Combinatorics: Homework 6

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Problem 14 (a). [2+]
Let $A_k(n)$ denote the number of k-element antichains in the boolean algebra B_n. Show that

(i) $A_1(n) = 2^n$

(ii) $A_2(n) = \dfrac{1}{2}(4^n - 2 \cdot 3^n + 2^n)$

(iii) $A_3(n) = \dfrac{1}{6}(8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n - 6 \cdot 3^n + 2 \cdot 2^n)$

Solution.

(i) Clearly any singleton set $\{A\}$ where $A \in 2^{[n]}$ is a 1-element antichain, so

$$A_1(n) = \#(2^{[n]}) = 2^n.$$

(ii) It is easiest to count ordered pairs (A, B) such that $A \not\subset B$ and $B \not\subset A$, and then divide by two to count the 2-element sets. By the Principle of Inclusion-Exclusion, we can start by counting all ordered pairs without restriction. For each $i \in [n]$, we can have

(1) $i \in A$ and $i \in B$,
(2) $i \in A$ and $i \notin B$,
(3) $i \notin A$ and $i \in B$, or
(4) $i \notin A$ and $i \notin B$.

Thus there are 4^n choices of sets without restriction.

However, we must remove the case where $A \subset B$ (respectively $B \subset A$) which means that for each $i \in [n]$ we can have

(1) $i \in A \cap B$
(2) $i \in A^c \cap B$ (respectively $i \in A \cap B^c$), or
(3) $i \in A^c \cap B^c$.

Thus there are $2 \cdot 3^n$ possibilities. However, this double-counts the case where $A \subset B$ and $B \subset A$, namely $A = B$, so we need to add these 2^n pairs back. Thus

$$A_2(n) = \frac{1}{2}(4^n - 2 \cdot 3^n + 2^n).$$

(iii) Again by the Principle of Inclusion-Exclusion, we will count ordered triples (A_1, A_2, A_3), that meet the criteria and then divide them by the number of permutations. There are nine cases, which correspond to subsets of ordered pairs. (All indices are distict):

$$\#\{(A_1, A_2, A_3) : A_i \in B_n\} = 8^n \tag{1}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i \subset A_j\} = 6 \cdot 6^n \tag{2}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i \subset A_j, A_i \subset A_k\} = 3 \cdot 5^n \tag{3}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i \subset A_j, A_k \subset A_j\} = 3 \cdot 5^n \tag{4}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i = A_j\} = 3 \cdot 4^n \tag{5}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i \subset A_j \subset A_k\} = 6 \cdot 5^n \tag{6}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i = A_j \subset A_k\} = 3 \cdot 3^n \tag{7}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i \subset A_j = A_k\} = 3 \cdot 3^n \tag{8}$$
$$\#\{(A_1, A_2, A_3) : A_i \in B_n, A_i = A_j = A_k\} = 2^n \tag{9}$$

(1) Start with one copy of each of the 8^n triples.

$$1 \cdot 8^n + ...$$

(2) For each of the six pairs (i, j) such that $A_i \subset A_j$, (1) adds it once, so subtract these off.

$$8^n + -1 \cdot 6 \cdot 6^n + ...$$

(3) For each of the three pairs $(i, \{j, k\})$ such that $A_i \subset A_j$ and $A_i \subset A_k$, (1) adds it once and (2) removes it twice, so we need to add it back once.

$$8^n - 6 \cdot 6^n + 1 \cdot 3 \cdot 5^n + ...$$

(4) For each of the three pairs $(\{i, k\}, j)$ such that $A_i \subset A_j$ and $A_k \subset A_j$, (1) adds it once and (2) removes it twice, so we need to add it back once.

$$8^n - 6 \cdot 6^n + 3 \cdot 5^n + 1 \cdot 3 \cdot 5^n + ...$$

(5) For each of the three unordered pairs $\{i, j\}$ such that $A_i = A_j$, (1) adds it once, (2) removes it twice, and the relation is not a subset of (3) or (4), so we need to add it back once

$$8^n - 6 \cdot 6^n + 6 \cdot 5^n + 1 \cdot 3 \cdot 4^n + ...$$

(6) For each of the six ordered triples (i, j, k) such that $A_i \subset A_j \subset A_k$, (1) adds it once, (2) removes it three times, and (3) and (4) both add it back once, so it doesn't affect the sum:

$$8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n + 0 \cdot 6 \cdot 5^n + ...$$

