Topology: Homework 1

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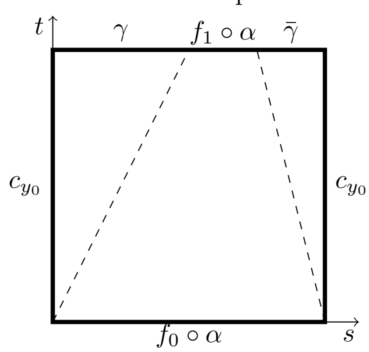
Problem 1.
Let the maps $f_0, f_1 \colon X \to Y$ be homotopic by a homotopy $H \colon X \times [0,1] \to Y$, and let γ be the path from $y_0 = f_0(x_0)$ to $z_0 = f_1(x_0)$ defined by $\gamma(t) = H(x_0, t)$.

(a) Let $\alpha \colon [0,1] \to X$ be a loop in X based at x_0. What is the path homotopy between the paths $f_0 \circ \alpha$ and $\gamma * ((f_1 \circ \alpha) * \bar{\gamma})$?

(b) Show that the homomorphisms $f_{0*} \colon \pi_1(X; x_0) \to \pi_1(Y; y_0)$ and $f_{1*} \colon \pi_1(X; x_0) \to \pi_1(Y; z_0)$ (induced by f_0 and f_1 respectively) are related by the property that $f_{0*} = T_\gamma \circ f_{1*}$, where T_γ is the usual change of basepoint isomorphism.

Proof.

(a) Here's the idea in a picture:



Here's the idea as a formula:

$$H(s,t) = \begin{cases} \gamma(2s) & 0 \le s \le \frac{1}{2}t \\ H\left(\alpha\left(\dfrac{s - t/2}{1 - 3t/4}\right), t\right) & \frac{1}{2}t \le s \le 1 - \frac{1}{4}t \\ \gamma(4 - 4s) & 1 - \frac{1}{4}t \le s \le 1 \end{cases}$$

Here's an explanation of why the formula is continuous: the three piecewise defined parts are compositions of continuous functions and so are continuous. So by the pasting lemma, it is enough to check that

(i) when $s = 0$, $H(0,t) = \gamma(0) = y_0$;
(ii) when $s = \frac{1}{2}t$, the first function evaluates to $\gamma(t)$ and the second to $H(\alpha(0), t) = H(x_0, t)$;
(iii) when $s = 1 - \frac{1}{4}t$, the second function evalues to $H(\alpha(1), t) = H(x_0, t)$ and the third to $\gamma(t)$; and
(iv) when $s = 1$, $H(1,t) = \gamma(0) = y_0$.

(b) To check the property that $f_{0*} = T_\gamma \circ f_{1*}$, it is enough to show that the image of $[\alpha] \in \pi_1(X, x_0)$ under both functions are equal. The maps are, respectively,

$$[\alpha] \xmapsto{f_{0*}} [f_0 \circ \alpha] \qquad \text{and} \qquad [\alpha] \xmapsto{T_\gamma \circ f_{1*}} [\gamma * (f_1 \circ \alpha) * \bar{\gamma}].$$

But the paths $f_0 \circ \alpha$ and $\gamma * (f_1 \circ \alpha) * \bar{\gamma}$ have already been shown to be homotopic by the homotopy G described in part (a). Thus $[f_0 \circ \alpha] = [\gamma * (f_1 \circ \alpha) * \bar{\gamma}]$ and $f_{0*} = T_\gamma \circ f_{1*}$.

