Problem 1.

Suppose you are given an $n \times m$ grid, and I then think of a rectangle with its corners on grid points. I then ask you to "black out" as many of the gridpoints as possible, in such a way that you can still guess my rectangle after I tell you all of the non-blacked out vertices that its corners lie on.

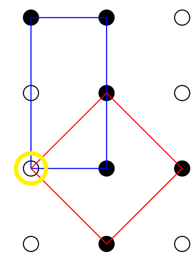


Figure 1: An example of an invalid "black out" for an 3×4 grid. The blue rectangle and the red rectangle have the same presentation, namely the gridpoint inside the yellow circle.

Question. How many vertices may be crossed out such that every rectangle can still be uniquely identified?

Related.

1. What if the interior of the rectangle is lit up instead?
2. What if all gridpoints that instersect the perimeter are lit up?
3. What if the rectangles must be square?
4. What if parallelograms are used instead of rectangles?
5. What if the rectangles must be horizontal or vertical?
6. What if the rectangles must be horizontal, vertical, or 45 diagonal?
7. What if this is done on a triangular grid with equilateral triangles?
8. What if this is done in more dimensions (e.g. with a rectangular prism or tetrahedron?)

Problem 2.

Jeremy Kun gives a canonical bijection between $\binom{n+1}{2}$ and a discrete tringle of length n, as seen in Figure 1.

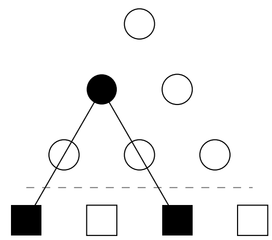


Figure 2: Bijection that maps a point on the triangle with side length 3 to a 2-subset of [3 + 1].

Question. Is there a similar "projection" that bijects a point on the discrete tetrahedron to a 3-subset of [n + 2]?

Note. Misha Lavrov gives a potential function to the question on Math Stack Exchange. (https://math.stackexchange.com/a/2468687/121988)

Related.

1. More generally is there a bijection from the k-simplex to a k-subset of [n + k - 1]?

Problem 3.

Let G be some $n \times m$ grid as in Figure 1, where each cell has two opposite diagonals connected (uniformly at random). A cell is chosen (also uniformly at random), and the segment given by the path of diagonals that goes through the selected cell is is inspected.

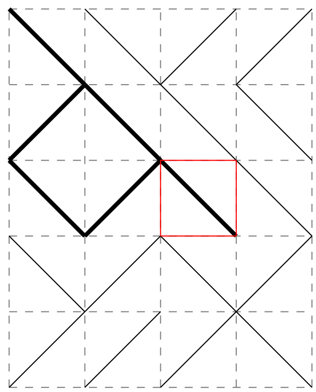


Figure 3: An example of a 4×5 grid, where a segment of size 6 has been selected.

Question. What is the expected length of the selected segment?

Related.

1. What is the expected number of segments in an $n \times m$ grid?
2. How long is the longest segment expected to be?
3. How does this change if the grid is toroidal, on a cylinder, on a Möbius strip, etc?

Problem 4.

Peter Winkler's Coins-in-a-Row game works as following:

> On a table is a row of fifty coins, of various denominations. Alice picks a coin from one of the ends and puts it in her pocket; then Bob chooses a coin from one of the (remaining) ends, and the alternation continues until Bob pockets the last coin.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed according to some probability distribution.

Question. For some fixed ω, what is the expected first player's score of Peter Winkler's Coins-in-a-Row game when played with $X_1(\omega), X_2(\omega), \ldots, X_3(\omega)$ where both players are using a min-max strategy?

Note. Let

$$e = E[X_2 + X_4 + \ldots + X_{2n}] \text{ and } o = E[X_1 + X_2 + \ldots + X_{2n-1}]$$

When played with $2n$ coins, the first player's score is bounded below by $\max(e, o) - \min(e, o)$ by the strategy outlined by Peter Winkler.

Trivially the first player's score is bounded above by the expected value of the n largest coins minus the expected value of the n smallest coins.

Related.

1. If all possible n-coin games are played with coins marked 0 and 1, how many games exist where both players have a strategy to tie.
2. How does this change when played according to the (fair) Thue-Morse sequence?
3. What if the players are cooperating to help the first player make as much as possible (with perfect logic)?
4. What is both players are using the greedy algorithm?
5. What if one player uses the greedy algorithm and the other uses min-max? (i.e. What is the expected value of the score improvement when using the min-max strategy?)
6. What if one player selects a coin uniformly at random, and the other player uses one of the above strategies?

Problem 5.

Let a "popsicle stick weave" be a configuration of lines segments, called "sticks", such that

(1) every stick has at least two sticks above it and one below or two sticks above and one below, and

(2) the removal of any stick results in a configuration that violates (1).

