# Complex Analysis: Homework 13

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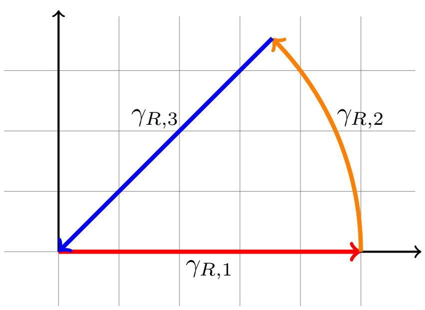
**Problem 3.** (page 206)
The formula (42) permits us to evaluate the *probability integral*

$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\int_0^\infty e^{-x}x^{-1/2}dx = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} \tag{1}$$

Use this result together with Cauchy's theorem to compute the *Fresnel integrals*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{1}{2}\sqrt{\pi/2} \tag{2}$$

*Proof.*
The plan is to use the construction from Wikipedia and integrate $f(z) = e^{-z^2}$ along the contour given by




$$\gamma_{R,1} = \{t + 0i \mid x \in [0, R]\} \tag{3}$$
$$\gamma_{R,2} = \{Re^{it} \mid t \in [0, \pi/4]\} \tag{4}$$
$$\gamma_{R,3} = \{te^{i\pi/4} \mid t \in [0, R]\}. \tag{5}$$

The first integral is known:

$$\lim_{R\to\infty}\int_{\gamma_{R,1}} f(z)\,dz = \int_0^\infty e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}. \tag{6}$$

The second integral vanishes in the limit,

$$\int_{\gamma_{R,2}} f(z)\,dz = iR\int_0^{\pi/4} \exp(-(Re^{it})^2)\cdot e^{it}\,dt. \tag{7}$$

So by looking at the modulus, we get

$$\left|\int_{\gamma_{R,2}} f(z)\,dz\right| \le R\int_0^{\pi/4} |\exp(-R^2e^{2it})|\cdot|e^{it}|\,dt \tag{8}$$
$$= R\int_0^{\pi/4} |e^{-R^2\cos(2t)}|\cdot\underbrace{|e^{-iR^2\sin(2t))}|}_{=1}\,dt. \tag{9}$$

Continuing now with the inequality $\cos(2t) \geq \pi/4 - t$ for $t \in [0, \pi/4]$,

$$\left| \int_{\gamma_{R,2}} f(z)\, dz \right| \leq R \int_0^{\pi/4} |e^{-R^2(\pi/4 - t)}|\, dt \tag{10}$$

$$= \frac{R}{e^{R^2\pi/4}} \int_0^{\pi/4} |e^{R^2 t}|\, dt \tag{11}$$

$$= \frac{R}{e^{R^2\pi/4}} \left[ \frac{e^{R^2 t}}{R^2} \right]_0^{\pi/4} \tag{12}$$

$$= \frac{R}{e^{R^2\pi/4}} \left[ \frac{e^{R^2\pi/4}}{R^2} - \frac{1}{R^2} \right] \tag{13}$$

$$= \frac{1}{R} - \frac{1}{Re^{R^2\pi/4}} \tag{14}$$

$$\leq \frac{1}{R}. \tag{15}$$

Thus

$$\lim_{R\to\infty} \int_{\gamma_{R,2}} f(z)\, dz = 0. \tag{16}$$

Next, the third integral:

$$\int_{\gamma_{R,3}} f(z)\, dz = \int_R^0 e^{i\pi/4} \exp(-t^2 \underbrace{e^{i\pi/2}}_{=i})\, dt \tag{17}$$

$$= e^{i\pi/4} \int_R^0 e^{-it^2}\, dt \tag{18}$$

$$= e^{i\pi/4} \int_R^0 \cos(-t^2) + i\sin(-t^2)\, dt \tag{19}$$

$$\tag{20}$$

Because $f$ is entire, it follows from Cauchy's theorem that

$$\int_{\gamma_{R,1}} f(z)\, dz + \int_{\gamma_{R,2}} f(z)\, dz + \int_{\gamma_{R,3}} f(z)\, dz = 0, \tag{21}$$

