Complex Analysis: Homework 3

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Problem 6. (page 47)
Determine all values of

(a) 2^i,

(b) i^i,

(c) $(-1)^{2i}$.

Proof.

(a) $2^i = \exp(i \log(2)) = \cos(log(2)) + i \sin(log(2))$,

(b) $i^i = \exp(i \log(i)) = \exp(i\pi/2 + 2\pi ik)^i = \exp(i\pi/2 + 2\pi ik)^i = \exp(-\pi/2 - 2\pi k)$,

(c) $(-1)^{2i} = \exp(\pi i)^{2i} = \exp(\pi i + 2\pi ik)^{2i} = \exp(-2\pi - 4\pi k)$.

$\square$

Problem 7. (page 47)
Determine the real and imaginary parts of z^z.

Proof. Let $z = x + iy$ where $x, y \in \mathbb{R}$

$$\begin{aligned}
x^x &= \exp(\log(z))^z \\
&= \exp(\log|z| + i\arg(z))^z \\
&= \exp(\log|z|)^z + \exp(iz\arg(z)) \\
&= |z|^z \cdot \exp(iz\arg(z)) \\
&= |z|^x \cdot |z|^{iy} \cdot \exp(iz\arg(z)) \\
&= |z|^x \cdot |z|^{iy} \cdot \exp((ix - y)\arg(z)) \\
&= |z|^x \cdot |z|^{iy} \cdot \exp(ix\arg(z)) \cdot \exp(-y\arg(z)) \\
&= |z|^x \cdot \exp(iy\log|z|) \cdot \exp(ix\arg(z)) \cdot \exp(-y\arg(z)) \\
&= |z|^x \cdot \exp(-y\arg(z)) \cdot \exp(i(y\log|z| + x\arg(z)))
\end{aligned}$$

Therefore

$$\begin{aligned}
&\operatorname{Re}(z^z) = |z|^x e^{-y\arg(z)} \cos(y\log|z| + x\arg(z)) \text{ and} \\
&\operatorname{Im}(z^z) = |z|^x e^{-y\arg(z)} \sin(y\log|z| + x\arg(z)).
\end{aligned}$$

(Where the argument is up to integer multiples of 2π.) $\square$

Problem 1. (page 72)
Give a precise defintion of a single-valued branch of $\sqrt{1+z}+\sqrt{1-z}$ in a suitable region and prove that it is analytic.

Proof. Let

$$\begin{aligned}
w &= \sqrt{1+z}+\sqrt{1-z} \\
w^2 &= 2+2\sqrt{1-z^2} \\
\frac{w^2}{2}-1 &= \sqrt{1-z^2} \\
\left(\frac{w^2}{2}-1\right)^2 &= 1-z^2 \\
\frac{w^4}{4}-w^2 &= -z^2 \\
w\sqrt{1-\frac{w^2}{4}} &= z
\end{aligned}$$

Take $\Omega = \mathbb{C}\setminus[-1,1]$ where $[-1,1]\subset\mathbb{R}$. And let the branch be values in the first quadrant with a strictly positive real part. Then the derivative

$$\frac{dw}{dz} = \frac{1}{\left(\frac{dz}{dw}\right)} = \frac{\sqrt{-w^2(w^2-4)}}{w(w^2-2)} = \frac{\sqrt{(-2-2\sqrt{1-z^2})(-2+2\sqrt{1-z^2})}}{(\sqrt{1+z}+\sqrt{1-z})(2\sqrt{1-z^2})} = \frac{2z}{(\sqrt{1+z}+\sqrt{1-z})(2\sqrt{1-z^2})}$$

exists for all $z\notin\{0,\pm1\}$, in particular all $z\in\Omega$.

$\square$

Problem 3. (page 72)
Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω. Show that either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$ throughout Ω.

