Combinatorics: Homework 8

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Problem 1: 58 (a). [2]
For $u \in \mathfrak{S}_k$, let $s_u(n) = \#S_u(n)$ the number of permutations $w \in \mathfrak{S}_n$ avoiding u. If also $v \in \mathfrak{S}_k$, then write $u \sim v$ if $s_u(n) = s_v(n)$ for all $n \geq 0$.

Let $u, v \in \mathfrak{S}_k$. Suppose that the permutation matrix P_v can be obtained from P_u by one of the eight dihedral symmnetries of the square. Show that $u \sim v$.

We then say that u and v are equivalent by symmetry, denoted $u \approx v$. What are the $\approx$ equivalence classes for $\mathfrak{S}_3$?

Solution. Suppose that σ is an element of the dihedral group of the square and P_v and P_u are permutation matrices such that $P_v = \sigma P_u$ under the group action.
Then there is an "obvious" bijection between $S_u(n)$ and $S_v(n)$, namely $f\colon S_u(n) \to S_v(n)$ maps $P_u \mapsto \sigma P_u$. To go back, simply do the group action of σ^{-1}.

There are only two equivalence classes for $\mathfrak{S}_3$:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \approx \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \text{ and}$$
$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix} \approx \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \approx \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix} \approx \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}.$$

Problem 2. Find the recurrence and generating function formula for the "corner colored" paths $(0,0) \to (n,n)$ with steps $(1,0)$ or $(0,1)$, on or above the main diagonal such that every inner corner of the path of the kind $(a,b) \to (a+1,b) \to (a+1,b+1)$ can be colored in one of two possible colors.

Solution.
We'll use the same technique that was used to compute the Catalan numbers in class; namely, we'll sum over all positions where the Dyck paths first hit the diagonal (stictly above $(0,0)$) and take the products of the smaller Dyck paths below and this point.

$$\begin{aligned} f(0) &= 1 \\ f(n) &= f(n-1) + \sum_{k=1}^{n-1} 2f(k-1)f(n-k) \end{aligned}$$

We also do the same sort of generating function argument.

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} f(n)x^n \\ &= 1 + \sum_{n=1}^{\infty} f(n)x^n \\ &= 1 + \sum_{n=1}^{\infty} \left(f(n-1) + \sum_{k=1}^{n-1} 2f(k-1)f(n-k) \right) x^n \\ &= 1 + \underbrace{\sum_{n=1}^{\infty} f(n-1)x^n}_{xF(x)} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} 2f(k-1)f(n-k) \right) x^n. \end{aligned}$$

By reindexing the sums on the right with the standard trick, letting $n = k + j$ where j now runs from 0 to infinity, we have

$$\begin{aligned} F(x) &= 1 + xF(x) + \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} 2f(k-1)f(j) \right) x^{k+j} \\ &= 1 + xF(x) + 2x \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f(k)f(j)x^k x^j \\ &= 1 + xF(x) + 2xF(x)^2 \end{aligned}$$

which means that solving $0 = 2xF(x)^2 + (x-1)F(x) + 1$ for $F(x)$ by the quadratic formula yields

$$\frac{-(x-1) \pm \sqrt{(x-1)^2 - 4 \cdot 2x}}{2 \cdot 2x}.$$

When x is near zero, $F(x)$ should be close to $f(0)$, so the root we care about is

$$F(x) = \frac{1 - x - \sqrt{x^2 - 10x + 1}}{4x}$$

Problem 3. Find the number of 132-avoiding alternating permutations of length $2n$.

Proof. I will construct a bijection $\varphi\colon \mathfrak{S}_n^{(132)} \to \mathfrak{S}_{2n,\mathrm{alt}}^{(132)}$ between 132-avoiding permutations of length n and 132-avoiding alternating permutations of length n.

The map φ is simple, but its inverse is more complicated. In particular, φ takes in a word in $\mathfrak{S}_{2n,\mathrm{alt}}^{(132)}$ and outputs the relative order of the odd letters.

$$w_1w_2w_3\dots w_{2n-1}w_{2n} \overset{\varphi}{\longmapsto} \mathrm{order}(w_1w_3\dots w_{2n-1})$$

For example, if $w = 65748231$, then $\varphi(w) = \mathrm{order}(6783) = 2341$.

Going back is a bit trickier.

1. Start with the permutation $w' \in \mathfrak{S}_n^{(132)}$.
2. For i increasing from 1 to $n-1$, recursively insert $a = \min(w_1, w_2, \dots w_{2i})$ into the $2i$th position, then increment all letters that are greater than or equal to a, except for the newly inserted letter.
3. Increment everything and append 1.

For example, starting with $w' = 2341$ this algorithm recovers the original alternating permutation.

$$\begin{array}{cccccccc}
2 & \underbrace{\qquad}_{\min(2,3)} & 3 & & 4 & & 1 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
3 & 2 & 4 & \underbrace{\qquad}_{\min(2,3,4,5)} & 5 & & 1 & \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
4 & 3 & 5 & 2 & 6 & \underbrace{\qquad}_{\min(2,3,4,5,6,1)} & 1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
5 & 4 & 6 & 3 & 7 & 1 & 2 & \underbrace{\qquad}_{1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
6 & 5 & 7 & 4 & 8 & 2 & 3 & 1
\end{array}$$

In practice, this is a sequence of maps

$$\mathfrak{S}_n^{(132)} \xrightarrow{\psi_1} \mathfrak{S}_{n+1}^{(132)} \xrightarrow{\psi_2} \mathfrak{S}_{n+3}^{(132)} \xrightarrow{\psi_3} \dots \xrightarrow{\psi_{n-1}} \mathfrak{S}_{2n-1} \xrightarrow{\psi_n} \mathfrak{S}_{2n}$$

Where each ψ_i has the property that for all $i \neq n$, if the preimage is alternating for letters $w_1 < w_2 > w_3 < \dots > w_{2i-1}$, then the image will be alternating for letters $w_1 < w_2 > w_3 < \dots > w_{2i+1}$. (In the case of ψ_n, there is no w_{2n+1} letter, but this holds up to w_{2n}.) Also ψ_i preserves the relative order of all of the letters away from position i.
There is only one map ψ_i that satisfies this:

1. If the inserted letter is greater than either of the neighboring letters, then it fails to satify the alternating condition.
2. If the inserted letter is less than its left neighbor w_{2i-1}, but greater than some letter w_j in the prefix, then the subsequence w_j, w_{2i-1}, a is not 132-avoiding.
3. If the inserted letter, a, is less than $\min(w_1, w_2, \dots w_{2i})$, then there exists some letter w_k with $k > 2i$ such that the subsequence a, w_{2i}, w_k is not 132 avoiding.

Therefore the map $\phi^{-1} = \psi_n \circ \psi_{n-1} \circ \dots \circ \psi_1$ recovers the orginal sequence, and so $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = \mathrm{id}$, and ϕ is a bijection. Thus $\#\mathfrak{S}_n^{(132)} = \#\mathfrak{S}_{2n,\mathrm{alt}}^{(132)} = C_n$. $\square$