

# Complex Analysis: Homework 8

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**Problem 1.** (page 161)

Find the poles and residues of the following functions

(a)  $\frac{1}{z^2 + 5z + 6}$

(b)  $\frac{1}{(z^2 - 1)^2}$

(c)  $\frac{1}{\sin z}$

(d)  $\cot z$

(e)  $\frac{1}{\sin^2 z}$

(f)  $\frac{1}{z^m(1-z)^n}$  ( $m, n$  positive integers).

*Proof.*

(a) First note that the denominator factors as

$$x^4 + 5x^2 + 6 = (x^2 + 2)(x^2 + 3) = (x + i\sqrt{2})(x - i\sqrt{2})(x + i\sqrt{3})(x - i\sqrt{3})$$

Thus there are four poles:

$$-i\sqrt{2}, i\sqrt{2}, -i\sqrt{3}, \text{ and } i\sqrt{3}$$

Next, looking at  $B_1$  in the expansion  $f(z) = B_h(z - z_0)^{-h} + \dots + B_1(z - z_0)^{-1} + \varphi(z)$  immediately yields

(i)  $\text{Res}_{z=-i\sqrt{2}} f(z) = \frac{1}{(-i\sqrt{2} - i\sqrt{2})((-i\sqrt{2})^2 + 3)} = \frac{-1}{2i\sqrt{2}}$

(ii)  $\text{Res}_{z=i\sqrt{2}} f(z) = \frac{1}{(i\sqrt{2} + i\sqrt{2})((i\sqrt{2})^2 + 3)} = \frac{1}{2i\sqrt{2}}$

(iii)  $\text{Res}_{z=-i\sqrt{3}} f(z) = \frac{1}{(-i\sqrt{3} - i\sqrt{3})((-i\sqrt{3})^2 + 2)} = \frac{1}{2i\sqrt{3}}$

(iv)  $\text{Res}_{z=i\sqrt{3}} f(z) = \frac{1}{(i\sqrt{3} + i\sqrt{3})((i\sqrt{3})^2 + 2)} = \frac{-1}{2i\sqrt{2}}$

(b) First note that the denominator factors as

$$(z^2 - 1)^2 = (z + 1)^2(z - 1)^2$$

so there are two poles of order 2:  $-1$  and  $1$ . Thus the residues are

(i)  $\text{Res}_{z=-1} f(z) = \frac{d}{dz} \left[ \frac{1}{(z - 1)^2} \right]_{z=-1} = \frac{1}{4}$

$$(ii) \operatorname{Res}_{z=1} f(z) = \frac{d}{dz} \left[ \frac{1}{(z+1)^2} \right]_{z=1} = -\frac{1}{4}$$

- (c) First note that  $\sin z$  has zeros of order 1 precisely at  $z = 2\pi k$  for some  $k \in \mathbb{N}$ ; therefore  $1/\sin(z)$  has poles of order 1 when  $z = 2\pi k$ . Thus the residue at each pole is

$$\begin{aligned} \lim_{z \rightarrow 2\pi k} \frac{(z - 2\pi k)}{\sin z} &= \lim_{z \rightarrow 2\pi k} \frac{(z - 2\pi k)}{(z - 2\pi k) - (z - 2\pi k)^3/3! + \dots} \\ &= \lim_{z \rightarrow 2\pi k} \frac{1}{1 - (z - 2\pi k)^2/3! + \dots} \\ &= 1 \end{aligned}$$

- (d) Note that  $\cot z = \frac{\cos z}{\sin z}$  has the same poles as above, all of order 1:  $z = 2\pi k$ . Thus the residues at each pole are

$$\begin{aligned} \lim_{z \rightarrow 2\pi k} (z - 2\pi k)(\cot z) &= \lim_{z \rightarrow 2\pi k} (z - 2\pi k) \frac{1 - (z - 2\pi k)^2 + \dots}{(z - 2\pi k) - (z - 2\pi k)^3/3! + \dots} \\ &= \lim_{z \rightarrow 2\pi k} \frac{1 - (z - 2\pi k)^2 + \dots}{1 - (z - 2\pi k)^2/3! + \dots} \\ &= 1 \end{aligned}$$

- (e) Now the poles are at  $z = 2\pi k$ , but they are of order 2.

$$\frac{d}{dz} \left[ \frac{1}{\sin^2 z} \right]_{z=2\pi k}$$

- (f) Clearly this final function has a pole of order  $m$  at  $z = 0$ , and of  $n$  at  $z = 1$ . Because a curve around an isolated singularity  $z_0$

