Consider a triangular grid with each gridpoint having a label in $\{1, 2, \ldots, n\}$. What is the biggest triangle that can be formed such that no sub-triangle has corners that all have the same label?
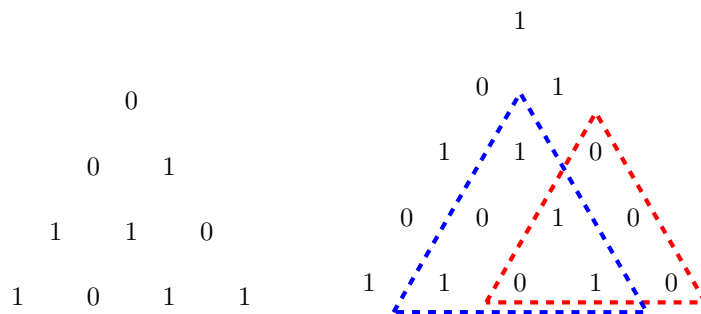

Figure 1: On the left is an example of a triangle on two labels that has no sub-triangles with equal corners. On the right is a non-example of such a triangle on two labels—it has two sub-triangles with equal corners.

Question. Given n labels, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

Related.

1. Given a triangle of side length k and labels up to n, what number of sub-triangles with equal corners must exist?

2. How many such triangles exist?

3. What if any orientation of equilateral triangles are not allowed to have equal corners?

4. What if this is done with hexagons instead of triangles?

5. What if this is done on a square grid?

6. What if for $n \geq 3$ no two corners are allowed to be equal?