Complex Analysis: Homework 3

Peter Kagey

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Problem 6. (page 108)
Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve in Ω.

Proof. $\square$

Problem 7. (page 108)

If $P(z)$ is a polynomial and C denotes the circle $|z-a|=R$, what is the value of of $\displaystyle\int_C P(z)d\bar{z}$?

Answer: $-2\pi i R^2 P'(a)$.

Proof. Parameterize the circle by $z(t) = Re^{it} + a$ for $\theta \in [0, 2\pi]$, then $\bar{z}(t) = \overline{Re^{it} + a} = Re^{-it} + \bar{a}$ and $d\bar{z} = -iRe^{-it}dt$.

$$\int_C P(z)d\bar{z} = -iR\int_0^{2\pi} P(Re^{it}+a)e^{-it}dt$$

Notice that the (finite) Taylor series expansion of P around a is

$$P(z) = P(a) + P'(a)(z-a) + \ldots + \frac{P^{(n)}(a)}{n!}(z-a)^n.$$

So the above integral becomes

$$\begin{aligned}
\int_C P(z)d\bar{z} &= -iR\int_0^{2\pi} e^{-it}\left(P(a) + P'(a)(Re^{it}) + \ldots + \frac{P^{(n)}(a)}{n!}(R^n e^{nit})\right)dt \\
&= -iR\int_0^{2\pi} P(a)e^{-it} + RP'(a) + \frac{P''(a)}{2}(Re^{it})\ldots + \frac{P^{(n)}(a)}{n!}(R^n e^{(n-1)it})dt \\
&= -iR\int_0^{2\pi} RP'(a)dt \\
&= -2\pi i R^2 P'(a)
\end{aligned}$$

because all terms except for the first derivative term vanish due to symmetry, (i.e. $c\int_0^{2\pi} e^{nit}dt = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$.) $\square$

Problem 1. (page 120)
Compute
$$\int_{|z|=1} \frac{e^z}{z} dz.$$

Proof. Because e^z is analytic in $\mathbb{C}$, and $\gamma = \{|z| = 1\}$ is a closed curve in the disk of radius 2 centered at the origin, Theorem 6 gives
$$\int_\gamma \frac{e^z}{z}\ dz = 2\pi i \cdot n(\gamma, 0) \cdot e^0.$$
The winding number of γ around the origin is 1 and $e^0 = 1$, so
$$\int_\gamma \frac{e^z}{z}\ dz = 2\pi i.$$
. □

Problem 3. (page 120)
Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}.$$

Proof. By partial fraction decomposition and Theorem 6,

$$\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2+1} &= \frac{1}{2}\int_{|z|=2} \frac{dz}{z-i} - \frac{1}{2}\int_{|z|=2} \frac{dz}{z+i} \\
&= \frac{1}{2} \cdot n(\gamma, i) \cdot 1 - \frac{1}{2} \cdot n(\gamma, -i) \cdot 1 \\
&= 0.
\end{aligned}$$

$\square$

Problem 2. (page 123)
Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Proof. Because $|f(z)| < |z|^n$, $0 \leq |f(z)/z^n| < 1$ for sufficiently large $|z|$, by Theorem 8,

$$\begin{aligned} f(z) &= f_{n+1}(z) \cdot (z)^{n+1} + \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(z)^k \\ 1 &> \left| \frac{f(z)}{z^n} \right| \\ &= \left| f_{n+1}(z) \cdot (z) + f^{(n)}(0) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}(z)^{k-n} \right| \end{aligned}$$

By taking large $|z|$, the absolute value of the sum can be made arbitrarily small. Therefore in order to stay bounded, $f_{n+1}(z) = 0$ for all z, so

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(z)^k$$

which is a polynomial. □

Problem 3. (page 123)
If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

Proof. By Equation (24),

$$\begin{aligned}
|f^{(n)}(z)| &= \frac{n!}{2\pi}\left|\int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta\right| && (1)\\
&\leq \frac{n!}{2\pi}\int_{|\zeta|=R} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}}|d\zeta| && (2)\\
&\leq \frac{n!M}{2\pi}\int_{|\zeta|=R} \frac{|d\zeta|}{|\zeta - z|^{n+1}} && (3)\\
&\leq \frac{n!M}{2\pi \cdot (R-\rho)^{n+1}}\int_{|\zeta|=R} |d\zeta| && (4)\\
&= \frac{n!2\pi r M}{2\pi \cdot (R-\rho)^{n+1}} && (5)\\
&= \frac{n!RM}{(R-\rho)^{n+1}}. && (6)
\end{aligned}$$

(2) is justified by Equation (9) in Ahlfors; (3) because $|f(\zeta)| \leq M$, by hypothesis; (4) because $|\zeta - z| \geq (R-\rho)$ for $|z| \leq \rho$; (5) by the arc length of the circle of radius R; and (6) by simplification. □

Problem 4. (page 123)
If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(z)|$ that Cauchy's inequality will yield.

Proof. Again using equation (24) and letting C be the circle of radius $0 < \rho < 1$ about the origin,

$$\begin{aligned}
|f^{(n)}(0)| &= \frac{n!}{2\pi}\left|\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| && (1)\\
&= \frac{n!}{2\pi}\left|\int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| && (2)\\
&\leq \frac{n!}{2\pi}\int_{|\zeta|=\rho} \frac{|f(\zeta)|}{|\zeta|^{n+1}}|d\zeta| && (3)\\
&= \frac{n!}{2\pi\rho^{n+1}}\int_{|\zeta|=\rho} |f(\zeta)||d\zeta| && (4)\\
&\leq \frac{n!}{2\pi\rho^{n+1}}\int_{|\zeta|=\rho} \frac{|d\zeta|}{1-|\zeta|} && (5)\\
&= \frac{n!}{2\pi\rho^{n+1}(1-\rho)}\int_{|\zeta|=\rho} |d\zeta| && (6)\\
&= \frac{n!2\pi\rho}{2\pi\rho^{n+1}(1-\rho)} && (7)\\
&= \frac{n!}{\rho^n(1-\rho)} && (8)
\end{aligned}$$

This last expression is minimized when $\rho^n(1-\rho)$ is maximized

$$\frac{d}{d\rho}[\rho^n(1-\rho)] = n\rho^{n-1} - (n+1)\rho^n = 0,$$

that is, $\rho = n/(n+1)$. In this case

$$\frac{n!}{\rho^n(1-\rho)} = (n+1)!\left(\frac{n+1}{n}\right)^n.$$

Therefore the best estimate of $|f^{(n)}(z)|$ is

$$|f^{(n)}(z)| \leq (n+1)!\left(\frac{n+1}{n}\right)^n.$$

□