

# Differential Geometry: Homework 2

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**Problem 1.** Let  $V$  be a vector space of a field  $k$  and let  $W \subset V$  be a subspace.

(a) Show that  $V/W$  is a vector space, with operations induced by those of  $V$  in the following sense: for  $\alpha$  and  $\beta$  in  $V/W$ , choose elements  $a$  and  $b$  with  $[a] = \alpha$ ,  $[b] = \beta$ , and define  $\alpha + \beta = [a + b]$  and  $c \cdot \alpha = [c \cdot a]$

*Proof.*

**Addition is well-defined.**

Let  $\alpha = [a_1] = [a_2]$ , and let  $\beta = [b_1] = [b_2]$ . Then by definition of the equivalence class  $\sim$ ,  $a_1 - a_2 \in W$  and  $b_1 - b_2 \in W$ . In order to show that  $\alpha + \beta = [a_1 + b_1] = [a_2 + b_2]$  is well-defined, it is sufficient to show that  $a_1 + b_1 \sim a_2 + b_2$ . By closure of  $W$  under addition,

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) - (b_1 - b_2) \in W$$

**Multiplication is well-defined.**

Let  $\alpha = [a_1] = [a_2]$  and  $c \in k$ . Then by definition of the equivalence class  $\sim$ ,  $a_1 - a_2 \in W$ . In order to show that  $c \cdot \alpha = [c \cdot a_1] = [c \cdot a_2]$  is well-defined, it is sufficient to show that  $c \cdot a_1 \sim c \cdot a_2$ . By distributivity laws and closure of  $W$  under scalar multiplication,

$$c \cdot a_1 - c \cdot a_2 = c(a_1 - a_2) \in W$$

**$V/W$  is an abelian group under  $+$ .** *Associativity.*

This is more-or-less inherited from the associativity of addition in  $V$ .

$$(\alpha + \beta) + \gamma = [(a + b) + c] = [a + (b + c)] = \alpha + (\beta + \gamma).$$

*Commutativity.*

This is more-or-less inherited from the commutivity of addition in  $V$ .

$$(\alpha + \beta) = [a + b] = [b + a] = \beta + \alpha.$$

*Identity element.*

Let  $0 = [\vec{0}]$  where  $\vec{0} \in W$ .

$$\alpha + 0 = [a + \vec{0}] = [a] = \alpha$$

$$0 + \alpha = [\vec{0} + a] = [a] = \alpha$$

*Inverse element.*

Let  $\alpha = [a]$ , then  $-\alpha = [-a]$ .

$$\alpha + -\alpha = [a + -a] = [\vec{0}] = e$$

$$-\alpha + \alpha = [-a + a] = [\vec{0}] = e$$

**Multiplication is well-behaved.**

$$(i) \quad 1 \cdot \alpha = [1 \cdot a] = [a] = \alpha.$$

- (ii)  $c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot [c_2 \cdot a] = [c_1 \cdot (c_2 \cdot a)] = [(c_1 c_2) \cdot a] = (c_1 c_2) \cdot \alpha.$
- (iii)  $c_1 \cdot (\alpha + \beta) = c_1 \cdot [a + b] = [c_1 \cdot (a + b)] = [c_1 \cdot a + c_1 \cdot b] = [c_1 \cdot a] + [c_1 \cdot b] = c_1 \cdot \alpha + c_1 \cdot \beta.$
- (iv)  $(c_1 + c_2) \cdot \alpha = [(c_1 + c_2) \cdot a] = [c_1 \cdot a + c_2 \cdot a] = [c_1 \cdot a] + [c_2 \cdot a] = c_1 \cdot \alpha + c_2 \cdot \alpha$

□

**Problem 1.** (b)

The quotient comes equipped with a natural linear map

$$\begin{aligned}\pi : V &\longrightarrow V/W \\ v &\longrightarrow [v] = v + W,\end{aligned}$$

called the *projection*, which has  $\ker \pi = W$  (check that  $\pi$  is linear and has kernel as desired.) Suppose  $V$  is finite-dimensional, and let  $U$  be a subspace complementary to  $W$ , that is a subspace such that  $V = W \oplus U$ . Show that the restriction of projection to  $U$

$$\pi_U : U \longrightarrow V/W$$

is an isomorphism.

*Proof.*

To see that  $\pi$  is linear, check its additivity and homogeneity. The additivity of  $\pi$  follows from

$$\pi(v + w) = [v + w] = [v] + [w] = \pi(v) + \pi(w),$$

and the homogeneity of  $\pi$  follows from

$$\pi(c \cdot v) = [c \cdot v] = c \cdot [v] = c \cdot \pi(v).$$

To check that  $\ker \pi = W$ , see that

$$\ker \pi = \{v \mid \pi(v) = 0\} = \{v \mid v + W = 0 + W\} = \{v \mid v \in W\} = W$$

because  $W$  is closed under addition.

