Differential Geometry: Homework 6

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Problem 1.

(a) Write in detail the construction of the canonical map $V^* \otimes W \xrightarrow{\alpha} \hom(V, W)$, and give a careful proof that it is an isomorphism if V and W are finite dimensional.

(b) Let $ev\colon V^* \otimes V \to \mathbb{R}$ be the linear map induced by the bilinear map $\overline{ev}\colon V^* \times V \to \mathbb{R}$, $(\phi, v) \mapsto \phi(v)$ by the universal property of the tensor product. Given a linear operator $T \in \hom(V, V)$ on a finite dimensional vector space define

$$tr(T) := ev(\alpha^{-1}(T)).$$

Show that this definition agrees with the usual definition of trace.

Proof.

(a) Firstly there exists a bilinear map $\eta\colon V^* \times W \to \hom_{\mathbb{R}}(V, W)$ given by

$$(v^*, \vec{w}) \mapsto (\vec{v} \mapsto \underbrace{v^*(\vec{v})}_{\in\mathbb{R}}\, \vec{w}).$$

Because $\hom(V, W)$ is an $\mathbb{R}$-vector space, by the universal property, there exists a unique linear map $\alpha = \bar{\eta}$ such that the diagram

$$\begin{tikzcd} V^* \times W \arrow[rr, "\eta"] \arrow[dr, "\phi"'] & & \hom(V,W) \\ & V^* \otimes W \arrow[ur, "\alpha = \bar{\eta}"'] & \end{tikzcd}$$

commutes. Because the universal property gives that α is a linear map, it is sufficient to check that α has a two-sided inverse. Let $\underline{v} = (v_1, \ldots, v_k)$ and $\underline{w} = (w_1, \ldots, w_r)$ be bases for V and W respectively, and let $\underline{v}^* = (v_1^*, \ldots, v_k^*)$ be the associated dual basis for V^*. Then $\{v_i^* \otimes w_j\}_{(j,i)}^{(k,r)}$ is a basis for $V^* \otimes W$, and $\{v_i^*(-)w_j\}_{(j,i)}^{(k,r)}$ is a basis for $\hom(V, W)$.

Then α^{-1} is the map that sends $v_i^*(-)w_j \mapsto v_i^* \otimes w_j$, extended by linearity, and thus α is an isomorphism.

(b) Let $\psi\colon V^* \times V \to V^* \otimes V$ map $(\phi, v) \mapsto \phi \otimes v$. Then by the universal property of tensor products, $\overline{ev} = ev \circ \psi$. That is, ev maps $\phi \otimes v \mapsto \phi(v)$.

Let $\underline{v} = (v_1, \dots, v_n)$ be a basis for V and $\underline{v}^* = (v_1^*, \dots, v_n^*)$ be the associated basis for V^* as above. Then

$$\begin{aligned} T(\vec{v}) &= \sum_{j=1}^{n} v_j^*(\vec{v}) T(v_j) \\ &= \sum_{j=1}^{n} v_j^*(\vec{v}) \left(\sum_{i=1}^{n} A_{ij} v_i \right) \\ &= \sum_{i,j=1}^{n} A_{ij} v_j^*(\vec{v}) v_i \end{aligned}$$

Applying a^{-1} yields

$$a^{-1}(T) = \sum_{i,j=1}^{n} A_{ij} v_j^* \otimes v_i,$$

and further applying ev gives

$$\begin{aligned} ev(a^{-1}(T)) &= \sum_{i,j=1}^{n} A_{ij} v_j^*(v_i) \\ &= \sum_{i,j=1}^{n} A_{ij} \delta_{ij} \\ &= \sum_{i}^{n} A_{ii} \\ &= tr(A) \end{aligned}$$

where $v_i^*(v_j) = \delta_{ij}$, the Kronecker delta, by construction of the associated dual basis for V^*.

$\square$

Problem 2. Exterior algebra 1.
Suppose that $\dim V = 3$ and $\underline{v} = (v_1, v_2, v_3)$ is a basis for V. Let $T\colon V \to V$ be the linear operator defined by

$$\begin{aligned}
T(v_1) &= av_1 + dv_2 + gv_3 \\
T(v_2) &= bv_1 + ev_2 + hv_3 \\
T(v_3) &= cv_1 + fv_2 + iv_3.
\end{aligned}$$

Derive a formula for $\det(T)$ in terms of a, b, c, d, e, f, g, h, and i.

