Complex Analysis: Homework 13

Peter Kagey

April 16, 2018

Problem 3. (page 206)
The formula (42) permits us to evaluate the *probability integral*

$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\int_0^\infty e^{-x}x^{-1/2}dx = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

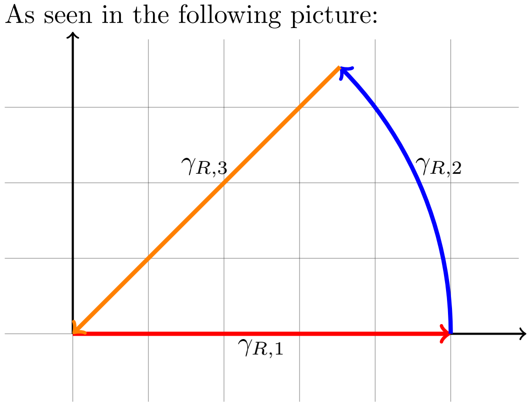
Use this result together with Cauchy's theorem to compute the *Fresnel integrals*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{1}{2}\sqrt{\pi/2}$$

Proof.
The plan is to use the construction from Wikipedia and integrate $f(z) = e^{-z^2}$ along the contour given by

$$\gamma_{R,1} = \{t + 0i \mid x \in [0, R]\} \tag{1}$$
$$\gamma_{R,2} = \{Re^{it} \mid t \in [0, \pi/4]\} \tag{2}$$
$$\gamma_{R,3} = \{te^{i\pi/4} \mid t \in [0, R]\}. \tag{3}$$

As seen in the following picture:



The first integral is known:

$$\lim_{R\to\infty}\int_{\gamma_{R,1}} f(z)\,dz = \int_0^\infty e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}.$$

The second integral vanishes in the limit:

$$\int_{\gamma_{R,2}} f(z)\,dz = iR\int_0^{\pi/4} \exp(-(Re^{it})^2)\cdot e^{it}\,dt \tag{4}$$

so by looking at the modulus, we get

$$\begin{aligned}
\left|\int_{\gamma_{R,2}} f(z)\,dz\right| &\leq R\int_0^{\pi/4} |\exp(-R^2 e^{2it})|\cdot|e^{it}|\,dt && (5)\\
&= R\int_0^{\pi/4} |e^{-R^2\cos(2t)}|\cdot\underbrace{|e^{-iR^2\sin(2t))}|}_{=1}\,dt && (6)\\
&\leq R\int_0^{\pi/4} |e^{-R^2(\pi/4-t)}|\,dt && (7)\\
&= \frac{R}{e^{R^2\pi/4}}\int_0^{\pi/4} |e^{R^2 t}|\,dt && (8)\\
&= \frac{R}{e^{R^2\pi/4}}\left[\frac{e^{R^2 t}}{R^2}\right]_0^{\pi/4} && (9)\\
&= \frac{R}{e^{R^2\pi/4}}\left[\frac{e^{R^2\pi/4}}{R^2}-\frac{1}{R^2}\right] && (10)\\
&= \frac{1}{R}-\frac{1}{Re^{R^2\pi/4}} && (11)\\
&\leq \frac{1}{R}. && (12)
\end{aligned}$$

Thus

$$\lim_{R\to\infty}\int_{\gamma_{R,2}} f(z)\,dz = 0.$$

Next, the third integral:

$$\begin{aligned}
\int_{\gamma_{R,3}} f(z)\,dz &= \int_R^0 e^{i\pi/4}\exp(-t^2\underbrace{e^{i\pi/2}}_{=i})\,dt\\
&= e^{i\pi/4}\int_R^0 e^{-it^2}\,dt\\
&= e^{i\pi/4}\int_R^0 \cos(-t^2)+i\sin(-t^2)\,dt
\end{aligned}$$

Because f is entire, it follows from Cauchy's theorem that

$$\int_{\gamma_{R,1}} f(z)\,dz + \int_{\gamma_{R,2}} f(z)\,dz + \int_{\gamma_{R,3}} f(z)\,dz = 0,$$

including in the limit, therefore

$$\begin{aligned}
\lim_{R\to\infty}\left(-\int_{\gamma_{R,3}} f(z)\,dz\right) &= e^{i\pi/4}\int_0^\infty \cos(-t^2)+i\sin(-t^2)\,dt\\
&= -\frac{1}{2}\sqrt{\pi}.
\end{aligned}$$

This means that

$$\begin{aligned}
\int_0^\infty \cos(-t^2)+i\sin(-t^2)\,dt &= \int_0^\infty \cos(t^2)-i\sin(t^2)\,dt && (13)\\
&= \frac{\sqrt{\pi}}{2e^{i\pi/4}} && (14)\\
&= \left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{\frac{\pi}{2}} && (15)
\end{aligned}$$

so by looking at the real and purely imaginary parts it follows that

$$\int_0^\infty \cos(t^2)\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \int_0^\infty \sin(t^2)\,dt. \tag{16}$$

□

Problem 2. (page 212)
Assume that $f(z)$ has genus zero so that

$$f(z) = z^m \prod_n \left(1 - \frac{z}{a_n}\right).$$

Compare $f(z)$ with

$$g(z) = z^m \prod_n \left(1 - \frac{z}{|a_n|}\right)$$

and show that the maximum modulus $\max\limits_{|z|=r} |f(z)|$ is less than or equal to the maximum modulus of g, and the minimum modulus of f is greater than or equal to the minimum modulus of g.

Proof.

□