Permutation statistics

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1 Fixed points

Let $F(n, m)$ denote the number of permutations $\pi \in S_n$ with exactly m fixed points. Then $F(1, 1) = 1$ and $F(1, m) = 0$ for $m \neq 1$. For $n > 1$, F satisfies the following recurrence relation

$$F(n, m) = \underbrace{F(n-1, m-1)}_{\text{(A)}} + \underbrace{(n-m-1)F(n-1, m)}_{\text{(B)}} + \underbrace{(m+1)F(n-1, m+1)}_{\text{(C)}} \tag{1}$$

where

(A) Append an n to the end of a word of length $n-1$ with $m-1$ fixed points. This increases both the length and the number of fixed points by one.

(B) Choose one of the $n-m-1$ non-fixed points of a word of length $n-1$ and m fixed points, and replace it with n, append the chosen letter to the end. This increases the length by one while preserving the number of fixed points.

(C) Choose one of the $m+1$ fixed points of a word of length $n-1$ and $m+1$ fixed points, replace it with n, append the chosen letter to the end. This decreases the number of fixed points by one and increases the length by one.

1.1 Take 1 (didn't work, skip to next subsection)

Theorem 1.1.1. *Let $F^{(a)}(n, m)$ denote the number of permutations $\pi \in S_n$ such that π has m fixed points and $\pi(1) = a$, and let $E[\pi^{n,m}_{fix,1}]$ denote the expected value of π for $\pi \in S_n$ with exactly m fixed points, then for $m \neq n-1$.*

$$E[\pi^{n,m}_{fix,1}] := \frac{1}{F(n, m)} \sum_{a=1}^{n} aF^{(a)}(n, m) = \frac{n-m+2}{2} \tag{2}$$

NOTE: Fix division by 0 when $m = n-1$.

Proof. By multiplying both sides (when $m \neq n-1$), it is equivalent to show that

$$F(n, m)E[\pi^{n,m}_{\text{fix},1}] = \sum_{a=1}^{n} aF^{(a)}(n, m). \tag{3}$$

I'll start by showing that both $F^{(1)}$ and $F^{(n)}$ with $b \neq 1$ satisfy essentially similar recurrences to F, namely

$$F^{(1)}(n,m) = \underbrace{F^{(1)}(n-1,m-1)}_{\text{(A)}} + \underbrace{(n-m-1)F^{(1)}(n-1,m)}_{\text{(B)}} + \underbrace{mF^{(1)}(n-1,m+1)}_{\text{(C')}} \tag{4}$$

$$= \underbrace{F(n-1,m-1)}_{\text{(D)}} \tag{5}$$

$$F^{(n)}(n,m) = \underbrace{F^{(n)}(n-1,m-1)}_{\text{(A)}} + \underbrace{(n-m-2)F^{(n)}(n-1,m)}_{\text{(B')}} + \underbrace{(m+1)F^{(n)}(n-1,m+1)}_{\text{(C)}} \tag{6}$$

$$= \underbrace{F(n-1,m) - F^{(1)}(n-1,m)}_{\text{(E)}} + \underbrace{F^{(1)}(n-1,m+1)}_{\text{(F)}} \tag{7}$$

where (A) has the exact same construction as above, and

(B') Choose one of the $n-m-2$ non-fixed points *that is not the first letter* of a word of length $n-1$ and m fixed points, and replace it with n, append the chosen letter to the end. This increases the length by one while preserving the number of fixed points.

(C') Choose one of the m fixed points *that is not the first letter* of a word of length $n-1$ and $m+1$ fixed points, replace it with n, append the chosen letter to the end. This decreases the number of fixed points by one and increases the length by one.

(D) Increase every letter by 1 and prepend a 1, this preserves the number of fixed points.

(E) Take a permutation not starting with 1, and replace its first letter with n, and put the previous first letter at the end of the word.

(F) Take a permutation starting with 1, and replace its first letter with n, and put the 1 at the end of the word.

