Topology: Homework 8

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Problem 1.
Suppose that A is homotopoy equivalent to a point. Show that $H_n(X, A)$ is isomorphic to $H_n(X)$ for every $n \geq 1$.

Proof.
By the snake lemma we can turn the short exact sequences

$$0 \to C(A) \to C(X) \to C(X, A) \to 0$$

into the long exact sequence

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \to \ldots.$$

Since A is homotopy equivalent to a point by hypothesis, for all $n > 0$, $H_n(A) = 0$. Therefore for $n > 1$

$$\underbrace{H_n(A)}_{0} \to H_n(X) \to H_n(X, A) \xrightarrow{\delta_n} \underbrace{H_{n-1}(A)}_{0}$$

and thus map $H_n(X) \to H_n(X, A)$ has a kernel of 0 so is injective, and it has an image of $H_n(X, A)$, so it's surjective, and thus $H_n(X) \cong H_n(X, A)$.

In the case of $n = 1$, the long exact sequence is

$$\ldots \to \underbrace{H_1(A)}_{0} \to H_1(X) \to H_1(X, A) \xrightarrow{\delta_1} \underbrace{H_0(A)}_{R} \to \ldots,$$

so the map $H_1(X) \to H_1(X, A)$ is injective. Thus it is enough to show that the map is surjective. However, take the equivalence class $[c] \in H_1(X, A) = H_1(X)/H_1(A)$, with representative $c \in H_1(X)$. So the quotient map $c \mapsto [c]$ is surjective. $\square$

Problem 2.
Suppose that X is homotopy equivalent to a point. Show that $H_n(X, A)$ is isomorphic to $H_{n-1}(A)$ for every $n \geq 2$. Show that this is in general false if $n = 1$.

Proof.
By the same construction above, we have the long exact sequence

$$\ldots \to H_n(X) \to H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \to H_{n-1}(X) \to \ldots.$$

If $n > 1$, then we have the short exact sequence

$$\underbrace{H_n(X)}_{0} \to H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \to \underbrace{H_{n-1}(X)}_{0}$$

so δ_n is an isomorphism.

In the case of $n = 1$, $H_1(X, A) = C_1(X)/B_1(X, A)$

$$\underbrace{H_1(X)}_{0} \to H_1(X, A) \xrightarrow{\delta_1} H_0(A) \to \underbrace{H_0(X)}_{R}.$$

The map $\delta_1 \colon H_1(X, A) \to H_0(A)$ is injective, so it is enough to show that δ_1 is not surjective. Let $X = 0$ and $X = A$. Then $X_1(X, A) = 0$ and $X_0(A) = R$, so this map cannot be surjective for all pairs (X, A) as shown by this counterexample. $\square$

Problem 3.
For $A \subset X$, suppose that the inclusion map $i \colon A \to X$ is a homotopy equivalence. Show that $H_n(X, A) = 0$ for every n.

Proof.
Firstly, we have the exact sequence

$$\ldots \to H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} H_{n-1}(X, A) \ldots.$$

where $\ker(i_*) = 0 = \operatorname{Im}(\delta_n)$, where the kernel is trivial because i_* is an isomorphism. Thus δ_n must be the zero map with kernel

$$\ker(\delta_n) = H_n(X, A).$$

Since $j_* \circ i_* \colon H_{n-1}(A) \to H_{n-1}(X, A)$ maps everything to the zero element in the quotient, j_* must be the zero map, so

$$0 = \operatorname{Im}(j_*) = \ker(\delta_n) = H_n(X, A).$$

$\square$

Problem 4.
Suppose that $X = X_1 \cup X_2$ for two subspaces $X_1, X_2 \subset X$. Let $C_n^{X_1X_2}(X) = C_n(X_1) + C_n(X_2) \subset C_n(X)$ consist of chains $c \in C_n(X)$ that can be written as a linear combination of simplices that are either completely contained in X_1 or completely contained in X_2. Let $H_n^{X_1X_2}$ denote the homology modules of the corresponding chain

Prove that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is an isomorphism for every n if and only if $X_1 - X_2$ can be excised from the pair (X, X_1).

a. Suppose that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is an isomorphism for every n. We want to show that $H_n(j)\colon H_n(X_2, X_1 \cap X_2) \to H_n(X, X_1)$ is surjective. For this, consider $[c] \in H_n(X, X_1)$ represented by $c \in C_n(X)$ with $\partial c \in C_{n-1}(X_1)$.
   (i) Let $c' = \partial c$. Considering the classes $[c'] \in H_n^{X_1X_2}(X)$ and $[c'] \in H_n(X)$, show that there exists $c_1 \in C_n(X_1)$ and $C_2 \in C_n(X_2)$ such that $c' = \partial c_1 + \partial c_2$.
   (ii) Show that there exists $c_1' \in C_n(X_1)$, $c_2' \in C_n(X_2)$ and $c' \in C_{n+1}(X)$ such that
   $$c - c_1 - c_2 = c_1' + c_2' + \partial c'.$$
   (iii) Show that $\partial(c_2 + c_2') \in C_{n-1}(X_1 \cap X_2)$, so that $c_2 + c_2'$ defines a class $[c_2 + c_2'] \in H_n(X_2, X_1 \cap X_2)$.
   (iv) Show that $H_n(j)([c_2 + c_2']) = [c] \in H_n(X, X_1)$, which concludes the proof that $H_n(j)$ is surjective.

