Differential Geometry: Homework 3

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Problem 1. Let $x_0 \in \mathbb{R}^n$ be a point and $r_1 < r_2$ positive real numbers. Construct (with proof) a C^∞ function $h\colon \mathbb{R}^n \to \mathbb{R}$ which equals 1 inside the ball of radius r_1 around x_0 and which equals 0 outside the ball of radius r_2 around x_0. Such functions are collectively called smooth bump functions.

Proof.
I'll construct a bump function from $\mathbb{R} \to \mathbb{R}$ following the Wikipedia construction[1]. First construct a smooth function f such that $f(0) = 0$ and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, but $f(1) \neq 0$. Let

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-1/x^2) & x > 0 \end{cases}.$$

Now if there is a differentiable function $p_k(x)$ such that

$$f^{(k)}(x) = \begin{cases} 0 & x < 0 \\ p_k(x)\exp(-1/x^2) & x \geq 0 \end{cases},$$

then by the product rule,

$$f^{(k+1)}(x) = \begin{cases} 0 & x < 0 \\ (p_k(x)2x^{-3} + p_k'(x))\exp(-1/x^2) & x \geq 0 \end{cases}.$$

Therefore $p_0(x) = 1$ and $p_{k+1}(x) = p_k(x)2x^{-3} + p_k'(x)$. And an inductive argument shows that by sufficiently many applications of L'Hôpital's rule,

$$\lim_{x\to 0} f^{(k)}(x) = 0$$

for all $k \in \mathbb{N}$ as desired. Now, given $\alpha < \beta$, let

$$b_{\alpha,\beta}(x) = \frac{f(x-\alpha)}{f(x-\alpha) + f(\beta - x)}$$

so that $b_{\alpha,\beta}(x_0) = 0$ for $x_0 \leq \alpha$ and $b_{\alpha,\beta}(x_1) = f(x_1 - \alpha)/f(x_1 - \alpha) = 1$ for $x_1 \geq \beta$.

The function has strictly positive denominator because f is nonnegative, $f(x - \alpha) > 0$ for $x > \alpha$, and $f(\beta - x) > 0$ for $x < \beta$. Therefore, as the quotient of a smooth function and a smooth positive function, $b_{\alpha,\beta}$ is smooth. Next, because the difference of smooth functions is smooth, let

$$g_{r_1,r_2}(x) = b_{-r_2,-r_1}(x) - b_{r_1,r_2}(x).$$

In order to make this work in $\mathbb{R}^n$, let $d_{x_0} : \mathbb{R}^n \to \mathbb{R}$ be the Euclidean distance between x and x_0.

Since d_{x_0} is smooth, and the composition of smooth functions is smooth, our desired function $h : \mathbb{R}^n \to \mathbb{R}$ can be defined by $h = g_{r_1,r_2} \circ d_{x_0}$. □

[1] https://en.wikipedia.org/wiki/Non-analytic_smooth_function#Smooth_transition_functions

Problem 2. Assume $U \in \mathbb{R}^m$ and $V \in \mathbb{R}^n$ are open sets and $f\colon U \to V$ is an immersion. Prove the immersion version of the implicit function theorem, assuming only the inverse function theorem: there exists a function $G\colon \tilde{V} \to Z$ where Z is an open set in $\mathbb{R}^n$.

Proof. The intuition here is that $G^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ (the inverse of the diffeomorphism G) looks very similar to $f : U \to \mathbb{R}^n$.

