Consider a system of linear recurrences $a_1, a_2, \ldots, a_t$ where

$$a_i(n) = b_{i,1}a_1(n-1) + b_{i,2}a_2(n-1) + \ldots + b_{i,t}a_t(n-1).$$

$$\begin{array}{ll}
a_1(1) = 0 & a_1(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + 2a_5(n-1) + a_7(n-1) \\
a_2(1) = 0 & a_2(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_3(1) = 0 & a_3(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_4(1) = 0 & a_4(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_5(1) = 0 & a_5(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + a_5(n-1) \\
a_6(1) = 0 & a_6(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_7(1) = 1 & a_7(n) = a_5(n-1) + a_7(n-1)
\end{array}$$

Figure 1: In this system of equations, the function $f(n) = a_5(n) + a_7(n)$ (which counts no-leaf subgraphs of the $2 \times n$ grid.) satisfies the recursion $f(n) = 5f(n-1) - 5f(n-2)$ for $n > 2$.

Question. Can any linear combination of these recurrences be turned into a single linear recurrence? If not, how far can it be "simplified"?

Related.

1. What are the number of terms in such a linear recurrence? (i.e. how "deep" does it go? In the example, it has a depth of 2.)
2. What if the initial recurrences have depth greater than 1? For example:

$$\begin{aligned} a_i(n) = {} & b_{i,1,1}a_1(n-1) + b_{i,1,2}a_1(n-2) + \ldots + b_{i,1,k_1}a_1(n-k_1) + \\ & b_{i,2,1}a_2(n-1) + b_{i,2,2}a_2(n-2) + \ldots + b_{i,2,k_2}a_2(n-k_2) + \ldots \\ & b_{i,t,1}a_t(n-1) + b_{i,t,2}a_t(n-2) + \ldots + b_{i,t,k_t}a_t(n-k_t) \end{aligned}$$

References.

A special case of Problem 64 is counted in the example.