Math 574: Homework 4

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Problem 1. From *Alternative version of #4, problem 1.*

Proof.

(a) A commutes with a cyclic matrix C because it commutes with a matrix with distinct eigenvalues. This means we can write $A = f(C)$.

(b) Since the set of cyclic matrices is open and dense, there exists a dense neighborhood of B containing cyclic matrices. Thus the set $\{c \in \mathbb{C} : B + cC \text{ is cyclic}\}$ is dense in $\mathbb{C}$. cyclic

(c) If (R, S) is a pair of commuting matrices with S cyclic, then we showed in homework 2 that $R = f(S)$. Now if we can find a diagonalizable matrix E that is ε close to R, then $D = f(E)$ will be $f(\varepsilon) = \varepsilon'$ close to S. By (b), take $c < \epsilon/n^2$ such that $E + cC$ is cyclic and call this S. Now S is ϵ-close to E and so R is ϵ'-close to D.

(d) Lastly since c is dense, and (D, E) is simultaneously diagonalizable by the last homework, (R, S) can be approximated arbitrarily closely by simultaneously diagonalizable matrices.

□

Problem 2. Let $A, B \in M_n(\mathbb{R})$ be skew symmetric.

Proof.

(a) Since A is skew symmetric, that is $A^\top = -A$, so

$$\langle Av, v\rangle = (Av)^\top v = v^\top A^\top v = \langle v, A^\top v\rangle = \langle v, -Av\rangle = -\langle v, Av\rangle.$$

by the Hermitian Property, $\langle v, Av\rangle = \overline{\langle Av, v\rangle}$, so when v is an eigenvector with corresponding eigenvalue λ,

$$\langle Av, v\rangle = -\overline{\langle Av, v\rangle} = \lambda\langle v, v\rangle = -\overline{\lambda}\langle v, v\rangle$$

since $\langle v, v\rangle > 0$ since v is a nonzero eigenvector, $\lambda = -\overline{\lambda}$. Writing $\lambda = a + bi$,

$$\underbrace{a + bi}_{\lambda} = \underbrace{-(a - bi)}_{-\overline{\lambda}} = -a + bi,$$

so $a = 0$, and λ is purely imaginary.
Moreover, since A is normal by

$$AA^* = AA^\top = A(-A) = -AA = A^\top A = A^* A,$$

it is diagonalizable by the Spectral Theorem.

(b) Since A and B are similar, they have the same eigenvalues. So the claim follows by Corollary 2.5.11(b) in Horn and Johnson:

> Two real skew-symmetric matrices are real orthogonally similar if and only if they have the same eigenvalues.

(c) Let

$$A = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & i \\ 0 & -\lambda & 1 \\ -i & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - i \begin{vmatrix} 0 & i \\ -\lambda & 1 \end{vmatrix} = -\lambda(\lambda^2 + 1) + \lambda = -\lambda^3$$

so A has all eigenvalues 0. But the dimension of the eigenspace corresponding to 0 is the nullity of A, which is 1, since

$$\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} t,$$

the dimension of the 0 eigenspace is 1. Thus A is not diagonalizable.

□

Problem 3.

Proof.

(a) I'll illustrate with an example. Suppose B is the first, fourth, and fifth rows/columns of A, a 5×5 matrix. Then

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^\top}_{w^\top} \underbrace{\begin{bmatrix} a_{11} & a_{14} & a_{15} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_2 \\ x_3 \end{bmatrix}^\top}_{v^\top} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_2 \\ x_3 \end{bmatrix}}_{v} > 0.$$

Zooming out now, if B is a $d \times d$ principal submatrix of A, then B must be Hermitian, because this is inherited from A, which is easy to see on "inspection", since the rows and columns that are deleted are symmetric with respect to reflection over the main diagonal. To see that B is positive definite, it follows from the example above: We can write

$$w^\top B w = v^\top A v > 0 \text{ for } v \neq 0,$$

where v has essentially the same entries as w, but with zeroes inserted in the positions of the deleted rows/columns. (See example.) Since $v = 0$ if and only if $w = 0$, B is positive definite.

Let $\lambda_{1,B}$ be the largest eigenvalue of B. Since A and B are Hermitian,

$$\lambda_1 = \max_{|v|=1} \langle Av, v \rangle \text{ and } \lambda_{1,B} = \max_{|w|=1} \langle Bw, w \rangle.$$

By the above construction, $\langle Bw, w \rangle = \langle Av, v \rangle$ for v with additional zeros (and so the same norm), thus as sets,

$$\{\langle Bw, w \rangle : |w| = 1\} \subset \{\langle Av, v \rangle : |v| = 1\},$$

so $\lambda_{1,B} \leq \lambda_1$.

