

Week 7 Quiz Solutions.

1. (5 points) Given the function

$$f(x, y, z) = \frac{x}{3y^2 + yz}$$

compute $f_{yx}(1, 2, 3)$ in whichever order you want.

Solution.

It is easiest to compute the derivative with respect to x first, and then compute the derivative with respect to y :

$$\begin{aligned} f_{xy}(x, y, z) &= \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial x} \right] \\ &= \frac{\partial}{\partial y} \left[\frac{1}{3y^2 + yz} \right] \\ &= \frac{-6y - z}{(3y^2 + yz)^2} \end{aligned}$$

So evaluating at $(1, 2, 3)$ gives

$$\frac{-6(2) - 3}{(3(4) + 6)^2} = \frac{-15}{18^2}.$$

2. (5 points) Use the
- chain rule**
- to determine
- $\frac{\partial T}{\partial \theta}$
- and
- $\frac{\partial^2 T}{\partial \theta \partial r}$
- when
- $r = 2$
- and
- $\theta = \pi/3$
- given that

$$\begin{aligned} T &= xe^y, \\ x &= r \cos \theta, \text{ and} \\ y &= r \sin \theta. \end{aligned}$$

Solution.

By the chain rule,

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \theta} \\ &= e^y(-r \sin \theta) + xe^y(r \cos \theta) \\ &= e^{r \sin \theta}(-r \sin \theta) + (r \cos \theta)e^{r \sin \theta}(r \cos \theta) \\ &= -ye^y + x^2e^y \end{aligned}$$

Evaluated at $r = 2$ and $\theta = \pi/3$ (and thus $x = 1$ and $y = \sqrt{3}$) gives

$$-\sqrt{3}e^{\sqrt{3}} + e^{\sqrt{3}} = e^{\sqrt{3}}(1 - \sqrt{3}).$$

Denote $\frac{\partial T}{\partial \theta} = T_\theta$. Then by switching the order of differentiation,

$$\begin{aligned}\frac{\partial^2 T}{\partial \theta \partial r} &= \frac{\partial T_\theta}{\partial r} \\ &= \frac{\partial T_\theta}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial T_\theta}{\partial y} \cdot \frac{\partial y}{\partial r} \\ &= (2xe^y)(\cos \theta) + (-e^y - ye^y + x^2e^y)(\sin \theta).\end{aligned}$$

Again evaluated at $r = 2$ and $\theta = \pi/3$ (and thus $x = 1$ and $y = \sqrt{3}$) gives

$$e^{\sqrt{3}} + (-e^{\sqrt{3}} - \sqrt{3}e^{\sqrt{3}} + e^{\sqrt{3}})\frac{\sqrt{3}}{2} = -\frac{e^{\sqrt{3}}}{2}.$$