Proof.
Let $\vec{w} = v_1 \wedge v_2 \wedge v_3$. Then

$$\begin{aligned}
T(\vec{w}) &= (av_1 + dv_2 + gv_3) \wedge (bv_1 + ev_2 + hv_3) \wedge (cv_1 + fv_2 + iv_3) \\
&= (av_1 \wedge (bv_1 + ev_2 + hv_3) + dv_2 \wedge (bv_1 + ev_2 + hv_3) + gv_3 \wedge (bv_1 + ev_2 + hv_3)) \\
&\qquad \wedge (cv_1 + fv_2 + iv_3) \\
&= (ab \underbrace{v_1 \wedge v_1}_{=0} + aev_1 \wedge v_2 + ahv_1 \wedge v_3) \\
&\qquad + (dbv_2 \wedge v_1 + de \underbrace{v_2 \wedge v_2}_{=0} + dhv_2 \wedge v_3) \\
&\qquad + (gbv_3 \wedge v_1 + gev_3 \wedge v_2 + gh \underbrace{v_3 \wedge v_3}_{=0})) \\
&\qquad \wedge (cv_1 + fv_2 + iv_3) \\
&= ((ae - db)v_1 \wedge v_2 + (ah - gb)v_1 \wedge v_3 + (dh - ge)v_2 \wedge v_3) \wedge (cv_1 + fv_2 + iv_3) \\
&= (aei - dbi)v_1 \wedge v_2 \wedge v_3 + (fah - fgb) \underbrace{v_2 \wedge v_1}_{=-v_1 \wedge v_2} \wedge v_3 + (cdh - cge)v_1 \wedge v_2 \wedge v_3) \\
&= (aei - dbi - fah + fgb + cdh - cge)v_1 \wedge v_2 \wedge v_3 \\
&= (aei - dbi - fah + fgb + cdh - cge)\vec{w}
\end{aligned}$$

Thus $\det(T) = (aei - dbi - fah + fgb + cdh - cge)$. $\square$

Problem 3. Exterior algebra 2.

(a) (i) Prove there is a canonical isomorphism $A^k(V) \cong \wedge^k V^* \cong (\wedge^k V)^*$.

(ii) Prove there is a canonical isomorphism $L^k(V) \cong (V^*)^{\otimes k} \cong (V^{\otimes k})^*$

(iii) Prove that under these isomorphisms, the natural map $A^k(V) \hookrightarrow L^k(V)$ is sent to the (dual of) the projection map $V^{\otimes k} \to \wedge^k V$.

(b) If V is finite-dimensional and V admits a linear symplectic form, prove that $n = \dim V$ is necessarily even, say $n = 2m$.

(c) Prove that $\omega \in \Lambda^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0 \in \Lambda^n V^*$ (where $n = 2m$).

Proof.

(a) (i) Let $f \in A^k(V)$ be multilinear map from k copies of V to $\mathbb{R}$, and let

$$\psi\colon \underbrace{V \times \ldots \times V}_{k} \to \wedge^k V \text{ send } (v_1, \ldots, v_k) \mapsto v_1 \wedge \ldots \wedge v_k.$$

Then by the universal property of $\wedge^k V$, there exists a unique linear map $\bar{f}\colon \wedge^k V \to \mathbb{R}$ such that $\bar{f} \circ \psi = f$. Thus the isomorphism $\phi\colon A^k(V) \to \wedge^k V^*$ is the map that sends $f \mapsto \bar{f}$ via the universal property. We can recover f by composing with ψ; that is, ϕ^{-1} maps $\bar{f} \mapsto \underbrace{\bar{f} \circ \psi}_{=f}$.