(b) In part (a), I showed that $\lambda_{1,B} \leq \lambda_1$ for any principal submatrix B, so in particular this holds for an $n - 1 \times n - 1$ principal submatrix. Thus it only remains to show that we can find some $\lambda_{1,B} \geq \lambda_2$.

Say that B is the $n-1 \times n-1$ principal submatrix with the ith row/column removed. Furthermore, since

$$\lambda_{1,B} = \max\{\langle Bw, w\rangle : |w| = 1\} = \max\{\langle Av, v\rangle : |v| = 1 \text{ and } v_i = 0\}$$

it suffices to construct a unit vector w such that $\langle Bw, w\rangle \geq \lambda_2$. Let v and u be eigenvectors corresponding to λ_1 and λ_2 respectively, and find α, β such that $\alpha v_i + \beta u_i$ so that $\alpha v + \beta u$ has a 0 in the ith position. Call this vector

$$w' = \underbrace{\alpha v}_{v'} + \underbrace{\beta u}_{u'},$$

which is designed so that

$$Aw' = A(v' + u') = Av' + Au' = \lambda_1 v' + \lambda_2 u'.$$

By scaling, we can choose α, β so that $|w'| = 1$. Let w be w' with its ith entry removed. Then

$$\begin{aligned} \langle Bw, w\rangle &= \langle Aw', w'\rangle \\ &= \langle A(v' + u'), v' + u'\rangle \\ &= \langle \lambda_1 v' + \lambda_2 u', v' + u'\rangle \\ &\geq \langle \lambda_2 v' + \lambda_2 u', v' + u'\rangle \\ &= \lambda_2 \underbrace{\langle v' + u', v' + u'\rangle}_{=1} \end{aligned}$$

as desired.

□

Problem 4. Let V be a finite dimensional normed vector space over $\mathbb{C}$, and let W be a proper subspace of V. Define a map $|\cdot|_q : V/W \to \mathbb{R}$ by

$$|v|_q = \inf_{w \in W} |v + w|.$$

Proof. (i) **Nonnegativity.** Firstly, let $v = 0 \in V/W$. Then

$$0 \leq \inf_{w \in W} |v + w| \leq |0 + 0| = 0,$$

so $|0|_q = 0$ since it is bounded above and below by 0.
Conversely, assume that $|v|_q = |v + w| = 0$, by nonnegativity of the norm on V, this implies $v + w = 0$. Of course this means that $v = -w \in W$, so $v = 0 \in W/V$.

(ii) **Scaling.** Let $\alpha \in \mathbb{C}$, and consider

$$|\alpha v|_q = \inf_{w \in W} |\alpha v + w| = \inf_{w \in W} |\alpha v + \alpha w| = \alpha \left(\inf_{w \in W} |v + w| \right) = \alpha |v|_q$$

because multiplying by α is surjective when $\alpha \neq 0$. (And the claim follows by (i) when $\alpha = 0$.)

(iii) **Triangle inequality.**

$$\begin{aligned} |v + u|_q &= \inf_{w \in W} |v + u + w| \\ &= \inf_{w \in W} |v + u + 2w| \\ &= \inf_{w \in W} |v + w + u + w| \\ &\leq \inf_{w \in W} (|v + w| + |u + w|) \\ &\leq \inf_{w \in W} |v + w| + \inf_{w' \in W} |u + w'| \\ &= |v|_q + |u|_q, \end{aligned}$$

where the second equality follows from the surjectivity of the scalar multiplication by 2.

□

Problem 5. Let $A, B \in M_n(K)$, and let $T(X) = AX - XB$.

Proof.

(a) Let x and y be eigenvectors of A and B respectively. Consider the induced transformation $T(x \otimes y) = (I_n \otimes A - B^\top \otimes I_n)(x \otimes y)$, where by Schur Decomposition, we can write two upper triangular matrices

$$\Delta_A = U^* A U$$
$$\Delta_B^\top = V^* B V$$

where U and V are unitary. Notice that $(I_n \otimes A)(B^\top \otimes I_n) = (B^\top \otimes I_n)(I_n \otimes A)$

(b) Now let $W = V \times U$ so that way $W^(I_n \otimes A) * W$ and $W^*(B^\top \otimes I_n)W$ are composed of blocks of Δ_A and $\Delta_B^\top$ respectively.

(c) Then $W^(I_n \otimes A - B^\top \otimes I_n) * W$ has blocks of $\Delta_A - \Delta_B^\top$. Since eigenvalues are preserved during the triangularization process, the eigenvalues of this transformation are precisely $a - b$ where a and b are eigenvalues of A and B respectively.

(d) If $A = B$, then for each eigenvalue λ of A, $\lambda - \lambda = 0$ is an eigenvalue of the transformation. There are are least n such corresponding eigenvectors or generalized eigenvectors, so $\dim \ker(T) \geq n$.

$\square$