Week 8 Quiz Solutions.

1. (*5 points*) Given the function
$$f(x,y,z) = \frac{x}{3y^2 + yz}$$
compute $f_{yx}(1,2,3)$ in whichever order you want.

Solution.
It is easiest to compute the derivative with respect to x first, and then compute the derivativ with respect to y:

$$\begin{aligned} f_xy(x,y,z) &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}\left[\frac{1}{3y^2+yz}\right] \\ &= \frac{-6y - z}{(3y^2+yz)^2} \end{aligned}$$

So evaluating at $(1,2,3)$ gives
$$\frac{-6(2)-3}{(3(4)+6)^2} = \frac{-15}{18^2}.$$

2. (*5 points*) Use the **chain rule** to determine $\frac{\partial T}{\partial \theta}$ and $\frac{\partial^2 T}{\partial \theta \partial r}$ when $r = 2$ and $\theta = \pi/3$ given that

$$\begin{aligned} T &= xe^y, \\ x &= r\cos\theta, \text{ and} \\ y &= r\sin\theta. \end{aligned}$$

Solution.
By the chain rule,

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \theta} \\ &= e^y(-r\sin\theta) + xe^y(r\cos\theta) \\ &= e^{r\sin\theta}(-r\sin\theta) + (r\cos\theta)e^{r\sin\theta}(r\cos\theta) \\ &= -ye^y + x^2e^y \end{aligned}$$

Evaluated at $r = 2$ and $\theta = \pi/3$ (and thus $x = 1$ and $y = \sqrt{3}$) gives
$$-\sqrt{3}e^{\sqrt{3}} + e^{\sqrt{3}} = e^{\sqrt{3}}(1-\sqrt{3}).$$

Denote $\dfrac{\partial T}{\partial \theta} = T_\theta$. Then by switching the order of differentiation,

$$\begin{aligned}\frac{\partial^2 T}{\partial \theta \partial r} &= \frac{\partial T_\theta}{\partial r}\\ &= \frac{\partial T_\theta}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial T_\theta}{\partial y} \cdot \frac{\partial y}{\partial r}\\ &= (2xe^y)(\cos\theta) + (-e^y - ye^y + x^2e^y)(\sin\theta).\end{aligned}$$

Again evaluated at $r = 2$ and $\theta = \pi/3$ (and thus $x = 1$ and $y = \sqrt{3}$) gives

$$e^{\sqrt{3}} + (-e^{\sqrt{3}} - \sqrt{3}e^{\sqrt{3}} + e^{\sqrt{3}})\frac{\sqrt{3}}{2} = -\frac{e^{\sqrt{3}}}{2}.$$