Topology: Homework 6

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Problem 1.
Let X be path connected and locally path connected, and assume that $\pi_1(X; x_0)$ is finite. Show that every map $f \colon X \to S^1$ from X to the unit circle is homotopic to a constant map. (Hint: consider the covering $p \colon \mathbb{R} \to S^1$ and use the Lifting Criterion.)

Proof.
Let $p \colon \mathbb{R} \to S^1$ be a covering map. Then by the path lifting criterion if there exists a map $\widetilde{f} \colon X \to \mathbb{R}$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(x_0) = p(f(x_0))$, then

$$f_* \pi_1(X; x_0) \subset p_* \pi_1(\mathbb{R}; r_0) = p_*(\mathbf{1}) = \mathbf{1} \text{ in } \pi_1(S^1; s_0).$$

First define $p \colon \mathbb{R} \to S^1$. Since X is path connected, there exists a path $\alpha_x \colon [0,1] \to X$ starting at x_0 and ending at x. Since f is continuous, $f \circ \alpha_x$ defines a path in S^1, call it $\beta_x = f \circ \alpha_x \colon [0,1] \to S^1$. By the path lifting property, we can choose any such path α_x (and thus β_x) and define its lift $\widetilde{\beta}_x \colon [0,1] \to \mathbb{R}$ based at $r_0 \in p^{-1}(f(x_0))$. Lastly, define $\widetilde{f}(x) = \widetilde{\beta}_x(1)$.

Since $\pi(X; x_0)$ is finite, the homomorphism $f_* \colon \pi(X; x_0) \to \pi(S^1; s_0) = \mathbb{Z}$ must be trivial. Thus if α_{x_0} is a loop in X, it is homotopic to a constant path in S^1, and so its lift is homotopic to a constant path in $\mathbb{R}$.

The constructed map satisfies the requirement that $p \circ \widetilde{f} = f$:

$$\begin{aligned}
p(\widetilde{f}(x)) &= p(\widetilde{\beta}_x(1)) \\
&= (p \circ \widetilde{\beta}_x)(1) \\
&= (\beta_x)(1) \\
&= (f \circ \alpha_x)(1) \\
&= f(\alpha_x(1)) \\
&= f(x).
\end{aligned}$$

So we can use the "contraction" homotopy in $\mathbb{R}$ to construct a homotopy of f, namely

$$\begin{aligned}
\widetilde{H}(t) &= (1-t)\widetilde{f}(x) + \widetilde{f}(x_0) \\
H(t) &= p((1-t)\widetilde{f}(x) + \widetilde{f}(x_0)).
\end{aligned}$$

Thus f is homotopic to the constant map with homotopy H. $\square$

Problem 2.
Let X be path connected and locally path connected. Show that if there exists a covering map $p\colon \widetilde{X} \to X$ and $\widetilde{x}_0 \in \widetilde{X}$ such that $\widetilde{X}$ is path connected and $\pi_1(\widetilde{X}; \widetilde{x}_0) = \mathbf{1}$, then X is semi-locally simply connected.

Proof.
In to show that X is semi-locally simply connected, it is sufficient to show that for all $x \in X$ there exists a neighborhood U around x such that all loops in U are null-homotopic in X.

Since $\widetilde{X}$ is the total space of the covering, there exists a neighborhood U_x around every point $x \in X$, such that

a. $p^{-1}(U_x) = \coprod_{i \in I} \widetilde{U}_i$ where $\widetilde{U}_i$ is an open subset of $\widetilde{X}$, and

b. the restriction $p|_{\widetilde{U}_i}\colon \widetilde{U}_i \to U$ is a homeomorphism.

Let $\alpha\colon [0,1] \to U_x$ be a loop in U_x. Choosing the $i \in I$ such that $\widetilde{x}_0 \in U_i$, we can lift α via

$$(p|_{\widetilde{U}_i})^{-1} \circ \alpha = \widetilde{\alpha}\colon [0,1] \to \widetilde{X}.$$

Since the fundamental group of $\widetilde{X}$ is trivial, $[\widetilde{\alpha}] = [\mathrm{id}_{x_0}] \in \pi_1(\widetilde{X}; \widetilde{x}_0) = \mathbf{1}$, thus there exists a homotopy $\widetilde{H}\colon [0,1] \times [0,1] \to \widetilde{X}$ from $\widetilde{\alpha}$ to id_{x_0}. This homotopy induces a homotopy $H\colon [0,1] \times [0,1] \to X$ by $p \circ \widetilde{H}$. Since p is a covering map, and thus continuous, H must be continuous, and thus defines a homotopy from $p \circ \widetilde{\alpha} = \alpha$ to $p \circ c_{\widetilde{x}_0} = c_{p(\widetilde{x}_0)}$.
Therefore there exists a neighborhood around every point $x \in X$ such that any loop in that neighborhood is nullhomotopic in X. $\square$

Problem 3.
Let X be the Hawaiian earrings. Show that X is not semi-locally simply connected.

Proof.
By defintion X is semi-locally simply connected if for every point $x \in X$ there exists a neighborhood U around x such that all loops in U are contractible in X.

