Math 510b: Midterm

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Problem 1.

Proof.

(a) (The idea here is that there is an increasing number of irreducible factors of each generator as you go down the chain. Since any element in R has a finite number of irreducible factors, no element can be be in all of the ideals.)

Since R is a PID, each of the ideals in the chain can be written as principal ideals, that is $I_i = (a_i)$ for some $a_i \in R$,

$$(a_1) \supsetneq (a_2) \supsetneq \dots$$

where $a_i \mid a_{i+1}$ for all $i \in \mathbb{N}_{>0}$, and thus $a_{i+1} = b_i a_i$ where b_i is not a unit (otherwise $(a_{i+1}) = (a_i)$).

Since a PID is a UFD, we can write each of these uniquely (up to units) as the product of irreducible elements of R, namely

$$a_n = \underbrace{b_{n,1} \dots b_{n,k_n}}_{b_n} \dots \underbrace{b_{2,1} \dots b_{2,k_2}}_{b_2} \underbrace{a_{1,1} \dots a_{1,k}}_{a_1}$$

(with $k_i \geq 1$) so the number of irreducible factors of a_n strictly increases as n increases. Since every $r \in R$ has finitely many irreducible factors (say m_r of them), r cannot be in the intersection of the ideals, because in particular $r \notin (a_{m_{r+2}})$. Thus the intersection of all of the ideals in the descending chain must be the zero ideal.

(b) Let $R = \mathbb{R}[x, y]$, which is a UFD, but not a PID. Then let $I_n = (x, y^n)$. It is clear that $I_{n+1} \subsetneq I_n$ and that $x \in I_n$ for all n, so $x \in \bigcup_{n=1}^{\infty} I_n \neq (0)$.

$\square$

Problem 2.

Proof.

(a) First, notice that $p(x,y,z) = x^2z^3 + (xy - y^2)z - x^3y$ is irreducible by the Eisenstein criteria viewed as a function of z over $\mathbb{Q}[x,y]$ with prime y. (In particular, $y \nmid x^2$, $y \mid (xy - y^2)$, $y \mid x^3y$, and $y^2 \nmid x^3y$.) So the polynomial is irreducible in $\mathbb{Q}[x,y][z]$. Similarly, viewed as a function of x with prime z it is irreducible over $\mathbb{Q}[y,z][x]$, and viewed as a function of y with prime x it is irreducible over $\mathbb{Q}[x,z][y]$. Thus it is irreducible in $\mathbb{Q}[x,y,z]$.

Next, suppose $[f(x,y,z)], [g(x,y,z)] \neq [0] \in \mathbb{Q}[x,y,z]/(p(x))$, then $[f(x,y,z)g(x,y,z)] \neq [0]$ because if $p(x,y,z) \nmid f(x,y,z)$ and $p(x,y,z) \nmid g(x,y,z)$, then $p(x,y,z) \nmid f(x,y,z) \cdot g(x,y,z)$ since $\mathbb{Q}[x,y,z]$ is a UFD, and in a UFD all irreducible elements are prime.

Thus S is an integral domain.

(b) This follows directly by Corollary 5.39 (iii) which states

> For any ideal I in $k[x_1, \ldots, x_n]$ where [...] k is a field, the quotient ring $k[x_1, \ldots, x_n]/I$ is noetherian.

This occurs because generators descend via the quotient map.

(c) Rotman defines a **Jacobson radical** $J(R)$ as the intersection of all maximal left (or right) ideals in R. Proposition 7.15 (i) states that $x \in J(R)$ if and only if $1 - rx$ has a left inverse for every $r \in R$. Since $p(x,y,z)$ is irreducible, $1 - rx$ having a left inverse is equivalent to $rx = 0$. However, because R/I is a UFD, so $rx = 0$ only when $x = 0$. Thus the only element in the intersection of all maximal ideals is 0.

(d) Let $[x] = x + R \in R/I$. Then $\mathfrak{m} = ([x-1], [y-1], [z-1])$ is a maximal ideal with $(R/I)/\mathfrak{m} = \mathbb{Q}$. Notice that in this quotient, $[x] = [y] = [z] = 1$, and this is well-defined since

$$p(1,1,1) = \underbrace{1^2 1^3}_{1} + \underbrace{(1 \cdot 1 - 1^2)1}_{0} - \underbrace{1^3 \cdot 1}_{1} = 0$$

so under this substitution map

$$s([f(x,y,z)]) = f(1,1,1) = f(1,1,1) + \underbrace{p(1,1,1)}_{0} q(x,y,z) = [f(x,y,z) + p(x,y,z)q(x,y,z)].$$

Therefore if $[[q]]$ is the image of $[q] = q + I$ under the quotient map (with respect to $\mathfrak{m}$) then $(R/I)/\mathfrak{m} = \mathbb{Q}$ with $[[q+I]] \mapsto q$ for all $q \in \mathbb{Q}$.