Now

$$F(n,m)E[\pi^{n,m}_{\text{fix},1}] = \sum_{a=1}^{n} aF^{(a)}(n,m) \tag{8}$$

$$= F^{(1)}(n,m) + \frac{(n+2)(n-1)}{2}F^{(n)}(n,m) \tag{9}$$

$$= F(n-1,m-1) + \frac{(n+2)(n-1)}{2}\Big(F(n-1,m) - F^{(1)}(n-1,m) + F^{(1)}(n-1,m+1)\Big) \tag{10}$$

$$= F(n-1,m-1) + \frac{(n+2)(n-1)}{2}(F(n-1,m) - F(n-2,m-1) + F(n-2,m)) \tag{11}$$

$$\tag{12}$$

Put everything on the right hand side into terms of $F(n-2,*)$,

$$\sum_{a=1}^{n} aF^{(a)}(n,m) = F(n-1,m-1) + \frac{(n+2)(n-1)}{2}(F(n-1,m) - F(n-2,m-1) + F(n-2,m)) \tag{13}$$

$$= F(n-2,m-2) + (n-m-1)F(n-2,m-1) + mF(n-2,m) \tag{14}$$

$$+ \frac{(n+2)(n-1)}{2}((n-m-1)F(n-2,m) + (m+1)F(n-2,m+1)) \tag{15}$$

$$= F(n-2,m-2) \tag{16}$$

$$+ (n-m-1)F(n-2,m-1) \tag{17}$$

$$+ \left(\frac{(n+2)(n-1)(n-m-1)}{2} + m\right)F(n-2,m) \tag{18}$$

$$+ \left(\frac{(n+2)(n-1)(m+1)}{2}\right)F(n-2,m+1) \tag{19}$$

Rewrite $F(n,m)$ in terms of $F(n-2,*)$,

$$F(n,m) = F(n-1,m-1) + (n-m-1)F(n-1,m) + (m+1)F(n-1,m+1) \tag{20}$$

$$= F(n-2,m-2) + (n-m-1)F(n-2,m-1) + mF(n-2,m) \tag{21}$$

$$+ (n-m-1)(F(n-2,m-1) + (n-m-2)F(n-2,m) + (m+1)F(n-2,m+1)) \tag{22}$$

$$+ (m+1)(F(n-2,m) + (n-m-3)F(n-2,m+1) + (m+2)F(n-2,m+2)) \tag{23}$$

$$= F(n-2,m-2) \tag{24}$$

$$+ 2(n-m-1)F(n-2,m-1) \tag{25}$$

$$+ (2m+1+(n-m-1)(n-m-2))F(n-2,m) \tag{26}$$

$$+ 2(1+m)(n-m-2)F(n-2,m+1) \tag{27}$$

$$+ (m+1)(m+2)F(n-2,m+2) \tag{28}$$

The problem, $E[\pi^{n,m}_{\text{fix},1}] = (n-m+2)/2$, and the algebra doesn't obviously work out. □

1.2 Take 2

For $n > 1$, F satisfies the following recurrence relation

$$F^{(1)}(n,m) = F(n-1,m-1) \tag{29}$$

$$F^{(a)}(n,m) = F^{(a)}(n-1,m-1) + \underbrace{(n-m-2)F^{(a)}(n-1,m)}_{\text{(B')}} + (m+1)F^{(a)}(n-1,m+1) \tag{30}$$

where the first recurrence comes from incrementing all of the letters and prepending 1 and

(B') Choose one of the $n-m-1-1$ non-fixed points *that is not the first letter* of a word of length $n-1$ and m fixed points, and replace it with n, append the chosen letter to the end. This increases the length by one while preserving the number of fixed points.

Theorem 1.2.1.