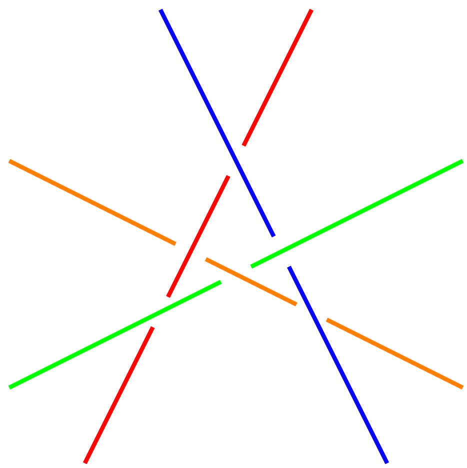


Figure 4: The unique example of a 4 stick crossing (up to reflection)

Question. How many distinct popsicle stick weaves exist for n sticks?

Related.

1. What if the sticks are only allowed to touch three other sticks?
2. What if the sticks are another geometric object (e.g. semicircles)?

Problem 6.

Let

$$C_n = \{f : [n] \to \mathbb{N} \mid \text{the convex hull around } \{(1, f(1)), \ldots, (n, f(n))\} \text{ forms an } n\text{-gon}\}$$

and then let $a(n)$ denote the least upper bound over all functions in C_n

$$a(n) = \min\{\max\{f(k) \mid k \in [n]\} \mid f \in C_n\}$$

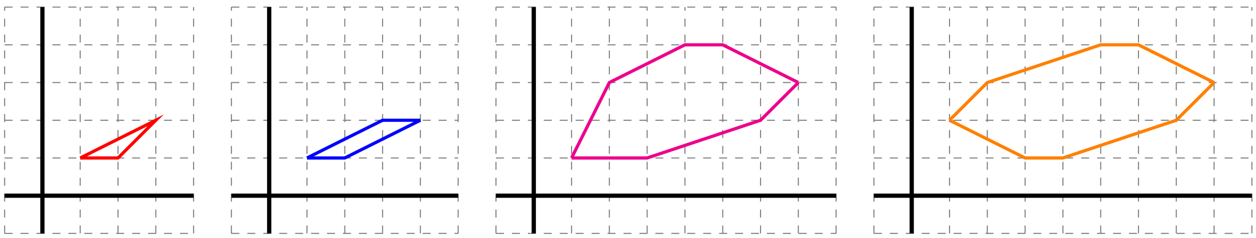


Figure 5: Examples of $a(3) = 2$, $a(4) = 2$, $a(7) = 4$, and $a(8) = 4$, where the polygons with an even number of vertices have rotational symmetry.

Question. Do these polygons converge to something asymptotically?

Related.

1. Does $a(2n) = a(2n - 1)$ for all n?
2. Do the minimal $2n$-gons always have a representative with rotational symmetry?
3. Are minimal $2n$-gons unique (up to vertical symmetry) with finitely many counterexamples?
4. What is the asymptotic growth of a(n)?

References.

A285521: "Table read by rows: the n-th row gives the lexicographically earliest sequence of length n such that the convex hull of (1, a(1)), ..., (n, a(n)) is an n-gon with minimum height." (https://oeis.org/A285521)

Problem 7.

Let $f_{n,m} : [n] \to [m]$ be a uniformly random function. Consider the convex hull around $\{(1, f(1)), \ldots (n, f(n))\}$

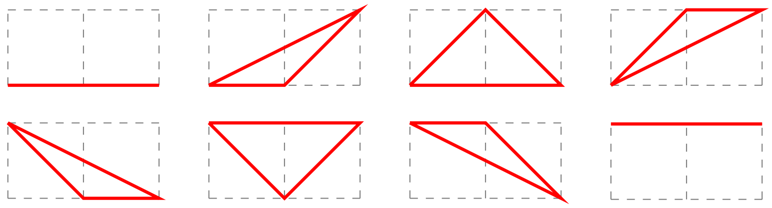


Figure 6: Examples of $f_{3,2}$. Here the expected number of vertices on a convex hull is 2.75

Question. What is the probability of seeing a k-gon (for some fixed k), when given a uniformly random function $f_{n,m}$?

Related.

1. What if $f_{n,n}$ is restricted to be a permutation?
2. What if $f_{n,m}$ is injective?

Problem 8.

Given an $n \times n$ grid, consider all convex polygons with grid points as vertices. Let $m(n)$ be the greatest integer k such that there exists a convex k-gon on the $n \times n$ grid.

