Math 533: Homework 3

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Problem 1.

Proof.

1.

2. A binary tree can be decomposed as a root and the left side and right side (which are themselves trees)

$$\begin{aligned} B(x) &= \underbrace{1}_{\text{root}} + \underbrace{B(x)}_{\text{left}}\underbrace{B(x)}_{\text{right}} \\ &= 1 + B(x)^2. \end{aligned}$$

3. Solving the quadratic $xB(x)^2 - B(x) + 1 = 0$ yields

$$B(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

   where we take minus, so that $\lim_{x\to 0} B(x) = 1$.

4. We can write

$$
\begin{aligned}
1-\sqrt{1-4x} &= 1-\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-4)^n x^n \\
&= -\sum_{n=1}^{\infty}\frac{(\frac{1}{2})(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}(-4)^n x^n \\
&= \sum_{n=1}^{\infty} 1\cdot 3\cdots(2(n-1)-1)\frac{(2x)^n}{n!} \\
&= \sum_{n=1}^{\infty}\frac{(2n-2)!}{2^{n-1}(n-1)!}\frac{(2x)^n}{n!} \\
\frac{1-\sqrt{1-4x}}{2x} &= \sum_{n=1}^{\infty}\frac{(2n-2)!}{2^{n-1}(n-1)!}2^{n-1}\frac{x^{n-1}}{n!} \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\binom{2n-2}{n-1}x^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\binom{2n}{n}x^n.
\end{aligned}
$$

Thus

$$b_n = \frac{1}{(n+1)!}\binom{2n}{n}.$$

□

Problem 2.

Proof.

(a) It is sufficient to evaluate both sides at $G(x)$:

$$
\begin{aligned}
\underbrace{F(G(x))}_{x} &= G(x)\cdot E(\underbrace{F(G(x))}_{x}) \\
x &= G(x)\cdot E(x) \\
&= \frac{x}{E(x)}\cdot E(x)
\end{aligned}
$$

(b)

□

Problem 3.

Proof. This falls nicely to a generating function approach. Let $r(n,j)$ be the number of rooted, ordered trees on n vertices with j leaves, and define the generating function

$$T(x,y) = \sum_{n,j\geq 0} r(n,j)x^n y^j.$$

Then $T(x,y)$ satisfies the recurrence

$$\begin{aligned} T(x,y) &= xy + xT(x,y) + xT(x,y)^2 + xT(x,y)^3 + \dots \\ &= xy + \frac{xT(x,y)}{1-T(x,y)} \\ T(x,y) - T(x,y)^2 &= xy(1-T(x,y)) + xT(x,y), \end{aligned}$$

and so it is sufficient to find the roots of the quadratic $T(x,y)^2 + (x - xy - 1)T(x,y) + xy$:

$$T(x,y) = \frac{1 + xy - x \pm \sqrt{(x-xy-1)^2 - 4xy}}{2}.$$

Then taking the root which subtracts the radical gives

$$\begin{aligned} T(x,y) &= \frac{1}{2}\left(1 + xy - x - \sqrt{1 - 2x + x^2 - 2xy - 2x^2y + x^2y^2}\right) \\ &= \frac{1}{2}\left(xy - x - \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-2x + x^2 - 2xy - 2x^2y + x^2y^2)^n\right) \\ &= \frac{1}{2}\left(xy - x - \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-2 + x - 2y - 2xy + xy^2)^n x^n\right) \\ &= \frac{1}{2}\left(xy - x - \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\sum_{m=0}^{n}\binom{n}{m}(-2-2y)^m(-2xy + xy^2 + x)^{n-m}x^n\right) \\ &= \frac{1}{2}\left(xy - x - \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\sum_{m=0}^{n}\binom{n}{m}(-2)^m(y+1)^m(y-1)^{2n-2m}x^{2n-m}\right). \end{aligned}$$

Then letting $n = m + j$,

$$\frac{1}{2}\left(1 + xy - x - \sum_{m,j\geq 0}^{\infty}\binom{\frac{1}{2}}{m+j}\binom{m+j}{m}(-2)^m(y+1)^m(y-1)^{2j}x^{m+2j}\right)$$

$\square$

Problem 4.

Proof. $\square$

Problem 5. Let B_{n+1} be the $n+1 \times n+1$ matrix with entries

$$b_{ij} = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ -1 & i = n+1 \text{ and } j \in \{1, n\} \\ -1 & j = n+1 \text{ and } i \in \{1, n\} \\ 0 & \text{otherwise} \end{cases}.$$

This is the Laplacian matrix for the cyclic graph C_{n+1}. Then by the matrix-tree theorem, deleting the $n+1$ row and column gives the number of spanning trees of the cyclic graph on $n+1$ vertices, C_{n+1}. The only way to get a spanning tree of C_{n+1} is to delete a vertex, so the number of spanning trees is equal to the number of vertices. Thus $\det(A_n) = n+1$.

Proof. □