Combinatorics: Homework 7

Peter Kagey

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Problem 21 (a). [2+]
Given numbers a_i for $i \in \mathbb{Z}$, with $a_i = 0$ for $i < 0$ and $a_0 = 1$, let $f(k) = \det[a_{j-i+1}]_1^k$. In particular, $f(0) = 1$. Show that

$$\sum_{k \geq 0} f(k)x^k = \frac{1}{1 - a_1 x + a_2 x^2 - \cdots}.$$

Solution. It is sufficient to show that

$$\begin{aligned}
1 &= (1 - a_1 x + a_2 x^2 - \cdots) \sum_{k \geq 0} f(k) x^k \\
&= \left(\sum_{n \geq 0} (-1)^n a_n x^n \right) \left(\sum_{k \geq 0} f(k) x^k \right) \\
&= \sum_{n,k \geq 0} (-1)^n a_n f(k) x^{n+k}.
\end{aligned}$$

This means is enough to show that the coefficient of x^m is zero for all $m > 0$:

$$\begin{aligned}
0 &= \sum_{i=0}^{m} (-1)^i a_i f(m-i) \\
f(m) &= \sum_{i=1}^{m} (-1)^{i-1} a_i f(m-i) \\
f(m) &= a_1 f(m-1) - a_2 f(m-2) + a_3 f(m-3) + \ldots + (-1)^{m-1} a_m \underbrace{f(0)}_{=1}.
\end{aligned}$$

This can be shown by Laplace expansion along the first row. Notice that $f(k) = \sum_{i=1}^{k} (-1)^{i-1} a_i \det(M_i)$ where M_i is the $k-1 \times k-1$ submatrix illustrated below.

$$\begin{vmatrix}
a_1 & a_2 & \ldots & a_{i-1} & a_i & a_{i+1} & \ldots & a_{k-1} & a_k \\
1 & a_1 & \ldots & a_{i-2} & a_{i-1} & a_i & \ldots & a_{k-2} & a_{k-1} \\
0 & 1 & \ldots & a_{i-3} & a_{i-2} & a_{i-1} & \ldots & a_{k-3} & a_{k-2} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & a_1 & a_2 & \ldots & a_{k-i} & a_{k-i+1} \\
0 & 0 & \ldots & 0 & 1 & a_1 & \ldots & a_{k-i-1} & a_{k-i} \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & a_{k-i-2} & a_{k-i-1} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & a_1
\end{vmatrix}$$

In particular, the structure of this matrix is

- the $f(k-i)$ matrix in the bottom $k-i$ rows and right $k-i$ columns,
- an upper triangular matrix in the top $i-1$ rows and left $i-1$ columns, and
- the zero matrix in the bottom $k-i$ rows and left $i-1$ columns.

Thus, the determinant of M_i can be computed by Laplace expansion, along the $i-1$ upper-left 1s, until the $f(k-i)$ matrix is reached. So

$$f(m) = \sum_{i=1}^{m} (-1)^{i-1} a_i f(m-i)$$

for all $m > 0$. Since $f(0) = 1$ when $m = 0$ (by hypothesis), the identity follows.

Problem 26. [2+]
Let $\pi \in \Pi_n$, the set of partitions of $[n]$. Let $S(\pi, r)$ denote the number of $\sigma \in \Pi_n$ such that $|\sigma| = r$ and $\#(A \cap B) \leq 1$ for all $A \in \pi$ and $B \in \sigma$. Show that

$$S(\pi, r) = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \prod_{A \in \pi} (i)_{\#A}.$$

Solution.
This can be counted with straightforward Inclusion-Exclusion. We will count the number of ordered partitions, and then divide by $r!$ to get unordered partitions.

We will first count the number of ways to partition $[n]$ into r-tuples that meet the above criteria *and where some of the parts can be empty.*
For sake of convenience, order π in the canonical way. Then the first element of the first set of π can go in any of the r "slots", because $\#(A \cap B) \leq 1$, the second element of the first set of π can go in $r - 1$ slots, the kth in $r - k + 1$. Thus the number of choices for the first set $A \in \pi$ is $(i)_{\#A}$.