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \operatorname{Res}_{z=z_0} f(z)$$

and

$$\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ \frac{1}{z^m} \right]_{z=1} = \frac{1}{2\pi i} \int_{\gamma} \frac{-z^{-m}}{(-1)^n (1-z)^n} dz$$

so

$$\operatorname{Res}_{z=1} f(z) = \frac{(-1)^n}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ \frac{1}{z^m} \right]_{z=1} = \frac{(-1)^n \cdot (-m)(-m+1) \cdots (-m+n-2)}{(n-1)!}$$

The other residue falls similarly

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[ \frac{1}{(1-z)^n} \right]_{z=0} = \frac{(-n)(-n+1) \cdots (-n+m-2)}{(m-1)!}.$$

□

**Problem 3.** (page 161)

Evaluate the following integrals by the method of residues:

(b)  $\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6}$

(g)  $\int_0^\infty \frac{x^{1/3}}{1+x^2} dx$

(h)  $\int_0^\infty (1+x^2)^{-1} \log x dx$

(i)  $\int_0^\infty \log(1+x^2) \frac{dx}{x^{1+\alpha}} \quad (0 < \alpha < 2).$

*Proof.*

(b) Because the integral is even, we can exploit that

$$\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6} = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6}$$

First note that the denominator  $x^4 + 5x^2 + 6 = (x^2 + 2)(x^2 + 3)$  has no real roots. Now integrating the complex function over  $\Gamma_R$ , the semicircle of radius  $R$  in the upper half plane centered at the origin gives

$$\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6} = \frac{1}{2} \left( \lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{z^2 dz}{z^4 + 5z^2 + 6} - \int_0^\pi \frac{(Re^{it})^2}{(Re^{it})^4 + 5(Re^{it})^2 + 6} \cdot ie^{it} dt \right)$$

where the final integral vanishes in the limit and the integral over  $\Gamma$  is given by

$$\begin{aligned} \int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6} &= \frac{1}{2} \cdot 2\pi i (\text{Res}_{z=i\sqrt{2}} f(z) + \text{Res}_{z=i\sqrt{3}} f(z)) \\ &= \frac{1}{2} \cdot 2\pi i \left( \frac{(i\sqrt{2})^2}{-2i\sqrt{2}} - \frac{i\sqrt{3}^2}{2i\sqrt{3}} \right) \\ &= \frac{\pi}{2} (\sqrt{3} - \sqrt{2}) \end{aligned}$$

using the residues calculated by a similar method to Problem 1(a).

(g) Here we want to avoid the branch cut at  $\theta = 2\pi/3$ , we will perform the substitution  $x = t^2$  transforming the integral into

$$2 \int_0^\infty \frac{t^{5/3}}{1+t^4} dt$$

and choosing the branch of  $t^{2/3}$  whose argument lies between  $-\pi/3$  and  $\pi$ . By Ahlfors argument,

$$\int_{-\infty}^\infty \frac{t^{5/3}}{1+t^4} dt = (1 - e^{2\pi i/3}) \int_0^\infty \frac{t^{5/3}}{1+t^4} dt$$

On the first integral, we can use the techniq from part (b), and take the residues from the poles in the upper half plane (which are  $z = e^{\pi i/4}$  and  $z = e^{3\pi i/4}$ ). These are simple enough to compute as

$$\frac{(e^{\pi i/4})^{5/3}}{(e^{\pi i/4} - e^{3\pi i/4})(e^{\pi i/4} - e^{5\pi i/4})(e^{\pi i/4} - e^{7\pi i/4})} \text{ and } \frac{(e^{3\pi i/4})^{5/3}}{(e^{3\pi i/4} - e^{\pi i/4})(e^{3\pi i/4} - e^{5\pi i/4})(e^{3\pi i/4} - e^{7\pi i/4})}.$$

Therefore

$$\int_0^\infty \frac{t^{5/3}}{1+t^4} dt = \frac{1}{1 - e^{2\pi i/3}} \left( \sum_{y>0} \text{Res}_{z=z_0} f(z) \right)$$

where the residues are the two long fractions above.

(h) Here we will integrate over the boundary of the half annulus in the upper half plane, with

$$\Gamma_1 = [\epsilon, R] \tag{1}$$

$$\Gamma_2 = \{Re^{it} : t \in [0, \pi]\} \tag{2}$$

$$\Gamma_3 = [-\epsilon, R] \tag{3}$$

$$\Gamma_4 = \{\epsilon e^{-it} : t \in [0, \pi]\} \tag{4}$$

and choosing the branch cut of  $\log$  to be the negative imaginary axis, so that  $\log z = \log|x| + i \arg z$  where  $-\pi/2 < \arg z < 3\pi/2$ .

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