Problem 1.

Suppose you are given an $n \times m$ grid, and I then think of a rectangle with its corners on grid points. I then ask you to "black out" as many of the gridpoints as possible, in such a way that you can still guess my rectangle after I tell you all of the non-blacked out vertices that its corners lie on.

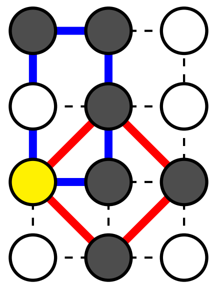


Figure 1: An example of an invalid "black out" for an 3×4 grid. The blue rectangle and the red rectangle have the same presentation, namely the gridpoint inside the yellow circle.

Question. How many vertices may be crossed out such that every rectangle can still be uniquely identified?

Related.

1. What if the interior of the rectangle is lit up instead?
2. What if all gridpoints that instersect the perimeter are lit up?
3. What if the rectangles must be square?
4. What if parallelograms are used instead of rectangles?
5. What if the rectangles must be horizontal or vertical?
6. What if the rectangles must be horizontal, vertical, or 45 diagonal?
7. What if this is done on a triangular grid with equilateral triangles?
8. What if this is done in more dimensions (e.g. with a rectangular prism or tetrahedron?)

Problem 2.

Jeremy Kun gives a canonical bijection between $\binom{n+1}{2}$ and a discrete tringle of length n, as seen in Figure 1.

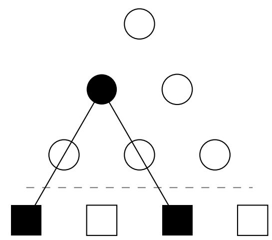


Figure 1: Bijection that maps a point on the triangle with side length 3 to a 2-subset of [3 + 1].

Question. Is there a similar "projection" that bijects a point on the discrete tetrahedron to a 3-subset of [n + 2]?

Note. Misha Lavrov gives a potential function to the question on Math Stack Exchange. (https://math.stackexchange.com/a/2468687/121988)

Related.

1. More generally is there a bijection from the k-simplex to a k-subset of [n + k - 1]?

Problem 3.

Let G be some $n \times m$ grid as in Figure 1, where each cell has two opposite diagonals connected (uniformly at random). A cell is chosen (also uniformly at random), and the segment given by the path of diagonals that goes through the selected cell is is inspected.

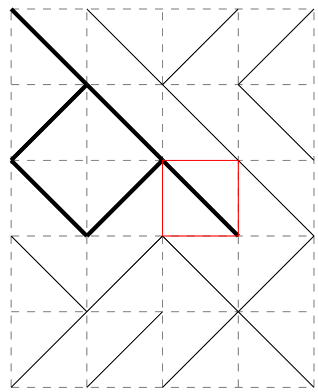


Figure 1: An example of a 4×5 grid, where a segment of size 6 has been selected.

Question. What is the expected length of the selected segment?

Related.

1. What is the expected number of segments in an $n \times m$ grid?
2. How long is the longest segment expected to be?
3. How does this change if the grid is toroidal, on a cylinder, on a Möbius strip, etc?

Problem 4.

Peter Winkler's Coins-in-a-Row game works as following:

> On a table is a row of fifty coins, of various denominations. Alice picks a coin from one of the ends and puts it in her pocket; then Bob chooses a coin from one of the (remaining) ends, and the alternation continues until Bob pockets the last coin.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed according to some probability distribution.

Question. For some fixed ω, what is the expected first player's score of Peter Winkler's Coins-in-a-Row game when played with $X_1(\omega), X_2(\omega), \ldots, X_3(\omega)$ where both players are using a min-max strategy?

Note. Let

$$e = E[X_2 + X_4 + \ldots + X_{2n}] \text{ and } o = E[X_1 + X_2 + \ldots + X_{2n-1}]$$

When played with $2n$ coins, the first player's score is bounded below by $\max(e, o) - \min(e, o)$ by the strategy outlined by Peter Winkler.

Trivially the first player's score is bounded above by the expected value of the n largest coins minus the expected value of the n smallest coins.

Related.

1. If all possible n-coin games are played with coins marked 0 and 1, how many games exist where both players have a strategy to tie.
2. How does this change when played according to the (fair) Thue-Morse sequence?
3. What if the players are cooperating to help the first player make as much as possible (with perfect logic)?
4. What is both players are using the greedy algorithm?
5. What if one player uses the greedy algorithm and the other uses min-max? (i.e. What is the expected value of the score improvement when using the min-max strategy?)
6. What if one player selects a coin uniformly at random, and the other player uses one of the above strategies?

Problem 5.

Let a "popsicle stick weave" be a configuration of lines segments, called "sticks", such that

(1) when you lift up any stick by the end, the structure supports itself (is in tension)

(2) the removal of any stick results in a configuration that no longer supports itself.

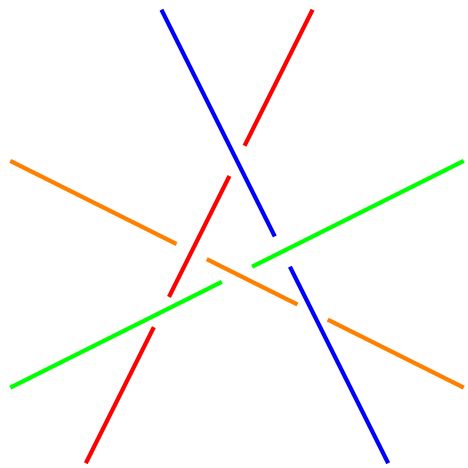


Figure 1: The unique example of a 4 stick crossing (up to reflection)

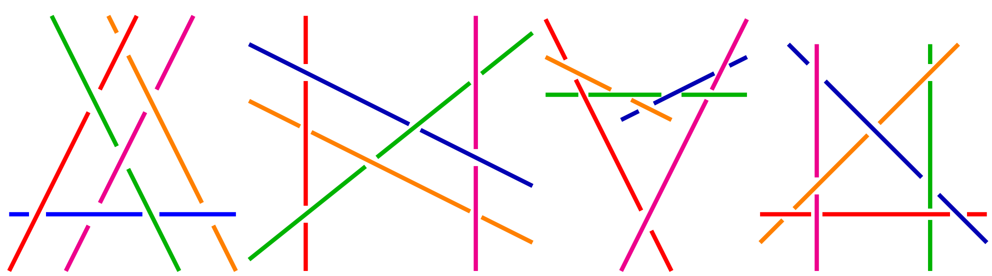


Figure 2: Four of five (?) known examples of five-stick crossings. Perhaps the fourth example shouldn't count, because shortening the blue stick to avoid the blue-red crossing results in a valid configuration (the remaining known five-stick crossing).

Question. How many distinct popsicle stick weaves exist for n sticks?

Related.

1. What if the sticks are only allowed to touch three other sticks?
2. What if the sticks are another geometric object (e.g. semicircles)?

Problem 6.

Let

$$C_n = \{f : [n] \to \mathbb{N} \mid \text{the convex hull around } \{(1, f(1)), \dots, (n, f(n))\} \text{ forms an } n\text{-gon}\}$$

and then let $a(n)$ denote the least upper bound over all functions in C_n

$$a(n) = \min\{\max\{f(k) \mid k \in [n]\} \mid f \in C_n\}$$

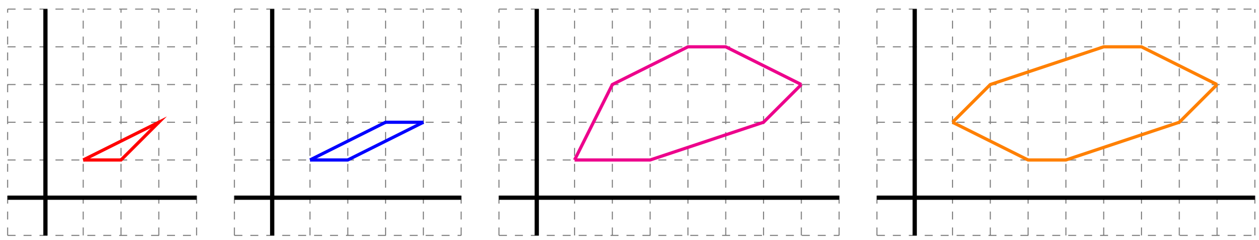


Figure 1: Examples of $a(3) = 2$, $a(4) = 2$, $a(7) = 4$, and $a(8) = 4$, where the polygons with an even number of vertices have rotational symmetry.

Question. Do these polygons converge to something asymptotically?

Related.

1. Does $a(2n) = a(2n - 1)$ for all n?
2. Do the minimal $2n$-gons always have a representative with rotational symmetry?
3. Are minimal $2n$-gons unique (up to vertical symmetry) with finitely many counterexamples?
4. What is the asymptotic growth of a(n)?

References.

A285521: "Table read by rows: the n-th row gives the lexicographically earliest sequence of length n such that the convex hull of (1, a(1)), ..., (n, a(n)) is an n-gon with minimum height." (https://oeis.org/A285521)

Problem 7.

Let $f_{n,m} : [n] \to [m]$ be a uniformly random function. Consider the convex hull around $\{(1, f(1)), \ldots (n, f(n))\}$

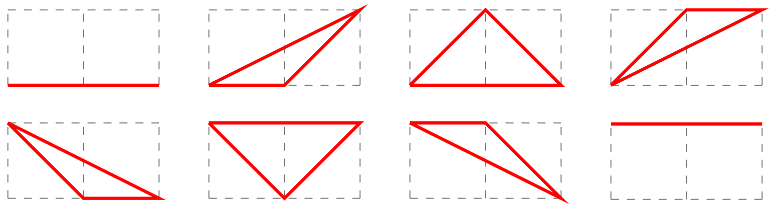


Figure 1: Examples of $f_{3,2}$. Here the expected number of vertices on a convex hull is 2.75

Question. What is the probability of seeing a k-gon (for some fixed k), when given a uniformly random function $f_{n,m}$?

Related.

1. What if $f_{n,n}$ is restricted to be a permutation?
2. What if $f_{n,m}$ is injective?

Problem 8.

Given an $n \times n$ grid, consider all convex polygons with grid points as vertices. Let $m(n)$ be the greatest integer k such that there exists a convex k-gon on the $n \times n$ grid.

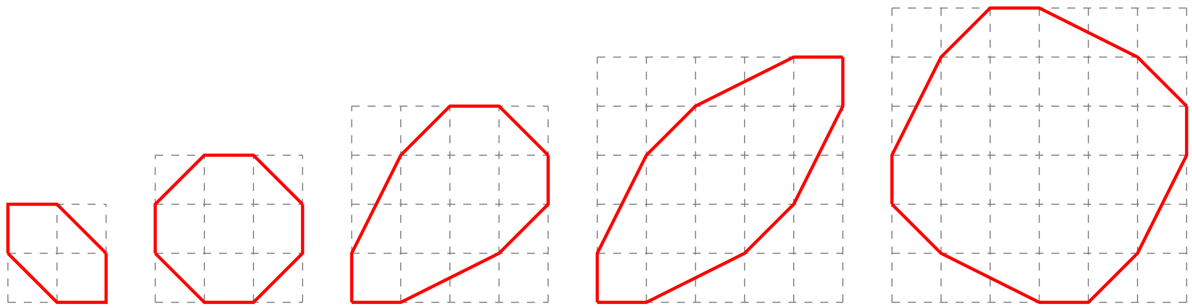


Figure 1: Examples that prove $m(3) = 6, m(4) = 8, m(5) \geq 9, m(6) \geq 10$, and $m(7) \geq (12)$

Question. What is $m(n)$?

Related.

1. What is a proof (or counterexample) that the examples shown are the best possible?
2. How does $m(n)$ grow asymptotically?
3. Do the shapes do anything interesting in the limit?
4. Are there finitely many maximal polygons without rotational symmetry (e.g. $m(5)$)?
5. How does this generalize to $m \times n$ grids?
6. See Problems 6 and 7.

Problem 9.

Given an $n \times n$ grid, consider all the ways that convex polygons with grid points as vertices can be nested.

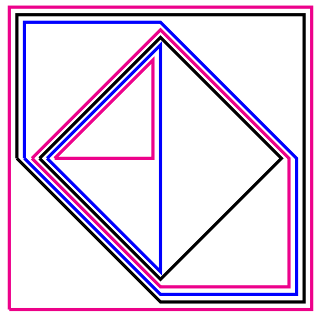


Figure 1: Seven nested convex polygons in the 3×3 grid.

Question. If we think of each polygon having the same height, what is the greatest volume that we can make by stacking the polygons this way?

Related.

1. What is the largest sum of the perimeters? The least?
2. What is the largest sum of the number of vertices? The least?
3. How many ways are there to stack $n^2 - 2$ polygons like this? Any number of polygons?
4. Does this generalize to polyhedra in the $n \times n \times n$ cube?
5. Does this generalize to polygons on a triangular grid?

Problem 10.

Consider all k-colorings of an $n \times n$ grid, where each row and column has $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ cells with each color.

