Math 574: Homework 3

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Friday, November 1, 2019

Problem 1. Let $A, B \in M_n(K)$ and assume that $AB = BA$.

(a) Show that A and B are simultaneously upper diagonalizable.

(b) Show that if A and B are diagonalizable, they are simultaneously diagonalizable.

Proof.

(a) If v is an eigenvector of B, then Av is also an eigenvector of B since

$$A(Bv) = A(\lambda v) = \lambda(Av) = B(Av),$$

so the eigenspace E_λ of B is invariant under multiplication by A. Thus by conjugating $U^{-1}AU$ to Jordan canonical form, $U^{-1}BU$ must be upper triangular.

(b) In the case of diagonalizable matrices, part (a) tells us that any eigenbasis of A is an eigenbasis of B, so by writing $U^{-1}AU = D_A$ where the columns of U are this common eigenbasis, we also get that $U^{-1}BU = D_B$ is diagonal.

□

Problem 2. Let $A \in M_n(\mathbb{R})$

Proof.

(a) Let v be a vector, and let $A^\top v = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$. Notice that $(A^\top v)^\top = v^\top A$, so

$$\begin{aligned} v^\top A A^\top v &= \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}^\top \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \\ &= a_1^2 + a_2^2 + \cdots + a_n^2 \\ &\geq 0. \end{aligned}$$

Since this argument holds for an arbitrary v, $AA^\top$ is semipositive definite. Moreover $AA^\top$ equals its own transpose,

$$(AA^\top)^\top = (A^\top)^\top A^\top = AA^\top,$$

so $AA^\top$ is symmetric. Notice that by singular value decomposition, $A = U\Sigma V^\top$, with U and V orthogonal and Σ diagonal, so

$$\begin{aligned} AA^\top &= U\Sigma V^\top (U\Sigma V^\top)^\top = U\Sigma \underbrace{V^\top V}_{I} \underbrace{\Sigma^\top}_{\Sigma} U^\top = U\Sigma^2 U^\top \\ A^\top A &= (U\Sigma V^\top)^\top U\Sigma V^\top = V \underbrace{\Sigma^\top}_{\Sigma} \underbrace{U^\top U}_{I} \Sigma V^\top = V\Sigma^2 V^\top \end{aligned}$$

Therefore $AA^\top$ is similar to $A^\top A$ with

$$AA^\top = U\Sigma^2 U^{-1} = U \underbrace{V^{-1}A^\top A V}_{\Sigma^2} U^{-1} = (UV^{-1})A^\top A(VU^{-1}).$$

(b) A counter example is $A = \begin{bmatrix} 0 & i \\ 0 & 1 \end{bmatrix}$. Notice

$$AA^\top = \begin{bmatrix} 0 & i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}$$

but

$$A^\top A = \begin{bmatrix} 0 & 0 \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which have rank 1 and 0 respectively, so cannot be similar.

□

Problem 3.

(a) Show that if A is a positive definite Hermitian matrix, then there is a unique B that is positive definite Hermitian with $B^2 = A$.

(b) Given an example of a complex square matrix A so that $A \neq B^2$ for any B.

Proof.

(a) If A is positive definite Hermitian, then all eigenvalues are real and positive, $\lambda_1, \lambda_2, \dots, \lambda_n$, and A is diagonalizable with $A = UD_AU^{-1}$ by letting $B = U \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})U^{-1}$ we see that $(UBU^{-1})^2 = UB^2U^{-1} = UD_AU^{-1} = A$.

