Permutation statistics

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This note explores the relationship between permutations $\pi \in S_n$ with some given, fixed permutation statistic, and the expected value of the first letter of π. Unless otherwise indicated, π will denote a permutation, and n will be left implicit.

Note 0.0.1. *A note on notation. Let* stat *be some permutation statistic. Then* $E_{n,m}^{stat}$ *denotes the expected value of the first letter over all permutations* $\pi \in S_n$ *such that* $\text{stat}(\pi) = m$.

1 Descents

The motivating example is that the number of descents has a nice, easy, explicit relationship with the expected value of the first letter of a permutation.

1.1 1-descents

Definition 1.1.1. *The number of* (1-)*descents of a permutation* $\pi = \pi_1\pi_2\cdots\pi_n$ *gives a permutation statistic, where*

$$\text{des}_1(\pi) = \{\pi_i < \pi_{i+1} : i \in [n-1]\}$$

Note 1.1.2. *For all* n *expected value of a permutation* $\pi \in S_n$ *with* k *descents is* $k+1$.

1.2 k-descents

Definition 1.2.1. *The number of* k*-descents of a permutation* $\pi = \pi_1\pi_2\cdots\pi_n$ *gives a permutation statistic, where*

$$\text{des}_\text{k}(\pi) = \{\pi_i < \pi_{i+k} : i \in [n-k]\}$$

Table 1.2.2. $E_{n,m}^{des_2}$ *gives the expected value of the first letter of* $\pi \in S_n$, *given that* $\text{des}_2(\pi) = m$.

		m						
		0	1	2	3	4	5	6
	2	3/2						
	3	4/3	8/3					
	4	5/3	5/2	10/3				
	5	3/2	27/10	33/10	9/2			
n	6	7/4	21/8	7/2	35/8	21/4		
	7	8/5	208/75	122/35	158/35	392/75	32/5	
	8	9/5	27/10	18/5	9/2	27/5	63/10	36/5
	9	5/3	105/37	355/99	14275/3117	16895/3117	635/99	265/37
	10	11/6	11/4	11/3	55/12	11/2	77/12	22/3
	11	12/7	240/83	23616/6475	193884/41797	283362/51317	332442/51317	307680/41797

Conjecture 1.2.3. *It appears that when n is even, the expected values are "nice", for example, the $2n$ row appears to be given by*

$$E^{des_2}_{2n,m} = \frac{2n+1}{n+1} + \frac{2n+1}{2n+2}m.$$

Note 1.2.4. *Since "flipping" the word (e.g 13425 $\mapsto$ 53241) changes all k-ascents to descents,*

$$E^{des_2}_{n,m} + E^{des_2}_{n,n-m-2} = n+1.$$

Table 1.2.5. *$E^{des_3}_{n,m}$ gives the expected value of the first letter of $\pi \in S_n$, given that $\mathrm{des}_3(\pi) = m$.*

		m							
		0	1	2	3	4	5	6	7
n	3	2/1							
	4	5/3	10/3						
	5	2/1	3/1	4/1					
	6	7/3	28/9	35/9	14/3				
	7	2/1	10/3	4/1	14/3	6/1			
	8	9/4	13/4	54/13	63/13	23/4	27/4		
	9	5/2	10/3	25/6	5/1	35/6	20/3	15/2	
	10	11/5	66/19	2497/585	7392/1475	8833/1475	3938/585	143/19	44/5
	11	12/5	222/65	1257/290	10722/2095	6/1	14418/2095	2223/290	558/65

Conjecture 1.2.6. $E^{des_3}_{2n+1,n-1}$

Conjecture 1.2.7.

$$E^{des_3}_{3n,m} = \frac{3n+1}{n+1} + \frac{3n+1}{3n+3}m$$

Conjecture 1.2.8. *The obvious extrapolation of these conjectures is that*

$$E^{des_k}_{kn,m} = \frac{kn+1}{n+1} + \frac{kn+1}{kn+k}m$$

1.3 k-continuous descents

Definition 1.3.1. *The number of k-continuous descents of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ gives a permutation statistic, where*

$$\mathrm{cdes_k}(\pi) = \{\pi_i < \pi_{i+1} < \cdots < \pi_{i+k} : i \in [n-k]\}$$

Table 1.3.2. *$E^{cdes_3}_{n,m}$ gives the expected value of the first letter of $\pi \in S_n$, given that $\mathrm{cdes}_3(\pi) = m$.*

		m						
		1	2	3	4	5	6	7
n	3	3/1						
	4	17/6	4/1					
	5	137/41	29/8	5/1				
	6	496/137	371/86	22/5	6/1			
	7	4157/1020	3605/803	891/167	31/6	7/1		
	8	3319/743	40885/8221	11071/2064	506/79	83/14	8/1	
	9	705025/143571	458983/86214	165997/28143	30977/4961	4481/597	107/16	9/1

2 Cycles

The big idea for this section is that the only way to affect the expected value of the first letter of a permutation is to affect the number of fixed-points.
Let $\mathrm{cyc}_k(\pi)$ denote the number of cycles of length k in a permutation π.

