The number of ways to draw a triangle on a triangular grid is given by

$$\sum_{k=1}^{n-1} k \cdot t(n-k) = \binom{n+2}{4} = A000332(n-2)$$

where $t(m)$ is the m-th triangular number, and A000332 is a figurate number based on the 4-simplex.

The number of ways to draw a square on a square grid is given by

$$\sum_{k=1}^{n-1} k \cdot (n-k)^2 = \frac{1}{6}\binom{n^2}{2} = n^2\left(\frac{n^2-1}{12}\right) = A002415(n)$$

where A002415 is a figurate number based on the 4-dimensional pyramid.

The number of ways to draw a hexagon on a hexagonal grid is given by

$$\sum_{k=1}^{n-1} k \cdot h(n-k) = n\binom{n+2}{3} = \frac{n^2(n+1)(n+2)}{6} = A002417(n-1).$$

where $h(m)$ is the m-th hexagonal number, and A002417 is a 4-dimensional figurate number.

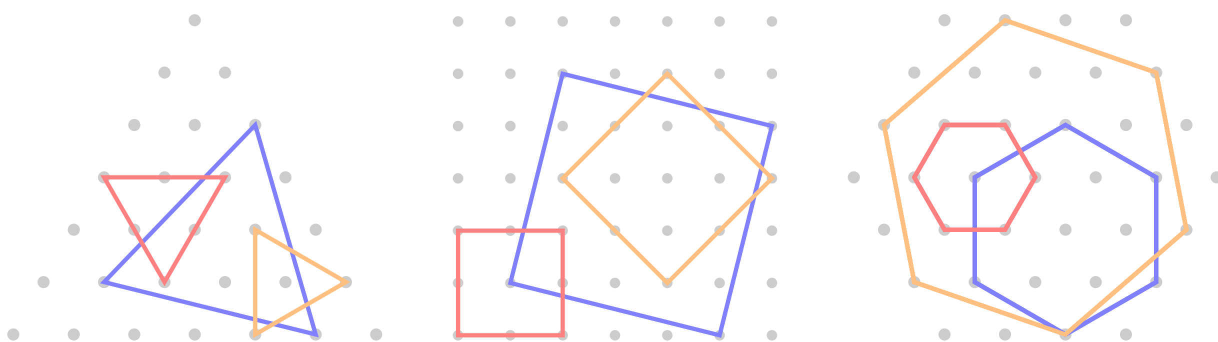

Figure 1: An illustration of three triangles on a triangular grid, three squares on a square grid, and three hexagons on a centered hexagonal grid.

Question. Is there a combinatorial explanation for why these numbers relate to 4-dimensional polytopes?

Related.

1. Can this be generalized to arbitrary regular n-gons in hyperbolic space?
2. How many triangles are on the "centered triangular number" grid?

References.

Problem 21.

`https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling`