Fall 2013: Complex Analysis Graduate Exam

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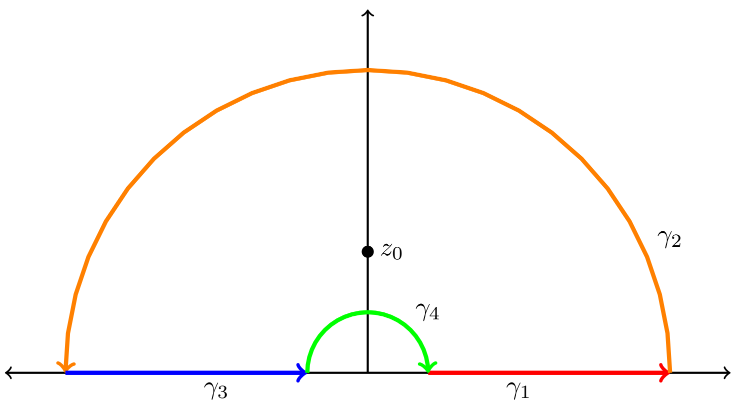
Problem 1. Compute
$$\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx.$$

Proof. For ease of notation, name the integrand f; that is,
$$f(z) = \frac{\log^2 z}{1+z^2}.$$

We will compute the integral by using the Residue Theorem together with (the limit of) a contour carefully designed to avoid the singularity at the origin, and including one of the simple poles of f:



$$\gamma_1 = \{t + 0i \mid x \in [\varepsilon, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [0, \pi]\} \tag{2}$$
$$\gamma_3 = \{0 + ti \mid t \in [-R, -\varepsilon]\} \tag{3}$$
$$\gamma_4 = \{\varepsilon e^{-it} \mid t \in [-\pi, 0]\}. \tag{4}$$

For small ϵ and large R, this contour encloses a single simple pole of f, namely $z_0 = i$.
$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 2\pi i \operatorname{Res}_i(f).$$

□

In the limit, the integrals over each arcs (γ_2 and γ_4) vanishes.
$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^\pi \frac{\log^2(Re^{it})}{1+R^2e^{2it}} iRe^{it}\,dt\right| \\
&\leq \int_0^\pi \left|\frac{\log^2(Re^{it})}{1+R^2e^{2it}} iRe^{it}\right|\,dt \\
&\leq \int_0^\pi \left|\frac{\log^2(Re^{it})}{R}\right|\,dt \\
&\leq \int_0^\pi \left|\frac{\log^2(R) + 2it\log(R) - t}{R}\right|\,dt
\end{aligned}$$

which vanishes by the ML inequality as $R \to \infty$. Similarly,

$$
\begin{aligned}
\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^\pi \frac{\log^2(\varepsilon e^{it})}{1+\varepsilon^2 e^{2it}} i\varepsilon e^{it}\,dt\right| \\
&\le \int_0^\pi \left|\frac{\log^2(\varepsilon e^{it})}{1+\varepsilon^2 e^{2it}} i\varepsilon e^{it}\right|\,dt \\
&\le \int_0^\pi \left|\frac{\log^2(\varepsilon e^{it})}{1} i\varepsilon e^{it}\right|\,dt \\
&\le \int_0^\pi \left|\varepsilon \log^2(\varepsilon e^{it})\right|\,dt \\
&\le \int_0^\pi \left|\varepsilon(\log^2(\varepsilon) + 2it\log(\varepsilon) + t)\right|\,dt,
\end{aligned}
$$

which also vanishes as $\varepsilon \to 0$ by the ML inequality, as can be seen by two applications of L'Hôpital's rule:

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon \log^2(\varepsilon) &= \lim_{\varepsilon\to 0} \frac{\log^2(\varepsilon)}{\varepsilon^{-1}} \\
&= \lim_{\varepsilon\to 0} \frac{2\log(\varepsilon)\varepsilon^{-1}}{-\varepsilon^{-2}} \\
&= \lim_{\varepsilon\to 0} \frac{2\log(\varepsilon)}{-\varepsilon^{-1}} \\
&= \lim_{\varepsilon\to 0} \frac{2\varepsilon^{-1}}{\varepsilon^{-2}} \\
&= \lim_{\varepsilon\to 0} 2\varepsilon \\
&= 0.
\end{aligned}
$$

This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 2\pi i \operatorname{Res}_i(f).$$

