Complex Analysis: Homework 1

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Problem 2. (page 2)
If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}.$$

Proof. (a)

$$\begin{aligned}
z^4 &= ((x+iy)^2)^2 \\
&= (x^2 - y^2 - 2xyi)^2 \\
&= (x^2-y^2)^2 - (2xy)^2 - 2(x^2-y^2)(2xy)i \\
&= x^4 - 6x^2y^2 + y^4 + 4xy(x^2-y^2)i
\end{aligned}$$

Therefore the real and imaginary parts are

$$\mathrm{Re}(z^4) = x^4 - 6x^2y^2 + y^4, \text{ and } \mathrm{Im}(z^4) = 4xy(x^2-y^2).$$

(b)

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{x+iy} \\
&= \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \\
&= \frac{x-iy}{x^2+y^2} \\
&= \frac{\bar{z}}{|z|^2}
\end{aligned}$$

Therefore the real and imaginary parts are

$$\mathrm{Re}\left(\frac{1}{z}\right) = x/(x^2+y^2), \text{ and } \mathrm{Im}\left(\frac{1}{z}\right) = -y/(x^2+y^2).$$

(c)

$$\begin{aligned}
\frac{z-1}{z+1} &= \frac{x-1+iy}{x+1+iy} \\
&= \frac{x-1+iy}{x+1+iy} \cdot \frac{x+1-iy}{x+1-iy} \\
&= \frac{x^2+y^2-1+2yi}{(x+1)^2+y^2}
\end{aligned}$$

Therefore the real and imaginary parts are

$$\mathrm{Re}\left(\frac{z-1}{z+1}\right) = \frac{x^2+y^2-1}{(x+1)^2+y^2}, \text{ and } \mathrm{Im}\left(\frac{z-1}{z+1}\right) = \frac{2y}{(x+1)^2+y^2}.$$

(d)

$$\begin{aligned}\frac{1}{z^2} &= \left(\frac{x-iy}{x^2+y^2}\right)^2 \\ &= \frac{x^2-y^2-2xyi}{x^4+2x^2y^2+y^4}\end{aligned}$$

Therefore the real and imaginary parts are

$$\operatorname{Re}\left(\frac{1}{z^2}\right) = \frac{x^2-y^2}{x^4+2x^2y^2+y^4}, \text{ and } \operatorname{Im}\left(\frac{1}{z^2}\right) = \frac{-2xy}{x^4+2x^2y^2+y^4}.$$

$\square$

Problem 1. (page 6)
Show that the system of all matrices of the special form

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

Proof. Define the mapping ϕ by

$$\phi\left(\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\right) = \alpha + \beta i.$$

Then ϕ a homomorphism with respect to matrix/complex addition.

$$\begin{aligned}
\phi\left(\begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} \alpha_0 + \alpha_1 & \beta_0 + \beta_1 \\ -(\beta_0 + \beta_1) & \alpha_0 + \alpha_1 \end{bmatrix}\right) \\
&= \alpha_0 + \alpha_1 + (\beta_0 + \beta_1)i \\
&= \alpha_0 + \beta_0 i + \alpha_1 + \beta_1 i \\
&= \phi\left(\begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}\right).
\end{aligned}$$

And similarly ϕ is a homomorphism with respect to matrix/complex multiplication.

$$\begin{aligned}
\phi\left(\begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} \alpha_0\alpha_1 - \beta_0\beta_1 & \alpha_0\beta_1 + \beta_0\alpha_1 \\ -(\alpha_0\beta_1 + \beta_0\alpha_1) & \alpha_0\alpha_1 - \beta_0\beta_1 \end{bmatrix}\right) \\
&= \alpha_0\alpha_1 - \beta_0\beta_1 + (\alpha_0\beta_1 + \beta_0\alpha_1)i \\
&= (\alpha_0 + \beta_0 i)(\alpha_1 + \beta_1 i) \\
&= \phi\left(\begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}\right) \phi\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}\right).
\end{aligned}$$

Lastly, ϕ is clearly a bijection with

$$\phi^{-1}(\alpha + \beta i) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

$\square$

Problem 3. (page 8)
Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Proof. Because of the identity

$$\left|\frac{a-b}{1-\bar{a}b}\right| = \frac{|a-b|}{|1-\bar{a}b|},$$

it is sufficient to show that $|a-b| = |1-\bar{a}b|$.

