Topology: Homework 3

Peter Kagey

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Problem 1.
In the free group $F_2 = \langle a, b; \rangle$, consider the elements

$$\begin{aligned}
A &= b^2a^{-1}b^{-3}ab^{-1}ab^{-2} \\
B &= b^2a^{-1}ba^{-1}bab^2 \\
C &= b^{-2}a^{-1}b^2a^3b \\
D &= b^{-2}a^{-1}b^2ab^{-2}a^3
\end{aligned}$$

(a) Compute $(AB)C$ and $A(BC)$ in the given order, and verify that $(AB)C = A(BC)$.

(b) Compute $(AB)D$ and $A(BD)$ in the given order, and verify that $(AB)D = A(BD)$.

Proof.

(a) First computing AB gives

$$\begin{aligned}
AB &= b^2a^{-1}b^{-3}ab^{-1}a\underbrace{b^{-2}\cdot b^2}_{\mathrm{id}_B}a^{-1}ba^{-1}bab^2 \\
&= b^2a^{-1}b^{-3}ab^{-1}\underbrace{aa^{-1}}_{\mathrm{id}_A}ba^{-1}bab^2 \\
&= b^2a^{-1}b^{-3}a\underbrace{b^{-1}b}_{\mathrm{id}_B}a^{-1}bab^2 \\
&= b^2a^{-1}b^{-3}\underbrace{aa^{-1}}_{\mathrm{id}_A}bab^2 \\
&= b^2a^{-1}\underbrace{b^{-3}b}_{b^{-2}}ab^2 \\
&= b^2a^{-1}b^{-2}ab^2.
\end{aligned}$$

Then computing $(AB)C$ gives

$$\begin{aligned}
(AB)C &= b^2a^{-1}b^{-2}a\underbrace{b^2\cdot b^{-2}}_{\mathrm{id}_B}a^{-1}b^2a^3b \\
&= b^2a^{-1}b^{-2}\underbrace{aa^{-1}}_{\mathrm{id}_A}b^2a^3b \\
&= b^2a^{-1}\underbrace{b^{-2}b^2}_{\mathrm{id}_B}a^3b \\
&= b^2\underbrace{a^{-1}a^3}_{a^2}b \\
&= b^2a^2b.
\end{aligned}$$

Similarly, computing BC gives

$$\begin{aligned} BC &= b^2a^{-1}ba^{-1}ba\underbrace{b^2\cdot b^{-2}}_{\mathrm{id}_B}a^{-1}b^2a^3b \\ &= b^2a^{-1}ba^{-1}b\underbrace{aa^{-1}}_{\mathrm{id}_A}b^2a^3b \\ &= b^2a^{-1}ba^{-1}\underbrace{bb^2}_{b^3}a^3b \\ &= b^2a^{-1}ba^{-1}b^3a^3b. \end{aligned}$$

Then computing $A(BC)$ gives

$$\begin{aligned} A(BC) &= b^2a^{-1}b^{-3}ab^{-1}a\underbrace{b^{-2}\cdot b^2}_{\mathrm{id}_B}a^{-1}ba^{-1}b^3a^3b \\ &= b^2a^{-1}b^{-3}ab^{-1}\underbrace{a\cdot a^{-1}}_{\mathrm{id}_A}ba^{-1}b^3a^3b \\ &= b^2a^{-1}b^{-3}a\underbrace{b^{-1}\cdot b}_{\mathrm{id}_B}a^{-1}b^3a^3b \\ &= b^2a^{-1}b^{-3}\underbrace{a\cdot a^{-1}}_{\mathrm{id}_A}b^3a^3b \\ &= b^2a^{-1}\underbrace{b^{-3}\cdot b^3}_{\mathrm{id}_B}a^3b \\ &= b^2\underbrace{a^{-1}\cdot a^3}_{a^2}b \\ &= b^2a^2b. \end{aligned}$$

