Complex Analysis: Homework 11

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April 4, 2018

Problem 1. (page 251)
If Ω is the punctured disk $0 < |z| < 1$ and if f is given by $f(\zeta) = 0$ for $|\zeta| = 1$, $f(0) = 1$ show that all functions $v \in \mathfrak{B}(f)$ are ≤ 0 in Ω.

Proof.

$\square$

Problem 2. (page 178)
Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\operatorname{Re} z > 1$ and represent its derivative in series form.

Proof.
Assume that $z = z + iy$ for $x > 1$. Then

$$\begin{aligned}
|\zeta(z)| &= \left| \sum_{n=1}^{\infty} \frac{1}{n^z} \right| && (1)\\
&\leq \sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right| && (2)\\
&= \sum_{n=1}^{\infty} \left| \frac{1}{n^{z+iy}} \right| && (3)\\
&= \sum_{n=1}^{\infty} \frac{1}{|n^x||n^{iy}|} && (4)\\
&= \sum_{n=1}^{\infty} \frac{1}{n^x|e^{iy\log(n)}|} && (5)\\
&= \sum_{n=1}^{\infty} \frac{1}{n^x} && (6)
\end{aligned}$$

which is known to converge by the integral test because $x > 1$.
The derivative of $\zeta(z)$ can be computed term by term

$$\begin{aligned}
\frac{d}{dz}\left[\zeta(z)\right] &= \frac{d}{dz}\left[\sum_{n=1}^{\infty} n^{-z}\right]\\
&= \sum_{n=1}^{\infty} \frac{d}{dz}\left[n^{-z}\right]\\
&= \sum_{n=1}^{\infty} \frac{d}{dz}\left[e^{-z\log(n)}\right]\\
&= \sum_{n=1}^{\infty} -e^{-z\log(n)}\log(n)\\
&= \sum_{n=1}^{\infty} -n^{-z}\log(n)
\end{aligned}$$

□

Problem 4. (page 178)
As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeros in Ω then $f(z)$ is either identically zero or has at most m zeros.

Proof.
Suppose for the sake of contradiction that $f(z)$ has $m+1$ zeros. Then we can form a $\square$

Problem 5. (page 184)
The Fibonacci numbers are defined by $c_0 = 0, c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}.$$

Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n.

Proof.
Let

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} c_n z^n \\
&= \sum_{n=1}^{\infty} c_n z^n \\
&= z + \sum_{n=2}^{\infty} c_n z^n \\
&= z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n \\
&= z + \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n \\
&= z + \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} \\
&= z + z \sum_{n=1}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n \\
&= z + z f(z) + z^2 f(z).
\end{aligned}$$

Then solving for $f(z)$ yields

$$f(z) = \frac{z}{1 - z - z^2},$$

thus f is a rational function. Now we must find a power series representation of f, and its coefficients will give us the terms of the Fibonacci sequence. Since f has roots $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$, partial fraction decomposition yields

$$\begin{aligned}
f(z) &= -\frac{z}{(z + \varphi)(z + \psi)} \\
&= \frac{A}{z + \varphi} + \frac{B}{z + \psi}
\end{aligned}$$

where A and B satisfy

$$-z = A(z + \psi) + B(z + \varphi)$$

and thus the system of equations

$$\begin{aligned}
A + B &= -1 \\
A\psi + B\varphi &= 0.
\end{aligned}$$

Solving for A and B gives

$$\begin{aligned}
f(z) &= \frac{1}{\sqrt{5}} \left(\frac{-\varphi}{z + \varphi} + \frac{\psi}{z + \psi} \right) \\
&= \frac{1}{\sqrt{5}} \left(-\frac{1}{z/\varphi + 1} + \frac{1}{z/\psi + 1} \right).
\end{aligned}$$

So the identity

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

together with $-1/\psi = \varphi$ and $-1/\varphi = \psi$ gives

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{-1}{\sqrt{5}}(-z/\varphi)^n + \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(-z/\psi)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left((-z/\psi)^n - (-z/\varphi)^n\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\varphi^n z^n - \psi^n z^n\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\varphi^n - \psi^n\right) z^n
\end{aligned}$$

Therefore

$$c_n = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

$\square$

Problem 4. (page 186)
Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}.$$

Calculate B_1, B_2, and B_3.

Proof.
(An idea from https://math.stackexchange.com/q/1006615/121988.)
We know that $f(z) = (e^z - 1)^{-1}$ has a pole at 0, so consider $f(z) = 1/(zg(z))$ where

$$g(z) = \frac{e^z - 1}{z} = \frac{1}{z}\left(-1 + \sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

which is analytic everywhere. When $z \neq 0$, $f(z) = h(z)/z$ where $h(z) = 1/g(z)$. (?)
Another idea:

$$f(z) = \frac{-1}{1 - e^z} = -\sum_{n=0}^{\infty} (e^z)^n = -\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)^n$$

$\square$

Problem 5. (page 186)
Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers.

Proof.

$\square$

Problem 1. (page 190)
Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}, \text{ and } \qquad \sum_{n=1}^{\infty} \frac{1}{n^6}$$

with a complete justifaction of the steps that are needed.

Proof.
By equation (10), we know that

$$\pi \cot \pi z = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

□

Problem 2. (page 190)
Express

$$\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form.

Proof.

$\square$