2.1 Fixed points (1-cycles)

Table 2.1.1. $E^{cyc_1}_{n,m}$ *gives the expected value of the first letter of* $\pi \in S_n$*, given that* $\mathrm{cyc}_1(\pi) = m$

		m									
		0	1	2	3	4	5	6	7	8	9
n	2	2/1									
	3	5/2	2/1								
	4	3/1	5/2	2/1							
	5	7/2	3/1	5/2	2/1						
	6	4/1	7/2	3/1	5/2	2/1					
	7	9/2	4/1	7/2	3/1	5/2	2/1				
	8	5/1	9/2	4/1	7/2	3/1	5/2	2/1			
	9	11/2	5/1	9/2	4/1	7/2	3/1	5/2	2/1		
	10	6/1	11/2	5/1	9/2	4/1	7/2	3/1	5/2	2/1	
	11	13/2	6/1	11/2	5/1	9/2	4/1	7/2	3/1	5/2	2/1

Theorem 2.1.2. $E^{cyc_1}_{n,m} = (n-m+2)/2$ *for* $m \neq n-1$.

2.2 2-cycles

Table 2.2.1. $E^{cyc_2}_{n,m}$ *gives the expected value of the first letter of* $\pi \in S_n$*, given that* $\mathrm{cyc}_2(\pi) = m$.

		m									
		0	1	2	3	4	5	6	7	8	9
n	2	1/1	2/1								
	3	2/1	2/1								
	4	13/5	2/1	3/1							
	5	3/1	3/1	3/1							
	6	101/29	18/5	3/1	4/1						
	7	4/1	4/1	4/1	4/1						
	8	1049/233	130/29	23/5	4/1	5/1					
	9	5/1	5/1	5/1	5/1	5/1					
	10	12809/2329	1282/233	159/29	28/5	5/1	6/1				

Conjecture 2.2.2.

$$E^{cyc_2}_{n,m} = \begin{cases} \dfrac{n}{2} + \dfrac{A161936(\frac{n}{2}-m)}{A000354(\frac{n}{2}-m)} & n \text{ is even} \\ \dfrac{n}{2} + \dfrac{1}{2} & \text{otherwise} \end{cases}$$

extending the domain of A161936 so that $A161936(0) := 1$.

Theorem 2.2.3. *Let* $C_2(n,m) = A114320(n,m)$ *be the number of permutations in* S_n *with* m *2-cycles, so that* $C_2(n,0) = A000266(n)$*. Then for* $m > 0$,

$$C_2(n,m) = \frac{n(n-1)}{2m} C_2(n-2, m-1).$$

Theorem 2.2.4. *The expected value of a permutation* $\pi \in S_n$ *with* m *2-cycles is given by*

$$E^{cyc_2}_{n,m} = \frac{n}{2} + 1 - \frac{nC_2(n-1,m)}{2C_2(n,m)}.$$

2.3 k-cycles

Table 2.3.1. $E^{cyc_3}_{n,m}$ *gives the expected value of the first letter of* $\pi \in S_n$*, given that* $\mathrm{cyc}_3(\pi) = m$.

		m			
		0	1	2	3
n	2	3/2			
	3	7/4	5/2		
	4	5/2	5/2		
	5	3/1	3/1		
	6	46/13	13/4	4/1	
	7	4/1	4/1	4/1	
	8	9/2	9/2	9/2	
	9	1159/232	131/26	19/4	11/2

Table 2.3.2. $E^{cyc_4}_{n,m}$ *gives the expected value of the first letter of* $\pi \in S_n$*, given that* $\mathrm{cyc}_4(\pi) = m$.

		m		
		0	1	2
n	2	3/2		
	3	2/1		
	4	7/3	3/1	
	5	3/1	3/1	
	6	7/2	7/2	
	7	4/1	4/1	
	8	113/25	13/3	5/1
	9	5/1	5/1	5/1
	10	11/2	11/2	11/2

Conjecture 2.3.3. $E^{cyc_k}_{n,m} = \dfrac{n+1}{2}$ *when* $k \nmid n$.

Table 2.3.4. $E^{cyc_k}_{n,1}$ *gives the expected value of the first letter of* $\pi \in S_n$*, given that* π *has exactly one* k*-cycle*

		k								
		1	2	3	4	5	6	7	8	9
n	3	2/1	2/1	5/2						
	4	5/2	2/1	5/2	3/1					
	5	3/1	3/1	3/1	3/1	7/2				
	6	7/2	18/5	13/4	7/2	7/2	4/1			
	7	4/1	4/1	4/1	4/1	4/1	4/1	9/2		
	8	9/2	130/29	9/2	13/3	9/2	9/2	9/2	5/1	
	9	5/1	5/1	131/26	5/1	5/1	5/1	5/1	5/1	11/2

Conjecture 2.3.5. $E^{cyc_k}_{n,1} = \dfrac{n+1}{2}$ *when* $k = 1$ *or* $k \nmid n$.

