Difficulty: 2/4 Interest: 3/4

OEIS sequence A261865 describes "$a(n)$ is the least $k \in \mathbb{N}$ such that some multiple of $\sqrt{k} \in (n, n+1)$." Clearly the asymptotic density of 2 in the sequence is $1/\sqrt{2}$.

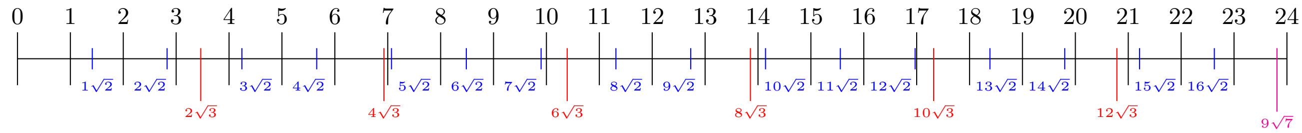

Figure 1: An illustration of $a(n)$ for $n \in \{1, 2, \dots, 23\}$.

Question. Let $S_\alpha \subset \mathbb{N}$ denote the squarefree integers strictly less than α. Is the asymptotic density of squarefree j given by

$$\frac{1}{\sqrt{j}} \prod_{s \in S_j} \left(1 - \frac{1}{\sqrt{s}}\right)?$$

Related.

1. Is there an algorithm to construct a value of n such that $a(n) > K$ for any specified K? (Perhaps using best Diophantine approximations or something?)
2. What is the asymptotic growth of the records?
3. Given some α what is the expected value of the smallest n such that $S_\alpha \subset \{a(1), \dots, a(n)\}$?
4. This sequence uses the "base sequence" of $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots\}$. On what other base sequences is this construction interesting?
5. What is the smallest $m \in \mathbb{N}$ such that $k2^{1/m} \in (n, n+1)$ for some $k \in \mathbb{N}$?
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References.

https://oeis.org/A261865