Math 510b: Homework 4

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Problem 8.7(i) (Rotman).

Proof. By definition,

$$U_{(p)}(n, M \oplus N) = \dim\left(\frac{p^n(M \oplus N)}{p^{n+1}(M \oplus N)}\right)$$

and

$$U_{(p)}(n, M) + U_{(p)}(n, N) = \dim\left(\frac{p^n M}{p^{n+1} M}\right) + \dim\left(\frac{p^n N}{p^{n+1} N}\right)$$

.

Consider the usual quotient map

$$p^n(M \oplus N) \cong p^n M \oplus p^n N \to \frac{p^n M}{p^{n+1} M} \oplus \frac{p^n N}{p^{n+1} N},$$

which has kernel $p^{n+1}M \oplus p^{n+1}N \cong p^{n+1}(M \oplus N)$, so by the first isomorphism theorem

$$\frac{p^n(M \oplus N)}{p^{n+1}(M \oplus N)} \cong \frac{p^n M}{p^{n+1} M} \oplus \frac{p^n N}{p^{n+1} N}$$

and thus

$$\dim\left(\frac{p^n(M \oplus N)}{p^{n+1}(M \oplus N)}\right) = \dim\left(\frac{p^n M}{p^{n+1} M} \oplus \frac{p^n N}{p^{n+1} N}\right) = \dim\left(\frac{p^n M}{p^{n+1} M}\right) + \dim\left(\frac{p^n N}{p^{n+1} N}\right)$$

as desired. □

Problem 8.31 (Rotman). We will start by computing the characteristic polynomials of the three matrices,

(a)

$$\begin{bmatrix} x-1 & 2 \\ 3 & x-4 \end{bmatrix} = x^2 - 5x - 2$$

which has roots α and $\overline{\alpha}$. Since the minimal and characteristic polynomials of A do not coincide (i.e. A does not satisfy $A = \alpha I$ or $A = \overline{\alpha} I$,) A is similar to the companion matrix of its characteristic polynomial, $\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$.

(b)

$$\begin{bmatrix} x-2 & 0 & 0 \\ 1 & x-2 & 0 \\ 0 & 0 & x-3 \end{bmatrix} = (x-2)^2(x-3) = x^3 - 7x^2 + 16x - 12$$

has only two proper divisors of degree 2, $(t-2)^2 = t^2 - 4t + 4$ and $(t-2)(t-3) = t^2 - 5t + 6$, and B does not satisfy either. Thus it is similar to the companion matrix of its characteristic polynomial,

$$\begin{bmatrix} 0 & 0 & 12 \\ 1 & 0 & -16 \\ 0 & 1 & 7 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} x-2 & 0 & 0 \\ 1 & x-2 & 0 \\ 0 & 1 & x-2 \end{bmatrix} = (x-2)^3 = x^3 - 6x^2 + 12x - 8$$

has only one proper divisors of degree 2, $(t-2)^2 = t^2 - 4t + 4$, and C does not satisfy it. Thus it is similar to the companion matrix of its characteristic polynomial,

$$\begin{bmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{bmatrix}.$$

Proof. □

Problem 8.39 (Rotman).

Proof. Let N be a nilpotent 6×6 matrix. Since $N^6 = 0$, its minimal polynomial is of the form x^k, for $1 \leq k \leq 6$ and its characteristic polynomial is of the form $x^k f(x)$ where $f(x)$ is a degree $n-k$ polynomial. $\square$

Problem 1.

Proof. First, note that if A is invertible, then AB and BA are similar (via $B(AB)B^{-1}$), so they have the same characteristic polynomial. If A is not invertible, then since invertible matrices are dense (Zariski topology), there is A_ε invertible in any neighborhood of A, so $A_\varepsilon B \simeq BA_\varepsilon$, and since this is true in any neighborhood, AB and BA must have the same characteristic polynomial. $\square$

Problem 2.

Proof.

(a) Per the hint, let $r \in R$ and define

$$p_r(x) = \prod_{g \in G} (x - g(r)).$$

It is sufficient to show that $p_r(x) \in R^G[x]$, that is p_r is fixed under the action of any $g' \in G$. Now let an arbitrary $g' \in G$ act on the coefficients of $p_r(x)$,

$$g' \cdot p_r(x) = \prod_{g \in G} (g'(1)x - g' \cdot g(r) = \prod_{g \in G} (x - g(r)) = p_r(x)$$

where $g'(x) = x$ because g' fixes all elements of $R^G[x]$.

(b)

$\square$

Problem 3.

Proof. Let $b|a$ with $a = bk$. There is a simple way to construct a submodule $N \subset M$ such that $(b) \in \exp(N)$ (and thus $(b) \subset \exp(N)$), namely let $N = kM$, so that $bkM = aM = \{0\}$.

Now for any arbitrary elements $c \in \exp(N), n \in N$, $cn = 0$ and $cn = ckm$ for some $m \in M$, thus $ck \in \exp(M) = (a)$, that is, there exists some h such that $ck = ah = bkh$. Thus since R is a PID, $c = bh$, and so $c \in (b)$, so $\exp(N) \subset (b)$.

Combining with the above, $\exp(N) = (b)$. $\square$

Problem 4.

Proof.

(a) ($\Longrightarrow$) Assume that A has order a power of p, namely $A^{p^e} = I$. This means that A satisfies

$$x^{p^e} - 1 \equiv (x-1)^{p^e} \bmod p,$$

and so the minimum polynomial looks like $m_A(x) = (x-1)^k$ for some $k \leq n$ (since the minimum polynomial also divides the characteristic polynomial which is at most degree n). Thus

$$(A-I)^n = \underbrace{(A-I)^k}_{m_A(A)=0}(A-I)^{n-k} = 0.$$

($\Longleftarrow$) Assume $(A-I)^n = 0$, then the minimum polynomial $m_A(x) = (x-1)^k$ for some $k \leq n$, and thus for some $p^e \geq k$,

$$m_A(x)(x-1)^{p^e-k} \equiv (x-1)^{p^e} \equiv x^{p^e} - 1 \pmod p$$

so

$$m_A(A)(A-I)^{p^e-k} = 0 = A^{p^e} - I$$

and $A^{p^e} = I$, as desired.

(b) This is very similar to the second part of the above argument. Assume $(A-I)^n = 0$, then the minimum polynomial $m_A(x) = (x-1)^k$ for some $k \leq n$, and thus $np > p \geq k$, so

$$m_A(x)(x-1)^{np-k} \equiv (x-1)^{np} \equiv x^{np} - 1 \pmod p$$

so

$$m_A(A)(A-I)^{np-k} = 0 = A^{np} - I$$

and $A^{np} = I$, as desired.

(c) We know that A is diagonalizable whenever its minimal polynomial splits over the field, and in this case, $(x-1)^k$ splits over the field.

$\square$