# Spring 2014: Complex Analysis Graduate Exam

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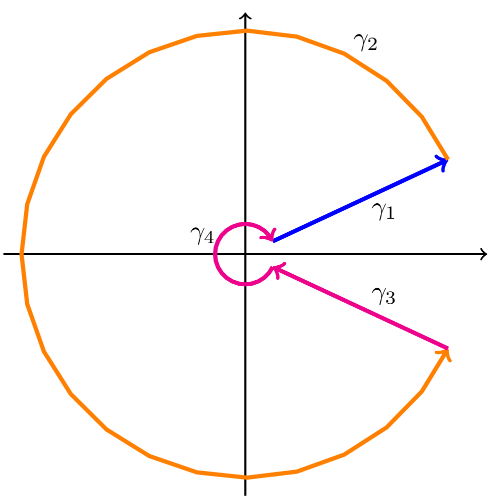
**Problem 1.** For $a > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(a+x)^3}\, dx$$

being careful to justify your methods.

*Proof.* First, call this integral $S$, and denote the integrand by $f(x)$. We will evaluate this integral using the usual trick: integrating $g(z) = f(z)\log(z)$ around a keyhole contour, with the branch cut of the logarithm along the positive real axis, so that the contour contains the pole at $z = -a$.




$$\gamma_1 = \{te^{i\rho} \mid t \in [\varepsilon, R]\} \tag{1}$$

$$\gamma_2 = \{Re^{it} \mid t \in [\rho, 2\pi - \rho]\} \tag{2}$$

$$\gamma_3 = \{-te^{(2\pi-\rho)i} \mid t \in [-R, -\varepsilon]\} \tag{3}$$

$$\gamma_4 = \{\varepsilon e^{-it} \mid t \in [-2\pi + \rho, -\rho]\}. \tag{4}$$

Then by the residue theorem,

$$\int_{\gamma_1} g(z)dz + \int_{\gamma_2} g(z)dz + \int_{\gamma_3} g(z)dz + \int_{\gamma_4} g(z)dz = \mathrm{Res}_{-a}(g).$$

As expected, the integrals along the arcs vanish in the limit.
Firstly, the large arc is bounded by

$$\begin{aligned}
\left|\int_{\gamma_2} g(z)dz\right| &\leq \int_0^{2\pi} \left|\frac{\log^2(Re^{i\theta})}{(a+Re^{i\theta})^3} Rie^{i\theta}\right| d\theta \\
s &\leq \int_0^{2\pi} \left|\frac{\log^2(Re^{i\theta})}{R^3 e^{3i\theta}} R\right| d\theta \\
&\leq \int_0^{2\pi} \left|\frac{\log^2 R + 2i\theta \log R - \theta}{R^2}\right| d\theta,
\end{aligned}$$

which vanishes as $R \to \infty$ by the ML inequality. Next, the small arc is bounded by

$$\begin{aligned}
\left|\int_{\gamma_4} g(z)dz\right| &\leq \int_0^{2\pi} \left|\frac{\log^2(\rho e^{i\theta})}{(a+\rho e^{i\theta})^3}\rho i e^{i\theta}\right| d\theta \\
&\leq \int_0^{2\pi} \left|\frac{\log^2(\rho e^{i\theta})}{(a/2)^3}\rho\right| d\theta \\
&\leq \frac{8}{a^3}\int_0^{2\pi} \left|\rho(\log^2\rho + 2i\theta\log\rho - \theta)\right| d\theta,
\end{aligned}$$

which vanishes as $\rho \to 0$ with the ML inequality together with two applications of L'Hôpital's rule

$$\lim_{\rho\to 0}\rho\log^2\rho = \lim_{\rho\to 0}\frac{\log^2\rho}{\rho^{-1}} = \lim_{\rho\to 0}\frac{2\rho^{-1}\log\rho}{-\rho^{-2}} = \lim_{\rho\to 0}\frac{2\log\rho}{-\rho^{-1}} = \lim_{\rho\to 0}\frac{2\rho^{-1}}{\rho^{-2}} = \lim_{\rho\to 0}2\rho = 0.$$

