Math 574: Homework 1

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Problem 1. Let V be a vector space over F of dimension n, with V_1 and V_2 subspaces of V.

(a) Prove that $V_1 + V_2$ is a subspace.

(b) Prove that $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

Proof.

(a) It is enough to show that $V_1 + V_2$ is closed under vector addition and scalar multiplication, because the other vector space properties are inherited from V. Closure with respect to addition follows from the commutativity of addition:

$$\underbrace{(v_1 + v_2)}_{V_1+V_2} + \underbrace{(v_1' + v_2')}_{V_1+V_2} = \underbrace{(v_1 + v_1')}_{V_1}\underbrace{(v_2 + v_2')}_{V_2},$$

and closure with respect to scalar multiplication follows from distributivity:

$$\underbrace{a(v_1 + v_2)}_{V_1+V_2} = \underbrace{av_1}_{V_1} + \underbrace{av_2}_{V_2}.$$

(b) The dimension of $V_1 + V_2$ is the cardinality of its basis. If V_1 has basis $\{u_i\}$, V_2 has basis $\{w_i\}$, and $V_1 \cap V_2$ has basis $\{v_i\}$ then each v_i can be written as both $v_i = \sum_k a_k u_k$ and $v_i = \sum_k b_k w_k$, where at least one a_k and at least one b_k are nonzero. Then solving for one such b_k in terms of the other a_is and b_is allows for b_k to be removed from the basis. This can be done inductively with each element the basis $\{v_i\}$ of $V_1 \cap V_2$ removing one element of the spanning set, ultimately resulting in a set of linearly independent vectors, so

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

as desired.

□

Problem 2.

(a) Prove that $V = V_1 \oplus V_2$ if and only if $n = n_1 + n_2$ and $V_1 \cap V_2 = 0$.

(b) Prove that $V = V_1 \oplus V_2$ if and only if each $v \in V$ can be written uniquely as $v_1 + v_2$.

Proof.

(a)

($\Longrightarrow$) Assume $V = V_1 \oplus V_2$. Then by definition, $V = V_1 + V_2$ and $V_1 \cap V_2 = 0$. So by Problem 1 (b),

$$\underbrace{\dim(V)}_{n} = \underbrace{\dim(V_1)}_{n_1} + \underbrace{\dim(V_2)}_{n_2} + \underbrace{\dim(V_1 \cap V_2)}_{0}.$$

($\Longleftarrow$) Assume $n = n_1 + n_2$ and $V_1 \cap V_2 = 0$. Then any basis of V_1 and any basis of V_2 are disjoint, call them $\{u_i\}$ and $\{w_i\}$ respectively, and their (disjoint) union forms a basis for V: $\{u_i\} \cup \{w_i\}$. Since any vector $v \in V$ can be written in terms of its basis elements,

$$v = \underbrace{\sum_i a_i u_i}_{V_1} + \underbrace{\sum_i b_i w_i}_{V_2}.$$

(b)

($\Longrightarrow$) Assume $V = V_1 \oplus V_2$. Then by definition, $V = V_1 + V_2$ so each v can be written as $v_1 + v_2$. Moreover, this sum is unique because if $v = v_1' + v_2'$, since $V_1 \cap V_2 = 0$, $v_1' - v_1 = 0$ and $v_2' - v_2 = 0$.

($\Longleftarrow$) Assume that each $v \in V$ can be uniquely written as $v_1 + v_2$, then $V \subseteq V_1 + V_2$ and $V_1 + V_2 \subseteq V$ (since they are subspaces). Next, arguing by contrapositive, suppose that there exists a nonzero $v \in V_1 \cap V_2$, then

$$\underbrace{(v_1 - u)}_{V_1} + \underbrace{(v_2 + u)}_{V_2} = v,$$

so the choice of vectors are not unique.

$\square$

Problem 3. Let $T\colon V \to V$ be a linear transformation with $\dim V = n$.

(a) Prove that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$ if and only if $V = T(V) \oplus \ker(T)$.

(b)

Proof. The rank of T is $\dim T(V)$.

(a)

($\Longrightarrow$) Assume $\operatorname{rank}(T) = \operatorname{rank}(T^2)$. Let $\{u_i\}$ be a basis for $T(V)$. By the hypothesis, $\{T(u_i)\}$ has the same dimension and is also a basis for $T(V)$. Since all basis elements of $\ker(T)$ are sent to 0 under T, $\ker(T) \cap T(V) = 0$ and moreover, every vector can be written as $v_1 + v_2$ with v_1 in the kernel and v_2 in the image.

