Topology: Midterm corrections

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Problem 1.
Let X be a subspace of the punctured plane $\mathbb{R}^2 - \{0\}$ such that the homomorphism $\pi_1(X; x_0) \to \pi_1(\mathbb{R}^2 - \{0\}; x_0)$ induced by the inclusion map is nontrivial. Show that X has a nonempty intersection with every half-line $L_{(a,b)} = \{(\lambda a, \lambda b) \in \mathbb{R}^2; \lambda > 0\}$ issued from the origin (with $(a, b) \neq (0, 0)$).

Proof.
Firstly, $\mathbb{R}^2 - L_{(a,b)}$ is star shaped with center $(-a, -b)$, so its fundamental group is trivial:

$$\pi_1(\mathbb{R}^2 - L_{(a,b)}; x_0) = \mathbf{1}.$$

Assume that X does not include L. Then consider the inclusion maps and the corresponding induced homomorphisms

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & \mathbb{R}^2 - L_{(a,b)} & \xrightarrow{i_2} & \mathbb{R}^2 - \{0\} \\ \pi_1(X; x_0) & \xrightarrow{i_{1*}} & \pi_1(\mathbb{R}^2 - L_{(a,b)}; x_0) & \xrightarrow{i_{2*}} & \pi_1(\mathbb{R}^2 - \{0\}; x_0) \\ \pi_1(X; x_0) & \xrightarrow{i_{1*}} & 1 & \xrightarrow{i_{2*}} & \mathbb{Z} \end{array}$$

Since $i_{2*} \colon \mathbf{1} \to \mathbb{Z}$ is trivial, the homomorphism induced from the inclusion map

$$i_{2*} \circ i_{1*} = (i_2 \circ i_1)_* \colon \pi_1(\mathbb{R}^2 - L_{(a,b)}; x_0) \to \pi_1(\mathbb{R}^2 - \{0\}; x_0)$$

must also be trivial.

This is a contradiction of the hypothesis, so X must intersect L. $\square$

Problem 2.
Let X be a path connected space whose fundamental group $\pi_1(X;x_0)$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$. Show that the change of basepoint isomorphism $T_\gamma\colon \pi_1(X;y_0)\to\pi_1(X;x_0)$ defined by $T_\gamma([\alpha])=[\gamma*\alpha*\overline{\gamma}]$ for some path from x_0 to y_0 depends only on the endpoints x_0 and y_0 and does not depend on the path γ.

Proof.
We will exploit the fact that $\pi_1(X;x_0)$ is abelian. Consider two different paths, γ_1 and γ_2, and an element of $[\alpha]\in\pi_1(X;y_0)$. It is suffient to show that $T_{\gamma_1}([\alpha])=T_{\gamma_2}([\alpha])\in\pi_1(X;x_0)$, which is to say, we want to show that

$$[\gamma_1*\alpha*\overline{\gamma}_1]\cdot[\gamma_2*\alpha*\overline{\gamma}_2]^{-1}=\mathrm{id}_{\pi_1(X;x_0)}\,.$$

Notice that $\overline{\gamma}_1*\gamma_2$ is a path from y_0 to x_0 concatenated with a path from x_0 to y_0, and thus $[\overline{\gamma}_1*\gamma_2]\in\pi_1(X;y_0)$. Since $\pi_1(X;y_0)\cong\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$ is abelian, the product can be rearranged as follows:

$$\begin{aligned}[\alpha*\overline{\gamma}_1*\gamma_2*\overline{\alpha}]&=[\alpha]\cdot[\overline{\gamma}_1*\gamma_2]\cdot[\overline{\alpha}]\\&=[\alpha]\cdot[\overline{\alpha}]\cdot[\overline{\gamma}_1*\gamma_2]\\&=[\alpha*\overline{\alpha}]\cdot[\overline{\gamma}_1*\gamma_2]\\&=[\overline{\gamma}_1*\gamma_2]\end{aligned}$$

so in particular, $\alpha*\overline{\gamma}_1*\gamma_2*\overline{\alpha}\cong\overline{\gamma}_1*\gamma_2$. Since $[\gamma*\alpha*\overline{\gamma}]^{-1}=[\gamma*\overline{\alpha}*\overline{\gamma}]$ by tracing the path backward, this is equivalent to