Therefore

$$(AB)C = b^2a^2b = A(BC).$$

(b) We already computed $AB = b^2a^{-1}b^{-2}ab^2$, so computing $(AB)D$

$$\begin{aligned} (AB)D &= b^2a^{-1}b^{-2}a\underbrace{b^2\cdot b^{-2}}_{\mathrm{id}_B}a^{-1}b^2ab^{-2}a^3 \\ &= b^2a^{-1}b^{-2}\underbrace{aa^{-1}}_{\mathrm{id}_A}b^2ab^{-2}a^3 \\ &= b^2a^{-1}\underbrace{b^{-2}b^2}_{\mathrm{id}_B}ab^{-2}a^3 \\ &= b^2\underbrace{a^{-1}a}_{\mathrm{id}_A}b^{-2}a^3 \\ &= \underbrace{b^2b^{-2}}_{\mathrm{id}_B}a^3 \\ &= a^3. \end{aligned}$$

Similarly, computing BD gives

$$\begin{aligned} BD &= b^2a^{-1}ba^{-1}ba\underbrace{b^2\cdot b^{-2}}_{\mathrm{id}_B}a^{-1}b^2ab^{-2}a^3 \\ &= b^2a^{-1}ba^{-1}b\underbrace{aa^{-1}}_{\mathrm{id}_A}b^2ab^{-2}a^3 \\ &= b^2a^{-1}ba^{-1}\underbrace{bb^2}_{b^3}ab^{-2}a^3 \\ &= b^2a^{-1}ba^{-1}b^3ab^{-2}a^3. \end{aligned}$$

Then computing $A(BC)$ gives

$$\begin{aligned}
A(BC) &= b^2a^{-1}b^{-3}ab^{-1}a\underbrace{b^{-2}\cdot b^2}_{\mathrm{id}_B}a^{-1}ba^{-1}b^3ab^{-2}a^3 \\
&= b^2a^{-1}b^{-3}ab^{-1}\underbrace{a\cdot a^{-1}}_{\mathrm{id}_A}ba^{-1}b^3ab^{-2}a^3 \\
&= b^2a^{-1}b^{-3}a\underbrace{b^{-1}\cdot b}_{\mathrm{id}_B}a^{-1}b^3ab^{-2}a^3 \\
&= b^2a^{-1}b^{-3}\underbrace{a\cdot a^{-1}}_{\mathrm{id}_A}b^3ab^{-2}a^3 \\
&= b^2a^{-1}\underbrace{b^{-3}\cdot b^3}_{\mathrm{id}_B}ab^{-2}a^3 \\
&= b^2\underbrace{a^{-1}\cdot a}_{\mathrm{id}_A}b^{-2}a^3 \\
&= \underbrace{b^2\cdot b^{-2}}_{\mathrm{id}_B}a^3 \\
&= a^3.
\end{aligned}$$

Therefore

$$(AB)D = a^3 = A(BD).$$

$\square$

Problem 2.
Consider the group $G = \langle a, b; a^2b^{-3} = 1\rangle = F(a,b)/\langle a^2b^{-3}\rangle$. Let $\tau \in \mathfrak{S}_3$ be the transposition $\tau = (1\ 2)$ and let ρ be the cyclic permutation $\rho = (1\ 2\ 3)$.

a. Show that there is a unique group homomorphism $\phi\colon G \to \mathfrak{S}_3$ sending a to τ and b to ρ. Conclude that G is not abelian.

b. Show that there is a surjective homomorphism $\psi\colon G \to \mathbb{Z}$. Conclude that G is infinite.

Proof.

a. First, $\langle a^2b^{-3}\rangle \subset \ker(\phi)$ because

$$\phi(a^2b^{-3}) = \phi(a^2)\phi(b^{-3}) = \underbrace{(1\ 2)(1\ 2)}_{\mathrm{id}_{\mathfrak{S}_3}}\underbrace{(1\ 2\ 3)(1\ 2\ 3)(1\ 2\ 3)}_{\mathrm{id}_{\mathfrak{S}_3}} = \mathrm{id}_{\mathfrak{S}_3}$$

This map is the unique homomorphism that sends $\tau \mapsto (1\ 2)$ and $\rho \mapsto (1\ 2\ 3)$, because it prescribes where to send all elements of $\langle a, b\rangle$, and so the quotient map inherits this uniqueness.

b. We will use the map defined by, $\psi(a) = 3$ and $\psi(b) = 2$. Since

$$\psi(a^2b^{-3}) = \psi(a^2)\psi(b^{-3}) = 2(3) + -3(2) = 0$$

we have that $\langle a^2b^{-3}\rangle \subset \ker(\psi)$, and so ψ defines a homomorphism. Thus it only remains to check that ψ is surjective: any even number $2n \in \mathbb{Z}$ can be written as $\psi([a^n])$, and similarly any odd number $2n+1 \in \mathbb{Z}$ can be written as $\psi([a^{n-1}b])$. Thus G is infinite.

