Combinatorics: Homework 2

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Problem 21. [2]
Fix $n \in \mathbb{P}$. In how many was can one choose a composition α of n, and then choose a composition of each part of α?

Solution.
There are $a(n) = 3^{n-1}$ ways. I will prove this by constructing an explicit bijection between $(n-1)$-letter words over a three letter alphabet and the number of ways of choosing a partition of n and then for each part, choosing a further partition.

We will illustrate this bijection for the case $n = 4$, $\alpha = (1, 3)$, and $\alpha' = ((1), (2, 1))$.
First, write n 1s (with $n - 1$ gaps between them):

$$1 \underbrace{\quad} 1 \underbrace{\quad} 1 \underbrace{\quad} 1.$$

Label each gap with a 1 as in the usual bijection for α:

$$1 \underbrace{\quad}_{1} 1 \underbrace{\quad} 1 \underbrace{\quad} 1.$$

For the next iteration of compositions, label each gap with a 2 in the usual way:

$$1 \underbrace{\quad}_{1} 1 \underbrace{\quad} 1 \underbrace{\quad}_{2} 1.$$

Label all the remaining gaps with 0s:

$$1 \underbrace{\quad}_{1} 1 \underbrace{\quad}_{0} 1 \underbrace{\quad}_{2} 1.$$

Thus the string 102_3 is in bijection with the (sub)composition $\alpha' = ((1), (2, 1))$.

This bijection has some nice properties.

1. If we "flatten" the sub-composition, then this corresponds to the composition made by changing all of the 2s to 1s. For example, $f((1), (2, 1)) = 102_3$ which corresponds to the composition $4 = 1 + 2 + 1$. And $f^{-1}(101_3) = ((1), (2), (1))$ which also corresponds to the composition $1 + 2 + 1$.

2. If we're interested in sub-subcompositions, then we can naturally extend this bijection to prove $n \mapsto 4^{n-1}$, and so on with sub-sub-subcompositions, etc.

Problem 29. [2]
Fix $k, n \in \mathbb{P}$. Show that

$$\sum a_1 \dots a_k = \binom{n+k-1}{2k-1},$$

where the sum ranges over all compositions $(a_1, \dots, a_k)$ of n into k parts.

Solution.
Denote the sum as

$$f(n,k) = \sum_{a_1+\dots+a_k=n} a_1 \dots a_k.$$

We'll let F_k be a generating function where

$$F_k(x) = \sum_n f(n,k) x^n,$$

and we'll show that

$$F_k(x) = \sum_n \binom{n+k-1}{2k-1} x^n.$$

We can write

$$\begin{aligned}
F_k(x) &= \sum_n \sum_{a_1+\dots+a_k=n} a_1 \dots a_k x^n \\
&= \sum_{a_1,\dots,a_k>0} a_1 \dots a_k x^{a_1+\dots+a_k} \\
&= \left(\sum_{a_1>0} a_1 x^{a_1}\right) \cdots \left(\sum_{a_k>0} a_k x^{a_k}\right).
\end{aligned}$$

And each of these sums simplifies nicely

$$\begin{aligned}
\sum_{j=1}^{\infty} j x^j &= x \sum_{j=1}^{\infty} j x^{j-1} \\
&= x \sum_{j=1}^{\infty} \frac{d}{dx}\left[x^j\right] \\
&= x \frac{d}{dx}\left[\sum_{j=1}^{\infty} x^j\right] \\
&= x \frac{d}{dx}\left[\frac{1}{1-x}\right] \\
&= \frac{x}{(1-x)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
F_k(x) &= \left(\frac{x}{(1-x)^2}\right)^k \\
&= x^k \frac{1}{(1-x)^{2k}}.
\end{aligned}$$

Using the multinomial coefficient trick from class yields

$$x^k \frac{1}{(1-x)^{2k}} = x^k \sum_{n-k=1}^{\infty} x^{n-k} \left(\!\binom{2k}{n-k}\!\right) = \sum_{n-k=1}^{\infty} \binom{n-k+2k-1}{n-k} x^n = \sum_{n-k=1}^{\infty} \binom{n+k-1}{2k-1} x^n$$

as desired.

Problem 33 (a). [2−]
Let $k, n \in \mathbb{P}$. Find the number of sequences $\emptyset = S_0, S_1, \ldots, S_k$ of subsets of $[n]$ if for all $1 \leq i \leq k$ we have either

(i) $S_{i-1} \subset S_i$ and $|S_i - S_{i-1}| = 1$, or

(ii) $S_i \subset S_{i-1}$ and $|S_{i-1} - S_i| = 1$.

