Thanksgiving week discussion problems.

Problem 13.3.7. Determine whether or not

$$\mathbf{F}(x,y) = \underbrace{(ye^x + \sin y)}_{P}\,\mathbf{i} + \underbrace{(e^x + x\cos y)}_{Q}\,\mathbf{j}$$

is a conservative vector field.
If it is, find a function f such that $\mathbf{F} = \nabla f$.

Solution.
In two dimensions, we can check if $\mathbf{F}$ is conservative by checking whether or not

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In this case these partial derivatives are both equal to

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = e^x + \cos y.$$

To recover the original function, we'll integrate P with respect to x and Q with respect to Y

$$\int ye^x + \sin y \, dx = ye^x + x\sin y + f_1(y)$$
$$\int e^x + x\cos y \, dy = ye^x + x\sin y + f_2(x)$$

Therefore $f_1(y) = f_2(x)$ are both constants, and

$$F(x,y) = \nabla(ye^x + x\sin y + c).$$

Problem 13.5.11. Determine whether or not

$$\mathbf{F}(x,y,z) = \underbrace{y^2z^3}_{P}\,\mathbf{i} + \underbrace{2xyz^3}_{Q}\,\mathbf{j} + \underbrace{3xy^2z^2}_{R}\,\mathbf{k}$$

is a conservative vector field.
If it is, find a function f such that $\mathbf{F} = \nabla f$.

Solution.
We'll start by computing curl $\mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\
&= \left(6xyz^2 - 6xyz^2\right)\mathbf{i} + \left(3y^2z^2 - 3y^2z^2\right)\mathbf{j} + \left(2yz^3 - 2yz^3\right)\mathbf{k} \\
&= 0.
\end{aligned}$$

Therefore $\mathbf{F}$ is conservative. In order to find f such that $F = \nabla f$

$$\begin{aligned}
f(x) = \int y^2z^3\,dx &= \int 2xyz^3\,dy &&= \int 3xy^2z^2\,dz \\
xy^2z^3 + f_1(y,z) &= xy^2z^3 + f_2(x,z) &&= xy^2z^3 + f_3(x,y)
\end{aligned}$$

So $f(x) = xy^2z^3 + c$.

Problem 13.5.13. Determine whether or not

$$\mathbf{F}(x,y,z) = \underbrace{3xy^2z^2}_{P}\,\mathbf{i} + \underbrace{2x^2yz^3}_{Q}\,\mathbf{j} + \underbrace{3x^2y^2z^2}_{R}\,\mathbf{k}$$

is a conservative vector field.
If it is, find a function f such that $\mathbf{F} = \nabla f$.

Solution.
We'll start by computing curl $\mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\
&= \left(6x^2yz^2 - 6x^2yz^2\right)\mathbf{i} + \left(6xy^2z - 6xy^2z^2\right)\mathbf{j} + \left(4xyz^3 - 6xyz^2\right)\mathbf{k} \\
&\neq \langle 0,0,0\rangle.
\end{aligned}$$

Therefore $\mathbf{F}$ is not conservative.

Problem 13.2.19. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $F(x, y) = \langle xy, 3y^2 \rangle$ and C is given by the function $\mathbf{r}(t) = \langle 11t^4, t^3 \rangle$ for $0 \leq t \leq 1$.

Solution.
We can rewrite the integral as

$$\int_{t=0}^{1} \langle 11t^7, 3t^6 \rangle \cdot \langle 44t^3, 3t^2 \rangle \, dt = \int_{t=0}^{1} 484t^{10} + 9t^8 \, dt = [44t^{11} + t^9]_0^1 = 45.$$

Problem 13.2.39. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $2, 1, 0$.

Solution.
Start by parameterizing the line segment

$$\mathbf{r}(t) = \langle 2t, t, 1 - t \rangle.$$

for $0 \leq t \leq 1$ Then then amount of work is the integral of the dot product of force and distance:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
&= \int_{t=0}^{1} \langle 2t - t^2, t - (1 - t)^2, 1 - t - 4t^2 \rangle \cdot \langle 2, 1, -1 \rangle \, dt \\
&= \int_{t=0}^{1} t^2 + 8t - 2 \, dt \\
&= [\frac{1}{3}t^3 + 4t^2 - 2t]_{t=0}^1 \\
&= \frac{7}{3}.
\end{aligned}$$

Problem 13.7.25. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ for

$$\mathbf{F}(x, y, z) = \langle x, -z, y \rangle$$

on the sphere $x^2 + y^2 + z^2 = 2^2$ in the first octant with orientation toward the origin.

