Complex Analysis: Homework 9

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Problem 1. (page 174)
Suppose f is analytic in the whole plane, real on the real axis, and purely imaginary on the imaginary axis. Show that f is odd.

Proof.
By Theorem 24, because the imaginary part of f vanishes on the real axis, $f(z) = \overline{f(\bar{z})}$, thus in the Taylor series

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \overline{a_n \bar{z}^n} = \sum_{n=0}^{\infty} \overline{a_n} z^n,$$

so each coefficient is real because $a_n = \bar{a}_n$. Now let y be real so that yi is purely imaginary. Then the Taylor series for $f(iy)$ can be split into its even and odd terms:

$$\begin{aligned} f(iy) &= \sum_{n=0}^{\infty} a_n i^n y^n \\ &= \sum_{n=0}^{\infty} a_{2n} i^{2n} y^{2n} + \sum_{n=0}^{\infty} a_{2n+1} i^{2n+1} y^{2n+1} \end{aligned}$$

This even/odd split also splits the sums into real and imaginary parts

$$f(iy) = \sum_{n=0}^{\infty} \underbrace{a_{2n}(-1)^n y^{2n}}_{\in\mathbb{R}} + i \sum_{n=0}^{\infty} \underbrace{a_{2n+1}(-1)^n y^{2n+1}}_{\in\mathbb{R}},$$

and by hypothesis, the image of $f(iy)$ is purely imaginary, so

$$\sum_{n=0}^{\infty} \underbrace{a_{2n}(-1)^n y^{2n}}_{\in\mathbb{R}} = 0$$

and thus f can be rewritten as

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1},$$

and so f is odd by inspection. $\square$

Problem 4. (page 174)
Use (66) to derive a formula for $f'(z)$ in terms of $u(z)$.

Proof.
(66) gives that

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC$$

By differentiating under the integral

$$\begin{aligned}
f'(z) &= \frac{d}{dz}\left[\frac{1}{2\pi i} \int\limits_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC\right] \\
&= \frac{1}{2\pi i} \int\limits_{|\zeta|=R} \frac{u(\zeta)}{\zeta} \cdot \frac{\partial}{\partial z}\left[\frac{\zeta + z}{\zeta - z}\right] d\zeta + \frac{d}{dz}[iC] \\
&= \frac{1}{2\pi i} \int\limits_{|\zeta|=R} \frac{u(\zeta)}{\zeta} \cdot \frac{2\zeta}{(\zeta - z)^2} d\zeta \\
&= \frac{1}{\pi i} \int\limits_{|\zeta|=R} \frac{u(\zeta)}{(\zeta - z)^2} d\zeta. \\
&= 2u'(z)
\end{aligned}$$

□

Problem 5. (page 174)
Suppose $u(z)$ is harmonic and $0 \leq u(z) \leq Ky$ for $y > 0$. Prove that $u = ky$ with $0 \leq k \leq K$.

Proof.
By the previous problem, $u'(z)$ reduces to a constant by Liouville's theorem, because it is bounded and analytic on $\mathbb{C}$.
Thus $f'(z) = 2u'(z) = k$ is also a constant, so $f(z) = kz$, and the real part is $u(z) = ky$.
Since $0 \leq u(z) = ky \leq Ky$, this implies that $0 \leq k \leq K$. $\square$

Problem 1. (page 244)
If E is a compact set in a region Ω, prove that there exists a constant M, depending only on E and Ω, such that every positive harmonic function $u(z)$ in Ω satisfies $u(z_2) \leq Mu(z_1)$ for any two points z_1, $z_2 \in E$.