b. Suppose that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is an isomorpishm for every n. We want to show that $H_n(j)\colon H_n(X_2, X_1 \cap X_2) \to H_n(X, X_1)$ is injective. For this, consider
   $$[c_2] \in \ker H_n(j) \subset H_n(X_2, X_1 \cap X_2)$$
   represented by $c_2 \in C_n(X_2)$ with $\partial c_2 \in C_{n-1}(X_1 \cap X_2)$.
   (i) Show that there exists $c_1 \in C_n(X_1), c_1' \in C_{n+1}(X_1)$ and $c_2' \in C_{n+1}(X_2)$ such that $c_2 = c_1 + \partial c_1' + \partial c_2'$. (Hint: use part a (i).)

Proof.

a. This part will assume that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is an isomorphism for every n, and prove that $H_n(j)\colon H_n(X_2, X_1 \cap X_2) \to H_n(X, X_1)$ is surjective.
   (i) Since $H_n(i)$ is a bijection, $H_n(i)^{-1}([c]) = [c_1 + c_2] \in H_n^{X_1X_2}(X)$. Therefore $c = c_1 + c_2 + \partial\tilde{c}$ where $\partial\tilde{c} \in \operatorname{Im}(\partial_{n+1})$. Moreover,
   $$c' = \partial c = \partial(c_1 + c_2 + \partial\tilde{c}) = \partial c_1 + \partial c_2 + \underbrace{\partial\partial\tilde{c}}_{0}.$$
   (ii) Similarly,
   $$H_n(i)^{-1}(\underbrace{[c - c_1 - c_2]}_{\in H_n(X)}) = [c_1' + c_2'] \in H_n^{X_1X_2}(X)$$
   which means that
   $$c - c_1 - c_2 = c_1' + c_2' + \underbrace{\partial c'}_{\in \operatorname{Im}(\partial)},$$
   by definition of the quotient.
   (iii) By the above
   $$\begin{aligned} c_2' + c_2 &= c - c_1 - c_1' - \partial c' \\ \partial(c_2' + c_2) &= \partial(c - c_1 - c_1' - \partial c') \\ &= \partial c - \partial(c_1 + c_1') \end{aligned}$$
   Since $\partial(c_1' + c_1) \in C_n(X_1)$, $\partial(c_2' + c_2) \in C_n(X_2)$, and $\partial c \in C_n(X_1)$ by hypothesis, the right hand side is in $C_n(X_1)$ and the left hand side is in $C_n(X_2)$, so both must be in $C_n(X_1 \cap X_2)$.

(iv) Using the above,

$$H_n(j)([c_2 + c_2']) = H_n(j)([c - c_1 - c_1' - \partial c']) = [j(c - c_1 - c_1' - \partial c')].$$

Since

$$\partial(c - c_1 - c_1' - \partial c') = \partial c - \underbrace{\partial(c_1 + c_1' + \partial c')}_{C_n(X_1)}$$

this equivalence class is

$$[j(c - c_1 - c_1' - \partial c')] = [c] \in H_n(X, X_1)$$

so $H_n(j)$ is surjective.

b. This part will assume that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is an isomorphism for every n, and prove that $H_n(j)\colon H_n(X_2, X_1 \cap X_2) \to H_n(X, X_1)$ is injective.

(i) Since $[c_2] \in \ker H_n(j)$, $[c_2] = 0 \in X_n(X, X_1) = Z(X, X_1)/B(X, X_1)$, so $c_2 \in B(X, X_1)$. By definition of relative boundary, this means that there exists some $c \in C_{n+1}(X)$ such that $c_2 - \partial c \in C_n(X_1)$. Let $c_1 = c_2 - \partial c \in C_n(X_1)$ so that $c_2 = c_1 + \partial c$. By part **a (i)**, $\partial c = \partial c_1' + \partial c_2'$ for some $c_1' \in C_{n+1}(X_1)$ and $c_2' \in C_{n+1}(X_2)$, so

$$\begin{aligned} c_2 &= c_1 + \partial c \\ &= c_1 + \partial c_1' + \partial c_2'. \end{aligned}$$

(ii) Solving for $c_1 + \partial c_1'$ yields

$$\underbrace{c_1 + \partial c_1'}_{\in C_n(X_1)} = \underbrace{c_2 - \partial c_2'}_{\in C_n(X_2)},$$

so $c_1 + \partial c_1' \in C_n(X_1 \cap X_2)$.