Case 1: Assume that $|a| = 1$.

$$|1-\bar{a}b| = \left|(1-\bar{a}b)\cdot\frac{a}{a}\right| = \left|\frac{a-|a|^2 b}{a}\right| = \left|\frac{a-b}{a}\right| = \frac{|a-b|}{|a|} = |a-b|.$$

Case 2: Assume that $|b| = 1$ (and so $|\bar{b}| = 1$.)

$$|1-\bar{a}b| = \left|(1-\bar{a}b)\cdot\frac{\bar{b}}{\bar{b}}\right| = \left|\frac{\bar{b}-\bar{a}|b|^2}{\bar{b}}\right| = \left|\frac{\bar{a}-\bar{b}}{\bar{b}}\right| = \frac{|\bar{a}-\bar{b}|}{|\bar{b}|} = |\bar{a}-\bar{b}| = |\overline{a-b}| = |a-b|.$$

Notice that if $\bar{a}b = 1$ (and thus $a = b$), then the quotient is not well-defined. $\square$

Problem 4. (page 8)
Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

Proof. Assuming that $a, b, c \in \mathbb{R}$. Denote z by $\alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$.
Then

$$\begin{aligned} az + b\bar{z} + c &= a(\alpha + \beta i) + b(\alpha - \beta i) + c \\ &= \alpha(a + b) + \beta i(a - b) + c \\ &= 0 \end{aligned}$$

So considering the real and imaginary parts separately

$$\begin{aligned} \operatorname{Im}(\alpha(a + b) + \beta i(a - b) + c) &= \beta(a - b) &= 0 \\ \operatorname{Re}(\alpha(a + b) + \beta i(a - b) + c) &= \alpha(a + b) + c &= 0. \end{aligned}$$

In order for the imaginary part to vanish, either $a = b$ or $\beta = 0$. However if $a = b$ then β can take on any value, so the equation has infinitely many solutions. Thus $\beta = 0$, $\alpha = -c/(a+b)$, and $z = \bar{z} = -c/(a+b)$. $\square$

Problem 1. (page 11)
Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Proof. It is sufficient to show that

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{(a-b)\overline{(a-b)}}{(1-\bar{a}b)(1-a\bar{b})} = \frac{|a|^2+|b|^2-a\bar{b}-\bar{a}b}{1+|a\bar{b}|^2-a\bar{b}-\bar{a}b} = \frac{|a|^2+|b|^2-2\operatorname{Re}(a\bar{b})}{1+|a\bar{b}|^2-2\operatorname{Re}(a\bar{b})} < 1.$$

The denominator is nonzero because $|\bar{a}b| < 1$, so $(1-\bar{a}b)^2 \neq 0$.

Since $|a| < 1$ and $|b| < 1$, it follows that $1-|a|^2 > 0$ and $1-|b|^2 > 0$. Thus

$$0 < (1-|a|^2)(1-|b|^2) = 1-|a|^2-|b|^2+|ab|^2, \text{ so}$$
$$|a|^2+|b|^2 < 1+|ab|^2.$$

Therefore

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 < 1$$

and the result follows. ∎

Problem 1. (page 17)
When does $az + b\bar{z} + c = 0$ represent a line?

Proof. As in problem 4 on page 8, let $z = \alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$, and as shown previously:

$$\begin{aligned}
\mathrm{Im}(\alpha(a+b) + \beta i(a-b) + c) &= \beta(a-b) &&= 0 \\
\mathrm{Re}(\alpha(a+b) + \beta i(a-b) + c) &= \alpha(a+b) + c &&= 0.
\end{aligned}$$

In order to satisfy the imaginary part either $\beta = 0$ or $a = b$. If $\beta = 0$ then α is determined by the equation: $\alpha = -c/(a+b)$. If $a = b$ then $\alpha = -c/(2a)$ and β can take on any value, and so the equation describes a line. $\square$

Problem 1. (page 20)
Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$.

