Algebraic Combinatorics: Homework 2

Peter Kagey

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Problem 1. Let $f(n)$ denote the number of structures on $[n]$, and let $e(n)$ denote the number of such structure for which the number of components is even. Prove

$$E(x) = \frac{1}{2}\left(F(x) + \frac{1}{F(x)}\right)$$

Proof. Let the sub-structure be denoted by $g(n)$, that is

$$\begin{aligned} f(\#S) &= \sum_{\{B_1,\ldots,B_k\}\in\Pi(S)} g(B_1)\cdots g(B_k) \\ F(x) &= \exp(G(x)) \end{aligned}$$

so that the "even" structure is

$$\begin{aligned} e(\#S) &= \sum_{\{B_1,\ldots,B_k\}\in\Pi(S)} g(B_1)\cdots g(B_k)h(k) \\ E(x) &= H(G(x)) \end{aligned}$$

where

$$h(k) = \begin{cases} 1 & k \text{ is even} \\ 0 & \text{otherwise} \end{cases},$$

and thus

$$H(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh(x) = \frac{1}{2}\left(e^x - e^{-x}\right).$$

And the desired identity follows:

$$\begin{aligned} E(x) &= H(G(x)) \\ &= \frac{1}{2}\left(\underbrace{\exp(G(x))}_{F(x)} - \exp(G(x))^{-1}\right) \\ &= \frac{1}{2}\left(F(x) - F(x)^{-1}\right). \end{aligned}$$

□

Problem 2. Let $T(x)$ be the exponential generating function for threshold graphs, and let $S(x)$ be the exponential generating function for threshold graphs with no isolated vertex.

(a) Find the first four coefficients of $T(x)$ and $S(x)$.

Solution.

(a)

$$\begin{aligned} T(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{8}{3!}x^3 \\ S(x) &= 1 \qquad\; + \frac{1}{2!}x^2 + \frac{4}{3!}x^3 \end{aligned}$$

(b) Let $t_k(n)$ be the number of threshold graphs with exactly k isolated vertices. Then $t_k(n) = \binom{n}{k}s(n-k)$ by choosing which k vertices are isolated, and imposing any non-isolated vertex structure on the remaining $n-k$ vertices.

$$t(n) = \sum_{k=0}^{n} t_k(n) = \sum_{k=0}^{n} \binom{n}{k} s(n-k).$$

Notice that since the coefficients of e^x are all 1,

$$e^x S(x) = \sum_{n=0}^{\infty} \Bigg(\underbrace{\sum_{k=0}^{n} \binom{n}{k} s(n-k)}_{t(n)} \Bigg) \frac{x^n}{n!} = T(x).$$

(c) When $n = 0$, there is one threshold graph, and it does not have an isolated vertex.

When $n = 1$, there is one threshold graph, and it has an isolated vertex.

When $n \geq 2$, given a threshold graph with an isolated vertex v, its complement has no isolated vertex—in particular, every vertex in the complement is connected to v. Taking the complement gives a 1-1 correspondence between threshold graphs with an isolated vertex and threshold graph with no isolated vertex.

Therefore $T(x) = 2S(x) + x - 1$, where the $+x - 1$ corrects for the $n = 0$ and $n = 1$ cases.

(d) Using

$$\begin{aligned} 2S(x) + x - 1 &= T(x) = e^x S(x) \\ 2S(x) - e^x S(x) &= 1 - x \\ S(x) &= \frac{1-x}{2-e^x} \end{aligned}$$

Then using $T(x) = e^x S(x)$ gives

$$T(x) = e^x \frac{1-x}{2-e^x}.$$

Problem 3. Let G be a simple graph on $[n]$ with k connected components. Prove that G is a forest if and only if G has $n - k$ edges.

