Difficulty: 2/4   Interest: 4/4

Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way that no end is "orphaned".

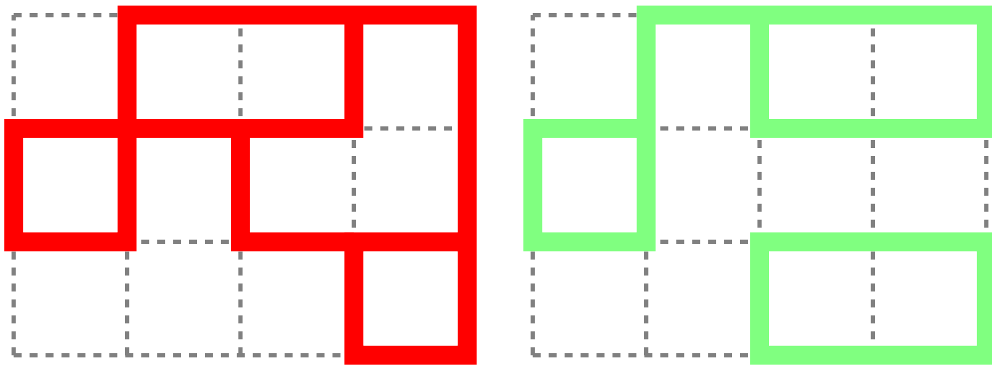

Figure 1: Two examples of a valid configurations on a 5×4 grid; the second has an "island" in the lower right corner.

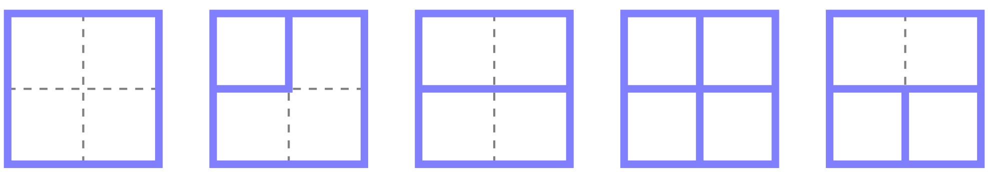

Figure 2: All(?) examples of valid configurations of 3×3 grids with border, up to dihedral action.

Question. Let $a_\ell(n)$ be the number of configurations on the $\ell \times n$ grid. What is a general formula for $a_\ell(n)$?

Related.

1. What if the matchsticks are of length k?
2. How does this generalize to a triangular/hexagonal lattice or to multiple dimensions?
3. What is the number of these configurations with rotational symmetry? Horizontal/vertical symmetry?
4. If such a configuration is chosen uniformly at random, what is the number of expected regions? (e.g. the first example has 4 interior regions.)
5. What if diagonals are allowed?
6. What if no gridpoint can have degree 0? Degree 2? 3? 4?
7. What if the entire border must be drawn?
8. What if the the graph must be connected (i.e. cannot have an "island".)
9. What if instead of horizontal/vertical lines, all edges have integer slope? What if these edges don't intersect except at vertices?

References.

The number of $2 \times n$ grids appears to be given by A093129.

The number of $3 \times n$ grids is given by A301976.