# Topology: Homework 5

Peter Kagey

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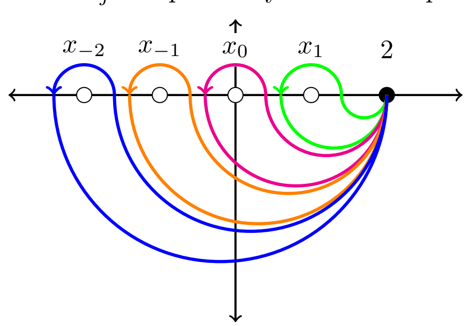
**Problem 1.**

a. Give a presentation for the fundamental groups $\pi_1(\mathbb{C} - \{-2, -1, 0, 1\}; 2)$ and $\pi_1(\mathbb{C} - \{0, 1\}; 4)$.

b. Consider the map
$$f\colon \mathbb{C} - \{-2, -1, 0, 1\} \to \mathbb{C} - \{0, 1\}$$
defined by $f(z) = z^2$. Compute the induced isomorphism
$$f_*\colon \pi_1(\mathbb{C} - \{-2, -1, 0, 1\}; 2) \to \pi_1(\mathbb{C} - \{0, 1\}; 4)$$
in terms of the generators in Part a.

*Proof.*

a. As shown in the previous homework, since $\mathbb{C} \simeq \mathbb{R}^2$, the fundmental groups are isomorphic to the free groups on four and two letters respectively:
$$\pi_1(\mathbb{C} - \{-2, -1, 0, 1\}; 2) \cong F_4(x_{-2}, x_{-1}, x_0, x_1)$$
$$\pi_1(\mathbb{C} - \{0, 1\}; 2) \cong F_2(y_0, y_1),$$
where $x_j$ is a positively oriented loop around $j$ as follows:




b. The generators of $\pi_1(\mathbb{C} - \{-2, -1, 0, 1\}; 2)$ map to the generators of $\pi_1(\mathbb{C} - \{0, 1\}; 2)$ as follows:

($x_1 \mapsto y_1$) The map $f$ maps the right-half plane to the entire plane $\mathbb{C}$ in a conformal, injective way. So loops map to loops, and the interiors, exteriors, and orientations are preserved. Thus $x_1$ maps to a loop going around $f(1) = 1$ once in the positive direction.

($x_0 \mapsto y_0^2$) Under $f$, a loop which is a circle around the origin maps to a circle which around the origin twice. Going from the base point to the circle along a path in the above diagram does not affect this.

($x_{-2} \mapsto \mathrm{id}$) Here the image of the path under $f$ does not go around any holes in $\mathbb{C} - \{0, 1\}$, and so it deformation retracts to a constant path.

($x_{-1} \mapsto y_0^{-1} y_1 y_0$) Instead of considering the loop directly, instead consider the loop that goes from 2 to 3, then completes a positively oriented circle of radius 3 around all holes, and then goes from 3 to 2. This is homotopic to the path $x_1 x_0 x_{-1} x_{-2}$. It's image goes from 4 to 9, then twice around a circle of radius 9, and the back from 9 to 4, so it is homotopic to $(y_1 y_0)^2 = y_1 y_0 y_1 y_0$. Since $f_*$ is a homomorphism,

$$\begin{aligned} f_*(x_1 x_0 x_{-1} x_{-2}) &= y_1 y_0 y_1 y_0 \\ f_*(x_1) f_*(x_0) f_*(x_{-1}) f_*(x_{-2}) &= y_1 y_0 y_1 y_0 \\ y_1 y_0^2 f_*(x_{-1}) \,\mathrm{id} &= y_1 y_0 y_1 y_0 \\ y_0 f_*(x_{-1}) \,\mathrm{id} &= y_1 y_0 \\ f_*(x_{-1}) \,\mathrm{id} &= y_0^{-1} y_1 y_0. \end{aligned}$$

Therefore $f_*$ maps

$$\begin{aligned} x_{-2} &\xmapsto{f_*} \mathrm{id}, \\ x_{-1} &\xmapsto{f_*} y_0 y_1 y_0^{-1}, \\ x_0 &\xmapsto{f_*} y_0^2, \text{ and} \\ x_1 &\xmapsto{f_*} y_1. \end{aligned}$$

$\square$

**Problem 2.**
Define

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} \text{ and}$$
$$B^2 = \{z \in \mathbb{C} : |z| \leq 1\},$$

and let $V_1$ and $V_2$ be two copies of $B^2 \times S^1$ whose boundries are identified to $S^1 \times S^1$. In $V_1$, let $D_1$ be the disk $B^2 \times \{1\}$ with boundary canonically identified to $\partial B^2 = S^1$

a. Let $X$ be obtained from the disjoint union of $D_1$ and $V_2$ by gluing the boundary $\partial D_1 = S^1$ to $\partial V_2 = S^1 \times S^1$ by the map $\phi\colon S^1 \to S^1 \times S^1$ defined by $\phi(u) = (u^a, u^b)$ for some $a, b \in \mathbb{Z}$. Compute the fundamental group of $X$.

