# Spring 2015: Real Analysis Graduate Exam

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**Problem 1.** Consider the sequence

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} cos\left(\frac{x}{n}\right), \quad n = 1, 2, \ldots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\ dx,$$

being careful to justify your answer.

*Proof.* The boundedness of cosine and non-negativity of $1 + \frac{x}{n}$ imply that for $n > 2$ and $x > 0$

$$|f_n(x)| \leq \left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} = g(x).$$

The function $g$ is integrable on $[0, \infty)$ because by the $p$-test

$$\int_0^\infty g(x)\ dx = \int_0^\infty \frac{dx}{1 + x + \frac{x^2}{4}} \leq 4 \int_0^\infty \frac{dx}{x^2} < \infty$$

Thus the Dominated Convergence Theorem with $g$ allows the limit to be moved inside the integral:

$$\lim_n \int_0^\infty f_n(x)\ dx = \int_0^\infty \lim_n f_n(x)\ dx = \int_0^\infty \frac{\lim_n \cos(x/n)}{\lim_n (1 + x/n)^n} dx = \int_0^\infty \frac{\cos(0)}{e^x} dx = \int_0^\infty e^{-x} dx = 1$$

$\square$

**Problem 2.** Suppose that $f : [0, \infty) \to \mathbb{R}$ is Lebesgue integrable.

(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.

(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or counterexample.

(iii) Suppose additionally that $f$ is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or counterexample.

*Proof.* (i) Suppose that there was no point $x_\varepsilon$ such that $|f(x_\varepsilon)| < \varepsilon$. Then $|f| \geq \varepsilon$ for all $\varepsilon > 0$. This means that

$$\int_0^\infty |f| \; dm \geq \int_0^\infty \varepsilon \; dm = \epsilon m((0, \infty)) = \infty,$$

which is a contradiction, because $f$ is integrable by hypothesis. Therefore for each $\varepsilon > 0$, there exists some $x_\varepsilon$ such that $f(x_\varepsilon) < \varepsilon$. Take a sequence such that $|f(x_n)| < 1/n$ for all $n$. This sequence converges to 0.

(ii) Let $f = \mathbb{1}_{\mathbb{Q}}$ be the indicator function for the rational numbers. Then $f$ does not converge to 0, but it is integrable:

$$\int_0^\infty f \; dm = m(\mathbb{Q}) = 0$$

(iii) The idea is that if $f \not\to 0$, then there exists some $\varepsilon$ such that for any $M$, there exists $x_0 > M$ such that $f(x_0) > \varepsilon$. Choose $M$ large enough that $|f'(x)| < 2$, then the area from $x_0$ to the next $x$-intercept must be greater than $\varepsilon$. But this means that the integral $\int_0^\infty |f| \; dm = \sum_{i=1}^\infty \varepsilon = \infty$.

$\square$

**Problem 3.** Define $f_n(x) = a\exp(-nax) - b\exp(-nbx)$ where $0 < a < b$.

(i) Show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)dx = 0$$

and

$$\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\, dx?$$

*Proof.* (i) Notice that

$$\int_0^{\infty} f_n(x)\ dx = \frac{\exp(-nbx)}{n} - \frac{\exp(-nax)}{n}\Big|_0^{\infty} = 0 - \left(\frac{-1}{n} + \frac{1}{n}\right) = 0$$

because $exp(-kx) \to 0$ as $x \to \infty$ for $k > 0$, and $\exp(0) = 1$. (from Daniel Douglas)

(ii)

$\square$

**Problem 4.** Assume that $f$ is integrable on $[0, 1]$ with respect to the Lebesgue measure $m$, and let $F(x) = \int_0^x f(t)dt$. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$\phi(x_1) - \phi(x_2) \leq C(x_1 - x_2), \quad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function $g$ which is integrable on $[0, 1]$ such that $\phi(F(x)) = \int_0^x g(t)\, dt$ for $x \in [0, 1]$.

**Note.** This is similar to problem # from the Spring 20## Real Analysis Exam.

*Proof.* Replace this text with the details of your proof or solution. $\square$