Combinatorics: Homework 11

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Problem 1. Prove that every group of $2n$ children in which every child is friends with at least n other children can be partitioned into pairs of friends in at least two different ways.

Proof.
If there is a Hamiltonian cycle through the friendship network, then the children can be placed in a circle based on this cycle and labeled $v_1, v_2, \ldots, v_{2n}$. Then the children can be paired up $\{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$ or $\{v_2v_3, v_4v_5, \ldots, v_{2n-2}v_{2n-1}, v_{2n}v_1\}$.

Firstly, this graph must be connected; if not, its smallest connected component has at most n vertices, which means that the degree of every vertex in the smallest connected component is at most $n - 1$.

Suppose $v_1v_2 \ldots v_k$ is the longest path in G, which is not a cycle. This means that all of v_k's and v_0's adjacent vertices are in $\{v_1, \ldots, v_{k-1}\}$. Since the $\deg(v_0) > n$ and $\deg(v_k) > n$, by the pigeonhole principle (since $k \leq 2n$), there exists some vertex $v_i \in \{v_1, \ldots, v_{k-1}\}$ such that $v_0v_{i+1} \in E$ and $v_iv_k \in E$. Then the path from v_1 to v_{i+1} to v_k to v_i back to v_1 must be a Hamiltonian cycle—if it's not, then it misses some vertex v_j, which can be used to create a contradiction: in particular a path from v_j to the cycle exists because the graph is connected, and this path followed by a walk around the cycle creates a longer path than the assumed longest path. □

Problem 2. Find the chromatic polynomial for the graph with

$$\begin{aligned}
&V = \{v_1, \ldots, v_n\}\\
&E = \{v_1v_2, v_2v_3, \ldots v_{n-1}v_n, v_1v_n\} \cup \{v_1v_3, v_1v_4, \ldots, v_1v_{n-1}\},
\end{aligned}$$

using the facts that the chromatic polynomial for the cyclic graph C_n is

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

and that the chromatic polynomial for any graph $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{v\}$ is

$$P_G(k) = \frac{1}{k} p_{G_1}(k) p_{G_2}(k).$$

Solution.
There are k ways to pick the color for v_1, $(k-1)$ ways to pick the color for v_2, since v_1 is adjacent to v_2, and then $(k-2)$ ways to pick the color for v_i as i goes from 3 to n, since v_i is adjacent to v_1 and v_{i-1}. Therefore if G_n is the graph with n vertices

$$P_{G_n}(k) = k(k-1)(k-2)^{n-2}.$$

In order to use the hint, contraction/deletion gives

$$P_{G_n}(k) = P_{G'_n}(k) - P_{G''_n}(k)$$

where $G'_n = G_n - v_{n-1}v_n$ and $G''_n = G_n / v_{n-1}v_n$.
With induction hypothesis given above, using the information that $G_3 = C_3$,

$$\begin{aligned}
P_{G_3}(k) &= P_{C_3}(k)\\
&= (k-1)^3 - (k-1)\\
&= (k-1)((k-1)^2 - 1)\\
&= k(k-1)(k-2)
\end{aligned}$$

so the base case is satisfied for $n = 3$.
The first graph in the contraction/deletion argument satisfies

$$G'_n = G_{n-1} \cup \underbrace{(\{v_1, v_n\}, \{v_1v_n\})}_{C_2}$$

where $G_{n-1} \cap C_2 = \{v_{n-1}\}$. Therefore, by the facts given along with the induction hypothesis, the chromatic polynomial of G'_n is given by

$$\begin{aligned}
P_{G'_n}(k) &= \frac{1}{k} \cdot \underbrace{k(k-1)(k-2)^{n-2}}_{P_{G_{n-1}}(k)} \cdot \underbrace{k(k-1)}_{P_{C_2}(k)}\\
&= k(k-1)^2(k-2)^{n-2}.
\end{aligned}$$

Similarly, $G''_n = G_{n-1}$, so by the induction hypothesis together with contraction deletion,

$$\begin{aligned}
P_{G_n}(k) &= P_{G'_n}(k) - P_{G''_n}(k)\\
&= k(k-1)^2(k-2)^{n-2} - k(k-1)(k-2)^{n-2}\\
&= k(k-1)(k-2)^{n-2}((k-1) - 1)\\
&= k(k-1)(k-2)^{n-1},
\end{aligned}$$

as desired.

Problem 3. Using the deletion/retraction recurrence on a graph, prove that the number of acyclic orientations of G is equal to $(-1)^{|V|}p_G(-1)$ where an acyclic orientation is an assignment of a direction to each edge such that there are no directed cycles.

Proof.
Let $A(G)$ be the number of acyclic orientations on G. By induction on $|V| + |E|$ with base case of the singleton graph $\mathbf{1}$, which has chromatic polynomial $P_{\mathbf{1}}(k) = k$. The only assignment of direction to each edge is the empty assignment, and

$$(-1)^{|\mathbf{1}|}P_1(-1) = (-1)^1(-1) = 1 = A(\mathbf{1})$$

as desired.

Recall the usual contraction/deletion recurrence

$$P_G(k) = P_{G'}(k) - P_{G''}(k)$$

with $G' = G - uv$ and $G'' = G/uv$.
Furthermore, the relation $A(G) = A(G') + A(G'')$ holds.

Applying the induction hypothesis together with the recurrence gives

$$\begin{aligned} A(G) &= A(G') + A(G'') \\ &= (-1)^{|V|}P_{G'}(-1) + (-1)^{|V|-1}P_{G'}(-1) \\ &= (-1)^{|V|}(P_{G'}(-1) - P_{G''}(-1)) \\ &= (-1)^{|V|}P_G(-1) \end{aligned}$$

as desired. $\square$

Problem 4. Let G be a planar connected bipartite graph $V = V_1 \coprod V_2$ and $E \subset V_1 \times V_2$ such that there is no 4-cycle and no vertex of degree 1. Show that $3(|V| - 2) \geq 2|E|$.

Solution.
Since G is planar, the Euler characteristic states that

$$v - 2 = e - f$$

Since G is bipartite, any cycle must have even length, so G does not contain any 3-cycles or 5-cycles. Since G is assumed to be simple (and therefore does not have multiple edges) G does not contain any 2-cycles. By assumption, G does not contain any 4-cycles. Thus any face of G must be adjacent to 6 or more edges, whenever G has at least 6 edges, and G must have at least 6 edges by the "no vertex of degree 1 criterion" coupled with having no cycles smaller than 6.

Since every edge is adjacent to at most two faces, and since $e \geq 6$,

$$\frac{f}{2} \leq \frac{e}{6}$$

and so in particular

$$\begin{aligned} v - 2 = e - f &\geq e - \frac{1}{3}e \\ 3(|V| - 2) &\geq 2|E|. \end{aligned}$$