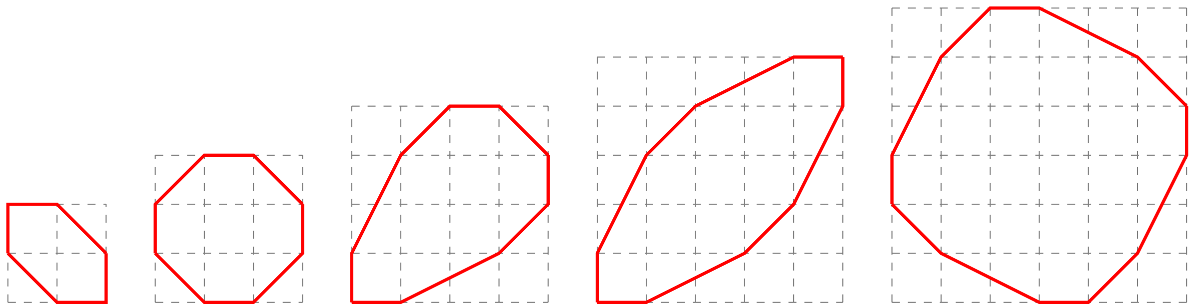


Figure 7: Examples that prove $m(3) = 6, m(4) = 8, m(5) \geq 9, m(6) \geq 10$, and $m(7) \geq (12)$

Question. What is $m(n)$?

Related.

1. What is a proof (or counterexample) that the examples shown are the best possible?
2. How does $m(n)$ grow asymptotically?
3. Do the shapes do anything interesting in the limit?
4. Are there finitely many maximal polygons without rotational symmetry (e.g. $m(5)$)?
5. How does this generalize to $m \times n$ grids?
6. See Problems 6 and 7.

Problem 9.

Given an $n \times n$ grid, consider all the ways that convex polygons with grid points as vertices can be nested.

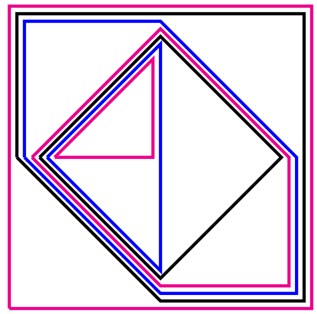


Figure 8: Seven nested convex polygons in the 3×3 grid.

Question. If we think of each polygon having the same height, what is the greatest volume that we can make by stacking the polygons this way?

Related.

1. What is the largest sum of the perimeters? The least?
2. What is the largest sum of the number of vertices? The least?
3. How many ways are there to stack $n^2 - 2$ polygons like this? Any number of polygons?
4. Does this generalize to polyhedra in the $n \times n \times n$ cube?
5. Does this generalize to polygons on a triangular grid?

Problem 10.

Consider all k-colorings of an $n \times n$ grid, where each row and column has $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ cells with each color.

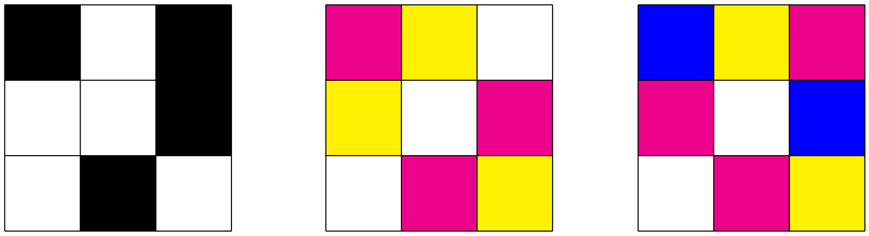


Figure 9: A valid 2-coloring, 3-coloring, and 4-coloring of an 3×3 grid.

Question. How many such k-colorings of the $n \times n$ grid?

Related.

1. What if there also must be a total of $\lfloor n^2/k \rfloor$ or $\lceil n^2/k \rceil$ cells of each color?
2. What if these are counted up to the dihedral action on the square D_4?
3. What if these are counted up to torus action?
4. What if these are counted up to permutation of the coloring?
5. Can this generalize to the cube? To a triangular tiling?

Problem 11.

Consider an $n \times n$ chess board, with pieces that can move integer distances, but only in diagonal directions—that is, they move like the hypotenuse of a Pythagorean triangle.

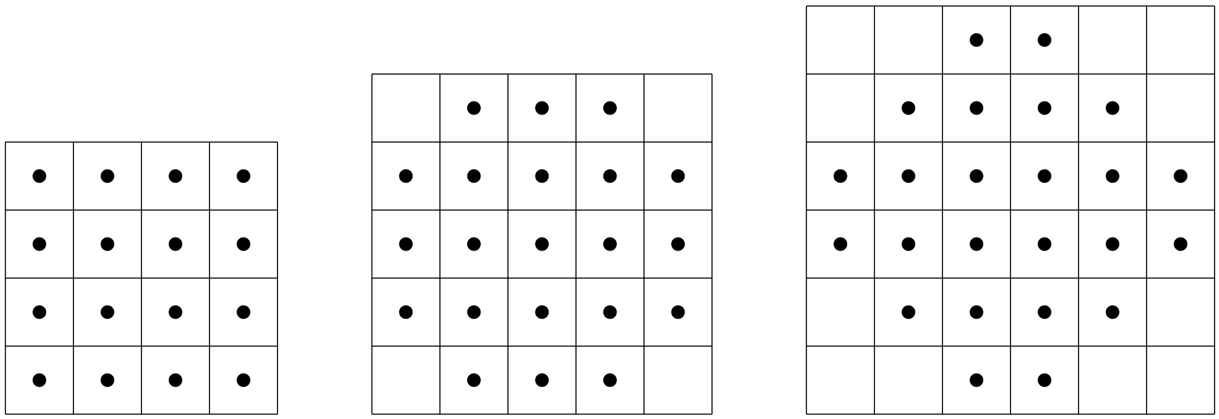


Figure 10: Valid configurations for 4×4, 5×5, and 6×6 grids, proving that $a(4) = 16$, $a(5) \geq 21$, and $a(6) \geq 24$.