(b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has rank 1. For sake of contradiction, assume there exists B with $B^2 = A$. Notice that B cannot have rank 2 (otherwise $\det(B)^2 \neq 0$), and B cannot have rank 0 (because squaring the zero matrix results in the zero matrix.) Therefore B must have rank 1 and so its second row must be a linear combination of the first.

$$B^2 = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}^2 = \begin{bmatrix} a^2 + cab & ab + cb^2 \\ ca^2 + c^2ab & cab + c^2b^2 \end{bmatrix}^2 = \begin{bmatrix} a(a+cb) & b(a+cb) \\ ac(a+cb) & cb(a+cb) \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A$$

since $b(a+cb) = 1$ and $cb(a+cb) = 0$, this implies that $c = 0$. By looking at the first entry, this means that $a^2 = 0 = a$, but this is a contradiction because if $a = c = 0$, then $b(a+cb) = 0 \neq 1$.

□

Problem 4. Show that the set of cyclic elements in $M_n(\mathbb{C})$ form an open dense subset of $M_n(\mathbb{C})$.

Proof. A matrix is cyclic if its Jordan normal form consists of blocks with distinct eigenvalues.
There are two things to show: (a) Given any cyclic $A \in M_n(\mathbb{C})$, there exists an open ball around A such that all matrices in the open ball are cyclic, and (b) Given a non-cyclic $B \in M_n(\mathbb{C})$, every open ball around B contains a cyclic matrix.

(a) Given a matrix A where each Jordan block has a distinct eigenvalue, changing each entry by less than ε means that each eigenvalue can change by at most $n\varepsilon$. Thus we can choose ε small enough that the eigenvalues remain distinct.

(b) Suppose that B is not cyclic. Then an arbitrarily small change to the entries of B will change the eigenvalue corresponding to a Jordan block, and therefore $B + \varepsilon a_{ij}$ will be cyclic. (Where a_{ij} is notation for a matrix that is all zeroes except for a 1 in the ij entry.)

□

Problem 5. Let A, B be Hermitian matrices with A positive definite.

(a) Show that if B is positive semidefinite then $A + B$ is positive definite.

(b) Show that the largest eigenvalue α of A is $\max\{\langle Av, v\rangle : ||v|| = 1\}$.

(c) Show that there exists an invertible matrix S such that $SAS^* = I_n$ and $SBS*$ is diagonal.

Proof.

(a) Let v be an arbitrary nonzero vector of suitable size. Then since A is positive definite and B is positive semidefinite,

$$v^\top(A+B)v = \underbrace{v^\top Av}_{>0} + \underbrace{v^\top Bv}_{\geq 0} > 0,$$

so $A + B$ is positive definite.

(b) Let α be the largest eigenvalue with corresponding unite eigenvector v_α, and notice that

$$\max\{\langle Av, v\rangle : ||v|| = 1\} \geq \alpha$$

since $\langle Av_\alpha, v_\alpha\rangle = \langle \alpha v_\alpha, v_\alpha\rangle = \alpha \underbrace{\langle v_\alpha, v_\alpha\rangle}_{1} = \alpha$. Because A is Hermitian, we can write an orthonormal basis of eigenvectors, $\{v_1, v_2, \ldots, v_n\}$. With corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then if we write $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, and $Av = \lambda_1a_1v_1 + \lambda_2a_2v_2 + \cdots + \lambda_na_nv_n$, then

$$\begin{aligned}
\langle v, Av\rangle &= \lambda_1a_1^2 + \lambda_2a_2^2 + \ldots + \lambda_na_n^2 \\
&\leq \lambda_1a_1^2 + \lambda_1a_2^2 + \ldots + \lambda_1a_n^2 \\
&= \lambda_1(a_1^2 + a_2^2 + \ldots + a_n^2) \\
&= \lambda_1
\end{aligned}$$

with equality when $a_1 = 1$.

(c) Since A is positive definite Hermitian, it can be factored as $U\Lambda U^*$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i > 0$. Then let $D = D^* = \mathrm{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_n^{-1/2})$ so that $D\Lambda D = I$. Next, letting $S = UD$

$$S\Lambda S^* = UD\Lambda D^*U^* = UIU^* = I.$$

The rest follows by Theorem 4.5.17 in Horn.

$\square$