Now the integral has been reduced to

$$\int_{\gamma_1} g(z)dz + \int_{\gamma_3} g(z)dz = \mathrm{Res}_{-a}(g).$$

Evaluating the remaining integrals yields

$$\begin{aligned}
\int_{\gamma_1} g(z)dz + \int_{\gamma_3} g(z) &= \lim_{\rho\to 0}\int_\varepsilon^R \frac{\log^2(te^{i\rho})}{(a+te^{i\rho})^3}e^{i\rho}dt + \int_R^\varepsilon \frac{\log^2(te^{i(2\pi-\rho)})}{(a+te^{i(2\pi-\rho)})^3}e^{i(2\pi-\rho)}dt \\
&= \int_\varepsilon^R \frac{\log^2(t)}{(a+t)^3}dt + \int_R^\varepsilon \frac{(\log(t)+2\pi i)^2}{(a+t)^3}dt \\
&= \int_\varepsilon^R \frac{4\pi^2 - 4\pi i\log(t)}{(a+t)^3}dt \\
&= \int_\varepsilon^R \frac{4\pi^2}{(a+t)^3}dt - 4\pi i\int_\varepsilon^R \frac{\log(t)}{(a+t)^3}dt.
\end{aligned}$$

Using ordinary techniques of integration, we can evaluate the integral

$$\int_0^\infty \frac{4\pi^2}{(a+t)^3}dt = \left[\frac{4\pi^2}{-2(a-t)^2}\right]_0^\infty = \frac{2\pi^2}{a^2}.$$

Computing the residue of $g$ at $z=-a$ just requires computing the Taylor series of $\log^2 z$ centered at $z=-a$ to a second order term, which requires computing the second derivative of $\log^2 z$.

$$\begin{aligned}
\frac{d}{dz}[\log^2 z] &= 2\log(z)z^{-1} \\
\frac{d^2}{dz^2}[\log^2 z] &= 2(-\log(z)z^{-2} + z^{-2})
\end{aligned}$$

Thus

$$\frac{\log^2 z}{(z+a)^3} = \frac{1}{(z+a)^3} = \left[\log^2(-a) + \frac{2\log(-a)}{-a}(z+a) + \frac{2(1-\log(-a))}{2a^2}(z+a)^2 + \dots\right].$$

So the residue at $z=-a$ is

$$\mathrm{Res}_{-a}(g) = \frac{1-\log(-a)}{a^2} = \frac{1-\log(a)-\log(-1)}{a^2} = \frac{1-\log(a)-\pi i}{a^2}.$$