Let $p \in U$, and $\tilde{U}$ a sufficiently small neighborhood of p so that $\tilde{V} = f(\tilde{U})$ is locally Euclidean. Let $\pi_{\text{sub}} : \mathbb{R}^n \to \mathbb{R}^m$ be the canonical submersion that "removes" the last $n - m$ coordinates of a point in $\mathbb{R}^n$. Denote $f\colon \tilde{U} \to \tilde{V} \subset \mathbb{R}^n$ coordinatewise by $f(x) = (f_1(x), \dots, f_n(x))$. Then

$$G^{-1}(x) = (\underbrace{f_1 \circ \pi_{\text{sub}}(x)}_{1}, \dots, \underbrace{f_m \circ \pi_{\text{sub}}(x)}_{m}, \underbrace{x_{m+1} + f_{m+1} \circ \pi_{\text{sub}}(x)}_{m+1}, \dots, \underbrace{x_n + f_n \circ \pi_{\text{sub}}(x)}_{n}),$$

that is, apply f to the first m coordinates and leave the other $n - m$ coordinates the same (perhaps after some permutation of coordinates).

This results in a derivative matrix that looks like (a permutation of) df in the first m columns, I in the $n - m \times n - m$ submatrix in the bottom right corner, and 0 elsewhere.

Because f is injective, the matrix df_p has full rank, so the rows can be permuted in such a way that e_{m+1} through e_n are not in the span of df_p.

G^{-1} is constructed in such a way that dG^{-1} has rank n. Thus by the inverse function theorem, $G = (G^{-1})^{-1}$ is a diffeomorphism with the property that $G \circ f = \pi_{\text{imm}}$. $\square$

Problem 3.

Proof.
($\Longrightarrow$) Assume that there exists an injective immersion $f : M^m \to Y \subset N^n$ onto Y, and let p_Y be a point in Y. The atlas of N^n contains an open set U around p_Y with a map $\phi\colon U \to \mathbb{R}^n$. Similarly because f is injective, the preimage of p_Y is a single point $f^{-1}(p_Y) = p_M \in M^m$, and there exists a chart $(V, \psi\colon V \to \mathbb{R}^m) \in \mathcal{A}_M$ centered at p_M.

Because (1) f is an injective immersion and (2) ψ^{-1} and ϕ are continuous bijections, $\phi \circ f \circ \psi^{-1}$ is an injective immersion. So by the implicit function theorem (immersion version) there exists a diffeomorphism G such that $G \circ \phi \circ f \circ \psi^{-1} = \pi_{imm}$.

Therefore let $\phi_G = G \circ \phi$. Then (U, ϕ_G) is a chart in N's maximal atlas $\mathcal{A}_N$, and $\phi_G \circ f(M^m) = \phi_G(U \cap Y)$ is the image of the model immersion, so $\phi_G(U \cap Y) = \phi_G(U) \cap (\mathbb{R}^m \times \{0\})$.

($\Longleftarrow$) Assume that there exists a subset $Y \subset N^n$ such that for each point $p \in Y$ there exists a chart (U_p, ϕ_p) with $\phi_p(U_p \cap Y) = \phi_p(U_p) \cap (\mathbb{R}^m \times \{0\})$. Composing with the model submersion $\pi_{\text{sub}} : \mathbb{R}^n \to \mathbb{R}^m$ yields $\pi_{\text{sub}} \circ \phi_p : Y \to \mathbb{R}^m$.

Therefore Y is a manifold with atlas $\mathcal{A}_Y = \{\, (U_p \cap Y, \pi_{\text{sub}} \circ \phi_p) \,\}_{p \in Y}$, and naturally has embedding via the inclusion map $i\colon Y \to \mathbb{R}^n$, which is (as shown in lecture) a injective immersion. $\square$

Problem 4. Prove the following result: if $f\colon M^m \to N^n$ is a submersion between two smooth manifolds, or more generally if f is simply a smooth map and $y \in N$ is a regular value of f, then $S = f^{-1}(y)$ has the structure of a smooth submanifold of M of dimension $m - n$.

Proof.
For each point $p \in S$, there exists a chart that contains p: $(U_p, \phi_p : M^m \to \mathbb{R}^m)$. Similarly, the maximal atlas of N contains many charts centered at y, namely $(f(U_p), \psi : N^n \to \mathbb{R}^n)$. Because f is a submersion by hypothesis, the implicit value theorem (submersion version) guarantees the existence of a diffeomorphism $F_p : \phi_p(U_p) \to \mathbb{R}^m$ such that $\psi \circ f \circ \phi_p^{-1} \circ F_p^{-1}$ is the model submersion $\pi_{\text{sub}} : F_p(\phi_p(U_p)) \to \mathbb{R}^n$.