Proof. Note that $F^{(a)}(n,m) = 0$ for $a > n$.

$$\begin{aligned}
F(n,m)E[\pi_{\text{fix},1}^{n,m}] &= \sum_{a=1}^{n} aF^{(a)}(n,m) &(31)\\
&= F(n-1,m-1) &(32)\\
&\quad - F^{(1)}(n-1,m-1) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m-1) &(33)\\
&\quad + (n-m-2)\left(-F^{(1)}(n-1,m) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m)\right) &(34)\\
&\quad + (m+1)\left(-F^{(1)}(n-1,m+1) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m+1)\right) &(35)
\end{aligned}$$

Now use the recurrence (from Take 1) that

$$F^{(1)}(n,m) = F^{(1)}(n-1,m-1) + (n-m-1)F^{(1)}(n-1,m) + mF^{(1)}(n-1,m+1) \tag{36}$$

$$\begin{aligned}
\sum_{a=1}^{n} aF^{(a)}(n,m) &= \sum_{a=1}^{n-1} aF^{(a)}(n-1,m-1) &(37)\\
&\quad + (n-m-2)\left(\sum_{a=1}^{n-1} aF^{(a)}(n-1,m)\right) + F^{(1)}(n-1,m) &(38)\\
&\quad + (m+1)\left(\sum_{a=1}^{n-1} aF^{(a)}(n-1,m+1)\right) - F^{(1)}(n-1,m+1) &(39)
\end{aligned}$$

By the induction hypothesis, these sums equal $F(n-1,m-1)E[\pi_{\text{fix},1}^{n-1,m-1}]$, $F(n-1,m)E[\pi_{\text{fix},1}^{n-1,m}]$, and $F(n-1,m+1)E[\pi_{\text{fix},1}^{n-1,m+1}]$ respectively, so

$$\begin{aligned}
\sum_{a=1}^{n} aF^{(a)}(n,m) &= \frac{n-m+2}{2}F(n-1,m-1) + \frac{(n-m-2)(n-m+1)}{2}F(n-1,m) &(40)\\
&\quad + \frac{(m+1)(n-m)}{2}F(n-1,m+1) + F^{(1)}(n-1,m) - F^{(1)}(n-1,m+1) &(41)
\end{aligned}$$

□

1.3 Take 3

(Mostly copy/pasted from Take 2.) For $n > 1$, F satisfies the following recurrence relation

$$\begin{aligned}
F^{(1)}(n,m) &= F(n-1,m-1) &(42)\\
F^{(a)}(n,m) &= F^{(a)}(n-1,m-1) + \underbrace{(n-m-2)F^{(a)}(n-1,m)}_{\text{(B')}} + (m+1)F^{(a)}(n-1,m+1) &(43)\\
F^{(n)}(n,m) &= F(n-1,m) - F(n-2,m-1) + F(n-2,m) &(44)
\end{aligned}$$

where the first recurrence comes from incrementing all of the letters and prepending 1, the last comes from Take 1, and

(B') Choose one of the $n-m-1-1$ non-fixed points *that is not the first letter* of a word of length $n-1$ and m fixed points, and replace it with n, append the chosen letter to the end. This increases the length by one while preserving the number of fixed points.

Theorem 1.3.1.

Proof. Note that $F^{(a)}(n,m) = 0$ for $a > n$.

$$\begin{aligned}\sum_{a=1}^{n} aF^{(a)}(n,m) = F(n-1,m-1) - F^{(1)}(n-1,m-1) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m-1) \\ +(n-m-2)\left(-F^{(1)}(n-1,m) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m)\right) \\ +(m+1)\left(-F^{(1)}(n-1,m+1) + \sum_{a=1}^{n-1} aF^{(a)}(n-1,m+1)\right)\end{aligned} \tag{45}$$

