Fall 2014: Complex Analysis Graduate Exam

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Problem 1. Let $a > 1$. Compute
$$\int_0^\pi \frac{d\theta}{a + \cos\theta}$$
being careful to justify your methods.

Proof. First, call this integral S, and begin with the standard trigonometic substitution,
$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}),$$
yielding
$$S = \int_0^\pi \frac{d\theta}{a + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}.$$
By exploiting the evenness of $a + \cos(\theta)$, this integral is equal to
$$S = \frac{1}{2}\int_{-\pi}^\pi \frac{d\theta}{a + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}.$$
Then by substituting $z = e^{i\theta}$ where the contour is the unit circle centered at the origin gives
$$S = \frac{1}{2}\int_{|z|=1} \frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$
where $dz/(iz)$ is the formal substitution for $d\theta$ because
$$\begin{aligned} e^{i\theta} &= z \\ i\theta &= \log z \\ d\theta &= -i\frac{dz}{z}. \end{aligned}$$
Some simplification of the integral results in
$$S = -i\int_{|z|=1} \frac{dz}{2az + (z^2+1)}.$$
By the quadratic formula, this integrand has poles at
$$\begin{aligned} \frac{-2a \pm \sqrt{4a^2-4}}{2} &= -a \pm \sqrt{a^2-1} \\ \alpha &= -a - \sqrt{a^2-1} \\ \beta &= -a + \sqrt{a^2-1} \end{aligned}$$

which are real because $a > 1$ by hypothesis. In particular, $\alpha = -a - \sqrt{a^2-1} < -a$, so clearly outside the contour. On the other hand,

$$\begin{aligned} a^2 &> a^2 - 1 && > a^2 - 2a + 1 \\ a &> \sqrt{a^2-1} && > a - 1 \\ 0 &> \underbrace{-a + \sqrt{a^2-1}}_{\beta} && > -1 \end{aligned}$$

so β is inside the contour.

Next, naming the integrand f, the residue theorem gives

$$S = -i \int_{|z|=1} \frac{dz}{2az + (z^2+1)} = -i(2\pi i \operatorname{Res}_\beta(f)) = 2\pi \operatorname{Res}_\beta(f).$$

Now, the residue is straightforward to compute:

$$\operatorname{Res}_\beta(f) = \lim_{z\to\beta} (z-\beta) \frac{1}{(z-\beta)(z-\alpha)} = \frac{1}{\beta - \alpha} = \frac{1}{2\sqrt{a^2-1}}.$$

Therefore

$$\int_0^\pi \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

$\square$

Problem 2. Find the number of zeros, counting multiplicity, of $z^8 - z^3 + 10$ inside the first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$.

Proof. For convenience name the aforementioned function: $f(z) = z^8 - z^3 + 10$. We'll repeatedly compare this function against $g(z) = z^8 + 10$.
Notice that anywhere on the circle $|z| = 2$, we have the inequality

$$|f - g| = |-z^3| = 8 < |z^8| - |z^3| - |10| = 256 - 8 - 10 < |f|,$$

which follows by the triangle inequality. By Rouché's Theorem, since g has all eight of its roots inside this circle, f also has all eight of its roots inside this circle. So when counting the roots of f inside the first quadrant, it is sufficient to count the roots of f inside the quarter circle of radius 2 in the first quadrant. Now, we will establish that $|f - g| < |f|$ on the boundary of this region.

Case 1: The arc.
We have already established that $|f - g| < |f|$ when $|z| = 2$, and this remains true when we restrict to the first quadrant.

Case 2: The real part.
We need to show that $|-x^3| < |10 + x^8 - x^3|$ for $x \in [0, 2]$, and because the right hand side is positive, this is equivalent to showing that $10 + x^8 - 2x^3 > 0$. This follows because

(a) $10 - 2x^3 > 0$ and $x^8 > 0$ for $x \in [0, \sqrt[3]{5})$, and

(b) $x^8 - 2x^3 > 0$ and $10 > 0$ for $x \in [\sqrt[3]{5}, 2]$.

Case 3: The purely imaginary part. We need to show that $|-i^3x^3| < |10 + i^8x^8 - i^3x^3|$ for $x \in [0, 2]$:

$$|f - g| = |-i^3x^3| = x^3 < \underbrace{|10 + x^8| - |i^3x^3|}_{10+x^8-x^3} \leq |f|,$$

and this follows by exactly the same argument as the second case.

Thus, by Rouché's theorem, f and g have the same number of zeros in the quarter circle (equivalently, the first quadrant). Since g has roots at $10^{1/8}e^{\pi i/8}\xi_8^k$ (where ξ_8 is an eigth root of unity), g has exactly two roots in each quadrant.

Therefore f has two roots in the first quadrant. $\square$

Problem 3. Assume that f and g are holomorphic in a punctured neighborhood of $z_0 \in \mathbb{C}$. Prove that if f has an essential singularity at z_0 and g has a pole at z_0, then $f(z)g(z)$ has an essential singularity at z_0.

Proof. $\square$

Problem 4.

(i) Suppose that f is holomorphic on $\mathbb{C}$, and assume that the imaginary part of f is bounded. Prove that f is constant.

(ii) Suppose that f and g are holomorphic on $\mathbb{C}$ and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists $\lambda \in \mathbb{C}$ such that $f = \lambda g$.

Proof. $\square$