In the game (tic-tac-toe)^2, each square is itself a smaller tic-tac-toe game; of course, one could imagine (tic-tac-toe)^3, where each of the squares in the smaller tic-tac-toe boards are themselves tic-tac-toe boards and so on. We're interested in counting (an abstraction of) the number of boards in this game, where two boards are considered the same if one is a rotation/reflection of another, or if any of the iterated games is a rotation/reflection of another, and so on.

In the simplest version, the boards are 2×2 and every square is filled in with an "X" or an "O", in perhaps unequal numbers.

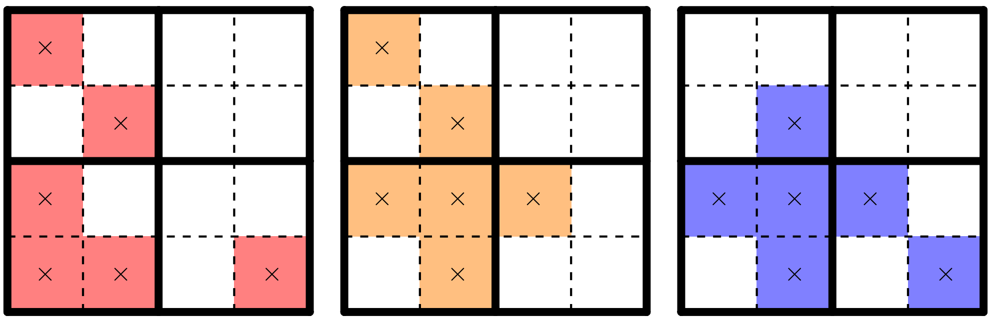

Figure 1: Three boards of depth 2 that are equivalent up to dihedral actions of both the smaller and larger boards.

Question. How many different boards of depth n are there?

Related.

1. What if these are counted up to interchanging all "Xs" and "Os"?
2. What if done on nested $m \times m$ boards?
3. What if the biggest board is $m_1 \times m_1$, the second level of boards are $m_2 \times m_2$, the i-th level of boards are $m_i \times m_i$ and so on?
4. How does this work in higher dimensions? On other grids? On polygons on the vertices of polygons (on the vertices of polygons)?

Note. It appears that the group that acts on an ordinary board is D_8. The group that acts on a depth-2 board is $D_8 \wr D_8$, a depth-3 board is $(D_8 \wr D_8) \wr D_8$ and so on.

References.

Problems 31, 61, 79, 86, and 91.

Wikipedia, Ultimate tic-tac-toe.