Combinatorics: Homework 11

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Problem 1. Show that $R_2(3,4) = 9$.

Proof.
$R_2(3,4) > 8$ because the coloring of K_8 in Figure 1 has no subgraphs K_3 which are colored blue, because the only blue edges are of neighbors around the border or opposite vertices. The coloring has no subgraphs of K_4 colored red because no edges around the border are colored red, so a choice of K_4 must include "every other" vertex around the border, and the two choices are constructed so that either choice has two blue edges.

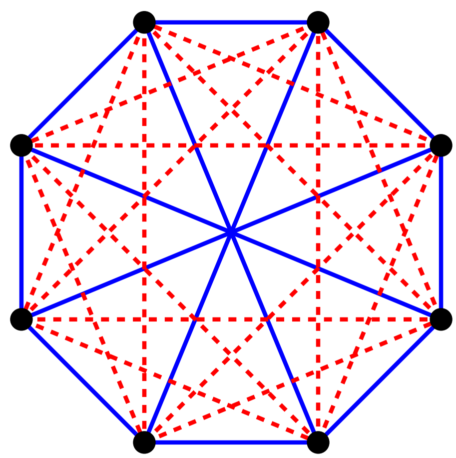

Figure 1: A 2-coloring of K_8 which has no red K_4 subgraphs or blue K_3 subgraphs.

Therefore it is sufficient to prove that $R_2(3,4) \leq 9$. We saw in class that $R_2(3,3) = 6$, and $R_2(2,4) = 4$. Therefore take any vertex v in K_9. Since v has degree 8, there are three cases to consider.

Case 1. Assume there are 5 red edges and 3 blue edges. Not all vertices can be in this case, because the sum of degrees over each vertex must be even, and if in the blue subgraph, all 9 vertices have degree 3, then the sum of degrees is 27, a contradiction. Thus, it is always possible to choose a vertex in one of the following two cases.

Case 2. Assume there are 6 or more edges which are colored red. In this case, take six vertices which are connected to v via a red edge, and look at the K_6 subgraph on these vertices. Because $R_2(3,3) = 6$ either K_6 has a blue K_3 (in which case K_9 does too, and we're done) or K_6 has a red K_3, in which case $K_3 \cup v$ is a complete graph on 4 red vertices.

Case 3. Assume there are 4 or more edges which are colored blue. In this case, take four vertices which are connected to v via a blue edge, and look at the K_4 subgraph on these vertices. Because $R_2(4,2) = 4$ either K_4 is all red (and thus has a red K_4 subgraph, and we're done) or K_4 has a blue K_2 subgraph, in which case $K_2 \cup v$ is a complete graph on 3 blue vertices. $\square$

Problem 2. Define M_n recursively by

$$\begin{aligned} M_1 &= 3 \\ M_m &= mM_{m-1} - m + 2. \end{aligned}$$

Show that

(a) $M_m = \lfloor m!\, e \rfloor + 1$

(b) If each edge on the complete graph K_{M_m} is colored in one of m colors, then there exists a monochromatic triangle.

Solution.

(a) By induction, the base case $m = 1$ is clear since $M_1 = \lfloor 1!\, e \rfloor + 1 = 2 + 1 = 3$. Thus using the identity

$$\begin{aligned} \lfloor m!\, e \rfloor &= \left\lfloor \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!} + \frac{m!}{(m+1)!} + \ldots \right\rfloor \\ &= \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!}. \end{aligned}$$

which holds for all $m \geq 1$ since

$$\frac{m!}{(m+1)!} + \frac{m!}{(m+2)!} + \ldots = \frac{1}{(m+1)} + \frac{1}{(m+1)(m+2)} + \ldots \leq e - 2 \approx 0.71828.$$

Therefore,

$$\begin{aligned} M_{m+1} &= (m+1)\lfloor m!\, e \rfloor + 2 \\ &= (m+1)\left\lfloor \frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!} + \ldots \right\rfloor + 2 \\ &= (m+1)\left(\frac{m!}{0!} + \frac{m!}{1!} + \ldots + \frac{m!}{m!} \right) + 2 \\ &= \frac{(m+1)!}{0!} + \frac{(m+1)!}{1!} + \ldots + \frac{(m+1)!}{m!} + 2 \\ &= \underbrace{\frac{(m+1)!}{0!} + \frac{(m+1)!}{1!} + \ldots + \frac{(m+1)!}{m!} + \frac{(m+1)!}{(m+1)!}}_{\lfloor (m+1)!\, e \rfloor} \underbrace{- \frac{(m+1)!}{(m+1)!}}_{-1} + 2 \\ &= \lfloor (m+1)!\, e \rfloor + 1. \end{aligned}$$

(b) By induction, where the base case $m = 1$ is clear.

Problem 3. Let $r \in \mathbb{P}$. Prove that any 2-coloring of the complete graph on $(r-1)^2+1$ vertices has either every red tree or every green tree in r vertices.

Proof.

$\square$