Differential Geometry: Homework 7

Peter Kagey

April 27, 2018

Problem 1.
Prove that the real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.

Proof.
By the hint, I'll start by showing that the antipodal map f on $S^n = \{(x_1, \ldots, x_{n+1}) : \sum x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ is orientation preserving if and only if n is odd. In particular this map is

$$(x_1, x_2, \ldots, x_{n+1}) \overset{f}{\mapsto} (-x_1, -x_2, \ldots, -x_{n+1})$$

Let $[\omega] = [dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}]$ be the standard orientation on $\mathbb{R}^{n+1}$, then

$$\begin{aligned} f_*([\omega]) &= [f^*\omega] \\ &= [d(-x_1) \wedge \ldots \wedge d(-x_{n+1})] \\ &= [(-1)^{n+1} d(x_1) \wedge \ldots \wedge d(x_{n+1})] \\ &= [(-1)^{n+1}\omega]. \end{aligned}$$

Thus the antipodal map is orientation preserving exactly when $(-1)^{n+1} = 1$, that is, when n is odd.

Recall that in Homework 4, problem 5b, we showed that $\mathbb{RP}^n \cong S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acted on S^n by the above map, $x \mapsto -x$. Thus when the map is orientation preserving, the resulting space is oriented; when the map is orientation preserving, the resulting space cannot be orientable. ☐

Problem 2.
Let S^2 denote the unit sphere in $\mathbb{R}^3$, $\{(r_1, r_2, r_3) \mid r_1^2 + r_2^2 + r_3^2 = 1\}$ with atlas $\mathcal{A} = \{(U_i^\pm, \pi_i^\pm)\}_{i=1,2,3}$ where

$$U_i^+ = \{r_i > 0\} \cap S^2, \text{ and } U_i^- = \{r_i < 0\} \cap S^2,$$

and $\pi_i^\pm$ is the projection away from the ith coordinate.

(a) Is $\mathcal{A}$ a Euclidean oriented atlas?

(b) Let

$$\sigma = \frac{r_1 dr_2 \wedge dr_3 - r_2 dr_1 \wedge dr_3 + r_3 dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}$$

be a two-form on $\mathbb{R}^3 \setminus \{0\}$. Prove that σ restricted to S^2 is closed.

(c) Prove that σ restricted to S^2 is not exact.

Proof.

(a) We will whether or not given any (U_α, ϕ_α) and (U_β, ϕ_β), the determinant of d of the transition function $\det(d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)}) > 0$ for all $p \in U_\alpha \cap U_\beta$. Notice that in particular, the inverse map of $\pi_1^- : U_1^- \to \mathbb{R}^2$ is

$$(x, y) \xmapsto{(\pi_1^-)^{-1}} (-\sqrt{1 - x^2 - y^2}, x, y).$$

So consider $\pi_2^+ \circ (\pi_1^-)^{-1}$

$$(x, y) \xmapsto{(\pi_1^-)^{-1}} (-\sqrt{1 - x^2 - y^2}, x, y) \xmapsto{\pi_2^+} (-\sqrt{1 - x^2 - y^2}, y),$$

which has Jacobian matrix

$$\det \begin{bmatrix} \dfrac{\partial(-\sqrt{1 - x^2 - y^2})}{\partial x} & \dfrac{\partial(-\sqrt{1 - x^2 - y^2})}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{bmatrix} = \det \begin{bmatrix} \dfrac{-x}{\sqrt{1 - x^2 - y^2}} & \dfrac{-y}{\sqrt{1 - x^2 - y^2}} \\ 0 & 1 \end{bmatrix} = \frac{-x}{\sqrt{1 - x^2 - y^2}}.$$

Since this is evaluated at $\pi_1^-(p)$ where $p \in U_1^- \cap U_2^+$, (the second coordinate of p and consequently the first coordinate of $\pi_1^-(p)$) the determinant is negative. Thus $\mathcal{A}$ is not orientation preserving.

