Topology: Homework 10

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Problem 1.

Proof.

(a) Composing the face maps with σ_1 and σ_2 respectively yields

$$\begin{aligned}
(\sigma_1 \circ F_0)(x_0, x_1) &= \sigma_1(0, x_0, x_1) = H(x_1, 1) &&= \beta(x_1)\\
(\sigma_1 \circ F_1)(x_0, x_1) &= \sigma_1(x_0, 0, x_1) = H(x_1, x_1) &&= f(x_1)\\
(\sigma_1 \circ F_2)(x_0, x_1) &= \sigma_1(x_0, x_1, 0) = H(0, x_1) &&= c_{\alpha(0)}(x_1)
\end{aligned}$$

$$\begin{aligned}
(\sigma_2 \circ F_0)(x_0, x_1) &= \sigma_2(0, x_0, x_1) = H(1, x_1) &&= c_{\alpha(1)}(x_1)\\
(\sigma_2 \circ F_1)(x_0, x_1) &= \sigma_2(x_0, 0, x_1) = H(x_1, x_1) &&= f(x_1)\\
(\sigma_2 \circ F_2)(x_0, x_1) &= \sigma_2(x_0, x_1, 0) = H(x_1, 0) &&= \alpha(x_1).
\end{aligned}$$

Since these are all functions of x_1,

$$\begin{aligned}
\partial_2(\sigma_1 - \sigma_2) &= \partial_2(\sigma_1) - \partial_2(\sigma_2)\\
&= ((\beta - f + c_{\alpha(0)}) - (c_{\alpha(1)} - f + \alpha)) \circ \pi_2\\
&= (\beta - \alpha + c_{\alpha(0)} - c_{\alpha(1)}) \circ \pi_2.
\end{aligned}$$

(b) It is sufficient to show that if α and β are path homotopic (i.e. $[\alpha] = [\beta] \in \pi_1(X; x_0)$) then

$$\rho([\alpha]) = [\alpha] = \rho([\beta]) = [\beta] \in H_1(X) = \ker(\partial_1)/\operatorname{Im}(\partial_2),$$

that is, to show that $\beta - \alpha \in \operatorname{Im}(\partial_2)$. Let σ_1, σ_2 be the simplices given above, and let $\sigma_{\alpha(0)}$, and $\sigma_{\alpha(1)}$ be the constant simplices.

$$\begin{aligned}
\partial(\sigma_1 - \sigma_2 - \sigma_{\alpha(0)} + \sigma_{\alpha(1)}) &= \partial(\sigma_1 - \sigma_2) - \partial(\sigma_{\alpha(0)}) + \partial(\sigma_{\alpha(1)})\\
&= \beta - \alpha + c_{\alpha(0)} - c_{\alpha(1)} - c_{\alpha(0)} + c_{\alpha(1)}\\
&= \beta - \alpha,
\end{aligned}$$

as desired.

(c) Use the simplex $\sigma\colon \Delta_2 \to X$ given by

$$(x_0, x_1, x_2) \mapsto \alpha * \beta(x_2 + x_1/2).$$

where $\alpha * \beta$ has the usual defintion,

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & t \in [0, 1/2] \\ \beta(2t - 1) & t \in [1/2, 1] \end{cases}.$$

Then the face maps are

$$\begin{aligned}
(x_0, x_1) &\xmapsto{F_0} (0, x_0, x_1) \xmapsto{\sigma} \alpha * \beta(x_1 + x_0/2) = \beta(x_1) \\
(x_0, x_1) &\xmapsto{F_1} (x_0, 0, x_1) \xmapsto{\sigma} \alpha * \beta(x_1) \\
(x_0, x_1) &\xmapsto{F_2} (x_0, x_1, 0) \xmapsto{\sigma} \alpha * \beta(x_1/2) = \alpha(x_1)
\end{aligned}$$

So $\partial_2(\sigma) = \beta - \alpha * \beta + \alpha = \alpha + \beta - \alpha * \beta$, as desired.

