Combinatorics: Homework 9

Peter Kagey

November 2, 2018

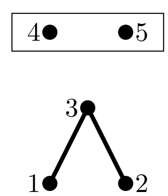
Problem 34. [2]
Find all nonisomorphic posets P such that

$$F(J(P), x) = (1 + x)(1 + x^2)(1 + x + x^2)$$

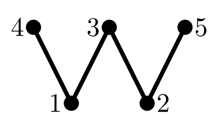
Solution. $F(J(P), x) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5$, so $J(P)$ has rank 5 so P has five elements, call them $\{1, 2, 3, 4, 5\}$

Since the coefficient of x in $F(J(P), x)$ is 2 there must be exactly two minimal elements in P. Name these 1 and 2. The remaining three elements of the poset are not minimal, so they must be greater than 1 or 2 or both. Without loss of generality, say that $1 \lessdot 3$.

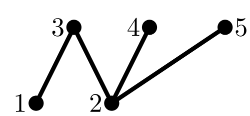
Case 1. Assume $1 \lessdot 3$ and $2 \lessdot 3$.
Thus the Hasse diagram looks like this (where the orders on 4 and 5 still need to be drawn):



This diagram (ignoring 4 and 5) has two one-element ideals (namely $\{1\}$ and $\{2\}$), but only one two-element ideal (namely $\{1, 2\}$). There are two ways (without loss of generalization) to put orders on 4 and 5 that add two two-element ideals but no one-element ideals



and



The poset P_{right} on the right has three 4-element ideals, $\langle 3, 4\rangle = \{1, 2, 3, 4\}$, $\langle 3, 5\rangle = \{1, 2, 3, 5\}$, and $\langle 4, 5\rangle = \{1, 2, 4, 5\}$, so the coefficient of x^4 in $F(J(P_{\text{right}}), x)$ is 3 not 2 as desired.

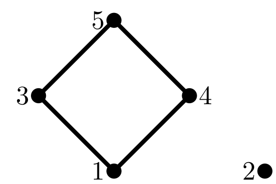
The poset P_{left} on the left has four 3-element ideals: $\langle 1, 4\rangle$, $\langle 1, 5\rangle$, $\langle 3\rangle$, and $\langle 4, 5\rangle$, so the coefficient of x^3 in $F(J(P_{\text{left}}), x)$ is 4 not 3 as desired. Thus there are no posets such that $1 \lessdot 3$ with the desired rank generating function.

Case 2. Assume $1 \lessdot 3$ and $2 \not\lessdot 3$.
Since there are no posets from Case 1 that work, any valid posets must be from case two. There is no element 3 such that $1 \lessdot 3$ and $2 \lessdot 3$, so the graph has two connected components $P_1 + P_2$. Assume without loss of generality that P_1 (which contains 1 and 3) has more elements than P_2, that is so P_1 has either 3 or 4 elements, and P_2 has 2 or 1 element respectively. If $P_2 = \{2\}$ has just one element, then 1 must be covered by one, two, or three elements.

1. If 1 is covered only by 3, then the poset has only two 2-element ideals, namely $\langle 1, 2\rangle$ and $\langle 3\rangle$.

2. If 1 is covered by two elements (call these 3 and 4 without loss of generality) then in order to avoid having too many 1, 2, or 3-element ideals, 5 must cover both 3 and 4.
3. If 1 is covered by $3, 4$, and 5, then the poset has only four 2-element ideals, namely $\langle 1, 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, and $\langle 5\rangle$.

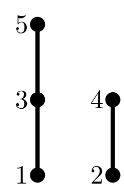
Thus if $P_2 = \{2\}$, then the only partial order that will work is



On the other hand, if $P_2 = \{2, 4\}$ with $2 \lessdot 4$, then we have two cases

1. If 1 is covered only by 3 (and thus $5 \lessdot 3$), then this satisfies the rank generating function.
2. If 1 is covered by two element, 3 and 5 and 5, then the poset has four 2-element ideals, namely $\langle 1, 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, and $\langle 5\rangle$.

So the poset $P = [3] + [2]$ also works:



Problem 46 a. [2]
Let $f(n)$ be the number of sublattices of rank n of the boolean algebra B_n. Show that $f(n)$ is also the number of partial orders P on $[n]$.

Solution.
Since B_n is a distributive lattice, all rank n sublattices of B_n must also be distributive lattices. Thus each sublattice of $L \subset B_n$ can be written as $L \cong J(P)$ for some poset P, where P is isomorphic to the set of join irreducibles of L. By proposition 3.4.5, since L has rank n, P must have n elements. So every distributive lattices of rank n gives a partial ordering of $[n]$.

This a bijection, because given some poset P, we can recover L via J (namely, $L \cong J(P)$). Each distinct P gives a distinct rank n sublattice of B_n because (1) $J(P)$ is a distributive lattice, (2) $J(P)$ contains the ideal $[n]$, (and no larger ideal) and, (3) $J(P)$ includes the ideal $\emptyset$.

Problem 53. [2]
Let P be a finite n-element poset. Simplify the two sums

$$f(P) = \sum_{I \subset J(P)} e(I)e(\overline{I}),$$

$$g(P) = \sum_{I \subset J(P)} \binom{n}{\#I} e(I)e(\overline{I}),$$

where $\overline{I}$ denotes the complement $P - I$ of the order ideal I.