Proof.
Harnack's inequality gives that

$$\frac{\rho - r}{\rho + r} u(z_1) \leq u(z_2) \leq \frac{\rho + r}{\rho - r} u(z_1).$$

where $|z_2 - z_1| = r < \rho$, and $u(z)$ is harmonic for $|z - z_1| < \rho$.
Because $E \subset \Omega$ is compact, it admits a finite subcover of balls of radius s_p, $B_{s_p}(p) \subset \Omega$. (Say that there are N such balls.) Then by "linking up" the balls, for any two points, we can find a polygonal chain through at most N balls composed of line segments between z_2 and p_1, p_1 and p_2, and so on (where p_k is a point at the "center" of the overlap of the balls)

$$z_2 \to p_1 \to p_2 \to \ldots \to p_{n-1} \to z_1$$

Then by the positivity of u,

$$\frac{u(z_2)}{u(p_1)} \cdot \frac{u(p_1)}{u(p_2)} \cdots \frac{u(p_{n-1})}{u(z_1)} = \frac{u(z_2)}{u(z_1)} \leq \left(\frac{\rho_1 + r_1}{\rho_1 - r_1}\right) \cdots \left(\frac{\rho_n + r_n}{\rho_n - r_n}\right).$$

Then just set M to be the *sup* of all such products, which is finite because there are a finite number of balls. $\square$

Problem 3. (page 248)
If v is continuous together with its partial derivatives up to the second order, prove that v is subharmonic if and only if $\Delta v \geq 0$.

Proof.
($\Longrightarrow$) Assume that v is subharmonic. Now if v is harmonic, then we're done, so assume that $\Delta v < 0$, which means that $\Delta(v + \varepsilon x^2) = \Delta v + 2\varepsilon$ and so $v + \varepsilon x^2$ is subharmonic for $\varepsilon \leq -\Delta v/2$

($\Longleftarrow$)
By contrapositive, assume that v is *not* subharmonic. That is, given some circle $\Omega' \subset \Omega$, on which u and v (with u harmonic), $u - v$ does *not* satisfy the maximum principle—that is, $|u - v|$ has a maximum in Ω. However, in order to have a maximum, $\partial^2/\partial x^2(v-u) \leq 0$ and $\partial^2/\partial y^2(v-u) \leq 0$, so $\Delta v = \Delta(v-u) \leq 0$. $\square$

Problem 4. (page 248)
Prove that a subharmonic function remains subharmonic if the independent variable is subjected to a conformal mapping.

Proof.
Let u be a subharmonic function and f a conformal mapping. Assume for the sake of contradiction that $u \circ f$ is *not* subharmonic—that is, there exists some z_0 and some corresponding harmonic function v such that $g = u \circ f - v$ takes a maximum at z_0. On same small disk centered at z_0, we can write

$$g(z) = u \circ f(z) - v \circ f^{-1} \circ f(z)$$

because v is harmonic and f and f^{-1} are conformal, $\hat{v} = v \circ f^{-1}$ is also harmonic. So choosing some $w_0 \in f^{-1}(z_0)$, we can write

$$g(w_0) = u(z_0) - \hat{v}(z_0)$$

is a maximal value of g. But this is a contradiction because both u and $\hat{v}$ are harmonic.

□

Problem 5. (page 248)
Formulate and prove a theorem to the effect that a uniform limit of subharmonic functions is subharmonic.

Proof.
Suppose that $u_1, u_2, \ldots$ is a sequence of functions that are subharmonic in Ω and converge uniformly to u in the sense that for any choice of $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|u_N(z) - u(z)| < \varepsilon \text{ for all } z \in \Omega.$$

Theorem 8 gives that a function f is subharmonic if and only if

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta$$

for every disk $|z - z_0| \leq r$ in Ω. Thus

$$\begin{aligned} u(z_0) &< \varepsilon + u_N(z_0) \\ &< \varepsilon + \frac{1}{2\pi} \int_0^{2\pi} u_N\left(z_0 + re^{i\theta}\right) d\theta \\ &< \varepsilon + \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) + \varepsilon \; d\theta \\ &< 2\varepsilon + \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta \end{aligned}$$

This holds for arbitrarily small ε, so u is subharmonic by Theorem 8.

□