Ron Graham's Sequence

and the On-Line Encylopedia of Integer Sequences

Peter Kagey

University of Southern California

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Genesis

Putnam 2013 A2

Consider choices of integers $b_1, b_2, \ldots, b_t$ such that $n < b_1 < b_2 < \cdots < b_t$ and $n \cdot b_1 \cdot b_2 \cdots b_t$ is a perfect square, and let $g(n)$ be the minimum of b_t over all such choices.

For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4,$ $2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $g(2) = 6$.

Show that the function g from $\mathbb{N}$ to $\mathbb{N}$ is one-to-one.

The On-Line Encyclopedia of Integer Sequences

- The On-Line Encyclopedia of Integer Sequences was started by Neil Sloane in 1964 as a graduate student at Cornell.
- In 1996, the database made its way to the internet.
- The OEIS contains over 300000 sequences.
- It is a good resource for finding references, generating functions, recurrences, plots, etc. about integer sequences.
- Roughly 100 new sequences are added every day after being reviewed by a team of volunteer editors.

Definition (OEIS A006255)

Let $g(n)$ be the smallest m for which there exists a strictly increasing **square sequence**:

$$n = b_1 < b_2 < \cdots < b_t = m$$

where

$$b_1 \cdot b_2 \dots b_t \text{ is square}$$

Examples

$$\begin{aligned}
g(2) &= 6 \qquad & 2 \cdot 3 \cdot 6 &= 6^2 \\
g(3) &= 8 \qquad & 3 \cdot 6 \cdot 8 &= 12^2 \\
g(4) &= 4 \qquad & 4 &= 2^2 \\
g(5) &= 10 \qquad & 5 \cdot 8 \cdot 10 &= 20^2 \\
&\dots \\
g(14) &= 21 \qquad & 14 \cdot 15 \cdot 18 \cdot 20 \cdot 21 &= 1260^2
\end{aligned}$$

Naïve, brute-force algorithm

Example: Computing $g(3)$

All strictly increasing sequences of numbers starting with 3 until a square sequence is reached:

$$
\begin{aligned}
\sqrt{3} &= \sqrt{3} \\
\sqrt{3 \cdot 4} &= 2\sqrt{3} \\
\sqrt{3 \cdot 5} &= \sqrt{15} \\
\sqrt{3 \cdot 4 \cdot 5} &= 2\sqrt{15} \\
\sqrt{3 \cdot 6} &= 3\sqrt{2} \\
\cdots & \\
\sqrt{3 \cdot 6 \cdot 8} &= 12
\end{aligned}
$$

There are heuristics can be used to filter out many sequences, but in general this procedure takes exponential time.

Background

- This problem first appeared as problem proposed by Ron Graham in the June 1987 edition of *MAA Mathematics Magazine*.
- The problem later appears in September 1988 in the first edition of *Concrete Mathematics* in by Ron Graham, Don Knuth, and Oren Patashnik.
- Appears in the OEIS in 1991 with 25 known terms.
- Extended to 54 terms in 1994, 70 terms in 2003, and 125 terms in 2014.
- At the end of 2014, the sequence was extended to 10,000 terms and six errors between the 84th and 104th terms were corrected.

Info

- $g(n) \leq 4n$ because $n \cdot 4n = (2n)^2$
- $g(n) \leq 2n$ for $n \geq 4$
  - For $n \geq 10$ there exists k with $n < 2k^2 < 2n$.
- In particular, $g(p) = 2p$ for any prime $p > 3$.
- $g(n)$ is a bijection from the natural numbers to the non-prime numbers.
- For any fixed integer k, $g(k \cdot p) = (k+1) \cdot p$ if p is a sufficiently large prime.

2013 Putnam A2: g is one-to-one.

Proof.

