

Topology: Homework 3

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Problem 1.

In the free group $F_2 = \langle a, b; \rangle$, consider the elements

$$A = b^2 a^{-1} b^{-3} a b^{-1} a b^{-2}$$

$$B = b^2 a^{-1} b a^{-1} b a b^2$$

$$C = b^{-2} a^{-1} b^2 a^3 b$$

$$D = b^{-2} a^{-1} b^2 a b^{-2} a^3$$

(a) Compute $(AB)C$ and $A(BC)$ in the given order, and verify that $(AB)C = A(BC)$.

(b) Compute $(AB)D$ and $A(BD)$ in the given order, and verify that $(AB)D = A(BD)$.

Proof.

(a) First computing AB gives

$$\begin{aligned} AB &= b^2 a^{-1} b^{-3} a b^{-1} \underbrace{a b^{-2}}_{\text{id}_B} \cdot b^2 a^{-1} b a^{-1} b a b^2 \\ &= b^2 a^{-1} b^{-3} a b^{-1} \underbrace{a a^{-1}}_{\text{id}_A} b a^{-1} b a b^2 \\ &= b^2 a^{-1} b^{-3} a \underbrace{b^{-1} b}_{\text{id}_B} a^{-1} b a b^2 \\ &= b^2 a^{-1} b^{-3} \underbrace{a a^{-1}}_{\text{id}_A} b a b^2 \\ &= b^2 a^{-1} \underbrace{b^{-3} b}_{b^{-2}} a b^2 \\ &= b^2 a^{-1} b^{-2} a b^2. \end{aligned}$$

Then computing $(AB)C$ gives

$$\begin{aligned} (AB)C &= b^2 a^{-1} b^{-2} a \underbrace{b^2 \cdot b^{-2}}_{\text{id}_B} a^{-1} b^2 a^3 b \\ &= b^2 a^{-1} b^{-2} \underbrace{a a^{-1}}_{\text{id}_A} b^2 a^3 b \\ &= b^2 a^{-1} \underbrace{b^{-2} b^2}_{\text{id}_B} a^3 b \\ &= b^2 \underbrace{a^{-1} a^3}_{a^2} b \\ &= b^2 a^2 b. \end{aligned}$$

Similarly, computing BC gives

$$\begin{aligned}
BC &= b^2 a^{-1} b a^{-1} b a \underbrace{b^2 \cdot b^{-2}}_{\text{id}_B} a^{-1} b^2 a^3 b \\
&= b^2 a^{-1} b a^{-1} b \underbrace{a a^{-1}}_{\text{id}_A} b^2 a^3 b \\
&= b^2 a^{-1} b a^{-1} \underbrace{b b^2}_{b^3} a^3 b \\
&= b^2 a^{-1} b a^{-1} b^3 a^3 b.
\end{aligned}$$

Then computing $A(BC)$ gives

$$\begin{aligned}
A(BC) &= b^2 a^{-1} b^{-3} a b^{-1} a \underbrace{b^{-2} \cdot b^2}_{\text{id}_B} a^{-1} b a^{-1} b^3 a^3 b \\
&= b^2 a^{-1} b^{-3} a b^{-1} a \cdot a^{-1} \underbrace{b a^{-1} b^3 a^3 b}_{\text{id}_A} \\
&= b^2 a^{-1} b^{-3} a \underbrace{b^{-1} \cdot b}_{\text{id}_B} a^{-1} b^3 a^3 b \\
&= b^2 a^{-1} b^{-3} a \cdot a^{-1} \underbrace{b^3 a^3 b}_{\text{id}_A} \\
&= b^2 a^{-1} \underbrace{b^{-3} \cdot b^3}_{\text{id}_B} a^3 b \\
&= b^2 \underbrace{a^{-1} \cdot a^3}_{a^2} b \\
&= b^2 a^2 b.
\end{aligned}$$

Therefore

$$(AB)C = b^2 a^2 b = A(BC).$$

(b) We already computed $AB = b^2 a^{-1} b^{-2} a b^2$, so computing $(AB)D$

$$\begin{aligned}
(AB)D &= b^2 a^{-1} b^{-2} a \underbrace{b^2 \cdot b^{-2}}_{\text{id}_B} a^{-1} b^2 a b^{-2} a^3 \\
&= b^2 a^{-1} b^{-2} a \underbrace{a a^{-1}}_{\text{id}_A} b^2 a b^{-2} a^3 \\
&= b^2 a^{-1} \underbrace{b^{-2} b^2}_{\text{id}_B} a b^{-2} a^3 \\
&= b^2 \underbrace{a^{-1} a}_{\text{id}_A} b^{-2} a^3 \\
&= \underbrace{b^2 b^{-2}}_{\text{id}_B} a^3 \\
&= a^3.
\end{aligned}$$

Similarly, computing BD gives

$$\begin{aligned}
BD &= b^2 a^{-1} b a^{-1} b a \underbrace{b^2 \cdot b^{-2}}_{\text{id}_B} a^{-1} b^2 a b^{-2} a^3 \\
&= b^2 a^{-1} b a^{-1} b \underbrace{a a^{-1}}_{\text{id}_A} b^2 a b^{-2} a^3 \\
&= b^2 a^{-1} b a^{-1} \underbrace{b b^2}_{b^3} a b^{-2} a^3 \\
&= b^2 a^{-1} b a^{-1} b^3 a b^{-2} a^3.
\end{aligned}$$

