Math 510b: Homework 2

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Problem 1 (Artin). Prove that the ideal $(x + y^2, y + x^2 + 2xy^2 + y^4)$ in $\mathbb{C}[x, y]$is a maximal ideal.

Proof. □

Problem 2 (Artin). Let I be the principal ideal of $\mathbb{C}[x, y]$ generated by the polynomial $y^2 + x^3 - 17$. Which of the following sets generate maximal ideals in the quotient ring $C[x, y]/I$?

(a) $(x - 1, y - 4)$

(b) $(x + 1, y + 4)$

(c) $(x^3 - 17, y^2)$

Proof. □

Problem 6 (Artin). Prove that the kernel of the homomorphism $\mathbb{Z} \to \mathbb{R}$ sending $x \mapsto 1 + \sqrt{2}$ is a principal ideal, and find a generator for this ideal.

Proof. $\square$

Problem 7 (Artin). Let f be an irreducible polynomial in $\mathbb{C}[x, y]$, and let g be another polynomial. Prove that if the variety of zeros of g in $\mathbb{C}^2$ contains the variety of zeros of f, then f divides g.

Proof.

$\square$

Problem 8 (Artin). Determine the points of intersection of the two complex plane curves in each of the following

(a) $y^2 - x^3 + x^2 = 1, x + y = 1$

(b) $x^2 + xy + y^2 = 1, x^2 + 2y^2 = 1$

(c) $y^2 = x^3, xy = 1$

(d) $x + y + y^2 = 0, x - y + y^2 = 0$

Proof. $\square$

Problem 9 (Artin). Prove that two quadratic polynomials f, g in two variables have at most four common zeros unless they have a non-constant factor in common.

Proof. □

Problem 10 (Artin). An algebraic curve $\mathcal{C}$ in $\mathbb{C}^2$ is called irreducible if it is the locus of zeros of an irreducible polynomial $f(x, y)$—one which cannot be factored as a product of nonconstant polynomials. A point $p \in \mathcal{C}$ is called a singular point of the curve if $\partial f/\partial x = \partial f/\partial y = 0$ at p. Otherwise p is a nonsingular point. Prove that an irreducible curve has only finitely many singular points.

Proof. $\square$

Extra Problem. Let $R = \mathbb{Z}(\sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Define $N \colon R \to \mathbb{Z}_{\geq 0}$ by sending $a + b\sqrt{-5} \mapsto a^2 + b^2$.

Show:

(a) $N(xy) = N(x)N(y)$ for all $x, y \in R$.

(b) If x is a unit in R then $N(x) = 1$. Thus the only units in R are ± 1.

(c) There does not exist $x \in R$ with $N(x) = 3$.

(d) If $N(x) = 9$ then x is irreducible in R.

(e) Note that $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$, and conclude that 3 is irreducible in R but not prime.

(f) Factorization into irreducible elements in R is not unique.

(g) Comparing this example to $\mathbb{Z}[i]$, what goes wrong here that works for $\mathbb{Z}[i]$?

(h) Find an ideal in R which is not principal.

Proof. $\square$