□

**Problem 1.** (c)

Let  $U$  denote the subspace of  $C^\infty(\mathbb{R})$  consisting of functions which vanish at 3 and 5

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(3) = f(5) = 0\}.$$

Prove that the quotient vector space  $C^\infty(\mathbb{R})/U$  is finite-dimensional. What is its dimension?

*Proof.*

Let  $\ell_5(x) = (5 - x)/2$  and  $\ell_3(x) = (x - 3)/2$ , so that  $\ell_5(5) = 0 = \ell_3(3)$  and  $\ell_5(3) = 1 = \ell_3(5)$ . Because  $\ell_5$  and  $\ell_3$  are lines, they are smooth.

Then for any function  $f \in C^\infty(\mathbb{R})$

$$\begin{aligned}f(x) - (f(3)\ell_5(x) + f(5)\ell_3(x)) &\in U \text{ because} \\ f(5) - (f(3)\ell_5(5) + f(5)\ell_3(5)) &= f(5) - (0 + f(5)) = 0 \text{ and} \\ f(3) - (f(3)\ell_5(3) + f(5)\ell_3(3)) &= f(3) - (f(3) + 0) = 0.\end{aligned}$$

Since any smooth function  $f$  is in a coset with a sum of two smooth functions,  $[f(x)] = f(3) \cdot [\ell_5(x)] + f(5) \cdot [\ell_3(x)]$ ,  $C^\infty(\mathbb{R})/U$  has dimension at most two.

Since  $\ell_5 \notin U$ , and  $\ell_3 + U \notin \text{span}\{0 + U, \ell_5 + U\}$ ,  $C^\infty(\mathbb{R})/U$  has dimension exactly two.

□

**Problem 1.** (d)

Let  $V$  be a vector space and  $W \subset V$  be a vector subspace. We denote the inclusion map by  $i : W \rightarrow V$ . Denote  $V^* = \text{Hom}_k(V, k)$ . There is a natural induced map  $i^* : V^* \rightarrow W^*$  dual to the inclusion sending a linear map  $\phi \mapsto \phi|_W$ . The kernel of  $i^*$  is called the *annihilator* of  $W$  and denoted

$$\text{Ann}(W) = \{\phi \in V^* \mid \phi|_W = 0 \in W^*\}.$$

It is the set of linear maps from  $V$  to  $k$  that return 0 on any element in  $W$ .

Prove that there is a canonical isomorphism

$$\text{Ann}(W) \cong (V/W)^*.$$

*Proof.*

The “obvious” maps to consider are

$$\begin{aligned} \psi : \text{Ann}(W) &\rightarrow (V/W)^* \text{ via } g \mapsto ([\vec{v}] \mapsto g(\vec{v})) \\ \psi^{-1} : (V/W)^* &\rightarrow \text{Ann}(W) \text{ via } f \mapsto (\vec{v} \mapsto f([\vec{v}])) \end{aligned}$$

It is sufficient to show that these maps (i) are well-defined, (ii) satisfy  $\psi \circ \psi^{-1} = \text{id}_{(V/W)^*}$ , and (iii) satisfy  $\psi^{-1} \circ \psi = \text{id}_{\text{Ann}(W)}$ .

**Proof of (i).** For the well-definedness of  $\psi$ , it is sufficient to show that if  $[\vec{v}] = [\vec{u}]$  then  $g(\vec{v}) = g(\vec{u})$ . Since  $\vec{v} - \vec{u} \in W$ ,  $g(\vec{v} - \vec{u}) = 0 = g(\vec{v}) - g(\vec{u})$ , so  $g(\vec{v}) = g(\vec{u})$ .

For the well-definedness of  $\psi^{-1}$ , it is sufficient to show that for all  $\varphi \in \text{Ann}(W)$  if  $[\vec{v}_1] = [\vec{v}_2]$  and the map  $\varphi(\vec{v}_1) = f([\vec{v}_1])$  then  $\varphi(\vec{v}_2) = f([\vec{v}_1])$ .

$$\begin{aligned} \varphi(\vec{v}_1 - \vec{v}_2) &= 0 \\ &= \varphi(\vec{v}_1) - \varphi(\vec{v}_2) \end{aligned}$$

So  $\varphi(\vec{v}_1) = \varphi(\vec{v}_2) = f([\vec{v}_1])$ .