Then computing $A(BC)$ gives

$$\begin{aligned}
A(BC) &= b^2 a^{-1} b^{-3} a b^{-1} a \underbrace{b^{-2} \cdot b^2}_{\text{id}_B} a^{-1} b a^{-1} b^3 a b^{-2} a^3 \\
&= b^2 a^{-1} b^{-3} a b^{-1} \underbrace{a \cdot a^{-1}}_{\text{id}_A} b a^{-1} b^3 a b^{-2} a^3 \\
&= b^2 a^{-1} b^{-3} a \underbrace{b^{-1} \cdot b}_{\text{id}_B} a^{-1} b^3 a b^{-2} a^3 \\
&= b^2 a^{-1} b^{-3} \underbrace{a \cdot a^{-1}}_{\text{id}_A} b^3 a b^{-2} a^3 \\
&= b^2 a^{-1} \underbrace{b^{-3} \cdot b^3}_{\text{id}_B} a b^{-2} a^3 \\
&= b^2 \underbrace{a^{-1} \cdot a}_{\text{id}_A} b^{-2} a^3 \\
&= \underbrace{b^2 \cdot b^{-2}}_{\text{id}_B} a^3 \\
&= a^3.
\end{aligned}$$

Therefore

$$(AB)D = a^3 = A(BD).$$

□

Problem 2.

Consider the group $G = \langle a, b; a^2b^{-3} = 1 \rangle = F(a, b)/\langle a^2b^{-3} \rangle$. Let $\tau \in \mathfrak{S}_3$ be the transposition $\tau = (1\ 2)$ and let ρ be the cyclic permutation $\rho = (1\ 2\ 3)$.

- Show that there is a unique group homomorphism $\phi: G \rightarrow \mathfrak{S}_3$ sending a to τ and b to ρ . Conclude that G is not abelian.
- Show that there is a surjective homomorphism $\psi: G \rightarrow \mathbb{Z}$. Conclude that G is infinite.

Proof.

- First, $\langle a^2b^{-3} \rangle \subset \ker(\phi)$ because

$$\phi(a^2b^{-3}) = \phi(a^2)\phi(b^{-3}) = \underbrace{(1\ 2)(1\ 2)}_{\text{id}_{\mathfrak{S}_3}} \underbrace{(1\ 2\ 3)(1\ 2\ 3)(1\ 2\ 3)}_{\text{id}_{\mathfrak{S}_3}} = \text{id}_{\mathfrak{S}_3}$$

This map is the unique homomorphism that sends $\tau \mapsto (1\ 2)$ and $\rho \mapsto (1\ 2\ 3)$, because it prescribes where to send all elements of $\langle a, b \rangle$, and so the quotient map inherits this uniqueness.

- We will use the map defined by, $\psi(a) = 3$ and $\psi(b) = 2$. Since

$$\psi(a^2b^{-3}) = \psi(a^2)\psi(b^{-3}) = 2(3) - 3(2) = 0$$

we have that $\langle a^2b^{-3} \rangle \subset \ker(\psi)$, and so ψ defines a homomorphism. Thus it only remains to check that ψ is surjective: any even number $2n \in \mathbb{Z}$ can be written as $\psi([a^n])$, and similarly any odd number $2n+1 \in \mathbb{Z}$ can be written as $\psi([a^{n-1}b])$. Thus G is infinite.

□

Problem 3.

Let X be a metric space with metric d_0 , and pick a base point $x_0 \in X$. Let $\Omega_{x_0}X$ denote the space of paths $\alpha: [0, 1] \rightarrow X$ with $\alpha(0) = \alpha(1) = x_0$.

- Define $x_2(X; x_0) = \pi_1(\Omega_{x_0}X; c_{x_0})$. Interpret $\pi_2(X; x_0)$ as a set of maps. What is the geometric interpretation of the group law in this context. $[0, 1] \times [0, 1] \rightarrow X$ modulo a certain equivalence relation.
- Show that $\pi_2(X; x_0)$ is an abelian group.

Solution.

- Let elements of $\pi_2(X, x_0)$ be equivalence classes of (continuous) maps $\alpha: [0, 1] \times [0, 1] \rightarrow X$ such that

$$\alpha(0, t) = \alpha(1, t) = x_0 = \alpha(s, 0) = \alpha(s, 1)$$

where two maps α_0 and α_1 are equivalent if there exists a continuous map $H: ([0, 1] \times [0, 1]) \times [0, 1] \rightarrow X$ between them such that

$$\begin{aligned} H(s, t, 0) &= \alpha_0(s, t), \\ H(s, t, 1) &= \alpha_1(s, t), \\ H(0, t, r) &= H(1, t, r) = x_0, \text{ and} \\ H(s, 0, r) &= H(s, 1, r) = x_0. \end{aligned}$$

The geometric interpretation of the group law is a path that starts and ends as the constant path at x_0 , and is a continuous collection of loops in between.

- We'll define the homotopy between $\alpha * \beta$ and $\beta * \alpha$ by $H: ([0, 1] \times [0, 1]) \times [0, 1] \rightarrow X$ as three maps H_1 , H_2 , and H_3 :

$$H(s, t, r) = \begin{cases} H_1(s, t, 3r) & r \in [0, 1/3] \\ H_2(s, t, 3r - 1) & r \in [1/3, 2/3] \\ H_3(s, t, 3r - 2) & r \in [2/3, 1] \end{cases}.$$

Define

$$H_1(s, t, r) = \begin{cases} x_0 & s \in [0, 1/2], t \in [0, r/2] \\ \alpha(2s, ?) & s \in [0, 1/2], t \in [r/2, 1] \\ \beta(2s - 1, ?) & s \in [1/2, 1], t \in [0, r/2] \\ x_0 & s \in [1/2, 1], t \in [r/2, 1] \end{cases}.$$

(I ran out of time.)