Week 12 Quiz Solutions.

1. (*7 points*) Let E be the region of the cone $x^2 + y^2 = z^2$ that is between the xy-plane and the plane $z = 1$. The density of the solid is given by $f(x, y, z) = x^2 + y^2 + z^2$.

   (a) Set up **but do not evaluate** the integral to find the mass of E using cylindrical coordinates.

   (b) Set up **but do not evaluate** the integral to find the mass of E using spherical coordinates.

   Solution.

   (a) Substituting r^2 for $x^2 + y^2$ in the integrand gives

   $$\int\int\int_E f(x, y, z)\, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r}^{1} r^3 + rz^2 \, dz \, dr \, d\theta$$

   (b) We know that $z = \rho \sin(\phi) \leq 1$ so $\rho \leq \frac{1}{\sin \phi}$. Substituting ρ^2 for $x^2 + y^2 + z^2$ in the integrand gives:

   $$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{1/\sin\phi} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

2. (*3 points*) Find the tangent plane to the surface parameterized by

$$\begin{aligned} x &= u + v \\ y &= 2u \\ z &= uv - 1 \end{aligned}$$

at the point $(1, 2, -1)$.

Solution.
Write the parameterization as $\vec{r}(u, v) = \langle u + v, 2u, uv - 1\rangle$. Then the partial derivatives with respect to u and v are

$$\begin{aligned} \vec{r}_u(u, v) &= \langle 1, 2, v\rangle \\ \vec{r}_v(u, v) &= \langle 1, 0, u\rangle. \end{aligned}$$

In particular, the only solution to $\vec{r}(u, v) = \langle 1, 2, -1\rangle$ is when $u = 1$ and $v = 0$. The normal vector to the tangent plane is given by

$$r_u(1, 0) \times r_v(1, 0) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \langle 2, -1, -2\rangle.$$

Therefore the plane through $(1, 2, -1)$ is

$$2(x - 1) - (y - 2) - 2(z + 1) = 0.$$