# Counting structures on the $n \times k$ grid graph

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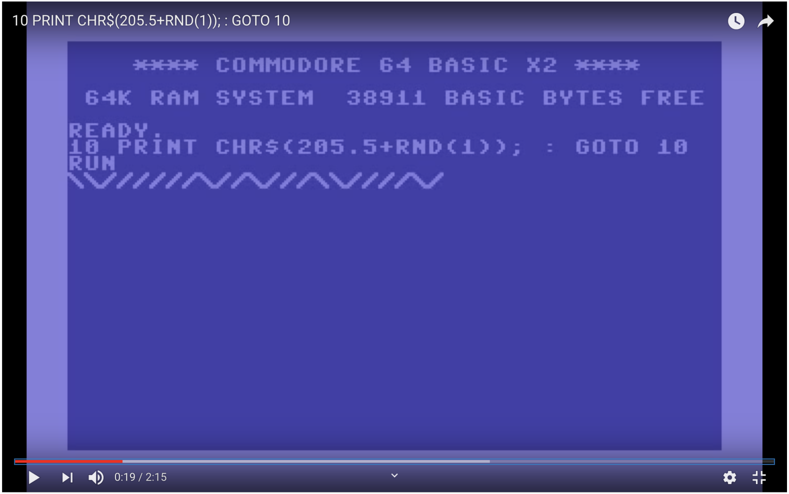
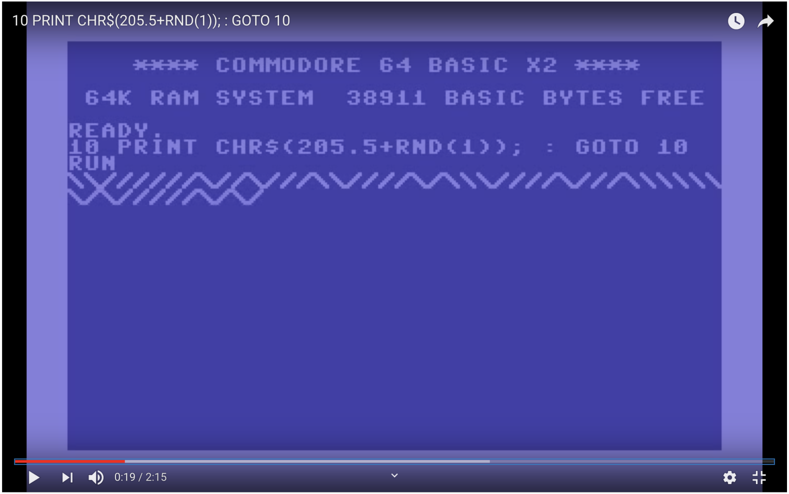
# Commodore 64 (1/3)

