

Topology: Homework 2

Peter Kagey

September 11, 2018

Problem 1.

Let

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1 \text{ by } x \mapsto (\cos x, \sin x) \\ \alpha: [0, 1] &\rightarrow S^1 \text{ by } t \mapsto p(2\pi t), \end{aligned}$$

let f be some map with the property that $f(-x, -y) = -f(x, y)$. And let $\tilde{\beta}: [0, 1] \rightarrow \mathbb{R}$ be a lift of the path $\beta = f \circ \alpha$.

- a.** Show that there exists an integer $n_0 \in \mathbb{Z}$ such that $\tilde{\beta}(\frac{1}{2}) = \tilde{\beta}(0) + 2n_0\pi + \pi$.

Proof.

We know that $p \circ \tilde{\beta} = f \circ \alpha$, which is to say,

$$(\cos(\tilde{\beta}(t)), \sin(\tilde{\beta}(t))) = f(\cos(2\pi t), \sin(2\pi t)).$$

And let's further decompose f as $f(x, y) = (f_x(x, y), f_y(x, y))$.

Thus, when $t = 0$, $\cos(\tilde{\beta}(0)) = f_x(1, 0)$. Similarly, when $t = \frac{1}{2}$, $\cos(\tilde{\beta}(\frac{1}{2})) = f_x(-1, 0) = -f_x(1, 0)$. Therefore

$$\cos(\tilde{\beta}(0)) + \cos(\tilde{\beta}(\frac{1}{2})) = 0$$

so $\tilde{\beta}(0)$ and $\tilde{\beta}(\frac{1}{2})$ differ by π up to some multiple of 2π :

$$\tilde{\beta}(\frac{1}{2}) = \tilde{\beta}(0) + 2n_0\pi + \pi.$$

□

- b.** Show that, for the integer n_0 of part **a**, $\tilde{\beta}(t) = \tilde{\beta}(t - \frac{1}{2}) + 2n_0\pi + \pi$ for every $t \in [\frac{1}{2}, 1]$.

Proof.

Writing things explicitly gives

$$\begin{aligned} (\cos(\tilde{\beta}(t)), \sin(\tilde{\beta}(t))) &= f(\cos(2\pi t), \sin(2\pi t)) \\ (\cos(\tilde{\beta}(t - \frac{1}{2})), \sin(\tilde{\beta}(t - \frac{1}{2}))) &= f((\cos(2\pi(t - \frac{1}{2})), \sin(2\pi(t - \frac{1}{2})))) \\ &= f((\cos(2\pi t - \pi), \sin(2\pi t - \pi))) \\ &= f((- \cos(2\pi t), - \sin(2\pi t))) \\ &= -f((\cos(2\pi t), \sin(2\pi t))) \end{aligned}$$

So similarly to part **a**,

$$\cos(\tilde{\beta}(t)) + \cos(\tilde{\beta}(t - \frac{1}{2})) = f_x(\cos(2\pi t), \sin(2\pi t)) - f_x(\cos(2\pi t), \sin(2\pi t)) = 0$$

the arguments $\tilde{\beta}(t)$ and $\tilde{\beta}(t - \frac{1}{2})$ must differ by π up to a multiple of 2π . By continuity, this multiple of 2π must be the same as in part **a**. Thus

$$\tilde{\beta}(t) = \tilde{\beta}(t - \frac{1}{2}) + 2n_0\pi + \pi.$$

Thus

$$\begin{aligned}\tilde{\beta}(1) &= \tilde{\beta}(\tfrac{1}{2}) + 2n_0\pi + \pi, \\ \tilde{\beta}(\tfrac{1}{2}) &= \tilde{\beta}(0) + 2n_0\pi + \pi, \text{ and so} \\ \tilde{\beta}(1) &= \tilde{\beta}(0) + 4n_0\pi + 2\pi,\end{aligned}$$

and because $4n_0 + 2 \neq 0$ for all $n_0 \in \mathbb{Z}$, $\tilde{\beta}(1) \neq \tilde{\beta}(0)$. \square

- c. Show that, if $x_0 = (1, 0)$ and $y_0 = f(1, 0)$, the induced homomorphism $f_*: \pi_1(S^1; x_0) \rightarrow \pi_1(S^1; y_0)$ is non-trivial.

Proof.

