Math 510b: Homework 4

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Monday, March 25, 2019

Problem 7.10 (Rotman). Let R be the ring of all 2×2 upper triangular matrices where $a_{11} \in \mathbb{Q}$ and $a_{12}, a_{22} \in \mathbb{R}$.

(a) Prove that R is right Artinian.

(b) Prove that R is not left Artinian.

(c) Find $J(R)$.

Proof.

(a)

(b) By the hint, consider the case where $V \subset \mathbb{R}$ is a vector space over $\mathbb{Q}$

$$\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} : v \in V \right\},$$

which is a left ideal because

$$\begin{bmatrix} q & r_1 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & qv \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

Then we can construct a descending chain without the descending chain condition, namely, we can come up with an infinite dimensional vector space $V \in \mathbb{R}$ over $\mathbb{Q}$ with basis $\{r_1, r_2, \dots\}$, and

$$(r_1, r_2, r_3, \dots) \supseteq (r_2, r_3, \dots) \supseteq (r_3, \dots) \supseteq \dots$$

is a descending chain that never stops.

(c) $J(R) = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$.

$\square$

Problem 7.17 (Rotman). Let I be a two-sided ideal of R. Prove that if $I \subseteq J(R)$, then

$$J(R/I) = J(R)/I.$$

Proof. This follows from the correspondence theorem for rings, which states that for any two-sided ideal I, there is a bijection φ between two-sided ideals in R that contain I and the set of ideals in R/I.

Then using the naive definition of Jacobson radical, $J(R/I)$ is the intersection of maximal ideals in R/I, and $J(R)/I$ is the intersection of all maximal ideals in R/I under the map φ^{-1}. □

Problem 7.26 (Rotman). Find $\mathbb{C}A_4$.

Proof. First, $|A_4| = 4!/2 = 12$, so $\mathbb{C}A_4$ is twelve-dimensional, and has four conjugacy classes corresponding to the identity, (acb), (abc), and $(ab)(cd)$. Since A_4 is finite, $\mathbb{C}A_4$ is semisimple, so we can decompose $\mathbb{C}A_4$ into direct sums of matrices over $\mathbb{C}$ by Artin-Wedderburn, and such decomopositions must have four summands by Theorem 7.58. Therefore

$$\mathbb{C}A_4 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}_2(\mathbb{C}),$$

because $1 + 1 + 1 + 9$ is the only quadruple of squares that sums to 12. □

Problem 7.54 (Rotman). Prove that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_4(\mathbb{R})$ as $\mathbb{R}$-algebras.

Proof. Since $\mathbb{H}$ is a central simple algebra, Lemma 7.48 (iv) gives that

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{op}} \cong \mathrm{Mat}_n(\mathbb{R}),$$

where $n = [\mathbb{H} : \mathbb{R}] = 4$, and $\mathbb{H} \cong \mathbb{H}^{\mathrm{op}}$ because there is only one 4-dimensional $\mathbb{R}$-algebra up to isomorphism. Thus

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_n(\mathbb{R}),$$

as desired. $\square$

Problem 7.57 (Rotman).

Proof. (i) Consider the ring homomorphism $\varphi\colon \mathbb{C}(x) \times \mathbb{C}(y) \to \mathbb{C}(x, y)$ which sends a pair of polynomials to their product, $(f, g) \overset{\varphi}{\longmapsto} fg$. First, the image of this map $\{f(x)g(y) : f \in \mathbb{C}(x), g \in \mathbb{C}(y)\}$ is a subring, inheriting its ring structure from $\mathbb{C}(x, y)$. It is easy to check that this map is middle linear:

$$\varphi(f + f', g) = fg + f'g = \varphi(f, g) + \varphi(f', g)$$
$$\varphi(f, g + g') = fg + fg' = \varphi(f, g) + \varphi(f, g')$$
$$\varphi(fz, g) = (fz)g = fzg = f(zg) = \varphi(f, zg)$$

Therefore by the universal property of tensor products, there exists a unique map $\widetilde{\varphi}\colon \mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y) \to \mathbb{C}(x, y)$ such that $\widetilde{\varphi} \circ i = \varphi$ where i is the standard projection onto $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$. Now $\widetilde{\varphi}$ is a surjective homomorphism onto its image, and it is injective via $f(x)g(x) \xmapsto{\varphi^{-1}} f \otimes g$ termwise, so it is a ring isomorphism.

(ii) Notice that the subring consisting of rational functions of the form $h(x, y)/f(x)g(y)$ is not a field, in particular, the inverse of an element $f(x)g(y)/h(x, y)$ can not necessarily be written as the quotient of a polynomial with a polynomial in x and a polynomial in y, for example, take $h(x, y)$ as an irreducible polynomial containing x and y and $f(x) = g(y) = 1$.

(iii) Exercise 7.7 states that a left artinian ring R with no left zero-divisors must be a division ring, then if Δ is Artinian with infinite dimension over its center, then $\Delta \otimes_{\Delta} \Delta = \Delta_{Z(\Delta)}$ is not Artinian, because we can construct a descending chain without DCC in the obvious way.