Assume that $m < n \leq g(m) = g(n)$ and denote the product sequences as:

$$A = \{n = a_1, a_2, \cdots, a_t = g(n)\}$$
$$B = \{m = b_1, b_2, \cdots, b_s = g(m)\}$$

Then the symmetric difference $A \triangle B$ defines a sequence which contradicts the hypothesis:

1. $\min(A \triangle B) = m$.
2. $\max(A \triangle B) < g(m)$.
3. $\mathrm{prod}(A \triangle B)$ is square.

$\square$

The idea

Consider the prime factorization of each of the elements in $\{n, n+1, \ldots, 2n\}$, and "pair up" the exponents.

Example

$$g(8) = 15 \text{ via } 8 \cdot 10 \cdot 12 \cdot 15$$
$$2^3 \cdot (2 \cdot 5) \cdot (2^2 \cdot 3) \cdot (3 \cdot 5)$$

The pairing up of the exponents can be realized as vector addition over $\mathbb{F}_2$.
In particular, we can define a function $v_k : \mathbb{N} \to \mathbb{F}_2^k$ to represent the parity of the exponents of an integer.

$$v_5(2 \cdot 3^2 \cdot 5^3) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 5 \\ \leftarrow 7 \\ \leftarrow 11 \end{array}$$

The algorithm

Set up a linear system of the powers of the exponents in the prime factorization so that we can pair them up.

Definition

Let $\mathcal{M}_n \in M_{\pi(2n)\times(n+1)}(\mathbb{F}_2)$ be defined as the following system of equations. (Here $v = v_{\pi(2n)}$, where $\pi(n)$ counts primes $\leq n$.)

$$\mathcal{M}_n = \left[\begin{array}{ccccc|c} | & | & & | & | & | \\ v(n+1) & v(n+2) & \cdots & v(2n-1) & v(2n) & v(n) \\ | & | & & | & | & | \end{array}\right]$$

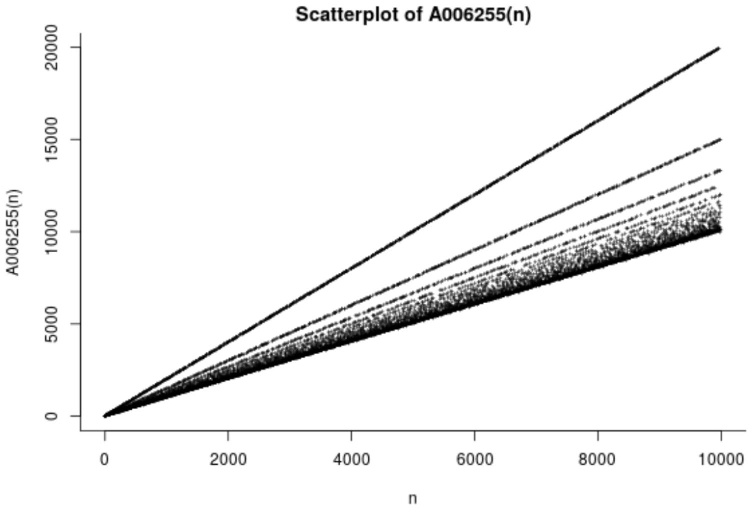
Example: $\mathcal{M}_8$

$$\mathcal{M}_8 = \begin{array}{c} \begin{array}{ccccccccc} 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 5 \\ \leftarrow 7 \\ \leftarrow 11 \\ \leftarrow 13 \end{array}$$

Example: rref$(\mathcal{M}_{14})$

$$
\begin{array}{c}
\begin{array}{ccccccccccccccc}
15 & & & 18 & & 20 & 21 & & & & & & & & 14 \\
\downarrow & & & \downarrow & & \downarrow & \downarrow & & & & & & & & \downarrow
\end{array} \\
\left[\begin{array}{cccccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

A scatterplot of A006255.



Counting valid sequences (OEIS A259527)

Example

For $n = 20$ there are four square sequences starting with 20 and ending with $g(20) = 30$:

$$
\begin{aligned}
20 \cdot 24 \cdot 30 &= 120^2 \\
20 \cdot 24 \cdot 25 \cdot 30 &= 600^2 \\
20 \cdot 21 \cdot 24 \cdot 27 \cdot 28 \cdot 30 &= 15120^2 \\
20 \cdot 21 \cdot 24 \cdot 25 \cdot 27 \cdot 28 \cdot 30 &= 75600^2
\end{aligned}
$$

How many square sequences in general?