Because ψ is centered at y,

$$F_p \circ \phi_p \circ f^{-1}(y) = \pi_{\text{sub}}^{-1} \circ \psi(y) = \{0 \in \mathbb{R}^n\} \times \mathbb{R}^{m-n}$$

Therefore by permuting coordinates and applying "Definition 2" of a submanifold, S is a submanifold of M^m with atlas

$$\mathcal{A} = \{\, (U_p \cap S, (\pi \circ F_p \circ \phi_p)\colon U_p \cap S \to \mathbb{R}^{m-n})\, \}$$

where π is the projection onto the last $m - n$ coordinates. $\square$

Problem 5. Prove that $S^n = \{ x_1^2 + \ldots + x_{n+1}^2 = 1 \} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by exhibiting it as the regular value of some smooth map between manifolds.

Proof.
Let $f(x) = x_1^2 + \ldots + x_n^2$, which is a smooth map from $\mathbb{R}^{n+1}$ to $\mathbb{R}$. Then

$$\begin{aligned} df(p) &= \left[\frac{\partial f}{\partial x_1}(p) \frac{\partial f}{\partial x_2}(p) \ldots \frac{\partial f}{\partial x_{n+1}}(p) \right] \\ &= [2p_1 2p_2 \ldots 2p_{n+1}]. \end{aligned}$$

So df has rank 1 for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$, so f is submersive for all $x \neq 0$. Therefore the preimage $f^{-1}(1)$ has the structure of a manifold of dimension $n + 1 - 1 = n$. $\square$

Problem 6.

(a) Show that $\mathrm{Sym}(n)$ is a submanifold of $M_n(\mathbb{R})$ (and in particular a manifold), and compute its dimension.

(b) Prove that $I \in \mathrm{Sym}(n)$ is a regular value of ϕ.

(c) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

(d) Prove that $O(n)$ is compact.

Proof.

Part (a). Let A_{ij} be an $n \times n$ matrix where the ij and ji entries are 1 and all other entries are 0. Then $\mathrm{Sym}(n)$ has basis $\{A_{ij} : i \leq j\}$, and has a (smooth) linear isomorphsim $\varphi : \mathrm{Sym}(n) \to \mathbb{R}^{n(n+1)/2}$. Similarly let $\psi : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ be the analogous linear isomorphism between $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$.

Then $\psi^{-1} \circ \pi_{\mathrm{imm}} \circ \varphi : \mathrm{Sym}(n) \hookrightarrow (\mathbb{R})$ is an embedding which demonstrates that $\mathrm{Sym}(n)$ is a submanifold of $(\mathbb{R})$.

Part (b). [2] It is sufficient to show that ϕ is a submersion for all points $p \in \phi^{-1}(I)$. So letting B be an arbitrary invertible matrix in $M_n(\mathbb{R})$, and taking the derivative

$$\phi'_p(A) = \lim_{h \to 0} \frac{\phi(A + hB) - \phi(A)}{h} B^{-1}.$$

It now must be shown that $\phi'(I)$ is a matrix of rank m.

Part (c). This follows from the corollary to the implicit function theorem. As shown in part (b), I is a regular value of ϕ, so $\phi^{-1}(I) = O(n)$ is a submanifold of $M_n(\mathbb{R})$ of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

Part (d). Because ϕ is continuous and the singleton set $\{I\}$ is closed, $\phi^{-1}(I) = O(n)$ is closed as well. Since $\phi(A) = 1$ for each $j \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{n} a_{ij}^2 = 1.$$

Since a_{ij}^2 is positive, each entry must be strictly less than 1, and therefore $O(n)$ is closed and bounded.

□

[2] https://math.stackexchange.com/a/383458/121988

Problem 7.