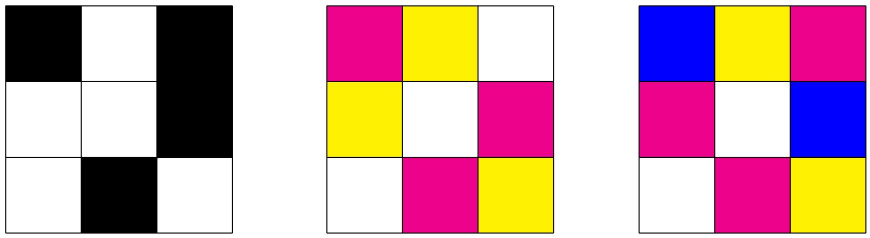


Figure 1: A valid 2-coloring, 3-coloring, and 4-coloring of an 3×3 grid.

Question. How many such k-colorings of the $n \times n$ grid?

Related.

1. What if there also must be a total of $\lfloor n^2/k \rfloor$ or $\lceil n^2/k \rceil$ cells of each color?
2. What if these are counted up to the dihedral action on the square D_4?
3. What if these are counted up to torus action?
4. What if these are counted up to permutation of the coloring?
5. Can this generalize to the cube? To a triangular tiling?

Problem 11.

Consider an $n \times n$ chess board, with pieces that can move integer distances, but only in diagonal directions—that is, they move like the hypotenuse of a Pythagorean triangle.

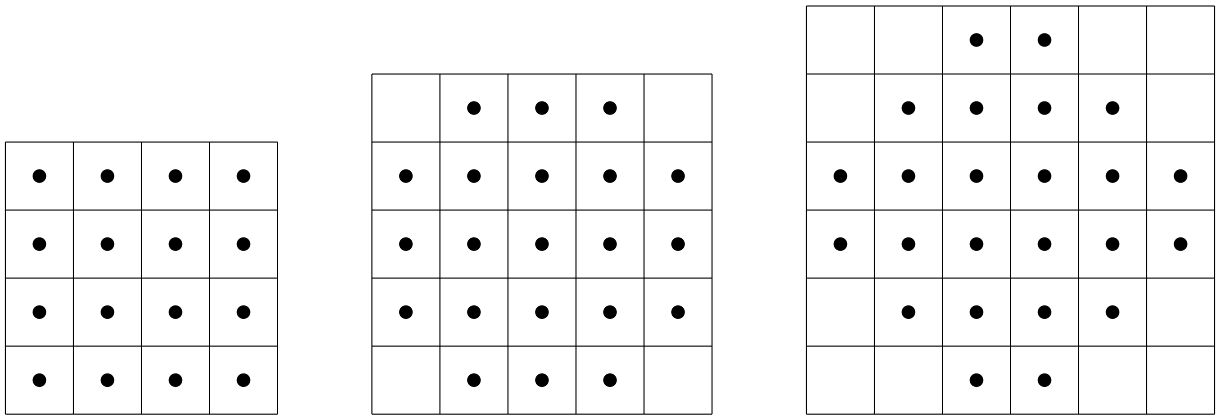


Figure 1: Valid configurations for 4×4, 5×5, and 6×6 grids, proving that $a(4) = 16$, $a(5) \geq 21$, and $a(6) \geq 24$.

Question. What is the greatest number of nonattacking pieces that can be placed on the board?

Related.

1. What is the board is $n \times m$?
2. What if pieces must move like *primitive* Pythagorean triples?
3. What if each piece can move k times?
4. What is the asymptotic growth of a?

Problem 12.

Consider Ron Graham's sequence for lcm, that is, look at sequences such that

$$n = a_1 < a_2 < \ldots < a_T = k \text{ and } \operatorname{lcm}(a_1, \ldots, a_T) \text{ is square.}$$

Question. What is the least k (as a function of n) such that such a sequence exists?

$$\begin{aligned}
a(1) &= 1 \quad \text{via } (1) \\
a(2) &= 4 \quad \text{via } (2,4) \\
a(3) &= 3 \quad \text{via } (3,9) \\
a(4) &= 4 \quad \text{via } (4) \\
a(5) &= 25 \text{ via } (5,25) \\
a(6) &= 12 \text{ via } (6,9,12) \\
a(7) &= 49 \text{ via } (7,49) \\
a(8) &= 16 \text{ via } (8,16)
\end{aligned}$$

Figure 1: Examples of $a(n)$ for $n \in \{1, 2, \ldots, 8\}$.

Related.

1. For what values n is $a(n)$ nonsquare?
2. For what values n does the corresponding sequence have three or more terms?
3. What is the analogous sequence for perfect cubes, etc?

Problem 13.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

There is a known way to efficiently compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a p-th power, where p is a prime and where any term appears at most $p - 1$ times.

Question. What is an efficient way to compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a c-th power, where c is composite?

$$\begin{aligned}
a_4(1) &= 1 \text{ via } 1 &&= 1^4 \\
a_4(2) &= 2 \text{ via } 2^2 \cdot 4 &&= 2^4 \\
a_4(3) &= 6 \text{ via } 3^2 \cdot 4 \cdot 6^2 &&= 6^4 \\
a_4(4) &= 4 \text{ via } 4^2 &&= 2^4 \\
a_4(5) &= 10 \text{ via } 5^2 \cdot 8^2 \cdot 10^2 &&= 20^4 \\
a_4(6) &= 9 \text{ via } 6^2 \cdot 8^2 \cdot 9 &&= 12^4 \\
a_4(7) &= 14 \text{ via } 7^2 \cdot 8^2 \cdot 14^2 &&= 28^4 \\
a_4(8) &\leq 15 \text{ via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 &&= 60^4 \\
a_4(9) &= 9 \text{ via } 9^2 &&= 3^4 \\
a_4(10) &\leq 18 \text{ via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 &&= 180^4
\end{aligned}$$

Figure 1: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

Related.

1. For what values n is $a(n) < A006255(n)$?
2. How many c-th power sequences have $a_T = a_c(n)$?
3. Do any such c-th power sequences exactly two distinct terms?

Problem 14.

Suppose you have a strip of toilet paper with n pieces, and you fold the paper evenly into d parts (divide by d) or fold the last k pieces in (subtract by k), until the length of the strip is less than k pieces.

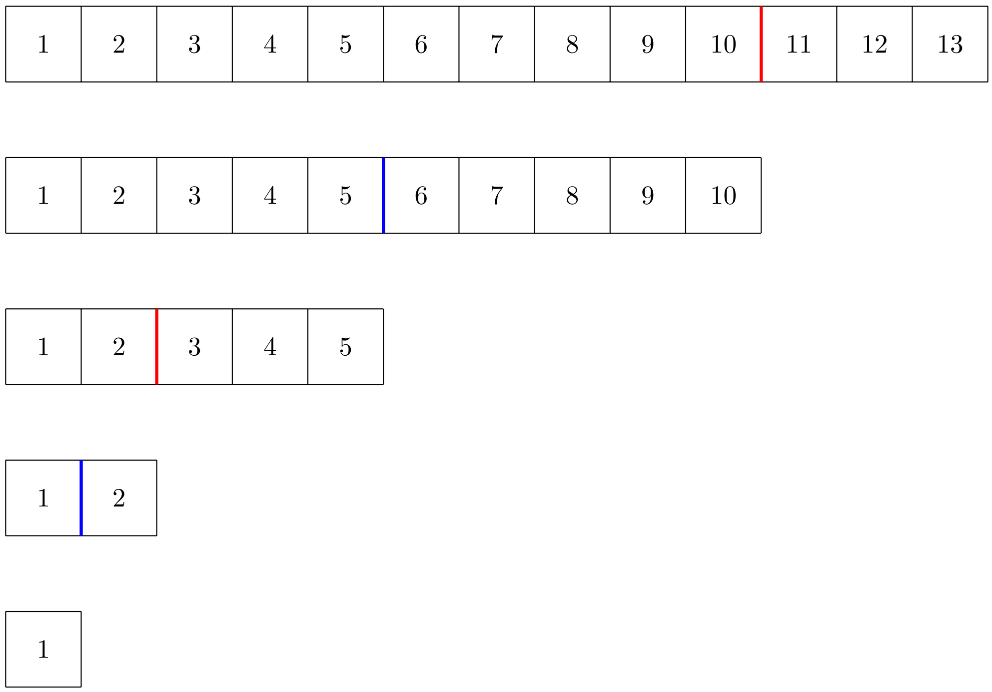


Figure 1: A folding of paper where $n = 13$, $d = 2$, and $k = 3$, showing that $a_{2,3}(13) \leq 4$. Where the red marks a subtraction by k and the blue marks a division by d.

Question. Is there an efficient way to compute $a_{d,k}(n)$?

Related.

1. What if you must keep folding until you cannot fold any longer?
2. What is the minimum number of terminal pieces? What is the minimum number of steps to this number?

Problem 15.

OEIS sequence A261865 describes "$a(n)$ is the least $k \in \mathbb{N}$ such that some multiple of $\sqrt{k} \in (n, n+1)$." Clearly the asymptotic density of 2 in the sequence is $1/\sqrt{2}$.

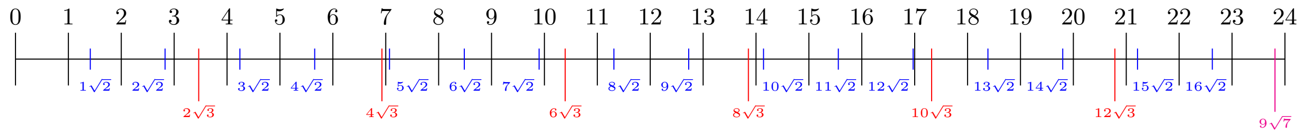


Figure 1: An illustration of $a(n)$ for $n \in \{1, 2, \ldots, 23\}$.

Question. Let $S_\alpha \subset \mathbb{N}$ denote the squarefree integers strictly less than α. Is the asymptotic density of squarefree j given by

$$\frac{1}{\sqrt{j}} \prod_{s \in S_j} \left(1 - \frac{1}{\sqrt{s}}\right)?$$

Related.

1. Is there an algorithm to construct a value of n such that $a(n) > K$ for any specified K? (Perhaps using best Diophantine approximations or something?)
2. What is the asymptotic growth of the records?
3. Given some α what is the expected value of the smallest n such that $S_\alpha \subset \{a(1), \ldots, a(n)\}$?
4. This sequence uses the "base sequence" of $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\}$. On what other base sequences is this construction interesting?

Problem 16.

Richard Guy beat me to this problem by a few years. (https://arxiv.org/abs/1207.5099).
John Conway described the "Subprime Fibonacci Sequence":

$$a(1) = a, a(2) = b, a(n+1) = \text{gpd}(a(n) + a(n-1)),$$

where $\text{gpd}(k)$ is the greatest proper divisor of k.
Conway then conjectured that regardless of the starting terms, the sequence ends in a handful of cycles.
Richard Guy found that there are more cycles than those that Conway conjectured.

Question. What are all of the different possible end behaviors of Conway's Subprime Fibonacci Sequence?

Problem 17.

Start with n piles with a single stone in each pile. If two piles have the same number of stones, then any number of stones can be moved between them.

$$
\begin{tikzcd}
 & 1,1,1,1,1 \arrow[d, dash] & \\
 & 2,1,1,1 \arrow[d, dash] & \\
 & 2,2,1 \arrow[dl, dash] \arrow[dr, dash] & \\
3,1,1 \arrow[d, dash] & & 4,1 \\
3,2 & &
\end{tikzcd}
$$

Figure 1: An illustration of all possible moves for $n = 5$.

Question. What is the greatest number of steps that can occur? Alternatively how many "levels" are in the tree of possible moves?

Related.

1. Let s be the total number of distinct states. (The example shows that $s(5) = 6$.)
2. Let c be the total number of states that *cannot* be acheived. (In the example, $c(5) = 1$ via the state (5).)
3. Is $c(p) = 1$ for all primes p?
4. Is $c(n) = 0$ if and only if n is a power of 2?
5. Let ℓ be the least number of steps to a terminal state. (In the example, $\ell(5) = 3$ ending in the state $(4, 1)$.)
6. Let g be the greatest number of steps to a terminal state. (In the example, $g(5) = 4$ ending in the state $(3, 2)$.)
7. Let p be the total number of paths. (In the example, $p(5) = 2$.)
8. Let t be the number of distinct *terminal* states. (In the example, $t(5) = 2$ with states $(4, 1)$ and $(3, 2)$.)