Solution.
Each set S_i differs from the set before it, S_{i-1} by one element. Thus for $i > 0$, we can construct $S_{i+1} = S_i \triangle j$ for $j \in [n]$ using the symmetric difference. At each step, there are n choices for the singleton set, so there are k^n such sequences.

Problem 38. [2]
Show that the number of permutations $w \in \mathfrak{S}_n$ fixed by the fundamental transformation $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ is the Fibonacci number F_{n+1}.

Solution.
I wrote a program to enumerate the first few cases, and it yielded

$$\omega_{1,1} = (1)$$

$$\begin{aligned}\omega_{2,1} &= (1)(2)\\ \omega_{2,2} &= (21)\end{aligned}$$

$$\begin{aligned}\omega_{3,1} &= (1)(2)(3)\\ \omega_{3,2} &= (21)(3)\\ \omega_{3,3} &= (1)(32)\end{aligned}$$

$$\begin{aligned}\omega_{4,1} &= (1)(2)(3)(4)\\ \omega_{4,2} &= (21)(3)(4)\\ \omega_{4,3} &= (1)(32)(4)\\ \omega_{4,4} &= (1)(2)(43)\\ \omega_{4,5} &= (21)(43)\end{aligned}$$

Let X_n be the set of permutations of $[n]$ that are fixed by the fundamental transformation. Then, the first few cases suggest that for $n \geq 3$, all members of X_n are either

$$\pi(k) = \begin{cases} \omega_{n-1}(k) & k \in [n-1] \\ n & k = n \end{cases}$$

or

$$\pi(k) = \begin{cases} \omega_{n-2}(k) & k \in [n-2] \\ n & k = n-1 \\ n-1 & k = n \end{cases},$$

for some fixed $\omega_{n-1} \in X_{n-1}$ or $\omega_{n-2} \in X_{n-2}$ which depends on π. (In other words, append n to a permutation from the previous generation, or $n(n-1)$ to a permutation from two generations ago.)

Now it is sufficient to show that the permutations fixed by the fundamental transformation must (i) consist only of 1-cycles and 2-cycles, and (ii) also represent a fixed permutation if the last cycle is removed. For the sake of contradiction, suppose we have a k-cycle k-cycle through positions $n, n+1, \cdots, n+k-1$. Then in order for this to be written in canonical notation, the biggest element in the cycle must be written first, so $\omega(n) = n+k-1$. However, this means that the last element in our cycle must be n because n maps to $n+k-1$. Thus $\omega(n) = n+k-1$ and $\omega(n+k-1) = n$, so we have a 2-cycle. This is a contradiction, so the longest cycle must be a 2-cycle.

Consider writing the permutation in cycle notation. The last cycle does not affect the positions of the initial permutation, so the permutation with the last cycle removed must be a fixed permutation if the original permutation is too.

Therefore we can enumerate the permutations inductively by appending a one cycle or a two cycle to the end, and there is only one (canonical) way to write each. Thus

$$f(1) = 1, f(2) = 2, f(n+2) = f(n+1) + f(n).$$

Problem 44 (a). [2]
Using generating functions, show that the total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$.

Solution.
An even permutation of $[n]$ is a permutation with an even number of even cycles, or equivalently, a permutation where the number of cycles has the same parity as the parity of $[n]$. Thus we can write the number total number of even permutations of $[n]$ minus the total number of odd functions of $[n]$ as

$$a_n = (-1)^n \sum_k (-1)^k k\, c(n,k)$$

where $c(n,k)$ is the signless Sterling number of the first kind, which counts the number of permutations of $[n]$ with exactly k cycles. By Proposition 1.3.7, we know that

$$\sum_k^n (-1)^k\, c(n,k) t^k = (t)(t-1)\cdots(t-n+1) = (t)_n.$$

and taking the derivative yields something that looks like our above function

$$\frac{d}{dt}\left[\sum_k^n (-1)^k\, c(n,k)t^k\right] = \sum_k^n (-1)^k k\, c(n,k) t^{k-1}$$

So

$$\begin{aligned}
\sum_k^n (-1)^k k\, c(n,k) t^k &= t\frac{d}{dt}[t(t-1)\dots(t-n+1)] \\
&= t\left((t)_{n-1} + (t-n+1)\frac{d}{dt}\left[(t)_{n-1}\right]\right) \\
&= t\left(\sum_k^{n-1}(-1)^k\, c(n-1,k)t^k + (t-n+1)a_{n-1}\right)
\end{aligned}$$

Thus setting $t=1$ (which we can do because this is a finite sum) yields

$$a_n = (1)(1-1)\dots(2-n) + (2-n)a_{n-1} = (2-n)a_{n-1}.$$

Since the base case $a_1 = 1$ is clear, inductively, we have

$$a_n = (-1)^n(n-2)!.$$