Math 510b: Homework 3

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Problem 5.57 (Rotman). If R is a commutative ring, then its **nilradical** $\mathrm{nil}(R)$ is defined to be the intersection of all of the prime ideals in R. Prove that $\mathrm{nil}(R)$ is the set of all the nilpotent elements in R:

$$\mathrm{nil}(R) = \{r \in R : r^m = 0 \text{ for some } m \geq 1\}$$

Proof.
($\Longleftarrow$) Let $\mathfrak{p}$ be an arbitrary prime ideal, and assume that $x \in \sqrt{(0)}$. Then $x^n \in \mathfrak{p}$ because $x^m = 0$ for some $m > 1$. Next, if $x^n \in \mathfrak{p}$, then $x \cdot x^{n-1} \in \mathfrak{p}$, meaning that either $x \in \mathfrak{p}$ (in which case we're done) or $x^{n-1} \in \mathfrak{p}$, which can be continued by induction to show that $x \in \mathfrak{p}$. Since $\mathfrak{p}$ was an arbitrary prime ideal, x is in the intersection of all prime ideals.

($\Longrightarrow$) Now to prove the converse, let $x \notin \sqrt{(0)}$. Next, let $\mathcal{S} = \{I \text{ ideal of } R : I \text{ has no element of the form } x^n\}$. This set is nonempty because it contains (0). Thus by Zorn's Lemma $\mathcal{S}$ has a maximal element $\mathfrak{m}$ because the union of all ideals in a chain is in $\mathcal{S}$ and serves as an upper bound. $\square$

Problem 10.27 (Rotman). If R is an integrally closed domain and $S \subseteq R$ is multiplicative, prove that $S^{-1}R$ is also integrally closed.

Proof.

Definition. A domain R is called an *integrally closed domain* if every element $\alpha \in \mathrm{Frac}(R)$ is the root of a monic polynomial in $R[x]$.

Let $f_1/f_2 \in \mathrm{Frac}(S^{-1}R)$. We want to show that f_1/f_2 can be written as the root of a monic polynomial with coefficients in $S^{-1}R$:

$$\left(\frac{f_1}{f_2}\right)^n + \frac{r_{n-1}}{s_{n-1}} \cdot \left(\frac{f_1}{f_2}\right)^{n-1} + \ldots + \frac{r_0}{s_0} = 0.$$

Let $s = s_{n-1} \ldots s_0$, since we know that since R is integrally closed and

$$\left(\frac{sf_1}{f_2}\right)^n + \frac{sr_{n-1}}{s_{n-1}} \cdot \left(\frac{sf_1}{f_2}\right)^{n-1} + \frac{s^2 r_{n-2}}{s_{n-2}} \cdot \left(\frac{sf_1}{f_2}\right)^{n-2} + \ldots + \frac{s^n r_0}{s_0} = 0$$

has coefficients in R, that $\frac{sf_1}{f_2}$ is a root of a polynomial with coefficients in R, and thus $\frac{f_1}{f_2}$ is a root of a polynomial with coefficients in $S^{-1}R$. Therefore $S^{-1}R$ is integrally closed. $\square$

Problem 10.39 (Rotman). Let k be a field and let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \dots, x_n]$. Prove that

$$\mathfrak{m} = (f_1(x_1), f_2(x_1, x_2), \dots, f_n(x_1, \dots, x_n)).$$

Proof. Consider $\mathfrak{m}' = \mathfrak{m} \cap k[x_1, \dots, x_{n-1}]$, which is maximal in $k[x_1, \dots, x_{n-1}]$. By Corollary 10.71, $\mathfrak{m} = (\mathfrak{m}', g_n(x_n))$ where f_n has coefficients in $k[x_1, \dots, x_{n-1}]$, so that $g_n(x_n) = f_n(x_1, \dots, x_n)$.

Then continuing by induction, $\mathfrak{m}^{(k)} = \mathfrak{m}^{(k-1)} \cap k[x_1, \dots, x_{n-k}]$, and so

$$\begin{aligned}
\mathfrak{m}^{(k)} &= (\mathfrak{m}^{(k+1)}, g_{n-k}(x_{n-k})) \\
&= (\mathfrak{m}^{(k+1)}, f_{n-k}(x_1, \dots, x_{n-k})) \\
\mathfrak{m} &= (\mathfrak{m}', f_n(x_1, \dots, x_n)) \\
&= (\mathfrak{m}'', f_{n-1}(x_1, \dots, x_{n-1}), f_n(x_1, \dots, x_n)) \\
&= \dots \\
&= (\mathfrak{m}^{(n)}, f_1(x_1), \dots, f_{n-1}(x_1, \dots, x_{n-1}), f_n(x_1, \dots, x_n))
\end{aligned}$$