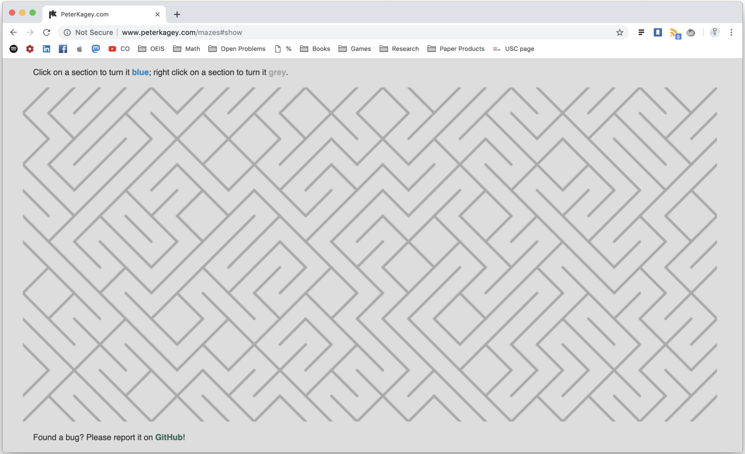


# Commodore 64 (2/3)



# Commodore 64 (3/3)

# Javascript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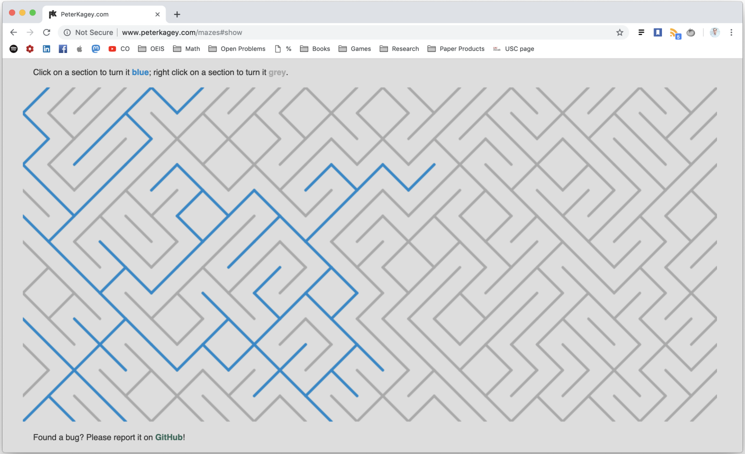
# Javascript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# Counting tilings of the grid

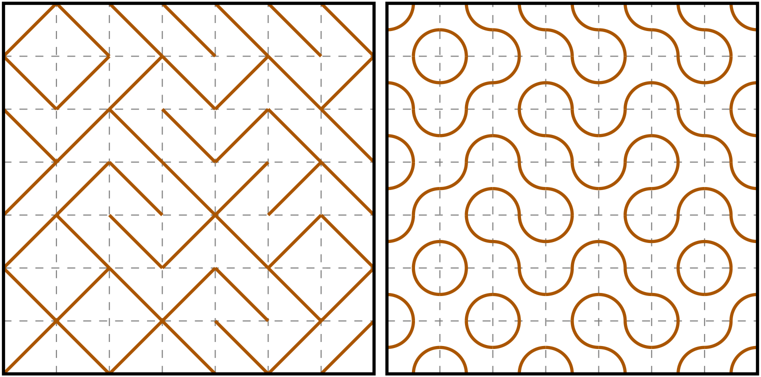


Figure 1: An illustration of the bijection between tiles with diagonal markings and tiles with quarter circles in opposite corners.

# A295229: Number of tilings of the $n \times n$ grid, using diagonal lines to connect the grid points.

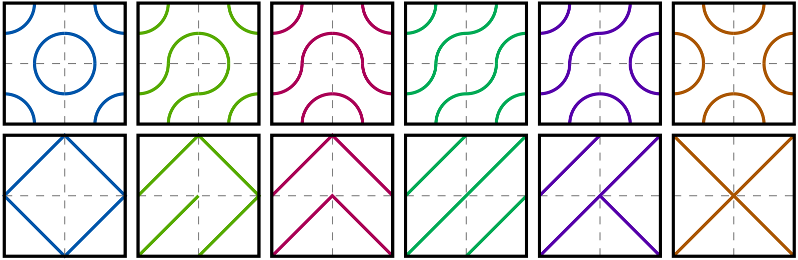


Figure 2: An example of the $a(2) = 6$ different ways to fill the $2 \times 2$ grid with diagonal tiles up to dihedral action of the square.

$$a(n) = \begin{cases} \frac{1}{8}\left(2^{n^2} + 2 \cdot 2^{n(n+1)/2} + 3 \cdot 2^{n^2/2} + 2 \cdot 2^{n^2/4}\right) & n \text{ even} \\ \frac{1}{8}\left(2^{n^2} + 2 \cdot 2^{n(n+1)/2} + 2^{(n^2+1)/2}\right) & n \text{ odd} \end{cases}$$

