

Spring 2013: Complex Analysis Graduate Exam

Peter Kagey

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Problem 1. Evaluate

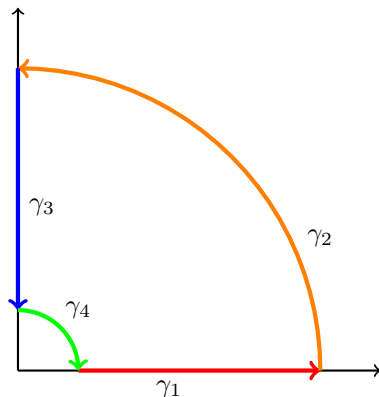
$$\int_0^\infty \frac{x^{1/3}}{1+x^4} dx$$

being careful to justify your answer.

Proof. For ease of notation, name the integrand f ; that is,

$$f(z) = \frac{z^{1/3}}{1+z^4}.$$

We will compute the integral by using the Residue Theorem together with (the limit of) the following contour:



$$\gamma_1 = \{t + 0i \mid t \in [\varepsilon, R]\} \quad (1)$$

$$\gamma_2 = \{Re^{it} \mid t \in [0, \pi/2]\} \quad (2)$$

$$\gamma_3 = \{0 + ti \mid t \in [\varepsilon, R]\} \quad (3)$$

$$\gamma_4 = \{\varepsilon e^{it} \mid t \in [0, \pi/2]\}. \quad (4)$$

For sufficiently small ϵ and large R , this contour encloses one singularity of f , namely $z_0 = e^{\pi i/4}$.

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

In the limit, both arcs (γ_2 and γ_4) vanish.

$$\begin{aligned} \left| \int_{\gamma_2} f(z) dz \right| &= \left| \int_0^{\pi/2} \frac{(Re^{it})^{1/3}}{1 + (Re^{it})^4} iRe^{it} dt \right| \\ &\leq \int_0^{\pi/2} \left| \frac{(Re^{it})^{1/3}}{1 + (Re^{it})^4} iRe^{it} dt \right| \\ &= \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{1 + R^4 e^{4it}} dt \right| \\ &\leq \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{R^4 e^{4it}} dt \right| = \frac{\pi}{2} R^{-8/3} \end{aligned}$$

which vanishes as $R \rightarrow \infty$. Similarly,

$$\begin{aligned}
\left| \int_{\gamma_4} f(z) dz \right| &= \left| \int_0^{\pi/2} \frac{(\varepsilon e^{it})^{1/3}}{1 + (\varepsilon e^{it})^4} i \varepsilon e^{it} dt \right| \\
&\leq \int_0^{\pi/2} \left| \frac{(\varepsilon e^{it})^{1/3}}{1 + (\varepsilon e^{it})^4} i \varepsilon e^{it} dt \right| \\
&= \int_0^{\pi/2} \left| i \varepsilon^{4/3} \frac{e^{4it/3}}{1 + \varepsilon^4 e^{4it}} dt \right| \\
&\leq \int_0^{\pi/2} \left| i \varepsilon^{4/3} \frac{e^{4it/3}}{1} dt \right| = \frac{\pi}{2} \varepsilon^{4/3}
\end{aligned}$$

which also vanishes as $\varepsilon \rightarrow 0$. This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_3} f(z) dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

And the right hand side further simplifies to

$$\begin{aligned}
\int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz + \int_R^{\varepsilon} \frac{(iz)^{1/3}}{1 + (iz)^4} i dz &= \int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz - i^{4/3} \int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz \\
&= (1 - i^{4/3}) \int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz.
\end{aligned}$$

So by the Residue Theorem, the integral evaluates to

$$\int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz = \frac{2\pi i \operatorname{Res}_{z_0}(f)}{1 - i^{4/3}} = \frac{2\pi i}{1 - e^{2\pi i/3}} \operatorname{Res}_{z_0}(f),$$

and it is enough to compute the residue:

$$\operatorname{Res}_{z_0}(f) = \frac{z_0^{1/3}}{(z_0^2 + i)(z_0 + z_0)} = \frac{e^{\pi i/12}}{(2e^{\pi i/2})(2e^{\pi i/4})} = \frac{1}{4} e^{-2\pi i/3}.$$

Therefore

$$\begin{aligned}
\int_{\varepsilon}^R \frac{z^{1/3}}{1 + z^4} dz &= \frac{2\pi i}{1 - e^{2\pi i/3}} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{2\pi i}{1 - (-1/2 + \sqrt{3}i/2)} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{4\pi i}{3 - \sqrt{3}i} \cdot \frac{1}{4} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\
&= \frac{\pi}{3 - \sqrt{3}i} \cdot i \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\
&= \frac{\pi}{2} \left(\frac{\sqrt{3} - i}{3 - \sqrt{3}i} \right) \\
&= \frac{\pi}{2\sqrt{3}}.
\end{aligned}$$

□

Problem 2. Assume that f is an entire function such that

$$|f(z)| \geq \frac{1}{1 + |z|} \text{ for all } z \in \mathbb{C}.$$

Prove that f is a constant function.

Proof.

□

Problem 3. Let f_n , $n \geq 1$, be a sequence of holomorphic functions on an open connected set D such that $|f_n(z)| \leq 1$ for all $z \in D$, $n \geq 1$. Let $A \subseteq D$ be the set of all $z \in D$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in D , then there exists a holomorphic function f on D such that $f_n \rightarrow f$ uniformly on every compact set of D as $n \rightarrow \infty$.

Proof.

□

Problem 4. Let $f(z)$ be meromorphic on \mathbb{C} , holomorphic for $\operatorname{Re} z > 0$ and such that $f(z+1) = zf(z)$ in its domain with $f(1) = 1$.

Show that f has the first order poles at $0, -1, -2, \dots$, and find the residues of f at these points.

Proof.

□