(iii) Therefore

$$c_2 - \partial c_2' = \underbrace{c_1 + \partial c_1'}_{C_n(X_1 \cap X_2)} \in C_n(X_1 \cap X_2)$$

so $c_2 \in B_n(X_2, X_1 \cap X_2)$, so $[c_2] = 0 \in H_n(X_2, X_1 \cap X_2) = Z_n(X_2, X_1 \cap X_2)/B_n(X_2, X_1 \cap X_2)$. Therefore $H_n(j)$ is injective.

c. This part will assume that $X_1 - X_2$ can be excised from the pair (X, X_1) and prove that $H_n(i)\colon H_n^{X_1X_2}(X) \to H_n(X)$ is injective.

(i) Since $[c_1 + c_2] \in \ker H_n(i)$ (and thus there exists $c \in C_n(X)$ such that $c_1 + c_2 = \partial c$) we can check

$$\begin{aligned} \partial(c_1 + c_2) &= \underbrace{\partial\partial c}_{0} \\ \partial c_1 &= -\partial c_2 \in C_{n-1}(X_2), \end{aligned}$$

So $\partial c_1 \in C_{n-1}(X_1) \cap C_{n-1}(X_2) = C_{n-1}(X_1 \cap X_2)$, and therefore c_2 defines a class $[c_2] \in H_n(X_2, X_1 \cap X_2)$.

(ii) By rearranging $c_1 + c_2 = \partial c$, it can be seent that $c_2 - \partial c = c_1 \in C_n(X_1)$, and so $c_2 \in B_n(X, X_1)$. Therefore $[c_2] = 0 \in Z_n(X, X_1)/B_n(X, X_1) = H_n(X, X_1)$. Since $H_n(j)$ is injective,

$$[c_2] = 0 \in H_n(X_2, X_1 \cap X_2) = Z_n(X_2, X_1 \cap X_2)/B_n(X_2, X_1 \cap X_2)$$

so $c_2 \in B_n(X_2, X_1 \cap X_2)$ and so there exists c_2' such that $c_2 - \partial c_2' \in C_n(X_1 \cap X_2)$. Name this element $c_{12} = c_2 - \partial c_2'$. Then rearranging,

$$c_2 = c_{12} - \partial c_2'.$$

(iii) By hypothesis $(X_2, X_1 \cap X_2) \to (X, X_1)$ is an excision so $H_n(j)$ is an isomorphism. We know by the first two parts that

$$\begin{aligned} \partial c &= c_1 + c_2, \text{ and} \\ c_2 &= c_{12} - \partial c_2' \end{aligned}$$

so it follows that $\partial(c + c_2') = c_1 + c_{12}$, and we can consider $[c - c_2'] \in H_{n+1}(X, X_1)$.
Because $H_n(j)$ is an isomorphism, by taking the inverse map, there exists $H_n(j)^{-1}([c - c_2']) = [c' + c''] \in H_{n+1}(X_2, X_1 \cap X_2)$, meaning there exists some $c'' \in C_{n+2}$ such that

$$c - c'2 = c_2'' + c_1' + \partial c'',$$

as desired.

(iv) From above, we can write

$$\begin{aligned} \partial(c - c_2') &= \partial(c_2'' + c_1' + \partial c'') \\ c_1 + c_2 &= \partial c_2' + \partial c_2'' + \partial c_1' \\ c_1 + c_2 &= \partial(c_2' + c_2'' + c_1') \end{aligned}$$

so $c_1 + c_2 \in \mathrm{Im}(\partial)$ and thus $[c_1 + c_2] \in H_n^{X_1 X_2}(X)$, and so $H_n(i)$ is injective.

d. This part will assume that $X_1 - X_2$ can be excised from the pair (X, X_1) and prove that $H_n(i)\colon H_n^{X_1 X_2}(X) \to H_n(X)$ is surjective.

Let $[c] \in H_n(X)$, which meaning that the representative $c \in C_n$ is in the kernel $\ker(\partial_n)$. We will construct $c_1 \in H_n(X_1)$ and $c_2 \in H_n(X_2)$ such that $c = c_1 + c_2$.
By the isomorpishm $H_n(j)$ there must exist $[c] \cong [c_2]$, namely $c = c_2 + \partial c_{12}$. Therefore $c = c_1 + c_2$ and

$$H_n(i)([c_1 + c_2]) = [c] \in H_n(X),$$

so $H_n(i)$ is surjective.

e. Given that $(X_2, X_1 \cap X_2) \to (X, X_1)$ is an excision implies that $H_n(j_1)$ is an isomorphism. Thus the previous two parts showed that $H_n(i)$ is also an isomorphism, so $c = c_1 + c_2$. Thus reversing the roles in the first two parts shows that $H_n(j_2)$ is also an isomorpism, meaning that $(X_1, X_1 \cap X_2) \to (X, X_2)$ is an excision.

$\square$