Combinatorics: Homework 4

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Problem 68. [2]
Let $n \geq 1$, and let $f(n)$ be the number of partitions of n such that for all k, the part k occurs at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $i(i+1)$, i.e., no parts equal to $2, 6, 12, 20, \ldots$. Show that $f(n) = g(n)$.

Solution.
We can write a generating function for f, by

$$\sum_{n=1}^{\infty} f(n)x^n = (1+x)(1+x^2+x^4)(1+x^3+x^6+x^9)\ldots = \prod_{i=1}^{\infty}\sum_{j=0}^{i} x^{ij},$$

where we choose at most one 1 in the partition, at most two 2s in the partition, etc. Also, the generating function for g is

$$\sum_{n=1}^{\infty} g(n)x^n = \frac{1}{(1-x)(1-x^2)\ldots}\prod_{i\geq 1} 1 - x^{i(i+1)} = \prod_{i\geq 1}\frac{1-x^{i(i+1)}}{1-x^i}.$$

It is enough to show that these two generating functions are equal.
In the generating function for f, we can write

$$\begin{aligned}\sum_{j=0}^{i} x^{ij} &= \sum_{j=0}^{\infty} x^{ij} - \sum_{j=(i+1)}^{\infty} x^{ij} \\ &= \frac{1}{1-x^i} - \frac{x^{i(i+1)}}{1-x^i} \\ &= \frac{1-x^{i(i+1)}}{1-x^i}.\end{aligned}$$

Thus the generating function for f can be rewritten as

$$\sum_{n=1}^{\infty} f(n)x^n = \prod_{i=1}^{\infty}\frac{1-x^{i(i+1)}}{1-x^i} = \sum_{n=1}^{\infty} g(n)x^n,$$

so f and g are equal.

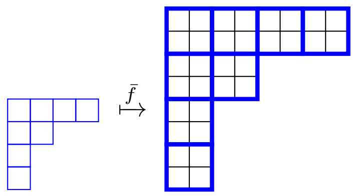
Problem 69. [2]
Let $f(n)$ denote the number of self-conjugate partitions of n all of whose parts are even. Express the generating function $\sum_{n\geq 0} f(n)x^n$ as a simple product.

Solution.
There exists an "obvious" bijection between self-conjugate partitions of n and self-conjugate partitions of n into even parts. Namely by looking at the Ferrer's diagram of a self-conjugate partition, then the bijection ϕ just "scales up" the diagram by a factor of two. In terms of the partition, each term is doubled and duplicated. (This is clearly injective; it is surjective because all self-conjugate partitions into even parts must have the parts come in "pairs", otherwise the conjugate would not have event parts.)

For example,

$$f((4,2,1,1)) = (8,8,4,4,2,2,2,2)$$

with the following diagram.



Therefore we can reuse the generating function (1.80) that Stanley gives for the number of self conguate partitions—only it must be scaled by a factor of four:

$$\sum_{n\geq 0} f(n)x^n = (1+x^4)(1+x^{12})(1+x^{20})\ldots = \prod_{n\geq 0} 1 + x^{4(2n+1)}.$$

Problem 84. [2]
Show that the number of partitions of n in which each part appears exactly $2, 3$, or 5 times is equal to the number of partitions of n into parts congruent to ± 2, ± 3, $6 \pmod{12}$.

Solution.
Let $f(n)$ denote that number of partitions of n in which each part appears exactly $2, 3$, or 5 times, and let $g(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, $6 \pmod{12}$. Then, we have that

$$\sum_{n=0}^{\infty} f(n)x^n = \prod_{n=1}^{\infty} 1 + x^{2n} + x^{3n} + x^{5n},$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} g(n)x^n &= \prod_{n=1}^{\infty} \left(1 + x^{2n} + x^{3n} + x^{6n} + x^{9n} + x^{10n} + x^{12n} + \dots\right) \\ &= \prod_{n=1}^{\infty} \left((1 + x^{2n} + x^{3n} + x^{6n} + x^{9n} + x^{10n})(1 + x^{12n} + x^{24n} + \dots)\right) \\ &= \prod_{n=1}^{\infty} \left(\frac{1 + x^{2n} + x^{3n} + x^{6n} + x^{9n} + x^{10n}}{1 - x^{12n}}\right)\end{aligned}$$

So if we multiply the generating function of f by this denominator,

$$\sum_{n=0}^{\infty} f(n)x^n = \prod_{n=1}^{\infty} \frac{(1 + x^{2n} + x^{3n} + x^{5n})(1 - x^{12n})}{1 - x^{12n}},$$

we can see it is enough to show equality of the generating functions

$$\prod_{n=1}^{\infty} (1 + x^{2n} + x^{3n} + x^{5n})(1 - x^{12n}) = \prod_{n=1}^{\infty} 1 + x^{2n} + x^{3n} + x^{5n} - x^{12n} - x^{14n} - x^{15n} - x^{17n}$$

and

$$\prod_{n=1}^{\infty} 1 + x^{2n} + x^{3n} + x^{6n} + x^{9n} + x^{10n}.$$

(I couldn't get farther than this.)

