Topology: Homework 7

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Problem 1.
Let X be path connected and locally path connected, and choose a base point $x_0 \in X$. Let $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ be two coverings with $p^{-1}(x_0) = p'^{-1}(x_0) = F$ (for some set F). Let $\rho, \rho'\colon \pi_1(X; x_0) \to \mathrm{Bij}(F)$ be the respective monodromy antihomomorphisms of p and p'. Show that the two coverings are isomorphic (by an isomorphism that is not necessarily the identity on F) if and only if there exists $\theta \in \mathrm{Bij}(F)$ such that

$$\rho'([\alpha]) = \theta \circ \rho([\alpha]) \circ \theta^{-1}$$

for every $[\alpha] \in \pi(X; x_0)$.

Proof.
($\Longrightarrow$) Suppose that the coverings are isomorphic, with isomorphism $\varphi\colon \widetilde{X} \to \widetilde{X}'$.
Since $p \circ \varphi^{-1} = p'$,

$$\begin{aligned} F = p'^{-1}(x_0) &= (p \circ \varphi^{-1})^{-1}(x_0) \\ &= \varphi(p'^{-1}(x_0)) \\ &= \varphi(F) \end{aligned}$$

and so the homeomorpism restricted to the fiber $\varphi|_F\colon F \to F$ is a bijection of F.
Setting $\theta = \varphi|_F\colon F \to F$ and letting $\widetilde{y} \in F$ be an arbitrary element in the fiber yields

$$\begin{aligned} \rho'([\alpha])(y) &= \widetilde{\alpha}'_y(1) \\ &= \varphi(\widetilde{\alpha}_{\varphi^{-1}(y)}(1)) \\ &= \varphi \circ \rho([\alpha]) \circ \varphi^{-1}(y) \end{aligned}$$

where $\widetilde{\alpha}'_y$ is a path from y to $\widetilde{x}'$, and $\widetilde{\alpha}_{\varphi^{-1}(y)}$ is a path from $\varphi^{-1}(y) \in \widetilde{X}$ to $\varphi^{-1}(\widetilde{x}')$.

($\Longleftarrow$) Suppose that there exists $\theta \in \mathrm{Bij}(F)$ such that

$$\rho'([\alpha]) = \theta \circ \rho([\alpha]) \circ \theta^{-1}$$

for every $[\alpha] \in \pi(X; x_0)$. We'll define the homeomorpism from $\widetilde{X} \to \widetilde{X}'$ as follows:

1. Let $\widetilde{\beta}\colon [0,1] \to \widetilde{X}$ be any path from $\widetilde{x}_0$ to $\widetilde{x}$.
2. Project this path down to X resulting in a map $p \circ \widetilde{\beta}\colon [0,1] \to X$, which describes a path from x_0 to $p(\widetilde{x})$.
3. Lift this path by to a path in X' based at $\theta(\widetilde{x}_0)$.

This map is continuous because it is the composition of lifts and continuous functions, and it is invertible by performing the procedure with the roles of $\widetilde{X}$ and $\widetilde{X}'$ switched. Thus it is a homeomorphism, $\varphi\colon \widetilde{X} \to \widetilde{X}'$, which respects composition by construction, $p' \circ \varphi = p$. $\square$

Problem 2.
Consider the torus $X = S^1 \times S^1$ and consider all coverings $p\colon \widetilde{X} \to X$ with fiber $F = \{1,2,3\}$. Up to covering isomorphism fixing F, how many such coverings are there? And how many up to covering isomorphsim not fixing F?

