Differential Geometry: Homework 2

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Problem 1. Let V be a vector space of a field k and let $W \subset V$ be a subspace.
(a) Show that V/W is a vector space, with operations induced by those of V in the following sense: for α and β in V/W, choose elements a and b with $[a] = \alpha, [b] = \beta$, and define $\alpha + \beta = [a + b]$ and $c \cdot \alpha = [c \cdot a]$

Proof.
Addition is well-defined.
Let $\alpha = [a_1] = [a_2]$, and let $\beta = [b_1] = [b_2]$. Then by definition of the equivalence class $\sim$, $a_1 - a_2 \in W$ and $b_1 - b_2 \in W$. In order to show that that $\alpha + \beta = [a_1 + b_1] = [a_2 + b_2]$ is well-defined, it is sufficient to show that $a_1 + b_1 \sim a_2 + b_2$. By closure of W under addition,

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) - (b_1 - b_2) \in W$$

Multiplication is well-defined.
Let $\alpha = [a_1] = [a_2]$ and $c \in k$. Then by definition of the equivalence class $\sim$, $a_1 - a_2 \in W$. In order to show that that $c \cdot \alpha = [c \cdot a_1] = [c \cdot a_2]$ is well-defined, it is sufficient to show that $c \cdot a_1 \sim c \cdot a_2$. By distributivity laws and closure of W under scalar multiplication,

$$c \cdot a_1 - c \cdot a_2 = c(a_1 - a_2) \in W$$

V/W is an abelian group under $+$. *Associativity.*
This is more-or-less inherited from the associativity of addition in V.

$$(\alpha + \beta) + \gamma = [(a + b) + c] = [a + (b + c)] = \alpha + (\beta + \gamma).$$

Commutivity.
This is more-or-less inherited from the commutivity of addition in V.

$$(\alpha + \beta) = [a + b] = [b + a] = \beta + \alpha.$$

Identity element.
Let $0 = [\vec{0}]$ where $\vec{0} \in W$.

$$\alpha + 0 = [a + \vec{0}] = [a] = \alpha$$
$$0 + \alpha = [\vec{0} + a] = [a] = \alpha$$

Inverse element.
Let $\alpha = [a]$, then $-\alpha = [-a]$.

$$\alpha + -\alpha = [a + -a] = [\vec{0}] = e$$
$$-\alpha + \alpha = [-a + a] = [\vec{0}] = e$$

Multiplication is well-behaved.

(i) $1 \cdot \alpha = [1 \cdot a] = [a] = \alpha$.

(ii) $c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot [c_2 \cdot a] = [c_1 \cdot (c_2 \cdot a)] = [(c_1 c_2) \cdot a] = (c_1 c_2) \cdot \alpha$.

(iii) $c_1 \cdot (\alpha + \beta) = c_1 \cdot [a + b] = [c_1 \cdot (a + b)] = [c_1 \cdot a + c_1 \cdot b] = [c_1 \cdot a] + [c_1 \cdot b] = c_1 \cdot \alpha + c_1 \cdot \beta$.

(iv) $(c_1 + c_2) \cdot \alpha = [(c_1 + c_2) \cdot a] = [c_1 \cdot a + c_2 \cdot a] = [c_1 \cdot a] + [c_2 \cdot a] = c_1 \cdot \alpha + c_2 \cdot \alpha$

$\square$

Problem 1. (b)
The quotient comes equipped with a natural linear map

$$\begin{aligned} \pi : V &\longrightarrow V/W \\ v &\longrightarrow [v] = v + W, \end{aligned}$$

called the *projection*, which has $\ker \pi = W$ (check that π is linear and has kernel as desired.) Suppose V is finite-dimensional, and let U be a subspace complementary to W, that is a subspace such that $V = W \oplus U$. Show that the restriction of projection to U

$$\pi_U : U \longrightarrow V/W$$

is an isomorphism.

Proof.
To see that π is linear, check its additivity and homogeneity. The additivity of π follows from

$$\pi(v + w) = [v + w] = [v] + [w] = \pi(v) + \pi(w),$$

and the homogeneity of π follows from

$$\pi(c \cdot v) = [c \cdot v] = c \cdot [v] = c \cdot \pi(v).$$

To check that $\ker \pi = W$, see that

$$\ker \pi = \{v \mid \pi(v) = 0\} = \{v \mid v + W = 0 + W\} = \{v \mid v \in W\} = W$$

because W is closed under addition.

