

# Spring 2013: Real Analysis Graduate Exam

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**Problem 1.** Suppose that  $\{f_n\}$  is a sequence of real valued continuously differentiable functions on  $[0, 1]$  such that

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n(x)| dx = 0 \text{ and } \lim_{n \rightarrow \infty} \int_0^1 |f'_n(x)| dx = 0.$$

Show that  $\{f_n\}$  converges to 0 uniformly on  $[0, 1]$ .

*Proof.*

□

**Problem 2.** Investigate the convergence of  $\sum_{n=0}^{\infty} a_n$ , where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x) dx$$

*Proof.*

□

**Problem 3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $f_n, f \in L^1(\mu)$ . Show that  $\int_X |f_n - f| d\mu \rightarrow 0$  as  $n \rightarrow \infty$  if and only if

$$\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \rightarrow 0$$

as  $n \rightarrow \infty$

*Proof.*

□

**Problem 4.** Let  $\mu$  and  $\nu$  be  $\sigma$ -finite positive measures,  $\mu \geq \nu$  and assume that  $\nu \ll \mu - \nu$  ( $\nu$  is absolutely continuous with respect to  $\mu - \nu$ ).

Prove that

$$\mu \left( \left\{ \frac{d\nu}{d\mu} = 1 \right\} \right) = 0.$$

*Proof.*

□