Consider the generating element $[\alpha] \in \pi_1(S^1; x_0)$, which maps to $[f \circ \alpha] = [p \circ \tilde{\beta}]$ under f_* . Since part **b** shows that $\tilde{\beta}(1) \neq \tilde{\beta}(0)$, we know that $p \circ \tilde{\beta}$ must describe a non-trivial loop around S^1 , and thus the homomorphism is non-trivial. \square

- d. Consider the 2-dimensional sphere S^2 and identify the unit circle S^1 to its equator.

- (i) Show that for every map $F: S^2 \rightarrow S^1$, the restriction $f: S^1 \rightarrow S^1$ which sends $(x, y) \mapsto F(x, y, 0)$ is homotopic to a constant map.

Proof.

This is similar to ... \square

- (ii) Show that there is no map $F: S^2 \rightarrow S^1$ such that $F(-x, -y, -z) = -F(x, y, z)$ for every $(x, y, z) \in S^2$.

Proof. This follows from parts **c** and **d** (i). Suppose that there were such a map. Then its restriction to the equator would satisfy the conditions for f above, and thus by part **c**, it would not be nullhomotopic. However, part **d** (i) showed that such a map *must* be nullhomotopic, a contradiction. Thus no map may exist. \square

- e. Let $f: S^2 \rightarrow \mathbb{R}^2$ be continuous. Show that there exists at least one pair of antipodal points that have the same image under g .

Proof. As per the hint, consider the map $F: S^2 \rightarrow S^1$ by

$$F(x, y, z) = \frac{g(x, y, z) - g(-x, -y, -z)}{\|g(x, y, z) - g(-x, -y, -z)\|}.$$

This function meets the criteria in part **d** (ii), namely

$$-F(x, y, z) = \frac{g(-x, -y, -z) - g(x, y, z)}{\|g(x, y, z) - g(-x, -y, -z)\|} = \frac{g(-x, -y, -z) - g(x, y, z)}{\|g(-x, -y, -z) - g(x, y, z)\|} = F(-x, -y, -z).$$

Therefore no such continuous function exists on all of S^2 , and so there exists some (x, y, z) such that $\|g(-x, -y, -z) - g(x, y, z)\| = 0$, and thus there exists some (x, y, z) such that $g(-x, -y, -z) = g(x, y, z)$. \square

- f. Let A and B be two bounded domains in the xy -plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

- (i) For each unit vector $\vec{u} \in S^2 \subset \mathbb{R}^3$, let $P_{\vec{u}}$ be the plane in \mathbb{R}^3 passing through the point $(0, 0, 1)$ and orthogonal to \vec{u} , and let $H_{\vec{u}}$ be the half-space delimited by $P_{\vec{u}}$ such that \vec{u} points toward the interior of $H_{\vec{u}}$.

Show that there exists $\vec{u} \in S^2$ such that $\text{area}(A \cap H_{\vec{u}}) = \frac{1}{2} \text{area}(A)$ and $\text{area}(B \cap H_{\vec{u}}) = \frac{1}{2} \text{area}(B)$.

Proof. As suggested by the hint, let $g: S^2 \rightarrow \mathbb{R}^2$ be defined by

$$g(\vec{u}) = (\text{area}(A \cap H_{\vec{u}}), \text{area}(B \cap H_{\vec{u}})).$$

Notice that $P_{-\vec{u}} = P_{\vec{u}}^c$, so

$$g(-\vec{u}) = (\text{area}(A) - \text{area}(A \cap H_{\vec{u}}), \text{area}(B) - \text{area}(B \cap H_{\vec{u}}))$$

and moreover, $g(\vec{u}) = g(-\vec{u})$ precisely when

$$\begin{aligned} \text{area}(A \cap H_{\vec{u}}) &= \frac{1}{2} \text{area}(A) \\ \text{area}(B \cap H_{\vec{u}}) &= \frac{1}{2} \text{area}(B). \end{aligned}$$

It takes some measure-theoretic argument to show that g is continuous, but taking that for granted, part **e** proves that there exists a pair of antipodal points that have the same image under g . Thus the desired vector \vec{u} is the one that satisfies the antipodal equality property. \square

- (ii) Show that there exists a line in \mathbb{R}^2 that divides each of A and B into halves of equal area.

Proof. Simply choose any vector \vec{u} that satisfies part **f** (i), and take the line which is the intersection of $P_{\vec{u}}$ and the xy -plane. \square