Week 9 Quiz Solutions.

1. (*5 points*) Suppose $f(1,2,3) = 9$, $f_x(1,2,3) = \sqrt{17}$, $f_y(1,2,3) = \ln 5$, $f_z(1,2,3) = e^2$. Use this information to approximate $f(0.99, 2.01, 2.98)$.

   Solution.
   Using the equation of the tangent plane to do a linear approximation gives

$$
\begin{aligned}
f(x,y,z) - \underbrace{f(1,2,3)}_{9} &= \underbrace{f_x(1,2,3)}_{\sqrt{17}}(x-1) + \underbrace{f_y(1,2,3)}_{\ln 5}(y-2) + \underbrace{f_z(1,2,3)}_{e^2}(z-3) \\
f(x,y,z) &= 9 + \sqrt{17}(x-1) + \ln(5)(y-2) + e^2(z-3) \\
f(0.99, 2.01, 2.98) &\approx 9 + \sqrt{17}(0.99-1) + \ln(5)(2.01-2) + e^2(2.98-3) \\
f(0.99, 2.01, 2.98) &\approx 9 + -0.01\sqrt{17} + 0.01\ln(5) + -0.02e^2
\end{aligned}
$$

2. (*5 points*) The temperature T at a location (x, y, z) is given by

$$T(x,y,z) = 100e^{-(x^2+\frac{y^2}{4}+\frac{z^2}{9})}$$

   . At $(1, 2, 3)$, what is the direction in which T increases most rapidly? What is the direction of most rapid decrease in T?

   Solution. The directional derivative of T in the direction $\vec{u}$ is given by

$$D_{\vec{u}}T(x,y,z) = \nabla T(x,y,z) \cdot \vec{u}$$

   where $\vec{u}$ is a unit vector. So this is maximized when $\nabla T(x,y,z)$ and $\vec{u}$ have the same direction. Namely,

$$\vec{u} = \frac{\nabla T(1,2,3)}{|\nabla T(1,2,3)|}.$$

   The gradient is given by

$$
\begin{aligned}
\nabla T(x,y,z) &= \langle T_x(x,y,z), T_y(x,y,z), T_z(x,y,z)\rangle \\
&= \left\langle -200xe^{-(x^2+\frac{y^2}{4}+\frac{z^2}{9})}, -50ye^{-(x^2+\frac{y^2}{4}+\frac{z^2}{9})}, -\frac{200}{9}ze^{-(x^2+\frac{y^2}{4}+\frac{z^2}{9})}\right\rangle \\
\nabla T(1,2,3) &= \langle -200e^{-3}, -(200/2)e^{-3}, -(200/3)e^{-3}\rangle \\
&= \frac{200}{6}e^{-3}\langle -6, -3, -2\rangle.
\end{aligned}
$$

   So the direction of the unit vector $\vec{u}$ where the increase in temperature is maximized is given by

$$\vec{u} = \frac{\langle -6, -3, -2\rangle}{\sqrt{36+9+4}} = \left\langle -\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right\rangle.$$

   The decrease in temperature occurs when moving in the opposite direction,

$$-\vec{u} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right\rangle.$$