Part (a). It is sufficient to show that (i) there exists an identity morphism for each object in Alg_k, (ii) the composition of two (composable) k-algebra homomorphisms is a k-algebra homomorphism, and (iii) k-algebra homomorphisms are associative.

(i) For each object $x \in \mathrm{ob}(\mathrm{Alg}_k)$, let $1_X \in \hom_{\mathrm{Alg}_k}(X, X)$ be the identity map that sends each element $x \in X$ to itself. Clearly 1_X is a k-algebra homomorphism because 1_X is a linear map of vector spaces which is compatible with the multiplication maps

$$1_X(\alpha \cdot \beta) = \alpha \cdot \beta = 1_X(\alpha) \cdot 1_X(\beta)$$

and preserves the identity elements ($1_X(1) = 1$.)
Also if $f \in \hom_{\mathrm{Alg}_k}(Z, X)$ is a k-algebra homomorphism,

$$1_X \circ f(\alpha) = 1_X(f(\alpha)) = f(\alpha),$$

and if $g \in \hom_{\mathrm{Alg}_k}(X, Y)$ is a k-algebra homomorphism

$$g \circ 1_X(\alpha) = g(1_X(\alpha)) = g(\alpha).$$

So indeed $1_X \circ f = f$ and $g \circ 1_X = g$, and therefore 1_X is an identity morphism.

(ii) Let $f \in \hom_{\mathrm{Alg}_k}(Z, X)$ and $g \in \hom_{\mathrm{Alg}_k}(X, Y)$.
Then $g \circ f$ is compatible with the multiplication maps

$$g \circ f(\alpha \cdot \beta) = g(f(\alpha) \cdot f(\beta)) = g(f(\alpha)) \cdot g(f(\beta)) = g \circ f(\alpha) \cdot g \circ f(\beta),$$

and $g \circ f$ preserves the identity elements

$$g \circ f(1) = g(1) = 1.$$

Therefore $g \circ f \in \hom_{\mathrm{Alg}_k}(Z, Y)$.

(iii) For each composable triple f, g, and h

$$h \circ (g \circ f) = (h \circ g) \circ f$$

because associativity is inherited from ordinary composition of functions.

Part (b). It is sufficient to show that (i) $C^0(X)$ is a vector space over $\mathbb{R}$, (ii) multiplication is bilinear, (iii) multiplication is associative, and (iv) there is a multiplicative identity.

(i) $C^0(X)$ is a vector space with pointwise addition and ordinary scalar multiplication. In general continuous functions are closed under addition. Multiplying (or dividing) every element in an open set U by a scalar a yields an open set aU, so if $f^{-1}(U)$ is open for every open set U, then $(af)^{-1}(aU)$ is also open. Therefore $C^0(X)$ is closed under scalar multiplication.
$C^0(X)$ inherits structure from $\mathbb{R}$ so that.

- Associativity and commutativity of addition follow from $\mathbb{R}$.
- The zero function (which is proven to be in $C^0(X)$ below) satisfies $f + 0 = f$ for all f.
- All elements are invertible with respect to addition: $f(x) + (-1) \cdot f(x) = 0$.
- The scalar $1 \in \mathbb{R}$ behaves as an identity element for scalar multiplication: $1f = f$.
- Everything distributes nicely: $a(b \cdot f) = (ab) \cdot f$, $a(f + g) = af + ag$, and $(a + b)f = af + bf$.

Lastly, it is important to check that continuous functions remain continuous after addition and scalar multiplication.