# Grids with other tiles

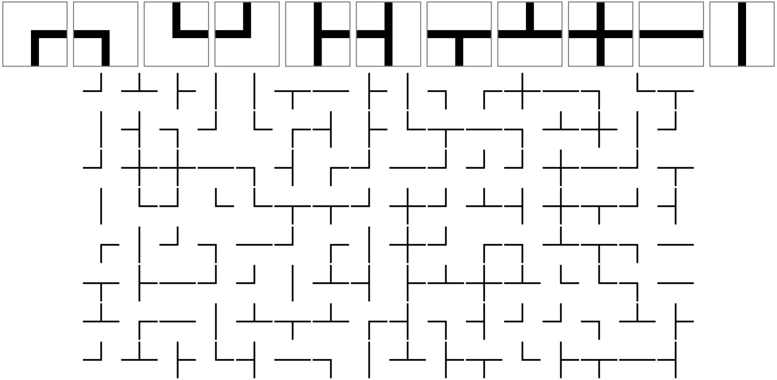


Figure 3: Eleven box-drawing characters placed on an $15 \times 8$ grid

# Baby's first corollary

## Corollary (of Burnside's Lemma)

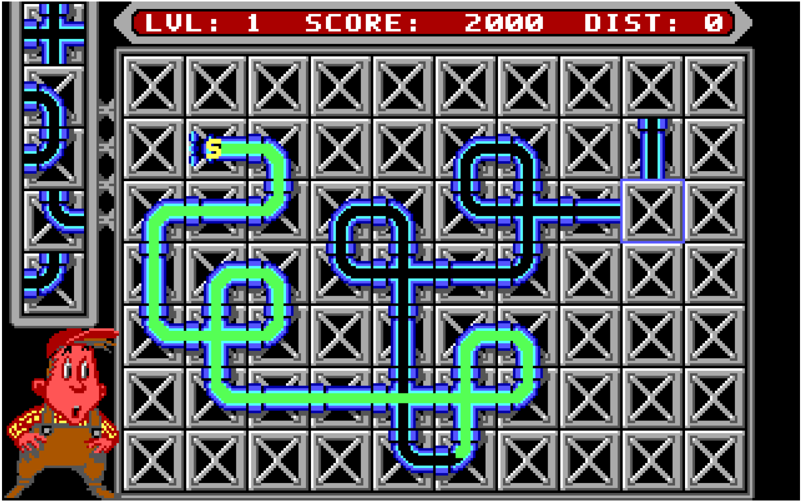
*Let*

- $t$ *be the number of tiles,*
- $q$ *be the number of tiles symmetric under a* $90^\circ$ *rotation,*
- $h$ *be the number of tiles symmetric under a* $180^\circ$ *rotation,*
- $d$ *be the number of tiles symmetric under a diagonal reflection, and*
- $v$ *be the number of tiles symmetric under a vertical reflection.*

*Then the number of tilings up to symmetries of the square is*

$$a(n) = \begin{cases} \frac{1}{8}\left(t^{n^2} + 2qt^{\frac{n^2-1}{4}} + ht^{\frac{n^2-1}{2}} + (v^n + d^n)t^{\frac{n^2-n}{2}}\right) & n \text{ odd} \\[2em] \frac{1}{8}\left(t^{n^2} + 3t^{\frac{n^2}{2}} + 2t^{\frac{n^2}{4}} + 2d^n t^{\frac{n^2-n}{2}}\right) & n \text{ even} \end{cases}$$

# Pipe Mania



# Leaf-free subgraphs of the grid graph

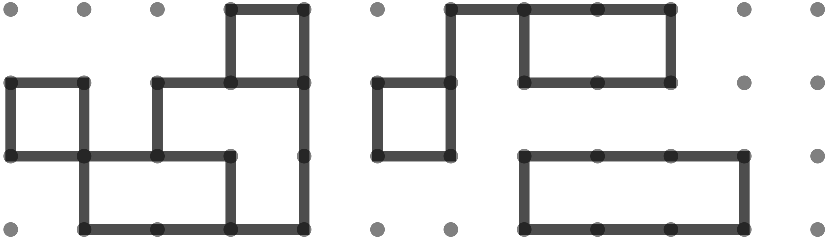


Figure 4: One of the $a_4(12) = 42650154782713601$ (42 quadrillion) subgraphs on the $12 \times 4$ grid graph $G_{12,4} = P_{12} \square P_4$.