Problem 18.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

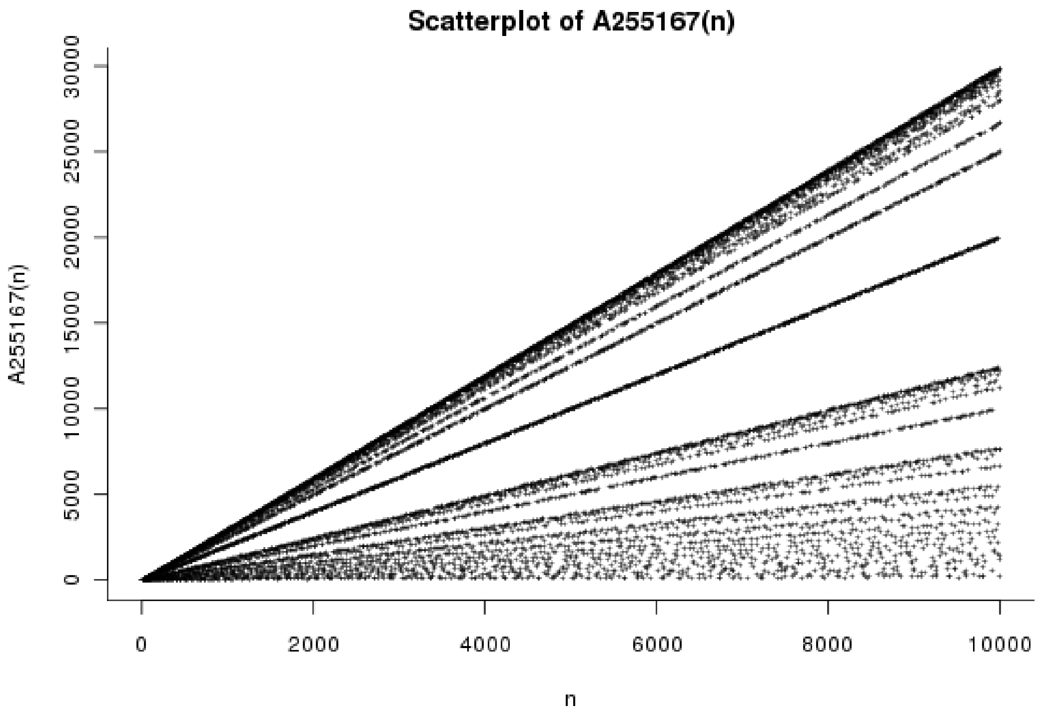
$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

A006255 is bounded above by A072905, the least $k > n$ such that $k \cdot n$ is square.

Question. Does there exist any n for which $A006255(n) = A072905(n)$. In other words, is there any non-square n for which $n \cdot A006255(n)$ is square?

Related.

1. Does the gap $A072905(n) - A006255(n)$ have a nonzero lower bound?



Problem 19.

Starting with 1 and working in a hexagonal spiral, repeatedly choose the smallest positive integer such that it won't be adjacent either itself (once) or to the same number twice.

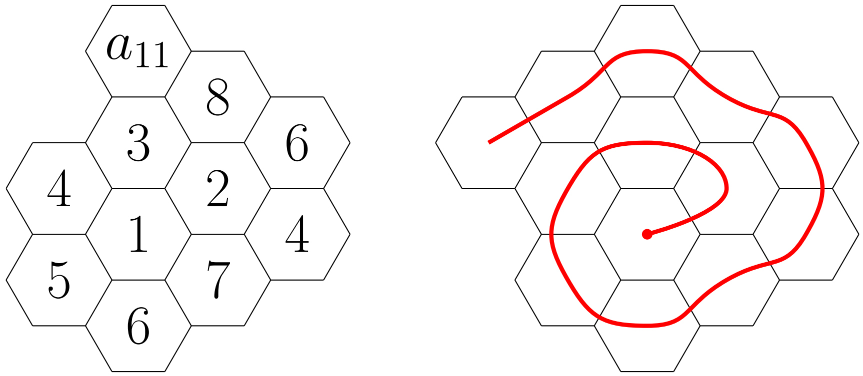


Figure 1: $a_{11} \neq 1$ because 3 is already adjacent to 1, $a_{11} \neq 2$ because 3 and 8 are already adjacent to 2, $a_{11} \neq 3$ because then a_{11} would be equal to its neighbor, $a_{11} \neq 4$ because 3 is already adjacent to 4, thus $a_{11} = 5$.

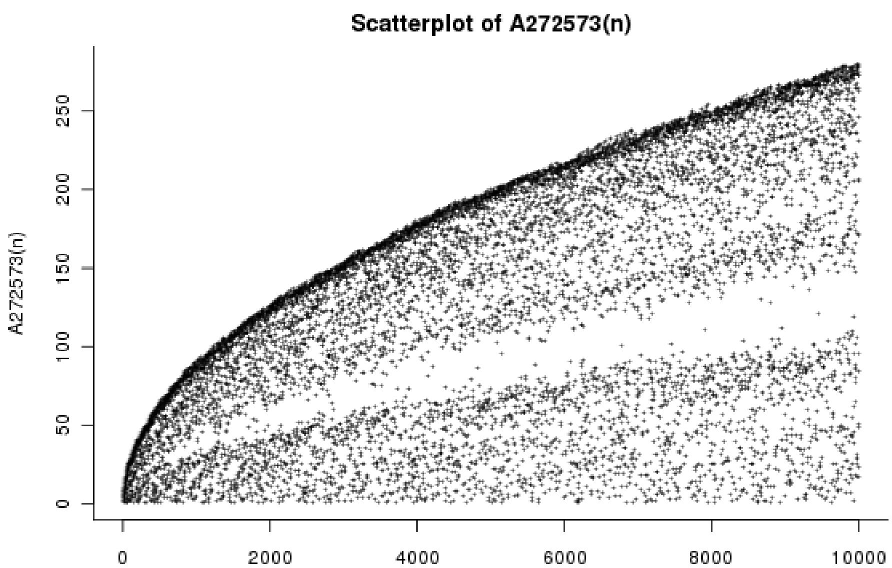


Figure 2: A plot of a_1 through a_{10000}.

Question. Why does a gap appear in the plot of the sequence?

Problem 20.

Let $a_3(n)$ be the least $k > n$ such that nk or nk^2 is a cube, and let $A299117$ be the image of $a_3(n)$.

$$\begin{aligned}
a_3(1) &= 8 \\
a_3(2) &= 4 \\
a_3(3) &= 9 \\
a_3(4) &= 16 \\
a_3(5) &= 25 \\
a_3(6) &= 36 \\
a_3(7) &= 49 \\
a_3(8) &= 27 \\
a_3(9) &= 24
\end{aligned}$$

Question. Is there another way to characterize what integers are in $A299117$?

Note. $A299117$ contains every cube, because $a(n^3) = (n+1)^3$.
$A299117$ contains the square of every prime, because $a(p) = p^2$.

Related.

1. Does $A299117$ contain every square?
2. Does $A299117$ contain any squarefree number?
3. What about the generalization: the image of a_β where $a_\beta(n)$ is the least $k > n$ such that $nk, nk^2, \ldots, nk^{\beta-2}$, or $nk^{\beta-1}$ is a β-th power? Prime β is an injection—is this well behaved?

Problem 21.

Consider placing any number of queens (of the same color) on an $n \times n$ chessboard in such a way as to maximize the number of legal moves available.

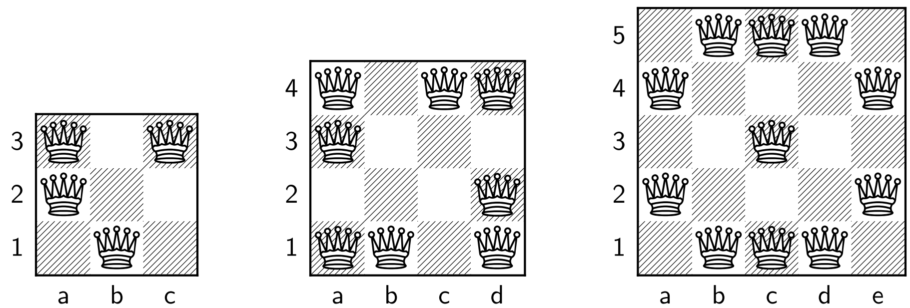


Figure 1: Examples of $a_q(3) = 17, a_q(4) = 40, a_q(5) = 76$.

Question. Is Alec Jones's conjecture true: $a_q(n) = 8(n-2)^2$ for $n \geq 6$, by placing the queens around the perimeter?

Related.

1. What about the analogous function for rooks (a_r) or bishops (a_b)?
2. What if the chessboard is a torus? Cylinder? Möbius strip?
3. What if the chessbaord is $n \times m$?
4. Is $a_b(n) = \lfloor a_q(n)/2 \rfloor$? for all n?
5. What if queens can attack?

References.

A278211: `http://oeis.org/A278211`

A278212: `http://oeis.org/A278212`

A275815: `http://oeis.org/A275815`

Problem 22.

Let U_n be the set of sequences of positive integers of length n such that no substring occurs twice.

$$(1,1,2,2,1,3,1) \in U_7 \tag{1}$$
$$(1,2,1,2,3) \notin U_5 \text{ because } (1,2) \text{ occurs twice.} \tag{2}$$
$$(1,1,1) \notin U_3 \text{ because } (1,1) \text{ occurs twice.} \tag{3}$$

Figure 1: An example and two non-examples of sequences with no repeated substrings.

Question. What is th enumber of sequences in U_n where the sum of terms is minimized?

Related.

1. What is the minimum least common multiple of a sequence in U_n? How many such minimal sequences?
2. What is the minimum product of a sequence in U_n? How many such minimal sequences?
3. What if substrings are considered forward and backward?
4. What if only substrings of length greater k are considered?
5. What is any term can appear at most ℓ times?

Problem 23.

Consider the fuction A285175(n) which is the lexicographically earliest sequence of positive integers such that no $k+2$ points are on a polynomial of degree k. (i.e. no two points are equal, no three points are colinear, no four points are on a parabola, etc.)

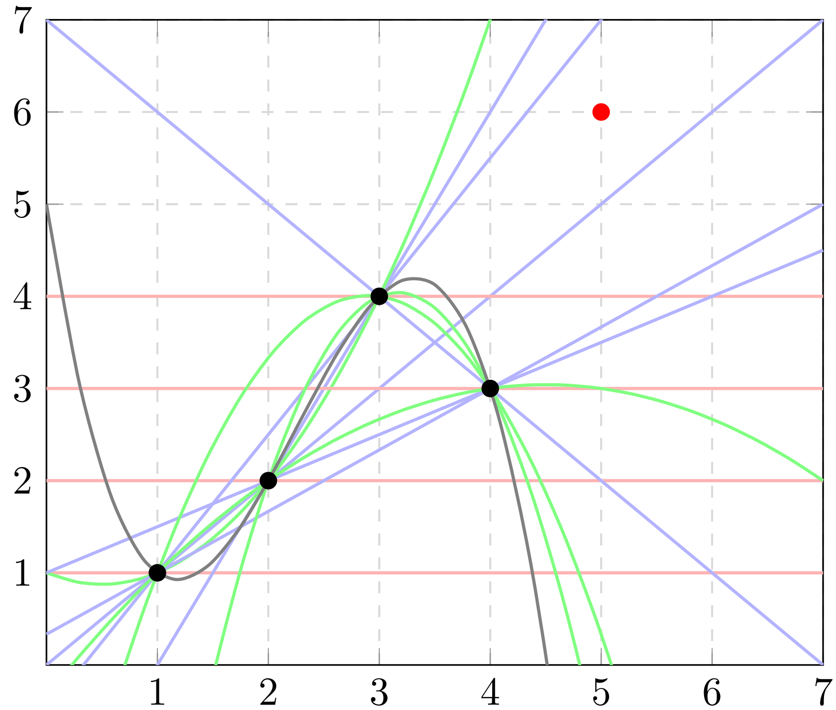


Figure 1: The first four points together with all interpolated polynomials. The red point marks the lowest integer coordinate $(5, k)$ that does not lie on an interpolated polynomial. (Degree 0 polynomials are plotted in red, degree 1 in blue, degree 2 in green and degree 3 in gray.)

Question. Do all positive integers occur in this sequence?

Related.

1. What is the asymptotic growth of this sequence?

Problem 24.

Let h be the maximum number of penny-to-penny connections on the vertices of a hexagonal lattice, and let $t(n)$ be the analogous sequence on the vertices of a triangular lattice.

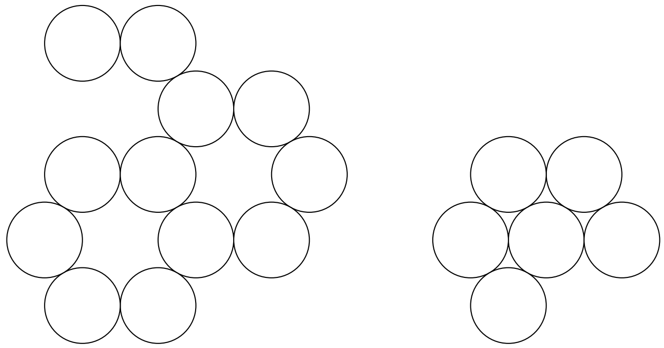


Figure 1: An example for $h(12) = 13$ and $t(6) = 9$

Question. What is a combinatorial proof that $h(2n) - t(n) = A216256(n)$.

Note. A216256 is

$$\underbrace{1}_{1}, \underbrace{2}_{1}, \underbrace{3, 3}_{2}, \underbrace{4, 4, 4}_{3}, \underbrace{5, 5, 5}_{3}, \underbrace{6, 6, 6, 6}_{4}, \underbrace{7, 7, 7, 7, 7}_{5}, \underbrace{8, 8, 8, 8, 8}_{5}, \underbrace{9, 9, 9, 9, 9, 9}_{6}, \dots$$

Problem 25.

Consider all rectangles with all corners on gridpoints on an $n \times m$ grid.

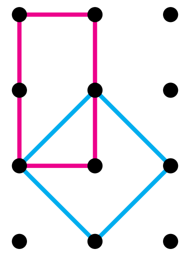


Figure 1: An example of two rectangles with all corners on gridpoints of a 3×4 grid.