(ii) Bilinearity follows from well-behaved distributivity on $\mathbb{R}$. Let $f, g, h \in C^0(X)$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
(f+g) \times h &= (f \cdot h) + (g \cdot h) = (f \times h) + (g \times h) \\
f \times (g+h) &= (f \cdot g) + (f \cdot h) = (f \times g) + (f \times h) \\
(\alpha \cdot f) \times g &= \alpha \cdot (f \cdot g) \qquad = \alpha \cdot (f \times g) \\
f \times (\alpha \cdot g) &= \alpha \cdot (f \cdot g) \qquad = \alpha \cdot (f \times g).
\end{aligned}$$

(iii) Associativity follows from associativity on $\mathbb{R}$.

$$(f \times g) \times h = (f \cdot g) \cdot h = f \cdot (g \cdot h) = f \times (g \times h).$$

(iv) The multiplicative identity is the constant function 1. Constant functions are in $C^0(X)$ because any open set that contains the constant has a preimage of X (which is an open set) and any set that does not contain the constant has a preimage of $\emptyset$ (which is also an open set.) For each function $f \in C^0(X)$ and each point $x \in X$

$$1(x) \times f(x) = 1 \cdot f(x) = f(x) = f(x) \cdot 1 = f(x) \times 1(x),$$

therefore $1 \times f = f = f \times 1$.

Part (c). Let f be a continuous map $f \in \hom_{\mathbf{Top}}(X, Y)$.
In order to prove that F is a contravariant functor, it is sufficient to show that (i) $F(f)$ is an $\mathbb{R}$-algebra homomorphism, (ii) $F\colon \hom_{\mathbf{Top}}(X,Y) \to \hom_{\mathbf{Alg}_{\mathbb{R}}}(F(Y), F(X))$ sends identity morphisms to identity morphisms, and (iii) $F(f) \circ F(g) = F(g \circ f)$ for composable morphisms.

(i) Let $g, h \in C^0(Y)$. Then $F(f)\colon C^0(Y) \to C^0(X)$ is an $\mathbb{R}$-algebra homomorphism because it is compatible with multiplication maps

$$F(f)(g \cdot h) = (g \cdot h) \circ f = (g \circ f) \cdot (h \circ f) = F(f)(g) \cdot F(f)(h)$$

and because it preserves the (multiplicative) identity element (the constant map 1)

$$F(f)(y \mapsto 1) = (y \mapsto 1) \circ f = (x \mapsto 1)$$

(ii) Let $\mathrm{id}_X \in \hom_{\mathbf{Top}}(X, X)$. Then for all $g \in F(X) = C^0(X)$,

$$F(\mathrm{id}_X)(g) = g \circ \mathrm{id}_X = g = \mathrm{id}_{C^0(X)}(g).$$

Therefore $F(\mathrm{id}_X) = \mathrm{id}_{C^0(X)}$.

(iii) Let $g \in \hom_{\mathbf{Top}}(Y, Z)$ and $h \in C^0(Z)$. Then

$$(F(f) \circ F(g))(h) = F(f)(F(g)(h)) = F(f)(h \circ g) = (h \circ g \circ f) = F(g \circ f)(h),$$

so $F(f) \circ F(g) = F(g \circ f)$.

Part (d). It is sufficient to show that (i) there exists a functor *Forget* from $\mathbf{Alg}_{\mathbb{R}}$ to **Set** (that maps algebras to their underlying sets and algebra homomorphisms to the corresponding map of sets), (ii) this functor is faithful, and (iii) this functor is not full.

1. *Forget* naturally maps identity morphisms to identity morphisms because the identity morphism on an $\mathbb{R}$-algebra is the same as the identity morphism on the underlying set, namely $x \mapsto x$. Composition is compatible because it is the same as the set theoretic function composition.

2. By definition of k-algebra homomorphism equality, if two k-algebra homomorphisms $\phi\colon A \to B$ and $\psi\colon A \to B$ have the same underlying map of sets, then they are equal. Therefore *Forget* is faithful.

3. Let $\phi\colon C^0(X) \to C^0(X)$ be the function $\phi(f) = (x \mapsto x+1) \circ f$. Then the unity element (the constant function $x \mapsto 1$) is not preserved under ϕ, so ϕ is not an $\mathbb{R}$-algebra homomorphism. Therefore ϕ is not in the image of *Forget*, and so *Forget* is not full.