Combinatorics: Homework 10

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Problem 70.

a. [2−] Let E_n denote the poset of all subsets of $[n]$ whose elements have even sum ordered by inclusion. Find $\#E_n$.

b. [2+] Compute $\mu(S,T)$ for all $S \leq T$ in E_n as a recursion

$$a_m = -\sum_{k<m} \binom{2m}{2k} a_k.$$

Solution.

a. Consider all 2^{n-1} subsets of $\{2, 3, \ldots, n\}$. There are a sets with an even sum, and b with an odd sum, where $a + b = 2^{n-1}$. This means that if we add 1 to each set, there are b new sets with an even sum and a odd sets with an odd sum.

   Therefore there are a subsets of $[n]$ with an even sum which do not contain 1, and b subsets of $[n]$ with an even sum which contain 1, totaling $a + b = 2^{n-1}$ subsets of $[n]$ with an even sum.

b. Using the identity

$$\mu(S,T) = \begin{cases} 1 & S = T \\ \displaystyle\sum_{S \leq U < T} -\mu(S,U) & S < T \end{cases}$$

Problem 89. [2]
For a finite lattice L, let $f_L(m)$ be the number of m-tuples $(t_1, \ldots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \ldots \wedge t_m = \hat{0}$. Prove via Möbius inversion that

$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t)(\#V_t)^m$$

where $V_t = \{s \in L : s \geq t\}$.

Solution.
Generalize $f_L(m)$ by defining

$$g_L^m(s) = \#\{t_1, \ldots, t_m : t_1 \wedge t_2 \wedge \ldots \wedge t_m = s\}.$$

In particular $g_L^m(\hat{0}) = f_L(m)$.

Notice that $t_1 \wedge t_2 \wedge \ldots \wedge t_m \in V_t$ if and only if $t_1, \ldots, t_m \in V_t$. Therefore if the number of tuples $(t_1, \ldots, t_m)$ with entries in V_s is exactly the number of tuples such that $t_1 \wedge t_2 \wedge \ldots \wedge t_m \in V_t$, that is:

$$(\#V_s)^m = \sum_{t \geq s} g_L^m(t).$$

Therefore by the dual form of the Möbius inversion formula,

$$g_L^m(s) = \sum_{t \geq s} \mu(\hat{0}, t)(\#V_t)^m,$$

so in particular when $s = \hat{0}$,

$$f_L(m) = g_L^m(\hat{0}) = \sum_{t \geq 0} \mu(\hat{0}, t)(\#V_t)^m = \sum_{t \in L} \mu(\hat{0}, t)(\#V_t)^m.$$