(d) In order to show that the map is a homomorphism, it is enough to show that

$$\rho([\alpha * \beta]) = \rho([\alpha]) + \rho([\beta]) \in H(X, \mathbb{Z}) = \ker(\partial_1)/\operatorname{Im}(\partial_2).$$

Using the usual abuse of notation, it is sufficient to show that

$$[\alpha] + [\beta] - [\alpha * \beta] = 0 \in H_1(X, \mathbb{Z}),$$

which is to say that

$$\alpha + \beta - \alpha * \beta \in \operatorname{Im}(\partial_2).$$

But this follows by part (c). In particular,

$$\partial_2(\sigma) = \alpha + \beta - \alpha * \beta.$$

(e) It is enough to check that

$$[G, G] = \langle [\alpha * \beta * \bar{\alpha} * \bar{\beta}] \rangle \subset \ker(\rho).$$

However

$$\begin{aligned}
\rho([\alpha * \beta * \bar{\alpha} * \bar{\beta}]) &= \rho([\alpha]) + \rho([\beta * \bar{\alpha} * \bar{\beta}]) \\
&= \dots \\
&= \rho([\alpha]) + \rho([\beta]) * \rho([\bar{\alpha}]) * \rho([\bar{\beta}]) \\
&= \rho([\alpha * \bar{\alpha}]) + \rho([\beta * \bar{\beta}]) \\
&= \rho(\mathrm{id}_G) + \rho(\mathrm{id}_G) \\
&= 0,
\end{aligned}$$

as desired.

$\square$

Problem 2.

Proof.

(a) There are two homeomorphisms and one deformation retract to prove.

(i) Claim: $U_n \simeq B^{2n}$.
Write $(z_0, \dots, z_n) = (a_0 + b_0 i, \dots, a_n + b_n i)$. Then

$$U_n = \left\{ (a_0, b_0, \dots, a_n, b_n) \in \mathbb{R}^{2n+2} - \{0\} : \sum_{i=0}^{n-1} a_i^2 + b_i^2 \le a_n^2 + b_n^2 \right\} / \sim$$

By the equivalence relation, we can choose the representative such that every coordinate is divided by $|z_n| = \sqrt{a_n^2 + b_n^2}$ (which is nonzero because the inequality would force the point to be zero if $|z_n|^2 = 0$.) This becomes

$$U_n' = \left\{ (a_0', b_0', \dots, a_{n-1}', b_{n-1}') \in \mathbb{R}^{2n} : \sum_{i=0}^{n-1} a_i^2 + b_i^2 \le 1 \right\} = B^{2n}.$$

(ii) The intersection $U_n \cap V_n \sim S^{2n-1}$ follows similarly,

$$\begin{aligned} U_n \cap V_n &= \left\{ (a_0, b_0, \dots, a_n, b_n) \in \mathbb{R}^{2n+2} - \{0\} : \sum_{i=0}^{n-1} a_i^2 + b_i^2 = a_n^2 + b_n^2 \right\} / \sim \\ &\simeq \left\{ (a_0', b_0', \dots, a_{n-1}', b_{n-1}') \in \mathbb{R}^{2n} : \sum_{i=0}^{n-1} a_i^2 + b_i^2 = 1 \right\} = S^{2n-1}. \end{aligned}$$

(iii) Lastly, the map $r \colon V_n \times [0,1] \to \mathbb{CP}^{n-1}$ which sends

$$r([(z_0, z_1, \dots, z_n)], t) \mapsto [(z_0, z_1, \dots, z_0(t-1))]$$

is a deformation retract, because it is continuous, $r(z, 0) = 1$, and maps V_n to the subspace when $t = 1$, where it is the identity when restricted to the subspace.

$$\begin{aligned} &\left\{ (z_0, z_1, \dots, z_{n-1}, 0) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n-1} |z_i|^2 \ge 0 \right\} / \sim \\ &\qquad \simeq \{(z_0, z_1, \dots, z_{n-1}) \in \mathbb{C}^n\} / \sim \\ &\qquad \simeq \mathbb{CP}^{n-1} \end{aligned}$$

(b) **Base case.**
First, consider the base case for $n = 0$. It follows from the defintions that $U_0 = \mathbb{CP}^0 \simeq \{x_0\}$ and $V_0 = \emptyset$. Therefore by Mayer–Vietoris:

$$\begin{aligned} H_0(U_0 \cap V_0) \to H_0(U_0) \oplus H_0(V_0) &\to H_0(\mathbb{CP}^0) \to 0 \\ H_0(\emptyset) \to H_0(\emptyset) \oplus H_0(\{x_0\}) &\to H_0(\mathbb{CP}^0) \to 0 \\ 0 \to 0 \oplus R &\to H_0(\mathbb{CP}^0) \to 0 \end{aligned}$$

so $H_0(\mathbb{CP}^0) = R$.