including in the limit, therefore

$$\lim_{R\to\infty} \left( - \int_{\gamma_{R,3}} f(z)\, dz \right) = e^{i\pi/4} \int_0^\infty \cos(-t^2) + i\sin(-t^2)\, dt \tag{22}$$

$$= \frac{1}{2}\sqrt{\pi}. \tag{23}$$

This means that

$$\int_0^\infty \cos(-t^2) + i\sin(-t^2)\, dt = \int_0^\infty \cos(t^2) - i\sin(t^2)\, dt \tag{24}$$

$$= \frac{\sqrt{\pi}}{2e^{i\pi/4}} \tag{25}$$

$$= \left( \frac{1}{2} - \frac{i}{2} \right) \sqrt{\frac{\pi}{2}} \tag{26}$$

so by looking at the real and purely imaginary parts it follows that

$$\int_0^\infty \cos(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \int_0^\infty \sin(t^2)\, dt. \tag{27}$$

$\square$

**Problem 2.** (page 212)
Assume that $f(z)$ has genus zero so that

$$f(z) = z^m \prod_n \left(1 - \frac{z}{a_n}\right). \tag{28}$$

Compare $f(z)$ with

$$g(z) = z^m \prod_n \left(1 - \frac{z}{|a_n|}\right) \tag{29}$$

and show that

$$\begin{aligned}
&\max_{|z|=r} |f(z)| \leq \max_{|z|=r} |g(z)|, \text{ and} \\
&\min_{|z|=r} |f(z)| \geq \min_{|z|=r} |g(z)|.
\end{aligned}$$

*Proof.*
The heuristic for why $g$ is more "extreme" is because all of the zeros lie on the same ray, the positive part of the real axis. In particular, by considering the ratio

$$\left|\frac{f(z_0)}{g(z_1)}\right| = \left|\frac{z_0^m \prod_n \left(1 - \frac{z_0}{a_n}\right)}{z_1^m \prod_n \left(1 - \frac{z_1}{|a_n|}\right)}\right| = \left|\frac{\prod_n (a_n - z_0)}{\prod_n (|a_n| - z_1)}\right| = \frac{\prod_n |a_n - z_0|}{\prod_n ||a_n| - z_1|}.$$

where $|z_0| = |z_1| = r$, it can be seen that it is enough to consider $\prod_n |a_n - z|$ and $\prod_n ||a_n| - z|$.

By the triangle inequality,

$$\begin{aligned}
&|a_n - z| \leq |a_n| + |z| = |a_n| + r \text{ and} \\
&|a_n - z| \geq |a_n| - |z| = |a_n| - r,
\end{aligned}$$

so in particular, a given linear factor $a_n - z$ is minimized at $z = r\frac{a_n}{|a_n|}$ and maximized at $z = -r\frac{a_n}{|a_n|}$:

$$\left|a_n - r\frac{a_n}{|a_n|}\right| = |a_n| \cdot \underbrace{\left|1 - \frac{r}{|a_n|}\right|}_{\text{real}} = |a_n|\left(1 - \frac{r}{|a_n|}\right) = |a_n| - r \leq |a_n - z| \tag{30}$$

$$\left|a_n + r\frac{a_n}{|a_n|}\right| = |a_n| \cdot \underbrace{\left|1 + \frac{r}{|a_n|}\right|}_{\text{real}} = |a_n|\left(1 + \frac{r}{|a_n|}\right) = |a_n| + r \geq |a_n - z| \tag{31}$$

Now, since $|a_n|$ is real for every $n$, $g$ is maximized at $z = -r$ and minimized at $z = r$, because that is where *every* linear term is maximized or minimized.

Now the result follows from the triangle inequality in the numerator

$$\begin{aligned}
\left|\frac{f(z_{f,\max})}{g(z_{f,\max})}\right| &= \frac{\prod_n |a_n - z_{f,\max}|}{\prod_n |a_n| + r} \leq \frac{\prod_n |a_n| + r}{\prod_n |a_n| + r} = 1 \\
\left|\frac{f(z_{f,\min})}{g(z_{f,\min})}\right| &= \frac{\prod_n |a_n - z_{f,\min}|}{\prod_n |a_n| + r} \geq \frac{\prod_n |a_n| + r}{\prod_n |a_n| + r} = 1.
\end{aligned}$$

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