(ii) Similarly, suppose $\eta \in L^k(V)$ is a map from k copies of V to $\mathbb{R}$, and let

$$\varphi\colon \underbrace{V \times \ldots \times V}_{k} \to V^{\otimes k} \text{ send } (v_1, \ldots, v_k) \mapsto v_1 \otimes \ldots \otimes v_k.$$

Then by the universal property of tensor products, there exists a map $\bar{\eta} : V^{\otimes k} \to \mathbb{R}$ such that $\bar{\eta} \circ \varphi = \eta$. Thus the isomorphism $\Phi : L^k(V) \to V^{\otimes k}$ is the map that sends $\eta \mapsto \bar{\eta}$ via the universal property. We similarly recover η by composing with φ; that is Φ^{-1} maps $\bar{\eta} \mapsto \underbrace{\bar{\eta} \circ \varphi}_{=\eta}$.

(iii) Call the above isomorphisms $\phi\colon A^k(V) \xrightarrow{\cong} (\Lambda^k V)^*$ and $\psi\colon L^k(V) \xrightarrow{\cong} (V^{\otimes k})^*$ respectively, and let $i\colon A^k(V) \to L^k(V)$ be the inclusion map.

(b) Consider $\omega(v_0, -) \in \hom_{\mathbb{R}}(V, \mathbb{R})$. Because ω is alternating multilinear, for all u, $\omega(v_0, u) - \omega(u, v_0) = 0$ and therefore must be of the form

$$\omega((v_{0,1}, \ldots, v_{0,n}), (u_1, \ldots, u_n)) = \sum_{i,j} \alpha_{ij} v_{0,i} u_j$$

where $\alpha_{ij} = -\alpha_{ji}$ to satisfy the alternating condition. This is equivalent to a skew-symmetric matrix, which is singular whenever the matrix has odd dimension by Jacobi's Theorem. Thus if $\omega(v_0, -)$ (i.e. does not have an underlying singular matrix) then $\dim V$ must be even.

(c) I don't know how to prove any of this, but I think the "nice form with respect to some basis" means that $\omega = \omega' + \sum_{(i,j) \in P} \alpha_{ij} v_i^* \wedge v_j^*$ where v_i are basis elements and P is a partition of $\{1, 2, \ldots, n\}$ into m equal pieces. Then by the binomial theorem, there will be some squarefree (hence nonzero) term.

$\square$

Problem 4. Give a careful construction of the exterior differentiation operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ using local coordinates; show that this definition is independent of local coordinates and is well-defined.

Proof.
We know how to compute $d\colon \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^{m+1})$, so the idea is to push forward to local coordinates, apply d there, and pull back to the manifold.
Let $\omega \in \Omega^k(M)$ be a k-form, and consider a chart $(U \subset M, \phi)$ around p. Then restrict to U so that

$$\omega_{\mathrm{loc},U} = (\phi^{-1})^*\omega = \sum_I f_I dx_I \in \Omega^k(\mathbb{R}^n)$$

is the corresponding (local) k-form. Then applying d, yields

$$d\omega_{\mathrm{loc},U} = d((\phi^{-1})^*\omega)$$

and pulling back to the manifold gives

$$d\omega = \phi^* d((\phi^{-1})^*\omega).$$

Now choosing another chart (V, ψ) around p, we want to show

$$\phi^*(d((\phi^{-1})^*\omega)) = d\omega = \psi^* d((\psi^{-1})^*\omega).$$

So we'll write $d(x_I \circ \phi)$ as shorthand for $d(x_{i_1} \circ \phi) \wedge \ldots \wedge d(x_{i_k} \circ \phi)$ and ω in local coordinates (with respect to ϕ) as

$$\omega = \sum_I f_I dx_I$$

so applying $(\phi^{-1})^*$ on the left gives

$$(\phi^{-1})^*\omega = \sum_I (f_I \circ \phi^{-1}) d(x_I \circ \phi^{-1})$$

then applying d gives

$$d((\phi^{-1})^*\omega) = \sum_I \sum_i \frac{\partial (f_I \circ \phi^{-1})}{\partial x_i} dx_i \wedge d(x_I \circ \phi^{-1}).$$

Finally pulling back to the mainfold,

$$\begin{aligned}
\phi^* d((\phi^{-1})^*\omega) &= \sum_I \sum_i \left(\frac{\partial (f_I \circ \phi^{-1})}{\partial x_i} \circ \phi \right) d(x_i \circ \phi) \wedge d(x_I \circ \phi^{-1} \circ \phi) \\
&= \sum_I \sum_i \left(\frac{\partial (f_I \circ \phi^{-1})}{\partial x_i} \circ \phi \right) d(x_i \circ \phi) \wedge dx_I.
\end{aligned}$$

□

Problem 5. Let M be a manifold. Prove that d satisfies the formula

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta)$$

where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$.

