Math 574: Homework 1

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Problem 1.

(a) Show that $A, B \in M_3(K)$ are similar if and only if they have the same minimal and characteristic polynomials.

Proof. A and B are similar if and only if they have the same Jordan normal form, so it is sufficient to compare the minimal and characteristic polynomials of matrices in Jordan normal form.

(a) A 3×3 matrix can have one of three Jordan normal forms

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \; A_2 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \text{or } A_3 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$

Notice that a matrix A

(i) similar to A_1 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly once,

(ii) similar to A_2 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly twice, and

(iii) similar to A_3 if and only if $(\lambda_1 - x)$ divides $m_A(x)$ exactly three times.

(b) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Both A and B have the same minimal and characteristic polynomials

$$m_A(x) = m_B(x) = (1 - x)^2$$
$$p_A(x) = p_B(x) = (1 - x)^4,$$

but A and B are not similar because they have different Jordan canonical forms.

$\square$

Problem 2. Fix $A \in M_n(K)$ and let $C(A) = \{B : BA = AB\}$.

Proof.

(a) Suppose A is cyclic, that is $p_A(x) = m_A(x)$, and moreover

$$A^n = c_0 I + c_1 A + \ldots + c_{n-1} A^{n-1}$$

and the Jordan normal form of A is a single Jordan block. Now look at what commutes

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ \lambda & 1 & & & \\ \lambda & 1 & & & \end{bmatrix}$$

(b) First consider A in Jordan canonical form.

$\square$

Problem 3.

(a) Show that if N is nilpotent then $N^n = 0$.

(b)

(c) How many similarity classes of 5×5 nilpotent matrices are there?

Proof.

(a) If $N^d = 0$, then there exists some $d' \leq n$ such that $N^{d'} = 0$ because the minimal polynomial $m_N(x) | x^d$, so the minimal polynomial is of the form $m_N(x) = x^{d'}$ with $d' <= n$, since the minimal polynomial has degree less than or equal to n.

(b)

(c) By (a), the characteristic polynomial of a 5×5 nilpotent matrix is $p(x) = x^5$, so in Jordan canonical form, $a_{ii} = 0$. Thus the Jordan canonical form of A has zeros along the diagonal, with possibly some ones on the superdiagonal:

$$\begin{bmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus similarity classes of nilpotent matrices are in bijection with the seven partitions of 5 via the size of the Jordan blocks:

$$\begin{aligned} &5 \\ &4+1 \\ &3+2 \\ &3+1+1 \\ &2+2+1 \\ &2+1+1+1 \\ &1+1+1+1+1 \end{aligned}$$

$\square$

Problem 4.

Proof. $\square$

Problem 5.

Proof.

(a) Let $A = U^{-1} J_A U$, where J_A is the Jordan canonical form of A, which can be written $J_A = D_A + N_A$, with D_A the diagonal entries of J_A, and N_A the superdiagonal entries of J_A.

$$\underbrace{\begin{bmatrix} \lambda_1 & * & & & \\ & \lambda_2 & * & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & * \\ & & & & \lambda_n \end{bmatrix}}_{J_A} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & & & \\ & \lambda_2 & 0 & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & 0 \\ & & & & \lambda_n \end{bmatrix}}_{D_A} = \underbrace{\begin{bmatrix} 0 & * & & & \\ & 0 & * & & \\ & & 0 & \ddots & \\ & & & \ddots & * \\ & & & & 0 \end{bmatrix}}_{N_A}$$

Notice that $p_{N_A}(x) = x^n$, so by Cayley-Hamilton, $N_A^n = 0$ and N_A. is nilpotent. Next, notice that

$$\begin{aligned} A &= U^{-1} J_A U \\ &= U^{-1}(D_A + N_A)U \\ &= U^{-1} D_A U + U^{-1} N_A U \end{aligned}$$

where $U^{-1} D_A U$ is clearly diagonalizable, and where $U^{-1} N_A U$ is nilpotent because

$$\begin{aligned} (U^{-1} N_A U)^n &= \underbrace{(U^{-1} N_A U)(U^{-1} N_A U) \dots (U^{-1} N_A U)}_{n} \\ &= U^{-1} \underbrace{N_A^n}_{0} U \\ &= 0. \end{aligned}$$

$\square$