Complex Analysis: Main ideas

Peter Kagey

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1. **Cauchy-Riemann Equations**
   Let $f(z) = u(x,y) + iv(x,y)$ then f is holomorphic if and only if $u_x = v_y$ and $u_y = -v_x$.
   Note: To remember which sign is which, check with $(x+iy)^2 = (x^2 - y^2) + i(2xy)$

2. **Cauchy's Integral Theorem**
   Let f be a holomorphic on a simply connected open set Ω, then the integral along any closed curve $\gamma \in \Omega$ vanishes:
   $$\oint_\gamma f\,dz = 0.$$

3. **Cauchy's Integral Formula**
   Let f be a holomorphic on a simply connected open set Ω, and let $\gamma \in \Omega$ be a simple closed curve around z_0. Then $f(z_0)$ is uniquely determined by the values on the boundary of the curve
   $$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z_0}\,d\xi,$$
   and by differentiating under the integral
   $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

4. **Defintion: Residue** The residue of f at z_0 is the coefficient of $(z-z_0)^{-1}$ in the Taylor expansion of f about z_0.

5. **Residue Theorem**
   Let Ω be a "nice" region, and f is holomorphic on $\Omega \setminus \{z_0\}$. Then
   $$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \mathrm{Res}_{z_0}(f).$$
   where γ is a nice curve in Ω.

6. **Schwarz Lemma**
   Assume that $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map such that $f(0) = 0$.

   Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. Also, if $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation.

7. **Rouché's Theorem**
   Let γ be a curve in Ω homologous to 0 with winding number at most 1. Then if f and g are analytic and satisfy the inequality $|f-g| < |f|$ for all points on γ, f and g enclose the same number of zeroes.

8. **Argument principle**
   Suppose f is a meromorphic function inside and on some closed contour C with no zeros or poles on C, then
   $$\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$$
   where N is the number of zeros and P is the number of poles inside C.

9. **Definition: Normal Family** (Ahlfors, p. 220)
A family $\mathfrak{F}$ is said to be normal in Ω if every sequence $\{f_n\}$ of functions $f_n \in \mathfrak{F}$ contains a subsequence which converges uniformly on every compact subset of Ω.