The number of leaf-free subgraphs of $G_{n,2}$ grid, obeys the recurrence

$$a_2(1) = 1,\ a_2(2) = 2$$
$$a_2(n) = 5a(n-1) - 5a(n-2).$$

# Leaf-free subgraphs: intermediate states

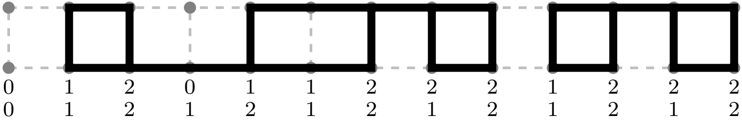



Figure 5: An example of a leaf-free subgraph with its states labeled

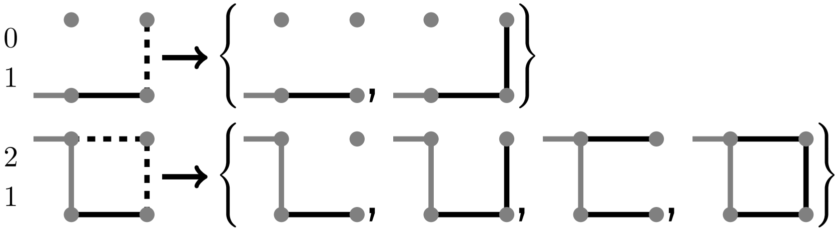



Figure 6: Two examples of transitions from states to their children

# Example: a system of linear difference equations

The states of leaf-free subgraphs on the $1 \times 2$ grid satisfy the initial conditions

$$\begin{aligned} a_{00}(1) &= a_{11}(1) = 1 \\ a_{10}(1) &= a_{01}(1) = a_{12}(1) = a_{21}(1) = a_{22}(1) = 0, \end{aligned}$$

and satisfy the system of first order homogeneous linear difference equations

$$\begin{aligned} a_{00}(n+1) &= a_{00}(n) + a_{22}(n) \\ a_{01}(n+1) &= a_{01}(n) + a_{21}(n) + a_{22}(n) \\ a_{10}(n+1) &= a_{10}(n) + a_{12}(n) + a_{22}(n) \\ a_{11}(n+1) &= a_{00}(n) + a_{11}(n) + a_{12}(n) + a_{21}(n) + 2a_{22}(n) \\ a_{12}(n+1) &= a_{01}(n) + a_{21}(n) + a_{22}(n) \\ a_{21}(n+1) &= a_{10}(n) + a_{12}(n) + a_{22}(n) \\ a_{22}(n+1) &= a_{11}(n) + a_{12}(n) + a_{21}(n) + a_{22}(n). \end{aligned}$$

# A single recurrence from a system of recurrences

Corollary (of Cayley–Hamilton theorem)

*In a system of first order homogeneous linear difference equations,*

$$\begin{aligned} a^{(1)}(n+1) &= \alpha_{11}a^{(1)}(n) + \ldots + \alpha_{1k}a^{(k)}(n) \\ \vdots \quad &= \qquad \qquad \vdots \\ a^{(k)}(n+1) &= \alpha_{k1}a^{(1)}(n) + \ldots + \alpha_{kk}a^{(k)}(n) \end{aligned}$$

*each equation satisfies the recurrence*

$$a^{(i)}(n) = -\beta_{k-1}a^{(i)}(n-1) - \ldots - \beta_1 a^{(i)}(n-k-1) - \beta_0 a^{(i)}(n-k)$$