Question. What is the greatest number of nonattacking pieces that can be placed on the board?

Related.

1. What is the board is $n \times m$?
2. What if pieces must move like *primitive* Pythagorean triples?
3. What if each piece can move k times?
4. What is the asymptotic growth of a?

Problem 12.

Consider Ron Graham's sequence for lcm, that is, look at sequences such that

$$n = a_1 < a_2 < \ldots < a_T = k \text{ and } \operatorname{lcm}(a_1, \ldots, a_T) \text{ is square.}$$

Question. What is the least k (as a function of n) such that such a sequence exists?

$$\begin{aligned}
a(1) &= 1 \quad \text{via } (1) \\
a(2) &= 4 \quad \text{via } (2,4) \\
a(3) &= 3 \quad \text{via } (3,9) \\
a(4) &= 4 \quad \text{via } (4) \\
a(5) &= 25 \text{ via } (5,25) \\
a(6) &= 12 \text{ via } (6,9,12) \\
a(7) &= 49 \text{ via } (7,49) \\
a(8) &= 16 \text{ via } (8,16)
\end{aligned}$$

Figure 11: Examples of $a(n)$ for $n \in \{1, 2, \ldots, 8\}$.

Related.

1. For what values n is $a(n)$ nonsquare?
2. For what values n does the corresponding sequence have three or more terms?
3. What is the analogous sequence for perfect cubes, etc?

Problem 13.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

There is a known way to efficiently compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a p-th power, where p is a prime and where any term appears at most $p - 1$ times.

Question. What is an efficient way to compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a c-th power, where c is composite?

$$\begin{aligned}
a_4(1) &= 1 \text{ via } 1 && = 1^4 \\
a_4(2) &= 2 \text{ via } 2^2 \cdot 4 && = 2^4 \\
a_4(3) &= 6 \text{ via } 3^2 \cdot 4 \cdot 6^2 && = 6^4 \\
a_4(4) &= 4 \text{ via } 4^2 && = 2^4 \\
a_4(5) &= 10 \text{ via } 5^2 \cdot 8^2 \cdot 10^2 && = 20^4 \\
a_4(6) &= 9 \text{ via } 6^2 \cdot 8^2 \cdot 9 && = 12^4 \\
a_4(7) &= 14 \text{ via } 7^2 \cdot 8^2 \cdot 14^2 && = 28^4 \\
a_4(8) &\leq 15 \text{ via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 && = 60^4 \\
a_4(9) &= 9 \text{ via } 9^2 && = 3^4 \\
a_4(10) &\leq 18 \text{ via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 && = 180^4
\end{aligned}$$

Figure 12: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

Related.

1. For what values n is $a(n) < A006255(n)$?
2. How many c-th power sequences have $a_T = a_c(n)$?
3. Do any such c-th power sequences exactly two distinct terms?

Problem 14.

Suppose you have a strip of toilet paper with n pieces, and you fold the paper evenly into d parts (divide by d) or fold the last k pieces in (subtract by k), until the length of the strip is less than k pieces.

