Consider an $n \times n$ grid with n marked cells, where each marked cell must be contained in exactly one of n distinct continuous n-cell regions.

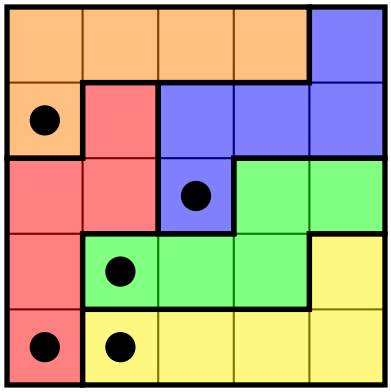

Figure 1: An example of a 5×5 grid with a unique solution.

Question. How many $n \times n$ boards exist with a unique solution?

Related.

1. How many $n \times n$ boards exist with no solution? Multiple solutions?
2. What board has the most solutions?
3. What if this is counted up to dihedral action?
4. What if this is done on an $n \times m$ board with k marked cells where $k|nm$ and each region has nm/k cells?
5. What if the board is a torus? Triangular/hexagonal grid? Multiple dimensions?
6. What if instead of marked cells there are marked regions? (e.g. two adjacent cells are both marked "1" and must be captured by the same region)
7. What if cells must must be rectangular? Symmetric?

References.

Problem 28

`https://math.stackexchange.com/q/3072735/121988`