(b) In order to show that σ is closed, we need to show that $d\sigma = 0$. Since σ is restricted to S^2, no division by zero will occur.

$$\begin{aligned}
d\sigma &= d\left(\frac{r_1 dr_2 \wedge dr_3}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) - d\left(\frac{r_2 dr_1 \wedge dr_3}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) + d\left(\frac{r_3 dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) \\
&= \frac{\partial}{\partial r_1}\left(\frac{r_1}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) dx_1 \wedge dr_2 \wedge dr_3 \\
&\quad - \frac{\partial}{\partial r_2}\left(\frac{r_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) dr_2 \wedge dr_1 \wedge dr_3 \\
&\quad + \frac{\partial}{\partial r_3}\left(\frac{r_3}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}\right) dr_3 \wedge dr_1 \wedge dr_2 \\
&= \frac{-2r_1^2 + r_2^2 + r_3^2}{(r_1^2 + r_2^2 + r_3^2)^{5/2}}\ dx_1 \wedge dr_2 \wedge dr_3 \\
&\quad + \frac{r_1^2 - 2r_2^2 + r_3^2}{(r_1^2 + r_2^2 + r_3^2)^{5/2}}\ dr_1 \wedge dr_2 \wedge dr_3 \\
&\quad + \frac{r_1^2 + r_2^2 - 2r_3^2}{(r_1^2 + r_2^2 + r_3^2)^{5/2}}\ dr_1 \wedge dr_2 \wedge dr_3 \\
&= 0\ dr_1 \wedge dr_2 \wedge dr_3
\end{aligned}$$

(c) The idea here is that because the equator of S^2 is a set of measure zero, we can approximate the integral over U_1^+ and U_2^- (and thus S^2) by (the limit of) compact sets V_r^+ and V_r^-, given by

$$V_r^\pm = \{(\pi_1^\pm)^{-1}(x,y) \mid x^2+y^2 \le r\}.$$

This is to say, we can compute the integral over S^2 by

$$\int_{S^2} \sigma = \lim_{r\to 1} \int_{V_r^+} \sigma + \int_{V_r^-} \sigma.$$

To make integration easier, let's break up σ into pieces

$$\sigma = f_1 dr_2 \wedge dr_3 + f_2 dr_1 \wedge dr_3 + f_3 dr_1 \wedge dr_2.$$

so

$$\int_{V_r^+} \sigma = \int_{\pi_1^+(V_r^+)} ((\pi_1^+)^{-1})^*\sigma = \int_{\pi_1^+(V_r^+)} ((\pi_1^+)^{-1})^*(f_1 dr_2 \wedge dr_3 + f_2 dr_1 \wedge dr_3 + f_3 dr_1 \wedge dr_2)$$

Now some intermediate computations of $((\pi_1^+)^{-1})^* dr_i$ (note that $r_i \colon \mathbb{R}^3 \to \mathbb{R}$ is the projection map to the i-th coordinate):

$$\begin{aligned}
((\pi_1^+)^{-1})^* dr_1 &= d(r_1 \circ (\pi_1^+)^{-1}) = d(\sqrt{1-x^2-y^2}) = -\frac{x}{\sqrt{1-x^2-y^2}}dx - \frac{y}{\sqrt{1-x^2-y^2}}dy \\
((\pi_1^+)^{-1})^* dr_2 &= d(r_2 \circ (\pi_1^+)^{-1}) = dx \\
((\pi_1^+)^{-1})^* dr_3 &= d(r_3 \circ (\pi_1^+)^{-1}) = dy
\end{aligned}$$

so by exploiting some cancellation,

$$\begin{aligned}
((\pi_1^+)^{-1})^* dr_1 \wedge dr_2 &= -\frac{y}{\sqrt{1-x^2-y^2}} dy \wedge dx \\
((\pi_1^+)^{-1})^* dr_1 \wedge dr_3 &= -\frac{x}{\sqrt{1-x^2-y^2}} dx \wedge dy \\
((\pi_1^+)^{-1})^* dr_2 \wedge dr_3 &= dx \wedge dy
\end{aligned}$$

