Topology: Homework 8

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Problem 1.

a. Express the map $\delta_i \circ F_j \colon \Delta_n \to \Delta_n \times [0,1]$ in terms of $(F_{j'} \times \mathrm{Id}_{[0,1]}) \circ \delta_{i'}$, $\delta_{i\pm 1} \circ F_{j'}$, $i_0 = \mathrm{Id}_{\Delta_n} \times 0$, or $i_1 = \mathrm{Id}_{\Delta_n} \times 1$.

b. Let $f_0, f_1 \colon X \to Y$ be homotopic by a homotopy $H \colon X \times [0,1] \to Y$. Define a linear map $K_n \colon C_n(X) \to C_{n+1}(Y)$ by
$$K_n(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i$$
for every simplex $\sigma \in C_n(X)$.
Show that
$$\partial_{n+1} \circ K_n + K_{n-1} \circ \partial_n = C_n(f_1) - C_n(f_0).$$

Proof.

a. There are six cases to consider:

(i) When $i = j = 0$
$$\begin{aligned}
&(t_0, t_1, \ldots, t_n) \xmapsto{F_0} (0, t_0, t_1, \ldots, t_n) \xmapsto{\delta_0} ((t_0, t_1, \ldots, t_n), \underbrace{t_0 + t_1 + \ldots + t_n}_{1}) \\
&(t_0, t_1, \ldots, t_n) \qquad\qquad\qquad\qquad\quad \xmapsto{i_1} ((t_0, t_1, \ldots, t_n), 1)
\end{aligned}$$
so $\delta_i \circ F_j = i_1$.

(ii) When $i = j > 0$
$$\begin{aligned}
&(t_0, t_1, \ldots, t_n) \xmapsto{F_i} (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \xmapsto{\delta_i} ((t_0, t_1, \ldots, t_{i-1}, 0 + t_i, \ldots t_n), t_i + \ldots + t_n) \\
&(t_0, t_1, \ldots, t_n) \xmapsto{F_i} (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \xmapsto{\delta_{i-1}} ((t_0, t_1, \ldots, t_{i-1} + 0, t_i, \ldots t_n), 0 + t_i + \ldots + t_n)
\end{aligned}$$
so $\delta_i \circ F_i = \delta_{i-1} \circ F_i$.

(iii) When $j - 1 = i < n$
$$\begin{aligned}
&(t_0, t_1, \ldots, t_n) \xmapsto{F_j} (t_0, t_1, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \xmapsto{\delta_{j-1}} ((t_0, t_1, \ldots, t_{j-1} + 0, t_j, \ldots t_n), 0 + t_j + \ldots + t_n) \\
&(t_0, t_1, \ldots, t_n) \xmapsto{F_j} (t_0, t_1, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \xmapsto{\delta_j} ((t_0, t_1, \ldots, t_{j-1}, 0 + t_j, \ldots t_n), t_j + \ldots + t_n)
\end{aligned}$$
so $\delta_i \circ F_j = \delta_{i+1} \circ F_j$.

(iv) When $j - 1 = i = n$
$$\begin{aligned}
&(t_0, t_1, \ldots, t_n) \xmapsto{F_{n+1}} (t_0, t_1, \ldots, t_n, 0) \xmapsto{\delta_n} ((t_0, t_1, \ldots t_n), 0) \\
&(t_0, t_1, \ldots, t_n) \qquad\qquad\qquad\qquad\quad \xmapsto{i_0} ((t_0, t_1, \ldots, t_n), 0)
\end{aligned}$$
so $\delta_n \circ F_{n+1} = i_0$.

(v) When $j-1>i$

$$\begin{aligned}
(t_0,t_1,\ldots,t_n) &\xmapsto{F_j} (t_0,\ldots,t_{j-1},0,t_j,\ldots,t_n)\\
&\xmapsto{\delta_i} ((t_0,\ldots,t_i+t_{i+1},\ldots,t_{j-1},0,t_j,\ldots,t_n),t_{i+1}+\ldots+t_n)\\
(t_0,t_1,\ldots,t_n) &\xmapsto{\delta_i} ((t_0,\ldots,t_i+t_{i+1},\ldots,t_n),t_{i+1}\ldots t_n)\\
&\xmapsto{F_{j-1}\times \mathrm{Id}_{[0,1]}} ((t_0,\ldots,t_i+t_{i+1},\ldots,t_{j-1},0,t_j,\ldots,t_n),t_{i+1}+\ldots+t_n)
\end{aligned}$$

so $\delta_i\circ F_j=(F_{j-1}\times \mathrm{Id}_{[0,1]})\circ\delta_i$.

