting the indices, so X_G is symmetric.

(b) By definition, the chromatic number of G is

$$\chi_G(n) = \#\{\kappa\colon V \to [n] : \kappa \text{ is a proper coloring}\}.$$

This is identically

$$\begin{aligned} X_G(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots) &= \sum_{\substack{\kappa \text{ proper} \\ \kappa(v) \leq n}} x_{\kappa(v_1)} \ldots x_{\kappa(v_n)} \\ &= \#\{\kappa\colon V \to [n] : \kappa \text{ is a proper coloring}\}, \end{aligned}$$

as desired.

$\square$