Solution.
Consider the connected components of G, which have $n_1, n_2, \dots, n_k$ vertices respectively. ($\Longrightarrow$) Assume that G is a forest with k connected components.
Each connected component is a tree, each of which has $n_i - 1$ edges. Then summing the edges gives the desired result:

$$\begin{aligned}(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) &= \underbrace{n_1 + n_2 + \ldots + n_k}_{n} \underbrace{-1 - 1 \ldots - 1}_{-k} \\ &= n - k\end{aligned}$$

($\Longleftarrow$) Assume that G has $n - k$ edges. Every connected graph on $[m]$ has at least at least $m - 1$ edges, so each connected components has $n_i + \ell_i$ edges where $\ell_i \geq -1$. Adding all of these up

$$\begin{aligned}(n_1 + \ell_1) + (n_2 + \ell_2) + \ldots + (n_k + \ell_k) &= n - k \\ \ell_1 + \ell_2 + \ldots + \ell_3 &= -k\end{aligned}$$

But since each $\ell_i \geq -1$, this means that $\ell_i = 1$ for all i. Thus the connected components are trees, and G is a forest.

Problem 4. Let $g(n,e)$ denote the number of connected, simple graphs on $[n]$ with e edges.

(a) Derive the mixed ordinary/exponential generating function

$$\sum_{n=1}^{\infty}\sum_{e} g(n,e)q^e\frac{x^n}{n!} = \log\sum_{n=0}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!}$$

(b) Use the formula to compute $\sum_e g(n,e)q^e$ for all $n \le 4$

(c) Verify by drawing all graphs in question.

Solution.

(a) First notice that

$$\sum_{n=0}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!} = 1 + x + (1+q)\frac{x^2}{2} + (1+q)^3\frac{x^3}{3!} + (1+q)^6\frac{x^4}{4!} + \dots$$

gives the number of (not necessarily connected) simple graphs, where q indexes the number of edges. (This is done by choosing or not choosing each of the $\binom{n}{2}$ edges on the labeled complete graph.) Then it is enough to exponentiate both sides to see that all simple graphs consist of connected components, so the usual $\exp(G(x)) = F(x)$ formula applies.

(b) Using the Taylor series for $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ yields

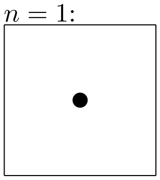
$$\begin{aligned}
\log\left(1+\sum_{n=1}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!}\right) &= \sum_{n=1}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!} - \frac{1}{2}\left(\sum_{n=1}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!}\right)^2 + \frac{1}{3}\left(\sum_{n=1}^{\infty}(1+q)^{\binom{n}{2}}\frac{x^n}{n!}\right)^3 - \dots \\
&= x + (1+q)\frac{x^2}{2} + (1+q)^3\frac{x^3}{3!} + (1+q)^6\frac{x^4}{4!} + O(x^5) \\
&\quad - \frac{1}{2}\left(x^2 + (1+q)x^3 + \frac{(1+q)^3}{3}x^4 + \frac{(1+q)^2}{4}x^4 + O(x^5)\right) \\
&\quad + \frac{1}{3}\left(x^3 + \frac{3}{2}(1+q)x^4 + O(x^5)\right) \\
&\quad - \frac{1}{4}\left(x^4 + O(x^5)\right).
\end{aligned}$$

Thus, simplifying, the first four terms (with respect to n) are

$$\sum_{n=1}^{4}\sum_{e} g(n,e)q^e\frac{x^n}{n!} = x + q\frac{x^2}{2!} + (3q^2+q^3)\frac{x^3}{3!} + (16q^3 + 15q^4 + 6q^5 + q^6)\frac{x^4}{4!}.$$

(c)

$n=1$:



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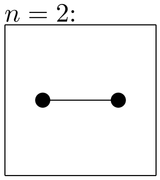
$n=2$:



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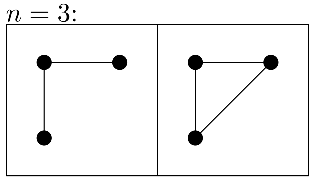
$n=3$:



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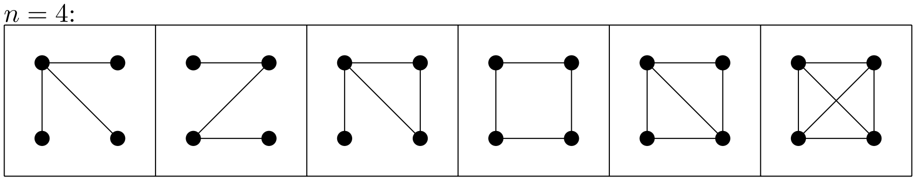
$n=4$:



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