# Combinatorics: Homework 3

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**Problem 1.**
Let $0 \leq k \leq 2$. Show that $n \geq 3$, the number of permutations $w \in S_n$ whose number of inversions is congruent to $k$ modulo 3 is independent of $k$. For instance, when $n = 3$, there are two permutations with 0 or 3 inversions, two with 1 inversion, and two with 2 inversions.

**Solution.**
We're going to prove this inductively. The problem statement establishes the base case: the claim is true for $n = 3$.

Suppose the claim is true for $3, 4, \ldots, n$, and I will show it is true for $n + 1$. I will illustrate with the case $n = 3$. Write the $n!$ permutations as words in lexicographic order, and then for each permutation, increment each element by 1 and prepend 1. This yields the first $n!$ permutations of the $n + 1$ case in lexicographic order.

$$\begin{aligned}
\mathrm{inv}(1234) &= 0 \\
\mathrm{inv}(1243) &= 1 \\
\mathrm{inv}(1324) &= 1 \\
\mathrm{inv}(1342) &= 2 \\
\mathrm{inv}(1423) &= 2 \\
\mathrm{inv}(1432) &= 3
\end{aligned}$$

Because 1 is smaller than all of the incremented elements, this does not introduce any new inversions, and so the first $n!$ permutations of $[n + 1]$ inherit the desired property from the $n$ case.

When we list the next $n!$ permutations (again in lexicographic order), the relative position of the last $n$ elements remains unchanged. Since every permutation starts with 2, this introduces exactly one new inversion.

$$\begin{aligned}
\mathrm{inv}(2134) &= 0 + 1 \\
\mathrm{inv}(2143) &= 1 + 1 \\
\mathrm{inv}(2314) &= 1 + 1 \\
\mathrm{inv}(2341) &= 2 + 1 \\
\mathrm{inv}(2413) &= 2 + 1 \\
\mathrm{inv}(2431) &= 3 + 1
\end{aligned}$$

Since we add the *same* number to each inversion count, this preserves the desired property. An identical arguement works for the $n!$ permutations that start with 3, the $n!$ permutations that start with 4, etc. Therefore the number of permutations $w \in S_n$ whose number is congruent to $k$ modulo 3 is independent of $k$.

**Problem 2.**
For any non-identity element $w \in S_n$, let $m_1(w)$ be the smallest element of the descent set $D(w)$. Set $m_1(\mathrm{id}) = 0$. Find the expected value $E_1(n)$ of $m_1(w)$ over all $w \in S_n$, chosen uniformly. Express your answer as a simple sum.

**Solution.**
We will first count the number of permutations such that the smallest element of the descent set is $k$. In particular, we'll choose the first $k+1$ terms of the sequence. The largest of these terms is $w_k$, which leaves $k$ remaining elements as choices for $w_{k+1}$. Permuting the remaining $n-k-1$ elements will give every possible sequence that satisfies

1. $w_1 < w_2 < \ldots < w_{k-1} < w_k$ and
2. $w_k > w_{k+1}$.

Thus

$$a_k(n) = \binom{n}{k+1} \cdot k \cdot (n-k-1)!.$$

Then summing over all choices of $k$, and multiplying each term by $k$ yields

$$a(n) = \sum_{k=1}^{n-1} k^2 \binom{n}{k+1} (n-k-1)!,$$

where the number of terms grows linearly.

The sequence begins

$$0, 1, 7, 37, 201, 1231, 8653, 69273, 623521, 6235291, \ldots.$$

Then the expected value is simply given by

$$E_1(n) = \frac{a(n)}{n!}.$$

I conjecture that

$$a(n+1) = (n+1)a(n) + n^2 \text{ for } n \geq 1$$

and that

$$\lim_{n\to\infty} \frac{a(n)}{n!} = e - 1.$$

**Problem 3.** (Exercise 53a) [2]
The *Eulerian Catalan number* is defined by $EC_n = A(2n+1, n+1)/(n+1)$. The first few Eulerian Catalan numbers, beginning with $EC_0 = 1$, are $1, 2, 22, 604, 31238$. Show that $EC_n = 2A(2n, n+1)$ (and thus $EC_n \in \mathbb{Z}$).