$\square$

Problem 1. Let G be a finite group and let $R = \mathbb{C}G$ be the group algebra.

(a) Show that the number of distinct group homomorphisms from G to $\mathbb{C}^*$ equals the number of copies of $\mathbb{C}$ in the Wedderburn-Artin decomposition of $\mathbb{C}G$

(b) Let $G = S_n$. Show that there are exactly two copies of $\mathbb{C}$ in the Wedderburn-Artin decomposition of $\mathbb{C}G$.

(c) Apply this to S_4 to find the the Wedderburn-Artin decomposition of $\mathbb{C}S_4$.

Proof.

(a)

(b) By part (a), it is sufficient to show that there are only two distinct group homomorphisms from G to $\mathbb{C}^*$, and this is well known: the only two group homomorphisms are $\omega \mapsto 1$ and $\omega \mapsto \operatorname{sgn}(\omega)$.

(c) There is only one 5-tuple of squares that sums to 24, so by earlier arguments

$$\mathbb{C}S_4 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C}).$$

$\square$

Problem 2. Let k be any field and let G be a finite group. In the group algebra kG, let $v = \sum_{g \in G} g$, that is, v is the sum of all the group elements.

(a) Show that $I = kv$ is a one-dimensional 2-sided ideal of kG.

(b) Show the converse of Maschke's theorem: that is, assume that $\operatorname{char} k = p > 0$, and p divides $|G|$. Then not all kG-modules are completely reducible. (hint: use part (a))

Proof.

(a) Firstly, I is one-dimensional with basis $\{v\}$. It's a two sided ideal because

$$v = \sum_{g \in G} g = \sum_{g \in G} g'g = \sum_{g \in G} gg',$$

that is, left (or right) multiplication by a fixed $g' \in G$ is a bijection of sets. Therefore

$$\begin{aligned}(k_1 g_1 + k_2 g_2 + \ldots + k_n g_n)\left(\sum_{g \in G} g\right) &= k_1 \left(\sum_{g \in G} g_1 g\right) + k_2 \left(\sum_{g \in G} g_2 g\right) + \ldots + k_n \left(\sum_{g \in G} g_n g\right) \\ &= \underbrace{(k_1 + k_2 + \ldots + k_n)}_{\in k} \underbrace{\left(\sum_{g \in G} g\right)}_{v} \\ &= \left(\sum_{g \in G} g\right)(k_1 g_1 + k_2 g_2 + \ldots + k_n g_n) \\ &\in I = kG,\end{aligned}$$

so I is indeed an two-sided ideal.

(b) The idea here is that the sum of group elements is always a one-dimensional two-sided ideal of kG.

$\square$

Problem 3.

Proof. 1. Since A is finite dimensional as a vector space, we know that for any descending chain $I_1 \supseteq I_2 \supseteq \ldots$, the dimensions of these subspaces must be weakly decreasing, and so the chain must stabilize. Thus A is Artinian and has only a finite number of maximal ideals, because otherwise the descending chain

$$\mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \subseteq \ldots$$

would violate DCC.

2. Choose $a \neq 0$ and b such that $a = aba$. Then

$$ab(abc - c) = \underbrace{aba}_{a} bc - abc = 0$$

so $abc = c$ and $ab = 1$ (and $1 \in R$). Doing this on the other side yields

$$(cab - c)ab = c \underbrace{aba}_{a} b - cab = 0$$

and so $ab = ba = 1$.

□

Problem 4.

Proof. Define the map $\varphi\colon R/I \times R/J \to R/(I+J)$ by sending

$$(r+I, s+J) \overset{\varphi}{\longmapsto} rs + (I+J).$$

Then it is quick to check that φ is middle linear

$$\begin{aligned}
\varphi((r_1+r_2)+I, s+J) &= r_1 s + r_2 s + (I+J) = \varphi(r_1+I, s+J) + \varphi(r_2+I, s+J)\\
\varphi(r+I, (s_1+s_2)+J) &= rs_1 + rs_2 + (I+J) = \varphi(r+I, s_1+J) + \varphi(r+I, s_2+J)\\
\varphi(rc+I, s+J) &= (rc)s + (I+J) = r(cs) + (I+J) = \varphi(r+I, cs+J).
\end{aligned}$$

Thus by the universal property of tensor products, there exists a unique $\widetilde{\varphi}\colon R/I \otimes_R R/J \to R/(I+J)$ satisfying $\widetilde{\varphi} \circ i = \varphi$.

Furthermore, it has inverse $\widetilde{\varphi}^{-1}(r+(I+J)) = (1+I) \otimes (r+J)$:

$$\begin{aligned}
\widetilde{\varphi} \circ \widetilde{\varphi}^{-1}(r+(I+J)) &= \widetilde{\varphi}((1+I) \otimes (r+J)) = r + (I+J)\\
\widetilde{\varphi}^{-1} \circ \widetilde{\varphi}((r+I) \otimes (s+J)) &= \widetilde{\varphi}^{-1}(rs+(I+J)) = (1+I) \otimes (rs+J) = (r+I) \otimes (s+J)
\end{aligned}$$

as desired. $\square$