$\square$

Problem 1. (c)
Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 3 and 5

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(3) = f(5) = 0\}.$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension?

Proof.
Let $\ell_5(x) = (5 - x)/2$ and $\ell_3(x) = (x - 3)/2$, so that $\ell_5(5) = 0 = \ell_3(3)$ and $\ell_5(3) = 1 = \ell_3(5)$. Because ℓ_5 and ℓ_3 are lines, they are smooth.

Then for any function $f \in C^\infty(\mathbb{R})$

$$\begin{aligned} &f(x) - (f(3)\ell_5(x) + f(5)\ell_3(x)) \in U \text{ because} \\ &f(5) - (f(3)\ell_5(5) + f(5)\ell_3(5)) = f(5) - (0 + f(5)) = 0 \text{ and} \\ &f(3) - (f(3)\ell_5(3) + f(5)\ell_3(3)) = f(3) - (f(3) + 0) = 0. \end{aligned}$$

Since any smooth function f is in a coset with a sum of two smooth functions, $[f(x)] = f(3) \cdot [\ell_5(x)] + f(5) \cdot [\ell_3(x)]$, $C^\infty(\mathbb{R})/U$ has dimension at most two.
Since $\ell_5 \notin U$, and $\ell_3 + U \notin \operatorname{span}\{0 + U, \ell_5 + U\}$, $C^\infty(\mathbb{R})/U$ has dimension exactly two. $\square$

Problem 1. (d)
Let V be a vector space and $W \subset V$ be a vector subspace. We denote the inclusion map by $i : W \to V$. Denote $V^* = \text{Hom}_k(V, k)$. There is a natural induced map $i^* : V^* \to W^*$ dual to the inclusion sending a linear map $\phi \mapsto \phi|_W$. The kernel of i^* is called the *annihilator* of W and denoted

$$\text{Ann}(W) = \{\phi \in V^* \mid \phi|_W = 0 \in W^*\}.$$

It is the set of linear maps from V to k that return 0 on any element in W.

Prove that there is a canonical isomorphism

$$\text{Ann}(W) \cong (V/W)^*.$$

Proof.
The "obvious" maps to consider are

$$\begin{aligned} \psi &: \text{Ann}(W) \to (V/W)^* \ \text{ via } g \mapsto ([\vec{v}] \mapsto g(\vec{v})) \\ \psi^{-1} &: (V/W)^* \ \to \text{Ann}(W) \text{ via } f \mapsto (\vec{v} \ \mapsto f([\vec{v}])) \end{aligned}$$

It is sufficient to show that these maps (i) are well-defined, (ii) satisfy $\psi \circ \psi^{-1} = \text{id}_{(V/W)^*}$, and (iii) satisfy $\psi^{-1} \circ \psi = \text{id}_{\text{Ann}(W)}$.

Proof of (i). For the well-definedness of ψ, it is sufficient to show that if $[\vec{v}] = [\vec{u}]$ then $g(\vec{v}) = g(\vec{u})$. Since $\vec{v} - \vec{u} \in W$, $g(\vec{v} - \vec{u}) = 0 = g(\vec{v}) - g(\vec{u})$, so $g(\vec{v}) = g(\vec{u})$

For the well-definedness of $\psi^{-}1$, it is sufficient to show that for all $\varphi \in \text{Ann}(W)$ if $[\vec{v}_1] = [\vec{v}_2]$ and the map $\varphi(\vec{v}_1) = f([\vec{v}_1])$ then $\varphi(\vec{v}_2) = f([\vec{v}_1])$.

$$\begin{aligned} \varphi(\vec{v}_1 - \vec{v}_2) &= 0 \\ &= \varphi(\vec{v}_1) - \varphi(\vec{v}_2) \end{aligned}$$

So $\varphi(\vec{v}_1) = \varphi(\vec{v}_2) = f([\vec{v}_1])$.

