Complex Analysis: Homework 6

Peter Kagey

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Problem 2. (page 129f)
Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Proof.
Let f be a function which is analytic in the whole plane and has a nonessential singularity at ∞.

Case 1 (removable singularity) Assume that f has a removable singularity at ∞. Then $\lim_{z\to\infty} f(z) \in \mathbb{C}$, and since f is analytic on the whole plane by Liouville's Theorem, f is constant.

Case 2 (pole) Assume f has a pole at ∞, and let $g(z) = f(1/z)$. In this case $\lim_{z\to\infty} f(z) = \infty$, so $\lim_{z\to 0} g(z) = \infty$. Then $h(z) = z^m \cdot g(z)$ is analytic in the whole plane, so by Taylor's Theorem expanded about 0,

$$z^m \cdot g(z) = h(0) + \frac{h'(0)}{1!}z + \ldots + \frac{h^{(m-1)}(0)}{(m-1)!}z^{m-1} + h_m(z)z^m.$$

Dividing by z^m yields

$$g(z) = \frac{h(0)}{z^m} + \frac{h'(0)}{1!z^{m-1}} + \ldots + \frac{h^{(m-1)}(0)}{(m-1)!z} + h_m(z),$$

and so

$$f(z) = g(1/z) = h(0)z^m + \frac{h'(0)}{1!}z^{m-1} + \ldots + \frac{h^{(m-1)}(0)}{(m-1)!}z + h_m(1/z).$$

So it is sufficient to show that h_m reduces to a constant. But this follows because

$$h_m(z) = g(z) - \left(\frac{h(0)}{z^m} + \frac{h'(0)}{1!z^{m-1}} + \ldots + \frac{h^{(m-1)}(0)}{(m-1)!z}\right)$$

is bounded and analytic on the whole plane, so must be a constant by Liouville's Theorem. $\square$

Problem 4. (page 129f)
Show that any function which is meromorphic in the extended plane is rational.

Proof.
Let f be a function which is meromorphic in the extended plane with poles $\{p_1, p_2, \ldots, p_n\} \subset \mathbb{C}$. Then if k_i is the order of p_i, let

$$g(z) = f(z) \cdot \prod_{i=1}^{n} (z - p_i)^{k_i}$$

which is analytic on $\mathbb{C}$. From the above question (Problem 2), we know $g(z)$ is rational, because any singularities at ∞ are nonessential. Therefore

$$f(z) = \frac{g(z)}{\prod_{i=1}^{n} (z - p_i)^{k_i}}$$

is a rational function. $\square$

Problem 5. (page 129f)
Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below.
(Hint: Apply a fractional linear transformation.)

Proof.
In any of these four cases, the half-plane can be mapped into the unit disk. For example, if $\operatorname{Re}(f(z)) > M$, then the transformation $z \mapsto z - M$ followed by $z \mapsto (z-1)/(z+1)$ yields a bounded function

$$g(z) = \frac{f(z) - M - 1}{f(z) - M + 1},$$

where $|g(z)| \leq 1$. Therefore g has a removable singularity. Extending g and applying the inverse linear transformation results in an analytic extension of f, showing that the singularity is removable. $\square$

Problem 6. (page 129f)
Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.
(Hint: f and e^f cannot have a common pole (why?). Now apply Theorem 9.)

Proof.
Firstly f and e^f cannot have a common pole because if f has a pole at z_0, then $g(z) = f(1/z)$ has a zero at z_0, and so $e^{g(z_0)} = 1$, and thus $e^f = 1$ at every pole of f.

In every neighborhood of a isolated singularity of f, $f(z)$ can be made arbitrarily close to ∞ (on the Riemann Sphere), and so $\exp(f(z))$ can be made arbitrarily close to any complex value $w = \alpha + \beta i$, by choosing z such that $f(z) = a + bi$ has a near α and b near $\beta + 2\pi k$ for some (large) k.

Therefore by Theorem 9, the pole at $f(z)$ is an essential singularity of $\exp f(z)$. $\square$

Problem 1. (page 133)
Determine explicitly the largest disk about the origin whose image under the mapping $f(z) = z^2 + z$ is one to one.

Proof.
By corollary 2, because $f'(z) = 2z + 1$ vanishes at $z = 1/2$, the disk must not contain the point $z = 1/2$. Thus, the disk can have radius at most $1/2$.

Suppose that $w \neq z$ but $w^2 + w = z^2 + z$. Then $w^2 - z^2 = z - w$, so $(w + z) = -1$. If w and z are in the disk of radius $1/2$, then $w + z \neq -1$, so f is indeed injective on the disk of radius $1/2$. $\square$

Problem 3. (page 133)
Apply the representation $f(z) = w_0 + \zeta(z)^n$ to $\cos z$ with $z_0 = 0$. Determine $\zeta(z)$ explicitly.

Proof.
Firstly $\cos(0) = 1$, so we can write
$$\cos(z) - 1 = z^2 \frac{\cos(z) - 1}{z^2}$$
so $g(z) = (\cos(z) - 1)/z^2$ and so by the double-angle formula
$$\begin{aligned} h(z) &= \sqrt{g(z)} \\ &= \sqrt{\frac{\cos(z) - 1}{z^2}} \\ &= \sqrt{\frac{-2\sin^2(z/2)}{z^2}} \\ &= \frac{i\sqrt{2}\sin(z/2)}{z}. \end{aligned}$$
And therefore $\zeta(z) = zh(z) = i\sqrt{2}\sin(z/2)$. $\square$