*for* $n > k$ *where* $A = \{\alpha_{ij}\}_{i,j=1}^{k}$ *is the coefficient matrix and*

$$m_A(x) = x^k + \beta_{k-1}x^{k-1} + \ldots + \beta_1 x + \beta_0$$

*is the minimal polynomial of* $A$*.*

# A single recurrence from a system of recurrences

$$\underbrace{\begin{bmatrix} a_{00}(n) \\ a_{01}(n) \\ a_{10}(n) \\ a_{11}(n) \\ a_{12}(n) \\ a_{21}(n) \\ a_{22}(n) \end{bmatrix}}_{\vec{a}(n)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^{n-1}}_{A^{n-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{a}(1)}$$

Let $x^k + \beta_{k-1}x^{k-1} + \ldots + \beta_1 x + \beta_0$ be the minimal polynomial of $A$. Then

$$\begin{aligned} A^k &= -\beta_{k-1}A^{k-1} - \ldots - \beta_1 A - \beta_0 \\ A^{n-1}\vec{a}(1) &= -\beta_{k-1}A^{n-2}\vec{a}(1) - \ldots - \beta_1 A^{n-k}\vec{a}(1) - \beta_0 A^{n-k-1}\vec{a}(1) \\ \vec{a}(n) &= -\beta_{k-1}\vec{a}(n-1) - \ldots - \beta_1\vec{a}(n-k+1) - \beta_0\vec{a}(n-k) \end{aligned}$$

# Some conjectural recurrences

For $k = 3, 4, 5$, $a_k(n)$ the number of leaf-free subgraphs of the $n \times k$ grid graph is conjectured to satisfy

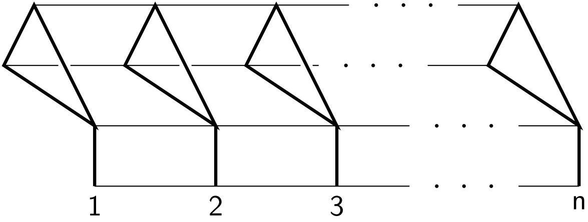
$$\begin{aligned} a_3(n) = {} & 12a_3(n-1) - 6a_3(n-2) - 20a_3(n-3) \\ & - 5a_3(n-4) \end{aligned}$$

$$\begin{aligned} a_4(n) = {} & 36a_4(n-1) - 7a_4(n-2) - 201a_4(n-3) \\ & + 49a_4(n-4) + 20a_4(n-5) - 5a_4(n-6) \end{aligned}$$

$$\begin{aligned} a_5(n) = {} & 103a_5(n-1) + 1063a_5(n-2) - 1873a_5(n-3) \\ & - 20274a_5(n-4) + 44071a_5(n-5) - 10365a_5(n-6) \\ & - 20208a_5(n-7) + 5959a_5(n-8) + 2300a_5(n-9) \\ & - 500a_5(n-10) \end{aligned}$$

For $k = 6$, this is the conjectured to be an $18$-order recurrence.

# Subgraphs which satisfy linear recurrences




### Theorem (Faase, 1994)

*Let $G$ be an arbitrary finite graph and let $H_n$ denote either the path graph $P_n$ or the cycle graph $C_n$ on $n$ vertices, and let $s(n)$ count the number of subgraphs $S$ of the Cartesian product $G \square H_n$ subject to any combination of the following properties:*

1. *Restrictions on degree*
2. *Connectivity*
3. *Acyclicity*

*Then $s(n)$ satisfies a linear recurrence.*

# Examples: subgraphs which satisfy linear recurrences

Let $G_{n,k}$ be a grid graph, then the following classes of subgraphs satisfy linear recurrences:

- Leaf-free subgraphs
  - Degree set $D = \{0, 2, 3, 4\}$
- Spanning tree (mazes)
  - Degree set $D = \{1, 2, 3, 4\}$
  - Connected
  - Acyclic
- Hamiltonian paths
  - Degree set $D = \{1, 2\}$
  - Connected
  - Acyclic
- Perfect matchings (domino tilings)
  - Degree set $D = \{1\}$

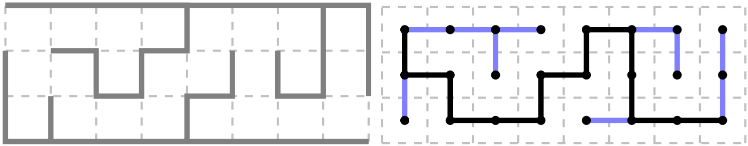


Figure 7: A correspondence between mazes and spanning trees

# Tiling with $3$-ominoes

Recall domino tilings were counted by subgraphs on the grid with degree set $D = \{1\}$.
Similarly, $3$-omino tilings are counted by subgraphs on the grid where each component has exactly three vertices.