Question. How many such rectangles exist?

Related.

1. How many squares exist? Rhombuses? Parallelograms? Kites? Quadrilaterals?
2. How many right triangles?
3. What if this is done on an $n \times m \times k$ grid?
4. What if the rectangles must be diagonal?
5. What if this is done on a triangular lattice?
6. How many tetrahedra are in an n-sided tetrahedra?

References.

See problem 1.

Problem 26.

The prime ant looks along the number line starting at 2. When she reaches a composite number, she divides by its least prime factor, and adds that factor to the previous term, and steps back.

2	3	4	5	6	7	8	9	10
2	5	2	5	6	7	8	9	10
2	5	2	7	3	7	8	9	10
2	5	2	7	3	9	4	9	10
2	5	2	7	6	3	4	9	10
2	5	2	9	2	3	4	9	10
2	5	5	3	2	3	4	9	10
2	5	5	3	2	3	7	3	10

Figure 1: An illustration of the prime ant's positions after the first 7 steps.

Question. Does the ant eventually stay to the right of any fixed position?

Related.

1. Are there any positions that stay permanently greater than 7? Than 11?
2. Does sequence of numbers converge in the long run? If so, what to? $(2, 5, 5, 3, 2, \ldots)$
3. Let S be a subset of $\mathbb{N}$ and let $f : S \times S^c \to \mathbb{N}^2$. For what "interesting" sets S and functions f can we answer the above questions?
   (In the example S is the prime numbers and f maps $(p, c) \mapsto (p + \mathrm{lpf}(c), \mathrm{gpf}(c))$.)

References.

https://codegolf.stackexchange.com/q/144695/53884

https://math.stackexchange.com/q/2487116/121988

https://oeis.org/A293689

Problem 27.

Consider polyominoes where each cell has one of n colors, and each distinct pair of colors is adjacent (horizontally or vertically) to each other somewhere in the polyomino. Let an n-minimum polyomino be one that has the minimum number of cells.

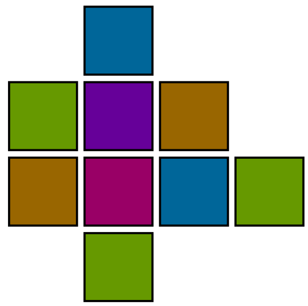


Figure 1: An example of a minimum polyomino for $n = 5$; $a(5) = 9$

Question. How many such n-minimum polyominoes exist?

Related.

1. What if the "distinct" restriction is lifted? (e.g. a blue label must somewhere be adjacent to another blue label.)
2. What is a way to determine the size of an n-minimum polyomino for large n?
3. What if this is done on a triangular or hexagonal grid?
4. What if this is done on a three dimensional cube lattice?

Problem 28.

Consider partitions of the $n \times m$ grid in which every piece has 180° rotational symmetry.

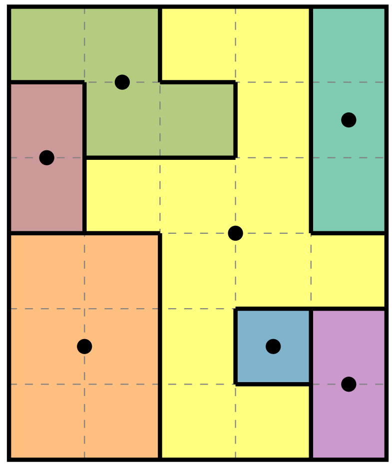


Figure 1: A partition of the 5×6 grid into 7 parts with rotational symmetry.

Question. How many such partitions of the $n \times n$ grid exist? Up to dihedral action?

Related.

1. How many partitions into exactly k parts?
2. How many partitions with other types of symmetry?
3. How many partitions of a torus? Cylinder? Möbius strip?
4. How many partitions of a triangular or hexagonal lattice?
5. How many partitions of an $n \times m \times p$ cuboid?

Problem 29.

Consider all rectangles composed of n squares such that the greatest common divisor of all the sidelengths is 1.

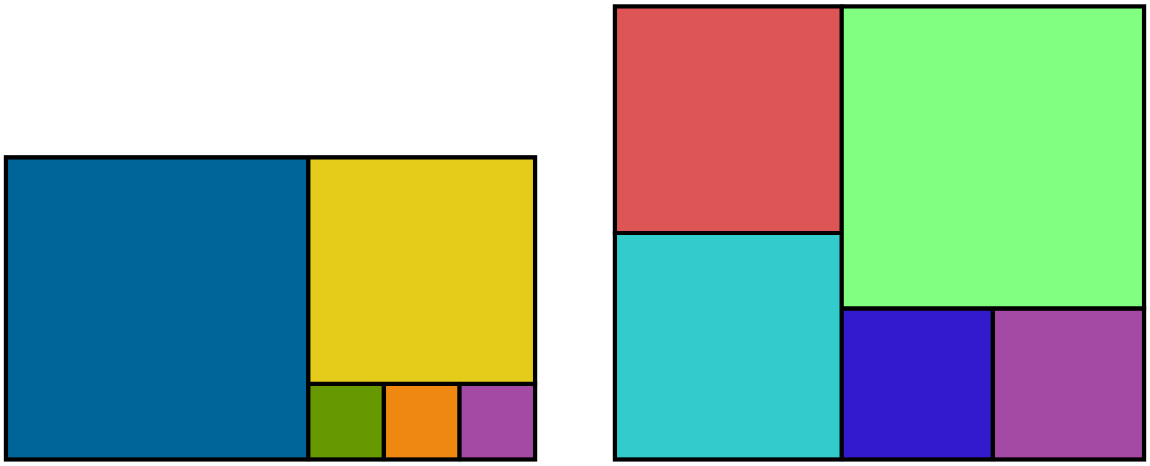


Figure 1: Two examples of rectangles made from $n = 5$ squares. In the first $\gcd(1, 1, 1, 3, 4) = 1$ and in the second $\gcd(2, 2, 3, 3, 4) = 1$.

Question. Given n squares, how many such rectangles exist?

Related.

1. How many ways are there to make convex polygons out of n equilateral triangles?
2. How many ways are there to make cuboids out of n cubes?

Problem 30.

Consider an n-coloring of a triangular grid such that no sub-triangle has corners all with the same color.

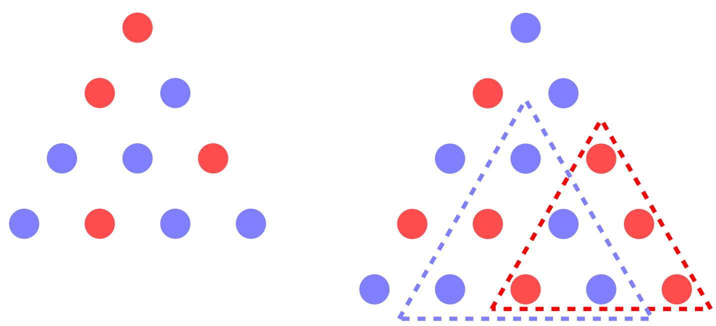


Figure 1: On the left is an example of a triangle on two labels that has no sub-triangles with equal corners. On the right is a non-example of such a triangle on two labels: it has two sub-triangles with equal corners.

Question. Given n labels, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

Related.

1. Given an n-coloring of a triangle of side length k, what number of sub-triangles with equal corners must exist?
2. How many such triangles exist?
3. What if diagonal equilateral triangles also are not allowed to have equal corners?
4. What if this is done with hexagons instead of triangles?
5. What if this is done on a square grid?
6. What if for $n \geq 3$ no *two* corners are allowed to be equal? (This is a bit like a peaceable queens problem on a hexagonal chessboard.)

References.

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

Problem 31.

A country has a strange legislative procedure. For each bill, the body is split up into k_1 committees of $\lfloor n/k_1 \rfloor$ or $\lfloor n/k_1 \rfloor$ legislators each, each of which picks a representative. These k_1 representatives are split up into k_2 sub-committees with $\lfloor k_1/k_2 \rfloor$ or $\lfloor k_1/k_2 \rfloor$ legislators each, which each elect a representative, and so on until $k_T = 1$ and the final committee votes on the bill.

There are a few rules:

1. Each committee (and subcommittee and so on) much have at least ℓ members.
2. Ties are settled by a coin toss.
3. The president does not get to vote, but she does get to choose the number of committees and who goes in each one.
4. There are α supporters who will always vote in the president's interests and $n - \alpha$ who will always vote against.

Let $a_\ell(n)$ be the minimum number of supporters (α) required for the president to be able to pass every bill.

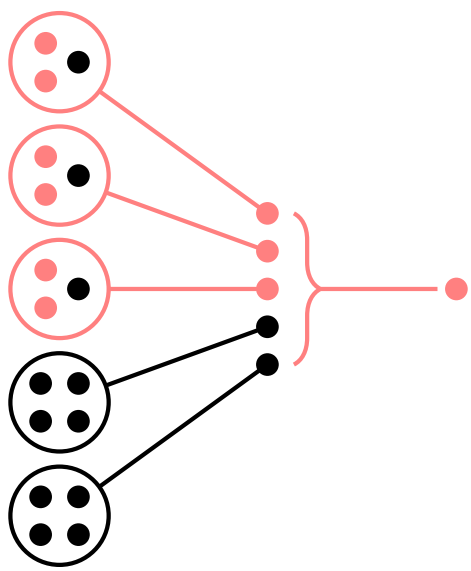


Figure 1: An example of $n = 17$ legislators with a minimum comittee size of $\ell = 3$, which demonstrates that $a_3(17) \leq 6$.

Question. What is an efficient way to compute $a_\ell(n)$ for general ℓ and n?

Related.

1. What if the president gets to choose who is on each committee but the opposition party gets to choose the committee size? Vice versa?
2. What if $k_1 \leq k_2 \leq \ldots \leq k_T$? Or $k_1 \geq k_2 \geq \ldots \geq k_T$?
3. What if ties go to the president? To the opposition?

References.

https://oeis.org/A290323

https://math.stackexchange.com/q/2395044/121988

Problem 32.

Consider tilings of the $n \times n$ grid up to D_8 action where the tiles are diagonals.

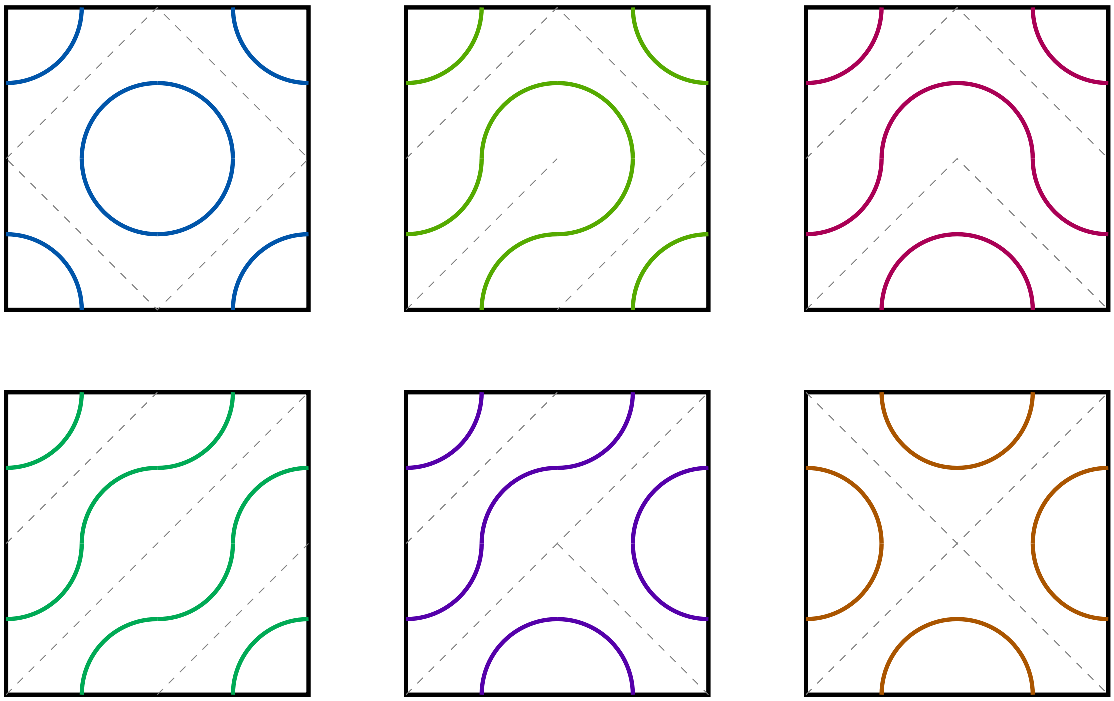


Figure 1: An example of the $a(2) = 6$ different ways to fill the 2×2 grid with diagonal tiles (up to dihedral action).

Question. How many such tilings exist?

Related.

1. What if grids are only counted up to C_4 (rotation) action?
2. What if this is counted on the torus/cylinder/Möbius strip?
3. What if each tile can have no diagonals or both diagonals?
4. What if tiles are black or white?
5. Is there an obvious bijection between the results on the $2n \times 2n$ grid for black/white versus diagonal tile types?