Proof. ($\Longrightarrow$) Assume that z and z' correspond to diametrically opposite points on the Riemann sphere. Then

$$z = \frac{x_1 + ix_2}{1 - x_3}, z' = \frac{-x_1 - ix_2}{1 + x_3}, \text{ and } z\bar{z}' = \frac{x_1 + ix_2}{1 - x_3} \cdot \frac{-x_1 + ix_2}{1 + x_3} = \frac{x_1^2 + x_2^2}{x_3^2 - 1}.$$

because (x_1, x_2, x_3) is on the unit sphere

$$x_1^2 + x_2^2 = -1(x_3^2 - 1) \Longrightarrow \frac{x_1^2 + x_2^2}{x_3^2 - 1} = -1.$$

($\Longleftarrow$) Assume that $z\bar{z}' = -1$. Then denote

$$z = \frac{x_1 + ix_2}{1 - x_3} \text{ and } \bar{z}' = \frac{\alpha_1 - i\alpha_2}{1 - \alpha_3}.$$

Then using the identity $z = 1/\bar{z}'$

$$|z|^2 = \frac{1 + x_3}{1 - x_3} = \frac{1}{|\bar{z}'|} = \frac{1 - \alpha_3}{1 + \alpha_3}.$$

and solving for x_3

$$\begin{aligned}
(1 + x_3)(1 + \alpha_3) &= (1 - x_3)(1 - \alpha_3) \\
1 + x_3 + \alpha_3 + x_3\alpha_3 &= 1 - x_3 - \alpha_3 + x_3\alpha_3 \\
2(x_3 + \alpha_3) &= 0 \\
x_3 &= -\alpha_3.
\end{aligned}$$

Then

$$\begin{aligned}
z\bar{z}' &= -1 \\
(x_1 + ix_2)(\alpha_1 - i\alpha_2) &= -(1 - x_3)(1 - \alpha_3) \\
\operatorname{Im}((x_1 + ix_2)(\alpha_1 - i\alpha_2)) &= -\operatorname{Im}((1 - x_3)(1 - \alpha_3)) \\
\alpha_1 x_2 - x_1\alpha_2 &= 0 \\
\frac{x_1}{\alpha_1} &= \frac{x_2}{\alpha_2}.
\end{aligned}$$

So combining the above fact that $x_3 = -\alpha_3$

$$\begin{aligned}
x_1^2 + x_2^2 + x_3^2 &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \\
x_1^2 + x_2^2 &= \alpha_1^2 + \alpha_2^2 \\
\frac{1}{x_1^2}(x_1^2 + x_2^2) &= 1 + \left(\frac{x_2}{x_1}\right)^2 = 1 + \left(\frac{\alpha_2}{\alpha_1}\right)^2 = \frac{1}{\alpha_1^2}(\alpha_1^2 + \alpha_2^2) \\
\frac{1}{x_1^2} &= \frac{1}{\alpha_1^2} \\
x_1 &= \alpha_1 \text{ or } -\alpha_1.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{1}{x_2^2}(x_1^2 + x_2^2) &= \left(\frac{x_1}{x_2}\right)^2 + 1 = \left(\frac{\alpha_1}{\alpha_2}\right)^2 + 1 = \frac{1}{\alpha_2^2}(\alpha_1^2 + \alpha_2^2) \\
\frac{1}{x_2^2} &= \frac{1}{\alpha_2^2} \\
x_2 &= \alpha_2 \text{ or } -\alpha_2.
\end{aligned}$$

By the identity $\alpha_1 x_2 = x_1 \alpha_2$, either $\alpha_1 = -x_1$ and $\alpha_2 = -x_2$, or $\alpha_1 = x_1$ and $\alpha_2 = x_2$. Since $x_3 \leq 1$,

$$x_1\alpha_1 + x_2\alpha_2 = x_3^2 - 1 \leq 0$$

and so $\alpha_1 = -x_1$, $\alpha_2 = -x_2$. Therefore z and z' correspond to antipodal points on the Riemann sphere. $\square$

Problem 3. (page 28)
Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

Proof. I'm assuming that $z = x + yi$ with $x, y \in \mathbb{R}$, so the function is real valued. Thus

$$f(x + yi) = ax^3 + bx^2y + cxy^2 + dy^3 = u(x + yi) + iv(x + yi)$$

where $u = f$ and $v = 0$. Taking the derivative:

$$\begin{aligned} u_{xx} &= 6ax + 2by, \\ u_{yy} &= 2cx + 6dy, \\ v_{xx} &= 0, \\ v_{yy} &= 0. \end{aligned}$$

