Combinatorics: Homework 11

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Problem 1. Prove that every group of $2n$ children in which every child is friends with at least n other children can be partitioned into pairs of friends in at least two different ways.

Proof.
If two children are friends, they can be paired up in exactly one way.
If four children each have two or more friends, then this is network is has a subgraph which is C_4, so it is sufficient to show that C_4 can be partitioned in two different ways. If the edges of C_4 are written $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, then take the pairs (v_1, v_2) and (v_3, v_4) or else take the pairs (v_4, v_1) and (v_2, v_3).
Take the child v with the fewest friends, and pair them up with their friend v' with the fewest friends. Sinc v is assumed to have the fewest friends, $\deg(v) \leq \deg(v')$.
Case 1. Assume $\deg(v') > n$. There are no mutual friends of v' and v'' with exactly n friends (otherwise this friend would be named v'), so removing v and v' everyone with at least $n - 1$ friends.
Case 2. Assume $\deg(v) = \deg(v') = n$. $\square$

Problem 2. Find the chromatic polynomial for the graph with

$$\begin{aligned} V &= \{v_1, \ldots, v_n\} \\ E &= \{v_1v_2, v_2v_3, \ldots v_{n-1}v_n, v_1v_n\} \cup \{v_1v_3, v_1v_4, \ldots, v_1v_{n-1}\}, \end{aligned}$$

using the facts that the chromatic polynomial for the cyclic graph C_n is

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

and that the chromatic polynomial for any graph $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{v\}$ is

$$P_G(k) = \frac{1}{k} p_{G_1}(k) p_{G_2}(k).$$

Solution.
There are k ways to pick the color for v_1, $(k-1)$ ways to pick the color for v_2, since v_1 is adjacent to v_2, and then $(k-2)$ ways to pick the color for v_i as i goes from 3 to n, since v_i is adjacent to v_1 and v_{i-1}. Therefore if G_n is the graph with n vertices

$$P_{G_n}(k) = k(k-1)(k-2)^{n-2}.$$

In order to use the hint, contraction/deletion gives

$$P_{G_n}(k) = P_{G'_n}(k) - P_{G''_n}(k)$$

where $G'_n = G_n - v_{n-1}v_n$ and $G''_n = G_n/v_{n-1}v_n$.
With induction hypothesis given above, using the information that $G_3 = C_3$,

$$\begin{aligned} P_{G_3}(k) &= P_{C_3}(k) \\ &= (k-1)^3 - (k-1) \\ &= (k-1)((k-1)^2 - 1) \\ &= k(k-1)(k-2) \end{aligned}$$

so the base case is satisfied for $n = 3$.
The first graph in the contraction/deletion argument satisfies

$$G'_n = G_{n-1} \cup \underbrace{(\{v_1, v_n\}, \{v_1v_n\})}_{C_2}$$

where $G_{n-1} \cap C_2 = \{v_{n-1}\}$. Therefore, by the facts given along with the induction hypothesis, the chromatic polynomial of G'_n is given by

$$\begin{aligned} P_{G'_n}(k) &= \frac{1}{k} \cdot \underbrace{k(k-1)(k-2)^{n-2}}_{P_{G_{n-1}}(k)} \cdot \underbrace{k(k-1)}_{P_{C_2}(k)} \\ &= k(k-1)^2(k-2)^{n-2}. \end{aligned}$$

Similarly, $G''_n = G_{n-1}$, so by the induction hypothesis together with contraction deletion,

$$\begin{aligned} P_{G_n}(k) &= P_{G'_n}(k) - P_{G''_n}(k) \\ &= k(k-1)^2(k-2)^{n-2} - k(k-1)(k-2)^{n-2} \\ &= k(k-1)(k-2)^{n-2}((k-1)-1) \\ &= k(k-1)(k-2)^{n-1}, \end{aligned}$$

as desired.

Problem 3. Using the deletion/retraction recurrence on a graph, prove that the number of acyclic orientations of G is equal to $(-1)^{|V|}p_G(-1)$ where an acyclic orientation is an assignment of a direction to each edge such that there are no directed cycles.

Proof.
Let $A(G)$ be the number of acyclic orientations on G. By induction on $|V| + |E|$ with base case of the singleton graph $\mathbf{1}$, which has chromatic polynomial $P_{\mathbf{1}}(k) = k$. The only assignment of direction to each edge is the empty assignment, and

$$(-1)^{|\mathbf{1}|}P_1(-1) = (-1)^1(-1) = 1 = A(\mathbf{1})$$

as desired.

Recall the usual contraction/deletion recurrence

$$P_G(k) = P_{G'}(k) - P_{G''}(k)$$

with $G' = G - uv$ and $G'' = G/uv$.
Furthermore, the relation $A(G) = A(G') + A(G'')$ holds.

Applying the induction hypothesis together with the recurrence gives

$$\begin{aligned}
A(G) &= A(G') + A(G'') \\
&= (-1)^{|V|}P_{G'}(-1) + (-1)^{|V|-1}P_{G'}(-1) \\
&= (-1)^{|V|}(P_{G'}(-1) - P_{G''}(-1)) \\
&= (-1)^{|V|}P_G(-1)
\end{aligned}$$

as desired. $\square$

Problem 4. Let G be a planar connected bipartite graph $V = V_1 \coprod V_2$ and $E \subset V_1 \times V_2$ such that there is no 4-cycle and no vertex of degree 1. Show that $3(|V| - 2) \geq 2|E|$.

Solution.
Since G is planar, the Euler characteristic states that

$$v - 2 = e - f$$

Since G is bipartite, any cycle must have even length, so G does not contain any 3-cycles or 5-cycles. Since G is assumed to be simple (and therefore does not have multiple edges) G does not contain any 2-cycles. By assumption, G does not contain any 4-cycles. Thus any face of G must be adjacent to 6 or more edges, whenever G has at least 6 edges, and G must have at least 6 edges by the "no vertex of degree 1 criterion" coupled with having no cycles smaller than 6.

Since every edge is adjacent to at most two faces, and since $e \geq 6$,

$$\frac{f}{2} \leq \frac{e}{6}$$

and so in particular

$$\begin{aligned} v - 2 = e - f &\geq e - \frac{1}{3}e \\ 3(|V| - 2) &\geq 2|E|. \end{aligned}$$