Math 533: Homework 4

Peter Kagey

Wednesday, February 27, 2019

Problem 1.

Proof.

(a) The number of partitions of n with no parts divisible by d is given by the generating function

$$\begin{aligned}
F(x) &= \frac{(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots}{(1+x^d+x^{2d}+\ldots)(1+x^2d+x^{4d}+\ldots)\ldots} \\
&= \frac{\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{ij}}{\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{dij}} = \prod_{i=1}^{\infty} \frac{1-x^{di}}{1-x^i} \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i} - \frac{x^{di}}{1-x^i}\right) \\
&= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots) - (x^{di}+x^{(d+1)i}+\ldots) \\
&= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots+x^{(d-1)i}),
\end{aligned}$$

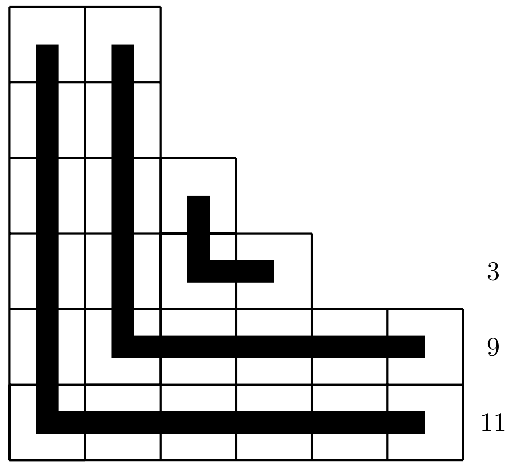
which is identically the generating function for the number of partitions of n with no part repeated d or more times.

(b) The number of paritions of n in which no part is a square is given by the generating function

$$\begin{aligned}
F(x) &= \frac{(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\ldots}{(1+x^1+x^2+\ldots)(1+x^4+x^8+\ldots)(1+x^9+x^{18}+\ldots)\ldots} \\
&= \frac{\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{ij}}{\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{i^2 j}} = \prod_{i=1}^{\infty} \frac{1-x^{i^2}}{1-x^i} \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i} - \frac{x^{i^2}}{1-x^i}\right) \\
&= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots) - (x^{i^2}+x^{i^2+i}+x^{i^2+2i}+\ldots) \\
&= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots+x^{i^2-i}),
\end{aligned}$$

which is the generating function for the number of partitions of n where no part j is repeated j or more times.

(c) Here it's easy to see from the Young Diagram, and best illustrated with an example:



Thus the self-conjugate partition $(6, 6, 4, 3, 2, 2)$ is in bijection with the partition $(11, 9, 3)$.

$\square$

Problem 2.

Proof. ($\Longrightarrow$) Assume $\lambda \leq \nu$
($\Longleftarrow$) Assume there is a sequence of partitions $\lambda = \mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k)} = \nu$ such that $\mu^{(m+1)}$ is a raise of $\mu^{(m)}$.
By hypothesis, $\mu^{(m+1)} = (\mu_1^{(m)}, \ldots, \mu_i^{(m)} + 1, \ldots, \mu_j^{(m)} - 1, \ldots \mu_\ell^{(m)})$. There are three cases of partial sums to consider:

(i) When the partial sums have $k < i$ terms, then $\mu_1^{(m+1)} + \mu_k^{(m+1)} = \mu_1^{(m)} + \mu_k^{(m)}$.

(ii) When the partial sums have $k \geq i$ terms but fewer than j terms, then

$$\begin{aligned}\mu_1^{(m+1)} + \ldots + \mu_k^{(m+1)} &= \mu_1^{(m)} + \ldots + \mu_i^{(m)} + 1 + \ldots + \mu_k^{(m)} \\ &= \mu_1^{(m)} + \ldots + \mu_k^{(m)} + 1.\end{aligned}$$

(iii) When the partial sums have $k > j$ terms, then

$$\begin{aligned}\mu_1^{(m+1)} + \ldots + \mu_k^{(m+1)} &= \mu_1^{(m)} + \ldots + \mu_i^{(m)} + 1 + \ldots + \mu_j^{(m)} - 1 + \ldots + \mu_k^{(m)} \\ &= \mu_1^{(m)} + \ldots + \mu_k^{(m)}.\end{aligned}$$

Therefore $\mu^{(m)} \leq \mu^{(m+1)}$ and

$$\lambda = \mu^{(0)} \leq \mu^{(1)} \leq \ldots \leq \mu^{(k)} = \nu,$$

and so $\lambda \leq \nu$. $\square$

Problem 3.

Proof. □

Problem 4.

Proof.

1. There are $n!$ ways to linearly order $[n]$. Then partitioning into sizes of $(1^{a_1}, 2^{a_2}, \dots k^{a_k})$ within each cycle of size n there are n equivalent ways to pick the first element. Similarly, if there are a_i cycles of size i, there are $a_i!$ equivalent ways to arrange the cycles. Thus there are

$$\frac{n!}{\underbrace{1\cdots 1}_{a_1} \cdot \underbrace{2\cdots 2}_{a_2} \cdots \underbrace{k\cdots k}_{a_k} a_1! a_2! \cdots a_k!} = \frac{n!}{z_\lambda}$$

2. Let S_n act on itself by conjugation. It is sufficient to compute the size of the stabilizer of w,

$$\mathrm{Stab}(w) = \{\sigma \in S_n : \sigma w \sigma^{-1} = w\} = \{\sigma \in S_n : \sigma w = w\sigma\},$$

   and the orbit is

$$\mathcal{O}(w) = \{\sigma w \sigma^{-1} : \sigma \in S_n\}.$$

   Since conjugation preserves cycle type, $\mathcal{O}(w)$ consists of all elements of cycle type λ and $\#\mathcal{O}(w) = n!/z_\lambda$, as above. Then by the orbit-stabilizer theorem,

$$\#\,\mathrm{Stab}(w) = \frac{\#S_n}{\#\mathcal{O}(w)} = \frac{n!}{n!/z_\lambda} = z_\lambda,$$

   as desired.

□

Problem 5.

Proof. □

Problem 6.

Proof.

(a) Notice that if $f_1, f_2 \in \bigoplus_n \Gamma^n$, then $f_1 f_2 \in \bigoplus_n \Gamma^n$ and $f_1 + f_2 \in \bigoplus_n \Gamma^n$. Thus in order to show that an element of Γ is in $\bigoplus_n \Gamma^n$, it is sufficient to show that the generators $p_1, p_3, p_5, \ldots$ are in $\bigoplus_n \Gamma^n$. Clearly $p_{2i-1} \in \Gamma^{2i-1}$, and furthermore,

$$\begin{aligned} p_{2i-1}(x_1, -x_1, x_3, x_4, \ldots) &= x_1^{2i-1} + (-x_1)^{2i-1} + x_3^{2i-1} + x_4^{2i-1} + \ldots \\ &= x_3^{2i-1} + x_4^{2i-1} + \ldots \\ &= p_{2i-1}(x_3, x_4, \ldots). \end{aligned}$$

Next it is necessary to show that any element of $\bigoplus_n \Gamma^n$ can be written as a sum of products of odd power sum symmetric functions.

(b) Since the power sum symmetric functions are multiplicative, it is sufficient to show that the number of partitions of n into distinct parts is equal to the number of partitions into odd parts. This is easy enough to show by generating function trickery. The generating function for partitions into distinct parts is given by

$$\begin{aligned} (1+x)(1+x^2)(1+x^3)\cdots &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots \\ &= (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^5 + x^{10} + \ldots) \cdots , \end{aligned}$$

which is the generating function for partitions into odd parts.

$\square$