$$\begin{aligned}[\gamma_1*\alpha*\overline{\gamma}_1]\cdot[\gamma_2*\alpha*\overline{\gamma}_2]^{-1}&=[\gamma_1*\alpha*\overline{\gamma}_1]\cdot[\gamma_2*\overline{\alpha}*\overline{\gamma}_2]\\&=[\gamma_1*\underbrace{\alpha*\overline{\gamma}_1*\gamma_2*\overline{\alpha}}_{\simeq\overline{\gamma}_1*\gamma_2}*\overline{\gamma}_2]\\&=[\gamma_1*\overline{\gamma}_1*\gamma_2*\overline{\gamma}_2]\\&=[c_{x_0}]\end{aligned}$$

Therefore $[\gamma_1*\alpha*\overline{\gamma}_1]\cdot[\gamma_2*\alpha*\overline{\gamma}_2]^{-1}=[c_{x_0}]$. Thus for every $[\alpha]\in\pi_1(X;y_0)$, $T_{\gamma_1}([\alpha])=T_{\gamma_2}([\alpha])$, the change of basepoint isomorphism does not depend on the path when the fundamental group is abelian. $\square$

Problem 3.
For positive integers $m, n \geq 1$, let $i_n \colon \mathbb{Z}_2 \to Z_{2n} = \langle a; a^{2n} = 1 \rangle$ denote the homomorphism that sends the nontrivial element of $\mathbb{Z}_2$ to a^n, and let $\mathbb{Z}_{2m} *_{\mathbb{Z}_2} \mathbb{Z}_{2n}$ be the amalgamated free product defined by the homomorphisms i_m and i_n.

a. Show that $\mathbb{Z}_2 *_{\mathbb{Z}_2} \mathbb{Z}_{2n}$ is abelian.

b. Construct a surjective group homomorphism

$$\varphi \colon \mathbb{Z}_{2m} *_{\mathbb{Z}_2} \mathbb{Z}_{2n} \to \mathbb{Z}_m * \mathbb{Z}_n.$$

c. Show that $\mathbb{Z}_{2m} *_{\mathbb{Z}_2} \mathbb{Z}_{2n}$ is not abelian when $m, n \geq 2$.

Proof.

a. Name the amalgamated product G and write it as $G = \langle a; a^2 = 1 \rangle *_{\mathbb{Z}_2} \langle b; b^{2n} = 1 \rangle$. By the maps $1 \overset{i_1}{\longmapsto} a$ and $1 \overset{i_n}{\longmapsto} b^n$, we can write G via the group presentation

$$G = \langle a, b; a = b^n, a^2 = b^{2n} = 1 \rangle \cong \mathbb{Z}_{2n},$$

which is abelian.

b. Name the amalgamated product G and write it as $G = \langle a; a^{2m} = 1 \rangle *_{\mathbb{Z}_2} \langle b; b^{2n} = 1 \rangle$. Similarly, by the maps $1 \overset{i_m}{\longmapsto} a^m$ and $1 \overset{i_n}{\longmapsto} b^n$, we can write G via the group presentation

$$G = \langle a, b; a^m b^{-n} = a^{2m} = b^{2n} = 1 \rangle.$$

By the universal property of free products, given the homomorphisms

$$\begin{aligned} j_A &\colon \langle a \rangle \to \langle c; c^m = 1 \rangle * \langle d; d^n = 1 \rangle \text{ which sends } a \mapsto d \\ j_B &\colon \langle b \rangle \to \langle c; c^m = 1 \rangle * \langle d; d^n = 1 \rangle \text{ which sends } b \mapsto c \end{aligned}$$

there exists a map

$$\varphi \colon \langle a \rangle * \langle b \rangle \to \langle c; c^m = 1 \rangle * \langle d; d^n = 1 \rangle$$

which sends $a \mapsto c$ and $b \mapsto d$ is a homomorphism. In particular, the relations of G are in the kernel of this map:

$$\begin{aligned} \varphi(a^m b^{-n}) = \varphi(a)^m \varphi(b^{-1})^n &= \underbrace{c^m}_{1 \in \mathbb{Z}_m} \underbrace{(d^{-1})^n}_{1 \in \mathbb{Z}_n} = 1 \\ \varphi(a^{2m}) = \varphi(a^2)^m &= (c^2)^m = 1 \in \mathbb{Z}_m \\ \varphi(b^{2n}) = \varphi(b^2)^n &= (d^2)^n = 1 \in \mathbb{Z}_n. \end{aligned}$$

