Fall 2014: Complex Analysis Graduate Exam

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Problem 1. Let $a > 1$. Compute
$$\int_0^\pi \frac{d\theta}{a + \cos\theta}$$
being careful to justify your methods.

Proof. First, call this integral S, and begin with the standard trigonometic substitution,
$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}),$$
yielding
$$S = \int_0^\pi \frac{d\theta}{a + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}.$$
By exploiting the evenness of $a + \cos(\theta)$, this integral is equal to
$$S = \frac{1}{2}\int_{-\pi}^\pi \frac{d\theta}{a + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}.$$
Then by substituting $z = e^{i\theta}$ where the contour is the unit circle centered at the origin gives
$$S = \frac{1}{2}\int_{|z|=1} \frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$
where $dz/(iz)$ is the formal substitution for $d\theta$ because
$$\begin{aligned} e^{i\theta} &= z \\ i\theta &= \log z \\ d\theta &= -i\frac{dz}{z}. \end{aligned}$$
Some simplification of the integral results in
$$S = -i\int_{|z|=1} \frac{dz}{2az + (z^2 + 1)}.$$
By the quadratic formula, this integrand has poles at
$$\begin{aligned} \frac{-2a \pm \sqrt{4a^2 - 4}}{2} &= -a \pm \sqrt{a^2 - 1} \\ \alpha &= -a - \sqrt{a^2 - 1} \\ \beta &= -a + \sqrt{a^2 - 1} \end{aligned}$$

which are real because $a > 1$ by hypothesis. In particular, $\alpha = -a - \sqrt{a^2-1} < -a$, so clearly outside the contour. On the other hand,

$$\begin{aligned} a^2 &> a^2 - 1 && > a^2 - 2a + 1 \\ a &> \sqrt{a^2-1} && > a - 1 \\ 0 &> \underbrace{-a + \sqrt{a^2-1}}_{\beta} && > -1 \end{aligned}$$

so β is inside the contour.

Next, naming the integrand f, the residue theorem gives

$$S = -i \int_{|z|=1} \frac{dz}{2az + (z^2+1)} = -i(2\pi i \operatorname{Res}_\beta(f)) = 2\pi \operatorname{Res}_\beta(f).$$

Now, the residue is straightforward to compute:

$$\operatorname{Res}_\beta(f) = \lim_{z\to\beta} (z-\beta) \frac{1}{(z-\beta)(z-\alpha)} = \frac{1}{\beta-\alpha} = \frac{1}{2\sqrt{a^2-1}}.$$

Therefore

$$\int_0^\pi \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

$\square$

Problem 2.

Proof. $\square$

Problem 3.

Proof. $\square$

Problem 4.

Proof. ∎