Differential Geometry: Homework 2

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Problem 1.

Proof. $\square$

Problem 2.

Proof. $\square$

Problem 3.

Proof. ∎

Problem 4.

Proof. □

Problem 5.

Proof. $\square$

Problem 6.

Proof. $\square$

Problem 7.

Proof.

Part (a). It is sufficient to show that (i) there exists an identity morphism for each object in Alg_k, (ii) the composition of two (composable) k-algebra homomorphisms is a k-algebra homomorphism, and (iii) k-algebra homomorphisms are associative.

(i) For each object $x \in \mathrm{ob}(\mathrm{Alg}_k)$, let $1_X \in \hom_{\mathrm{Alg}_k}(X, X)$ be the identity map that sends each element $x \in X$ to itself. Clearly 1_X is a k-algebra homomorphism because 1_X is a linear map of vector spaces which is compatible with the multiplication maps

$$1_X(\alpha \cdot \beta) = \alpha \cdot \beta = 1_X(\alpha) \cdot 1_X(\beta)$$

and preserves the identity elements ($1_X(1) = 1$.)
Also if $f \in \hom_{\mathrm{Alg}_k}(Z, X)$ is a k-algebra homomorphism,

$$1_X \circ f(\alpha) = 1_X(f(\alpha)) = f(\alpha),$$

and if $g \in \hom_{\mathrm{Alg}_k}(X, Y)$ is a k-algebra homomorphism

$$g \circ 1_X(\alpha) = g(1_X(\alpha)) = g(\alpha).$$

So indeed $1_X \circ f = f$ and $g \circ 1_X = g$, and therefore 1_X is an identity morphism.

(ii) Let $f \in \hom_{\mathrm{Alg}_k}(Z, X)$ and $g \in \hom_{\mathrm{Alg}_k}(X, Y)$.
Then $g \circ f$ is compatible with the multiplication maps

$$g \circ f(\alpha \cdot \beta) = g(f(\alpha) \cdot f(\beta)) = g(f(\alpha)) \cdot g(f(\beta)) = g \circ f(\alpha) \cdot g \circ f(\beta),$$

and $g \circ f$ preserves the identity elements

$$g \circ f(1) = g(1) = 1.$$

Therefore $g \circ f \in \hom_{\mathrm{Alg}_k}(Z, Y)$.

(iii) For each composable triple f, g, and h

$$h \circ (g \circ f) = (h \circ g) \circ f$$

because associativity is inherited from ordinary composition of functions.

Part (b). It is sufficient to show that (i) $C^0(X)$ is a vector space over $\mathbb{R}$, (ii) multiplication is bilinear, (iii) multiplication is associative, and (iv) there is a multiplicative identity.

(i) $C^0(X)$ is a vector space with pointwise addition and ordinary scalar multiplication. In general continuous functions are closed under addition. Multiplying (or dividing) every element in an open set U by a scalar a yields an open set aU, so if $f^{-1}(U)$ is open for every open set U, then $(af)^{-1}(aU)$ is also open. Therefore $C^0(X)$ is closed under scalar multiplication.
$C^0(X)$ inherits structure from $\mathbb{R}$ so that.

- Associativity and commutivity of addition follow from $\mathbb{R}$.
- The zero function (which is proven to be in $C^0(X)$ below) satisfies $f + 0 = f$ for all f.
- All elements are invertible with respect to addition: $f(x) + (-1) \cdot f(x) = 0$.
- The scalar $1 \in \mathbb{R}$ behaves as an identity element for scalar multiplication: $1f = f$.
- Everything distributes nicely: $a(b \cdot f) = (ab) \cdot f$, $a(f + g) = af + ag$, and $(a + b)f = af + bf$.

Lastly, it is important to check that continuous functions remain continuous after addition and scalar multiplication.