Solution.
We can parameterize the eigth sphere in cartesian coordinates with over the quarter disk D by

$$\begin{aligned} z &= \sqrt{4 - x^2 - y^2} \\ 0 \leq y &\leq \sqrt{4 - x^2} \\ 0 \leq x &\leq 2 \end{aligned}$$

so $r(x, y) = (x, y, \sqrt{4 - x^2 - y^2})$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle x, -\sqrt{4 - x^2 - y^2}, y \rangle \cdot (r_y(x, y) \times r_x(x, y))\, dA$$

Where the order of the cross product is given by the right hand rule, and

$$\begin{aligned} r_y(x, y) &= \left\langle 0, 1, -\frac{y}{\sqrt{4 - x^2 - y^2}} \right\rangle \\ r_x(x, y) &= \left\langle 1, 0, -\frac{x}{\sqrt{4 - x^2 - y^2}} \right\rangle \\ r_y(x, y) \times r_x(x, y) &= \left\langle -\frac{x}{\sqrt{4 - x^2 - y^2}}, -\frac{y}{\sqrt{4 - x^2 - y^2}}, -1 \right\rangle. \end{aligned}$$

Evaluating the dot product gives

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \frac{-x^2}{\sqrt{4 - x^2 - y^2}}\, dA.$$

Then switching to polar coordinates, this is

$$\begin{aligned} \iint_D \frac{-x^2}{\sqrt{4 - x^2 - y^2}}\, dA &= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \frac{-r^2 \cos^2(\theta)}{\sqrt{4 - r^2}} r\, dr\, d\theta \\ &= \left(\int_{\theta=0}^{\pi/2} \cos^2(\theta)\, d\theta \right) \left(\int_{r=0}^{2} \frac{-r^3}{\sqrt{4 - r^2}}\, dr \right) \end{aligned}$$

which can be evaluated with ordinary methods to be $-4\pi/3$.

Problem 13.7.27. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ for

$$\mathbf{F}(x, y, z) = \langle 0, y, -z\rangle$$

on the paraboloid $y = x^2 + z^2$ for $0 \leq y \leq 1$ and on the disk $x^2 + z^2 \leq 1$ in the plane $y = 1$.

Solution.
We can parameterize the paraboloid cartesian coordinates over the unit circle D in the xz-plane by

$$r(x, z) = \langle x, x^2 + z^2, z\rangle.$$

We can parameterize the disk over the unit circle in the xz-plane by

$$s(x, z) = \langle x, 1, z\rangle.$$

Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle 0, \underbrace{x^2 + z^2}_{y}, -z\rangle \cdot (r_x(x, y) \times r_z(x, y))\, dA + \iint_D \langle 0, 1, -z\rangle \cdot (s_z(x, y) \times s_x(x, y))\, dA$$

Where the order of the cross product is given by the right hand rule so that the orientation points out:

$$\begin{aligned} r_x(x, z) &= \langle 1, 2x, 0\rangle \\ r_z(x, z) &= \langle 0, 2z, 1\rangle \\ r_x(x, z) \times r_z(x, z) &= \langle 2x, -1, 2z\rangle, \end{aligned}$$

and

$$\begin{aligned} s_z(x, z) &= \langle 0, 0, 1\rangle \\ s_x(x, z) &= \langle 1, 0, 0\rangle \\ s_z(x, z) \times s_x(x, z) &= \langle 0, 1, 0\rangle. \end{aligned}$$

Evaluating the dot products gives

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -(x^2 + y^2) - 2z^2\, dA + \underbrace{\iint_D 1\, dA}_{\pi},$$

where the first integral can be evaluated by polar coordinates with $x = r\cos\theta$ and $z = r\sin\theta$,

$$\begin{aligned} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (-(r^2) - 2r^2 \sin^2\theta) r\, dr\, d\theta &= -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^3 + 2r^3 \sin^2\theta\, dr\, d\theta \\ &= -\left(\int_{\theta=0}^{2\pi} 1 + 2\sin^2\theta\, d\theta\right)\left(\int_{r=0}^{1} r^3\, dr\right) \\ &= (-4\pi)(1/4) = -\pi. \end{aligned}$$

Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\pi + \pi = 0.$$