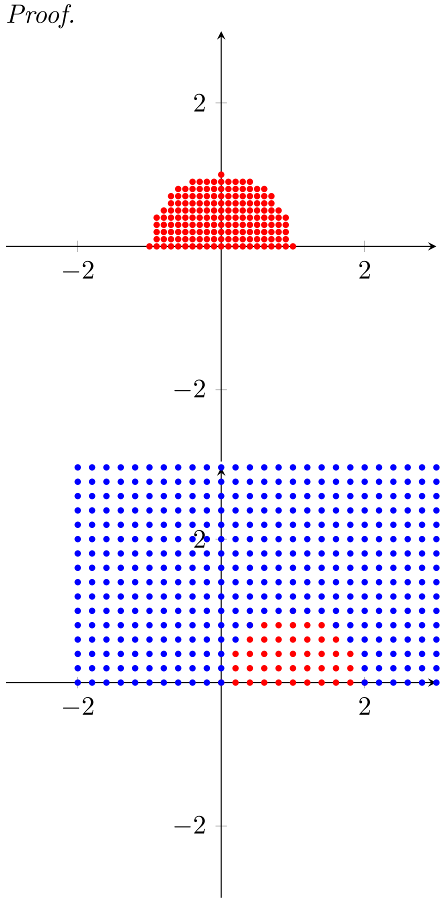
Complex Analysis: Homework 4

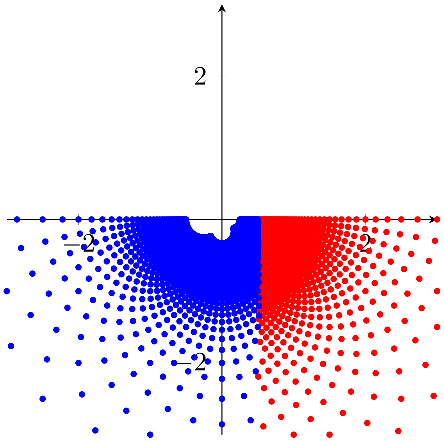
Peter Kagey

February 6, 2018

Problem 3. (page 96)
Construct a map f from the set $\{x+yi : \; y \geq 0, x^2+y^2 \neq 1\}$ onto to the set $\{z : \; |z| > 1\}$ so that $f(\infty) = \infty$.

Proof.




2
−2
2
−2

□

Problem 4. (page 96)
Construct a map f from $\{x + yi : y^2 > 2px\}$ onto $\{z : |z| < 1\}$ so that $f(0) = 1$ and $f(-p/2) = 0$.

Proof.

$\square$

Problem 7. (page 96)
Construct a map f from $\{x+yi : (x/a)^2+(y/b)^2 > 1\}$ onto $\{z : |z| < 1\}$ such that symmetries are preserved.

Proof.

$\square$

Problem 1. (page 99)
Describe the Riemann surface associated with the function

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Proof.
Fixing w_0 and solving for z yields a quadratic equation $z^2 - 2wz + 1 = 0$ with two solutions

$$\frac{w \pm \sqrt{w^2 - 1}}{2}.$$

The discriminant is 0 only when $w = 1$ or $w = -1$; that is, when $z = 1$ or $z = -1$. Therefore $z = 1$ and $z = -1$ are the only points in both copies of the plane. $\square$

Problem 1. (page 108)
Compute
$$\int_\gamma x dz$$
where γ is the directed line segment from 0 to 1 + i.

Proof.
Let $z(t) = t(1+i)$ on $[0,1]$. Then
$$\int_\gamma z\ dz = \int_\gamma \mathrm{Re}(z)\ dz = \int_0^1 \mathrm{Re}(z(t))z'(t)\ dt = (1+i)\int_0^1 t\ dt = \frac{1+i}{2}$$
$\square$

Problem 2. (page 108)
Compute

$$\int_{|z|=r} x \, dz,$$

for the positive sense of the circle in two ways: first by use of a parameter, and second, by observing that $x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

Proof.

1. Let $z(t) = r\cos(t) + ir\sin(t)$ for $t \in [0, 2\pi]$. Then

$$r^2 \int_{|z|=r} x \, dz = \int_0^{2\pi} \cos(t)(-\sin(t) + i\cos(t)) \, dt = r^2 \int_0^{2\pi} -\cos(t)\sin(t) \, dt + ir^2 \int_0^{2\pi} \cos^2(t) \, dt.$$

The first integral can be evaluated via substitution. Let $u = \cos(t)$ and $du = -\sin(t)$. Then

$$\int_{t=0}^{2\pi} u \, du = \left[\frac{u^2}{2}\right]_{t=0}^{2\pi} = \left[\frac{\cos^2(t)}{2}\right]_{t=0}^{2\pi} = 0.$$

The second integral can be evaluated via a trigonometic substitution.

$$ir^2 \int_0^{2\pi} \cos^2(t) dt = ir^2 \int_0^{2\pi} \frac{\cos(2t) + 1}{2} dt = ir^2 \left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_0^{2\pi} = ir^2\pi.$$

Thus

$$\int_{|z|=r} x \, dz = ir^2\pi.$$

2.

□

Problem 5. (page 108)
Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary.

Proof.

$\square$