Proof.
Note that D_3 is isomorphic to the dihedral group $D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$. The fundamental group of the torus is $\pi_1(X; x_0) = \langle a, b\rangle$. So we will consider all monodromy group antihomomorphisms of the covering $\widetilde{X} \to X$ with fiber $F = \{1,2,3\}$, namely antihomomorpishms of the form

$$\rho\colon \langle a, b; aba^{-1}b^{-1} = 1\rangle \to D_3$$

where D_3 is the symmetric group. In particular, by the universal property of free products, the antihomomorphism must satisfy

$$\rho(aba^{-1}b^{-1}) = \rho(a^{-1}) \circ \rho(b^{-1}) \circ \rho(a) \circ \rho(b)) = 1.$$

Thus there are eighteen maps:

$$a \mapsto \mathrm{id}_{D_3} \qquad b \mapsto \mathrm{id}_{D_3} \tag{1}$$
$$a \mapsto \mathrm{id}_{D_3} \qquad b \mapsto \tau \tag{2}$$
$$a \mapsto \mathrm{id}_{D_3} \qquad b \mapsto \sigma \tag{3}$$
$$a \mapsto \tau \qquad b \mapsto \mathrm{id}_{D_3} \tag{4}$$
$$a \mapsto \sigma \qquad b \mapsto \mathrm{id}_{D_3} \tag{5}$$
$$a \mapsto \tau \qquad b \mapsto \tau \tag{6}$$
$$a \mapsto \sigma \qquad b \mapsto \sigma \tag{7}$$
$$a \mapsto \sigma \qquad b \mapsto \sigma^{-1} \tag{8}$$

where (1) gives one maps, (2) gives two maps, (3) gives three maps, (4) gives two maps, (5) gives three maps, (6) gives three maps, (7) gives two maps, and (8) gives two maps. Thus there are $1+2+3+2+3+3+2+2 = 18$ different antihomomorphisms and eight conjugacy classes. So there are 18 covering spaces up to isomorphism fixing F.

By Problem 1, the number of conjugacy classes counts the number of isomorphisms not necessarily fixing F, so there are eight isomorphisms. $\square$

Problem 3.
Consider the Klein bottle K (with $\pi_1(K; x_0) = \langle a, b; aba^{-1} = b^{-1}\rangle$) and consider all coverings $p\colon \widetilde{X} \to X$ with fiber $F = \{1, 2, 3\}$. Up to covering isomorphism fixing F, how many such coverings are there? And how many up to covering isomorphsim not fixing F?

Proof.
Note that D_3 is isomorphic to the dihedral group $D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$. There are eighteen antihomomorphisms

$$
\begin{aligned}
&a \mapsto \mathrm{id}_{D_3} && b \mapsto \mathrm{id}_{D_3} && (1)\\
&a \mapsto \mathrm{id}_{D_3} && b \mapsto \tau && (2)\\
&a \mapsto \tau && b \mapsto \mathrm{id}_{D_3} && (3)\\
&a \mapsto \tau && b \mapsto \tau && (4)\\
&a \mapsto \tau && b \mapsto \sigma && (5)\\
&a \mapsto \sigma && b \mapsto \mathrm{id}_{D_3} && (6)
\end{aligned}
$$

where (1) gives one map, (2) gives three maps, (3) gives three maps, (4) gives three maps, (5) gives six maps, and (6) gives two maps, totaling $1 + 3 + 3 + 3 + 6 + 2 = 18$ maps with six conjugacy classes. Thus there are eighteen covering isomorphisms fixing F.

By Problem 1, the number of conjugacy classes counts the number of isomorphisms not necessarily fixing F, so there are six isomorphisms. □

Problem 4.
Let $\rho\colon \pi_1(X;x_0) \to \mathrm{Bij}(\mathbb{Z})$ be the monodromy antihomomorphism of the parking structure covering. Compute $\rho([\alpha]), \rho([\beta]), \rho([\gamma]) \in \mathrm{Bij}(\mathbb{Z})$ for the paths α, β, γ shown on the picture.

Proof.
When you traverse α, you go up one floor. When you traverse β you go up two floors, and when you traverse γ you go down one floor. Thus the map sends

$$\begin{aligned}
\rho([\alpha]) &= (n \mapsto n+1) \\
\rho([\beta]) &= (n \mapsto n+2) \\
\rho([\gamma]) &= (n \mapsto n-1).
\end{aligned}$$

$\square$