

**Difficulty:** 3/4    **Interest:** 4/4

Consider convex polygons with integer coordinates. The notion of a best Diophantine approximation can be generalized to equilateral triangles by saying that a triangle is a better diophantine approximation if the ratio of the largest side to the smallest side is less than the ratio of any other triangle with smaller perimeter.

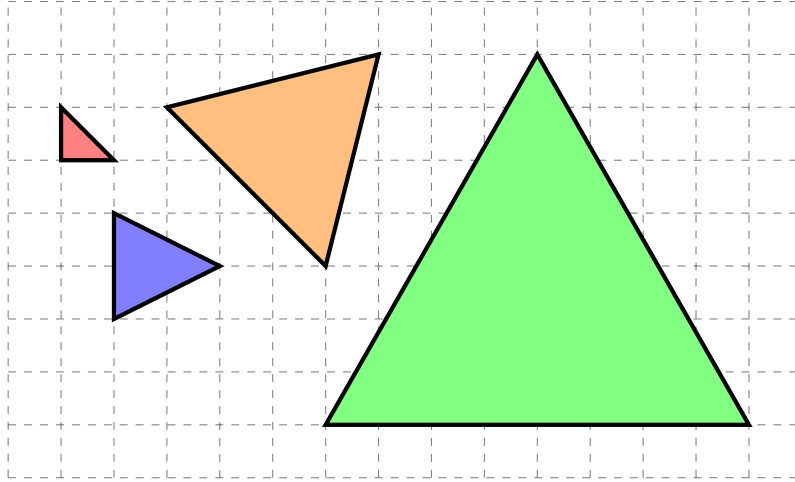


Figure 1: Four best (?) Diophantine approximations of an equilateral triangle. The red triangle has a ratio of  $\sqrt{2/1} \approx 1.41$ , the blue has a ratio of  $\sqrt{5/4} \approx 1.118$ , the orange has a ratio of  $\sqrt{18/17} \approx 1.029$ , and the green has a ratio of  $\sqrt{64/63} \approx 1.008$ .

**Question.** What is the growth of the perimeter of the  $k$ -th best Diophantine approximation of an equilateral triangle as a function of  $k$ ?

**Related.**

1. How can this be generalized in a reasonable way to regular  $n$ -gons? (Just looking at side lengths isn't enough—angles can behave badly.)
2. What if this is done on tetrahedra?

**References.**

<https://math.stackexchange.com/q/2251555/121988>