Topology: Homework 4

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Problem 1.
Let $x_0, x_1, \ldots, x_p$ be p distinct points in $\mathbb{R}^n$. Compute the fundamental group

$$\pi_1(\mathbb{R}^n - \{x_1, x_2, \ldots, x_p\}; x_0).$$

Proof.
If $p = 1$, then we know that $\mathbb{R}^n - \{x_1\}$ has a deformation retract to a circle, so its fundamental group is isomorphic to the free group on one letter:

$$\pi_1(\mathbb{R}^n - \{x_1\}; x_0) \cong \mathbb{Z} \cong \mathbb{F}_1(x_1).$$

Now our inductive hypothesis is that if we remove p points, then the fundamental group is isomorphic to the free group on p letters:

$$\pi_1(\mathbb{R}^n - \{x_1, x_2, \ldots, x_p\}; x_0) \cong \mathbb{F}_p(x_1, x_2, \ldots, x_p).$$

The idea is that when we remove one more point, the resulting set can be partitioned into two overlapping sets with one part homeomorphic to $\mathbb{R}^n - \{x_1, x_2, \ldots, x_p\}$, the other part homeomorphic to $\mathbb{R}^n - \{x_1\}$, and overlap homeomorphic to $\mathbb{R}^n$. Thus, because $\pi_1(\mathbb{R}^n; x_0')$ is trivial and the free product of the free group on n letters with the free group on m letters results in the free group on $n + m$ letters

$$\begin{aligned}
\pi_1(\mathbb{R}^n - \{x_1, x_2, \ldots, x_p, x_{p+1}\}; x_0) &\cong \pi_1(\mathbb{R}^n - \{x_1, x_2, \ldots, x_p, x_{p+1}\}; x_0) *_{\pi_1(\mathbb{R}^n)} \pi_1(\mathbb{R}^n - \{x_1\}; x_0) \\
&\cong \mathbb{F}_p(x_1, x_2, \ldots, x_p) *_1 \mathbb{F}_1(x_{p+1}) \\
&\cong \mathbb{F}_{p+1}(x_1, x_2, \ldots, x_p, x_{p+1}).
\end{aligned}$$

$\square$

Problem 2.
Let X_1 and X_2 be two surfaces whose boundaries are homeomorphic to the circle. Choose an arbitrary homeomorphism $\phi\colon \partial X_1 \to \partial X_2$, and let X be the surface obtained by gluing X_1 to X_2 using ϕ. Give a presentation for the fundamental group of X.

Proof.
The surface X_1 "clearly" has a deformation retract to a figure-eight, so its fundamental group is isomorphic to the free group on two letters $\mathbb{F}_2 = \langle a, b \rangle$. Since X_2 is star-shaped, its fundamental group is trivial. If we call a clockwise loop around the left "a" and a clockwise loop around the right "b", then a (positively oriented) loop around the boundary of X_2 would be a^2b^2, and the image is homeomorphic to S^1, so $\pi_1(\phi(\partial X_1) \cap X_2); x_0) = \langle a^2b^2 \rangle$.

Then by van Kampen's theorem,

$$\begin{aligned}\pi_1(X, [(x_0, \phi(x_0)]) &\cong \pi_1(X_1, x_0) *_{\pi_1(\phi(\partial X_1) \cap X_2); x_0)} \pi_1(X_2, \phi^{-1}(x_0)) \\ &\cong \mathbb{F}_2(a, b) *_{\mathbb{Z}} \mathbf{1}.\end{aligned}$$

Where the homomorphisms are

$$\begin{aligned} &i_1 \colon \mathbb{Z} \to \mathbb{F}_2(a, b) \text{ by } 1 \mapsto a^2b^2 \text{ and} \\ &i_2 \colon \mathbb{Z} \to \mathbf{1} \qquad\qquad \text{by } x \mapsto e. \end{aligned}$$