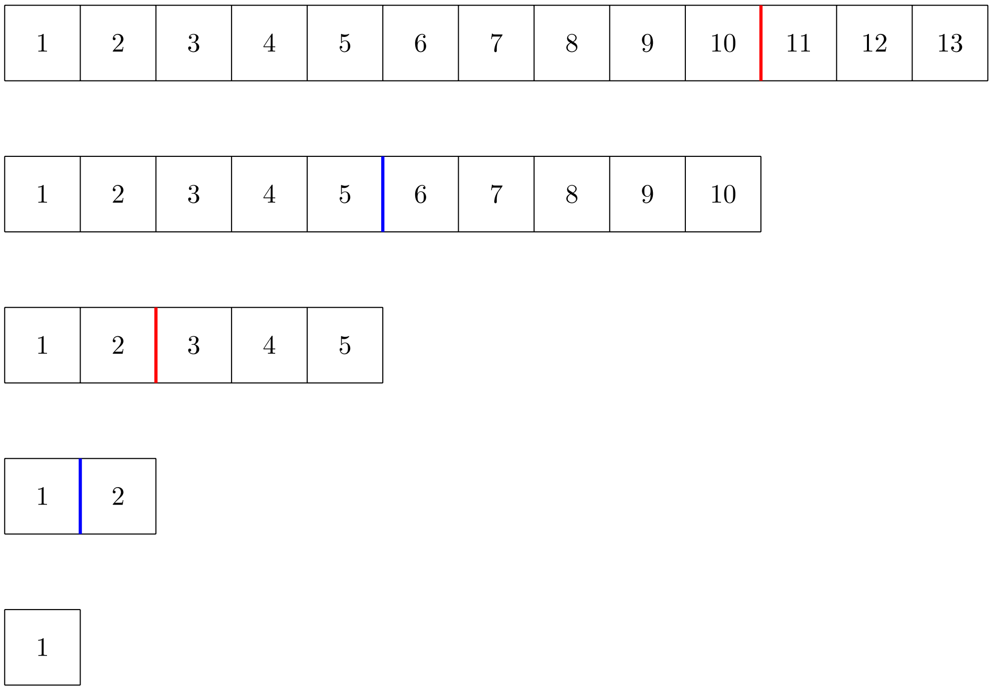


Figure 13: A folding of paper where $n = 13$, $d = 2$, and $k = 3$, showing that $a_{2,3}(13) \leq 4$. Where the red marks a subtraction by k and the blue marks a division by d.

Question. Is there an efficient way to compute $a_{d,k}(n)$?

Related.

1. What if you must keep folding until you cannot fold any longer?
2. What is the minimum number of terminal pieces? What is the minimum number of steps to this number?

Problem 15.

OEIS sequence A261865 describes "$a(n)$ is the least $k \in \mathbb{N}$ such that some multiple of $\sqrt{k} \in (n, n+1)$." Clearly the asymptotic density of 2 in the sequence is $1/\sqrt{2}$.

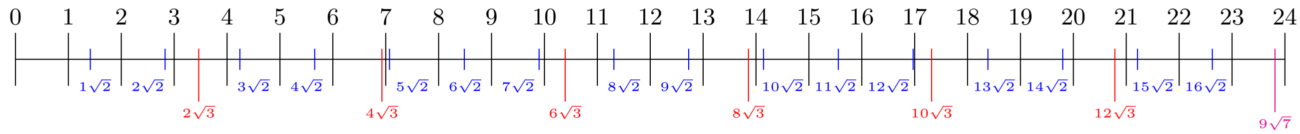


Figure 14: An illustration of $a(n)$ for $n \in \{1, 2, \ldots, 23\}$.

Question. Let $S_\alpha \subset \mathbb{N}$ denote the squarefree integers strictly less than α.
Is the asymptotic density of squarefree j given by

$$\frac{1}{\sqrt{j}} \prod_{s \in S_j} \left(1 - \frac{1}{\sqrt{s}}\right)?$$

Related.

1. Is there an algorithm to construct a value of n such that $a(n) > K$ for any specified K? (Perhaps using best Diophantine approximations or something?)
2. What is the asymptotic growth of the records?
3. Given some α what is the expected value of the smallest n such that $S_\alpha \subset \{a(1), \ldots, a(n)\}$?
4. This sequence uses the "base sequence" of $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\}$. On what other base sequences is this construction interesting?

Problem 16.

Richard Guy beat me to this problem by a few years. (https://arxiv.org/abs/1207.5099).
John Conway described the "Subprime Fibonacci Sequence":

$$a(1) = a, a(2) = b, a(n+1) = \operatorname{gpd}(a(n) + a(n-1)),$$

where $\operatorname{gpd}(k)$ is the greatest proper divisor of k.
Conway then conjectured that regardless of the starting terms, the sequence ends in a handful of cycles.
Richard Guy found that there are more cycles than those that Conway conjectured.

Question. What are all of the different possible end behaviors of Conway's Subprime Fibonacci Sequence?

Problem 17.

Start with n piles with a single stone in each pile. If two piles have the same number of stones, then any number of stones can be moved between them.

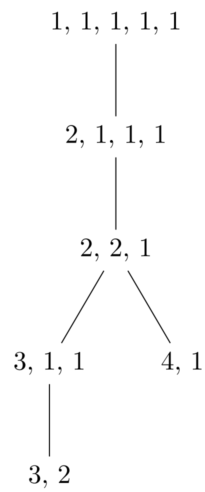


Figure 15: An illustration of all possible moves for $n = 5$.

Question. What is the greatest number of steps that can occur? Alternatively how many "levels" are in the tree of possible moves?

Related.

1. Let s be the total number of distinct states. (The example shows that $s(5) = 6$.)
2. Let c be the total number of states that *cannot* be acheived. (In the example, $c(5) = 1$ via the state (5).)
3. Is $c(p) = 1$ for all primes p?
4. Is $c(n) = 0$ if and only if n is a power of 2?
5. Let ℓ be the least number of steps to a terminal state. (In the example, $\ell(5) = 3$ ending in the state $(4, 1)$.)
6. Let g be the greatest number of steps to a terminal state. (In the example, $g(5) = 4$ ending in the state $(3, 2)$.)
7. Let p be the total number of paths. (In the example, $p(5) = 2$.)
8. Let t be the number of distinct *terminal* states. (In the example, $t(5) = 2$ with states $(4, 1)$ and $(3, 2)$.)