Problem 85. [2+]
Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions of n into parts not congruent to $\pm 1 \pmod 6$.

Solution.
Let $f(n)$ be the number of partitions of n in which no part appears exactly once, and let $g(n)$ be the number of partitions of n into parts not congruent to $\pm 1 \pmod 6$. Then the generating function for f is

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)x^n &= \prod_{n=1}^{\infty}(1 + x^{2n} + x^{3n} + \ldots) \\
&= \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n} - x^n\right) \\
&= \prod_{n=1}^{\infty}\left(\frac{1-x^n+x^{2n}}{1-x^n}\right) \\
&= \prod_{n=1}^{\infty}\frac{\left(\frac{1+x^{3n}}{1+x^n}\right)}{1-x^n} \\
&= \prod_{n=1}^{\infty}\frac{1+x^{3n}}{1-x^{2n}} \\
&= \left(\prod_{n=1}^{\infty}\frac{1}{1-x^{2n}}\right)\left(\prod_{n=1}^{\infty}1+x^{3n}\right) \\
&= \left(\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^6}\right)\left(\frac{1}{1-x^8}\right)\cdots\right)\left(\prod_{n=1}^{\infty}1+x^{3n}\right).
\end{aligned}$$

Now we use Grant Bowling's trick, and break this product up based on congruence class,

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)x^n &= \left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-4}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-2}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n}}\right)\left(\prod_{n=1}^{\infty}1+x^{3n}\right) \\
&= \left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-4}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-2}}\right)\left(\prod_{n=1}^{\infty}\frac{1+x^{3n}}{1-x^{6n}}\right) \\
&= \left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-4}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-2}}\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{3n}}\right).
\end{aligned}$$

This describes the generating function for g on inspection.

$$\sum_{n=0}^{\infty} g(n)x^n = \underbrace{\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-4}}\right)}_{\lambda_i \equiv 2 \pmod 6}\underbrace{\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{6n-2}}\right)}_{\lambda_i \equiv 4 \pmod 6}\underbrace{\left(\prod_{n=1}^{\infty}\frac{1}{1-x^{3n}}\right)}_{\lambda_i \equiv 3 \pmod 6}.$$

(Use the following procedure: independently choose the number of parts equal to $2, 8, 14, \ldots$, then choose the number of parts equal to $4, 10, 16, \ldots$, and lastly, choose the number of parts equal to $3, 6, 9, \ldots$. Once all of these choices are made, there is only one way to order the partition.)

Problem 102.

(a) [2] Let x and y be variables satisfying the commutation relation $yx = qxy$, where q commutes with x and y. Show that
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}.$$

(b) [2] Generalize to $(x_1 + x_2 + \ldots + x_m)^n$ where $x_i x_j = q x_j x_i$ for $i > j$.

(c) [2+] Generalize further to $(x_1 + x_2 + \ldots + x_m)^n$ where $x_i x_j = q_j x_j x_i$ for $i > j$, and where the q_js are variables commuting with all the x_is and with each other.

Solution.
(I couldn't make progress on this one.)

Problem 125. [2+]
Find the number $f(n)$ of binary sequences $w = a_1a_2 \dots a_k$ (where k is arbitrary) such that $a_1 = 1$, $a_k = 0$ and $\text{inv}(w) = n$. For instance, $f(4) = 5$, corresponding to the sequences $10000, 11110, 10110, 10010, 1100$. How many of these sequences have exactly j 1s?

Solution.
The key insight here is that each binary sequence that meets the criteria for $f(n)$ uniquely describes a partition of n.

For example, if we look at the sequences in the example, and we mark the number of 0s that come after each 1, we get

$$1\ 0\ 0\ 0\ 0 \Rightarrow 4 \tag{1}$$
$$1\ 1\ 1\ 1\ 0 \Rightarrow 1+1+1+1 \tag{2}$$
$$1\ 0\ 1\ 1\ 0 \Rightarrow 2+1+1 \tag{3}$$
$$1\ 0\ 0\ 1\ 0 \Rightarrow 3+1 \tag{4}$$
$$1\ 1\ 0\ 0 \Rightarrow 2+2. \tag{5}$$

Therefore $f(n)$ is what Stanley calls $p(n)$, the number of partitions of n, and the number of sequences with exactly j parts is what Stanley calls $p_j(n)$, the number of partitions of n with exactly j parts.