$\square$

Problem 2.
Let X be a metric space with metric d_0, and pick two points $x_0, y_0 \in X$. Let $\Omega_{x_0 y_0} X$ denote the space of paths $\alpha\colon [0,1] \to X$ going from x_0 to y_0. Endow $\Omega_{x_0 y_0} X$ with the distance

$$d_1(\alpha, \beta) = \sup_{t \in [0,1]} d_0(\alpha(t), \beta(t))$$

a. Let $H\colon [0,1] \times [0,1] \to X$ be a path homotopy from $\alpha \in \Omega_{x_0 y_0} X$ to $\beta \in \Omega_{x_0 y_0} X$. For every $t \in [0,1]$, let $h_t \in \Omega_{x_0 y_0} X$ be the path defined by $h_t(s) \coloneqq H(t,s)$.
Show that the map $h\colon [0,1] \to \Omega_{x_0 y_0} X$ defined by $h(t) = h_t$ is a path in $\Omega_{x_0 y_0} X$ going from $h(0) = \alpha$ to $h(1) = \beta$.

b. Conversely, let $h\colon [0,1] \to \Omega_{x_0 y_0} X$ be a path going from $h(0) = \alpha$ to $h(1) = \beta$ in $\Omega_{x_0 y_0} X$. Define $H\colon [0,1] \times [0,1] \to X$ by the property that $H(s,t) = h_t(s)$ where $h_t = h(t)$.
Show that H is a path homotopy from α to β.

Solution.

a. For this first part, there's no need to appeal to the metric space. By hypothesis $H\colon [0,1] \times [0,1] \to X$ is a homotopy from α to β, and thus a continuous function. The function $h_t\colon [0,1] \to X$ is a restriction of H, and thus is continuous too—in particular, it is a continuous map from $[0,1]$ to X, where $h_t(0) = H(0,t) = x_0$ and $h_t(1) = H(1,t) = x_1$. Therefore h_t is a path from x_0 to y_0 for all $t \in [0,1]$.

b. In this next part, we will use the uniform continuity hint. In order to show that $H(s,t) = h_t(s)$ defines a homotopy, we must show that

(i) $H(0,t) = x_0$, $H(1,t) = y_0$,
(ii) $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, and
(iii) H is continuous.

The first two conditions come by hypothesis: $h(t) \in \Omega_{x_0 y_0} X$, and h is a path from α to β. Thus it only remains to check that H is continuous.

Because $[0,1]$ is compact, by the Heine-Cantor theorem, h is uniformly continuous. So in particular, for each $\delta/2 > 0$, we can find $\varepsilon > 0$ such that for all $|t - t'| < \varepsilon$, $d_1(h_t, h_{t'}) < \delta/2$. Similarly, for each h_t and $\delta/2 > 0$ we can find $\varepsilon' > 0$ such that $d_0(h_t(s) - h_t(s')) < \delta/2$.

Thus, by the triangle inequality, there is a ball of radius $\min(\varepsilon, \varepsilon')$ centered at (s,t) such that for all (s', t') in the ball, $d_0(H(s,t), H(s',t')) < \delta$. Therefore H is continuous.

Problem 3.
Let $f\colon X \to Y$ be a map such that there exists maps $h, k\colon Y \to X$ such that $h \circ f \simeq \mathrm{Id}_X$, and $f \circ k \simeq \mathrm{Id}_Y$. Show that f is a homotopy equivalence, in the sense that there exists a single map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$.

Solution. Firstly, we have that $k \simeq h$:

$$k \simeq \underbrace{h \circ f}_{\mathrm{Id}_X} \circ k = h \circ \underbrace{f \circ k}_{\mathrm{Id}_Y} \simeq h.$$

Thus, let $g = h$. By hypothesis, $h \circ f \simeq \mathrm{Id}_X$, so it is enough to construct a homotopy between $f \circ h$ and Id_Y. We will do this by combining the homotopy H_1 from h to k and the homotopy H_2 from $f \circ k$ to Id_Y.

In particular,

$$H(y,t) = \begin{cases} f(H_1(y,2t)) & t \in [0,1/2] \\ H_2(y,2t-1)) & t \in [1/2,1] \end{cases}$$

To check continuity,

(i) $H(y,0) = f(H_1(y,0)) = f(h(y))$

(ii) $H(y,1/2) = f(H_1(y,1)) = f(k(y)) = H_2(y,0)$

(iii) $H(y,1) = f(H_2(y,1)) = \mathrm{Id}_Y(y)$.

Thus H is a homotopy between between $f \circ h$ and Id_Y, so $f \circ h \simeq \mathrm{Id}_Y$, and f is a homotopy equivalence.