Math 533: Homework 5

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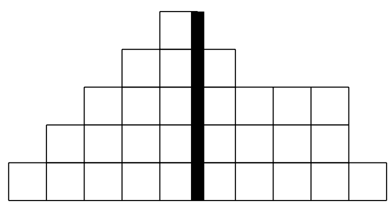
Problem 1.

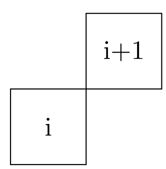
Proof.

(a) Given some filling, remove as many primes as possible, and call this the minimal filling. This can be done unambiguously, because there is at most one i' per row. Now there are $2^\ell(\gamma)$ ways of priming elements. In particular, each row has at least one "primeable" element, all independent. Since there are $\ell(\gamma)$ rows, this means that for each minimal filling, there are a multiple of $2^\ell(\gamma)$ fillings with the same weight.

(b) When γ is a strict partition, the shifted tableau can be partitioned into a "staircase" and a Young tableau.



Now consider a modified Bender-Knuth involution: hold constant any pairs (ignoring primes)



Now all that remains is horizontal strips of the form

$$\underbrace{i, i, \ldots, i}_{a}, \underbrace{i+1, i+1, \ldots, i+1}_{b}$$

map these strips to

$$\underbrace{i, i, \ldots, i}_{b}, \underbrace{i+1, i+1, \ldots, i+1}_{a}$$

adding primes when vertical dominoes force it and removing primes when horizontal dominoes force it.

(c) From the previous homework, it is enough to show that $Q_\gamma(x_1, -x_1, x_3, x_4, \ldots) = Q_\gamma(x_3, x_4, \ldots)$

$\square$

Problem 2.

Proof.

(c) For an involution, the inner structure is cycles of length one or two, which has exponential generating function $x + x^2/2$, and the outer structure puts such cycles together, so the appropriate generating function is $\exp(x + x^2/2)$.

$\square$

Problem 4.

Proof.

(a) $a(n)$ is equivalent to a Catalan walk: insert $\{1, 2, \ldots, 2n\}$ in order, i can only go in the second row if there are strictly more elements in the first row. This is equivalent to a walk that stays above the diagonal. (Placing a number in the first row is a step up, and placing a number in the second row is a step right.) Thus

$$a(n) = \frac{1}{n+1}\binom{2n}{n}$$

(b) $b(n)$ the number of n-step walks that stay above the diagonal. This is because we can always add to the second row if strictly more letters (respectively vertical steps) have been added to the first row. This is given by

$$b(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

(c) There is a bijection between pairs of standard Young Tableaux of shape λ and standard young Tableaux of shape (n, n). The idea is to "flip" the second standard Young Tableau, and reindex by mapping all entries to $k \mapsto 2n + 1 - k$. For example,

$$\begin{array}{|c|c|ccc}
\hline
2 & 5 & \multicolumn{3}{c}{} \\
\hline
1 & 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} \\
\hline
\end{array}$$

$$\begin{array}{|c|c|ccc}
\hline
4 & 7 & \multicolumn{3}{c}{} \\
\hline
1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & 5 & 9 & 10 & 12 & 13 & 14 \\
\hline
1 & 3 & 4 & 6 & 7 & 8 & 11 \\
\hline
\end{array}$$

To recover the original pair: For the first tableau, take the sub-tableau that consists of entries up to n; for the second, take the complement, flip it, and re-apply the map. Therefore

$$c(n) = a(n) = \frac{1}{n+1}\binom{2n}{n}.$$

$\square$

Problem 5. Prove that the crystal string operators

$$S_i(b) = \begin{cases} f_i^{\mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}}(b) & \mathrm{wt}(b)_i \geq \mathrm{wt}(b)_{i+1} \\ e_i^{\mathrm{wt}(b)_{i+1} - \mathrm{wt}(b)_i}(b) & \text{otherwise} \end{cases}$$

satisfy the relations

(a) $S_i^2(b) = b$,

(b) $S_i S_j(b) = S_j S_i(b)$ for $|i - j| > 1$, and

(c) $S_i S_{i+1} S_i(b) = S_{i+1} S_i S_{i+1}(b)$.