Problem 33.

Consider the rectangles from Problem 29: those composed of n squares such that the greatest common divisor of all the sidelengths is 1. If rectangles are measured by the longest side, the smallest rectangles are given by $A295753$.

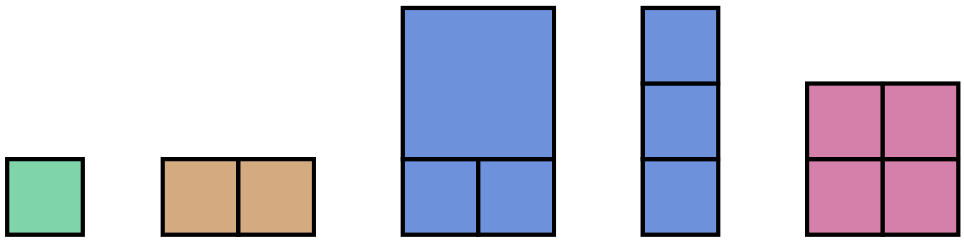


Figure 1: Examples of $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, and $a(4) = 1$.

Question. How many distinct rectangles composed of n squares have a longest side of $A295753(n)$?

Related.

1. Is the largest rectangle (as measured by smallest side) unique for large n?
2. What if smallest rectangle is measured by perimeter?

Note. Largest rectangles might be Fibonacci spirals, or they might be similar to the second example or the examples in the References.

References.

https://en.wikipedia.org/wiki/Squaring_the_square

Problem 34.

Consider all configurations of nonattacking rooks on an $n \times n$ board up to dihedral action.

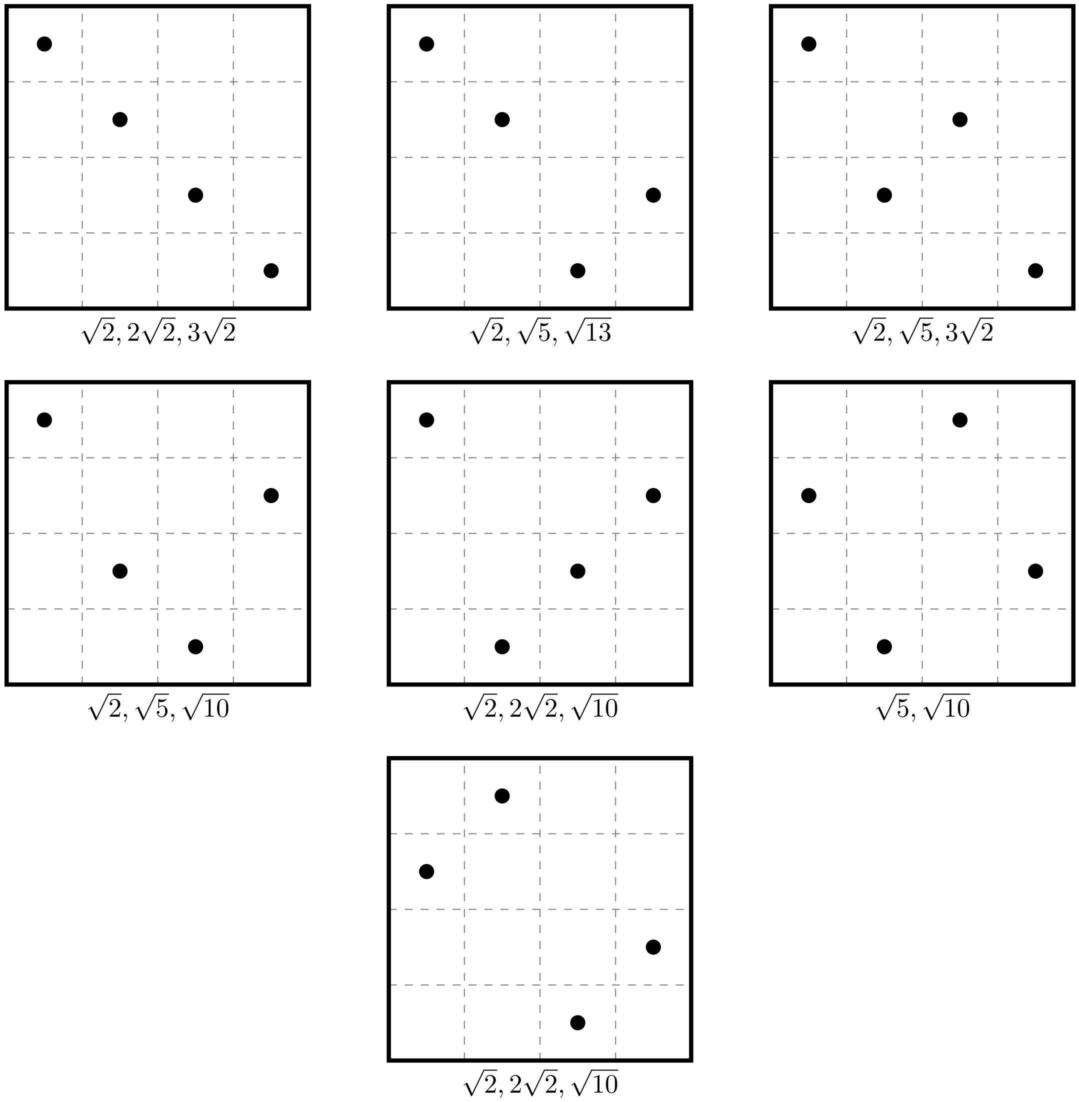


Figure 1: Each figure is marked with the distinct distances between pieces.

Question. What is the minimum number of distinct distances on such a figure?

Related.

1. What if rooks are allowed to be in attacking positions?
2. How many configurations of nonattacking rooks on the torus?
3. Are any configurations of nonattacking rooks on the torus that can be meaningfully called a "generalized Costas array"?

References.

https://en.wikipedia.org/wiki/Costas_array

Problem 35.

Consider square, triangular, and hexagonal grids that are filled in with with tiles of different patterns.

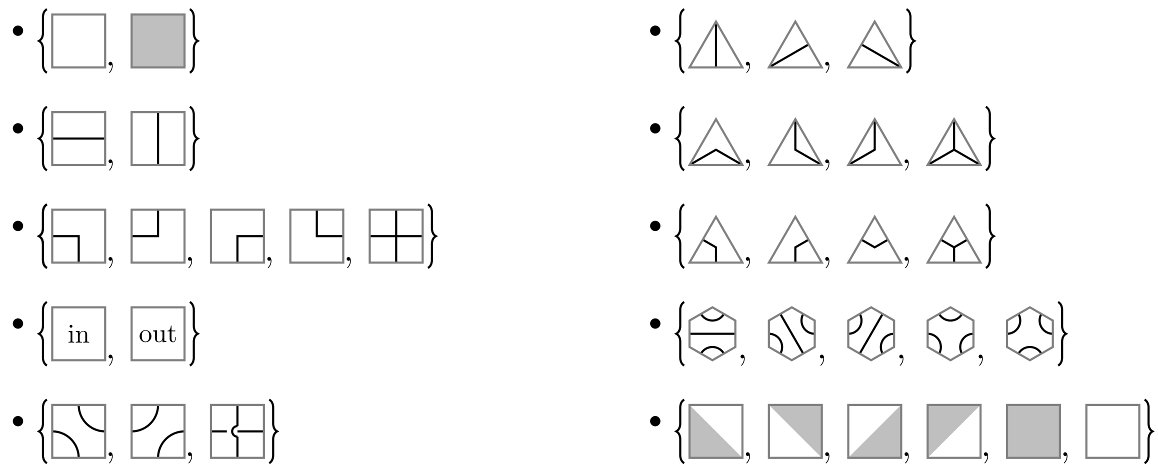


Figure 1: Ten examples of different tiles.

Question. How many essentially different grids of size n exist with these tiles? (Up to dihedral action? Up to cyclic action?)

Related.

1. The square grid can be $n \times n$ or $n \times m$.
2. The hexagonal grid can have triangles with side length n or hexagons with side length n.
3. The triangular grid can have triangles with side length n or hexagons with side length n.
4. The square grid can be quotiented to be a cylinder, torus, or Möbius strip.
5. What if shapes have to "match-up" (e.g. the lines in the third example or colors in the last example have to be "smooth".)
6. How many distinct regions, as in Question 3?

References.

Question 3.

Question 32.

`https://en.wikipedia.org/wiki/Burnside%27s_lemma`

Problem 36.

Starting with an $n \times m$ grid, remove one corner at a time (uniformly at random) until the grid is gone.

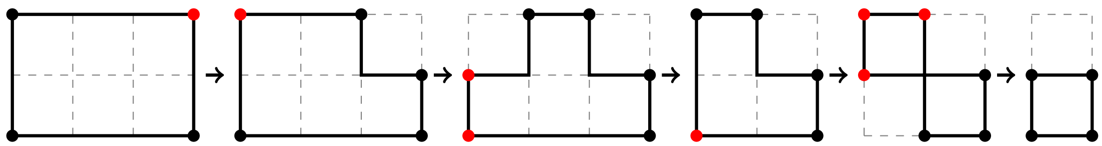


Figure 1: An example of a process starting with a 2×3 grid.

Question. If a stopping point is chosen randomly, how many corners are expected?

Related.

1. What if the deletion is uniform with respect to faces instead of vertices?
2. How many sides are expected?
3. If all polygons in the process are considered, what is the expected number of corners on the polygon with the greatest number of corners?
4. What figure produces the greatest number of corners?
5. How many possible processes exist (up to, say, dihedral action)?
6. What if each figure must stay path connected?
7. What if paths cannot travel through corners? (e.g. the second-to-last figure is illegal.)

Problem 37.

Consider all of the ways to stack "blocks" of different shapes on a platform of length n.

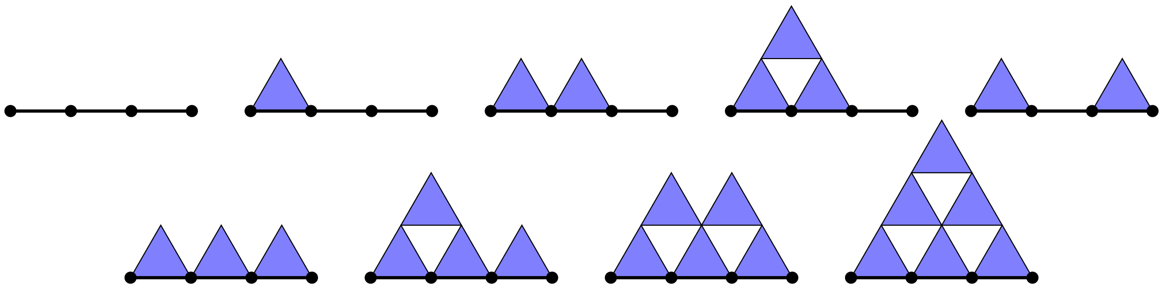


Figure 1: All towers of equilateral triangles on a platform of width 3.

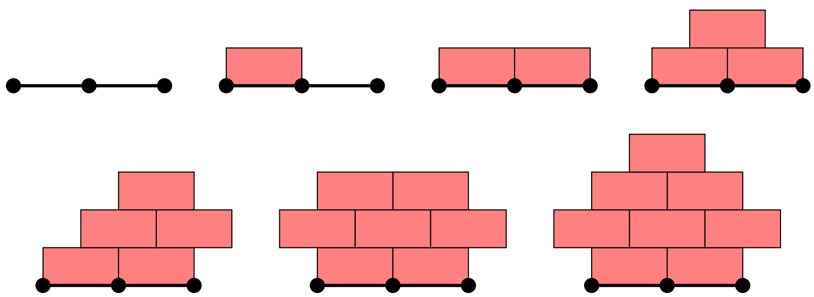
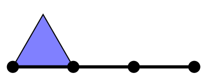
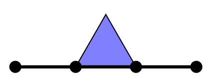
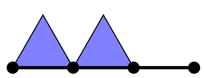
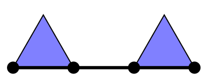


Figure 2: All seven towers of 2×1 bricks on a length 2 platform.

Question. How many different stacks exist for these shapes?

Related.

1. What if  and  are considered to be distinct?
2. What if  and  are considered to be the same (because one turns into the other by "sliding".)
3. What if "upside-down" triangles can be placed in the gaps?
4. What if "upside-down" triangles *must* be placed in the gaps in order to stack on top?
5. What about bricks of length 3?
6. What about tetrahedrons and cuboids?

Note. If cantilevers are not allowed, the brick stacking problem reduces to the triangle stacking problem.

Problem 38.

Consider ways to partition the $n \times m$ grid so that no three tiles of the same partition fall on a line.

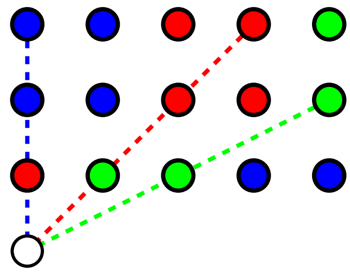


Figure 1: A 3 partition of the 5×3 grid. The white circle cannot be in any of the existing partitions, otherwise three circles of the same color would fall on the same line.

Question. How many colors are required to satisfy the "no three in a row" criterion?

Related.

