How many non-intersecting walks from $(1, 1)$ to (n, m) with steps up and to the right exist on the $n \times m$ torus?

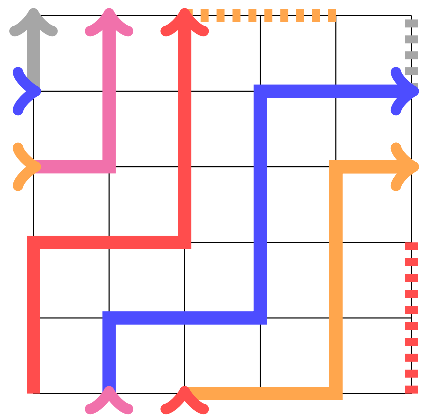

Figure 1: An example of a walk on a 5×5 torus that touches every lattice point.

Question. How many such walks exist?

Related.

1. What if the walks must touch every lattice point?
2. What if the walks must wrap around the torus exactly k times? (For $k = 1$ and $m = n$, this is the number of walks along the edges of an $n \times n$ non-toroidal grid.)
3. What if there always must be weakly more "up" steps than "right" steps? (generalization of staying above the diagonal) Strongly more?
4. What if this is done on a cylinder? Möbius strip? More dimensions?
5. What if walks can intersect at a right angle? What if there must be exactly k intersections? Only at $(0, 0)$?
6. What if more general loops were counted? (i.e. any walk from $(0, 0)$ to $(0, m)$, $(n, 0)$ or (n, m).)

References.

Problem 92.

`https://oeis.org/A324603`