$\square$

Problem 3.
Let X be a metric space with metric d_0, and pick a base point $x_0 \in X$. Let $\Omega_{x_0} X$ denote the space of paths $\alpha\colon [0,1] \to X$ with $\alpha(0) = \alpha(1) = x_0$.

a. Define $x_2(X; x_0) = \pi_1(\Omega_{x_0} X; c_{x_0})$. Interpret $\pi_2(X; x_0)$ as a set of maps. What is the geometric interpretation of the group law in this context. $[0,1] \times [0,1] \to X$ modulo a certain equivalence relation.

b. Show that $\pi_2(X; x_0)$ is an abelian group.

Solution.

a. Let elements of $\pi_2(X, x_0)$ be equivalence classes of (continuous) maps $\alpha\colon [0,1] \times [0,1] \to X$ such that

$$\alpha(0,t) = \alpha(1,t) = x_0 = \alpha(s,0) = \alpha(s,1)$$

where two maps α_0 and α_1 are equivalent if there exists a continuous map $H\colon ([0,1] \times [0,1]) \times [0,1] \to X$ between them such that

$$\begin{aligned} H(s,t,0) &= \alpha_0(s,t), \\ H(s,t,1) &= \alpha_1(s,t), \\ H(0,t,r) &= H(1,t,r) = x_0, \text{ and} \\ H(s,0,r) &= H(s,1,r) = x_0. \end{aligned}$$

The geometric interpretation of the group law is a path that starts and ends as the constant path at x_0, and is a continuous collection of loops in between.

b. We'll define the homotopy between $\alpha * \beta$ and $\beta * \alpha$ by $H\colon ([0,1] \times [0,1]) \times [0,1] \to X$ as three maps H_1, H_2, and H_3:

$$H(s,t,r) = \begin{cases} H_1(s,t,3r) & r \in [0,1/3] \\ H_2(s,t,3r-1) & r \in [1/3,2/3] \\ H_3(s,t,3r-2) & r \in [2/3,1] \end{cases}.$$

Define

$$H_1(s,t,r) = \begin{cases} x_0 & s \in [0,1/2], t \in [0,r/2] \\ \alpha(2s,?) & s \in [0,1/2], t \in [r/2,1] \\ \beta(2s-1,?) & s \in [1/2,1], t \in [0,r/2] \\ x_0 & s \in [1/2,1], t \in [r/2,1] \end{cases}.$$

(I ran out of time.)

c. Let elements of $\pi_n(X, x_0)$ be equivalence classes of (continuous) maps $\alpha\colon [0,1]^n \to X$ such that

$$\begin{aligned} \alpha(0,t_2,t_3,\ldots,t_n) = \alpha(1,t_2,t_3,\ldots,t_n) &= \alpha(t_1,0,t_3,\ldots,t_n) = \alpha(t_1,1,t_3,\ldots,t_n) \\ &= \ldots \\ &= \alpha(t_1,t_2,\ldots,t_{n-1},0) = \alpha(t_1,t_2,\ldots,t_{n-1},1) \\ &= x_0 \end{aligned}$$

where two maps α_0 and α_1 are equivalent if there exists a continuous map $H\colon ([0,1]^n) \times [0,1] \to X$

between them such that

$$
\begin{aligned}
H(t_1, t_2, \ldots, t_n, 0) &= \alpha_0(t_1, t_2, \ldots, t_n), \\
H(t_1, t_2, \ldots, t_n, 1) &= \alpha_1(t_1, t_2, \ldots, t_n), \\
H(0, t_2, t_3, \ldots, t_n, r) &= x_0 \\
H(1, t_2, t_3, \ldots, t_n, r) &= x_0 \\
H(t_1, 0, t_3, \ldots, t_n, r) &= x_0 \\
H(t_1, 1, t_3, \ldots, t_n, r) &= x_0 \\
&\cdots \\
H(t_1, t_2, \ldots, t_{n-1}, 0, r) &= x_0 \\
H(t_1, t_2, \ldots, t_{n-1}, 1, r) &= x_0.
\end{aligned}
$$

d. Here we use a similar procedure to part b, but a higher-dimensional analog.