(vi) When $i>j$

$$\begin{aligned}
(t_0,t_1,\ldots,t_n) &\xmapsto{F_j} (t_0,\ldots,t_{j-1},0,t_j,\ldots,t_n)\\
&\xmapsto{\delta_i} ((t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{i-1}+t_i,\ldots,t_n),t_i+\ldots+t_n)\\
(t_0,t_1,\ldots,t_n) &\xmapsto{\delta_{i-1}} ((t_0,\ldots,t_{i-1}+t_i,\ldots,t_n),t_i\ldots t_n)\\
&\xmapsto{F_j\times \mathrm{Id}_{[0,1]}} ((t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{i-1}+t_i,\ldots,t_n),t_i+\ldots+t_n)
\end{aligned}$$

so $\delta_i\circ F_j=(F_j\times \mathrm{Id}_{[0,1]})\circ\ \delta_{i-1}$.

b. The two terms of the sum can be written as

$$\partial_{n+1}(K_n(\sigma))=\partial_{n+1}\left(\sum_{i=0}^{n}(-1)^iH\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ\delta_i\right)=\sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}H\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ\delta_i\circ F_j$$

and

$$\begin{aligned}
K_{n-1}(\partial_n(\sigma)) &= \sum_{i=0}^{n-1}(-1)^iH\circ(\partial_n(\sigma)\times\mathrm{Id}_{[0,1]})\circ\delta_i && (1)\\
&= \sum_{i=0}^{n-1}(-1)^iH\circ\left(\sum_{j=0}^{n}(-1)^j\sigma\circ F_j\times\mathrm{Id}_{[0,1]}\right)\circ\delta_i && (2)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}H\circ\left(\sigma\circ F_j\times\mathrm{Id}_{[0,1]}\right)\circ\delta_i && (3)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}H\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ(F_j\times\mathrm{Id}_{[0,1]})\circ\delta_i && (4)
\end{aligned}$$

This final sum (4) can be split based on cases.

$$\begin{aligned}
K_{n-1}(\partial_n(\sigma)) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}H\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ(F_j\times\mathrm{Id}_{[0,1]})\circ\delta_i && (5)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j}H\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ(F_j\times\mathrm{Id}_{[0,1]})\circ\delta_i && (6)\\
&\quad+\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}(-1)^{i+j}H\circ(\sigma\times\mathrm{Id}_{[0,1]})\circ(F_j\times\mathrm{Id}_{[0,1]})\circ\delta_i && (7)
\end{aligned}$$

Then these sums can be reindexed based on the above identities

$$K_{n-1}(\partial_n(\sigma)) = \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i-1+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ (F_j \times \mathrm{Id}_{[0,1]}) \circ \delta_{i-1} \tag{8}$$

$$+ \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ (F_{j-1} \times \mathrm{Id}_{[0,1]}) \circ \delta_i \tag{9}$$

$$= \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i-1+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{10}$$

$$+ \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{11}$$

Then adding this to $\partial_{n+1}(K_n(\sigma))$ yields

$$\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{12}$$

$$- \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{13}$$

$$- \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j. \tag{14}$$

By swapping the sums on the first term (12), and splitting the second (13) and third (14) sums to their overlap ($1 \le i \le n-1$) gives

$$\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{15}$$

$$- \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{16}$$

$$- \sum_{j=0}^{n-1}(-1)^{n+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_j \tag{17}$$

$$- \sum_{j=2}^{n+1}(-1)^{j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_j \tag{18}$$

$$- \sum_{i=1}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j. \tag{19}$$

Then the second (16)and fifth (19) terms sum over $1 \le i \le n-1$, but miss values where $j = 1$ or

$j = i + 1$, so we'll add these back in

$$\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{20}$$

$$- \sum_{i=1}^{n-1} \sum_{j=0}^{n+1} (-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{21}$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i}^{i+1} (-1)^{i+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_j \tag{22}$$

$$- \sum_{j=0}^{n-1} (-1)^{n+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_j \tag{23}$$

$$- \sum_{j=2}^{n+1} (-1)^{j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_j. \tag{24}$$