**Proof of (ii).** Let  $f : V/W \rightarrow k$  be a linear function.

$$\psi(\psi^{-1}(f)) = \psi(\vec{v} \mapsto f([\vec{v}])) = [\vec{v}] \mapsto f([\vec{v}]) = f$$

Therefore  $\psi \circ \psi^{-1} = \text{id}_{(V/W)^*}$ .

**Proof of (iii).** Let  $g : \text{Ann}(W)$  be a linear function.

$$\psi^{-1}(\psi(g)) = \psi^{-1}([\vec{v}] \mapsto g(\vec{v})) = \vec{v} \mapsto g(\vec{v}) = g$$

Therefore  $\psi^{-1} \circ \psi = \text{id}_{\text{Ann}(W)}$ .

□

**Problem 2.** Let

$$S^n = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}.$$

Prove that  $S^n$  has the structure of a smooth manifold, using charts associated to the cover  $U_N = \{x_1 \neq 1\}$ ,  $U_S = \{x_1 \neq -1\}$ .

*Proof.* Let  $f_p$  be a parameterization of a line that begins at  $(1, 0, \dots, 0)$  and equals  $p$  at time 1:

$$f_p(t) = (1-t)(1, 0, \dots, 0) + t(x_1, \dots, x_{n+1}).$$

This line intersects the subspace  $T = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 = 0\} \cong \mathbb{R}^n$  when

$$\begin{aligned} 0 &= (1-t) + tx_1 \\ t &= 1/(1-x_1). \end{aligned}$$

Thus we can define  $\phi_N : U_N \rightarrow T$  by

$$\begin{aligned} p &\mapsto f_p\left(\frac{1}{1-\pi_1(p)}\right) \\ (x_1, \dots, x_{n+1}) &\mapsto \frac{1}{1-x_1}(0, x_2, \dots, x_{n+1}). \end{aligned}$$

Similarly  $\phi_S : U_S \rightarrow T$  is defined by

$$(x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+x_1}(0, x_2, \dots, x_{n+1}).$$

The functions  $\phi_N$  and  $\phi_S$  are smooth because the first coordinate is the constant map 0, and the other coordinates are being multiplied by a constant scalar.

The inverse  $\phi_N^{-1}$  is constructed in a similar way. The line  $f_p$  intersects  $S^n$  when

$$\begin{aligned} 1 &= (1-t)^2 + t^2x_1^2 + \dots + t^2x_{n+1}^2 \\ 2t &= t^2(1+x_1^2 + \dots + x_{n+1}^2) \\ t &= \frac{2}{1+\|p\|^2} \end{aligned}$$

Thus  $\phi_N^{-1} : T \rightarrow U_N$  is defined by

$$p = (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+\|p\|^2}(\|p\|^2 - 1, 2x_2, 2x_3, \dots, 2x_{n+1}),$$

and  $\phi_S^{-1} : T \rightarrow U_S$  is defined by

$$p = (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1+\|p\|^2}(1 - \|p\|^2, 2x_2, 2x_3, \dots, 2x_{n+1}).$$

The square of the Euclidean norm,  $\|\cdot\|^2$ , is smooth, so  $(1 - \|p\|^2)/(1 + \|p\|^2)$ , and  $2x_i/(1 + \|p\|^2)$  is smooth. Therefore  $\phi_N^{-1}$  and  $\phi_S^{-1}$  are smooth.

Therefore  $\phi_S \circ \phi_N^{-1}$  and  $\phi_N \circ \phi_S^{-1}$  are both smooth on  $U_N \cap U_S$ , and  $S^n$  is a smooth manifold.  $\square$

**Problem 3.** Prove that the product of two smooth manifolds

$$(M^m, \mathcal{A}_M = \{(U_\alpha, \phi_\alpha : U_\alpha \rightarrow \mathbb{R}^m)\}_{\alpha \in I}), \text{ and} \\ (N^n, \mathcal{A}_N = \{(U_\beta, \psi_\beta : U_\beta \rightarrow \mathbb{R}^n)\}_{\beta \in J})$$

naturally has the structure of a smooth manifold, with atlas given by

$$\mathcal{A}_{M \times N} = \{(U_\alpha \times V_\beta, (\phi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \rightarrow \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n})\}_{(\alpha, \beta) \in I \times J}.$$

*Proof.*

□

**Problem 4.** Prove that the antipodal map  $S^n \rightarrow S^n$ ,  $x \mapsto -x$  is a diffeomorphism of manifolds.

*Proof.*

□

**Problem 5.** Finish the proof from class that  $\mathbb{R}P^n$  is a smooth manifold.

*Proof.*

□

**Problem 6.** Finish the proof from class that  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  is a smooth 2-manifold.

*Proof.*

□