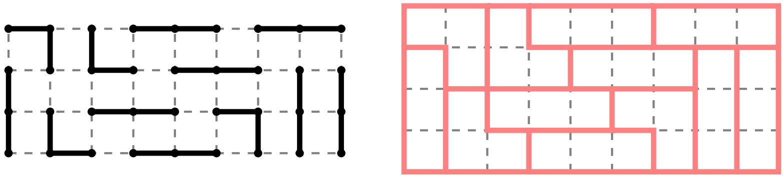


Figure 8: Correspondence between subgraphs and triomino tilings

# More subgraphs which satisfy linear recurrences

Theorem

*Let $G$ be an arbitrary finite graph, and let $s(n)$ count the number of subgraphs of $G\square P_n$ subject to any combination of the following properties:*

1. *Exactly/fewer than/more than $m$ vertices of degree $d$*
2. *Exactly/fewer than/more than $m$ connected components on exactly/fewer than/more than $d$ vertices.*
3. *Has $k \pmod{m}$ vertices of degree $d$*
4. *Exactly/fewer than/more than $\ell$ connected components containing exactly/fewer than/more than $v$ vertices.*

*Then $s(n)$ satisfies a linear recurrence.*

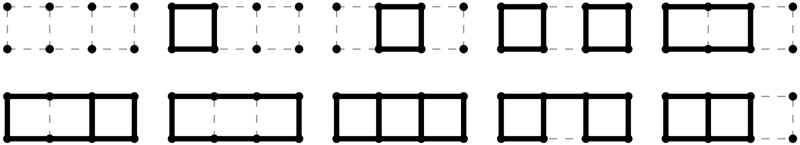
# Counting up to symmetry

The number of no-leaf subgraphs of the $2 \times n$ grid satisfies the two term recurrence

$$a_2(n) = 5a_2(n-1) - 5a_2(n-2).$$

The number of no-leaf subgraphs of the $2 \times n$ grid up to horizontal/vertical reflection is conjectured to satisfy the eight term recurrence

$$\begin{aligned} s(n) = 8s(n-1) - 16s(n-2) - 20s(n-3) + 95s(n-4) \\ - 60s(n-5) - 80s(n-6) + 100s(n-7) - 25s(n-8) \end{aligned}$$



# Subgraphs up to group action

### Theorem

*Suppose $H$ is a group and the finite graph $X$ is an $H$-set which is fixed under the action of $H$, and consider $P_n$ under the action of $\mathbb{Z}_2$ which is either the identity on $P_n$ or reverses $P_n$. Next let $s(n)$ count the number of subgraphs $S \subset X \square P_n$ such that $S$ is fixed under the action of $(h, d)$ for $(h, d) \in H \times \mathbb{Z}_2$ Then $s(n)$ is defined by a linear recurrence.*

### Corollary

*The number of subgraphs of $X \square P_n$ (with appropriate vertex degree, connectivity, or acyclicity restrictions) counted up to the group action of $H \times \mathbb{Z}_2$ satisfies a linear recurrence.*

# Example: Möbius ladder (Guy, 1967)

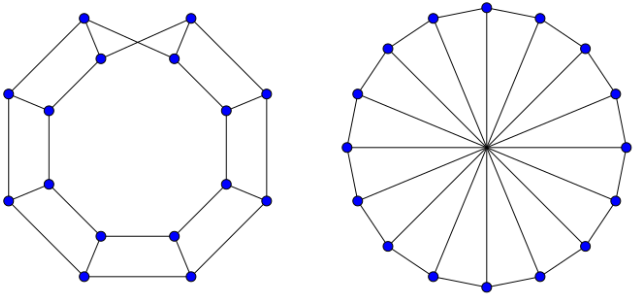


Figure 9: From Wikipedia: the Möbius ladder $M_{16}$ on $16$ vertices.

