Complex Analysis: Main ideas

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August 16, 2018

Measures

1. **Definition: σ-ring**
   A σ-ring is a collection of sets that are closed under countable unions and finite differences.

2. **Definition: σ-field**
   A σ-algebra or σ-field is a collection of sets that are closed under countable unions and complements.

3. **Definition: Set measures**
   A measure on a set X equipped with a σ-algebra $\mathcal{M}$ is a function $\mu\colon \mathcal{M} \to [0, \infty]$ such that

   (a) $\mu(\emptyset) = 0$, and

   (b) $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$ given that $\{E_j\}_1^{\infty}$ is a sequence of disjoint sets.

4. **Definition: Outer measure**
   An outer measure on a set X is a function $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ such that

   (a) $\mu^*(\emptyset) = 0$,

   (b) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$, and

   (c) $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

5. **Definition: Premeasure**
   A premeasure on a set X equipped with an algebra $\mathcal{A}$ is a function $\mu_0\colon \mathcal{A} \to [0, \infty]$ such that

   (a) $\mu_0(\emptyset) = 0$, and

   (b) $\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j)$ given that $\{A_j\}_1^{\infty}$ is a sequence of disjoint sets with its union in $\mathcal{A}$.

6. **Definition: Construction of measures on $\mathbb{R}^n$**

7. **Definition: Signed measure**
   A signed measure on $(X, \mathcal{M})$ is a function $\nu\colon \mathcal{M} \to (-\infty, \infty]$ or $\nu\colon \mathcal{M} \to [-\infty, \infty)$ such that

   (a) $\nu(\emptyset) = 0$,

   (b) $\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ when $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets.

8. **Definition: Complex measure**
   A complex measure on $(X, \mathcal{M})$ is a function $\nu\colon \mathcal{M} \to \mathbb{C}$ such that

   (a) $\nu(\emptyset) = 0$,

   (b) If $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets, then $\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$, where the sum converges absolutely.

9. **Definition: Mutually singular measures**
   Two signed measures μ and ν are called mutually singular (denoted $\mu \perp \nu$) on $(X, \mathcal{M})$ if there exists a partition of X into $E, F \in \mathcal{M}$ such that E and F are null for μ and ν respectively.

10. **Definition: Variation of signed measures**
    For any signed measure ν, the Jordan Decomposition theorem guarantees unique, positive, mutually singular measures $\nu^+ \perp \nu^-$ such that $\nu = \nu^+ - \nu^-$, called the positive and negative variations of ν.

11. **Definition: Positive set**
    Suppose ν is a signed measure on $(X, \mathcal{M})$. A set $E \in \mathcal{M}$ is called positive if for any subset of E in $\mathcal{M}$ has nonnegative measure.

12. **Hahn decomposition theorem**
    Suppose ν is a signed measure on $(X, \mathcal{M})$. There exists a positive set P and a negative set N such that $P \cup N = X$ and $P \cap N = \emptyset$. This is unique up to null sets.

13. **Definition: Absolute continuity**
    Suppose ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ (denoted $\nu \ll \mu$) if $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \mathcal{M}$. This is the "opposite" of mutual singularity.

14. **Definition: Product measures**

15. **Definition: Regular measures**
    A Borel measure ν on $\mathbb{R}^n$ is called regular if
    (a) $\nu(K) < \infty$ for every compact K;
    (b) $\nu(E) = \inf\{\nu(U) \mid U \text{ open}, E \subset U\}$ for every $E \in \mathcal{B}_{\mathbb{R}^n}$.

16. **Definition: Measurable functions**
    A function $f\colon X \to Y$ is called measurable if $f^{-1}(E) \in \mathcal{M}_X$ for any choice of $E \in \mathcal{N}_Y$.

Integration.

1. **Definition: L^+**
   The space of all measurable functions from X to $[0, \infty]$ is denoted by L^+.

2. **Definition: L^1**
   The space of all functions f from X to $\mathbb{C}$ such that $\int |f| < \infty$ is denoted by L^1.

3. **Lebesgue's dominated convergence theorem**
   Let $\{f_n\}$ be a sequence in L^1 such that
   (a) $f_n \to f$ almost everywhere, and
   (b) there exists a nonnegative $g \in L^1$ such that $|f_n| \leq g$ almost everywhere for all n.

   Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

4. **Levi's monotone convergence theorem**
   Let $\{f_n\}$ be an increasing sequence of positive measurable functions. Then
   $$\lim_{n\to\infty} \int f_n = \int f.$$

5. **Radon-Nikodym theorem**
   Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist σ-finite signed measures λ, ρon $(X, \mathcal{M})$ such that
   $$\lambda \perp \mu,\ p \ll \mu,\ \text{and}\ \nu = \lambda + \rho.$$

6. **Fubini's theorem**
   If $f \in L^1(\mu \times \nu)$, then
   (a) $f_x \in L^1(\nu)$ for almost every $x \in X$,
   (b) $f^y \in L^1(\mu)$ for almost every $y \in Y$,
   (c) $g(x) = \int f_x \, d\nu \in L^1(\mu)$,
   (d) $h(y) = \int f^y \, d\mu \in L^1(\nu)$, and
   (e) $\displaystyle \int f d(\mu \times \nu) = \int \left[\int f(x,y) d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y) d\mu(x) \right] d\nu(y).$

7. **Tonelli's theorem**
   If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x \, d\nu$ and $h(y) = \int f^y \, d\mu$ are in $L^+(X)$ and $L^+(Y)$ respectively, and
$$\int f d(\mu \times \nu) = \int \left[\int f(x,y) d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y) d\mu(x) \right] d\nu(y).$$

8. **Convolution**

9. **The n-dimensional Lebesgue integral**

10. **Polar coordinates**

Convergence.

1. **Defintion: Almost everywhere convergence**
   Convergence almost everywhere means that
$$\mu\left(\left\{x : \lim_{n \to \infty} |f_n(x) - f(x)| > 0\right\}\right) = 0.$$

2. **Defintion: Uniform Convergence**
   Uniform convergence means that for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,
$$\{x : |f_n(x) - f(x)| > \epsilon\} = \emptyset.$$

3. **Defintion: Almost Uniform Convergence**
   Almost uniform convergences means that for each $\varepsilon > 0$, $\delta > 0$, there exists E_δ with $\mu(E_\delta) < \delta$ and $N \in \mathbb{N}$ such that for all $n > N$,
$$\{x : |f_n(x) - f(x)| > \varepsilon\} \subset E_\delta.$$

4. **Defintion: Convergence in measure**
   Convergence in measure means that for each $\varepsilon > 0$, $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,
$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \leq \delta.$$

5. **Defintion: Convergence in L^1**
   We say that $f_n \to f$ in L^1 if $\int |f_n - f| \to 0$.

6. **Egoroff's Theorem**
   Suppose that X has finite measure and $\{f_k \colon X \to \mathbb{C}\}_{k \in \mathbb{N}}$ are measurable functions that converge almost everywhere to f. Then we can find an exceptional set E with arbitrary small measure, such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

7. **Lusin's Theorem**
   If $f \colon [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.