Complex Analysis: Homework 14

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Problem 2. (page 227)
Show that the functions z^n, n a nonnegative integer, form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point on the unit circle.

Proof.
By Arzela's theorem, it is enough to show that for $\Omega_< = \{z : |z| < 1\}$ and $\Omega_> = \{z : |z| > 1\}$, (i) $\mathfrak{F}$ is equicontinuous on every compact set $E \subset \Omega$, and (ii) for any $z \in \Omega$ and $f \in \mathfrak{F}$, $f(z)$ lies in some compact subset of $\mathbb{C}$; and that at least one of these hypothesis fails when Ω contains a point z such that $|z| = 1$.

Equicontinuity of $\mathfrak{F}$ on $\Omega_<$.
Suppose that E is a compact subset of $\Omega_<$. There exists some closed ball $\overline{B_r}(0)$ of radius $r < 1$ centered at zero around E. Then given some $\varepsilon > 0$, we can construct a δ such that $|z^n - z_0^n| < \epsilon$ whenever $|z - z_0| < \delta$. Notice that

$$\begin{aligned}
|z^n - z_0^n| &= \left|(z - z_0)\sum_{k=0}^{n-1} z^k z_0^{n-1-k}\right| \\
&\leq |z - z_0| \left|\sum_{k=0}^{n-1} r^{n-1}\right| \\
&= |z - z_0| \cdot \underbrace{nr^{n-1}}_{\to 0} \\
&\leq |z - z_0| \cdot \max_n \left(nr^{n-1}\right)
\end{aligned}$$

where $\max_n \left(nr^{n-1}\right) < \infty$. Thus taking

$$\delta < \frac{\varepsilon}{\max\limits_n \left(nr^{n-1}\right)}$$

is sufficient.

Boundedness of $\mathfrak{F}$ on $\Omega_<$.
It is easy enough to see that $f(z) \in \overline{B_r}(0)$ for all $f \in \mathfrak{F}$:

$$|f(z)| = |z^n| = |z|^n \leq r^n \leq r.$$

Theorem 17 on $\Omega_>$.
Theorem 17 says that $\mathfrak{F}$ is normal in the classical sense if and only if

$$\rho(f_n) = \frac{2n|z|^{n-1}}{1 + |z|^{2n}}$$

is locally bounded. Suppose that $z \in E$, a compact subset of $\Omega_>$, which is to say that there exists some r

such that $|z| > r$ for all $z \in E$. Then we have the bound

$$\begin{aligned}
\rho(f_n) &= \frac{2n|z|^{n-1}}{1+|z|^{2n}} \\
&< \frac{2n|z|^{n-1}}{|z|^{2n}} \\
&= \frac{2n}{|z|^{n+1}} \\
&\leq \frac{2n}{r^{n+1}} \\
&\leq \max_n \frac{2n}{r^{n+1}} < \infty
\end{aligned}$$

which is bounded since $r > 1$.

Theorem 17 on $|z| = 1$.
This follows from the above argument. When $|z| = 1$, we have

$$\rho(f_n) = \frac{2n}{1+1} = n,$$

which is unbounded. Thus by the "only if" of Theorem 17, $\mathfrak{F}$ is not normal on on any compact set that intersects the boundary of the unit disk. $\square$

Problem 3. (page 227)
If $f(z)$ is analytic in the whole plane, show that the family $\mathfrak{F}$ formed by all functions $f(kz)$ with constant $k \in \mathbb{R}$ is normal in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.

Proof.
Because f is entire, we can write

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \text{ and } f(kz) = \sum_{j=0}^{\infty} a_j k^j z^j.$$

($\Longrightarrow$)
By contrapositive, assume that f is *not* a polynomial, which is to say that $f(z) = a_0 + a_1 z + \ldots$ with infinitely many nonzero coefficients. By Theorem 17 it is enough to check that

$$\rho(f_k) = \frac{2|f_k'(z)|}{1 + |f(kz)|^2} = \frac{2\left|\sum_{j=1}^{\infty} j a_j k^j z^{j-1}\right|}{1 + \left|\sum_{j=0}^{\infty} a_j (kz)^j\right|^2}$$

is *not* locally bounded, which is to say, it can be made arbitrarily large on some compact set.