Now use the recurrence (from Take 1) that

$$F^{(1)}(n,m) = F^{(1)}(n-1,m-1) + (n-m-1)F^{(1)}(n-1,m) + mF^{(1)}(n-1,m+1) \tag{46}$$

$$\sum_{a=1}^{n} aF^{(a)}(n,m) = \sum_{a=1}^{n-1} aF^{(a)}(n-1,m-1) \tag{47}$$

$$+ (n-m-2)\left(\sum_{a=1}^{n-1} aF^{(a)}(n-1,m)\right) + F^{(1)}(n-1,m) \tag{48}$$

$$+ (m+1)\left(\sum_{a=1}^{n-1} aF^{(a)}(n-1,m+1)\right) - F^{(1)}(n-1,m+1) \tag{49}$$

By the induction hypothesis, these sums equal $F(n-1,m-1)E[\pi_{\text{fix},1}^{n-1,m-1}]$, $F(n-1,m)E[\pi_{\text{fix},1}^{n-1,m}]$, and $F(n-1,m+1)E[\pi_{\text{fix},1}^{n-1,m+1}]$ respectively, so

$$\sum_{a=1}^{n} aF^{(a)}(n,m) = \frac{n-m+2}{2}F(n-1,m-1) + \frac{(n-m-2)(n-m+1)}{2}F(n-1,m) \tag{50}$$

$$+ \frac{(m+1)(n-m)}{2}F(n-1,m+1) + F^{(1)}(n-1,m) - F^{(1)}(n-1,m+1) \tag{51}$$

□

2 Descents

2.1 1-Descents

Let $D(n,m)$ denote the number of permutations on S_n with m descents, and let $D^{(a)}(n,m)$ denote the number of permutations starting with the letter a on S_n with m descents. Then the important recurrence relations hold for $n > 1$:

$$D(n,m) = \underbrace{(n-m)D(n-1,m-1)}_{\text{(A)}} + \underbrace{(m+1)D(n-1,m)}_{\text{(B)}} \tag{52}$$

where

(A) Increase the number of descents by one by appending n to the beginning or any non-descent. There are $n-1-m+1$ such positions.

(B) Preserve the number of descents by appending n between any descent, or at the end. There are $m+1$ ways of doing this.

Similarly, for $n > 1$:

$$D^{(n)}(n,m) = D(n-1,m-1) \tag{53}$$
$$D^{(a)}(n,m) = \underbrace{(n-m-1)D^{(a)}(n-1,m-1)}_{\text{(C)}} + \underbrace{(m+1)D^{(a)}(n-1,m)}_{\text{(D)}} \text{ for } a \in [n-1]. \tag{54}$$

where the first recurrence comes from prepending n which increments the number of descents, and

(C) Increase the number of descents by placing n between any non-descent. There are $n-1-m$ such positions.

(D) Preserve the number of descents by appending n between any descent, or at the end. There are $m+1$ ways of doing this.

Definition 2.1.1. *Let $E[\pi_{des}^{m,n}]$ denote the expected value of the first letter of a permutation $\pi \in S_n$ with m descents. That is*

$$E[\pi_{des}^{m,n}] = \frac{\sum_{a=1}^{n} aD^{(a)}(n,m)}{D(n,m)}. \tag{55}$$

Theorem 2.1.2. *The expected value of the first letter of a permutation $\pi \in S_n$ with m descents is $m+1$*

$$E[\pi_{des}^{m,n}] = m+1 \tag{56}$$

for $m \in \{0,1,2,\ldots,n-1\}$.

Proof. By induction on n with induction hypothesis $D(n,m)E[\pi_{\mathrm{des}}^{m,n}] = \sum_{a=1}^{n} aD^{(a)}(n,m)$, and base case clear for $n=2$.

$$\sum_{a=1}^{n} aD^{(a)}(n,m) = nD^{(n)}(n,m) + \sum_{a=1}^{n-1} aD^{(a)}(n,m) \tag{57}$$
$$= nD(n-1,m-1) + \sum_{a=1}^{n-1} a\Big((n-m-1)D^{(a)}(n-1,m-1) + (m+1)D^{(a)}(n-1,m)\Big) \tag{58}$$
$$= nD(n-1,m-1) + (n-m-1)\sum_{a=1}^{n-1} aD^{(a)}(n-1,m-1) + (m+1)\sum_{a=1}^{n-1} aD^{(a)}(n-1,m) \tag{59}$$
$$\tag{60}$$