Proof. First, notice that by construction, $S_i(b)$ has the same weight as b, but with the values in the i and $i+1$ positions switched. That is, if $\mathrm{wt}(b) = (w_1, w_2, \ldots, w_n)$ then $\mathrm{wt}(S_i(b)) = (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_n)$

(a) If $\mathrm{wt}(b)_i = \mathrm{wt}(b)_{i+1}$, then $S_i = \mathrm{id}$, so $S_i^2 = \mathrm{id}^2 = \mathrm{id}$.

If $\mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1} = k > 0$, then $S_i(b) = f_i^k(b)$, where $\mathrm{wt}(S_i(b))_{i+1} - \mathrm{wt}(S_i(b))_i = k$, so $S_i(S_i(b)) = e_i^k(f_i^k(b))$, then since function composition is commutative, and $f_i^k(b) \neq 0$, cancelling from the middle yields

$$e_i^{k-1} \circ e_i \circ f_i \circ f_i^{k-1}$$

where $e_i \circ f_i$ is the identity.

If $\mathrm{wt}(b)_{i+1} - \mathrm{wt}(b)_i = k > 0$, then $S_i(b) = e_i^k(b)$, where $\mathrm{wt}(S_i(b))_i - \mathrm{wt}(S_i(b))_{i+1} = k$, so $S_i(S_i(b)) = f_i^k(e_i^k(b))$, then since function composition is commutative, and $e_i^k(b) \neq 0$, cancelling from the middle yields

$$f_i^{k-1} \circ f_i \circ e_i \circ e_i^{k-1}$$

where $f_i \circ e_i$ is the identity.

(b) I can't show that these functions commute, but I can show that they both result in elements with the same weight: the values in positions i and then $i+1$ and then switching those in positions j and $j+1$ is the same as doing it in the other order:

$$\begin{aligned} (w_1, \ldots, w_i, w_{i+1}, w_{i+2}, \ldots, w_n) &\xmapsto{S_i} (w_1, \ldots, w_{i+1}, w_i, \ldots, w_j, w_{j+1}, \ldots, w_n) \\ &\xmapsto{S_j} (w_1, \ldots, w_{i+1}, w_i, \ldots, w_{j+1}, w_j, \ldots, w_n) \\ (w_1, \ldots, w_i, w_{i+1}, w_{i+2}, \ldots, w_n) &\xmapsto{S_j} (w_1, \ldots, w_i, w_{i+1}, \ldots, w_{j+1}, w_j, \ldots, w_n) \\ &\xmapsto{S_i} (w_1, \ldots, w_{i+1}, w_i, \ldots, w_{j+1}, w_j, \ldots, w_n), \end{aligned}$$

So the commutator relation satisfies the weight condition.

(c) The same thing happens with the braid case

$$\begin{aligned} (w_1, \ldots, w_i, w_{i+1}, w_{i+2}, \ldots, w_n) &\xmapsto{S_i} (w_1, \ldots, w_{i+1}, w_i, w_{i+2}, \ldots, w_n) \\ &\xmapsto{S_{i+1}} (w_1, \ldots, w_{i+1}, w_{i+2}, w_i, \ldots, w_n) \\ &\xmapsto{S_i} (w_1, \ldots, w_{i+2}, w_{i+1}, w_i, \ldots, w_n) \\ (w_1, \ldots, w_i, w_{i+1}, w_{i+2}, \ldots, w_n) &\xmapsto{S_{i+1}} (w_1, \ldots, w_i, w_{i+2}, w_{i+1}, \ldots, w_n) \\ &\xmapsto{S_i} (w_1, \ldots, w_{i+2}, w_i, w_{i+1}, \ldots, w_n) \\ &\xmapsto{S_{i+1}} (w_1, \ldots, w_{i+2}, w_{i+1}, w_i, \ldots, w_n), \end{aligned}$$

that is, the braids satisfy the weight condition.

$\square$