Fall 2012: Complex Analysis Graduate Exam

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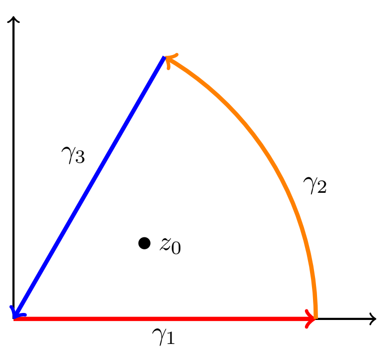
Problem 1. Evaluate the integral
$$\int_0^\infty \frac{dx}{1+x^n}\,dx$$
being careful to justify your methods.

Proof.
First notice that the integrand $f(z) = (1+x^n)^{-1}$ has poles at
$$\begin{aligned} 1 + x^n &= 0 \\ x^n &= e^{\pi i + 2\pi i k} \\ z_k &= e^{(2k+1)\pi i/n} \text{ where } 0 \leq k < n. \end{aligned}$$

The idea is to draw a contour around the first pole $z_0 = e^{\pi i/n}$ along an n-th root of unity, and then compute the integral via the Residue Theorem. In particular, we will use the contour given by:



$$\gamma_1 = \{t + 0i \mid x \in [0, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [0, 2\pi/n]\} \tag{2}$$
$$\gamma_3 = \{te^{2\pi i/n} \mid t \in [0, R]\} \tag{3}$$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

In the limit, the integral over γ_2 vanishes.
$$\begin{aligned} \left| \int_{\gamma_2} f(z)\,dz \right| &= \left| \int_0^{2\pi/n} \frac{dt}{1+(Re^{it})^n} iRe^{it} \right| \\ &\leq \int_0^{2\pi/n} \left| \frac{iRe^{it}}{1+R^n e^{tni}} \right| dt \\ &\leq \int_0^{2\pi/n} \left| \frac{iRe^{it}}{R^n e^{tni}} \right| dt \\ &= \frac{1}{R^{n-1}} \int_0^{2\pi/n} dt \\ &= \frac{2\pi}{nR^{n-1}} \end{aligned}$$

which vanishes as $R \to \infty$. This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

Also the integral over γ_3 is a multiple of the integral over γ_1,

$$\begin{aligned} \int_R^0 \frac{1}{1+(te^{2\pi i/n})^n} e^{2\pi i/n}\,dt &= -e^{2\pi i/n} \int_0^R \frac{dt}{1+t^n} \\ &= -e^{2\pi i/n} \int_{\gamma_1} f(z)\,dz, \end{aligned}$$

so the equation further simplifies to

$$\int_{\gamma_1} f(z)\,dz - e^{2\pi i/n} \int_{\gamma_1} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

So by the Residue Theorem, the integral evaluates to

$$\int_{\gamma_1} f(z)\,dz = \frac{2\pi i \operatorname{Res}_{z_0}(f)}{1-e^{2\pi i/n}},$$

and it is enough to compute the residue:

$$\operatorname{Res}_{z_0}(f) = \lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} \frac{1}{\left(\dfrac{1+z^n}{z-z_0}\right)} = \frac{1}{\dfrac{d}{dz}\left[1+z^n\right]_{z=z_0}} = \frac{1}{nz_0^{n-1}}$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{dx}{1+x^n} &= \frac{2\pi i}{nz_0^{n-1}(1-e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{e^{\pi i(n-1)/n}(1-e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{\underbrace{e^{\pi i}}_{=-1} e^{-\pi i/n}(1-e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{-e^{-\pi i/n}+e^{\pi i/n}} \\ &= \frac{\pi}{n} \cdot \left(\frac{e^{\pi i/n}-e^{-\pi i/n}}{2i}\right)^{-1} \\ &= \frac{\pi}{n\sin(\pi/n)} \end{aligned}$$

$\square$

Problem 2.

Proof. $\square$

Problem 3.

Proof. $\square$

Problem 4.

Proof. □