We can do a similar argument for the remaining sets in π, resulting in

$$\prod_{A \in \pi} (r)_{\#A}$$

possible tuples. However, as stated earlier, this includes r-tuples where some of the parts are empty. So we must subtract these off

$$\prod_{A \in \pi} (r)_{\#A} - \binom{r}{1} \prod_{A \in \pi} (r-1)_{\#A},$$

where $\binom{r}{1}$ is the number of ways to choose the empty position.
However, this subtracts off too many r-tuples. In particular, it double-counts those where two or more parts are empty, so we must add these back in, and so on by the Principle of Inclusion-Exclusion

$$\prod_{A \in \pi} (r)_{\#A} - \binom{r}{1} \prod_{A \in \pi} (r-1)_{\#A} + \binom{r}{2} \prod_{A \in \pi} (r-2)_{\#A} + \ldots + \binom{r}{r} \prod_{A \in \pi} (0)_{\#A} = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \prod_{A \in \pi} (i)_{\#A}.$$

Lastly, since the parts of the r-tuple are distinct, we can divide by the number of permutations, so

$$S(\pi, r) = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \prod_{A \in \pi} (i)_{\#A},$$

as desired.

Problem 3. Let E_{2n} be the number of "alternating" permuations, $w \in S_{2n}$ such that

$$w_1 > w_2 < w_3 > w_4 < \ldots > w_{2n}$$

Show that
$$\sum_n E_{2n} \frac{x^{2n}}{(2n)!} = \frac{1}{1 - x^2/2! + x^4/4! - x^6/6! + \ldots} = \frac{1}{\cos(x)}.$$

Proof.
Using the technique from class, we'll instead count the case where each $<$ is replaced with no relation. Then since the $<$ are at all even positions, we can write

$$\underbrace{w_1 > w_2}_{s_1=2}, \underbrace{w_3 > w_4}_{s_2-s_1=2}, \ldots \underbrace{w_{2n-1} > w_{2n}}_{s_n-s_{n-1}=2}.$$

In particular, $s_i - s_j = 2(j-i)$. So using the determinant technique:

$$E_{2n} = (2n)! \begin{vmatrix} \frac{1}{s_1!} & \frac{1}{s_2!} & \frac{1}{s_3!} & \ldots & \frac{1}{s_{n-1}!} & \frac{1}{s_n!} \\ 1 & \frac{1}{(s_2-s_1)!} & \frac{1}{(s_3-s_1)!} & \ldots & \frac{1}{(s_{n-1}-s_1)!} & \frac{1}{(s_n-s_1)!} \\ 0 & 1 & \frac{1}{(s_3-s_2)!} & \ldots & \frac{1}{(s_{n-1}-s_2)!} & \frac{1}{(s_n-s_2)!} \\ 0 & 0 & 1 & \ldots & \frac{1}{(s_{n-1}-s_3)!} & \frac{1}{(s_n-s_3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \frac{1}{(s_n-s_{n-1})!} \end{vmatrix} = (2n)! \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \ldots & \frac{1}{(2n-2)!} & \frac{1}{(2n)!} \\ 1 & \frac{1}{2!} & \frac{1}{4!} & \ldots & \frac{1}{(2n-4)!} & \frac{1}{(2n-2)!} \\ 0 & 1 & \frac{1}{2!} & \ldots & \frac{1}{(2n-6)!} & \frac{1}{(2n-4)!} \\ 0 & 0 & 1 & \ldots & \frac{1}{(2n-8)!} & \frac{1}{(2n-6)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \frac{1}{2!} \end{vmatrix}$$

This satisfies the property in problem 1, namely $\frac{E_{2n}}{(2n)!} = \det[a_{j-i+1}]_1^n$ with

$$a_i = \begin{cases} 0 & i < 0 \\ 1 & i = 0 \\ \frac{1}{(2i)!} & i > 0 \end{cases}.$$

Therefore substituting x^2 for x in problem 1 yields

$$\sum_n \frac{E_{2n}}{(2n)!} x^{2n} = \frac{1}{1 - (1/2!)x^2 + (1/4!)x^4 - (1/6!)x^6 + \ldots}$$

as desired. $\square$