And the left-hand side further simplifies to

$$
\begin{aligned}
\int_\varepsilon^R \frac{\log^2 z}{1+z^2}\,dz + (-1)\int_R^\varepsilon \frac{\log^2(-z)}{1+(-z)^2}\,dz &= \int_\varepsilon^R \frac{\log^2 z + \log^2(-z)}{1+z^2}\,dz \\
&= \int_\varepsilon^R \frac{\log^2 z + (\log(z)+\log(-1))^2}{1+z^2}\,dz \\
&= 2\int_{\gamma_1} f(z)\,dz + \int_\varepsilon^R \frac{2\pi i\log(z)}{1+z^2}\,dz + \int_\varepsilon^R \frac{-\pi^2}{1+z^2}\,dz
\end{aligned}
$$

So by the Residue Theorem, the integral evaluates to

$$\int_0^\infty \frac{\log^2 z}{1+z^2}\,dz = \pi i \operatorname{Res}_i(f) - \underbrace{\pi i \int_0^\infty \frac{\log(z)}{1+z^2}\,dz}_{\text{purely imaginary}} - \frac{1}{2}\int_0^\infty \frac{-\pi^2}{1+z^2}\,dz,$$

and by only considering the real part, it is enough to compute the residue and the last integral:

$$\operatorname{Res}_i(f) = \frac{\log^2(i)}{2i} = \frac{(\pi i/2)^2}{2i} = \frac{i\pi^2}{8},$$

and

$$-\pi^2 \int_\varepsilon^R \frac{1}{1+z^2}\,dz = -\frac{\pi^3}{2}$$

Therefore

$$\begin{aligned}
\int_0^\infty \frac{\log^2 z}{1+z^2}\,dz &= \pi i\left(\frac{i\pi^2}{8}\right) - \frac{1}{2}\left(-\frac{\pi^3}{2}\right) \\
&= -\frac{\pi^3}{8} + \frac{\pi^3}{4} \\
&= \frac{\pi^3}{8}.
\end{aligned}$$

Problem 2. Find the number of *distinct* zeros of $f(z) = z^6 + (10-i)z^4 + 1$ inside $(-1,1) \times (-1,1)$.

Proof. First, we will use Rouché's Theorem to establish a bound on the number of roots (with multiplicity) inside of the region $D = (-1,1) \times (-1,1)$.

For a lower bound, we will count the number of roots inside $|z| = 1$ and for an upper bound, we will count the number of roots inside $|z| = \sqrt{2}$. In both cases we will compare against the function $g(z) = (10-i)z^4 + 1$.

(Case 1: $|z| = 1$**)** Notice that when $|z| = 1$,

$$\begin{aligned} |f - g| = |z^6| = 1 & \\ &< |(10-i)z^4| - |z^6| - 1 = |10-i| - 2 \\ &< |f|, \end{aligned}$$

by the triangle inequality. So f and g have the same number of roots inside the unit disk, and g has all four roots inside the unit disk:

$$\begin{aligned} g(z) &= (10-i)z^4 + 1 = 0 \\ |z| &= \left|\frac{-1}{10-i}\right|^{1/4} < 1. \end{aligned}$$

Thus f has at least four roots in D.

(Case 2: $|z| = \sqrt{2}$**)** When $|z| = \sqrt{2}$,

$$\begin{aligned} |f - g| = |z^6| = 8 & \\ &< |(10-i)z^4| - |z^6| - 1 = 4|10-i| - 8 - 1 \\ &< |f|, \end{aligned}$$

by the triangle inequality. And since g has all four roots inside the unit disk, it certainly has all roots inside the disk of radius $\sqrt{2}$.

Now that we have established that f has four roots inside D, it remains to check multiplicity, which can be done by comparing the roots of f and f' inside of D.

Notice that $f'(z) = 6z^5 + 4(10-i)z^3$ factors as $f'(z) = z^3(6z^2 + 40 - 4i)$. Clearly f does not have any roots at $z = 0$, so it is enough to check the roots of $6z^2 + 40 - 4i$.

$$\begin{aligned} z^2 &= \frac{40-4i}{6} \\ |z| &= \left|\frac{40-4i}{6}\right|^{1/2} > \sqrt{6}. \end{aligned}$$

Therefore $f'(z)$ does not share any roots with $f(z)$ inside D, and so all roots inside D are distinct. Thus f has exactly four distinct roots inside D. $\square$

Problem 3.

Proof. $\square$

Problem 4.

Proof. □