**Difficulty: 3/4   Interest: 3/4**

Ron Graham's Sequence (A006255) is the least k for which there exists a strictly increasing sequence

$$n = b_1 < b_2 < \ldots < b_t = k \text{ where } b_1 \cdot \ldots \cdot b_t \text{ is square.}$$

There is a known way to efficiently compute analogous functions $a_p$ where $a_p(n)$ is the least integer such that there exists a sequence

(a) $n = b_1 \leq b_2 \leq \ldots \leq b_t = a_p(n)$,

(b) any term appears at most $p - 1$ times, and

(c) $b_1 \cdot b_2 \cdot \ldots \cdot b_t$ is a $p$-th power.

**Question.** An efficient way to compute $a_p$ is known when $p$ is prime. What is an efficient way to compute $a_c$ when $c$ is composite?

$$\begin{aligned}
a_4(1) &= 1 && \text{via } 1 && = 1^4\\
a_4(2) &= 2 && \text{via } 2^2 \cdot 4 && = 2^4\\
a_4(3) &= 6 && \text{via } 3^2 \cdot 4 \cdot 6^2 && = 6^4\\
a_4(4) &= 4 && \text{via } 4^2 && = 2^4\\
a_4(5) &= 10 && \text{via } 5^2 \cdot 8^2 \cdot 10^2 && = 20^4\\
a_4(6) &= 9 && \text{via } 6^2 \cdot 8^2 \cdot 9 && = 12^4\\
a_4(7) &= 14 && \text{via } 7^2 \cdot 8^2 \cdot 14^2 && = 28^4\\
a_4(8) &\leq 15 && \text{via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 && = 60^4\\
a_4(9) &= 9 && \text{via } 9^2 && = 3^4\\
a_4(10) &\leq 18 && \text{via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 && = 180^4
\end{aligned}$$

Figure 1: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

**Related.**

1. For what values $n$ is $a_4(n) < A006255(n)$?
2. Given some integers $k, c$, how many terms have $a_c(n) = k$? (e.g. $a_4(6) = a_4(9) = 9$.)
3. Does $a_c$ contain arbitrarily many copies of the same value?
   (i.e. does there exist a sequence such that $a_4(n_1) = a_4(n_2) = \ldots = a_4(n_m)$ for arbitrarily large $m$?)
4. How many times does $k$ appear in the image of $a_c$? (e.g. 9 appears twice, as $a(6)$ and $a(9)$.)
5. What integers are in the image of $a_c$?

**References.**

https://oeis.org/A300516