($\Longleftarrow$) Assume $V = T(V) \oplus \ker(T)$. Let $\{u_i\}$ be a basis for the image of T and $\{w_i\}$ be a basis for the kernel of T, so that for each $v \in V$

$$v = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n-m} b_i w_i$$

then

$$T(v) = T\Big(\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n-m} b_i w_i\Big) = \sum a_i T(u_i) + \sum b_i \underbrace{T(w_i)}_{0}.$$

Since $T(V) \cap \ker(T) = 0$, $\sum a_i u_i$ is in $\ker(T)$ if and only if $a_1 = \ldots = a_m = 0$. Therefore $\{T(u_i)\}$ forms a basis for $T(V)$ and

$$\operatorname{rank}(T) = \dim\{u_1, \ldots, u_m\} = \dim\{T(u_1), \ldots, T(u_m)\} = \operatorname{rank}(T^2).$$

(b) If $T^2 = T$, then clearly $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, so $V = \ker(T) \oplus T(V)$. Furthermore, every element in the image is fixed under a successive transformation, so $T(V) = \{v \in V \mid T(v) = v\}$.

$\square$

Problem 4. Let V be the space of complex valued C^∞ functions on $\mathbb{R}$, and let $D\colon V \to V$ be differentiation.

(a) Find all eigenvalues and eigenvectors for D.

(b) Find $\ker(D^n)$.

Proof.

(a) This is equivalent to solving the differential equation $D(f(x)) = af(x)$. By the existence and uniqueness of ODEs, the only solution to this equation is $b\exp(ax)$, and $D(b\exp(ax)) = ab\exp(ax)$, which has eigenvalue a.

(b) Firstly, note that the only functions f such that $Df = 0$ are the constant functions, $f(x) = c_1$. By induction with the hypothesis that $\ker(D^n)$ consists of polynomials of degree $n-1$. Certainly $\ker(D^1)$ consists of 0-degree polynomials, so assuming that $\ker(D^{n-1}) = \{a_n x^{n-1} + a_{n-1}x^{n-2} + \ldots a_2 x + a_1\}$, and integrating,

$$\begin{aligned}\ker(D^n) &= \left\{ \frac{a_n}{n}x^n + \frac{a_{n-1}}{n-1}x^{n-1} + \ldots \frac{a_2}{2}x^2 + a_1 x + a_0 \,\middle|\, a_i \in \mathbb{C} \right\} \\ &= \left\{ b_n x^n + b_{n-1}x^{n-1} + \ldots b_2 x^2 + b_1 x + b_0 \,\middle|\, b_i \in \mathbb{C} \right\}\end{aligned}$$

gives the set of polynomials of degree n.

$\square$

Problem 5.

Proof.

(a)

($\Longrightarrow$) Arguing by contrapositive, assume that A_1 has a nonzero kernel (that is, A_1 is not invertible), and let $u \in \ker(A_1)$ be a nonzero vector. Then

$$(A_1 \otimes A_2)(u \otimes v) = (\underbrace{A_1(u)}_{0} \otimes A_2(v)) = 0,$$

so $A_1 \otimes A_2$ has a nonzero kernel and therefore is not invertible.
($\Longleftarrow$) Assume A_1 and A_2 are invertible. Then

$$\begin{aligned}(A_1^{-1} \otimes A_2^{-1})(A_1 \otimes A_2)(u \otimes v) &= (A_1^{-1} \otimes A_2^{-1})(A_1(u) \otimes A_2(v)) \\ &= (A_1^{-1}(A_1(u)) \otimes A_2^{-1}(A_2(v))) \\ &= (u, v),\end{aligned}$$

so $(A_1 \otimes A_2)^{-1} = (A_1^{-1} \otimes A_2^{-1})$.

(b) This follows almost directly by looking at the Kronecker product. Suppose that a_{ii} gives the ith entry on the main diagonal of A_1. Then reading down the main diagonal of the Knocker product gives

$$\begin{aligned}\operatorname{trace}(A_1 \otimes A_2) &= a_{11}\operatorname{trace}(A_2) + a_{22}\operatorname{trace}(A_2) + \ldots + a_{nn}\operatorname{trace}(A_2) \\ &= (a_{11} + a_{22} + \ldots + a_{nn})\operatorname{trace}(A_2) \\ &= \operatorname{trace}(A_1)\operatorname{trace}(A_2),\end{aligned}$$

as desired.

$\square$