Similarly intermediate computations of $((\pi_1^+)^{-1})^* f_i = f_i \circ (\pi_1^+)^{-1}$:

$$\begin{aligned}
(x,y) &\xmapsto{(\pi_1^+)^{-1}} (\sqrt{1-x^2-y^2}, x, y) \xmapsto{f_1} \frac{\sqrt{1-x^2-y^2}}{((\sqrt{1-x^2-y^2})^2+x^2+y^2)^{3/2}} = \sqrt{1-x^2-y^2} \\
(x,y) &\xmapsto{(\pi_1^+)^{-1}} (\sqrt{1-x^2-y^2}, x, y) \xmapsto{f_2} \frac{-x}{((\sqrt{1-x^2-y^2})^2+x^2+y^2)^{3/2}} = -x \\
(x,y) &\xmapsto{(\pi_1^+)^{-1}} (\sqrt{1-x^2-y^2}, x, y) \xmapsto{f_3} \frac{y}{((\sqrt{1-x^2-y^2})^2+x^2+y^2)^{3/2}} = y
\end{aligned}$$

Using these computations shows that

$$\begin{aligned}
\int_{V_r^+} \sigma &= \int_{\pi_1^+(V_r^+)} \underbrace{\sqrt{1-x^2-y^2}\, dx\, dy}_{((\pi_1^+)^{-1})^*(f_1 dr_2 \wedge dr_3)} + \underbrace{\frac{x^2}{\sqrt{1-x^2-y^2}}\, dx\, dy}_{((\pi_1^+)^{-1})^*(f_2 dr_1 \wedge dr_3)} + \underbrace{\frac{y^2}{\sqrt{1-x^2-y^2}}\, dx\, dy}_{((\pi_1^+)^{-1})^*(f_3 dr_1 \wedge dr_2)} \\
&= \int_{\pi_1^+(V_r^+)} \frac{1}{\sqrt{1-x^2-y^2}}\, dx\, dy \\
&= \int_0^{2\pi} \int_0^r \frac{s}{\sqrt{1-s^2}}\, ds\, d\theta = \int_0^{2\pi} \left[-\sqrt{1-s^2}\right]_0^r d\theta = 2\pi(1-\sqrt{1-r^2})
\end{aligned}$$

The integral on V_r^- follows very similarly, only the resulting integral has the opposite sign. These integrals do *not* cancel each other, because the orientation is opposite.
Therefore σ restricted to S^2 is *not* exact because the integral does not vanish:

$$\int_{S^2} \sigma = 4\pi \neq 0.$$

$\square$

Problem 3.
Suppose that $M = M_1 \coprod M_2$. Prove that

$$H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2)$$

Proof.
The cheapest way to see this is to appeal to the Mayer-Vietoris sequence:

$$\begin{array}{llll}
\ldots \to \Omega^{k-1}(M_1 \cap M_2) & \to H^k(M) \xrightarrow{\phi} H^k(M_1) \oplus H^k(M_2) \to \Omega^k(M_1 \cap M_2) & \to \ldots \\
\ldots \to \Omega^{k-1}(\emptyset) & \to H^k(M) \xrightarrow{\phi} H^k(M_1) \oplus H^k(M_2) \to \Omega^k(\emptyset) & \to \ldots \\
\ldots \to 0 & \to H^k(M) \xrightarrow{\phi} H^k(M_1) \oplus H^k(M_2) \to 0 & \to \ldots
\end{array}$$

Because the sequence is exact,

$$0 \to H^k(M) \xrightarrow{\phi} H^k(M_1) \oplus H^k(M_2)$$

implies that ϕ is injective and

$$H^k(M) \xrightarrow{\phi} H^k(M_1) \oplus H^k(M_2) \to 0$$

imples that ϕ is surjective. Thus ϕ is an isomorphism, and

$$H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2).$$

□

Problem 4.
Use the Mayer-Vietoris sequence to prove that

$$H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases},$$

and then prove by induction that

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

Proof.
Base case.
Inductive step.
The inducitve step requires three ingredients:

(i) The Mayer-Vietoris sequence is exact:

$$H^{k-1}(\mathbb{R}) \oplus H^{k-1}(\mathbb{R}) \to H^{k-1}(S^{n-1} \times (-\varepsilon, \varepsilon)) \to H^k(S^n) \to H^k(\mathbb{R}) \oplus H^k(\mathbb{R}).$$

