Math 510b: Homework 2

Peter Kagey

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Problem 1 (Artin). Prove that the ideal $I = (x + y^2, y + x^2 + 2xy^2 + y^4)$ in $\mathbb{C}[x, y]$ is a maximal ideal.

Proof. It is sufficient to show that $R = \mathbb{C}[x, y]/I$ is a field. Notice that in R, $x + y^2 = 0$, so $x = -y^2$. Substituting into the second polynomial gives

$$y + (-y^2)^2 + 2(-y^2)y^2 + y^4 = y + y^4 + -2y^4 + y^4 = y = 0,$$

so $y = 0$ in R. Thus $x = y = 0$ and $\mathbb{C}[x, y]/I \cong \mathbb{C}$, a field, so I is maximal. □

Problem 2 (Artin). Let I be the principal ideal of $\mathbb{C}[x, y]$ generated by the polynomial $y^2 + x^3 - 17$. Which of the following sets generate maximal ideals in the quotient ring $R = \mathbb{C}[x, y]/I$?

(a) $(x - 1, y - 4)$

(b) $(x + 1, y + 4)$

(c) $(x^3 - 17, y^2)$

Proof.

1. In this ideal, we have that $\bar{x} = \bar{1}$ and $\bar{y} = \bar{4}$. In I we have that $\bar{y}^2 + \bar{x}^3 = 17$. So $R/(x - 1, y - 4) \cong \mathbb{C}$, a field. Thus $(x - 1, y - 4)$ is a maximal ideal.
2. In this ideal, we have that $\bar{x} = -\bar{1}$ and $\bar{y} = -\bar{4}$. In I we have that $\bar{y}^2 + \bar{x}^3 = 17$, which is not satisfied by the above substitution. Thus $R/(x + 1, y + 4)$ is not a field, and $(x - 1, y - 4)$ is not a maximal ideal.
3. Notice that $I \subset (x^3 - 17, y^2)$ since $y^2 + x^3 - 17 = (x^3 - 17) + (y^2)$, and so $\mathbb{C}[x, y]/I \cong \mathbb{C}[x, y]/(x^3 - 17, y^2)$. This is not a field because $y^2 = 0$, but $y \neq 0$, and fields do not have zero divisors.

□

Problem 6 (Artin). Prove that the kernel of the homomorphism $f\colon \mathbb{Z}[x] \to \mathbb{R}$ sending $x \mapsto 1+\sqrt{2}$ is a principal ideal, and find a generator for this ideal.

Proof. Notice that the polynomial $x^2 - 2x - 1 \in \mathbb{Z}[x]$ has roots $1 \pm \sqrt{2}$, so is irreducible in $\mathbb{Z}[x]$. Notice that $x - (1+\sqrt{2})$ is a factor of every polynomial $f \in \ker f \subset \mathbb{Z}[x] \subset \mathbb{R}[x]$ via the natural inclusion map, so $x^2 - 2x - 1$ must be a factor of every polynomial in the kernel, when looking at coefficients in $\mathbb{Z}$. Thus $\ker f = (x^2 - 2x - 1)$. $\square$

Problem 7 (Artin). Let f be an irreducible polynomial in $\mathbb{C}[x, y]$, and let g be another polynomial. Prove that if the variety of zeros of g in $\mathbb{C}^2$ contains the variety of zeros of f, then f divides g.

Proof.
We know that for any two nonzero polynomials $f, g \in \mathbb{C}[x, y]$, $V(f, g)$ only differs from $V(\gcd(f, g))$ by a finite set. Therefore the variety

$$V\left(\frac{f}{\gcd(f,g)}, \frac{g}{\gcd(f,g)}\right)$$

is empty. Thus $\frac{f}{\gcd(f,g)}$ is constant and $\gcd(f, g) = f$ (up to unit) and $f \mid g$. $\square$

Problem 8 (Artin). Determine the points of intersection of the two complex plane curves in each of the following

(a) $y^2 - x^3 + x^2 = 1, x + y = 1$

(b) $x^2 + xy + y^2 = 1, x^2 + 2y^2 = 1$

(c) $y^2 = x^3, xy = 1$

(d) $x + y + y^2 = 0, x - y + y^2 = 0$

Proof.

