Differential Geometry: Homework 1

Peter Kagey

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Problem 1. Show that the induced topology (for a subset $X \subset Y$ of a topological space Y) and the quotient topology (for a surjection $X \twoheadrightarrow Y$ from a topological space X onto a set Y) satisfy the axioms of a topological space.

Proof.
Induced topology. Let $\mathcal{T}_X = \{i^{-1}(U_y) | U_y \in \mathcal{T}_Y\}$.

- $\emptyset, X \in \mathcal{T}_{\mathcal{X}}$: $i^{-1}(Y) = X$ and $i^{-1}(\emptyset) = \emptyset$. ($Y, \emptyset \in \mathcal{T}_Y$.)
- Closed under arbitrary union:
  Let $\{U_{X,j}\}_{j \in I}$ be a collection of open sets in $\mathcal{T}_X$. Then
  $$\bigcup_{j \in I} U_{X,j} = \bigcup_{j \in I} i^{-1}(U_{Y,j}) = i^{-1}(\bigcup_{j \in I} U_{Y,j})$$
  where $\bigcup_{j \in I} U_{Y,j} \in \mathcal{T}_Y$ because Y is a topological space.
- Closed under finite intersection.
  Similarly
  $$\bigcap_{j=1}^{N} U_{X,j} = \bigcap_{j=1}^{N} i^{-1}(U_{Y,j}) = i^{-1}(\bigcap_{j=1}^{N} U_{Y,j})$$
  where $\bigcap_{j=1}^{N} U_{Y,j} \in \mathcal{T}_Y$ because Y is a topological space.

Quotient topology. Let $\mathcal{T}_Y = \{V \subset Y \mid p^{-1}(V) \in \mathcal{T}_X\}$.

- $\emptyset, Y \in \mathcal{T}_{\mathcal{Y}}$:
  Because p is surjective, $p^{-1}(Y) = X \in \mathcal{T}_X$. Also $p^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$.
- Closed under arbitrary union:
  Let $\{U_{Y,j}\}_{j \in I}$ be a collection of open sets in $\mathcal{T}_Y$. Then $p^{-1}(U_{Y,j}) \in \mathcal{T}_X$ for all $j \in I$, so
  $$\bigcup_{j \in I} p^{-1}(U_{Y,j}) = p^{-1}(\bigcup_{j \in I} U_{Y,j}) \in \mathcal{T}_X$$
  because $\mathcal{T}_X$ is closed under union. Therefore $\bigcup_{j \in I} U_{Y,j} \in \mathcal{T}_Y$.
- Closed under finite intersection:
  Similarly
  $$\bigcap_{j=1}^{N} p^{-1}(U_{Y,j}) = p^{-1}\left(\bigcap_{j=1}^{N} U_{Y,j}\right) \in \mathcal{T}_X$$
  because $\mathcal{T}_X$ is closed under finite intersection. Therefore $\bigcap_{j=1}^{N} U_{Y,j} \in \mathcal{T}_Y$.

□

Problem 2. Show that the topological spaces $S^1 \subset \mathbb{R}^2$ (with topology induced by the inclusion into $\mathbb{R}^2$) and $[0,1]/\{0,1\}$ (with the quotient topology from the topology on $[0,1] \subset \mathbb{R}$) are homeomorphic.

Proof. Let $f : [0,1] \to S^1$ be the map $t \overset{f}{\mapsto} (\cos(2\pi t), \sin(2\pi t))$. Because sin and cos are continuous functions with respect to the standard topology on $\mathbb{R}^2$, f is continuous, so it is sufficient to show that f has a two-sided inverse.

Let

$$f^{-1}(x,y) = \begin{cases} 0 \sim 1 & (x,y) = (0,0) \\ 1/2 & (x,y) = (-1,0) \\ \frac{1}{2\pi}\cos^{-1}(x) & y > 0 \\ 1 - \frac{1}{2\pi}\cos^{-1}(x) & y < 0 \end{cases}.$$

The continuity of f^{-1} follows from the continuity of $\cos^{-1}$ and the equality of f^{-1} at the "handoff" points. Then

$$f(f^{-1}(x,y)) = \begin{cases} (\sin(0), \cos(0)) = (\sin(2\pi), \cos(2\pi)) = (0,0) & (x,y) = (0,0) \\ (\cos(\pi), \sin(\pi)) = (-1,0) & (x,y) = (-1,0) \\ (\cos(\cos^- 1(x)), \sin(\cos^- 1(x))) = (x, \sqrt{1-x^2}) = (x,y) & y > 0 \\ (\cos(-\cos^{-1}(x)), \sin(-\cos^{-1}(x))) = (x,y) & y < 0 \end{cases}.$$