(ii) By a lemma of Poincaré, the k-th de Rham cohomology group of Euclidean space is trivial

$$H^k(\mathbb{R}) = 0 \text{ for } k \neq 0$$

(iii) The product of the n-sphere with an interval is homotopic to the n-sphere

$$S^{n-1} \times (-\varepsilon, \varepsilon) \simeq S^{n-1}$$

Case 1. $(k = 0)$
Since S^n is connected, this follows from Tuesday's Lemma that $H^0(M) = \mathbb{R}$ for any connected manifold M.

Case 2. $(k = n \neq 1)$
This follows by induction. We know that $H^{n-1}(S^{n-1}) = \mathbb{R}$, so we have the exact sequence

$$\underbrace{H^{n-1}(\mathbb{R}^n) \oplus H^{n-1}(\mathbb{R}^n)}_{0\times 0 \text{ by (ii)}} \to \underbrace{H^{n-1}(S^{n-1} \times (-\varepsilon, \varepsilon))}_{=H^{n-1}(S^{n-1}) \text{ by (iii)}} \xrightarrow{\delta} H^n(S^n) \to \underbrace{H^n(\mathbb{R}^n) \oplus H^n(\mathbb{R}^n)}_{0\times 0 \text{ by (ii)}}$$

$$0 \to \mathbb{R} \xrightarrow{\delta} H^n(S^n) \to 0.$$

which means that δ must be both injective and surjective, and thus $H^n(S^n) = \mathbb{R}$.

Case 2′. $(k = n = 1)$
We're allowed to assume this input, as it was shown in class that $H^1(S^1) = \mathbb{R}$.

Case 3. $(k \notin \{0, 1, n\})$

$$\underbrace{H^{k-1}(S^{n-1} \times (-\varepsilon, \varepsilon))}_{=H^{k-1}(S^{n-1})=0} \to H^k(S^n) \to \underbrace{H^k(\mathbb{R}^n) \oplus H^k(\mathbb{R}^n)}_{0\times 0 \text{ by (ii)}}$$

$$0 \to H^k(S^n) \to 0.$$

Therefore $H^k(S^n) = 0$

Case 3′. $(0 \neq k \neq n = 1)$
Similar to Case 2′, we are allowed to assume that when $n = 1$, $H^k(S^1) = 0$ when $k \notin \{0, 1\}$.

Therefore the relationship holds by induction:

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

□

Problem 5.
Give a careful computation of the de Rham cohomology (and hence, the Euler characteristic) of a genus g surface Σ_g.

Proof.
Let $\widehat{\Sigma}_g$ be the punctured genus g surface. We will procede by induction on the genus of the surface.

Base case.
Consider the punctured torus $\widehat{\Sigma}_1$, which is homotopic to the "figure 8".
Because the punctured torus is connected, $H^0(\widehat{\Sigma}_1) = \mathbb{R}$. Because we can decompose the figure 8 into two copies of S^1 that intersect at a point. So using the Mayer-Vietoris sequence

$$\begin{aligned} H^0(\mathbb{R}^0) \to H^1(\widehat{\Sigma}_1) &\to H^1(S^1) \oplus H^1(S^1) \to H^1(\mathbb{R}^0) \\ 0 \to H^1(\widehat{\Sigma}_1) &\to \mathbb{R}^2 \qquad\qquad\qquad \to 0 \end{aligned}$$

shows that $H^1(\widehat{\Sigma}_1) = \mathbb{R}^2$. Similary, the Mayer-Vietoris sequence

$$\begin{aligned} H^1(\mathbb{R}^0) \to H^2(\widehat{\Sigma}_1) &\to H^2(S^1) \oplus H^2(S^1) \\ 0 \to H^2(\widehat{\Sigma}_1) &\to 0 \end{aligned}$$

shows that $H^2(\widehat{\Sigma}_1) = 0$.