(a) Notice that $y = 1 - x$ by the second equation, so substituting this into the first yields

$$\begin{aligned} -x^3 + x^2 - 2x + 1 &= 1 \\ x(-x^2 + x - 2) &= 0 \end{aligned}$$

So $x = 0$ or $x = \frac{1}{2}(1 \pm i\sqrt{7})$, and so the solutions are

$$(0,1), \left(\frac{1+i\sqrt{7}}{2}, \frac{1-i\sqrt{7}}{2}\right), \text{ and } \left(\frac{1-i\sqrt{7}}{2}, \frac{1+i\sqrt{7}}{2}\right).$$

(b) Subtracting the second equation from the first gives

$$y(y - x) = 0$$

so either $y = 0$ or $x = y$. If $y = 0$, $x = \pm 1$. If $x = y$, $3x^2 = 1$ so $x = y = \pm\sqrt{1/3}$.

(c) By the second equation, $x = 1/y$, so the first yields $y^2 = 1/y^3$. Multiplying both sides by y^3 gives $y^5 = 1$. Thus $y = \exp(\frac{2}{5}\pi ki)$ for some $k \in \{1, 2, 3, 4\}$ (i.e. y is a fifth root of unity). To satisfy the second equation, $x = \exp(-\frac{2}{5}\pi ki)$, the complex conjugate. Thus any k such that there exists $j, n \in \mathbb{N}$ satisfying

$$\frac{4}{5}\pi ki = -\frac{6}{5}j\pi i + 2\pi n.$$

Taking $k = j = n$ always works, so all fifth roots of unity are points of intersection.

(d) Subtracting the second equation from the first yields $2y = 0$, so $y = 0$. Then $x + 0 + 0^2 = 0$, so $x = 0$. The point $(0, 0)$ satisfies both equations, and furthermore is the only point of intersection.

$\square$

Problem 9 (Artin). Prove that two quadratic polynomials f, g in two variables have at most four common zeros unless they have a non-constant factor in common.

Proof. We know that

1. $V(f, g)$ is a finite set since f, g are relatively prime, and
2. specializing y results in polynomials with at most two roots by the fundamental theorem of algebra.

If f, g have fewer than four common zeros, we're done, so assume that they have at least four common zeros. Since no three can fall on the same line (otherwise we could subtract off by the line, which could contradict 2.) we know that the points must noncolinear. Thus there can be at most four common zeros. □

Problem 10 (Artin). An algebraic curve $\mathcal{C}$ in $\mathbb{C}^2$ is called irreducible if it is the locus of zeros of an irreducible polynomial $f(x, y)$—one which cannot be factored as a product of nonconstant polynomials. A point $p \in \mathcal{C}$ is called a singular point of the curve if $\partial f/\partial x = \partial f/\partial y = 0$ at p. Otherwise p is a nonsingular point. Prove that an irreducible curve has only finitely many singular points.

Proof. It is sufficient to show that the variety $V(\partial f/\partial x, \partial f/\partial y)$ is finite, which by the Nullstellensatz means that it's sufficient to show that $\mathbb{C}[x, y]/(\partial f/\partial x, \partial f/\partial y)$ is finite. $\square$

Extra Problem. Let $R = \mathbb{Z}(\sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Define $N\colon R \to \mathbb{Z}_{\geq 0}$ by sending $a + b\sqrt{-5} \mapsto a^2 + 5b^2$.

Show:

(a) $N(xy) = N(x)N(y)$ for all $x, y \in R$.

(b) If x is a unit in R then $N(x) = 1$. Thus the only units in R are ± 1.

(c) There does not exist $x \in R$ with $N(x) = 3$.

(d) If $N(x) = 9$ then x is irreducible in R.

(e) Note that $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$, and conclude that 3 is irreducible in R but not prime.

(f) Factorization into irreducible elements in R is not unique.