Similarly

$$f^{-1}(f(t)) = \begin{cases} f^{-1}(0,0) = (0 \sim 1) & t = (0 \sim 1) \\ f^{-1}(-1,0) = 1/2 & t = 1/2 \\ f^{-1}(\cos(2\pi t), \sin(2\pi t)) = t & t \in (0, 1/2) \\ f^{-1}(\cos(2\pi t), \sin(2\pi t)) = t & t \in (1/2, 1) \end{cases}.$$

Thus f is a homeomorphism, so the two spaces are homeomorphic. $\square$

Problem 3. Prove that S^1, with either topology considered above, is a topological manifold.

Proof. Consider the quotient topology $[0,1]$

(i) (Hausdorff)
Let a, b be distinct points in $[0,1]/(0 \sim 1)$.
Case 1: $a, b \neq (0 \sim 1)$. Then take $r = |a-b|/2$.

$$B_r(a) \cap B_r(b) = \emptyset.$$

Case 2: Without loss of generality, assume $a = (0 \sim 1)$. Let $r = \min(b, 1-b)/2$. Then

$$B_r(a) \cap B_r(b) = \emptyset.$$

(ii) (Second countable)
A countable basis for $[0,1]/(0 \sim 1)$ is:

$$\{B_r(x) \cap (0,1) \mid r \in \mathbb{Q}, x \in \mathbb{Q}\} \cup \{[0,r) \cup (1-r,1] \mid r \in \mathbb{Q}\}.$$

(iii) (Locally Euclidean)
Assume that $x \neq (0 \sim 1)$. Then the ball with radius $r = \min(x, 1-x)$ maps in the obvious way to $B_r(x) \subset (0,1) \subset \mathbb{R}$.

Assume that $x = (0 \sim 1)$. Then the open set $[0,1/4) \cup (3/4,1]$ is homeomorphic to $(0,1/2)$ via

$$f(x) = \begin{cases} x - 3/4 & x \in (3/4,1) \\ 1/4 & x = (0 \sim 1) \\ x + 1/4 & x \in (0,1/4) \end{cases}.$$

Where f is continuous and has two sided inverse

$$f^{-1}(p) = \begin{cases} p + 3/4 & p \in (0,1/4) \\ (0 \sim 1) & p = 1/4 \\ p - 1/4 & p \in (1/4,1/2) \end{cases}.$$

$\square$

Problem 4. Show that the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ if it exists at a point $a \in \mathbb{R}^m$ is unique.

Proof. I will prove this starting only from the definition that if the derivative of a function exists at a point $a \in \mathbb{R}^n$ then the derivative is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\vec{h} \to 0} \frac{|f(a+\vec{h}) - f(a) - L(\vec{h})|}{|\vec{h}|} = 0.$$

Suppose for the sake of contradiction that there is another linear map $L_2 : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies the above property. By the triangle inequality

$$0 = \lim_{\vec{h} \to 0} \frac{|f(a+\vec{h}) - f(a) - L(\vec{h})|}{|\vec{h}|} + \frac{|-f(a+\vec{h}) + f(a) + L_2(\vec{h})|}{|\vec{h}|} \geq \lim_{\vec{h} \to 0} \frac{|L_2(\vec{h}) - L(\vec{h})|}{|\vec{h}|} \geq 0.$$

Thus

$$\lim_{\vec{h} \to 0} \frac{|L_2(\vec{h}) - L(\vec{h})|}{|\vec{h}|} = 0.$$

Notice that the limit does not depend on $\vec{h}$ being small. Let $\vec{h} = \varepsilon \vec{v}$. By linearity

$$\frac{|L_2(\vec{h}) - L(\vec{h})|}{|\vec{h}|} = \frac{|\varepsilon L_2(\vec{v}) - \varepsilon L(\vec{v})|}{|\varepsilon \vec{v}|} = \frac{|L_2(\vec{v}) - L(\vec{v})|}{|\vec{v}|}.$$

Thus if the the ratio vanishes in the limit, it must vanish everywhere. So $L_2 = L$ for all $\vec{v}$, and the derivative is unique. $\square$

Problem 5. Produce, with proofs, examples of the following topological spaces which are not topological manifolds:

a) A space X which is locally Euclidean and second countable, but not Hausdorff.

b) A space X which is Hausdorff and second countable, but not locally Euclidean.