By the induction hypothesis, these sums are $D(n-1,m-1)E[\pi_{\text{des}}^{m-1,n-1}] = mD(n-1,m-1)$ and $D(n-1,m)E[\pi_{\text{des}}^{m,n-1}] = (m+1)D(n-1,m)$ respectively, so

$$\sum_{a=1}^{n} aD^{(a)}(n,m) = \underbrace{(mn - m^2 - m - n)}_{(n-m)(m+1)} D(n-1,m-1) + (m+1)^2 D(n-1,m) \tag{61}$$

$$= (m+1)((n-m)D(n-1,m-1) + (m+1)D(n-1,m)) \tag{62}$$

$$= (m+1)D(n,m). \tag{63}$$

Thus it directly follows that

$$E[\pi_{\text{des}}^{m,n}] = \frac{\sum_{a=1}^{n} aD^{(a)}(n,m)}{D(n,m)} = \frac{(m+1)D(n,m)}{D(n,m)} = m+1. \tag{64}$$

□

3 Cycles

Definition 3.0.1. *Let* $\operatorname{cyc}_k(\pi)$ *denote the number of* k*-cycles in* π*.*

Definition 3.0.2. *Let* $C_k(n,m)$ *be the number of permutations* $\pi \in S_n$ *such that* $\operatorname{cyc}_k(\pi) = m$*.*

Lemma 3.0.3. $C_k^{(1)}(n,m) = C_k(n-1,m)$ *for all* $k \geq 2$*.*

Proof. Writing π as a word, consider the map $\pi_1\pi_2\ldots\pi_n \mapsto (\pi_2 - 1)\ldots(\pi_n - 1)$. Since $\pi_1 = 1$, the inverse map is clear. □

Lemma 3.0.4. $C_k^{(2)}(n,m) = \cdots = C_k^{(n)}(n,m)$*.*

Proof. It is enough to show that $C_k^{(a)}(n,m) = C_k^{(b)}(n,m)$ for all $a,b > 1$. Since the permutations under consideration do not fix 1, conjugation by (ab) is an isomorphism which takes all words starting with a to words starting with b without changing the cycle structure. □

Lemma 3.0.5. *For all* $2 \leq a \leq n$*,*

$$C_k^{(a)}(n,m) = \frac{C_k(n,m) - C_k(n-1,m)}{n-1}. \tag{65}$$

Proof. Since

$$C_k(n,m) = C_k^{(1)}(n,m) + C_k^{(2)}(n,m) + \cdots + C_k^{(n)}(n,m) \tag{66}$$

using Lemma 3.0.4, for all values $2 \leq a \leq n$, this can be rewritten as

$$C_k(n,m) = C_k^{(1)}(n,m) + (n-1)C_k^{(a)}(n,m) \tag{67}$$

solving for $C_k^{(a)}(n,m)$ and using the substitution from Lemma 3.0.3 gives the desired result:

$$C_k^{(a)}(n,m) = \frac{C_k(n,m) - C_k(n-1,m)}{n-1}. \tag{68}$$

□

Note 3.0.6. *It appears that* $C_k(n,0)$ *is given by the expansion of the exponential generating function*

$$\frac{\exp(-x^k/k)}{(1-x)}, \tag{69}$$

and moreover, it appears that

$$C_k(n,0) = \sum_{i=0}^{\lfloor n/k \rfloor} \frac{n!(-1)^i}{i!\, k^i} = A122974(n,k). \tag{70}$$

These appear in the OEIS as

$$C_1(n,0) = A000166(n) \tag{71}$$
$$C_2(n,0) = A000266(n) \tag{72}$$
$$C_3(n,0) = A000090(n) \tag{73}$$
$$C_4(n,0) = A000138(n) \tag{74}$$
$$C_5(n,0) = A060725(n) \tag{75}$$
$$C_6(n,0) = A060726(n) \tag{76}$$

