Complex Analysis: Homework 5

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Problem 6. (page 108)
Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve in Ω.

Proof.
Because $|f(z) - 1| < 1$, $w = f(z)$ stays strictly in the right half plane for $z \in \Omega$. Therefore the principal branch of log is analytic for $z \in \Omega$, and so $f'(z)/f(z)$ is the derivative of an analytic function in Ω and thus the integral only depends on its endpoints. Since γ is a closed curve, the integral must vanish. $\square$

Problem 7. (page 108)
If $P(z)$ is a polynomial and C denotes the circle $|z-a|=R$, what is the value of of $\int_C P(z)d\bar{z}$?
Answer: $-2\pi i R^2 P'(a)$.

Proof. Parameterize the circle by $z(t) = Re^{it} + a$ for $\theta \in [0, 2\pi]$, then $\bar{z}(t) = \overline{Re^{it} + a} = Re^{-it} + \bar{a}$ and $d\bar{z} = -iRe^{-it}dt$.

$$\int_C P(z)d\bar{z} = -iR\int_0^{2\pi} P(Re^{it}+a)e^{-it}dt$$

Notice that the (finite) Taylor series expansion of P around a is

$$P(z) = P(a) + P'(a)(z-a) + \ldots + \frac{P^{(n)}(a)}{n!}(z-a)^n.$$

So the above integral becomes

$$\begin{aligned}
\int_C P(z)d\bar{z} &= -iR\int_0^{2\pi} e^{-it}\left(P(a) + P'(a)(Re^{it}) + \ldots + \frac{P^{(n)}(a)}{n!}(R^n e^{nit})\right)dt \\
&= -iR\int_0^{2\pi} P(a)e^{-it} + RP'(a) + \frac{P''(a)}{2}(Re^{it})\ldots + \frac{P^{(n)}(a)}{n!}(R^n e^{(n-1)it})dt \\
&= -iR\int_0^{2\pi} RP'(a)dt \\
&= -2\pi i R^2 P'(a)
\end{aligned}$$

because all terms except for the first derivative term vanish due to symmetry, (i.e. $c\int_0^{2\pi} e^{nit}dt = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.) $\square$

Problem 1. (page 120)
Compute

$$\int_{|z|=1} \frac{e^z}{z} dz.$$

Proof. Because e^z is analytic in $\mathbb{C}$, and $\gamma = \{|z| = 1\}$ is a closed curve in the disk of radius 2 centered at the origin, Theorem 6 gives

$$\int_\gamma \frac{e^z}{z}\, dz = 2\pi i \cdot n(\gamma, 0) \cdot e^0.$$

The winding number of γ around the origin is 1 and $e^0 = 1$, so

$$\int_\gamma \frac{e^z}{z}\, dz = 2\pi i.$$

. $\square$

Problem 3. (page 120)
Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition that $|a| \neq \rho$.

Proof. By the hint, we can rewrite the integral as

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z|z-a|^2} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\overline{z-a})} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\bar{z}-\bar{a})} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\rho^2/z-\bar{a})} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\bar{a}z)}
\end{aligned}$$

Now using partial fraction decomposition,

$$\frac{1}{(z-a)(\rho^2-\bar{a}z)} = \frac{A}{(z-a)} + \frac{B}{(\rho^2-\bar{a}z)}.$$

By the system of equations

$$\begin{aligned}
-A\bar{a} + B &= 0 \\
A\rho^2 - aB &= 1
\end{aligned}$$

yields $A = (\rho^2 - |a|^2)^{-1}$ and $B = \bar{a}(\rho^2 - |a|^2)^{-1}$. Therefore

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{1}{\rho^2-|a|^2} \cdot \frac{1}{(z-a)} + \frac{1}{\rho^2-|a|^2} \cdot \frac{\bar{a}}{(\rho^2-\bar{a}z)} dz \\
&= -\frac{i\rho}{\rho^2-|a|^2} \left(\int_{|z|=\rho} \frac{dz}{(z-a)} + \int_{|z|=\rho} \frac{\bar{a}}{(\rho^2-\bar{a}z)} dz \right)
\end{aligned}$$

Finally there are two cases:

1. $|a| < \rho$ and $\rho^2 a/|a|^2 > \rho$. The first integral has a singularity, so the second integral vanishes. Therefore

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{2\pi\rho}{\rho^2-|a|^2}.$$

2. $|a| > \rho$ and $\rho^2 a/|a|^2 < \rho$. The second integral has a singularity, so the first integral vanishes. Therefore

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{2\pi\rho}{\rho^2-|a|^2}.$$

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In either case, the integrals are equal.