2.4 Number of cycles

Table 2.4.1. *$C_{\#}(n, m)$ gives the expected value of the first letter of $\pi \in S_n$, given that π has exactly m cycles.*

		m								
		1	2	3	4	5	6	7	8	9
n	1	1/1								
	2	2/1	1/1							
	3	5/2	2/1	1/1						
	4	3/1	29/11	2/1	1/1					
	5	7/2	16/5	19/7	2/1	1/1				
	6	4/1	512/137	10/3	47/17	2/1	1/1			
	7	9/2	179/42	907/232	24/7	14/5	2/1	1/1		
	8	5/1	1735/363	299/67	3907/967	7/2	65/23	2/1	1/1	
	9	11/2	4028/761	147719/29531	1234/267	4429/1069	32/9	37/13	2/1	1/1

2.5 Longest cycle

Table 2.5.1. *$C_{long}(n, m)$ gives the expected value of the first letter of $\pi \in S_n$, given that π has longest cycle of length m.*

		m								
		1	2	3	4	5	6	7	8	9
n	3	1/1	2/1	5/2						
	4	1/1	7/3	5/2	3/1					
	5	1/1	13/5	3/1	3/1	7/2				
	6	1/1	3/1	17/5	7/2	7/2	4/1			
	7	1/1	37/11	53/14	4/1	4/1	4/1	9/2		
	8	1/1	413/109	195/46	31/7	9/2	9/2	9/2	5/1	
	9	1/1	1219/291	595/127	131/27	5/1	5/1	5/1	5/1	11/2

Conjecture 2.5.2. *$C_{long}(n, m) = \dfrac{n+1}{2}$ for $\frac{n}{2} < m < n$.*

Note 2.5.3. *Instead considering the "expected value of the first letter of $\pi \in S_n$, given that π has longest cycle of length greater (resp. less) than m" does not have any obvious structure.*

Note 2.5.4. *Shortest cycle analogs don't seem to have interesting structure.*

3 Other permutation statistics

3.1 Inversion number

Table 3.1.1. *$I(n, m)$ gives the expected value of the first letter of $\pi \in S_n$, given that* $\mathrm{inv}(\pi) = m$.

		m								
		1	2	3	4	5	6	7	8	9
n	2	2/1								
	3	3/2	5/2	3/1						
	4	4/3	9/5	5/2	16/5	11/3	4/1			
	5	5/4	14/9	29/15	49/20	3/1	71/20	61/15	40/9	19/4
	6	6/5	10/7	49/29	2/1	169/71	253/90	330/101	377/101	377/90
	7	7/6	27/20	76/49	87/49	343/169	86/37	954/359	1374/455	1807/531
	8	8/7	35/27	111/76	95/58	628/343	615/301	2191/961	3598/1415	549/194

Conjecture 3.1.2. *For $n \geq 4$,*

$$I(n,3) = \frac{A005286(n-2)}{A005286(n-3)} = \frac{n^3+3n^2-4n-6}{n^3-7n}.$$

Conjecture 3.1.3. *For sufficiently large n,*

$$I(n,k) = \frac{A008302(n+1,k)}{A008302(n,k)}$$

3.2 Major index

Table 3.2.1. *$M(n,m)$ gives the expected value of the first letter of $\pi \in S_n$, given that* $\mathrm{maj}(\pi) = m$.

		m								
		0	1	2	3	4	5	6	7	8
n	1	1/1								
	2	1/1	2/1							
	3	1/1	5/2	3/2	3/1					
	4	1/1	3/1	9/5	5/2	16/5	2/1	4/1		
	5	1/1	7/2	19/9	8/3	33/10	3/1	27/10	10/3	35/9
	6	1/1	4/1	17/7	84/29	25/7	246/71	16/5	330/101	377/101
	7	1/1	9/2	11/4	22/7	27/7	659/169	946/259	1296/359	1756/455
	8	1/1	5/1	83/27	129/38	120/29	211/49	1236/301	3838/961	5872/1415
	9	1/1	11/2	17/5	135/37	419/95	736/157	933/205	1378/313	8081/1803
	10	1/1	6/1	41/11	121/31	187/40	1797/356	3817/766	21649/4489	39182/8095

3.3 Longest increasing subsequence

Table 3.3.1. *$S_<(n,m)$ gives the expected value of the first letter of $\pi \in S_n$, given that π has exactly m increasing subsequences.*

		m								
		0	1	2	3	4	5	6	7	8
n	1	1/1								
	2	1/1	2/1							
	3	1/1	2/1	3/1						
	4	1/1	2/1	31/10	4/1					
	5	1/1	2/1	204/65	21/5	5/1				
	6	1/1	2/1	1117/354	1033/240	37/7	6/1			
	7	1/1	2/1	5616/1775	9869/2264	153/28	89/14	7/1		
	8	1/1	2/1	27217/8590	82043/18687	58711/10538	11471/1736	89/12	8/1	
	9	1/1	2/1	65356/20617	629781/142804	752945/133584	133199/19661	2111/273	127/15	9/1