Problem 18.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

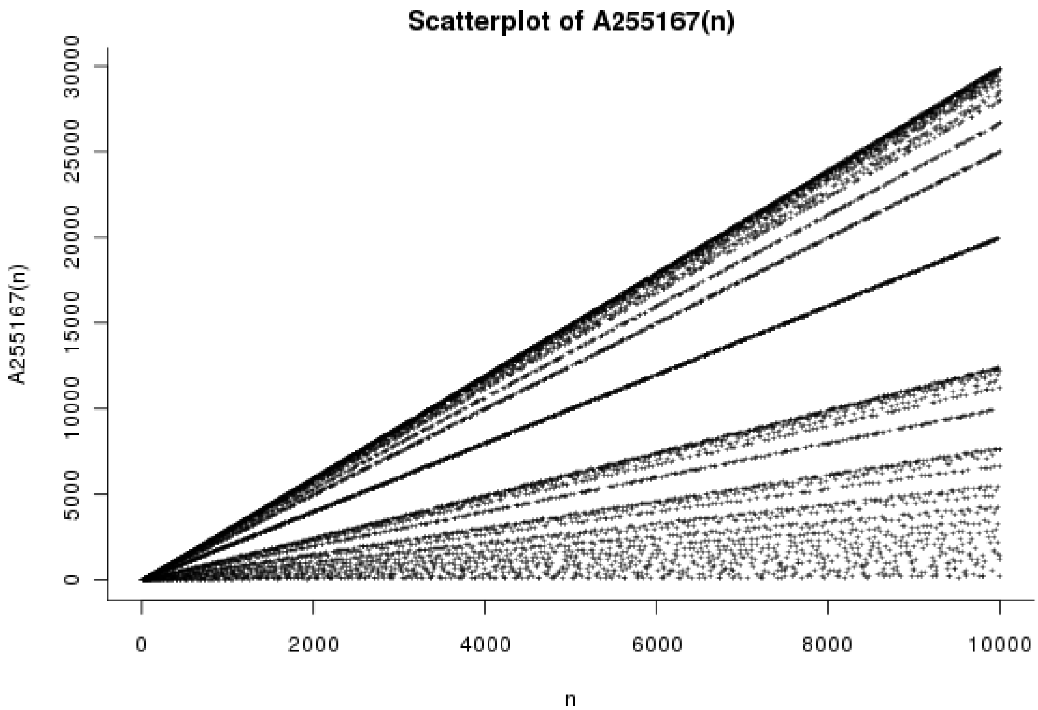
$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

A006255 is bounded above by A072905, the least $k > n$ such that $k \cdot n$ is square.

Question. Does there exist any n for which $A006255(n) = A072905(n)$. In other words, is there any non-square n for which $n \cdot A006255(n)$ is square?

Related.

1. Does the gap $A072905(n) - A006255(n)$ have a nonzero lower bound?



Problem 19.

Starting with 1 and working in a hexagonal spiral, repeatedly choose the smallest positive integer such that it won't be adjacent either itself (once) or to the same number twice.

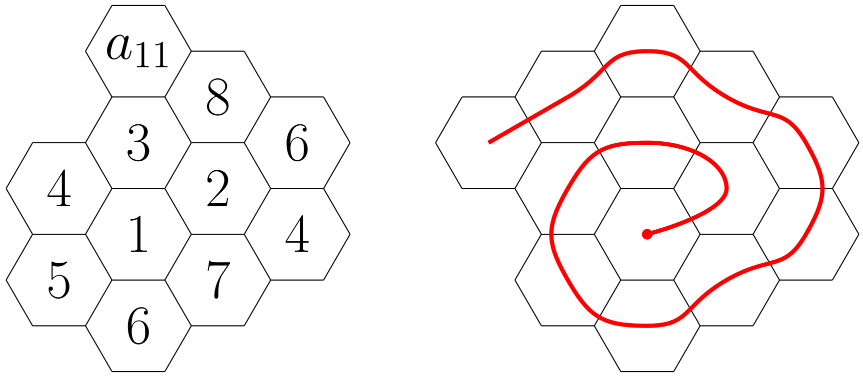


Figure 16: $a_{11} \neq 1$ because 3 is already adjacent to 1, $a_{11} \neq 2$ because 3 and 8 are already adjacent to 2, $a_{11} \neq 3$ because then a_{11} would be equal to its neighbor, $a_{11} \neq 4$ because 3 is already adjacent to 4, thus $a_{11} = 5$.