Proof. The pre-image $\sigma^{-1}([j])$ of a linear extension σ is a j-element ideal. In particular, every ideal can be written as $\sigma^{-1}([j])$ for some linear extension. Therefore each linear extension has $n+1$ associated ideals, namely, $\sigma^{-1}([k])$ for $k = \{0, 1, \ldots, n\}$ with the convention that $[0] = \emptyset$.
Therefore the sum over all k-element ideals will give $e(P)$. So

$$\begin{aligned} f(P) &= \sum_{k=0}^{n} \sum_{\substack{I \subset J(P) \\ \#I = k}} e(I)e(\bar{I}) \\ &= \sum_{k=0}^{n} e(P) \\ &= (n+1)e(P) \end{aligned}$$

And similarly

$$\begin{aligned} g(P) &= \sum_{k=0}^{n} \sum_{\substack{I \subset J(P) \\ \#I = k}} \binom{n}{k} e(I)e(\bar{I}) \\ &= \sum_{k=0}^{n} \binom{n}{k} e(P) \\ &= 2^n e(P) \end{aligned}$$

□

Problem 57.

a. [2] Let P be an n-element poset. If $t \in P$, then set $\lambda_t = \#\{s \in P : s \leq t\}$. Show that

$$e(P) \geq \frac{n!}{\prod_{t \in P} \lambda_t}.$$

b. [2+] Show that the equality holds if and only if every component of P is a rooted tree.

Proof. a. By induction, the base case is clear. When $n = 1$, there is only one poset with one linear extension.

$$e([1]) = \frac{1!}{\lambda_1} \geq 1.$$

Thus given some poset P, we can take the subposet $P - m$ (where m is any maximal element in P) which is a disjoint union of posets $P_1 + P_2 + \ldots + P_k$ with $n_1, n_2, \ldots, n_k$ elements respectively. There are $n - \lambda_m + 1$ choices of labels for m. After choosing one, n_i labels for each P_i can be chosen, and each can be permuted in $e(P_i)$ ways, so

$$\begin{aligned}
e(P) &= (n - \lambda_m + 1)\binom{n-1}{n_1, n_2, \ldots, n_k} e(P_1)e(P_2)\ldots e(P_k) \\
&\geq (n - \lambda_m + 1)\frac{(n-1)!}{n_1!n_2!\ldots n_k!} \cdot \frac{n_1!}{\prod_{t \in P_1} \lambda_t} \cdot \frac{n_2!}{\prod_{t \in P_2} \lambda_t} \cdots \frac{n_k!}{\prod_{t \in P_k} \lambda_t} \\
&= (n - \lambda_m + 1)\frac{(n-1)!}{\prod_{t \in P - m} \lambda t}
\end{aligned}$$

Since λm must take on values in $[n]$, $\lambda_m(n - \lambda_m + 1)$ is concave down with respect to λ_m and so has minima at $\lambda_m = 1$ and $\lambda_m = n$, in particular $n - \lambda_m + 1 \geq n/\lambda_m$ so substituting this into the inequality above gives

$$e(P) \geq \frac{n}{\lambda_m} \cdot \frac{(n-1)!}{\prod_{t \in P - m} \lambda t} = \frac{n!}{\prod_{t \in P} \lambda t},$$

as desired.

b. ($\Longrightarrow$) Assume that the equality holds. This means that in the above argument, for each maximal element m, $\lambda_m = 1$ or $\lambda_m = n$. If $\lambda_m = 1$, then the component containing m is the singleton poset and thus is a rooted tree. If $\lambda_m = n$, then m must be the unique maximal element of its component, $m = \hat{1}_{P_i}$.
By the above argument, this can be repeated inductively; that is, after removing any maximal element, the new maximal elements have the same property. Therefore P_i is a rooted tree for all components P_i of P.

($\Longleftarrow$)
By induction, the base case is clear. When $n = 1$, there is only one poset (which is a rooted tree) with one linear extension.

$$e([1]) = \frac{1!}{\lambda_1} = 1.$$

Thus given some rooted tree P, we can take the subposet $P - \hat{1}$, which is a disjoint union of rooted trees $P_1 + P_2 + \ldots + P_k$ with $n_1, n_2, \ldots, n_k$ elements respectively. Since P has a unique maximum, it must be labeled with n, then we can then choose which letters go in each sub-tree (using a multinomial coefficient), and then there are $e(P_i)$ ways to order the n_i labels for each P_i. Therefore

$$\begin{aligned}
e(P) &= \binom{n-1}{n_1, n_2, \ldots, n_k} e(P_1)e(P_2)\ldots e(P_k) \\
&= \binom{n-1}{n_1, n_2, \ldots, n_k} \frac{n_1!}{\prod_{t \in P_1} \lambda_t} \frac{n_2!}{\prod_{t \in P_2} \lambda_t} \cdots \frac{n_k!}{\prod_{t \in P_k} \lambda_t} \\
&= \left(\frac{(n-1)!}{n_1!n_2!\ldots n_k!}\right) \frac{n_1!n_2!\ldots n_k!}{\prod_{t \in P - \hat{1}} \lambda_t} \\
&= \frac{(n-1)!}{\prod_{t \in P - \hat{1}} \lambda_t}
\end{aligned}$$

Since all n elements of P are less than or equal to $\hat{1}$, $\lambda_{\hat{1}} = n$,

$$n \prod_{t \in P - \hat{1}} \lambda_t = \prod_{t \in P} \lambda_t$$

and thus

$$e(P) = \frac{n(n-1)!}{n \prod_{t \in P - \hat{1}} \lambda_t} = \frac{n!}{\prod_{t \in P} \lambda_t}$$

as desired.

$\square$