b. Let $L$ be obtained by the disjoint union of $V_1$ and $V_2$ by gluing the boundary $\partial V_1$ to $\partial V_2$ by the map

$$\phi\colon \underbrace{S^1 \times S^1}_{\partial V_1} \to \underbrace{S^1 \times S^1}_{\partial V_2} \text{ which sends } \phi(u, v) = (u^a v^c, u^b v^d)$$

for some integers $a, b, c, d \in \mathbb{Z}$.

*Proof.*

a. First note that $D_1$ is homeomorphic to a disk and $V_2$ is homeomorphic to a solid torus. Thus $D_1$ deformation retracts to a point, and $V_2$ deformation retracts to $S^1$. Therefore by van Kampen's theorem, we can write $\pi_1(X)$ as

$$\begin{aligned}\pi_1(X; [(x_1, x_2)]) &\cong \underbrace{\pi_1(D_1; x_1)}_{\mathbf{1}} \underbrace{*_{\pi_1(\phi(\partial D_1) \cap V_2; (\phi(x_1), x_2))}}_{*_{\mathbb{Z}}} \underbrace{\pi_1(V_2; \phi(x_1))}_{\mathbb{Z}} \\ &\cong \mathbf{1} *_{\mathbb{Z}} \mathbb{Z},\end{aligned}$$

There is only one choice of map (and thus homomorphism) for $i_A\colon \mathbb{Z} \to \mathbf{1}$, namely sending everything to the identity. Thus, it only remains to determine $i_B\colon \mathbb{Z} \to \mathbb{Z}$. Since $V_2 = B^2 \times S^1$ deformation retracts to $S^1$ by the map $(b, s) \overset{d}{\mapsto} s$, and so $d \circ \phi(u) = d(u^a, u^b) = u^b$. Therefore the desired map is

$$i_B(u) = d \circ \phi(u) = u^b$$

so the fundamental group is simply the cyclic group

$$\pi_1(X; [(\phi(x_1), x_2)]) \cong \langle x; x^b = 1\rangle \cong \mathbb{Z}/a\mathbb{Z}.$$

b. Here we do a similar argument to part **a.**: By van Kampen's theorem, we can write

$$\pi_1(X; [(x_1, x_2)]) \cong \underbrace{\pi_1(V_1; x_1)}_{\mathbb{Z} = \langle v_1; \rangle} \underbrace{*_{\pi_1(\psi(\partial V_1) \cap \partial V_2; (\psi(x_1), x_2))}}_{F_2 = \langle y_1, y_2; \rangle} \underbrace{\pi_1(V_2; \psi(x_1))}_{\mathbb{Z} = \langle v_2; \rangle}.$$

Where $v_1$ and $v_2$ are loops around $S^1 \subset B^2 \times S^1$ in $V_1$ and $V_2$ respectively, and $y_1$ and $y_2$ are loops around $\partial B^2$, and $S^1$ respectively.
Then the homomorphism $i_A\colon \langle y_1, y_2\rangle \to \langle v_1\rangle$ is given by

$$\begin{aligned} y_1 &\mapsto 1 \\ y_2 &\mapsto v_1, \end{aligned}$$

because the "obvious" deformation retract on $B^2 \subset V_1$ means that $y_1$ is nullhomotopic, and so maps to the identity. The loop $y_2$ around $S^1 \subset \partial V_1$ maps into a single loop around $S^1 \subset V_1$.

Similarly, the homomorphism $i_B \colon \langle y_1, y_2 \rangle \to \langle v_2 \rangle$ is given by

$$\begin{aligned} y_1 &\mapsto v_2^b \\ y_2 &\mapsto v_2^d, \end{aligned}$$

because given any loop $\alpha \colon [0,1] \to S^1 \times S^1$ can be decomposed and mapped to $\partial V_2$ by

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) \overset{\psi}{\mapsto} (\alpha_1^a(t)\alpha_2^c(t), \alpha_1^b(t)\alpha_2^d(t)) \overset{d}{\mapsto} \alpha_1^b(t)\alpha_2^d(t).$$

Therefore if $a_1$ is a curve that wraps around $\partial B^2$ once, then $\psi \circ \alpha_1 \colon [0,1] \to V_2$ is homotopic to a curve via the deformation retract $d$) that wraps around $S^1 \subset V_2$ $b$ times. Thus the fundamental group has the presentation:

$$\pi_1(X; [(x_1, x_2)]) \cong \langle v_1, v_2; v_1 = v_2^a, v_2^b = 1 \rangle.$$

$\square$