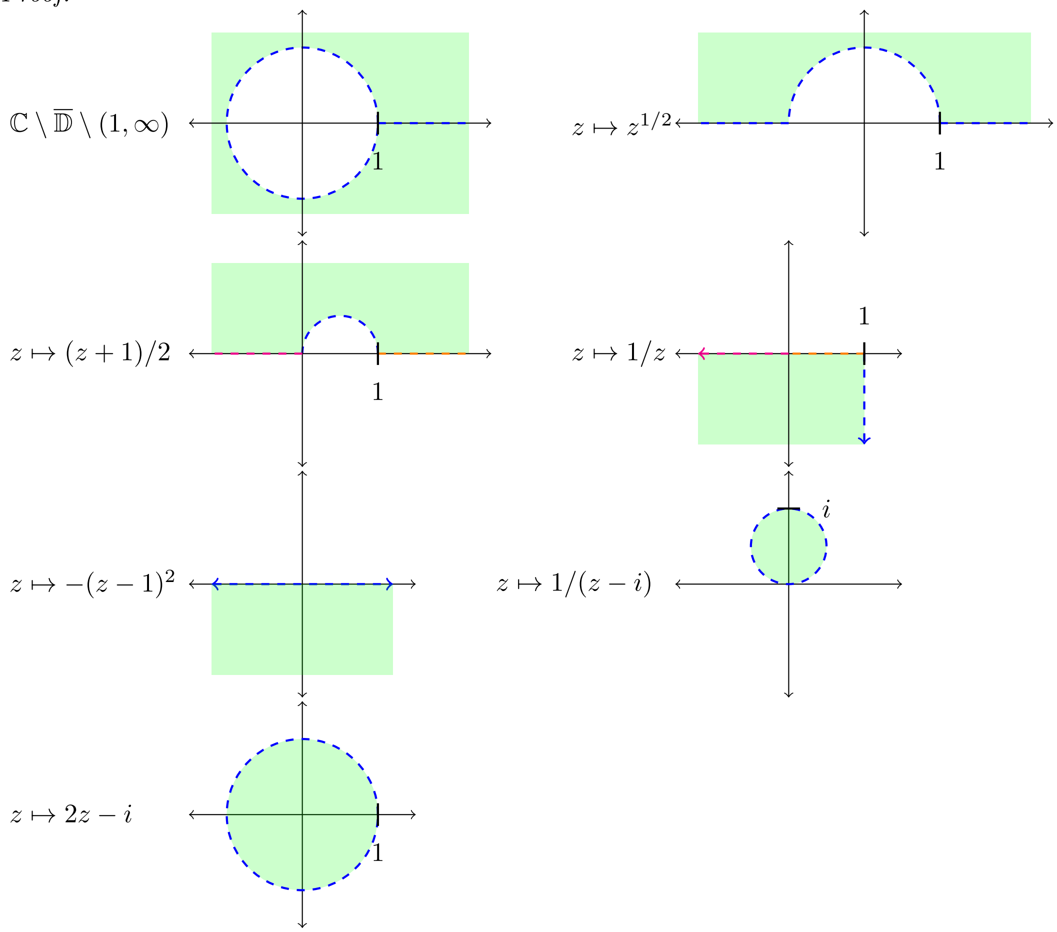
Now we have all of the ingredients to compute the integral:

$$\begin{aligned}
\frac{2\pi^2}{a^2} - 4\pi i\int_0^\infty \frac{\log(t)}{(a+t)^3}dt &= 2\pi i\left(\frac{1-\log(a)-\pi i}{a^2}\right) \\
\int_0^\infty \frac{\log(t)}{(a+t)^3}dt &= \frac{1}{4\pi i}\left(\frac{2\pi^2}{a^2} - \frac{2\pi i(1-\log(a)-\pi i)}{a^2}\right) = \frac{\log(a)-1}{2a^2}.
\end{aligned}$$

□

**Problem 2.** Find a conformal mapping of the region $\{z : |z| > 1\} \setminus (1, \infty)$ onto the open unit disk $\{z : |z| < 1\}$.

*Proof.*




□

**Problem 3.** Suppose that $f_n$ are analytic functions on a connected open set $U \subset \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of $U$. In each case indicate the main steps in the proofs of the following standard results.

(i) $f$ is analytic in $U$;

(ii) $f_n' \to f'$ uniformly on compact subsets of $U$;

(iii) if $f_n(z) \neq 0$ for all $n$ and all $z \in U$, then either $f(z) \neq 0$ for all $z \in U$ or else $f \equiv 0$.

*Proof.* $\square$

**Problem 4.**

(a) Suppose that $f$ is analytic on the open unit disk $\{z : |z| < 1\}$ and that there exists a constant $M$ such that $|f^k(0)| \leq k^4 M^k$ for all $k \geq 0$. Show that $f$ can be extended to be analytic on $\mathbb{C}$.

(b) Suppose that $f$ is analytic on the open unit disk and that there exists a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$. Show that $f$ is identically zero.

*Proof.* $\square$