**Fact**

*The number of leaf-free subgraphs on the Möbius ladder on $2n$ vertices is equal to the number of leaf-free subgraphs on the $(n+2) \times 2$ grid graph.*

# Cycling around (a generalization of the Möbius ladder)

**Theorem**

*Let $G$ be an arbitrary finite graph with vertex set $V = \{v_1, v_2, \ldots, v_m\}$, and let $E \subseteq V \times V$. Then let $H_n$ be the graph Cartesian product $G \square P_n$ together with the edges $\{((1, v_i), (n, v_j)) : (v_i, v_j) \in E\})$*
*Next let $s(n)$ count the number of subgraphs of $H_n$ subject to any combination of the following properties:*

1. *Restrictions on degree*
2. *Connectivity*
3. *Acyclicity*

*Then $s(n)$ satisfies a linear recurrence.*

Note that $E = \{(v_1, v_1), (v_2, v_2), \ldots, (v_m, v_m)\}$ recovers $G \square C_n$ and $E = \emptyset$ recovers $G \square P_n$.

# Example: king graph

Let $P_5$ be the path labeled $v_1, v_2, \ldots, v_5$ in the obvious way, and let $E = \{(v_i, v_{i+i}) : i < 5\} \cup \{(v_i, v_i) : i \in [5]\} \cup \{(v_i, v_{i-i}) : i > 1\}$.

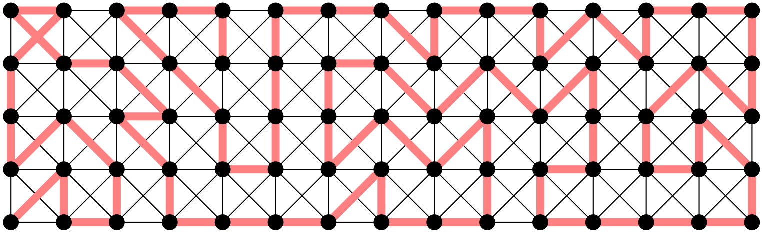


Figure 10: The number of ways a king can tour every square of the $5 \times 15$ chessboard is given by the number of Hamiltonian subgraphs in the $5 \times 15$ king graph

The number of kings tours on a $k \times n$ chessboard is a linear recurrence in $n$.

# More exotic connections

**Theorem**

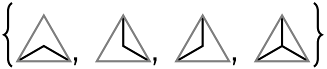
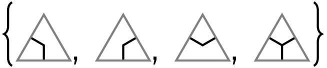
*Let $G$ be an arbitrary finite graph with vertex set $V = \{v_1, v_2, \ldots, v_m\}$, and let $E \subseteq V \times V$. Then let $H_n$ be the disjoint union $\underbrace{G \sqcup G \sqcup \ldots \sqcup G}_{n \text{ times}}$ together with the edges*

$$\{((k, v_i), (k+1, v_j)) : (v_i, v_j) \in E, k \in [n-1]\}.$$

*Next let $s(n)$ count the number of subgraphs of $H_n$ subject to any combination of the properties mentioned in the previous theorems. Then $s(n)$ satisfies a linear recurrence.*

Note that $E = \{(v_1, v_1), (v_2, v_2), \ldots, (v_m, v_m)\}$ recovers $G \square C_n$.

# Open questions

- Do there exist general nice algorithms for counting subgraphs of the $n \times n$ grid graph satisfying particular properties?
- Given some graph and some properties on subgraphs, how do you find the order of the smallest recurrence that counts the number of such subgraphs?
- What is the expected value of a randomly selected component in the diagonal "maze" construction shown in the Commodore 64 program? What's the expected number of connected components?
- What if we want to count structures on other tiles, e.g.:
  - 
  - 
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