(ii) Bilinearity follows from well-behaved distributivity on $\mathbb{R}$. Let $f, g, h \in C^0(X)$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} (f+g) \times h &= (f \cdot h) + (g \cdot h) = (f \times h) + (g \times h) \\ f \times (g+h) &= (f \cdot g) + (f \cdot h) = (f \times g) + (f \times h) \\ (\alpha \cdot f) \times g &= \alpha \cdot (f \cdot g) \qquad = \alpha \cdot (f \times g) \\ f \times (\alpha \cdot g) &= \alpha \cdot (f \cdot g) \qquad = \alpha \cdot (f \times g). \end{aligned}$$

(iii) Associativity follows from associativity on $\mathbb{R}$.

$$(f \times g) \times h = (f \cdot g) \cdot h = f \cdot (g \cdot h) = f \times (g \times h).$$

(iv) The multiplicative identity is the constant function 1. Constant functions are in $C^0(X)$ because any open set that contains the constant has a preimage of X (which is an open set) and any set that does not contain the constant has a preimage of $\emptyset$ (which is also an open set.) For each function $f \in C^0(X)$ and each point $x \in X$

$$1(x) \times f(x) = 1 \cdot f(x) = f(x) = f(x) \cdot 1 = f(x) \times 1(x),$$

therefore $1 \times f = f = f \times 1$.

Part (c). Let f be a continuous map $f \in \hom_{\mathbf{Top}}(X, Y)$.
In order to prove that F is a contravariant functor, it is sufficient to show that (i) $F(f)$ is an $\mathbb{R}$-algebra homomorphism, (ii) $F\colon \hom_{\mathbf{Top}}(X, Y) \to \hom_{\mathbf{Alg}_{\mathbb{R}}}(F(Y), F(X))$ sends identity morphisms to identity morphisms, and (iii) $F(f) \circ F(g) = F(g \circ f)$ for composable morphisms.

(i) Let $g, h \in C^0(Y)$. Then $F(f)\colon C^0(Y) \to C^0(X)$ is an $\mathbb{R}$-algebra homomorphism because it is compatible with multiplication maps

$$F(f)(g \cdot h) = (g \cdot h) \circ f = (g \circ f) \cdot (h \circ f) = F(f)(g) \cdot F(f)(h)$$

and because it preserves the (multiplicative) identity element (the constant map 1)

$$F(f)(y \mapsto 1) = (y \mapsto 1) \circ f = (x \mapsto 1)$$

(ii) Let $\mathrm{id}_X \in \hom_{\mathbf{Top}}(X, X)$. Then for all $g \in F(X) = C^0(X)$,

$$F(\mathrm{id}_X)(g) = g \circ \mathrm{id}_X = g = \mathrm{id}_{C^0(X)}(g).$$

Therefore $F(\mathrm{id}_X) = \mathrm{id}_{C^0(X)}$.

(iii) Let $g \in \hom_{\mathbf{Top}}(Y, Z)$ and $h \in C^0(Z)$. Then

$$(F(f) \circ F(g))(h) = F(f)(F(g)(h)) = F(f)(h \circ g) = (h \circ g \circ f) = F(g \circ f)(h),$$

so $F(f) \circ F(g) = F(g \circ f)$.

Part (d). It is sufficent to show that (i) there exists a functor *Forget* from $\mathbf{Alg}_{\mathbb{R}}$ to **Set** (that maps algebras to their underlying sets and algebra homomorphisms to the corresponding map of sets), (ii) this functor is faithful, and (iii) this functor is not full.

1. *Forget* naturally maps identity morphisms to identity morphisms because the identity morphism on an $\mathbb{R}$-algebra is the same as the identity morphism on the underlying set, namely $x \mapsto x$. Composition is compatible because it is the same as the set theoretic function composition.

2. (?) Let $A = \mathbb{R}$ with the ordinary multiplication $a \times b = ab$, and let $B = \mathbb{R}$ with the multiplication $a \times b = ab/2$. Then $A \neq B$ as $\mathbb{R}$-algebras, but $A = B$ as sets.

3. Let $\phi\colon C^0(X) \to C^0(X)$ be the function $\phi(f) = (x \mapsto x + 1) \circ f$. Then the unity element (the constant function $x \mapsto 1$) is not preserved under ϕ, so ϕ is not an $\mathbb{R}$-algebra homomorphism. Therefore ϕ is not in the image of *Forget*, and so *Forget* is not full.

□