Where $\mathfrak{m}^{(n)}$ is a maximal ideal in k, a field, so it can be omitted. Thus

$$\mathfrak{m} = (\mathfrak{m}^{(n)}, f_1(x_1), \dots, f_{n-1}(x_1, \dots, x_{n-1}), f_n(x_1, \dots, x_n)),$$

as desired. $\square$

Problem 10.40 (Rotman). Prove that if R is Noetherian then $\operatorname{nil}(R)$ is a nilpotent ideal.

Proof. We'll do this combinatorially. Notice that R is Noetherian so the nilpotent ideal is finitely generated, $\operatorname{nil}(R) = (a_1, \ldots, a_n)$, and thus every element can be written as

$$\operatorname{nil}(R) = \sum_{i=1}^{n} a_i R.$$

Therefore

$$\operatorname{nil}(R)^k = \sum_{i=1}^{N} \left(\prod_{j=1}^{k} a_{i_j} R \right).$$

Then let $p = \max\{p_i : a_i^{p_i} = 0\}$ and let $k = np$. Then by the pigeonhole principal, some a_i must show up at least p times in the product, so $\prod_{j=1}^{k} a_{i_j} R = 0R = (0)$. $\square$

Problem 4 (Artin).

(a) Determine the prime ideals of the polynomial ring $\mathbb{C}[x, y]$ in two variables.

(b) Show that unique factorization of ideals does not hold in the ring $\mathbb{C}[x, y]$.

Proof. □

Problem 15 (Artin). Determine the singular points of $x^3 + y^3 - 3xy = 0$.

Proof. Singular points occur when

$$\frac{\partial}{\partial x} = x^3 + y^3 - 3xy = 3x^2 - 3y = 0$$
$$\frac{\partial}{\partial y} = x^3 + y^3 - 3xy = 3y^2 - 3x = 0$$

Then solving for y in the first equation yields $y = x^2$, and substituting it into the second equation gives $3x^4 - 3x = 0 = 3x(x^3 - 1)$ which has roots at $x = 0$ and $x = 1$. Using the relation $y = x^2$ gives roots $(0, 0)$ and $(1, 1)$. □

Problem 16 (Artin).

(a) Consider the map $\psi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$ which sends $f(x,y) \mapsto f(t^2,t^3)$. Prove that its kernel is a principal ideal and that its image is the set of polynomials $p(t)$ such that $p'(0)=0$.

(b) Consider the map $\phi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$ which sends $f(x,y) \mapsto f(t^2-t,t^3-t^2)$. Prove that its kernel is a principal ideal and that its image is the set of polynomials $p(t)$ such that $p(0)=p(1)$. Give an intuitive explanation in terms of the geometry of the variety $\{f=0\}$ in $\mathbb{C}^2$.

Proof.

(a) The $\ker(\psi)$ consists of polynomials of the form

$$\psi\left(\sum_{n=0}^{N}\sum_{i=0}^{n} c_{n,i}x^i y^{n-i}\right) = \sum_{n=0}^{N}\sum_{i=0}^{n} c_{n,i}(t^2)^i(t^3)^{n-i} = 0,$$

and thus $\ker(\psi)$ is principal.
Now polynomials with $p'(0)=0$ are exactly of the form

$$p(t) = c^n t^n + \ldots + c_3t^3 + c_2t^2 + c_0,$$

that is, those with vanishing linear term. Notice that all non-negative integers besides 1 can be written as $2k+3\ell$ for some k and ℓ, so all polynomials with $p'(0)=0$ are in $\operatorname{Im}(\psi)$. Similarly, $1 \neq 2k+3\ell$ for any $k,\ell \in \mathbb{N}_{\geq 0}$, so $p(t)$ has no linear terms and $p'(0)=0$.

(b) The $\ker(\phi)$ consists of polynomials of the form

$$\begin{aligned}\phi\left(\sum_{n=0}^{N}\sum_{i=0}^{n} c_{n,i}x^i y^{n-i}\right) &= \sum_{n=0}^{N}\sum_{i=0}^{n} c_{n,i}(t^2-t)^i(t^3-t^2)^{n-i} = 0\\ &= \sum_{n=0}^{N}\sum_{i=0}^{n} c_{n,i}(t-1)^n t^{2n-i}.\end{aligned}$$

□