Proof. Suppose for the sake of contradiction that $\operatorname{Re}(f) = 0$ at some point z_0. Then $f(z_0) = 0 + iv(z_0)$ where v is real valued, and so $f(z_0)^2 = -v(z_0)^2$. But then $|f(z)^2 - 1| = | - v(z_0)^2 - 1| = |v(z_0)^2 + 1| \geq 1$ because $v(z_0) \in \mathbb{R}$, which is a contradiction.

Because f is analytic on Ω, it is continuous. Therefore if $\operatorname{Re} f(z_0) > 0$, then $\operatorname{Re}(f) > 0$ on Ω, and if $\operatorname{Re} f(z_0) < 0$, then $\operatorname{Re}(f) < 0$ on Ω. $\square$

Problem 3. (page 78)
Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by a reflection in the real axis.

Proof.
Let S be a linear transformation that maps $0 \mapsto 0$, $\infty \mapsto \infty$, and $1 \mapsto e^{i\alpha}$ for some $\alpha \in \mathbb{R}$ (in order to preserve the unit distance between $S(1)$ and the origin.)

Then S^{-1} maps $0 \mapsto 0$, $\infty \mapsto \infty$, and $e^{i\alpha} \mapsto 1$, so S^{-1} is described the the cross ratio

$$S^{-1}z = (z, e^{i\alpha}, 0, \infty) = \frac{z}{e^{i\alpha}} = ze^{-i\alpha}.$$

and has inverse

$$(S^{-1})^{-1}z = \frac{e^{i\alpha}z - 0}{-0z + 1} = e^{i\alpha}z = Sz.$$

Therefore the corresponding matrix in $GL_2(\mathbb{C})$ is the rotation matrix

$$\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{bmatrix}.$$

Also because both S and complex conjugation preserve distance, the map $x \mapsto S\bar{z}$ is also a (not linear) transformation that leaves the origin fixed and preserves all distances. $\square$

Problem 4. (page 78)
Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Proof.
Let S be the linear transformation that maps the real axis into itself. Thus $0 \mapsto r_0$, and $1 \mapsto r_1$ with $r_0, r_1 \in \mathbb{R}$.

Thus $S^{-1}z$ can be written as the cross ratio (z, r_1, r_0, z_3) for some $z_3 \in \mathbb{C}$. Let $z \in \mathbb{R} \setminus \{r_0, r_1\}$. Then $(z, r_1, r_0, z_4) \in \mathbb{R}$ (because S^{-1} maps the real axis into itself.) Then by Theorem 13, z_4 is on the same line as z, r_1, and r_0, that is, $z_4 = \infty$ or $z_4 \in \mathbb{R}$.

Case 1. Assume $z_4 = \infty$. Then

$$S^{-1}z = \frac{z - r_0}{r_1 - r_0}$$

is written with real coefficients (so S is too.)

Case 2. Assume $z_4 \in \mathbb{R}$. Then

$$S^{-1}z = \frac{z - r_0}{z - z_4} \cdot \frac{r_1 - z_4}{r_1 - r_0} = \frac{(r_1 - z_4)z - (r_1 - z_4)r_0}{(r_1 - r_0)z - (r_1 - r_0)z_4}$$

is written with real coefficients (so S is too.)

$\square$

Problem 3. (page 80)
If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

and interpret the result geometrically.

Proof. By Theorem 13, the cross ratio $(z_1, z_2, z_3, z_4) = r$ for some real r because the points lie on a circle.

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} = \frac{z_1 z_2 - z_1 z_4 - z_2 z_3 + z_3 z_4}{z_1 z_2 - z_1 z_3 - z_2 z_4 + z_3 z_4} = r \in \mathbb{R}$$