Note on the cohomology of $\widehat{\Sigma}_g$ versus Σ_g.
First notice that we can "patch" the puncture on $\widehat{\Sigma}_g$ with a surface homotopic to $\mathbb{R}^2$ and with overlap homotopic to S^1. Doing so gives the Mayer-Vietoris sequence

$$\begin{array}{llllllll} 0 \to H^0(\Sigma_g) & \to H^0(\widehat{\Sigma}_g) \oplus H^0(\mathbb{R}^2) & \to H^0(S^1) & \to H^1(\Sigma_g) & \to H^1(\widehat{\Sigma}_g) \oplus H^1(\mathbb{R}^2) & \to H^1(S^1) & \to H^2(\Sigma_g) & \to 0 \\ 0 \to \mathbb{R} & \to \mathbb{R}^2 & \to \mathbb{R} & \to H^1(\Sigma_g) & \to H^1(\widehat{\Sigma}_g) \oplus 0 & \to \mathbb{R} & \to \mathbb{R} & \to 0 \end{array}$$

By repeated use of the rank-nullity theorem,

$$1 + 2 - 1 + \dim(H^1(\Sigma_g)) - \dim(H^1(\widehat{\Sigma}_g)) + 1 - 1 = 0$$

so $\dim(H^1(\Sigma_{g+1})) = \dim(H^1(\widehat{\Sigma}_g))$.

Induction step.
The induction hypothesis is that $H^1(\Sigma_g) = H^1(\widehat{\Sigma}_g) = \mathbb{R}^{2g}$. Here we will decompose the genus Σ_{g+1} surface into a punctured genus g surface and a puctured torus, with intersection homotopic to S^1.

$$\begin{array}{llllllll} 0 \to H^0(\Sigma_{g+1}) & \to H^0(\widehat{\Sigma}_g) \oplus H^0(\widehat{\Sigma}_1) & \to H^0(S^1) & \to H^1(\Sigma_{g+1}) & \to H^1(\widehat{\Sigma}_g) \oplus H^1(\widehat{\Sigma}_1) & \to H^1(S^1) & \to H^2(\Sigma_{g+1}) & \to 0 \\ 0 \to \mathbb{R} & \to \mathbb{R}^2 & \to \mathbb{R} & \to H^1(\Sigma_{g+1}) & \to \mathbb{R}^{2g} \oplus \mathbb{R}^2 & \to \mathbb{R} & \to \mathbb{R} & \to 0 \end{array}$$

By repeated use of the rank-nullity theorem,

$$1 + 2 - 1 + \dim(H^1(\Sigma_{g+1})) - (2g + 2) + 1 - 1 = 0$$

so $\dim(H^1(\Sigma_{g+1})) = 2g + 2$, and $H^1(\Sigma_{g+1}) = \mathbb{R}^{2g+2}$. $\square$

Problem 6.

(a) Let $S^1 = \mathbb{R}/\mathbb{Z}$, and fix an orientation on S^1. For every $k \in \mathbb{Z}$ there is a map $F_k \colon S^1 \to S^1$ by $[z] \mapsto [kz]$. Compute the degree of F_k.

(b) Compute the degree of the reflection map $f \colon S^n \to S^n$

$$(x_1, \ldots, x_{n+1}) \mapsto (-x_1, \ldots, x_{n+1}).$$

Proof.

(a) F_k can be thought of as the map that winds the circle around itself k times. Let's first construct an oriented atlas for S^1

$$\begin{aligned} \mathcal{A} &= \{(U_\varepsilon, \phi), (V_\varepsilon, \psi)\} \\ U_\varepsilon &= \{[x] : \varepsilon < x < 1-\varepsilon\} \text{ and } \phi([x]) = x - \lfloor x \rfloor \\ V_\varepsilon &= \{[x] : -2\varepsilon < x < 2\varepsilon\} \text{ and } \psi([x]) = \phi([x - 1/2]). \end{aligned}$$

