Spring 2013: Real Analysis Graduate Exam

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Problem 1. Suppose that $\{f_n\}$ is a sequence of real valued continuously differentiable functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|dx = 0 \text{ and } \lim_{n\to\infty} \int_0^1 |f_n'(x)|dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0, 1]$.

Proof. $\square$

Problem 2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x) dx$$

Proof. Because the integrand $sin(\pi x)x^n/(1-x)$ is positive, Tonelli's theorem gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{1-x} \sin(\pi x) dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \sin(\pi x) dx && (1)\\
&= \int_0^1 \frac{\sin(\pi x)}{1-x} \sum_{n=0}^{\infty} x^n dx && (2)\\
&= \int_0^1 \frac{\sin(\pi x)}{1-x} \cdot \frac{1}{1-x} dx && (3)\\
&= \int_0^1 \frac{\sin(\pi x)}{(1-x)^2} dx. && (4)
\end{aligned}$$

Notice that (2) implies (3) because the bounds of the integral ensure that x is m-almost everwhere within the radius of convergece of the sum.

Because the integrand is non-negative, by elementary calculus

$$\int_0^1 \frac{\sin(\pi x)}{(1-x)^2} dx \geq \int_{1/2}^1 \frac{\sin(\pi x)}{(1-x)^2} dx \geq \int_{1/2}^1 \frac{1-x}{(1-x)^2} dx = \int_{1/2}^1 (1-x)^{-1} dx = \infty.$$

Therefore $\sum_{n=0}^{\infty} a_n = \infty$. □

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n,\ f \in L^1(\mu)$. Show that $\int_X |f_n - f| d\mu \to 0$ as $n \to \infty$ if and only if

$$\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \to 0$$

as $n \to \infty$

Proof. ($\Longrightarrow$) Assume that $\int_X |f_n - f| d\mu \to 0$ as $n \to \infty$; in particular choose sufficiently large n so that $\int_X |f_n - f| d\mu < \varepsilon$. Then for any set $A \subset X$

$$\varepsilon = \int_X |f_n - f| d\mu = \int_A |f_n - f| d\mu + \int_{A^c} |f_n - f| d\mu \geq \int_A |f_n - f| d\mu \geq \left| \int_A f_n - f \ d\mu \right| .$$

Since this holds for all $A \in \mathcal{M}$, it also holds for the supremum.

$\square$

Problem 4. Let μ and ν be σ-finite positive measutures, $\mu \geq \nu$ and assume that $\nu \ll \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).
Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

Proof. $\square$