Then the group presentation for the fundamental group of X is given by

$$\pi_1(X, [(x_0, \phi(x_0)]) \cong \langle a, b; a^2b^2 = 1 \rangle.$$

$\square$

Problem 3.
Let $X_n \subset \mathbb{R}^2$ be the circle of radius $\frac{1}{n}$ centered at the point $(\frac{1}{n}, 0)$, and let $X = \bigcup_{n=1}^{\infty} X_n$, with the subspace topology induced from the standard topology on $\mathbb{R}^2$.

a. Show that the map $r_n \colon X \to X_n$ defined by

$$r_n(x) = \begin{cases} x & x \in X_n \\ (0,0) & x \notin X_n \end{cases}$$

is continuous.

b. Let $\varepsilon = (\varepsilon_n) n \in \mathbb{N}$ be a sequence valued in $\{\pm 1\}$. Consider the map $\gamma_\varepsilon \colon [0,1] \to X$ defined in such a way that the loop turns once around the circle X_n for $s \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, clockwise or counterclockwise depending on whether $\varepsilon_n = 1$ or $\varepsilon_n = -1$.

c. Show that every distinct sequence results in a distinct loop.

d. Conclude that $\pi_1(X; x_0)$ is not countable.

Solution.

a. Suppose that we have an open set in $S \subset X_n$.

Case 1. If this open set contains $(0,0)$, then $r_n^{-1}(S) = X - (X_n - S)$.
Since S is open in X_n, $X_n - S$ is closed in X_n and thus in X, Therefore its complement, $r_n^{-1}(S)$, is open in X.

Case 2. If this open set does not contain $(0,0)$, then we can find some $\varepsilon > 0$ such that the (open) set of all points within ε of S (namely, $U = \bigcup\limits_{s \in S} B_\varepsilon(s) \subset \mathbb{R}^2$) does not contain any points of S_k for $k \neq n$. Thus $r_n^{-1}(S) \subset (U \cap X) = (U \cap X_n)$, and so $r_n^{-1}(S)$ is open in X by defintion of the subspace topology.

b. Since γ_ε is piecewise defined as two functions which are the composition of continuous functions, all that must be checked is (i) $\gamma_\varepsilon(\frac{1}{n}) = \gamma_\varepsilon(\frac{1}{n+1})$ and (ii) $\gamma_\varepsilon(s) \to (0,0)$ as $s \to 0$.

(i) By evaluating the function at $\frac{1}{n}$, it is clear that

$$\gamma_\varepsilon\left(\frac{1}{n}\right) = (0,0) = \gamma_\varepsilon\left(\frac{1}{n+1}\right) \text{ for all } n \in \mathbb{N},$$

so the "handoffs" are continuous by the pasting lemma.

(ii) Also, $\gamma_\varepsilon(s) \to (0,0)$ as $s \to 0$ because as $s \to 0$, $n \to \infty$, and since sine and cosine are bounded (in absolute value) above by 1, the x and y coordinates are bounded (in absolute value) above by $2s$ and s respectively. Therefore by the squeeze theorem, $\gamma_\varepsilon(s) \to (0,0)$ as $s \to 0$.

c. Since the map r_n is continuous, $r_n \circ \gamma_\varepsilon$ is homotopic to a loop in X_n, and in particular is clockwise if and only if $\varepsilon_n = 1$. In particular, because X_n is homeomorphic to S^1, this path is not homeomorphic to this path traversed the other way. Therefore if $\varepsilon \neq \varepsilon'$, then $\gamma_\varepsilon \not\cong \gamma_{\varepsilon'}$

d. There is a clear bijection between sequences $\{a_n = \pm 1\}_{n \in \mathbb{N}}$ and elements of the power set $2^{\mathbb{N}}$: namely take $\{a_n = 1 : n \in \mathbb{N}\}$. To go from a sequence $S \in 2^{\mathbb{N}}$, simply define

$$a_n = \begin{cases} 1 & n \in S \\ -1 & n \notin S \end{cases}.$$

Since $2^{\mathbb{N}}$ is uncountable, $\pi_1(X, x_0)$ must have an uncountable number of elements. This is incompatible with having a countable set of generators: because elements in the free group can be realized as finite strings, a free group with a countable number of generators must itself be countable.