Consider pairs subsets of $[n] = \{1, 2, 3, \dots, n\}$ such that the arithmetic mean of the subsets is equal. How many different pairs of subsets can we find, up to some sort of depedence, where two pairs are equivalent if there exists a linear transformation that takes one pair to the other, or if there exists a "chain" of subsets that implies equality.

$$\left(\frac{1+5}{2} = \frac{1+3+5}{3}\right) \cong \left(\frac{3+5}{2} = \frac{3+4+5}{3}\right) \tag{1}$$

$$\left(\frac{1+5}{2} = \frac{3}{1}\right) \cong \left(\frac{3}{1} = \frac{2+4}{2}\right) \implies \left(\frac{1+5}{2} = \frac{2+4}{2}\right) \tag{2}$$

Figure 1: The first equalities are considered equivalent under the linear transformation $x \mapsto \frac{1}{2}(x+5)$. The equality $\frac{1}{2}(1+5) = \frac{1}{2}(2+4)$ is a combination of equations (1) and (2), and so is not an independent equation.

Question. Is there some notion of a "basis" for these pairs of subsets, from which we can work out all pairs with equal means?

Related.

1. What is the minimal "basis" that can describe all pairs of subsets with equal means?
2. What if the subsets in the pair need to be disjoint?
3. Is there a way to combine two pairs of subsets into another pair?
4. Can this generalize to multisets?