Proof of (ii). Let $f : V/W \to k$ be a linear function.

$$\psi(\psi^{-1}(f)) = \psi(\vec{v} \mapsto f([\vec{v}])) = [\vec{v}] \mapsto f([\vec{v}]) = f$$

Therefore $\psi \circ \psi^{-1} = \text{id}_{(V/W)^*}$

Proof of (iii). Let $g : \text{Ann}(W)$ be a linear function.

$$\psi^{-1}(\psi(g)) = \psi^{-1}([\vec{v}] \mapsto g(\vec{v})) = \vec{v} \mapsto g(\vec{v}) = g$$

Therefore $\psi^{-1} \circ \psi = \text{id}_{\text{Ann}(W)}$.

$\square$

Problem 2. Let
$$S^n = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}.$$
Prove that S^n has the structure of a smooth manifold, using charts associated to the cover $U_N = \{x_1 \neq 1\}$, $U_S = \{x_1 \neq 1\}$.

Proof. Let f_p be a parameterization of a line that begins at $(1, 0, \dots, 0)$ and equals p at time 1:
$$f_p(t) = (1-t)(1, 0, \dots, 0) + t(x_1, \dots, x_{n+1}).$$
This line intersects the subspace $T = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 = 0\} \cong \mathbb{R}^n$ when
$$\begin{aligned} 0 &= (1-t) + tx_1 \\ t &= 1/(1-x_1). \end{aligned}$$
Thus we can define $\phi_N : U_N \to T$ by
$$\begin{aligned} p &\mapsto f_p\left(\frac{1}{1-\pi_1(p)}\right) \\ (x_1, \dots, x_{n+1}) &\mapsto \frac{1}{1-x_1}(0, x_2, \dots, x_{n+1}). \end{aligned}$$
Similarly $\phi_S : U_S \to T$ is defined by
$$(x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+x_1}(0, x_2, \dots, x_{n+1}).$$
The functions ϕ_N and ϕ_S are smooth because the first coordinate is the constant map 0, and the other coordinates are being multiplied by a constant scalar.

The inverse ϕ_N^{-1} is constructed in a similar way. The line f_p intersects S^n when
$$\begin{aligned} 1 &= (1-t)^2 + t^2x_1^2 + \dots + t^2x_{n+1}^2 \\ 2t &= t^2(1 + x_1^2 + \dots + x_{n+1}^2) \\ t &= \frac{2}{1 + ||p||^2} \end{aligned}$$
Thus $\phi_N^{-1} : T \to U_N$ is defined by
$$p = (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+||p||^2}(||p||^2 - 1, 2x_2, 2x_3, \dots, 2x_{n+1}),$$
and $\phi_S^{-1} : T \to U_S$ is defined by
$$p = (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+||p||^2}(1 - ||p||^2, 2x_2, 2x_3, \dots, 2x_{n+1}).$$
The square of the Euclidean norm, $|| \cdot ||^2$, is smooth, so $(1-||p||^2)/(1+||p||^2)$, and $2x_i/(1+||p||^2)$ is smooth. Therefore ϕ_N^{-1} and ϕ_S^{-1} are smooth.

Therefore $\phi_S \circ \phi_N^{-1}$ and $\phi_N \circ \phi_S^{-1}$ are both smooth on $U_N \cap U_S$, and S^n is a smooth manifold. $\square$

Problem 3. Prove that the product of two smooth manifolds

$$(M^m,\ \mathcal{A}_M = \{(U_\alpha,\ \phi_\alpha : U_\alpha \to \mathbb{R}^m)\}_{\alpha\in I}),\ \text{and}$$
$$(N^n,\ \mathcal{A}_N = \{(U_\beta,\ \psi_\beta : U_\beta \to \mathbb{R}^n)\}_{\beta\in J})$$

naturally has the structure of a smooth manifold, with atlas given by

$$\mathcal{A}_{M\times N} = \{(U_\alpha \times V_\beta,\ (\phi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n})\}_{(\alpha,\beta)\in I\times J}.$$

Proof.
It is clear enough (I think!) that if $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ are *topological* manifolds then $M \times N$ is too. Thus it is sufficient to show that for every $(U \times V)_i, (U \times V)_j \in M \times N$ (with $i, j \in I \times J$), that the transition map

$$\omega_j \circ \omega_i^{-1} : \omega_i((U \times V)_i \cap (U \times V)_j) \to \omega_j((U \times V)_i \cap (U \times V)_j)$$

is smooth.

