Spring 2013: Complex Analysis Graduate Exam

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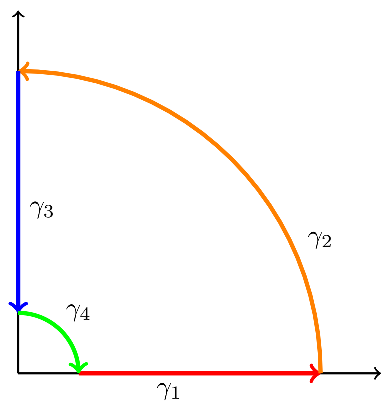
Problem 1. Evaluate
$$\int_0^\infty \frac{x^{1/3}}{1+x^4}\,dx$$
being careful to justify your answer.

Proof. For ease of notation, name the integrand f; that is,
$$f(z) = \frac{z^{1/3}}{1+z^4}.$$
We will compute the integral by using the Residue Theorem together with (the limit of) the following contour:



$$\gamma_1 = \{t + 0i \mid x \in [\varepsilon, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [0, \pi/2]\} \tag{2}$$
$$\gamma_3 = \{0 + ti \mid t \in [\varepsilon, R]\} \tag{3}$$
$$\gamma_4 = \{\varepsilon e^{it} \mid t \in [0, \pi/2]\}. \tag{4}$$

For sufficiently small ϵ and large R, this contour encloses one singularity of f, namely $z_0 = e^{\pi i/4}$.
$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

□

In the limit, both arcs (γ_2 and γ_4) vanish.
$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^{\pi/2} \frac{(Re^{it})^{1/3}}{1+(Re^{it})^4} iRe^{it}\,dt\right| \\
&\le \int_0^{\pi/2} \left|\frac{(Re^{it})^{1/3}}{1+(Re^{it})^4} iRe^{it}\,dt\right| \\
&= \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{1+R^4e^{4it}}\,dt\right| \\
&\le \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{R^4e^{4it}}\,dt\right| = \frac{\pi}{2}R^{-8/3}
\end{aligned}$$

which vanishes as $R \to \infty$. Similarly,

$$\begin{aligned}
\left| \int_{\gamma_4} f(z)\, dz \right| &= \left| \int_0^{\pi/2} \frac{(\varepsilon e^{it})^{1/3}}{1 + (\varepsilon e^{it})^4} i\varepsilon e^{it}\, dt \right| \\
&\leq \int_0^{\pi/2} \left| \frac{(\varepsilon e^{it})^{1/3}}{1 + (\varepsilon e^{it})^4} i\varepsilon e^{it}\, dt \right| \\
&= \int_0^{\pi/2} \left| i\varepsilon^{4/3} \frac{e^{4it/3}}{1 + \varepsilon^4 e^{4it}}\, dt \right| \\
&\leq \int_0^{\pi/2} \left| i\varepsilon^{4/3} \frac{e^{4it/3}}{1}\, dt \right| = \frac{\pi}{2}\varepsilon^{4/3}
\end{aligned}$$

which also vanishes as $\varepsilon \to 0$. This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z)\, dz + \int_{\gamma_3} f(z)\, dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

And the right hand side further simplifies to

$$\begin{aligned}
\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz + \int_R^\varepsilon \frac{(iz)^{1/3}}{1+(iz)^4} i\, dz &= \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz - i^{4/3} \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz \\
&= (1 - i^{4/3}) \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz.
\end{aligned}$$

So by the Residue Theorem, the integral evaluates to

$$\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz = \frac{2\pi i \operatorname{Res}_{z_0}(f)}{1 - i^{4/3}} = \frac{2\pi i}{1 - e^{2\pi i/3}} \operatorname{Res}_{z_0}(f),$$

and it is enough to compute the residue:

$$\operatorname{Res}_{z_0}(f) = \frac{z_0^{1/3}}{(z_0^2 + i)(z_0 + z_0)} = \frac{e^{\pi i/12}}{(2e^{\pi i/2})(2e^{\pi i/4})} = \frac{1}{4} e^{-2\pi i/3}.$$

Therefore

$$\begin{aligned}
\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz &= \frac{2\pi i}{1 - e^{2\pi i/3}} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{2\pi i}{1 - (-1/2 + \sqrt{3}i/2)} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{4\pi i}{3 - \sqrt{3}i} \cdot \frac{1}{4} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \\
&= \frac{\pi}{3 - \sqrt{3}i} \cdot i \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \\
&= \frac{\pi}{2} \left(\frac{\sqrt{3} - i}{3 - \sqrt{3}i} \right) \\
&= \frac{\pi}{2\sqrt{3}}.
\end{aligned}$$

Problem 2.

Proof. ∎

Problem 3.

Proof. $\square$

Problem 4.

Proof. □