($\Longleftarrow$)
Assume that f is a polynomial, $f(z) = a_0 + a_1 z + \ldots + a_n z^n$. By Theorem 17 it is enough to check that

$$\rho(f_k) = \frac{2|f_k'(z)|}{1 + |f(kz)|^2}$$

is locally bounded—which means that it is sufficient to check that $\rho(f)$ is *totally* bounded. In particular, look at the function $g(z) = f(1/z)$

$$\rho(g) = \frac{2\left|\sum_{j=1}^{n} \frac{j a_j}{z^{j-1}}\right|}{1 + \left|\sum_{j=0}^{n} \frac{a_j}{z^j}\right|^2}$$

Then multiplying the numerator and denominator by $|z|^{2n}$ yields

$$\rho(g) = \frac{2\left|\sum_{j=1}^{n} j a_j z^{n-j+1}\right|}{|z|^{2n} + \left|\sum_{j=0}^{n} a_j z^{n-j}\right|^2}$$

which is continuous in a delta ball around 0. So $|g(z)| \leq M$ for $z < \delta$, and $|f(z)| \leq M$ for $z > 1/\delta$. Since f is bounded for bounded z (in particular, $|z| \leq 1/\delta$), f is totally bounded, and so if f is a polynomial, then $\mathfrak{F} = \{f(kz)\}_{k \in \mathbb{C}}$ is a normal family.

$\square$

Problem 1. (page 232)
If z_0 is real and Ω is symmetric with respect to the real axis, prove that f satisfies with the symmetry relation $f(\bar{z}) = \overline{f(z)}$ using the uniqueness condition in Theorem 1.

Proof.
First notice that the map $g(z) = \overline{f(\bar{z})}$ is holomorphic: write $f(z) = u(x,y) + iv(x,y)$ so that $g(z) = u(x,-y) - iv(x,-y)$. Since g is continuous (being the sum/composition of continuous functions), it only remains to check that the Cauchy-Riemann Equations are satisfied:

$$\frac{\partial}{\partial x}\left[u(x,-y)\right] = \frac{\partial u}{\partial x}(x,-y) \tag{1}$$
$$\frac{\partial}{\partial y}\left[u(x,-y)\right] = -\frac{\partial u}{\partial y}(x,-y) \tag{2}$$
$$\frac{\partial}{\partial x}\left[-v(x,-y)\right] = -\frac{\partial v}{\partial x}(x,-y) \tag{3}$$
$$\frac{\partial}{\partial y}\left[-v(x,-y)\right] = \frac{\partial v}{\partial y}(x,-y), \tag{4}$$

where the equality of (1) and (4) along with (2) and (3) follows by the Cauchy-Riemann Equations on f. Thus g is holomorphic.

Now it just needs to be shown that g is conformal. Notice that the above equations together with the knowledge that f is conformal show that the derivative of g never vanishes. Since g is the composition of a bijection on Ω, followed by f, followed by a bijection on $\mathbb{D}$, g is also a one-to-one surjection onto the disk.

Next, notice that g maps z_0 to zero: $g(z_0) = \overline{f(\bar{z_0})} = \overline{f(z_0)} = \bar{0} = 0$. Also the derivative at $g'(z_0)$ is positive because z_0 has imaginary part of zero:

$$g'(z_0) = g'(x_0 + 0i) = \frac{\partial u}{\partial x}(x_0, 0) = f'(z_0) > 0$$

Therefore Theorem 17 guarantees that g is identically f, and the symmetry relation follows.

$$f(z) = \overline{f(\bar{z})}$$
$$\overline{f(z)} = f(\bar{z}).$$

$\square$

Problem 2. (page 232)
What is the corresponding conclusion if Ω is symmetric with respect to the point z_0?

Proof.
Suppose Ω is symmetric with respect to the point z_0, that is (i) $f(z_0) = 0$ and (ii) if $z_1 \in \Omega$, then $z_0 - (z_1 - z_0) = 2z_0 - z_1 \in \Omega$. Denote this point by $\widetilde{z_1}$.

Define $g\colon \Omega \to \mathbb{D}$ by $g(z) = -f(\tilde{z})$. Notice now that

1. $g(z_0) = -f(\widetilde{z_0}) = -f(z_0 - (z_0 - z_0)) = -f(z_0) = -0 = 0$
2. The map g is conformal because it is the composition of conformal maps.
3. The derivative g' is positive at z_0

$$g'(z_0) = \frac{d}{dz}\left[-f(\hat{z})\right]_{z=z_0} = -f'(\hat{z_0})\frac{d}{dz}\left[2z_0 - z\right]_{z=z_0} = f'(\hat{z_0}) = f'(z_0) > 0.$$

Because f was the unique function with these properties, $g(z) = f(z)$, so

$$f(\widetilde{z_1}) = -f(z_1).$$

□