Complex Analysis: Homework 8

Peter Kagey

March 8, 2018

Problem 1. (page 161)
Find the poles and residues of the following functions

(a) $\dfrac{1}{z^2 + 5z + 6}$

(b) $\dfrac{1}{(z^2 - 1)^2}$

(c) $\dfrac{1}{\sin z}$

(d) $\cot z$

(e) $\dfrac{1}{\sin^2 z}$

(f) $\dfrac{1}{z^m(1-z)^n}$ $(m,\ n$ positive integers).

Proof.

(a) First note that the denominator factors as

$$x^4 + 5x^2 + 6 = (x^2+2)(x^2+3) = (x+i\sqrt{2})(x-i\sqrt{2})(x+i\sqrt{3})(x-i\sqrt{3})$$

Thus there are four poles:

$$-i\sqrt{2}, i\sqrt{2}, -i\sqrt{3}, \text{ and } i\sqrt{3}$$

Next, looking at B_1 in the expansion $f(z) = B_h(z-z_0)^{-h} + \ldots + B_1(z-z_0)^{-1} + \varphi(z)$ immediately yields

(i) $\operatorname{Res}_{z=-i\sqrt{2}} f(z) = \dfrac{1}{(-i\sqrt{2}-i\sqrt{2})((-i\sqrt{2})^2+3)} = \dfrac{-1}{2i\sqrt{2}}$

(ii) $\operatorname{Res}_{z=i\sqrt{2}} f(z) = \dfrac{1}{(i\sqrt{2}+i\sqrt{2})((i\sqrt{2})^2+3)} = \dfrac{1}{2i\sqrt{2}}$

(iii) $\operatorname{Res}_{z=-i\sqrt{3}} f(z) = \dfrac{1}{(-i\sqrt{3}-i\sqrt{3})((-i\sqrt{3})^2+2)} = \dfrac{1}{2i\sqrt{3}}$

(iv) $\operatorname{Res}_{z=i\sqrt{3}} f(z) = \dfrac{1}{(i\sqrt{3}+i\sqrt{3})((i\sqrt{3})^2+2)} = \dfrac{-1}{2i\sqrt{2}}$

(b) First note that the denominator factors as

$$(z^2-1)^2 = (z+1)^2(z-1)^2$$

so there are two poles of order 2: -1 and 1. Thus the resides are

(i) $\operatorname{Res}_{z=-1} f(z) = \dfrac{d}{dz}\left[\dfrac{1}{(z-1)^2}\right]_{z=-1} = \dfrac{1}{4}$

(ii) $\mathrm{Res}_{z=1} f(z) = \dfrac{d}{dz}\left[\dfrac{1}{(z+1)^2}\right]_{z=1} = -\dfrac{1}{4}$

(c) First note that $\sin z$ has zeros of order 1 precisely at $z = 2\pi k$ for some $k \in \mathbb{N}$; therefore $1/\sin(z)$ has poles of order 1 when $z = 2\pi k$. Thus the residue at each pole is

$$\begin{aligned}
\lim_{z\to 2\pi k} \frac{(z-2\pi k)}{\sin z} &= \lim_{z\to 2\pi k} \frac{(z-2\pi k)}{(z-2\pi k) - (z-2\pi k)^3/3! + \ldots} \\
&= \lim_{z\to 2\pi k} \frac{1}{1-(z-2\pi k)^2/3! + \ldots} \\
&= 1
\end{aligned}$$

(d) Note that $\cot z = \dfrac{\cos z}{\sin z}$ has the same poles as above, all of order 1: $z = 2\pi k$. Thus the residues at each pole are

$$\begin{aligned}
\lim_{z\to 2\pi k}(z-2\pi k)(\cot z) &= \lim_{z\to 2\pi k}(z-2\pi k)\frac{1-(z-2\pi k)^2+\ldots}{(z-2\pi k)-(z-2\pi k)^3/3!+\ldots} \\
&= \lim_{z\to 2\pi k} \frac{1-(z-2\pi k)^2+\ldots}{1-(z-2\pi k)^2/3!+\ldots} \\
&= 1
\end{aligned}$$

(e) Now the poles are at $z = 2\pi k$, but they are of order 2.

$$\frac{d}{dz}\left[\frac{1}{\sin^2 z}\right]_{z=2\pi k}$$

(f) Clearly this final function has a pole of order m at $z = 0$, and of n at $z = 1$. Because a curve around an isolated singularity z_0