Choose the point $x = (0, 0) \in X$. Then for any open neighborhood U containing the origin, we can find a ball of radius ε centered at the origin and contained in U, and this ball contains X_n for $1/n < \varepsilon/2$. As shown in problem 3 of homework 4, the loop around X_n is not contractible in X, so X must not be semi-locally simply connected. $\square$

Problem 4.
Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ path connected and with base points $\widetilde{x}_0 \in \widetilde{X}$ and $x_0 = p(\widetilde{x}_0) \in X$. Consider the right quotient set $\pi_1(X;x_0)/p_*(\pi_1(\widetilde{X};\widetilde{x}_0))$.

a. Show that there is a well-defined map

$$f\colon p^{-1}(x_0) \to \pi_1(X;x_0)/p_*(\pi_1(\widetilde{X};\widetilde{x}_0))$$

such that if $\widetilde{x} \in p^{-1}(x_0)$, then $f(\widetilde{x})$ is the equivalence class of $[p \circ \widetilde{\alpha}] \in \pi_1(X;x_0)$ for every path $\widetilde{\alpha}\colon [0,1] \to \widetilde{X}$ with $\widetilde{\alpha}(0) = \widetilde{x}$ and $\widetilde{\alpha}(1) = \widetilde{x}_0$.

b. Show that f is surjective.

c. Show that f is injective.

Proof.

a. Suppose that $\widetilde{\alpha}, \widetilde{\beta}\colon [0,1] \to \widetilde{X}$ are two paths in X such that $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x}$ and $\widetilde{\alpha}(1) = \widetilde{\beta}(1) = \widetilde{x}_0$. Then $f(\widetilde{x}) = [[p \circ \widetilde{\alpha}]]$ and $f(\widetilde{x}) = [[p \circ \widetilde{\beta}]]$, so in order for f to be well defined, it is sufficient to show that $[p \circ \widetilde{\alpha}] \sim [p \circ \widetilde{\beta}]$. But this follows because

$$\begin{aligned}
\underbrace{[p \circ \widetilde{\alpha}]^{-1}}_{a^{-1}} \cdot \underbrace{[p \circ \widetilde{\beta}]}_{b} &= [p \circ \overline{\widetilde{\alpha}}] \cdot [p \circ \widetilde{\beta}] \\
&= [(p \circ \overline{\widetilde{\alpha}}) * (p \circ \widetilde{\beta})] \\
&= [p \circ (\overline{\widetilde{\alpha}} * \widetilde{\beta})] \\
&= p_*[\overline{\widetilde{\alpha}} * \widetilde{\beta}] \\
&\in p_*(\pi_1(\widetilde{X};\widetilde{x}_0)).
\end{aligned}$$

b. Let $[[\alpha]] \in \pi_1(X;x_0)/p_*(\pi_1(\widetilde{X};\widetilde{x}_0))$ be an arbitrary element of the right quotient set. Consider any underlying path $\alpha\colon [0,1] \to X$. By the path lifting property there exists a lift $\widetilde{\alpha}\colon [0,1] \to \widetilde{X}$ based at $\widetilde{x}_0 \in p^{-1}(x_0)$. Since $p \circ \widetilde{\alpha} = \alpha$ is a loop, $\widetilde{\alpha}(1) \in p^{-1}(x_0)$, and so $f(\widetilde{\alpha}(1)) = [[\alpha]]$, and so f is surjective.

c. Assume that $f(\widetilde{x}) = f(\widetilde{x}')$. It is sufficient to show that $\widetilde{x}_0 = \widetilde{x}_0'$. By defintion, $f(\widetilde{x})$ is the equivalence class of paths $[p \circ \widetilde{\alpha}]$ from $\widetilde{x}$ to $\widetilde{x}_0$ and $f(\widetilde{x}')$ is the equivalence class of paths $[p \circ \widetilde{\alpha}']$ from $\widetilde{x}'$ to $\widetilde{x}_0$. So consider a choice of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ from the respective equivalence classes. Since $[p \circ \widetilde{\alpha}] \sim [p \circ \widetilde{\alpha}']$,

$$\begin{aligned}
\underbrace{[p \circ \widetilde{\alpha}]^{-1}}_{a^{-1}} \cdot \underbrace{[p \circ \widetilde{\alpha}']}_{b} &= [p \circ \overline{\widetilde{\alpha}}] \cdot [p \circ \widetilde{\alpha}'] \\
&= [(p \circ \overline{\widetilde{\alpha}}) * (p \circ \widetilde{\alpha}')] \\
&= [p \circ (\overline{\widetilde{\alpha}} * \widetilde{\alpha}')] \\
&= p_*[\overline{\widetilde{\alpha}} * \widetilde{\alpha}'] \\
&\in p_*(\pi_1(\widetilde{X};\widetilde{x}_0)).
\end{aligned}$$

In order $*$ to be well-defined, $\overline{\widetilde{\alpha}}(1) = \widetilde{\alpha}(0) = \widetilde{\alpha}'(0)$, so α and α' are paths with the same base point, namely, $\widetilde{x} = \widetilde{x}'$. Thus the map f is injective.

□