Complex Analysis: Homework 2

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Problem 5. (page 37)
Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

Proof. When $x = 0$, each term in the sum is 0, so the series converges to 0. Let $|x| = \varepsilon > 0$ in the denominator yields the inequality

$$\begin{aligned}
\left| \sum_{n=N}^{\infty} \frac{x}{n(1+nx^2)} \right| &= \varepsilon \sum_{n=N}^{\infty} \frac{1}{n(1+n\varepsilon^2)} \\
&= \sum_{n=N}^{1/\varepsilon^2} \frac{1}{n(1+n\varepsilon^2)} + \sum_{n=1+1/\varepsilon^2}^{\infty} \frac{1}{n(1+n\varepsilon^2)} \\
&\geq \sum_{n=N}^{1/\varepsilon^2} \frac{1}{n(1+n\varepsilon^2)} + \frac{1}{2\varepsilon^2} \sum_{n=1+1/\varepsilon^2}^{\infty} \frac{1}{n^2}
\end{aligned}$$

which can be made arbitrarily large by choosing small enough ε
Thus the sum is not uniformly convergent on $\mathbb{R}$ (particularly near 0). $\square$

Problem 3. (page 41)
Find the radius of convergence of the following power series:

(a) $\sum n^p z^n$

(b) $\sum \frac{z^n}{n!}$

(c) $\sum n! z^n$

(d) $\sum q^{n^2} z^n$ where $|q| < 1$

(e) $\sum z^{n!}$

Proof. (a) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n\to\infty} |n^p|^{1/n} = \limsup_{n\to\infty} |n^p|^{1/n} = \limsup_{n\to\infty} n^{p/n} = 1$$

So the radius of convergence is $R = 1$.

(b) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} < \limsup_{n\to\infty} \left(\frac{1}{N^n}\right)^{1/n} = \frac{1}{N}.$$

Because $R > N$ for all N, the radius of convergence is ∞.

(c) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} (n!)^{1/n} > \limsup_{n\to\infty} (N^n)^{1/n} = N.$$

Because $R < 1/N$ for all N, the radius of convergence is 0.

(d) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n\to\infty} (q^{n^2})^{1/n} = \limsup_{n\to\infty} q^n = 0 \text{ for } |q| < 1.$$

Thus the radius of convergence is ∞.

(e) Notice that $|z^{n!}| \geq |z^n|$ for $|z| \geq 1$, and $|z^{n!}| \leq |z^n|$ for $|z| < 1$.

$$\left|\sum z^{n!}\right| \leq \left|\sum z^n\right| < \infty \text{ for } |z| < 1$$
$$\left|\sum z^{n!}\right| \geq \left|\sum z^n\right| = \infty \text{ for } |z| \geq 1$$

Thus the radius of convergence is 1.

□

Problem 8. (page 41)
For what values of z is
$$\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$$
convergent?

Proof. The sum $\sum_{n=0}^{\infty} w^n$ is convergent for $|w| < 1$. The sum in the problem is convergent when
$$\left|\frac{z}{1+z}\right| < 1 \Longrightarrow |z| < |1+z|.$$
Letting $z = a + bi$, and comparing the squares of the absolute values:
$$\begin{aligned} a^2 + b^2 &< (1+a)^2 + b^2 \\ a^2 &< 1 + 2a + a^2 \\ -2a &< 1 \\ a &> -1/2 \end{aligned}$$
Thus the sum converges when $\mathrm{Re}(z) > -1/2$. □

Problem 3. (page 44)
Use the addition formulas to separate $\cos(x+iy)$ and $\sin(x+iy)$ in real and imaginary parts.

Proof. First note the identites of sin and cos with purely imaginary inputs:

$$\cos(iy) = 1 - \frac{i^2y^2}{2!} + \frac{i^4y^4}{4!} - \cdots = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots = \cosh(y)$$

$$\sin(iy) = iy - \frac{i^3y^2}{3!} + \frac{i^5y^5}{5!} - \cdots = i(y + \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots) = i\sinh(y)$$

Then using the addition formulas

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \text{ and}$$
$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b),$$

it is clear that

$$\cos(x+iy) = \cos(x)\cosh(y) + i\sin(x)\sinh(y) \text{ and}$$
$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

□

Problem 4. (page 44)
Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x)$$

Proof. Starting with the proof of cos. From the above problem

$$\cos(z) = \cos(x)\cosh(y) + i\sin(x)\sinh(y)$$

so the square of the absolute value of $\cos(z)$ is

$$\begin{aligned}
|\cos z|^2 &= (\cos(x)\cosh(y))^2 + (\sin(x)\sinh(y))^2 \\
&= (\cos(x)\cosh(y))^2 + (\sin(x)\sinh(y))^2 + (\cos(x)\sinh(y))^2 - (\cos(x)\sinh(y))^2 \\
&= \sinh^2(y)(\sin^2(x) + \cos^2(x)) + \cos^2(x)(\cosh^2(y) - \sinh^2(y)) \\
&= \sinh^2(y) + \cos^2(x)
\end{aligned}$$

$$\begin{aligned}
|\cos z|^2 &= (\cos(x)\cosh(y))^2 + (\sin(x)\sinh(y))^2 \\
&= (\cos(x)\cosh(y))^2 + (\sin(x)\sinh(y))^2 + (\sin(x)\cosh(y))^2 - (\sin(x)\cosh(y))^2 \\
&= \cosh^2(y)(\sin^2(x) + \cos^2(x)) + \sin^2(x)(\sinh^2(y) - \cosh^2(y)) \\
&= \cosh^2(y) - \sin^2(x)
\end{aligned}$$

Thus adding the two different values together yields

$$\begin{aligned}
2|\cos z|^2 &= \sinh^2(y) + \cos^2(x) + \cosh^2(y) - \sin^2(x) \\
&= (\sinh^2(y) + \cosh^2(y)) + (\cos^2(x) - \sin^2(x)) \\
&= \cosh(2y) + \cos(2x)
\end{aligned}$$

Similarly for sin, from the above problem

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

so the square of the absolute value of $\cos(z)$ is

$$\begin{aligned}
|\sin z|^2 &= (\sin(x)\cosh(y))^2 + (\cos(x)\sinh(y))^2 \\
&= (\sin(x)\cosh(y))^2 + (\cos(x)\sinh(y))^2 + (\sin(x)\sinh(y))^2 - (\sin(x)\sinh(y))^2 \\
&= \sinh^2(y)(\sin^2(x) + \cos^2(x)) + \sin^2(x)(\cosh^2(y) - \sinh^2(y)) \\
&= \sinh^2(y) + \sin^2(x)
\end{aligned}$$

$$\begin{aligned}
|\sin z|^2 &= (\sin(x)\cosh(y))^2 + (\cos(x)\sinh(y))^2 \\
&= (\sin(x)\cosh(y))^2 + (\cos(x)\sinh(y))^2 + (\cos(x)\cosh(y))^2 - (\cos(x)\cosh(y))^2 \\
&= \cosh^2(y)(\sin^2(x) + \cos^2(x)) + \cos^2(x)(\sinh^2(y) - \cosh^2(y)) \\
&= \cosh^2(y) - \cos^2(x)
\end{aligned}$$

Thus adding the two different values together yields

$$\begin{aligned}
2|\sin z|^2 &= \sinh^2(y) + \sin^2(x) + \cosh^2(y) - \cos^2(x) \\
&= (\sinh^2(y) + \cosh^2(y)) - (\cos^2(x) - \sin^2(x)) \\
&= \cosh(2y) - \cos(2x)
\end{aligned}$$

$\square$