Subtracting (21) from (20), and splitting (22) into two sums gives

$$\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = \sum_{j=0}^{n+1} (-1)^{j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_j \tag{25}$$

$$+ \sum_{j=0}^{n+1} (-1)^{n+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_j \tag{26}$$

$$+ \sum_{i=1}^{n-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_i \tag{27}$$

$$- \sum_{i=1}^{n-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_{i+1} \tag{28}$$

$$- \sum_{j=0}^{n-1} (-1)^{n+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_j \tag{29}$$

$$- \sum_{j=2}^{n+1} (-1)^{j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_j. \tag{30}$$

Then subtracting (30) from (25) and (29) from (26) gives

$$\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = \sum_{j=0}^{1} (-1)^{j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_j \tag{31}$$

$$+ \sum_{j=n}^{n+1} (-1)^{n+j} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_j \tag{32}$$

$$+ \sum_{i=1}^{n-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_i \tag{33}$$

$$- \sum_{i=1}^{n-1} H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_i \circ F_{i+1}. \tag{34}$$

Then using the substitution $\delta_i \circ F_{i+1} = \delta_{i+1} \circ F_{i+1}$ when $i < n$, subtracting (34) from (33), and writing everything termwise gives

$$
\begin{aligned}
\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) = {} & H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \underbrace{\delta_0 \circ F_0}_{i_1} && (35)\\
& - H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_0 \circ F_1 && (36)\\
& + H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_n && (37)\\
& - H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \underbrace{\delta_n \circ F_{n+1}}_{i_0} && (38)\\
& + H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \underbrace{\delta_1 \circ F_1}_{\delta_0 \circ F_1} && (39)\\
& - H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ \delta_n \circ F_n. && (40)
\end{aligned}
$$

So (36) cancels with (39) and (37) cancels with (40) leaving

$$
\begin{aligned}
\partial_{n+1}(K_n(\sigma)) + K_{n-1}(\partial_n(\sigma)) &= H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ i_1 - H \circ (\sigma \times \mathrm{Id}_{[0,1]}) \circ i_0 \\
&= H \circ (\sigma \times 1) - H \circ (\sigma \times 0) \\
&= C_n(f_1)(\sigma) - C_n(f_0)(\sigma).
\end{aligned}
$$

□

Problem 2.
Let X be a topological space. For all n, let $C_n(X)$ be the usual R-module of singular n-chains in X with coefficients in the ring R. In particular, $C_0(X) = \left\{\sum_{i=1}^k a_i x_i : a_i \in R, x_i \in X\right\}$ consists of all linear combinations of points in X. Consider the homomorphism $\widetilde{\partial}_0 \colon C_0(X) \to R$ defined by the property that $\widetilde{\partial}_0\left(\sum_{i=1}^k a_i x_i\right) = \sum_{i=1}^k a_i$
For $n \in \mathbb{Z}$, define

$$\widetilde{C}_n(X) = \begin{cases} C_n(X) & n \geq 0 \\ R & n = -1 \\ 0 & n \leq -2 \end{cases}$$

and define $\widetilde{\partial}_n \colon \widetilde{C}_n(X) \to \widetilde{C}_{n-1}(X)$ by the property that

$$\widetilde{\partial}_n = \begin{cases} \partial_n & n > 0 \\ \widetilde{\partial}_0 & n = 0 \\ 0 & n < 0. \end{cases}$$

Finally, let $\widetilde{H}_n(X) = \ker(\widetilde{\partial}_n)/\operatorname{Im}(\widetilde{\partial}_{n+1})$

a. Show that $\widetilde{H}_n(X) = H_n(X)$ when $n \neq 0$.

b. Show that $\widetilde{H}_0(X) = 0$ if X is path connected.

c. Show that $\widetilde{H}_0(X) \cong R^{n-1}$ if X has n path-connected components.

