Complex Analysis: Homework 2

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Problem 5. (page 37)
Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

Proof. $\square$

Problem 3. (page 41)
Find the radius of convergence of the following power series:

(a) $\sum n^p z^n$

(b) $\sum \frac{z^n}{n!}$

(c) $\sum n! z^n$

(d) $\sum q^{n^2} z^n$ where $|q| < 1$

(e) $\sum z^{n!}$

Proof. (a) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n\to\infty} |n^p|^{1/n} = \limsup_{n\to\infty} |n^p|^{1/n} = \limsup_{n\to\infty} n^{p/n} = 1$$

So the radius of convergence is $R = 1$.

(b) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} < \limsup_{n\to\infty} \left(\frac{1}{N^n}\right)^{1/n} = \frac{1}{N}.$$

Because $R > N$ for all N, the radius of convergence is ∞.

(c) Let N be an arbitrarily large integer, then by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} (n!)^{1/n} > \limsup_{n\to\infty} (N^n)^{1/n} = N.$$

Because $R < 1/N$ for all N, the radius of convergence is 0.

(d) By Hadamard's formula, let

$$\frac{1}{R} = \limsup_{n\to\infty} (q^{n^2})^{1/n} = \limsup_{n\to\infty} q^n = 0 \text{ for } |q| < 1.$$

Thus the radius of convergence is ∞.

(e) Notice that $|z^{n!}| \geq |z^n|$ for $|z| \geq 1$, and $|z^{n!}| \leq |z^n|$ for $|z| < 1$.

$$\left|\sum z^{n!}\right| \leq \left|\sum z^n\right| < \infty \text{ for } |z| < 1$$
$$\left|\sum z^{n!}\right| \geq \left|\sum z^n\right| = \infty \text{ for } |z| \geq 1$$

Thus the radius of convergence is 0.

$\square$

Problem 8. (page 41)
For what values of z is
$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$
convergent?

Proof. The sum $\sum_{n=0}^{\infty} w^n$ is convergent for $|w| < 1$. The sum in the problem is convergent when
$$\left| \frac{z}{1+z} \right| < 1 \Longrightarrow |z| < |1+z|.$$
Letting $z = a + bi$, and comparing the squares of the absolute values:
$$\begin{aligned} a^2 + b^2 &< (1+a)^2 + b^2 \\ a^2 &< 1 + 2a + a^2 \\ -2a &< 1 \\ a &> -1/2 \end{aligned}$$
Thus the sum converges when $\mathrm{Re}(z) > -1/2$. $\square$

Problem 3. (page 44)
Use the addition formulas to separate $\cos(x + iy)$ and $\sin(x + iy)$ in real and imaginary parts.

Proof. $\square$

Problem 4. (page 44)
Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x)$$

Proof. □

Problem 6. (page 47)
Determine all values of

(a) 2^i,

(b) i^i,

(c) $(-1)^{2i}$.

Proof. $\square$

Problem 7. (page 47)
Determine the real and imaginary parts of z^z.

Proof. $\square$

Problem ?. (page ?)

Proof. ∎

Problem ?. (page ?)

Proof. $\square$

Problem ?. (page ?)

Proof. □