For two positive real numbers $a > b \in \mathbb{R}_+$ consider the part of the two spirals parameterized by

$$\vec{x}_a(t) = (at\cos(2\pi t), at\sin(2\pi t)) \text{ for } t \in \left[0, \frac{b}{a-b}\right] \text{ and}$$
$$\vec{x}_b(t) = (bt\cos(2\pi t), bt\sin(2\pi t)) \text{ for } t \in \left[0, \frac{a}{a-b}\right],$$

which lie within a circle of radius $r = \dfrac{ab}{a-b}$

If we look at the area between the curves, we can see that the area between the curves is precisely $\frac{1}{3}\pi r^2$. Moreover, if we look at the area of this region that lies inside of a circle of radius r_* for $r_* \in (0, r]$, we find the area is $\pi r_*^2 \left(1 - \frac{2}{3}\frac{r_*}{r}\right)$

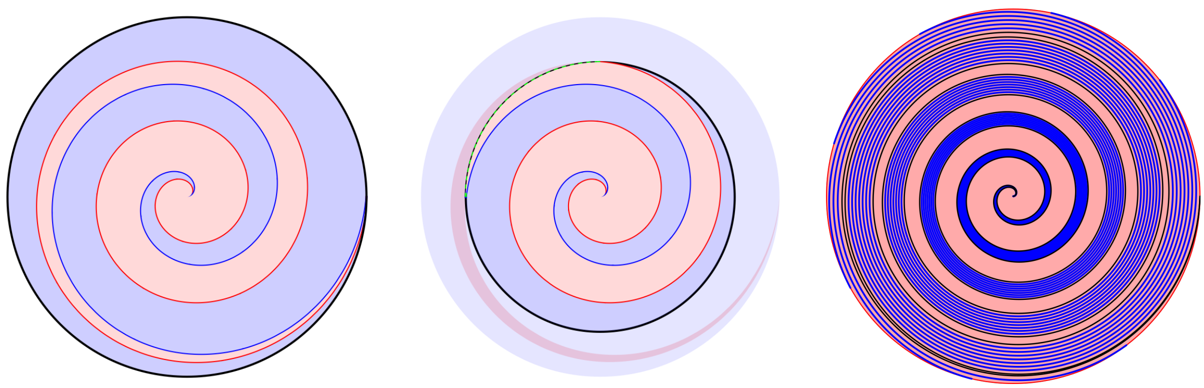

Figure 1: The area between the two curves is drawn in red, and is 1/3 the area of the circle. In the second illustration the area between the two curves is drawn in red, and the region inside of the circle of radius $\frac{3}{4}r$ is precisely half of that circle.

Question. Is there an elementary proof of this?

Related.

1. Is there a natural way to extend $r^* > r$?
2. Are there analogous results if $\vec{c}_f = f(t)\,(\cos(2\pi t), \cos(2\pi t))$ for some well-behaved pair of functions f_1 and f_2?
3. Are there analogous ways to chop up the (hyper)sphere?

References.

Pappus on Archimedes' spiral translated by Henry Mendell, Cal State LA.

Note. We can prove this in two ways with multivariable calculus:

1. Green's theorem to set up an vector line integral over the boundary of the region where the vector field is $\vec{F}(x, y) = (-y/2, x/2)$, and
2. Parameterizing the points in the complementary region by $\vec{x}_s(t)$ for $s \in [a, b]$ and $t \in \left[0, \dfrac{r^* ab}{rs(a-b)}\right]$.