Because f is harmonic for all x, y,

$$\begin{aligned} 6ax + 2by + 2cx + 6dy &= 0 \\ (6a + 2c)x + (2b + 6d)y &= 0. \end{aligned}$$

Thus $c = -3a$ and $b = -3d$. So the most general form of f is

$$f(x + yi) = u(x + yi) = ax^3 - 3dx^2y - 3axy^2 + dy^3.$$

To determine the conjugate harmonic function via integration

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3ax^2 - 6dxy - 3ay^2 &&= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -3dx^2 - 6axy + 3dy^2 &&= -\frac{\partial v}{\partial x} \end{aligned}$$

so

$$\begin{aligned} v(x + yi) &= \int 3ax^2 - 6dxy - 3ay^2 \; \partial y \\ &= 3ax^2y - 3dxy^2 - ay^3 + \psi(x) \end{aligned}$$

and similarly

$$\begin{aligned} v(x + yi) &= \int 3dx^2 + 6axy - 3dy^2 \; \partial x \\ &= dx^3 + 3ax^2y - 3dxy^2 + \xi(y). \end{aligned}$$

Thus $v(x + yi) = dx^3 + 3ax^2y - 3dxy^2 + ay^3$, and the corresponding analytic function g is

$$g(x + iy) = (ax^3 - 3dx^2y - 3axy^2 + dy^3) + (dx^3 + 3ax^2y - 3dxy^2 + ay^3)i.$$

Using the formal method,

$$\begin{aligned} g(z) &= 2u(z/2, z/2i) - u(0, 0) \\ &= 2\left(a\frac{z^3}{8} - 3d\frac{z^2}{4} \cdot \frac{z}{2i} - 3a\frac{z}{2} \cdot \frac{z^2}{-4} + d\frac{z^3}{-8i}\right) \\ &= \frac{a}{4}z^3 - \frac{3d}{4i}z^3 + \frac{3a}{4}z^3 - \frac{d}{4i}z^3 \\ &= z^3\left(\frac{ai - 3d + 3ai - d}{4i}\right) = z^3\left(\frac{ai - d}{i}\right) = z^3(a + di) \end{aligned}$$

□

Problem 4. (page 28)
Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Proof. If $|f(z)| = 0$ then $f(z) = 0$, which is analytic, so suppose that $|f(z)| = c > 0$. Then

$$\frac{1}{f(z)} = \frac{\overline{f(z)}}{|z|^2} = \frac{\overline{f(z)}}{c^2}.$$

Clearly 1 is analytic, and $f(z) \neq 0$ so $\frac{1}{f(z)}$ is analytic too.

However, since $\overline{f(z)}$ is analytic, it has derivative 0 with respect to z. therefore $\overline{f(z)}$ (and thus $f(z)$) is a constant. $\square$

Problem 5. (page 28)
Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

Proof. It is sufficient to show that $f(z)$ being analytic implies $\overline{f(\bar{z})}$ is analytic because $\overline{\overline{f(\bar{\bar{z}})}} = f(z)$.
From section 1.2, if $f(z) = u(z) + iv(z)$ is analytic, then $u(x, y)$ and $v(x, y)$ have partial derivatives which satisfy the Cauchy-Riemann equations, and the converse is also true. Thus it is sufficient to show that $u(x, -y)$ and $-v(x, -y)$ have partial derivatives which satisfy the Cauchy-Riemann equations.

$$\begin{aligned}
\frac{\partial}{\partial x} u(x, -y) &= u_x \\
\frac{\partial}{\partial y} u(x, -y) &= -u_y \\
\frac{\partial}{\partial x} [-v(x, -y)] &= -v_x \\
\frac{\partial}{\partial y} [-v(x, -y)] &= -(-v_y) = v_y
\end{aligned}$$

Because f is analytic

$$\begin{aligned}
\frac{\partial}{\partial x} u(x, -y) = u_x = v_y &= \frac{\partial}{\partial y} [-v(x, -y)] \\
-\frac{\partial}{\partial y} u(x, -y) = u_y = -v_x &= \frac{\partial}{\partial x} [-v(x, -y)]
\end{aligned}$$

So $f(z)$ is analytic if and only if $\overline{f(\bar{z})}$ is analytic.

□