Given some set of functions $\{p_i\colon [n] \to [k]\}_{i=1}^m$ consider the semigroup generated by $\langle p_1, p_2, \ldots, p_m \rangle$.

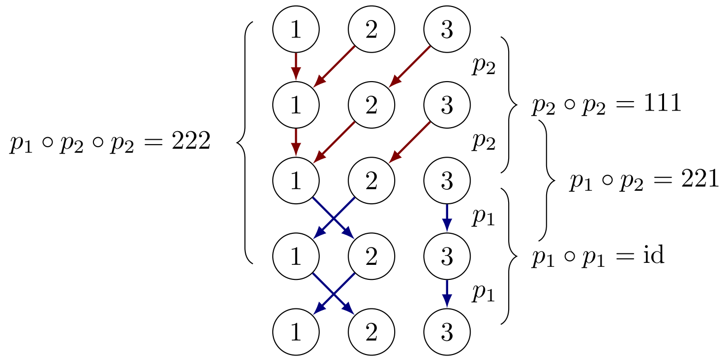

Figure 1: Given $p_1 = 213$ and $p_2 = 112$, there are six distinct functions that can be made from nonempty compositions of these functions: p_1, $p_1 \circ p_1$, $p_1 \circ p_2$, $p_1 \circ p_2 \circ p_2$, p_2, and $p_2 \circ p_2$.

Question. If the p_is are chosen uniformly at random, what is the expected size of the semigroup?

Related.

1. What's the expected number of functions m such that $|\langle p_1, p_2, \ldots, p_m \rangle| = k^n$? The minimum number of functions?

2. What is the largest semigroup as a function of n, k, and m?

3. Can you make any size semigroup with the right parameters? If not, what sizes can you make?

References.

Problem 6.