1. What if this is generalized to k in a row?
2. What if this is generalized to a triangular or hexagonal grid?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if this only queen moves or rook moves are considered?
5. How many distinct configurations exist with a minimal number of partitions?
6. How many distinct configurations exist with k partitions?

References.

Problem 30.

Problem 39.

Consider integer functions f from an n-element subset of $\mathbb{N}$ such that no k of the points $\{(j_1, f(j_1)), \ldots, (j_n, f(j_n))\}$ fall on a $k-2$-degree polynomial.

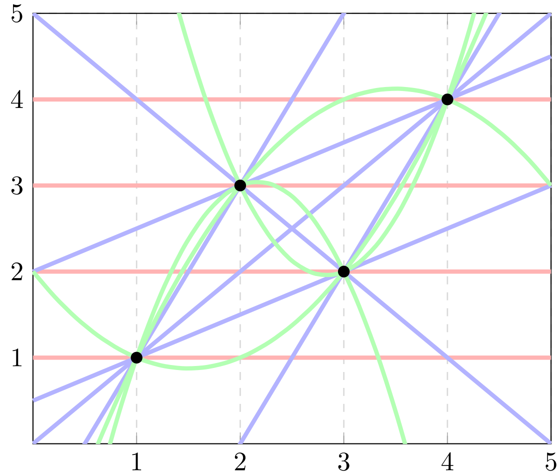


Figure 1: An example that shows that $a(4) = 4$. (Degree 0 polynomials are plotted in red, degree 1 in blue, and degree 2 in green.)

Question. What is $a(n)$, the least N such that there exists a function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, N\}$ with the above property?

Note. Trivially, $a(n)$ is bounded above by the function described in problem 23.

Related.

1. What is the least M such that there exists a subset $S \subset \{1, 2, \ldots, M\}$ and a surjection $g : S \to \{1, 2, \ldots, n\}$ with the aforementioned property?
2. How many such functions exist when N and M are minimized respectively?

References.

Problem 23.

Problem 40.

Consider an n-coloring of a triangular grid such that no upright sub-triangle has the same coloring as any other (up to rotation).

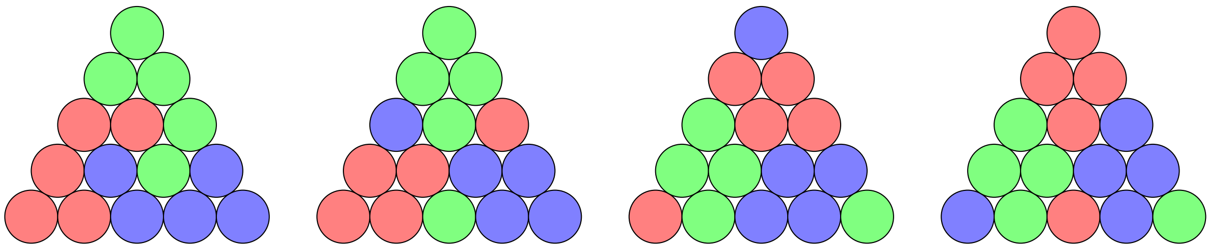


Figure 1: Four examples of 3-colorings of the length 5 triangle. In all cases, 10 different colorings appear exactly once. In the first example, starting from the top: (1) GGG, (2) RRG, (3) RGG, (4) RRB, (5) RGB, (6) GGB, (7) RRR, (8) RBB, (9) GBB, and (10) BBB. (Incidentally, this is *all* of the colorings, so $a(3) = 5$.)

Question. Given n colors, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

Related.

1. What if inverted triangles are counted too?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. How many patterns exist for a triangle of length k with the minimum number of labels?
4. What if diagonal equilateral triangles are also considered? (For example, take the second circle on every side as measured clockwise from each corner.)
5. What if this is done on a square grid?
6. What if this is done on hexagonal shapes?
7. What if this is done on tetrahedra or cuboids?
8. Consider the lexicographically earliest infinite case. Does every triangle eventually appear?

References.

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

Problem 41.

Start with an $n \times n$ grid of boxes and place lines through gridpoints at the border. A box is considered "on" if a line travels through its interior.

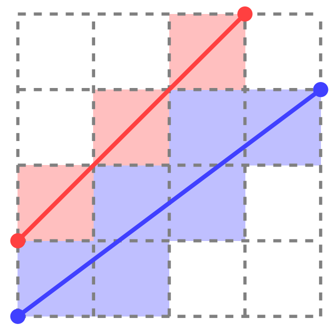


Figure 1: An example of two lines drawn on a grid. The seven white squares still require a line to be drawn through them.

Question. What is the minimum number of lines required to turn on all of the squares in and $n \times n$ grid?

Related.

1. What if touching the corner of a square also turns it on?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. What if no two lines can be parallel? If no two line segments can be congruent?
4. What if no two lines can intersect?
5. How many fully "on" grids exist? How many such minimal grids? (A grid is minimal if removing any line results in a square turning off.)
6. Suppose a grid is on if an even number of lines pass through it and off if an odd number of lines pass through it. How many such grids?
7. How about on an $n \times m$ grid?
8. What if this is done on a triangular grid?
9. What if this is done on a cuboid? On a cuboid with planes passing through the cubes?

Note. In the case where "grid is on if an even number of lines pass through it and off if an odd number of lines pass through it", there exist 2^k configurations. If we further restrict to dihedral-symmetric grids, there are 2^j configurations.
It appears that $k = 5n^2 - 14n + 9$, the 12-gonal numbers. Is there a bijection between the basis elements and the 12-gonal numbers?

Problem 42.

There is a well known magic trick called "Communicating the Card" in which a spectator draws k cards from an n-card deck and shows them to the magician's assistant. He then shows $k-1$ of them to the magician in a particular order, after which she (the magician) can deduce the remaining card. In this variation, the largest possible deck is $k!+k-1$ cards.

$f(1,2)=\{1,2,3\}$	$f(4,8)=\{1,4,8\}$	$f(7,2)=\{2,4,7\}$	$f(8,3)=\{3,5,8\}$
$f(2,1)=\{1,2,4\}$	$f(5,1)=\{1,5,6\}$	$f(8,2)=\{2,4,8\}$	$f(3,6)=\{3,6,7\}$
$f(1,5)=\{1,2,5\}$	$f(5,7)=\{1,5,7\}$	$f(2,5)=\{2,5,6\}$	$f(6,3)=\{3,6,8\}$
$f(1,6)=\{1,2,6\}$	$f(5,8)=\{1,5,8\}$	$f(5,2)=\{2,5,7\}$	$f(7,3)=\{3,7,8\}$
$f(1,7)=\{1,2,7\}$	$f(6,7)=\{1,6,7\}$	$f(8,5)=\{2,5,8\}$	$f(4,5)=\{4,5,6\}$
$f(1,8)=\{1,2,8\}$	$f(6,8)=\{1,6,8\}$	$f(6,2)=\{2,6,7\}$	$f(5,4)=\{4,5,7\}$
$f(1,3)=\{1,3,4\}$	$f(7,8)=\{1,7,8\}$	$f(8,6)=\{2,6,8\}$	$f(8,4)=\{4,5,8\}$
$f(3,1)=\{1,3,5\}$	$f(2,3)=\{2,3,4\}$	$f(8,7)=\{2,7,8\}$	$f(4,6)=\{4,6,7\}$
$f(6,1)=\{1,3,6\}$	$f(3,2)=\{2,3,5\}$	$f(3,4)=\{3,4,5\}$	$f(6,4)=\{4,6,8\}$
$f(7,1)=\{1,3,7\}$	$f(2,6)=\{2,3,6\}$	$f(4,3)=\{3,4,6\}$	$f(7,4)=\{4,7,8\}$
$f(8,1)=\{1,3,8\}$	$f(2,7)=\{2,3,7\}$	$f(3,7)=\{3,4,7\}$	$f(5,6)=\{5,6,7\}$
$f(1,4)=\{1,4,5\}$	$f(2,8)=\{2,3,8\}$	$f(3,8)=\{3,4,8\}$	$f(6,5)=\{5,6,8\}$
$f(4,1)=\{1,4,6\}$	$f(2,4)=\{2,4,5\}$	$f(3,5)=\{3,5,6\}$	$f(7,5)=\{5,7,8\}$
$f(4,7)=\{1,4,7\}$	$f(4,2)=\{2,4,6\}$	$f(5,3)=\{3,5,7\}$	$f(7,6)=\{6,7,8\}$

Figure 1: An example of an encoding where $k=3$ and $n=k!+k-1=8$.

Question. What if the assistant can show any number of cards less than k, and the magician must guess all of the remaining cards?

Related.

1. How many different encodings exist (up to relabeling)?
2. What if the magician just needs to guess one of the remaining cards?
3. What if there are ℓ identical copies of a deck, how many cards can the original trick support?
4. If the assistant shows $k-2$ cards to the magician, what is the biggest deck that this trick can be done with? $k-j$?

References.

http://oeis.org/A030495

https://www.reddit.com/r/math/comments/7l1t84/a_combinatorists_card_trick/

https://web.northeastern.edu/seigen/11Magic/Articles/Best%20Card%20Trick.pdf

Problem 43.

Consider triangles with vertices on grid points and sides of equal length *according to the Taxicab metric*—in particular, those with no smaller, similar triangle.

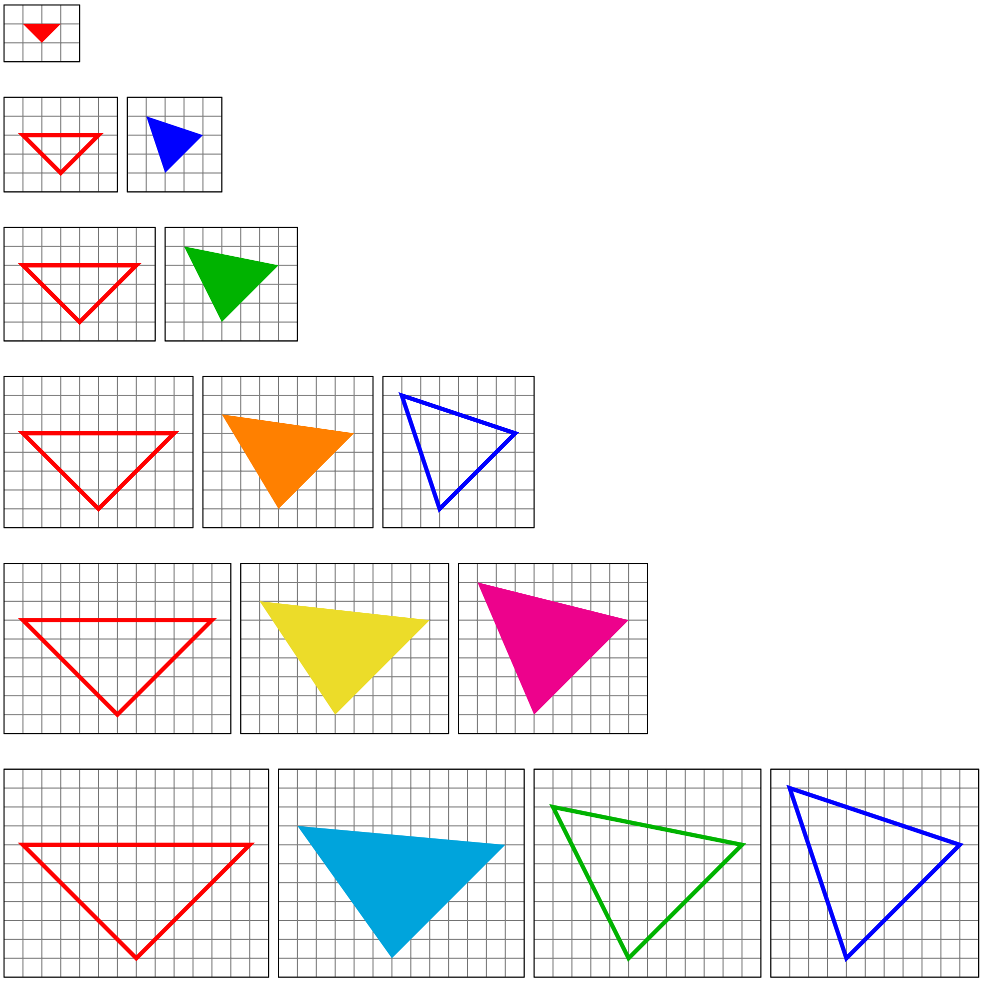


Figure 1: An example of $a(1) = 1$, $a(2) = 1$, $a(3) = 1$, $a(4) = 1$, $a(5) = 2$, and $a(6) = 1$.

Question. How many triangles of side length $2n$ exist?

Note. The answer is probably that $a(1) = 1$ and $a(n) = \lfloor n/2 \rfloor + 1 - \sum_{d|n} a(d)$, which appears to be *A*023022.

Related.

1. What is the related sequence for triangles measured under the d_∞ metric?
2. How does this generalize to equilateral n-gons? Convex n-gons?
3. How does this generalize to a Taxicab-like metric on a triangular grid?

References.

`http://oeis.org/A023022`

Problem 44.

This one is based on correspondence from Alec Jones: Consider all of the ways of partitioning the complete graph on n vertices into smaller complete graphs.