Theorem 3.0.7. *For all $k > 0, m > 0$*

$$mC_k(n,m) = (k-1)!\binom{n}{k} C_k(n-k, m-1). \tag{77}$$

Proof. As an abuse of notation, let $C_k(n,m) = \{\pi \in S_n \mid \mathrm{cyc}_k(\pi) = m\}$. Then consider the two sets, whose cardinalities match the left- and right-hand sides of the equation above:

$$S^L_{n,m,k} = \{(\pi, c) \mid \pi \in C_k(n,m), c \text{ a distinguished } k\text{-cycle of } \pi\} \tag{78}$$
$$S^R_{n,m,k} = \{(\sigma, d) \mid \pi \in C_k(n-m, m-1), d \text{ an } n\text{-ary necklace of length } k\} \tag{79}$$

The first set, $S^L_{n,m,k}$, is constructed by taking a permutation in $C_k(n,m)$ and choosing one of its m k-cycles to be distinguished, so $S^L_{n,m,k} = mC_k(n,m)$.

In second set, $S^R_{n,m,k}$, the two parts of the tuple are independent. There are $C_k(n-k, m-1)$ choices for σ and $(k-1)!\binom{n}{k}$ choices for d. Thus $S^R_{n,m,k} = (k-1)!\binom{n}{k}C_k(n-k, m-1)$.

Now, consider the map $\phi\colon S^L_{n,m,k} \to S^R_{n,m,k}$ which in cycle notation does the following

$$(\pi_1\pi_2 \dots \pi_\ell, \pi_1) \mapsto (\pi'_2 \dots \pi'_\ell, \pi_1) \tag{80}$$

where π'_i is π_i after relabeling.

By construction, σ has one fewer k-cycle and k fewer letters than π. □

Example 3.0.8. *Suppose $\pi = (18)(\mathbf{37})(254)$ in cycle notation with (37) distinguished. Then*

$$((18)(37)(254), (37)) \mapsto ((16)(243), (37)) \tag{81}$$

under this bijection.

Theorem 3.0.9. *For $k > 1$, the expected value of a permutation $\pi \in S_n$ with m k-cycles is given by*

$$E^{cyc_k}_{n,m} = \frac{n}{2}\left(1 - \frac{C_k(n-1,m)}{C_k(n,m)}\right) + 1. \tag{82}$$

Proof. By definition,

$$E^{\mathrm{cyc}_k}_{n,m} = \frac{\sum_{a=1}^{n} aC_k^{(a)}(n,m)}{C_k(n,m)}. \tag{83}$$

Using Lemma 3.0.4, we can consolidate all but the first term of the numerator

$$\sum_{a=1}^{n} aC_k^{(a)}(n,m) = C_k^{(1)}(n,m) + \sum_{a=2}^{n} aC_k^{(n)}(n,m) \tag{84}$$

$$= C_k^{(1)}(n,m) + C_k^{(n)}(n,m)\sum_{a=2}^{n} a \tag{85}$$

$$= C_k^{(1)}(n,m) + \frac{(n-1)(n+2)}{2}C_k^{(n)}(n,m) \tag{86}$$

$$\tag{87}$$

Now using the recurrences in Lemmas 3.0.3 and 3.0.5

$$\sum_{a=1}^{n} aC_k^{(a)}(n,m) = C_k(n-1,m) + \frac{(n-1)(n+2)}{2}\left(\frac{C_k(n,m) - C_k(n-1,m)}{n-1}\right) \tag{88}$$

$$= \left(\frac{n}{2}+1\right)C_k(n,m) - \frac{n}{2}C_k(n-1,m). \tag{89}$$

Lastly, dividing by the numerator yields the result

$$E_{n,m}^{\mathrm{cyc}_k} = \frac{\left(\frac{n}{2}+1\right)C_k(n,m) - \frac{n}{2}C_k(n-1,m)}{C_k(n,m)} = \frac{n}{2}\left(1 - \frac{C_k(n-1,m)}{C_k(n,m)}\right) + 1. \tag{90}$$

