Combinatorics: Homework 5

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September 25, 2018

Problem 47 (a). [2]
Let D be the operator $\frac{d}{dx}$. Show that

$$(xD)^n = \sum_{k=0}^{n} S(n,k)x^k D^k.$$

(Hint: $(xD)^n f = (xD)^{n-1}(xf')$.)

Solution.
It seems the most natural way to prove this is by induction. For the case $n = 1$, we have

$$(xD)^1 = \sum_{k=0}^{1} S(1,k)x^k D^k = \underbrace{S(1,0)}_{=0} x^0 + \underbrace{S(1,1)}_{=1} x^1 D^1 = xD$$

Thus

$$\begin{aligned}
(xD)^n &= xD((xD)^{n-1}) \\
&= xD\left(\sum_{k=0}^{n-1} S(n-1,k)x^k D^k\right) \\
&= \sum_{k=0}^{n-1} S(n-1,k)xD(x^k D^k) \\
&= \sum_{k=0}^{n-1} S(n-1,k)(kx^k D^k + x^{k+1}D^{k+1}) \\
&= \sum_{k=0}^{n-1} kS(n-1,k)x^k D^k + \sum_{k=1}^{n} S(n-1,k-1)x^k D^k.
\end{aligned}$$

Next, substituting the identity

$$\begin{aligned}
S(n,k) &= S(n-1,k-1) + kS(n-1,k) \\
S(n-1,k-1) &= S(n,k) - kS(n-1,k)
\end{aligned}$$

yields

$$
\begin{aligned}
(xD)^n &= \sum_{k=0}^{n-1} kS(n-1,k)x^k D^k + \sum_{k=1}^{n} (S(n,k) - kS(n-1,k))x^k D^k \\
&= \sum_{k=0}^{n-1} kS(n-1,k)x^k D^k + \sum_{k=1}^{n} S(n,k)x^k D^k - \sum_{k=1}^{n} kS(n-1,k)x^k D^k \\
&= \underbrace{0 \cdot S(n-1,0)x^0 D^0}_{0} + \sum_{k=1}^{n} S(n,k)x^k D^k - n\underbrace{S(n-1,n)}_{0} x^n D^n \\
&= \sum_{k=1}^{n} S(n,k)x^k D^k = \sum_{k=1}^{n} S(n,k)x^k D^k + \underbrace{S(n,0)}_{0} = \sum_{k=0}^{n} S(n,k)x^k D^k.
\end{aligned}
$$

So the identity holds.

Problem 133b. [2+]
Let $A_n(x)$ be the Eulerian polynomial. Give a combinatorial proof that

$$\frac{A_n(x)}{x} = \sum_{k=0}^{n-1}(n-k)!S(n,n-k)(x-1)^k.$$

(Note: $(n-k)!S(n,n-k)$ is the number of ordered partitions of an n-set into $n-k$ blocks.)

Solution.
By defintion

$$\frac{A_n(x)}{x} = \sum_{k=1}^{n} A(n,k)x^{k-1} = \frac{A_n(x)}{x} = \sum_{k=0}^{n-1} A(n,k+1)x^k$$

where $A(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with $k-1$ descents. So when $x \in \mathbb{P}$, $A_n(x)/x$ is the number of x-descent-colorings of all of the permutations.

I will describe a bijection from permutations with k descents where each descent has an x-coloring to ordered set partitions into $n-j$ blocks where each element greater than the smallest number in the block (j of them total) has an $(x-1)$-coloring.

The bijection is best described with an example. In the example, let $n = 9$, $k = 4$, and $x = 3$, and suppose we have the permutation

$$\tau = 7 \mid_2 3\ 5 \mid_1 1\ 6\ 9 \mid_3 8 \mid_1 2\ 4$$

written as a word, where each descent is labeled "$|_a$" with $a \in [x]$.

Then we can turn this into an ordered set partition by placing additional "bars" between pairs $a_i < a_{i+1}$ as follows

$$7 \mid_2 3 \mid 5 \mid_1 1 \mid 6 \mid 9 \mid_3 8 \mid_1 2 \mid 4$$

then split the word along the bars, unless the bar is labeled "$|_a$" with $a \in [x-1]$

$$\{7 \mid_2 3\}, \{5 \mid_1 1\}, \{6\}, \{9\}, \{8 \mid_1 2\}, \{4\}$$
$$(\{7,3\}, \{5,1\}, \{6\}, \{9\}, \{8,2\}, \{4\}, (2,1,1))$$

Which results in an ordered set partition with labels in $[x-1]$. It's easy enough to go back: just concatenate the elements of the set into a word and place "$|_x$" between any unmarked descent.

This function is surjective because the algorithm works "backward", and it's bijective because the "backward" algorithm works as an inverse.

Since this idea works for any $x \in \mathbb{P}$, and since both sides are polynomials of degree at most $n-1$, the two polynomials must be equal.