**Solution.**
$A(2n+1, n+1)$ is the number of permutations of $w \in \mathfrak{S}_{2n+1}$ with exactly $n$ descents.

We want to show

$$2(n+1)A(2n, n+1) = A(2n+1, n+1).$$

The Eulerian numbers $A(2n, n+1)$ and $A(2n+1, n+1)$ count the number of permutations $w \in \mathfrak{S}_{2n}$ and $w \in \mathfrak{S}_{2n+1}$ respectively with exactly $n$ descents.

Notice that map $f \colon \mathfrak{S}_{2n} \to \mathfrak{S}_{2n}$ where the permutation (written as a word) is reversed has the number of descents given by

$$\operatorname{des}(f(w)) = 2n - 1 - \operatorname{des}(w).$$

So in particular, $f$ defines a bijection between permutations of $w \in \mathfrak{S}_{2n}$ with exactly $n$ descents and permutations of $w \in \mathfrak{S}_{2n}$ with exactly $n-1$ descents.

Thus it is enough to define a method of taking a permutation $w \in \mathfrak{S}_{2n}$ with $n$ or $n-1$ descents, and producing from it $n+1$ permutations $w_1, \ldots, w_{n+1} \in \mathfrak{S}_{2n+1}$ with $n$ descents.

Given some permutation in $\mathfrak{S}_{2n}$ with $n$ descents (written as a word), we can insert $2n+1$ after any descent position, or at the end of the word. For example, in the following permutation ($n = 4$):

$$2 \quad 6 \underbrace{\qquad} 1 \quad 8 \underbrace{\qquad} 4 \quad 7 \underbrace{\qquad} 6 \underbrace{\qquad} 5 \underbrace{\qquad}.$$

Conversely, given some permutation in $\mathfrak{S}_{2n}$ with $n-1$ descents (written as a word), we can insert $2n+1$ before any of the $n+1$ non-descent positions. For example, in the following permutation ($n = 4$):

$$\underbrace{\qquad} 5 \underbrace{\qquad} 6 \underbrace{\qquad} 7 \quad 4 \underbrace{\qquad} 8 \quad 1 \underbrace{\qquad} 6 \quad 2.$$

Since this procedure preserves the order of the elements in $[2n]$, all of the resulting elements in $\mathfrak{S}_{2n+1}$ are distinct. Furthermore, since permutations with $n$ descents can only have "parent" permutations with $n-1$ or $n$ descents, this procedure enumerates all of the permutations counted by $A(2n+1, n+1)$.

Therefore

$$2A(2n, n+1) = \frac{A(2n+1, n+1)}{(n+1)} = EC_n,$$

and $EC_n \in \mathbb{Z}$.

**Problem 4.** (Exercise 54) [2]
How many $n$-element multisets on $[2m]$ are there satisfying

(i) $1, 2, \ldots, m$ appear at most once each, and

(ii) $m+1, m+2, \ldots, 2m$ appear an even number of times each?

**Solution.**
This problem has a very clean solution using generating functions. To choose the elements satisfying the first condition, we can choose any subset of $[m]$, and to choose the elements satisfying the second condition, we can choose any multiset from $m+1, m+2, \ldots, 2m$ and "double" it.

Call our counting function $g(n,m)$, and our generating function for $m$ $f_m(x)$. Thus

$$f_m(x) = \sum_{n=0}^{\infty} g(n,m)x^n = \sum_{k=0}^{m}\sum_{j=0}^{\infty} \binom{m}{k} \left(\!\binom{m}{j}\!\right) x^k x^{2j}$$

Because $m$ and $j$ are independent in the sum on the right, this can be split into

$$\begin{aligned}
\sum_{k=0}^{m}\sum_{j=0}^{\infty} \binom{m}{k} \left(\!\binom{m}{j}\!\right) x^k x^{2j} &= \left(\sum_{k=0}^{m} \binom{m}{k} x^k\right) \left(\sum_{j=0}^{\infty} \left(\!\binom{m}{j}\!\right) x^{2j}\right) \\
&= (1+x)^m \left(\frac{1}{1-x^2}\right)^m \\
&= \frac{(1+x)^m}{(1-x)^m(1+x)^m} \\
&= \frac{1}{(1-x)^m} \\
&= \sum_{n=0}^{\infty} \left(\!\binom{m}{n}\!\right) x^n.
\end{aligned}$$

Thus

$$g(n,m) = \left(\!\binom{m}{n}\!\right).$$