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \mathrm{Res}_{z=z_0} f(z)$$

and

$$\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{z^m}\right]_{z=1} = \frac{1}{2\pi i}\int_\gamma \frac{-z^{-m}}{(-1)^n(1-z)^n}\,dz$$

so

$$\mathrm{Res}_{z=1} f(z) = \frac{(-1)^n}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{z^m}\right]_{z=1} = \frac{(-1)^n\cdot(-m)(-m+1)\cdots(-m+n-2)}{(n-1)!}$$

The other residue falls similarly

$$\mathrm{Res}_{z=0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[\frac{1}{(1-z)^n}\right]_{z=0} = \frac{(-n)(-n+1)\cdots(-n+m-2)}{(m-1)!}.$$

□

Problem 3. (page 161)
Evaluate the following integrals by the method of residues:

(b) $\displaystyle\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6}$

(g) $\displaystyle\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx$

(h) $\displaystyle\int_0^\infty (1+x^2)^{-1} \log x \, dx$

(i) $\displaystyle\int_0^\infty \log(1+x^2)\frac{dx}{x^{1+\alpha}}$ $(0<\alpha<2)$.

Proof.

(b) Because the integral is even, we can exploit that

$$\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2 dx}{x^4+5x^2+6}$$

First note that the denominator $x^4+5x^2+6=(x^2+2)(x^2+3)$ has no real roots. Now integrating the complex function over Γ_R, the semicircle of radius R in the upper half plane centered at the origin gives

$$\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6} = \frac{1}{2}\left(\lim_{R\to\infty}\int_{\Gamma_R}\frac{z^2 dz}{z^4+5z^2+6} - \int_0^\pi \frac{(Re^{it})^2}{(Re^{it})^4+5(Re^{it})^2+6}\cdot ie^{it}\, dt\right)$$

where the final integral vanishes in the limit and the integral over Γ is given by

$$\begin{aligned}\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6} &= \frac{1}{2}\cdot 2\pi i(\operatorname{Res}_{z=i\sqrt{2}} f(z) + \operatorname{Res}_{z=i\sqrt{3}} f(z))\\ &= \frac{1}{2}\cdot 2\pi i\left(\frac{(i\sqrt{2})^2}{-2i\sqrt{2}} - \frac{i\sqrt{3})^2}{2i\sqrt{3}}\right)\\ &= \frac{\pi}{2}\left(\sqrt{3}-\sqrt{2}\right)\end{aligned}$$

using the residues calculated by a similar method to Problem 1(a).

(g) Here we want to avoid the branch cut at $\theta = 2\pi/3$, we will perform the substitution $x=t^2$ transforming the integral into

$$2\int_0^\infty \frac{t^{5/3}}{1+t^4} dt$$

and choosing the branch of $t^{2/3}$ whose argument lies between $-\pi/3$ and π. By Ahlfors argument,

$$\int_{-\infty}^\infty \frac{t^{5/3}}{1+t^4} dt = (1-e^{2\pi i/3})\int_0^\infty \frac{t^{5/3}}{1+t^4} dt$$

On the first integral, we can use the techniq from part (b), and take the residues from the poles in the upper half plane (which are $z=e^{\pi i/4}$ and $z=e^{3\pi i/4}$). These are simple enough to compute as

$$\frac{(e^{\pi i/4})^{5/3}}{(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{5\pi i/4})(e^{\pi i/4}-e^{7\pi i/4})} \text{ and } \frac{(e^{3\pi i/4})^{5/3}}{(e^{3\pi i/4}-e^{\pi i/4})(e^{3\pi i/4}-e^{5\pi i/4})(e^{3\pi i/4}-e^{7\pi i/4})}.$$

Therefore

$$\int_0^\infty \frac{t^{5/3}}{1+t^4}\, dt = \frac{1}{1-e^{2\pi i/3}}\left(\sum_{y>0}\operatorname{Res}_{z=z_0} f(z)\right)$$

where the residues are the two long fractions above.

(h) Here we will integrate over the boundary of the half annulus in the upper half plane, with

$$\Gamma_1 = [\epsilon, R] \tag{1}$$
$$\Gamma_2 = \{Re^{it} : t \in [0, \pi]\} \tag{2}$$
$$\Gamma_3 = [-\epsilon, R] \tag{3}$$
$$\Gamma_4 = \{\epsilon e^{-it} : t \in [0, \pi]\} \tag{4}$$

and choosing the branch cut of log to be the negative imaginary axis, so that $\log z = \log |x| + i \arg z$ where $-\pi/2 < \arg z < 3\pi/2$. Now using the residue theorem, the integral around the boundary of the half annulus vanishes. Also, the integrals of the contours around the semicircles vanishes as $\epsilon \to 0$ and $R \to \infty$, and the integral of the negative real axis also vanishes. Thus the entire integral must vanish.

$\square$