Similarly

$$\begin{aligned}
(z_1, z_3, z_2, z_4) &= \frac{z_1 - z_2}{z_1 - z_4} \cdot \frac{z_3 - z_4}{z_3 - z_2} \\
&= \frac{z_1 z_3 - z_1 z_4 - z_2 z_3 + z_2 z_4}{z_1 z_2 - z_1 z_3 - z_2 z_4 + z_3 z_4} \\
&= \frac{(z_1 z_2 - z_1 z_4 - z_2 z_3 + z_3 z_4) - (z_1 z_2 - z_1 z_3 - z_2 z_4 + z_3 z_4)}{z_1 z_2 - z_1 z_3 - z_2 z_4 + z_3 z_4} \\
&= (z_1, z_2, z_3, z_4) - 1
\end{aligned}$$

Therefore

$$(z_1 - z_3)(z_2 - z_4) = r(z_1 - z_4)(z_2 - z_3)$$

and

$$(z_1 - z_2)(z_3 - z_4) = (r - 1)(z_1 - z_4)(z_3 - z_2)$$

thus

$$\begin{aligned}
(z_1 - z_3)(z_2 - z_4) &= (z_1 - z_4)(z_2 - z_3) + (r - 1)(z_1 - z_4)(z_3 - z_2) \\
&= (z_1 - z_4)(z_2 - z_3) + (z_1 - z_2)(z_3 - z_4)
\end{aligned}$$

The geometric interpretation is that the sum of the products of the lengths of the opposite sides is equal to the sum of the lengths of the diagonals. (This is Ptolemy's Theorem.) $\square$

Problem 4. (page 80)
Show that any four distinct points can be carried by a linear transformation to positions $1, -1, k, -k$, where the value of k depends on the points. How many solutions are there? How are they related?

Proof. Let $c_1, c_2, c_3, c_4 \in \mathbb{C}$ be four arbitrary (distinct points).
Construct a linear map S that maps $c_1 \mapsto 1$, $c_2 \mapsto 0$, and $c_3 \mapsto \infty$.

$$Sz = (z, c_1, c_2, c_3) = \frac{z - c_2}{z - c_3} \cdot \frac{c_1 - c_3}{c_1 - c_2} \text{ with corresponding matrix } \begin{bmatrix} c_1 - c_3 & c_2(c_3 - c_1) \\ c_1 - c_2 & c_3(c_2 - c_1) \end{bmatrix}.$$

The construct another map T that maps $1 \mapsto 1$, $-1 \mapsto 0$, and $k \mapsto \infty$.

$$Tz = (z, c_1, c_2, c_3) = \frac{z + 1}{z - k} \cdot \frac{1 - k}{2} \text{ with corresponding matrix } \begin{bmatrix} 1 - k & 1 - k \\ 2 & -2k \end{bmatrix}.$$

These are constructed to so that $T^{-1} \circ S$ maps $c_1 \mapsto 1$, $c_2 \mapsto -1$, and $c_3 \mapsto k$. Now we must simply pick the value of k so that $c_4 \mapsto -k$ under $T^{-1} \circ S$.

$$\begin{bmatrix} -2k & k - 1 \\ -2 & 1 - k \end{bmatrix} \begin{bmatrix} c_1 - c_3 & c_2(c_3 - c_1) \\ c_1 - c_2 & c_3(c_2 - c_1) \end{bmatrix}$$

which corresponds to

$$\frac{((k-1)(c_1 - c_2) - 2k(c_1 - c_3))z + (k-1)(c_1 - c_2)c_3 - 2kc_2(c_3 - c_1)}{((1-k)(c_1 - c_2) - 2(c_1 - c_3))z + -(1-k)(c_1 - c_2)c_3 - 2c_2(c_3 - c_1)}$$

Therefore when $z = c_4$, solving for k

$$\frac{((k-1)(c_1 - c_2) - 2k(c_1 - c_3))c_4 + (k-1)(c_1 - c_2)c_3 - 2kc_2(c_3 - c_1)}{((1-k)(c_1 - c_2) - 2(c_1 - c_3))c_4 + -(1-k)(c_1 - c_2)c_3 - 2c_2(c_3 - c_1)} = -k$$

multiplying by the denominator will result in a quadratic equation with two complex roots.Thus there are two solutions for k related by the complex conjugate. $\square$