Permutation statistics

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1 Symmetric Polynomials

1.1 Monomial

Definition 1.1.1 (Monomial symmetric polynomial). *Let $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ with be a partition, then the monomial symmetric polynomial m_λ on $m \geq \ell$ variables is defined to be*

$$m_\lambda(x_1, x_2, \cdots, x_m) = \sum_{\sigma \in S_\infty} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(\ell)}^{\lambda_\ell}.$$

Definition 1.1.2 (Monomial basis). *Let $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ be a partition of n, then all degree n symmetric polynomials can be written with basis*

$$\left\{ m_\lambda(x_1, x_2, \cdots) = \sum_{\sigma \in S_m} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(\ell)}^{\lambda_\ell} : |\lambda| = n \text{ and } \ell \leq m \right\}.$$

Example 1.1.3. *The symmetric polynomial*

$$f(x_1, x_2, x_3) = -x_1^2 + 4x_1x_2 + 4x_1x_3 - x_2^2 + 4x_2x_3 - x_3^2$$

can be written in the monomial basis as

$$f(x_1, x_2, x_3) = 4m_{(1,2)} - m_{(2)}.$$

Exercise 1.1.4. *Show that*

$$\mathfrak{B}_m = \{m_\lambda(x) : \lambda \text{ a partition of } n \text{ with at most } m \text{ rows}\}$$

is indeed a basis for The space of homogenous symmetric polynomials of degree n in m variables.

Proof. Of course, $m_\lambda(x)$ is itself a symmetric polynomial, and the difference of symmetric polynomials is itself symmetric. Thus the proof will proceed inductively: let $f(x)$ be an arbitrary symmetric polynomial, and look at the leading term

$$f(x) = \alpha x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell} + \cdots.$$

subtract $\alpha m_\lambda(x)$, and you get a polynomial with strictly fewer terms, by symmetry. Since a symmetric polynomial only has a finite number of terms, after a finite number of steps, this constructs f as sums of polynomials in $\mathfrak{B}_m$.

It is clear that $\mathfrak{B}_m$ spans the space of such symmetric polynomials, so the only thing left to check that $\mathfrak{B}_m$ is linearly independent. However, since every $m_\lambda(x)$ is nonzero when λ has fewer then m rows, and any linear combination of nonzero monomial symmetric polynomials is distinct "on inspection", $\mathfrak{B}_m$ is linearly independent. □

1.2 Schur

Definition 1.2.1 (Schur polynomial)**.** *Fulton defines the Schur polynomial to be*

$$s_\lambda(x) = \sum_{T \text{ shape } \lambda} x^T,$$

where the sum is over all Semi-standard Young Tableaux.

Example 1.2.2.

$$\begin{aligned}
s_{(3,1)}(x_1,x_2,x_3) = {} & \underbrace{x_1^3x_2}_{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1^3x_3}_{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_1^2x_2^2}_{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1^2x_2x_3}_{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_1^2x_2x_3}_{\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1^2x_3^3}_{\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_1x_2^3}_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1x_2^2x_3}_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array}} \\
& + \underbrace{x_1x_2^2x_3}_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1x_2x_3^2}_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_1x_2x_3^2}_{\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}} + \underbrace{x_1x_3^3}_{\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_2^3x_3}_{\begin{array}{|c|c|c|}\hline 2&2&2\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_2^2x_3^2}_{\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 3\\\cline{1-1}\end{array}} + \underbrace{x_2x_3^3}_{\begin{array}{|c|c|c|}\hline 2&3&3\\\hline 3\\\cline{1-1}\end{array}} \\
= {} & x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1^2x_2^2 + x_1x_2^3 + 2x_1x_2^2x_3 \\
& + 2x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 + x_2^2x_3^2 + x_2x_3^3 + x_2x_3^3 \\
= {} & m_{(3,1)} + 2m_{(2,1,1)} + m_{(2,2)}
\end{aligned}$$