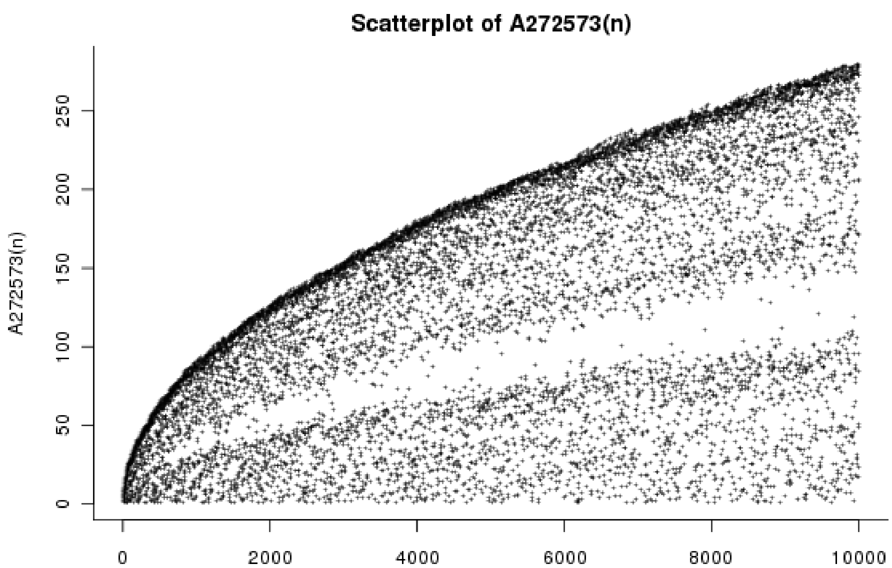


Figure 17: A plot of a_1 through a_{10000}.

Question. Why does a gap appear in the plot of the sequence?

Problem 20.

Let $a_3(n)$ be the least $k > n$ such that nk or nk^2 is a cube, and let $A299117$ be the image of $a_3(n)$.

$$\begin{aligned}
a_3(1) &= 8 \\
a_3(2) &= 4 \\
a_3(3) &= 9 \\
a_3(4) &= 16 \\
a_3(5) &= 25 \\
a_3(6) &= 36 \\
a_3(7) &= 49 \\
a_3(8) &= 27 \\
a_3(9) &= 24
\end{aligned}$$

Question. Is there another way to characterize what integers are in $A299117$?

Note. $A299117$ contains every cube, because $a(n^3) = (n+1)^3$.
$A299117$ contains the square of every prime, because $a(p) = p^2$.

Related.

1. Does $A299117$ contain every square?
2. Does $A299117$ contain any squarefree number?
3. What about the generalization: the image of a_β where $a_\beta(n)$ is the least $k > n$ such that $nk, nk^2, \ldots, nk^{\beta-2}$, or $nk^{\beta-1}$ is a β-th power? Prime β is an injection—is this well behaved?

Problem 21.

Consider placing any number of queens (of the same color) on an $n \times n$ chessboard in such a way as to maximize the number of legal moves available.

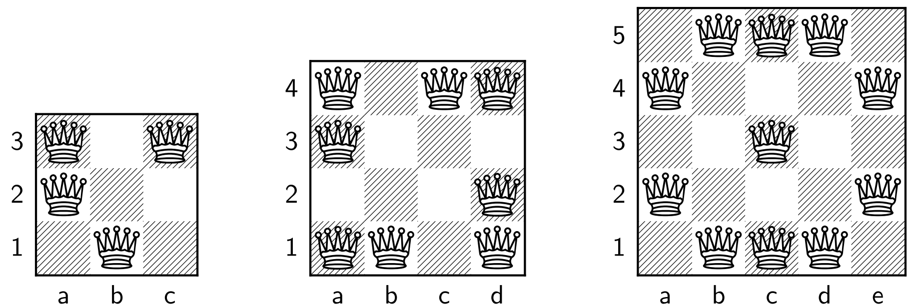


Figure 18: Examples of $a_q(3) = 17, a_q(4) = 40, a_q(5) = 76$.

Question. Is Alec Jones's conjecture true: $a_q(n) = 8(n-2)^2$ for $n \geq 6$, by placing the queens around the perimeter?

Related.

1. What about the analogous function for rooks (a_r) or bishops (a_b)?
2. What if the chessboard is a torus? Cylinder? Möbius strip?
3. What if the chessbaord is $n \times m$?
4. Is $a_b(n) = \lfloor a_q(n)/2 \rfloor$? for all n?
5. What if queens can attack?

Problem 22.

Let U_n be the set of sequences of positive integers of length n such that no substring occurs twice.

$$(1,1,2,2,1,3,1) \in U_7 \tag{1}$$
$$(1,2,1,2,3) \notin U_5 \text{ because } (1,2) \text{ occurs twice.} \tag{2}$$
$$(1,1,1) \notin U_3 \text{ because } (1,1) \text{ occurs twice.} \tag{3}$$

Figure 19: An example and two non-examples of sequences with no repeated substrings.

Question. What is th enumber of sequences in U_n where the sum of terms is minimized?

Related.

1. What is the minimum least common multiple of a sequence in U_n? How many such minimal sequences?
2. What is the minimum product of a sequence in U_n? How many such minimal sequences?
3. What if substrings are considered forward and backward?
4. What if only substrings of length greater k are considered?
5. What is any term can appear at most ℓ times?

Problem 23.

