Week 10 Quiz Solutions.

1. (*5 points*) Use the second derivatives test to identify local maxima, local minima, and saddle points for the function
$$f(x,y) = \frac{x^3}{3} + 8x + 3y^2 - 6xy.$$

Solution.
First find the critical points of f, by setting the gradient equal to zero
$$\nabla f(x,y) = \langle x^2 + 8 - 6y, 6y - 6x\rangle = \langle 0,0\rangle.$$
By the second equation this occurs when $x = y$. By substituting in the first equation, this gives $x^2 - 6x + 8 = (x-2)(x-4) = 0$. So the critical points occur at $(2,2)$ or $(4,4)$. By the second derivative test
$$\begin{aligned} D &= f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 \\ &= (2x)(6) - (-6)^2 \\ &= 12x - 36. \end{aligned}$$
When $(x,y) = (2,2)$, $D = -12 < 0$, so this is a saddle point. When $(x,y) = (4,4)$, $D = 12 > 0$ and $f_{xx} = 8$, so this is a local minimum.

2. (*5 points*) Use Lagrange multipiers to find the extreme values of the above function $f(x,y)$ subject to the constraint that $x - y = 2$.

Solution. Name the constraint $g(x,y) = x - y$. Then we have
$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ \langle x^2 + 8 - 6y, 6y - 6x\rangle &= \lambda\langle 1, -1\rangle \end{aligned}$$
which gives the following system of equations
$$x^2 + 8 - 6y = \lambda \tag{1}$$
$$6y - 6x = -\lambda \tag{2}$$
$$x - y = 2 \tag{3}$$
Combining equations (2) and (3) gives that $\lambda = 12$. Then solving for y in equation (3) and using equation (1) gives
$$\begin{aligned} x^2 + 8 - 6(x-2) &= 12 \\ x^2 - 6x + 8 &= 0 \\ (x-2)(x-4) &= 0. \end{aligned}$$
So $x = 2$ or $x = 4$, corresponding to the points $(2,0)$ and $(4,2)$ respectively. The function f does not have a global maximum subject to the constraint $x - y = 2$, but since
$$f(2,0) = \frac{8}{3} + 16 = \frac{56}{3} > f(4,2) = \frac{64}{3} + 32 + 12 - 48 = \frac{52}{3}$$
$(2,0)$ is a local max and $(4,2)$ is a local min.