(7) For each of the three pairs $(\{i, j\}, k)$ such that $A_i = A_j \subset A_k$, (1) adds it once, (2) removes it four times, (3) adds it twice, (4) adds it once, and (5) adds it once, so we need to subtract if off once:

$$8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n + -1 \cdot 3 \cdot 3^n + ...$$

(8) For each of the three pairs $(i, \{j, k\})$ such that $A_i \subset A_j = A_k$, (1) adds it once, (2) removes it four times, (3) adds it once, (4) adds it twice, and (5) adds it once, so we need to subtract it off once:

$$8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n - 3 \cdot 3^n + -1 \cdot 3 \cdot 3^n ...$$

(9) Lastly, for the single unordered triple $\{i, j, k\}$ such that $A_i = A_j = A_k$, (1) adds it once, (2) removes it six times, (3) and (4) each add it three times, (5) adds it three times, (6) does not affect the sum, (7) and (8) each subtract it once, resulting in it needing to be added back in twice:

$$8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n - 6 \cdot 3^n + 2 \cdot 2^n$$

This gives us our desired sum.

Problem 14 (b). [2+]
Show that for fixed $k \in \mathbb{P}$ there exist integers $a_{k,2}, a_{k,3}, \ldots, a_{k,2^k}$ such that

$$A_k(n) = \frac{1}{k!} \sum_{i=2}^{2^k} a_{k,i} i^n.$$

Show in particular that

(i) $a_{k,2^k} = 1$,

(ii) $a_{k,i} = 0$ if $3 \cdot 2^{k-2} < i < 2^k$, and

(iii) $a_{k,3\cdot 2^{k-2}} = k(k-1)$.

Solution.
This can be proven by a similar argument to above: counting ordered k-tuples by the Principle of Inclusion-Exclusion, where we add and subtract all possible sets of set inclusions.

$$\{(A_1, A_2, \ldots, A_k) : A_i \subset A_j \text{ for all } (i,j) \in S \subset [k] \times [k]\} \subset B_n^k$$

For some given set of set inclusions, this constrains the element-wise truth table of possible values. That is, for all $i \in [n]$, we can write a list of possible ways to put i in each set, as illustrated in part **a (ii)**. Since each list cooresponds to a subset of $[k]$, there are at most 2^k ways to place i in each subset. Since there are n is, this means that there are at most 2^{kn} such sets. Since we can choose to put i in every set or in no set, there are at least 2^n such sets.

(i) We begin with all possible sets, so there are 2^k possible choices for each of the n positions, thus the "no relation" k-tuple that we start from has $1 \cdot 2^{nk}$ elements.

(ii) Since the above part takes care of the "no relation" tuples, we must have at least one restriction, namely $A_i \subset A_j$. This means that for each $m \in [n]$ we can have

- $m \in A_i$ and $m \in A_j$,
- $m \notin A_i$ and $m \in A_j$, or
- $m \notin A_i$ and $m \notin A_j$.

And once these choices are made, there are no restrictions on choosing whether m is in the other $k-2$ sets. Thus, with the minimal positive number of relations (one) there are at least $3 \cdot 2^{k-2}$ choices, so there is no relation corresponding to ℓ choices where $3 \cdot 2^{k-2} < \ell < 2^k$.

(iii) Similarly, the coefficient is given by the number of (ordered) ways to choose two positions in the k-tuple for the relation $A_i \subset A_j$. Namely,

$$\#\{(i,j) | i \neq j, i \in [k], j \in [k]\} = k(k-1).$$

Since we subtract these off, the coefficient is $-k(k-1)$, as desired.