Consider the fuction A285175(n) which is the lexicographically earliest sequence of positive integers such that no $k+2$ points are on a polynomial of degree k. (i.e. no two points are equal, no three points are colinear, no four points are on a parabola, etc.)

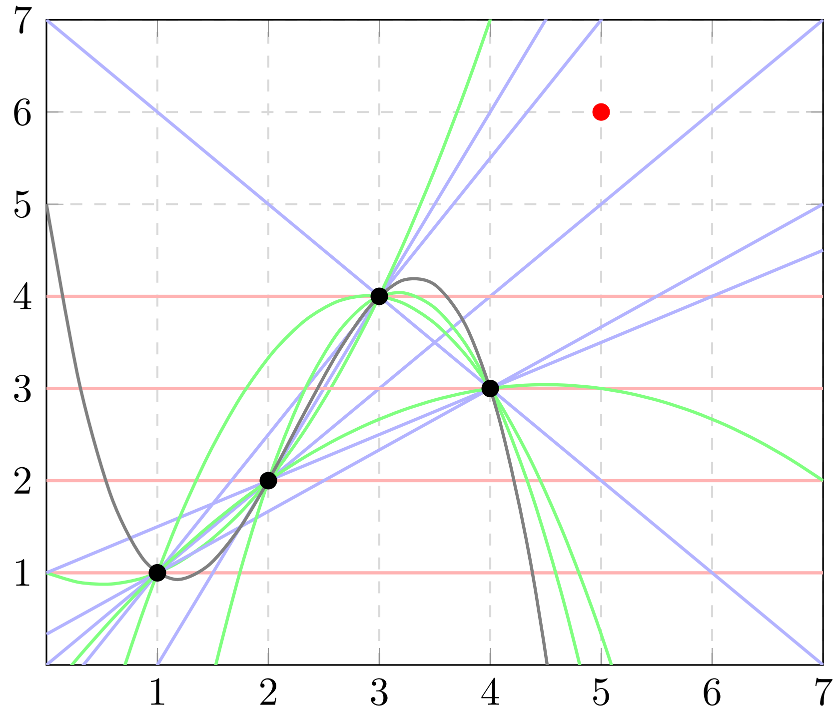


Figure 20: The first four points together with all interpolated polynomials. The red point marks the lowest integer coordinate $(5, k)$ that does not lie on an interpolated polynomial. (Degree 0 polynomials are plotted in red, degree 1 in blue, degree 2 in green and degree 3 in gray.)

Question. Do all positive integers occur in this sequence?

Related.

1. What is the asymptotic growth of this sequence?

Problem 24.

Let h be the maximum number of penny-to-penny connections on the vertices of a hexagonal lattice, and let $t(n)$ be the analogous sequence on the vertices of a triangular lattice.

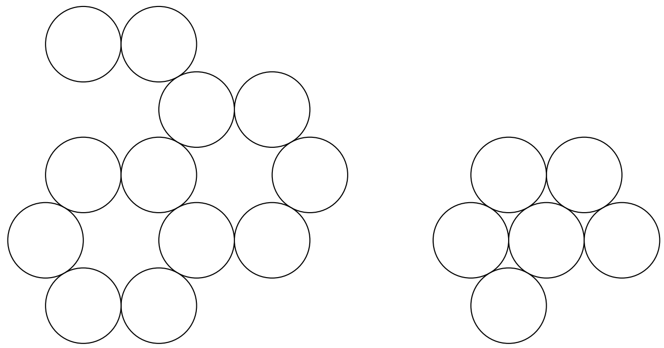


Figure 21: An example for $h(12) = 13$ and $t(6) = 9$

Question. What is a combinatorial proof that $h(2n) - t(n) = A216256(n)$.

Note. A216256 is

$$\underbrace{1}_{1}, \underbrace{2}_{1}, \underbrace{3, 3}_{2}, \underbrace{4, 4, 4}_{3}, \underbrace{5, 5, 5}_{3}, \underbrace{6, 6, 6, 6}_{4}, \underbrace{7, 7, 7, 7, 7}_{5}, \underbrace{8, 8, 8, 8, 8}_{5}, \underbrace{9, 9, 9, 9, 9, 9}_{6}, \ldots$$

Problem 25.

Consider all rectangles with all corners on gridpoints on an $n \times m$ grid.

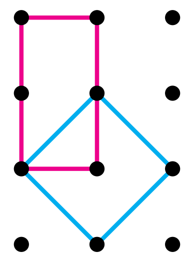


Figure 22: An example of two rectangles with all corners on gridpoints of a 3×4 grid.

Question. How many such rectangles exist?

Related.

1. How many squares exist? Rhombuses? Parallelograms? Kites? Quadrilaterals?
2. How many right triangles?
3. What if this is done on an $n \times m \times k$ grid?
4. What if the rectangles must be diagonal?
5. What if this is done on a triangular lattice?
6. How many tetrahedra are in an n-sided tetrahedra?

References.

See problem 1.