$\square$

Problem 3.

Proof. Let R_0 be the subalgebra of R generated by the entries of all the matrices A_i, and let $S_0 = M_n(R_0)$, as per the hint. The number of entries in all of the m matrices is at most mn^2, so R_0 is a finitely generated commutative k-algebra. Since k is (presumably) a field and thus Noetherian, thus by Hilbert's Basis Theorem, since R_0 is a finitely generated commutative k-algebra, it is also a Noetherian ring.

Thus it is sufficient to show that S_0 is finitely generated, because a finitely generated module over a Noetherian ring is a Noetherian module. If R_0 includes 1 (that is $R_0 = R$), then S_0 is finitely generated because each matrix in S_0 can be written as

$$\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix} = r_{11} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + r_{12} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \dots + r_{nn} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

so S_0 is generated by n^2 elements in M. But if R_0 does not include 1, we can exploit the fact that each entry in S_0 is finitely generated, so each element of S_0 can be written as

$$\begin{aligned}
&r_{11}^{(1)} \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} && + r_{12}^{(1)} \begin{bmatrix} 0 & a_1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \dots + r_{nn}^{(1)} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{bmatrix} \\
&+ r_{11}^{(2)} \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} && + r_{12}^{(2)} \begin{bmatrix} 0 & a_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \dots + r_{nn}^{(2)} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_2 \end{bmatrix} \\
&+ \dots + \\
&+ r_{11}^{(m)} \begin{bmatrix} a_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} && + r_{12}^{(m)} \begin{bmatrix} 0 & a_m & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \dots + r_{nn}^{(m)} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{bmatrix}
\end{aligned}$$

where $R_0 = (a_1, \dots, a_m)$. Thus S_0 is finitely generated over a Noetherian ring and thus is a Noetherian module and algebra. $\square$

Problem 4.

Proof. The **Hilbert's Basis Theorem** states simply that that if R is a Noetherian ring, then $R[x]$ is also a Noetherian ring.

(a) I chose this theorem because it allows us to construct lots of natural examples of Noetherian rings which turn out to be very familiar and important, like $\mathbb{R}[x, y, z]$, $\mathbb{C}[x, y]$ and $k[x]$ (where k is a field).

The Wikipedia page for Hilbert's Basis Theorem lists two specific applications

(i) "Since any affine variety over R^n may be written as the locus of an ideal $\mathfrak{a} \subset R[X_0, \ldots, X_{n-1}]$ and further as the locus of its generators, it follows that every affine variety is the locus of finitely many polynomials—i.e. the intersection of finitely many hypersurfaces."

(ii) "If A is a finitely-generated R-algebra, then we know that $A \simeq R[X_0, \ldots, X_{n-1}]/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal. The basis theorem implies that $\mathfrak{a}$ must be finitely generated, [...] i.e. A is finitely presented."

At nLab it's mentioned that one reason to care about a ring being Noetherian at all is because it allows for induction over its ideals since, by one definition, a noetherian ring is one that satisfies the ascending chain condition on ideals, i.e. for any chain

$$I_1 \subseteq \ldots \subseteq I_j \subseteq I_{j+1} \subseteq \ldots$$

there exists some large N such that $I_n = I_{n+1}$ for all $n \geq N$

(b) The proof assumes that R is noetherian and J is a nonzero ideal of $R[x]$, and shows that J is finitely generated.

First one considers all ideals of F defined by $I_m = \{r \in R : rx^m + a_{m-1}x^m - 1 + \ldots + a_0 \in J\}$ and notes that I_j is an ideal of R and $I_j \subseteq I_{j+1}$, so that we can use the ascending chain condition on R—this means that eventually $I_n = I_{n+1}$ for all $n \geq N$. Since R is noetherian, all rings are finitely generated so $I_N = (a_1, \ldots, a_m)$. Next the proof constructs an ideal $J' = (f_1, \ldots, f_m)$ where each f_i is in I and has leading coefficient a_i, which by definition is contained in J.

The remainder of the proof shows (by contradiction) that J is also contained in J'. In particular, it assumes there is some polynomial $g \in J \setminus J'$, chosen to be of minimal degree, and then constructs a polynomial of smaller degree: In particular, it constructs another function of degree $\deg(g)$ in I with the same leading coefficient, and the difference of these two functions is a polynomial in I of strictly smaller degree.

$\square$