This atlas is oriented because the transition map

$$\psi \circ \phi^{-1} \colon \underbrace{(\varepsilon, 2\varepsilon) \cup (1-2\varepsilon, 1-\varepsilon)}_{\phi(U \cap V)} \to \underbrace{\left(\frac{1}{2} - 2\varepsilon, \frac{1}{2} - \varepsilon\right) \cup \left(\frac{1}{2} + \varepsilon, \frac{1}{2} + 2\varepsilon\right)}_{\phi(U \cap V)}$$

given by

$$x \xmapsto{\phi - 1} [x] \xmapsto{\psi} \begin{cases} x + \frac{1}{2} & \phi(x) \in (\varepsilon, 2\varepsilon) \\ x - \frac{1}{2} & \phi(x) \in (1-2\varepsilon, 1-\varepsilon) \end{cases}$$

has derivative 1 at every point.

Now, the cohomological definition can be used to compute $\deg(F_k)$

$$\deg(F_k) = \frac{\displaystyle\int_{S^1} F_k^* \omega}{\displaystyle\int_{S^1} \omega}.$$

Although the map $t \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ given by $[x] \overset{t}{\mapsto} x$ is not a single-valued function, dt is well-defined because t is injective on any sufficiently small neighborhood of the codomain. Then by computing

$$\int_{S^1} dt = \lim_{\varepsilon \to 0} \int_{V_\varepsilon} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} (\phi^{-1})^* dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} d(t \circ \phi^{-1}) = \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} dx = \lim_{\varepsilon \to 0} 1 - 2\varepsilon = 1.$$

Similarly

$$\int_{S^1} F_K^* dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} d(t \circ F_K^* dt \circ \phi^{-1}) = \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} d(kx) = \lim_{\varepsilon \to 0} k(1 - 2\varepsilon) = k.$$

Thus $\deg(F_k) = k$.

(b) Since f is a diffeomorphism, $y = (1, 0, \ldots, 0)$ is a regular value, and its preimage has only one point, call it p,

$$f^{-1}(y) = \{(-1, 0, \ldots, 0)\}.$$

Using the geometric defintion of degree,

$$\deg(f) = \sum_{q \in f^{-1}(y)} \deg_q(f) = \deg_p(f).$$

Therefore the degree of f is 1 if df_p is orientation preserving, otherwise the degree of f is -1. But a manual computation shows that df_p is orientation reversing,

$$df_p = \left[\frac{\partial f_i}{\partial x_j}(p)\right] = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

which has determinant of -1. Therefore $\deg(f) = -1$.

$\square$

Problem 7.
Let $M^n \subset \mathbb{R}^{n+1}$ be a compact oriented n-dimensional submanifold of $\mathbb{R}^{n+1}$ without boundary. For each point $x \in \mathbb{R}^{n+1} \setminus M^n$, define $\sigma_x \colon M^n \to S^n$ as the map

$$p \mapsto \frac{p-x}{||p-x||}.$$

(a) Prove that if x and y are in the same component of $\mathbb{R}^n + 1 \setminus M^n$, then σ_x is smoothly homotopic to σ_y.

(b) Prove that x is in the bounded component if and only if $\deg(\sigma_x) = \pm 1$

(c) Prove that x is in the unbounded component if and only if σ_x is homotopic to the constant function.

Proof.

(a) If x and y are in the same component of $\mathbb{R}^n + 1 \setminus M^n$, then by connectivity, there is some smooth curve $\gamma \colon [0,1] \to \mathbb{R}^n + 1 \setminus M^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then the map $\sigma_{\gamma(t)} \colon M^n \times [0,1]_t \to S^n$ will serve as our homotopy.

(b) We will use the geometric defintion of homology and choose some regular value of σ_x, call it y. Then

$$\sum_{p \in f^{-1}(y)} \deg_p(f)$$

. Since the manifold is oriented, all of the "folds" of the manifold cancel each other out, so $\deg(f) = \pm 1$.

(c) Because degree is preserved under homotopy, we can choose our point y arbitrarily far away from the manifold. In particular, we can make the image of σ_y bounded in an arbitrarily small neighborhood of S^n by choosing y sufficiently far away, and thus place the image of σ_y within a chart on S_n. Then we can use the fact that $\mathbb{R}^n$ is null-homotopic to see that σ_y is homotopic to a constant function.

$\square$