Counting valid sequences (OEIS A259527)

Once g(n) has been computed, a similar algorithm can be used to compute the number of sequences:

$$n = b_1 < b_2 < \ldots < b_t = g(n)$$

Again, we set up a system of equations (where $v = v_{\pi(g(n))}$):

$$\mathcal{C}_n = \begin{bmatrix} | & | & & | & | \\ v(n) & v(n+1) & \cdots & v(g(n)-1) & v(g(n)) \\ | & | & & | & | \end{bmatrix}$$

And the number of valid sequences is:

$$2^{\mathrm{nul}(C_n)-1}$$

Counting valid sequences (Example)

$$
\mathcal{C}_{20} = \begin{array}{c}
\begin{array}{ccccccccccc}
20 & 21 & & & & \cdots & & & & 29 & 30 \\
\downarrow & \downarrow & & & & & & & & \downarrow & \downarrow
\end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

A basis for the null space of $\mathcal{C}_{20}$

$$\left\{ \begin{bmatrix}1\\0\\0\\0\\1\\0\\0\\0\\0\\0\\1\end{bmatrix} \begin{array}{l} \leftarrow \; 20\\ \\ \\ \\ \leftarrow \; 24\\ \\ \\ \\ \\ \\ \leftarrow \; 30\end{array}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\1\\1\\0\\0\end{bmatrix} \begin{array}{l} \\ \leftarrow \; 21\\ \\ \\ \\ \\ \\ \leftarrow \; 27\\ \leftarrow \; 28\\ \\ \\ \end{array}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow \; 25\\ \\ \\ \\ \\ \\ \end{array} \right\}$$

The basis can be constructed in such a way that only one vector has 1 as its first and last entry.

Number of square sequences: $2^{\mathrm{nul}(C_n)-1}$

Generalize to cubes (OEIS A227494)

Let $g_3(n)$ be the least m for which there exists a sequence

$$n = b_1 < b_2 \leq b_3 \leq \cdots \leq b_t = m$$

where

$$b_1 \cdot b_2 \cdot b_3 \cdot \ldots \cdot b_t \text{ is a cube.}$$

Example

$$\begin{aligned}
g_3(1) &= 1 && \to 1 && = 1^3 \\
g_3(2) &= 4 && \to 2 \cdot 4 && = 2^3 \\
g_3(3) &= 6 && \to 3 \cdot 4^2 \cdot 6^2 && = 12^3 \\
g_3(4) &= 9 && \to 4 \cdot 6 \cdot 9 && = 6^3 \\
g_3(5) &= 10 && \to 5 \cdot 6 \cdot 9 \cdot 10^2 && = 30^3
\end{aligned}$$

Generalize to cubes (OEIS A227494)

$$\mathcal{M}_{3,3} = \begin{array}{cc} \begin{array}{cccc} 4 & 5 & 6 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} & \\ \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right] & \begin{array}{c} \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 5 \end{array} \end{array}$$

$$\mathrm{rref}(\mathcal{M}_{3,3}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus $g_3(3) = 6$ via $3 \cdot 4^2 \cdot 6^2$ given by the solution

$$\begin{bmatrix} 2 & 0 & 2 & 1 \end{bmatrix}^T$$

Generalizations

- Powers greater than 3.
  - Powers that are prime follow a similar construction.
  - The above construction does not work for composite powers.
- Counting number of subsequences of a finite sequence with a square product.
- Sequences such that the LCM is square (OEIS A300516).
- Sequences such that no proper nonempty subsequence has a square product.

Final Conjecture

Conjecture (Robert G. Wilson v, 2002)

$$n \cdot g(n) \text{ is nonsquare for all nonsquare } n \in \mathbb{N}$$