(g) Comparing this example to $\mathbb{Z}[i]$, what goes wrong here that works for $\mathbb{Z}[i]$?

(h) Find an ideal in R which is not principal.

Proof. (a) On the right hand side:

$$\begin{aligned} N((a_1 + a_2\sqrt{-5})(b_1 + b_2\sqrt{-5})) &= N(a_1b_1 + a_1b_2\sqrt{-5} + a_2b_1\sqrt{-5} + a_2b_2\sqrt{-5}^2) \\ &= N(a_1b_1 + 5a_2b_2 + (a_1b_2 + a_2b_1)\sqrt{-5}) \\ &= (a_1b_1 - 5a_2b_2)^2 + 5(a_1b_2 + a_2b_1)^2 \\ &= a_1^2b_1^2 - 10a_1a_2b_1b_2 + 25a_2^2b_2^2 + 5a_1^2b_2^2 + 10a_1a_2b_1b_2 + 5a_2^2b_1^2 \\ &= a_1^2b_1^2 + 5a_1^2b_2^2 + 5a_2^2b_1^2 + 25a_2^2b_2^2 \end{aligned}$$

On the left hand side:

$$\begin{aligned} N(a_1 + a_2\sqrt{-5})N(b_1 + b_2\sqrt{-5}) &= (a_1^2 + 5a_2^2)(b_1^2 + 5b_2^2) \\ &= a_1^2b_1^2 + 5a_1^2b_2^2 + 5a_2^2b_1^2 + 25a_2^2b_2^2 \end{aligned}$$

(b) $N(r) \in \mathbb{Z}_0^+$ for $r \in R$, so if there exists some $r^{-1} \in R$ such that $r(r^{-1}) = 1$, then $N(r)N(r^{-1}) = N(1) = 1$, so $N(r) = N(r^{-1}) = 1$.

(c) If $b = 0$, then $N(a) = a^2 = 3$ does not have a solution in the integers. If $b > 0$, then $N(a + b\sqrt{-5}) \geq 5$. Thus there is no $r \in R$ with $N(r) = 3$.

(d) Suppose for the sake of contradiction that $x = rs$, with r, s non-unit. Since 9 only has factors of $1, 3, 9$, and $N(x) = N(r)N(s)$ this implies one of the following contradictions

   (i) $N(r) = 1$, in which case r is a unit,

   (ii) $N(r) = 3$, a contradiction by (c), or

   (iii) $N(r) = 9$, in which case s is a unit.

   Thus if $N(x) = 9$, x is irreducible.

(e) Since $N(3) = 9$, 3 is irreducible by part (d). However, 3 is not prime because $3 \nmid (2 \pm \sqrt{-5})$.

(f) $N(2 + \sqrt{-5}) = N(2 - \sqrt{-5}) = 2^2 + 5(1^2) = 9$, so 9 factors into irreducibles in two different ways.

(g) In the case of $\mathbb{Z}[i]$, all irreducible elements are prime, so (e) fails.

(h) Consider the ideal $I = (3, 2 + \sqrt{-5})$. Notice that if I is principal with generator g, then $N(3) = 9 = N(rg) = N(r)N(g)$.

(i) If $N(g) = 9$, then r must be a unit, and so $r = \pm 1$ and $g = \pm 3$. However, this contradicts $2 + \sqrt{-5} \in I$.

(ii) If $N(g) = 3$, this contradicts (c).

(iii) If $N(g) = 1$, then $g = \pm 1$, so $I = R$, and 1 can be expressed as

$$\begin{aligned} 1 &= 3(a_1 + a_2\sqrt{-5}) + (2 + \sqrt{-5})(b_1 + b_2\sqrt{-5}) \\ &= 3a_1 + 2b_1 - 5b_2 + (3a_2 + b_1 + 2b_2)\sqrt{-5} \end{aligned}$$

Then looking modulo 3, this means that $2b_1 + b_2 \cong 1 \bmod 3$ and $b_1 + 2b_2 \cong 1 \bmod 3$. Adding this together yields $0 \cong 1 \bmod 3$, a contradiction.

Therefore I cannot be principal. $\square$