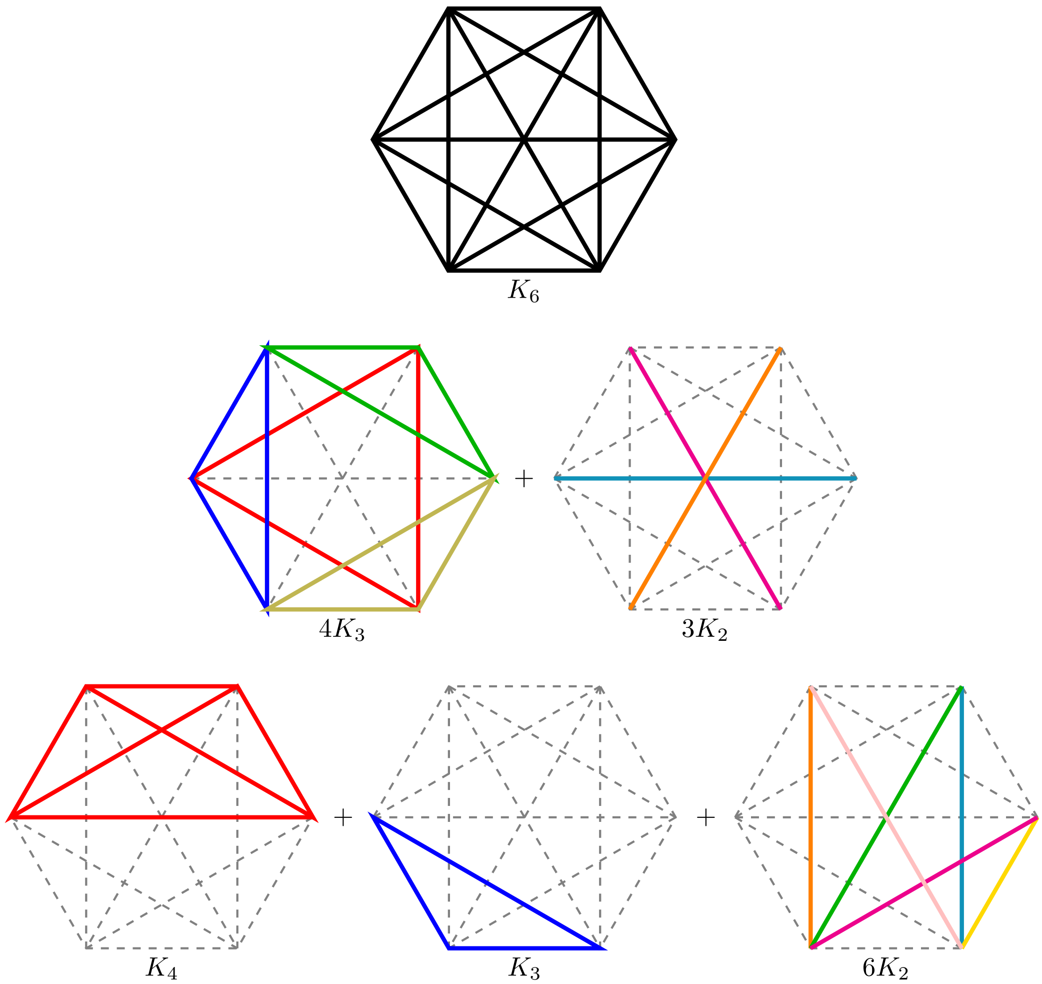


Figure 1: An example three ways to partition K_6 into complete graphs: the trivial partition, a partition into 4 copies of K_3 and 3 copies of K_2, and a partition into 1 copy of K_4, 1 copy of K_3, and 6 copies of K_2.

Question. How many such partitions exist, up to graph isomorphism?

Related.

1. What if the union of K_j graphs cannot contain a K_{j-1} subgraph?
2. What if the partition can only consist of two "sizes" of complete graphs, as in the second example?
3. How many such partitions exist up to dihedral action?

Problem 45.

From correspondence with Alec Jones: Consider a game played on the $m \times n$ rectangular grid, where players take turns placing their pieces onto the board. Each player gets a point for each 3-in-a-row that they make.

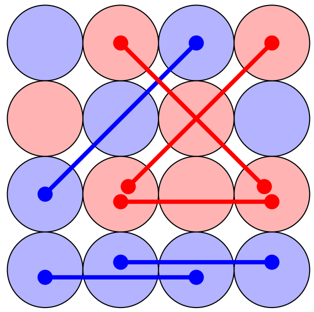


Figure 1: In this game on a 4×4 board, the red player and blue player tie with three points each.

Question. Which player has a winning strategy?

Related.

1. What is the score differential under perfect play?
2. If players cooperate, what is the greatest score differential?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if the game is played with k players or requires ℓ-in-a-row?
5. What if the game is played on a triangular grid?
6. What if the game is played in d dimensions?

References.

Problem 38.

Problem 46.

A problem inspired by a Project Euler problem: suppose an n-robot takes steps that are $1/n$ of a circle, and turns right or left after every step.

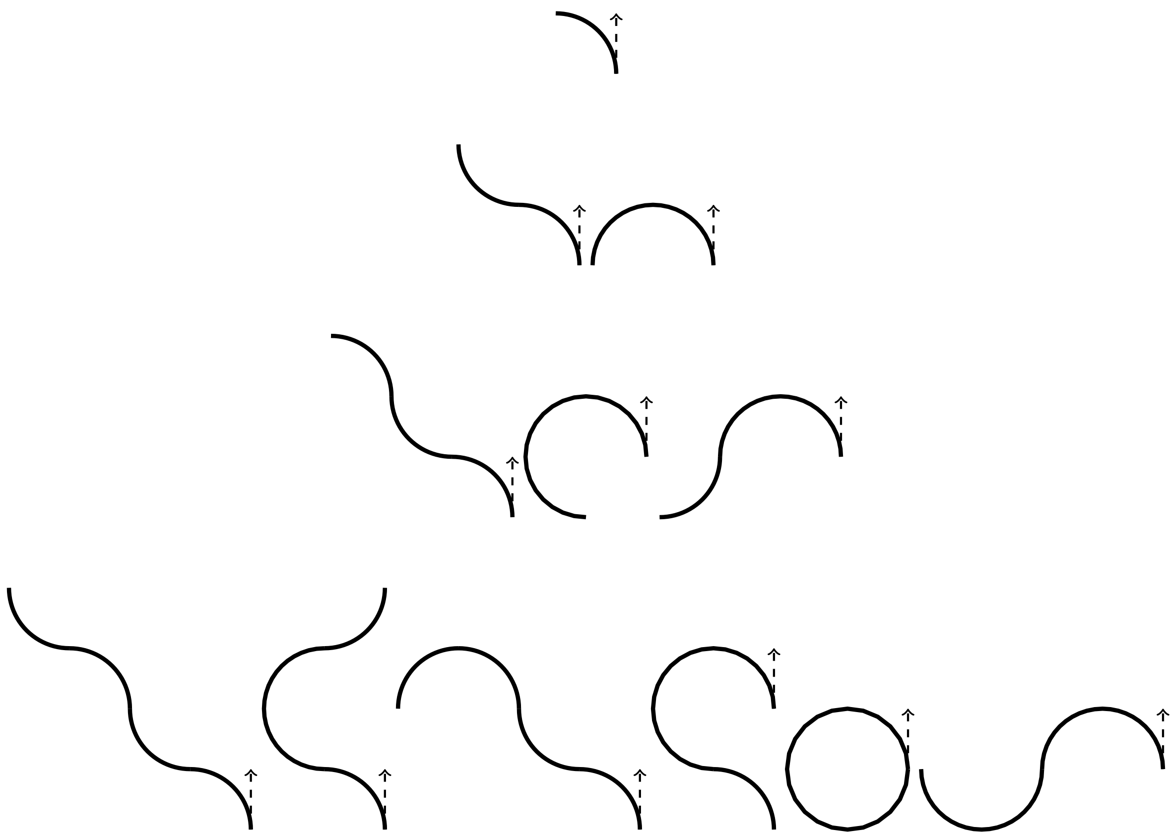


Figure 1: An example of distinct paths of k steps (up to dihedral action) for a 4-robot. $a(1) = 1$, $a(2) = 2$, $a(3) = 3$, and $a(4) = 6$.

Question. How many distinct paths exist for an n-robot, where the robot never retraces its steps?

Related.

1. What if the robot is allowed to retrace its steps?
2. What is the smallest radius that can contain a k-step walk if the robot cannot retrace its steps?
3. What if only smooth loop paths are counted? (The robot returns to where it started in the same direction that it started.)
4. Can smooth loop paths occur when the number of steps is not a multiple of n?
5. What if the orientation of the path matters (i.e. *not* counted up to dihedral action)?
6. What if this is done on a torus, cylinder, or Möbius strip?
7. What if the robot cannot cross its own path?

References.

`https://projecteuler.net/index.php?section=problems&id=208`

Problem 47.

Consider walks in a city, starting mid-block, where (1) at each intersection the walker goes left right or straight, (2) at each mid-block, the walker decides whether or not to turn around, and (3) she ends up back at her apartment.

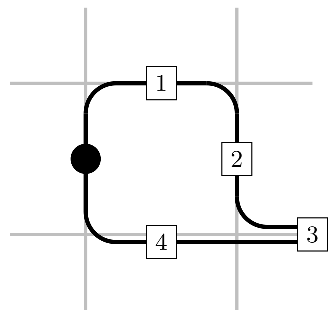


Figure 1: An example of a 5-step walk returning to the apartment.

Question. How many n-block walks can the walker take?

Related.

1. What if the walker does not want to walk along the same strip of road twice?
2. What if the walker does not want to walk along the same *side* of the same strip of road? (Suppose she always walks on the right side of the street.)
3. What if the walker never wants to revisit the same intersection?
4. How many walks up to dihedral action?
5. What if the walker does not turn around?
6. What if the walker never goes straight? Never goes right?

References.

Problem 46.

Problem 48.

Consider labeled rooted trees where the sum of the labels of the branches connected to a vertex is less than the "top" label is n.

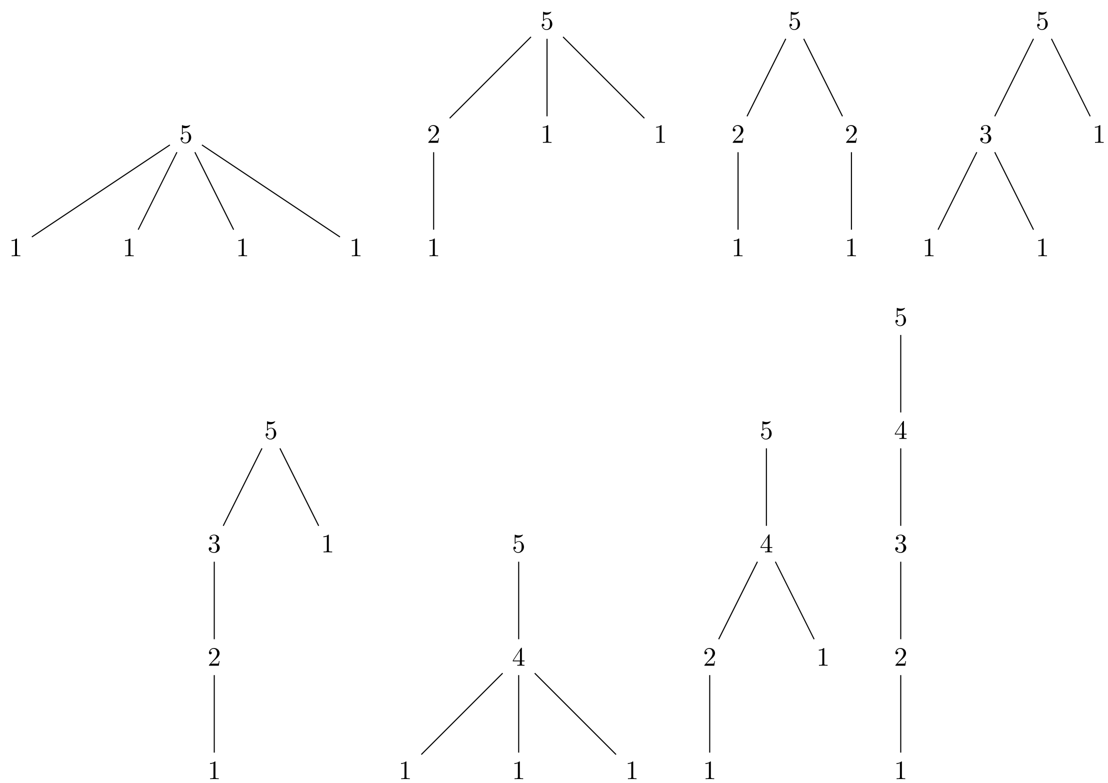


Figure 1: Eight examples of trees with a greatest label of 5: $a(5) = a(1)^4 + a(2)a(1)^2 + a(2)^2 + a(3)a(1) + a(4) = 8$

Question. How many such labels exist?

Related.

1. What if the same label cannot appear multiple times in the same row?
2. What if labels are strictly greater than 1 and the *product* of the branches is less than their parent?

References.

A196545

Problem 49.