Therefore we can write our desired homomorphism as $\varphi' \colon G \to \langle c; c^m = 1 \rangle * \langle d; d^n = 1 \rangle$ which sends $a \overset{\varphi'}{\longmapsto} c$ and $b \overset{\varphi'}{\longmapsto} d$.
This map is surjective because we can write any element

$$c^{i_1} d^{i_2} c^{i_3} d^{i_4} \ldots c^{i_{n-1}} d^{i_n} \in \langle c; c^m = 1 \rangle * \langle d; d^n = 1 \rangle$$

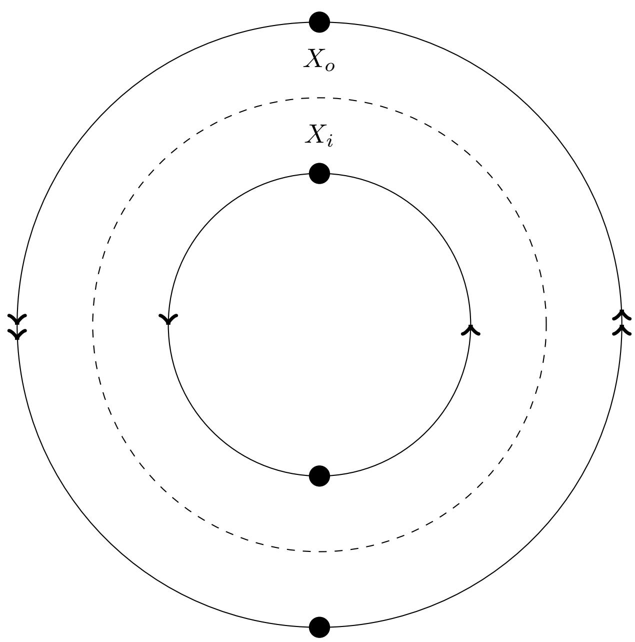
as $\varphi(a^{i_1} \cdot b^{i_2} \cdot a^{i_3} \cdot b^{i_4} \ldots \cdot a^{i_{n-1}} \cdot b^{i_n})$.

c. The surjective homomorphism φ, which is described above, maps $\mathbb{Z}_{2m} *_{\mathbb{Z}_2} \mathbb{Z}_{2n}$ onto the free group on two letters, which is not commutative. Since homomorphisms preserve commutivity, $\mathbb{Z}_{2m} *_{\mathbb{Z}_2} \mathbb{Z}_{2n}$ is not abelian.

$\square$

Problem 4.
Let X be the quotient space of the annulus $S^1 \times [0,1]$ by the equivalence relation $\sim$ which identifies each point $(z, 0)$ to $(-z, 0)$ and each point $(z, 1)$ to $(-z, 1)$. Compute the fundamental group of X.

Proof. We will use van Kampen with the a cut along $S^1 \times \{1/2\} \simeq S^1$:



Let X_o be the outer part of the annulus with the outer boundary identified, and let X_i be the inner part of the annulus with the inner boundary identified. Since both X_o and X_i have a deformation retract to their respective boundaries, $\pi_1(X_o; x_0) \cong \pi_1(X_i; x_0) \cong \mathbb{Z}$, where x_0 is a point on $S^1 \times [0,1]$. Then the fundamental group is given by the amalgamated product

$$\begin{aligned} \pi_1(X; x_0) &= \pi_1(X_o; x_0) *_{\pi_1(S_1; x_0)} \pi_1(X_i; x_0) \\ &\cong \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} \\ &\cong \langle a \rangle *_{\langle b \rangle} \langle c \rangle \end{aligned}$$

with maps

$$\begin{aligned} i_o &\colon \langle b \rangle \to \langle a \rangle \qquad \text{sending} \qquad b \mapsto a^2 \\ i_i &\colon \langle b \rangle \to \langle c \rangle \qquad \text{sending} \qquad b \mapsto c^2. \end{aligned}$$

Since a loop around the interior of the annulus maps to a loop twice around the boundary under the deformation retract.
Therefore, the resulting amalgamated product is

$$\pi_1(X; x_0) = \langle a, c; a^2 = c^2 \rangle.$$

□