□

Problem 2. (page 123)
Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Proof. Because $|f(z)| < |z|^n$, $0 \leq |f(z)/z^n| < 1$ for sufficiently large $|z|$, by Theorem 8,

$$\begin{aligned} f(z) &= f_{n+1}(z) \cdot (z)^{n+1} + \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(z)^k \\ 1 &> \left|\frac{f(z)}{z^n}\right| \\ &= \left| f_{n+1}(z) \cdot (z) + f^{(n)}(0) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}(z)^{k-n} \right| \end{aligned}$$

By taking large $|z|$, the absolute value of the sum can be made arbitrarily small. Therefore in order to stay bounded, $f_{n+1}(z) = 0$ for all z, so

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(z)^k$$

which is a polynomial. □

Problem 3. (page 123)
If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

Proof. By Equation (24),

$$\begin{aligned}
|f^{(n)}(z)| &= \frac{n!}{2\pi} \left| \int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| &(1)\\
&\leq \frac{n!}{2\pi} \int_{|\zeta|=R} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| &(2)\\
&\leq \frac{n!M}{2\pi} \int_{|\zeta|=R} \frac{|d\zeta|}{|\zeta - z|^{n+1}} &(3)\\
&\leq \frac{n!M}{2\pi \cdot (R-\rho)^{n+1}} \int_{|\zeta|=R} |d\zeta| &(4)\\
&= \frac{n! 2\pi r M}{2\pi \cdot (R-\rho)^{n+1}} &(5)\\
&= \frac{n!RM}{(R-\rho)^{n+1}}. &(6)
\end{aligned}$$

(2) is justified by Equation (9) in Ahlfors; (3) because $|f(\zeta)| \leq M$, by hypothesis; (4) because $|\zeta - z| \geq (R-\rho)$ for $|z| \leq \rho$; (5) by the arc length of the circle of radius R; and (6) by simplification. □

Problem 4. (page 123)
If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \le 1/(1-|z|)$, find the best estimate of $|f^{(n)}(z)|$ that Cauchy's inequality will yield.

Proof. Again using equation (24) and letting C be the circle of radius $0 < \rho < 1$ about the origin,

$$\begin{aligned}
|f^{(n)}(0)| &= \frac{n!}{2\pi}\left|\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| &(1)\\
&= \frac{n!}{2\pi}\left|\int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| &(2)\\
&\le \frac{n!}{2\pi}\int_{|\zeta|=\rho} \frac{|f(\zeta)|}{|\zeta|^{n+1}}|d\zeta| &(3)\\
&= \frac{n!}{2\pi\rho^{n+1}}\int_{|\zeta|=\rho} |f(\zeta)||d\zeta| &(4)\\
&\le \frac{n!}{2\pi\rho^{n+1}}\int_{|\zeta|=\rho} \frac{|d\zeta|}{1-|\zeta|} &(5)\\
&= \frac{n!}{2\pi\rho^{n+1}(1-\rho)}\int_{|\zeta|=\rho} |d\zeta| &(6)\\
&= \frac{n!2\pi\rho}{2\pi\rho^{n+1}(1-\rho)} &(7)\\
&= \frac{n!}{\rho^{n}(1-\rho)} &(8)
\end{aligned}$$

This last expression is minimized when $\rho^n(1-\rho)$ is maximized

$$\frac{d}{d\rho}[\rho^n(1-\rho)] = n\rho^{n-1} - (n+1)\rho^n = 0,$$

that is, $\rho = n/(n+1)$. In this case

$$\frac{n!}{\rho^n(1-\rho)} = (n+1)!\left(\frac{n+1}{n}\right)^n.$$

Therefore the best estimate of $|f^{(n)}(z)|$ is

$$|f^{(n)}(z)| \le (n+1)!\left(\frac{n+1}{n}\right)^n.$$

□