Problem 25 (a). [2]
Let $f_i(m,n)$ be the number of $m \times n$ matrices 0s and 1s with at least one 1 in every row and column, and with a total of i 1s. Use the Principle of Inclusion-Exclusion to show that

$$\sum_i f_i(m,n)t^i = \sum_{k=0}^{n}(-1)^k \binom{n}{k}((1+t)^{n-k}-1)^m.$$

Proof.
First, I will count a related function: let $g_i(m,n)$ be the number of $m \times n$ matrices 0s and 1s with at least one 1 in every row, and with a total of i 1s.
Then by the Principle of Inclusion-Exclusion, we can start with all matrices that have i ones, and remove those that have no 1s in the kth row, add back those that have no 1s in the k_1th and k_2th row, etc

$$g_i(m,n) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} \underbrace{\binom{n(m-j)}{i}}_{\substack{\text{Ways to place } i \text{ 1s in} \\ n\times m-j \text{ submatrix}}}$$

From here, we can do an analougous process for $f_i(m,n)$, only with columns instead of rows

$$\begin{aligned} f_i(m,n) &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} g_i(m,n-k) \\ &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(\sum_{j=0}^{m}(-1)^j \binom{m}{j}\binom{(n-k)(m-j)}{i}\right) \end{aligned}$$

so the generating function

$$\sum_i f_i(m,n)t^i = \sum_i \left(\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(\sum_{j=0}^{m}(-1)^j \binom{m}{j}\binom{(n-k)(m-j)}{i}\right)\right) t^i.$$

By rearranging the sums, it is enough to show that for some arbitrary $n, k \in \mathbb{P}$,

$$\begin{aligned} \sum_i \sum_{j=0}^{m}(-1)^j \binom{m}{j}\binom{(n-k)(m-j)}{i} t^i &= ((1+t)^{n-k}-1)^m \\ &= \sum_{j=0}^{m}\binom{m}{j}((1+t)^{n-k})^{m-j}(-1)^j \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j}\left(\sum_i \binom{(n-k)(m-j)}{i} t^i\right) \\ &= \sum_i \sum_{j=0}^{m}(-1)^j \binom{m}{j}\binom{(n-k)(m-j)}{i} t^i. \end{aligned}$$

Therefore the desired identitity holds. $\square$

Problem 25 (b). [2]
Show that

$$\sum_{m,n\geq 0}\sum_{i\geq 0} f_i(m,n)t^i \frac{x^m y^n}{m!n!} = e^{-x-y}\sum_{i\geq 0}\sum_{j\geq 0}(1+t)^{ij}\frac{x^i y^j}{i!j!}.$$

Proof.
I'll use Apoorva Shah's technique and substitute the sum from part **a.**:

$$\sum_{m,n\geq 0}\sum_{i\geq 0} f_i(m,n)t^i \frac{x^m y^n}{m!n!} = \sum_{m,n\geq 0}\left(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}((1+t)^{n-k}-1)^m\right)\frac{x^m y^n}{m!n!}.$$

First, expanding the binomial $((1+t)^{n-k}-1)^m$ yields

$$\sum_{m,n\geq 0}\left(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\left(\sum_{0\leq j\leq m}\binom{m}{j}(-1)^j((1+t)^{n-k})^{m-j}\right)\right)\frac{x^m y^n}{m!n!}.$$

Letting $n=p+k$, and $m=q+j$ and summing over p and q yields

$$\sum_{p,q\geq 0}\left(\sum_{0\leq k}(-1)^k\binom{p+k}{k}\left(\sum_{0\leq j}\binom{q+j}{j}(-1)^j((1+t)^{p+k-k})^{q+j-j}\right)\right)\frac{x^{q+j}y^{p+k}}{(q+j)!(p+k)!},$$

which by simplifying, rearranging, and expanding the binomial coefficients gives

$$\sum_{p,q,k,j\geq 0}\left((-1)^k(-1)^j\frac{(p+k)!}{p!k!}\frac{(q+j)!}{q!j!}(1+t)^{pq}\right)\frac{x^{q+j}y^{p+k}}{(q+j)!(p+k)!}.$$

Since the indicies are independent, this can be split as

$$\left(\sum_k(-1)^k\frac{y^k}{k!}\right)\left(\sum_j(-1)^j\frac{x^j}{j!}\right)\left(\sum_{p,q\geq 0}(1+t)^{pq}\frac{x^q y^p}{q!p!}\right).$$

Since the first two factors evaluate to e^{-y} and e^{-x} respectively, by renaming the indices p and q to i and j gives the desired identity. $\square$