Inheriting from the original manifolds,

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \text{ and}$$
$$\psi_j \circ \psi_i^{-1} : \psi_i(V_i \cap V_j) \to \psi_j(V_i \cap V_j)$$

are smooth. So the product $(\phi_j \circ \phi_i^{-1}) \times (\psi_j \circ \psi_i^{-1}) = \omega_j \circ \omega_i^{-1}$ is therefore smooth.

Thus the product of two smooth manifolds is itself a smooth manifold. □

Problem 4. Prove that the antipodal map $S^n \to S^n$, $x \mapsto -x$ is a diffeomorphism of manifolds.

Proof.
It is enough to check that (i) $f(x) = -x$ is a smooth map, and (ii) it admits a smooth two-sided inverse.

Proof of (i).
Clearly f is a map, due to the symmetry of S^n, if $p \in S^n$ then $-p \in S^n$.
The constant function -1 and the identity function id_{S^n} are smooth. Since the product of smooth functions is smooth, $f = -1 \cdot id_{S^n}$ is smooth.
Proof of (ii).
Existence of the inverse is easy: $f(f(x)) = -(-x) = x$ so $f^{-1} = f$ which has been shown by (i) to be smooth. $\square$

Problem 5. Finish the proof from class that $\mathbb{R}P^n$ is a smooth manifold.

Proof.
We left off with the atlas $\mathcal{A} = \{U_k, \phi_k\}$ where $U_k = \{x_k \neq 0\}/\sim$ and

$$\phi_k\left(\left[\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k}\right]\right) = \left(\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k}\right).$$

It is now sufficient to show that (i) U_k is an open subset of $\mathbb{R}P^n$, and (ii) $\phi_j \circ \phi_k^{-1}$ is smooth.

Proof of (i).
Let $q : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/\sim$ be the projection map from $\mathbb{R}^{n+1}$ to $\mathbb{R}P^n$. Then

$$q^{-1}(U_k) = \{p \in \mathbb{R}^{n+1} \mid \pi_k(p) \neq 0\}$$

which is an open set in $\mathbb{R}^{n+1}$, as can be seen by placing an open ball of radius $\pi_k(p)$ around any point.

Proof of (ii).

$$\begin{aligned}(\phi_j \circ \phi_k^{-1})((x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n)) &= \phi_j([x_0, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n]) \\ &= \phi_j\left(\left[\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{k-1}}{x_j}, \frac{1}{x_j}, \frac{x_{k+1}}{x_j}, \dots, \frac{x_n}{x_j}\right]\right) \\ &= \left(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{k-1}}{x_j}, \frac{1}{x_j}, \frac{x_{k+1}}{x_j}, \dots, \frac{x_n}{x_j}\right)\end{aligned}$$

Where each coordinate is smooth because each coordinate is either

1. A smooth constant function $(x \mapsto 1)$ divided by a projection $(x \mapsto x_j)$, which is smooth or
2. a projection divided by another projection.

Thus the composition $\phi_j \circ \phi_k^{-1}$ is smooth. $\square$

Problem 6. Finish the proof from class that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth 2-manifold.

Proof. We left off with the atlas

$$\mathcal{A} = \{(U_{\vec{x}} = p(B_{1/2}(\vec{x})),\ \mathrm{id}_{\vec{x}} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2)\}_{\vec{x}\in\mathbb{R}^2}$$

where p is the surjective map from $\mathbb{R}^2$ to $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$.
It is sufficient to show that for two points $t, s \in \mathbb{T}^2$,

$$\mathrm{id}_t \circ \mathrm{id}_s^{-1} : \mathrm{id}_s(U_s \cap U_t) \to \mathrm{id}_t(U_s \cap U_t)$$

is smooth.

Because we have shown that id_s and id_s are bijections onto their domains, and the identity map is smooth, $\mathrm{id}_t \circ \mathrm{id}_s^{-1}$ is smooth. In particular, $\mathrm{id}_s(U_s \cap U_t) = \mathrm{id}_t(U_s \cap U_t) = V \subset \mathbb{R}^2$ and $\mathrm{id}_t \circ \mathrm{id}_s^{-1} = \mathrm{id}_{\mathbb{R}^2}|_V$. $\square$