Proof.

a. For $n > 0$, $\widetilde{\partial}_n = \partial_n$ and $\widetilde{C}_n(X) = C_n(X)$, so in particular, $\ker(\widetilde{\partial}_n) = \ker(\partial_n)$ and $\operatorname{Im}(\widetilde{\partial}_n) = \operatorname{Im}(\partial_n)$. Therefore
$$\widetilde{H}_n(X) = \ker(\widetilde{\partial}_n)/\operatorname{Im}(\widetilde{\partial}_{n+1}) = \ker(\partial_n)/\operatorname{Im}(\partial_{n+1}) = H_n(X)$$
for $n > 0$.
When $n < 0$, $\widetilde{\partial}_n = 0$, so $\ker(\widetilde{\partial}_n) = \ker(0) = 0$. In particular, $\widetilde{H}_n(X) = \ker(0)/\operatorname{Im}(\widetilde{\partial}_{n+1}) = 0 = H_n(X)$ with the last equality by convention.

b. First note that
$$\partial_1(\sigma_i) = \sum_{j=0}^{1}(-1)^j \sigma_i \circ F_j = \sigma_i \circ F_0 - \sigma_i \circ F_1$$
so if $\sigma_i(0,1) = x_0$ and $\sigma_i(1,0) = x_1$, then $\partial_1(\sigma_i)$ is the constant map from the 0-simplex to the difference of the end points of σ_i, namely $1 \mapsto x_0 - x_1$.
Let $c = \sum_i c_i \sigma_i$ be an element of $C_1(X)$. Then
$$\partial_1\left(\sum_i c_i \sigma_i\right) = \sum_i c_i \partial_1(\sigma_i) = \sum_i c_i (x_{i,0} - x_{i,1}).$$
Then any element in $\operatorname{Im}(\partial_1)$ maps to 0 under $\widetilde{\partial}_0$
$$\begin{aligned} \widetilde{\partial}_0\left(\sum_i c_i(x_{i,0} - x_{i,1})\right) &= \widetilde{\partial}_0\left(\sum_i c_i x_{i,0} - \sum_i c_i x_{i,1}\right) \\ &= \widetilde{\partial}_0\left(\sum_i c_i x_{i,0}\right) - \widetilde{\partial}_0\left(\sum_i c_i x_{i,1}\right) \\ &= \sum_i c_i - \sum_i c_i \\ &= 0, \end{aligned}$$

which shows that $\operatorname{Im}(\partial_1) \subset \ker(\widetilde{\partial}_0)$.
Next I will show that $\operatorname{Im}(\partial_1) \subset \ker(\widetilde{\partial}_0)$:
Let $c \in \ker(\widetilde{\partial}_0) \subset C_0(X)$ be written as $c = \sum_i c_i x_i$. Then, since X is path-connected, for each x_i, there exists a path path $\sigma_i \colon \Delta_1 \to X$ from x_i to some designated basepoint x_0. Then let $c_1 \in C_1(X)$ be defined as $\sum_i c_i \sigma_i$. Then

$$\begin{aligned}
\partial_1(c_1) &= \partial_1\left(\sum_i c_i \sigma_i\right) \\
&= \sum_i c_i (x_i - x_0) \\
&= \sum_i c_i x_i - \sum_i c_i x_0 \\
&= \underbrace{\sum_i c_i x_i}_{c} - \underbrace{\left(\sum_i c_i\right)}_{0} x_0 \\
&= c.
\end{aligned}$$

Since each set contains the other, $\operatorname{Im}(\partial_1) = \ker(\widetilde{\partial}_0)$ and $\widetilde{H}_0(X) = \ker(\widetilde{\partial}_0)/\operatorname{Im}(\partial_1) = 0$.

c. Suppose X has n path-connected components $X_1, X_2, \ldots, X_n$. Then $C_n(X)$ is freely generated

$$C_n(X) = \bigoplus_{i=1}^{n} C_n(X_i).$$

The above result that $\operatorname{Im}(\partial_1) \subset \ker(\widetilde{\partial}_0)$ did not depend on path-connectedness, so it still holds. Also the image of ∂_1 can be written as the disjoint union of boundaries of components

$$\operatorname{Im}(\partial_1) = \coprod_{i=1}^{n} \operatorname{Im}(\partial_1|_{C_1(X_i)}).$$

I'm not sure what to do next with this:

$$\bigoplus_{i=1}^{n} C_n(X_i) \xrightarrow{\partial_1} \bigoplus_{i=1}^{n} C_n(X_i) \xrightarrow{\widetilde{\partial}_0} R.$$

$\square$