Combinatorics: Homework 8

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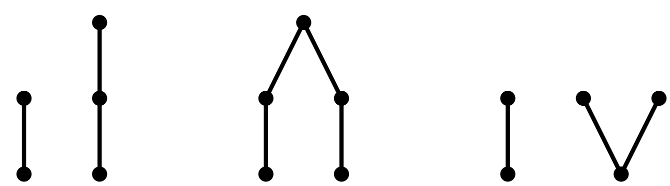
Problem 34. [2]
Find all nonisomorphic posets P such that

$$F(J(P), x) = (1 + x)(1 + x^2)(1 + x + x^2)$$

Solution. $F(J(P), x) = 1 + 2x + 3x^3 + 3x^4 + 2x^5 + x^6$, so $J(P)$ has rank 5 so P has five elements, call them $\{1, 2, 3, 4, 5\}$

Since $J(P)$ has two elements of rank 1, P must have two minimal elements 1 and 2 without loss of generality, one must be connected to one of the remaining elements, say $1 \lessdot 3$. If also $2 \lessdot 3$, then $J(P)$ would have only one ideal with two elements, namely $\langle 1, 2 \rangle$, not the required two.
Therefore again without loss of generality, $2 \not\lessdot 3$ and $2 \lessdot 4$. Since this is symmetric, there are only three possible cases.



In the first case, $J(P) = [3] \times [4]$, so this works.
In the second case, $J(P)$ has only two ideals with three elements, so this doesn't work.
In the third case, $J(P)$, has four ideals of with two elements, so this doesn't work.

Thus $P = [2] + [3]$.

Problem 46 a. [2]
Let $f(n)$ be the number of sublattices of rank n of the boolean algebra B_n. Show that $f(n)$ is also the number of partial orders P on $[n]$.

Solution.
Since B_n is a distributive lattice, all rank n sublattices of B_n must also be distributive lattices. Thus each sublattice of $L \subset B_n$ can be written as $L \cong J(P)$ for some poset P, where P is isomorphic to the set of join irreducibles of L. By proposition 3.4.5, since L has rank n, P must have n elements. So every distributive lattices of rank n gives a partial ordering of $[n]$.

This a bijection, because given some poset P, we can recover L via J (namely, $L \cong J(P)$). Each distinct P gives a distinct rank n sublattice of B_n because (1) $J(P)$ is a distributive lattice, (2) $J(P)$ contains the ideal $[n]$, (and no larger ideal) and, (3) $J(P)$ includes the ideal $\emptyset$.

Problem 53. [2]
Let P be a finite n-element poset. Simplify the two sums

$$f(P) = \sum_{I \subset J(P)} e(I) e(\overline{I}),$$

$$g(P) = \sum_{I \subset J(P)} \binom{n}{\#I} e(I) e(\overline{I}),$$

where $\overline{I}$ denotes the complement $P - I$ of the order ideal I.

Proof. $\square$

Problem 57.

a. [2] Let P be an n-element poset. If $t \in P$, then set $\lambda_t = \#\{s \in P : s \leq t\}$. Show that

$$e(P) \geq \frac{n!}{\prod_{t\in P} \lambda_t}.$$

b. [2+] Show that the equality holds if and only if every component of P is a rooted tree.

Proof. a. By induction, the base case is clear. When $n = 1$, there is only one poset with one linear extension.

$$e([1]) = \frac{1!}{\lambda_1} \geq 1.$$

Thus given some poset P, we can take the subposet $P - m$ (where m is any maximal element in P) which is a disjoint union of posets $P_1 + P_2 + \ldots + P_k$ with $n_1, n_2, \ldots, n_k$ elements respectively. We have several choices of labels for m, but if we choose one of those arbitrarily, then we can choose the n_i labels for P_i and each can be permuted in $e(P_i)$ ways, so

$$\begin{aligned} e(P) &\geq \binom{n-1}{n_1, n_2, \ldots, n_k} e(P_1)e(P_2)\ldots e(P_k) \\ &\geq \frac{(n-1)!}{n_1!n_2!\ldots n_k!} \cdot \frac{n_1!}{\prod_{t\in P_1}\lambda_t} \cdot \frac{n_2!}{\prod_{t\in P_2}\lambda_t} \cdots \frac{n_k!}{\prod_{t\in P_k}\lambda_t} \\ &= \ldots \end{aligned}$$

(I was unable to finish this part, but I think the second part is right.)

b. By induction, the base case is clear. When $n = 1$, there is only one poset (which is a rooted tree) with one linear extension.

$$e([1]) = \frac{1!}{\lambda_1} = 1.$$

Thus given some rooted tree P, we can take the subposet $P - \hat{1}$, which is a disjoint union of rooted trees $P_1 + P_2 + \ldots + P_k$ with $n_1, n_2, \ldots, n_k$ elements respectively. Since P has a unique maximum, it must be labeled with n, then we can then choose which letters go in each sub-tree (using a multinomial coefficient), and then there are $e(P_i)$ ways to order the n_i labels for each P_i. Therefore

$$\begin{aligned} e(P) &= \binom{n-1}{n_1, n_2, \ldots, n_k} e(P_1)e(P_2)\ldots e(P_k) \\ &= \binom{n-1}{n_1, n_2, \ldots, n_k} \frac{n_1!}{\prod_{t\in P_1}\lambda_t} \frac{n_2!}{\prod_{t\in P_2}\lambda_t} \cdots \frac{n_k!}{\prod_{t\in P_k}\lambda_t} \\ &= \left(\frac{(n-1)!}{n_1!n_2!\ldots n_k!}\right) \frac{n_1!n_2!\ldots n_k!}{\prod_{t\in P-\hat{1}}\lambda_t} \\ &= \frac{(n-1)!}{\prod_{t\in P-\hat{1}}\lambda_t} \end{aligned}$$

Since all n elements of P are less than or equal to $\hat{1}$, $\lambda_{\hat{1}} = n$,

$$n \prod_{t\in P-\hat{1}} \lambda_t = \prod_{t\in P} \lambda_t$$

and thus

$$e(P) = \frac{n(n-1)!}{n\prod_{t\in P-\hat{1}}\lambda_t} = \frac{n!}{\prod_{t\in P}\lambda_t}$$

as desired.

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