Proof.
We can write α and β in local coordinates as

$$\alpha = \sum_I g_I dx_I$$
$$\beta = \sum_J g_J dx_J$$

so by linearity

$$\begin{aligned} d(\alpha \wedge \beta) &= d\left(\sum_I g_I dx_I \wedge \sum_J g_J dx_J\right) \\ &= d\left(\sum_{I,J} g_I g_J dx_I \wedge dx_J\right) \\ &= \sum_{I,J} d(g_I g_J dx_I \wedge dx_J) \end{aligned}$$

And continuing by definition of d and the product rule:

$$\begin{aligned} &= \sum_{I,J} \sum_i \frac{\partial g_I g_J}{\partial x_i} dx_i \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} \sum_i \left(\frac{\partial g_I}{\partial x_i} g_J + \frac{\partial g_J}{\partial x_i} g_I\right) dx_i \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} \left(\sum_i \frac{\partial g_I}{\partial x_i} g_J dx_i \wedge dx_I \wedge dx_J + \sum_i \frac{\partial g_J}{\partial x_i} g_I dx_i \wedge dx_I \wedge dx_J\right) \\ &= \sum_{I,J} \left(\underbrace{\sum_i \frac{\partial g_I}{\partial x_i} dx_i \wedge dx_I}_{d(\alpha)} \wedge \underbrace{g_J dx_J}_{\beta} + \sum_i \frac{\partial g_J}{\partial x_i} dx_i \wedge \underbrace{g_I dx_I}_{\alpha} \wedge dx_J\right) \end{aligned}$$

Then by performing k transpositions to move dx_i to be between dx_I and dx_J, we can see that

$$dx_i \wedge dx_I \wedge dx_J = (-1)^k dx_I \wedge dx_i \wedge dx_J$$

And so splitting up the above sum, we get

$$\begin{aligned} &= \sum_{I,J} \left(\underbrace{\sum_i \frac{\partial g_I}{\partial x_i} dx_i \wedge dx_I}_{d(\alpha)} \wedge \underbrace{g_J dx_J}_{\beta} + (-1)^k \underbrace{g_I dx_I}_{\alpha} \wedge \underbrace{\sum_i \frac{\partial g_J}{\partial x_i} dx_i \wedge dx_J}_{d(\beta)}\right) \\ &= d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta) \end{aligned}$$

$\square$

Problem 6. Prove that d commutes with pullback; that is, $d \circ f^* = f^* \circ d$ for any smooth $f\colon M \to N$.

Proof.
Let's try chasing definitions. Suppose we have a form $\omega \in \Omega^k(M)$ where $\omega = \sum_I g_I dx_I$ in local coordinates. Then

$$\begin{aligned} d(f^*(\omega)) &= d\left(f^*\left(\sum_I g_I dx_I\right)\right) \\ &= d\left(\sum_I (g_I \circ f) d(x_I \circ f)\right) \\ &= \left(\sum_I d((g_I \circ f) d(x_I \circ f))\right) \\ &= \sum_I \sum_i \frac{\partial (g_I \circ f)}{\partial x_i} dx_i \wedge d(x_I \circ f), \end{aligned}$$

and

$$\begin{aligned} f^*(d\omega) &= f^* \circ d\left(\sum_I g_I dx_I\right) \\ &= f^*\left(\sum_I dg_I \wedge dx_I\right) \\ &= f^*\left(\sum_I \left(\sum_i \frac{\partial g_I}{\partial x_i} dx_i\right) \wedge dx_I\right) \\ &= \sum_I \sum_i \left(\frac{\partial g_I}{\partial x_i} \circ f\right) d(x_i \circ f) \wedge d(x_I \circ f). \end{aligned}$$

□