Proof.

a) Let $X = \mathbb{R} \cup \{\star\}$, and let the topology $\mathcal{T}$ on X be defined by

$$\mathcal{T} = \mathcal{T}_{\mathbb{R},\text{std}} \cup \{U \setminus \{0\} \cup \{\star\} \mid U \in \mathcal{T}_{\mathbb{R},\text{std}} \text{ and } 0 \in U\}.$$

Locally Euclidean: Suppose that $x \neq \star$. Then the open ball of radius $r > 0$ centered at x in X is homeomorphic to $(-r, r) \in \mathcal{T}_{\mathbb{R},\text{std}}$ in the obvious way.
Suppose then that $x = \star$, then the open ball of radius 1 centered at $\star$ is homeomorphic to $(-1, 1)$ via

$$f(x) = \begin{cases} 0 & x = \star \\ x & \text{otherwise} \end{cases} \text{ with inverse } f^{-1}(x) = \begin{cases} \star & x = 0 \\ x & \text{otherwise} \end{cases}.$$

Second countable: A countable basis for $\mathcal{T}$ is

$$\{B_r(x) \mid r \in \mathbb{Q}, x \in \mathbb{Q} \cup \{\star\}\}.$$

Not Hausdorff: For every open set which contains 0 or $\star$ contains points near 0, so 0 and $\star$ do not have distinct neighborhoods.
Let $U \in \mathcal{T}$ be a set that contains 0. Then there exists some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \in U$. At the same time, let $V \in \mathcal{T}$ be a set that contains $\star$. By construction, there exists some $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(0) \setminus \{0\} \in V$.

Thus for all such open sets U and V, the point $(\varepsilon_0 + \varepsilon_1)/2 \in U \cap V$. So 0 and $\star$ have no disjoint neighborhoods.

b) Let $X = [0, 1]$ with the induced topology from $\mathbb{R}$ (with the standard topology).
Hausdorff: This is inherited from the standard topology on $\mathbb{R}$.
Second countable: This is also inherited from the standard topology on $\mathbb{R}$.
Not locally Euclidean: X is compact, which is not true of any open subset of $\mathbb{R}$. Thus X cannot be homeomorphic to any open subset of $\mathbb{R}$, because this property is preserved under continuous maps.

□

Problem 6. Let h be a continuous real-valued function on $S^1 = \{x^2 + y^2 = 1 \subset \mathbb{R}^2\}$ satisfying $h(0,1) = h(1,0) = 0$ and $h(-x_1, -x_2) = -h(x_1, x_2)$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} ||x||h(x/||x||) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a) Show that f is continuous at $(0,0)$, that the partial derivatives of f at $(0,0)$ are defined, and that more generally all directional derivatives of f are defined.

b) Show that f is not differentiable at $(0,0)$ except if h is identically zero.

Proof.

a) S^1 is a compact subspace of R^2, so the continuity of h implies that the image of S^1 under h is bounded by some N. Let $\varepsilon > 0$. Then for all x with $||x|| < \varepsilon/N$,

$$||f(x)|| < ||x||N < \varepsilon.$$

So f is continuous at 0.

The existence of partial derivatives follows from the existence of directional derivatives, so it is sufficient to show that all directional derivatives are defined.

$$df(0)(\vec{a}) = \lim_{k \to 0} \frac{f(0 + k\vec{a}) + f(0)}{k} = \lim_{k \to 0} \frac{f(k\vec{a})}{k} = \lim_{k \to 0} \frac{||k\vec{a}||h\left(\frac{k\vec{a}}{||k\vec{a}||}\right)}{k} = ||\vec{a}||h\left(\frac{\vec{a}}{||\vec{a}||}\right).$$

b) If f is differentiable at a point, all derivatives agree with the linear map. Because $df(0)(\vec{e}_1) = df(0)(\vec{e}_2) = 0$, this means that the directional derivative in the direction $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ must vanish:

$$df(0)(\vec{a}) = a_1 df(0)(\vec{e}_1) + a_2 df(0)(\vec{e}_2) = 0a_1 + 0a_2 = 0.$$

This implies that if f is differentiable at $(0,0)$, then for all $\vec{a}$,

$$df(0)(\vec{a}) = ||\vec{a}||h\left(\frac{\vec{a}}{||\vec{a}||}\right) = 0.$$

Thus h (and consequently f) must be identically 0.

$\square$