Consider a puzzle on a (blank) $n \times m$ board, where each column and row has a number denoting the number of markers that should go in that column or row. The player's goal is to fill in the grid in such a way that the row/column "histograms" are satisfied.

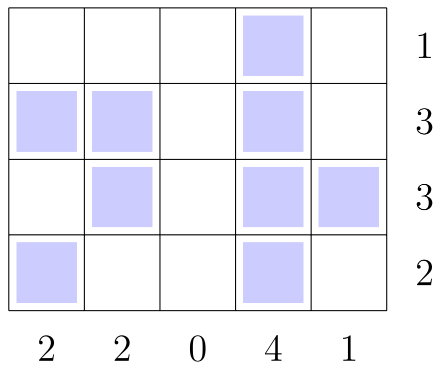


Figure 1: Example of a solution to the puzzle $(2, 2, 0, 4, 1) \times (1, 3, 3, 2)$. Is the solution unique?

Question. What is a procedure for determining if a grid has a solution? If it has a unique solution?

Related.

1. What if the game is played on a d-dimensional hypercube?
2. What if the game is played on a triangle? Tetrahedron?
3. What is the greatest amount of ambiguity a non-unique board can have? (i.e. what is the greatest number of solutions?)
4. How many maximally ambiguous boards exist?
5. How many distinct boards exist up to dihedral action? Up to torus action?
6. What if multiple markers can be put in each cell?

References.

`https://oeis.org/A297077`

Problem 50.

Say that a number M is (n, k)-constructible if there exists an $n \times n$ board with M k-in-a-row markers.

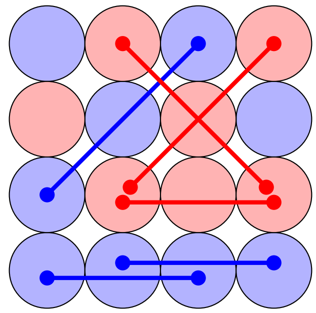


Figure 1: The number 6 is $(4, 3)$-constructible because the above 4×4 board has 6 sets of markers that are placed 3-in-a-row. (This figure was borrowed from Problem 45.)

Question. What is a procedure for determining if a grid has a solution? If it has a unique solution?

Related.

1. What if there are ℓ colors of pieces?
2. What numbers have the greatest number of constructions? Up to dihedral action?
3. What is the smallest number that is (n, k)-constructible?
4. What if this is done on a hypercube or a triangular grid?

References.

Problem 45.

Problem 51.

From Alec Jones. Let $a_k(n)$ count the number of k-gons with vertices on the $n \times n$ grid.

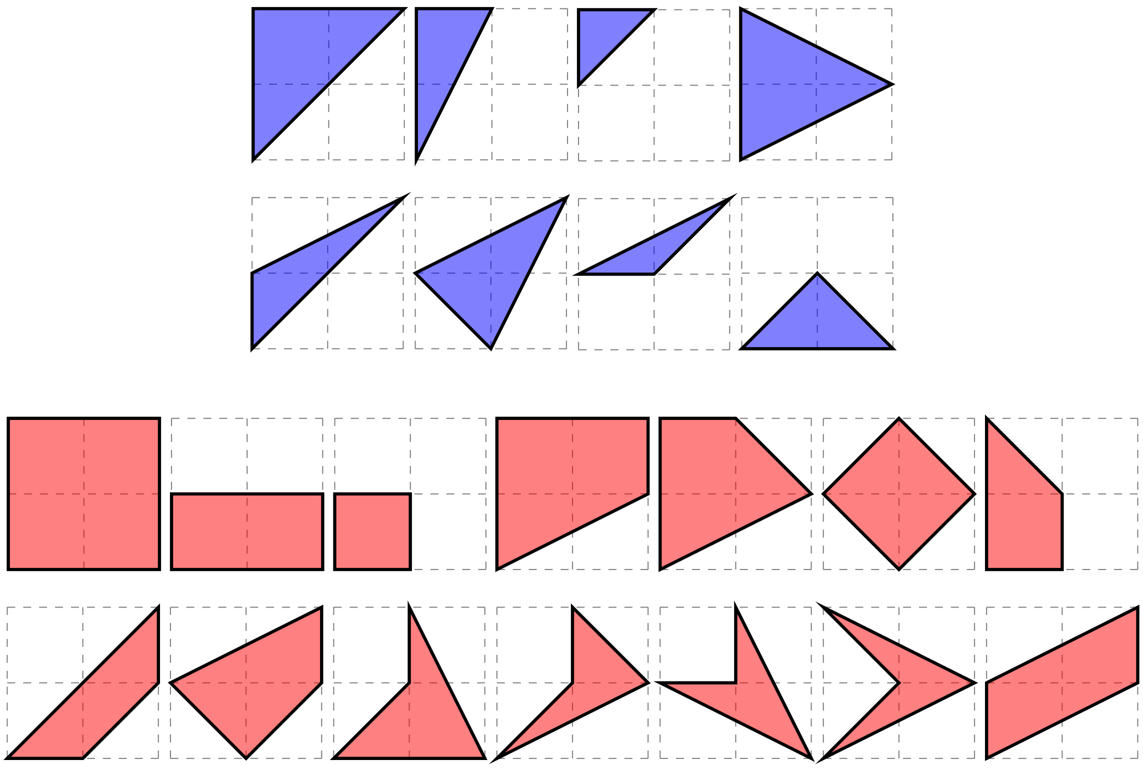


Figure 1: An example showing that $a_3(2) \geq 8$ and $a_4(2) \geq 14$.

Question. What is $a_k(n)$?

Related.

1. For a fixed n, what is the value of k such that $a_k(n)$ is maximized?
2. Here two polygons are considered equivalent if they are congruent. What if two polygons are considered equivalent if they are similar? If they are the same under dihedral action? If they are the same over linear transformation? (e.g. stretching/skewing)
3. What if concave polygons are excluded?
4. What if this is done on an $n \times m$ grid?
5. What if we don't deduplicate based on congruence?
6. What if this is done on a hypercube or a triangular grid?

Problem 52.

A polyform counting problem from Alec Jones: let $a_k(n)$ count the number of polyabolos with n faces and k exposed edges.

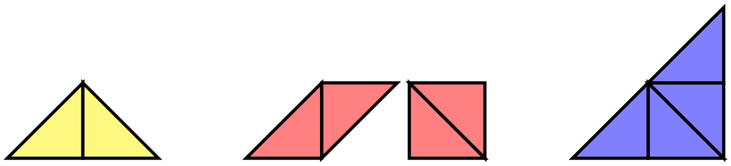


Figure 1: An example in yellow showing that $a_3(2) \geq 1$, two examples in red showing that $a_4(2) \geq 2$, and an example in blue showing that $a_3(4) \geq 1$.

Question. What is the smallest k such that for some fixed n, $a_k(n) > 0$?

Related.

1. What is the largest k such that for some fixed n, $a_k(n) > 0$?
2. What if $\hat{a}_k(n)$ counts polyiamonds instead?
3. What if concave polygons are excluded?
4. Is the following function well-defined?

$$b(k) = \max\{\, a_k(n) : n \in \mathbb{N} \,\}$$

5. Is the following function interesting?

$$c(n) = \max\{\, a_k(n) : k \in \mathbb{N} \,\}$$

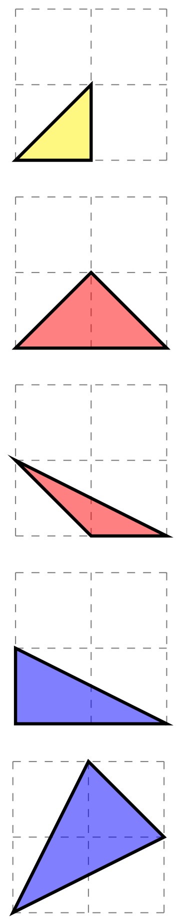
References.

`https://en.wikipedia.org/wiki/Polyiamond`

`https://en.wikipedia.org/wiki/Polyabolo`

Problem 53.

Define an n-triangle to be a triangle with integer coordinates and perimeter in $[n, n+1)$.



$$1 + 1 + \sqrt{2} \in [3, 4)$$

$$2 + \sqrt{2} + \sqrt{2} \in [4, 5)$$

$$1 + \sqrt{2} + \sqrt{5} \in [4, 5)$$

$$1 + 2 + \sqrt{5} \in [5, 6)$$

$$\sqrt{2} + \sqrt{5} + \sqrt{5} \in [5, 6)$$

Figure 1: An example in yellow showing that $a(3) = 1$, and example in red showing that $a(4) = 2$, and an example in blue showing that $a(5) = 3$.

Question. Let $a(n)$ count n-triangles up to dihedral action. What is the asymptotic growth of $a(n)$?

Related.

1. How many tetrahedra?
2. How many quadrilaterals?

References.

`https://oeis.org/A298079` counts the number up to congruence.

Problem 51

Problem 54.

Define an n-triangle to be a triangle with integer coordinates and perimeter in $[n, n+1)$.

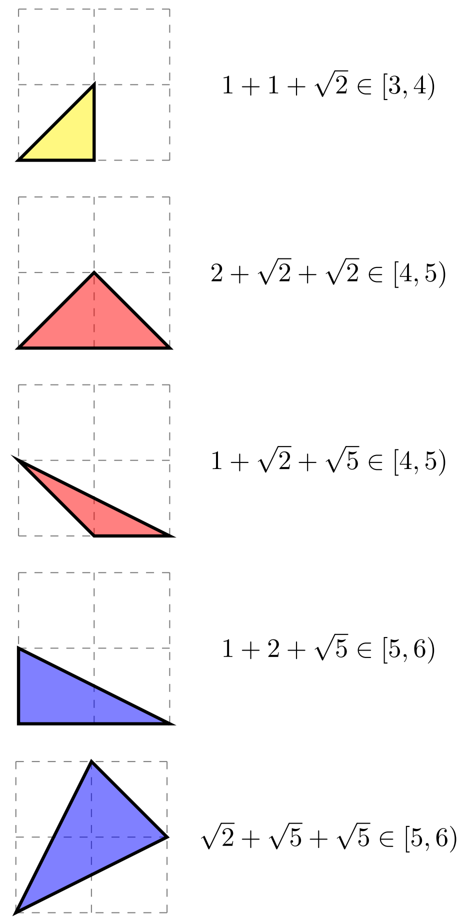


Figure 1: An example in yellow showing that $a(3) = 1$, and example in red showing that $a(4) = 2$, and an example in blue showing that $a(5) = 3$.

Question. Let $a(n)$ count n-triangles up to dihedral action. What is the asymptotic growth of $a(n)$?

Related.

1. How many tetrahedra?
2. How many quadrilaterals?

References.

`https://oeis.org/A298079` counts the number up to congruence.

Problem 51

Problem 70.

Consider walks in a city, starting mid-block, where (1) at each intersection the walker goes left right or straight, (2) at each mid-block, the walker decides whether or not to turn around, and (3) she ends up back at her apartment.

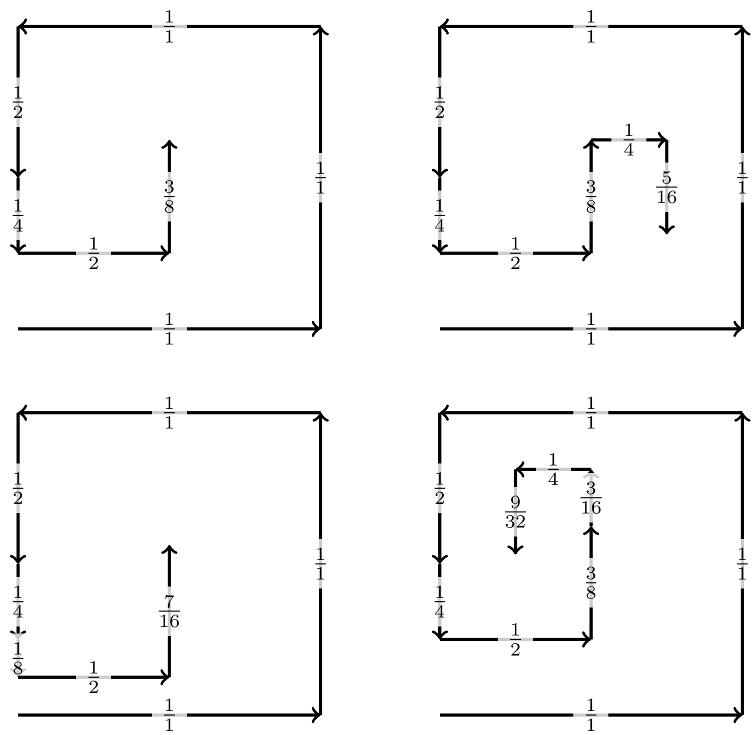


Figure 1: An example of a 5-step walk returning to the apartment.

Question. How many n-block walks can the walker take?

Related.

1. What if the walker does not want to walk along the same strip of road twice?
2. What if the walker does not want to walk along the same *side* of the same strip of road? (Suppose she always walks on the right side of the street.)
3. What if the walker never wants to revisit the same intersection?
4. How many walks up to dihedral action?
5. What if the walker does not turn around?
6. What if the walker never goes straight? Never goes right?

References.

Problem 46.