□

Note 3.0.10. *Why does this proof fail when $k = 1$?*

Note 3.0.11. *In some sense, this theorem is all we could hope for, since between Note 3.0.6 and Theorem 3.0.7 this recurrence is easy to compute.*

3.1 2-cycles

Note 3.1.1. $C_2(n,m) = A114320(n,m)$

Definition 3.1.2. $A161936(n)$ *gives the number of direct isometries that are derangements of the $(n-1)$-dimensional facets of an n-cube. (A direct isometry is a rotation of the hypercube.)*

Definition 3.1.3. $A000354(n)$ *gives the expansion of the exponential generating function*

$$\frac{\exp(-x)}{1-2x}. \tag{91}$$

which is also the number of $(n-1)$-dimensional facet derangements for the n-dimensional hypercube.

Note 3.1.4. *Heuristically, about half of the isometries which derange faces should be direct isometries, and the other half should involve a reflection.*

Conjecture 3.1.5.

$$E_{n,m}^{cyc_2} = \begin{cases} \frac{n+1}{2} + \frac{(-1)^{n/2-m}}{2A000354(\frac{n}{2}-m)} & n \text{ is even} \\ \frac{n+1}{2} & \text{otherwise} \end{cases}. \tag{92}$$

Conjecture 3.1.6.

$$E_{n,m}^{cyc_2} = \begin{cases} \frac{n}{2} + \frac{A161936(\frac{n}{2}-m)}{A000354(\frac{n}{2}-m)} & n \text{ is even} \\ \frac{n}{2} + \frac{1}{2} & \text{otherwise} \end{cases} \tag{93}$$

extending the domain of A161936 so that $A161936(0) := 1$.

Conjecture 3.1.7. *This is equivalent to the conjecture that*

$$\frac{A161936(n-m)}{A000354(n-m)} + n\frac{C_2(2n-1,m)}{C_2(2n,m)} = 1 \tag{94}$$

and

$$(2n+1)C_2(2n,m) = C_2(2n+1,m). \tag{95}$$

Note 3.1.8. *The "odd" part of the conjecture follows from Theorem 3.0.9 and the fact that* $nC_2(n-1,m) = C_2(n,m)$ *when* n *is odd. However, I don't know how to show the latter part, and ideally I'd like a bijective proof.*

3.2 3-cycles

Conjecture 3.2.1.

$$E_{n,m}^{cyc_3} = \begin{cases} \frac{n+1}{2} + \frac{(-1)^{n/3-m}}{2A000180(\frac{n}{3}-m)} & 3 \mid n \\ \frac{n+1}{2} & \textit{otherwise} \end{cases}. \tag{96}$$

Note 3.2.2. *A000180 begins* $1, 2, 13, 116, 1393, 20894, 376093, 7897952, 189550849, \ldots$

Note 3.2.3. *Does this relate to wreath products the way the 2-cycle version relates to hypercube derangements?*

3.3 k-cycles

Conjecture 3.3.1. *For* $k > 1$,

$$E_{n,m}^{cyc_k} = \begin{cases} \frac{n+1}{2} + \frac{(-1)^{n/k-m}}{2A320032(\frac{n}{k}-m,k)} & k \mid n \\ \frac{n+1}{2} & \textit{otherwise} \end{cases}, \tag{97}$$

where $A320032(n,k)$ *is the expansion of the exponential generating function*

$$\frac{\exp(-x)}{1-kx}. \tag{98}$$

Note 3.3.2. *These conjectures allow us to compute* $C_k^{(1)}(n,k)$ *and* $C_k^{(\alpha)}(n,k)$ *for* $\alpha \neq 1$.