For $p > 0$

$$\begin{aligned} H_p(U_0) \oplus H_p(V_0) &\to H_p(\mathbb{CP}^0) \to H_{p-1}(U_0 \cap V_0) \\ H_p(\{x_0\}) \oplus H_p(\emptyset) &\to H_p(\mathbb{CP}^0) \to H_{p-1}(\emptyset) \\ 0 \oplus 0 &\to H_p(\mathbb{CP}^0) \to 0 \end{aligned}$$

so $H_p(\mathbb{CP}^0) = 0$ for $p > 0$.

Inductive step.
Now, using Mayer–Vietoris:
Case 1. Assume $p > 2n$ so that $p = 2n + k$ for some $k \geq 1$. Then

$$\begin{aligned}
H_{2n+k}(U_n) \oplus H_{2n+k}(V_n) &\to H_{2n+k}(\mathbb{CP}^n) \to H_{2n+k-1}(U_n \cap V_n) \\
H_{2n+k}(B^{2n}) \oplus H_{2n+k}(\mathbb{CP}^{n-1}) &\to H_{2n+k}(\mathbb{CP}^n) \to H_{2n+k-1}(S^{2n-1}) \\
0 \oplus 0 &\to H_{2n+k}(\mathbb{CP}^n) \to 0,
\end{aligned}$$

so $H_{2n+k}(\mathbb{CP}^n) = 0$ for $k > 0$.

Case 2. Assume $p = 2n$.

$$\begin{array}{lllll}
H_{2n}(U_n) \oplus H_{2n}(V_n) & \to H_{2n}(\mathbb{CP}^n) \to H_{2n-1}(U_n \cap V_n) & \to H_{2n-1}(U_n) & \oplus H_{2n-1}(V_n) \\
H_{2n}(B^{2n}) \oplus H_{2n}(\mathbb{CP}^{n-1}) & \to H_{2n}(\mathbb{CP}^n) \to H_{2n-1}(S^{2n-1}) & \to H_{2n-1}(B^{2n}) & \oplus H_{2n-1}(\mathbb{CP}^{n-1}) \\
0 \oplus 0 & \to H_{2n}(\mathbb{CP}^n) \to R & \to 0 & \oplus 0
\end{array}$$

so $H_{2n}(\mathbb{CP}^n) = R$, as desired.

Case 3. Assume $0 < p < 2n$, where p is even, that is $p = 2(n-k)$ for some $0 < k < n$.

$$\begin{array}{llll}
H_{2(n-k)}(U_n \cap V_n) \to H_{2(n-k)}(U_n) & \oplus H_{2(n-k)}(V_n) & \to H_{2(n-k)}(\mathbb{CP}^n) \to H_{2(n-k)-1}(U_n \cap V_n) \\
H_{2(n-k)}(S^{2n-1}) \to H_{2(n-k)}(B^{2n}) & \oplus H_{2(n-k)}(\mathbb{CP}^{n-1}) & \to H_{2(n-k)}(\mathbb{CP}^n) \to H_{2(n-k)-1}(S^{2n-1}) \\
0 \to 0 & \oplus R & \to H_{2(n-k)}(\mathbb{CP}^n) \to 0
\end{array}$$

so $H_{2n-2k}(\mathbb{CP}^n) = R$, as desired.

Case 4. Assume $0 < p < 2n$, where p is odd.

$$\begin{aligned}
H_p(U_n) \oplus H_p(V_n) &\to H_p(\mathbb{CP}^n) \to H_{p-1}(U_n \cap V_n) \\
H_p(B^{2n}) \oplus H_p(\mathbb{CP}^{n-1}) &\to H_p(\mathbb{CP}^n) \to H_{p-1}(S^{2n-1}) \\
0 \oplus 0 &\to H_p(\mathbb{CP}^n) \to 0
\end{aligned}$$

so $H_p(\mathbb{CP}^n) = 0$, as desired.

Case 5. Assume $p = 0$. Thus there is a short exact sequence

$$\underbrace{H_1(\mathbb{CP}^n)}_{0} \to \underbrace{H_0(S^{2n-1})}_{R} \to \underbrace{H_0(U_n)}_{R} \oplus \underbrace{H_0(V_n)}_{R} \to H_0(\mathbb{CP}^n) \to 0.$$

so $H_0(\mathbb{CP}^n) = R$, as desired.

□