# Open problem collection

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This is a catalog of open problems that I began in late 2017 to keep tabs on different problems and ideas I had been thinking about.

Each problem consists of an introduction, a figure which illustrates an example, a question, and a list related questions. Some problems also have references which refer to other problems, to the OEIS, or to other web references.

## 1 Rating

Each problem is rated both in terms of how difficult and how interesting I think the problem is.

### 1.1 Difficulty

The difficulty score follows the convention of ski trail difficulty ratings.

| | | |
|---|---|---|
| ● | Easiest | The problem should be solvable with a modest amount of effort. |
| ■ | Moderate | Significant progress should be possible with moderate effort. |
| ◆ | Difficult | Significant progress will be difficult or take substantial insight. |
| ◆◆ | Most difficult | The problem may be intractable, but special cases may be solvable. |

### 1.2 Interest

The interest rating follows a four-point scale. Each roughly describes what quartile I think it belongs in with respect to my interest in it.

| | | |
|---|---|---|
| ★☆☆☆ | Least interesting | Problems have an interesting idea, but may feel contrived. |
| ★★☆☆ | More interesting | Either a somewhat complicated or superficial question. |
| ★★★☆ | Very interesting | Problems that are particularly natural or simple or cute. |
| ★★★★ | Most interesting | These are the problems that I care the most about. |

Be patient towards all that is unsolved in your heart and try to love the questions themselves as if they were locked rooms or books written in a very foreign language. Don't search for the answers, which could not be given to you now because you would not be able to live them. And the point is to live everything. Live the questions now. Perhaps then, someday far in the future, you will gradually, without even noticing it, live your way into the answer.

Rainer Maria Rilke, *Letters to a Young Poet*

My love for this teaching and the seriousness with which I take it rests in part on my deep reverence for the gravity and the power of questions in human life. I think that this is undervalued in a culture that is in love with the form of words that is an answer — and the way with words that is an argument.

But I also find a question to be a mighty form of words, and I have learned a few things about questions. I have learned that questions elicit answers in their likeness — that answers rise or fall to the questions they meet. We've all seen this. We've all experienced it. It's very hard to respond to a combative question with anything but a combative answer. It's almost impossible to transcend a simplistic question with anything but a simplistic answer. But the opposite is also true: it's hard to resist a generous question. This is a skill that needs relearning, but I believe that we all have it in us to ask questions that invite, that draw forth searching in dignity and revelation. There is something redemptive and lifegiving about asking a better question.

And it is a deep truth in science, and also in each of our lives, that we are shaped as much by the quality of the questions we're asking at any given point as by the answers we have it in us to give. Those moments in our lives when a better quality of question rises up in us, stops us in our tracks — those are pivot points. Those are moments where discovery and new possibility break in.

Krista Tippett, *On Being*, "Living the Questions".

# Problem 1.

Suppose you are given an $n \times m$ grid, and I then think of a rectangle with its corners on grid points. I then ask you to "black out" as many of the gridpoints as possible, in such a way that you can still guess my rectangle after I tell you all of the non-blacked out vertices that its corners lie on.

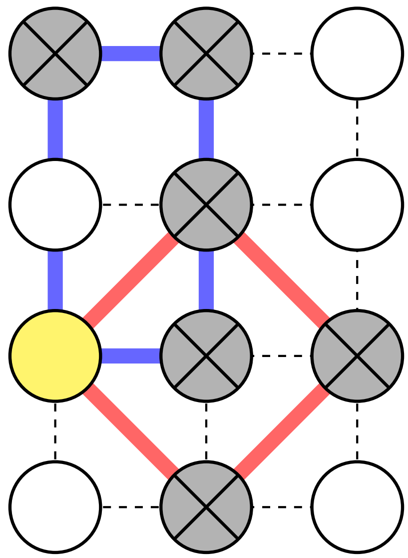


Figure 1: An example of an invalid "black out" for an $4 \times 3$ grid. The blue rectangle and the red rectangle have the same presentation, namely the gridpoint inside the yellow circle.

**Question.** How many vertices may be crossed out such that every rectangle can still be uniquely identified?

**Related.**

1. What if the interior of the rectangle is lit up instead?
2. What if all gridpoints that instersect the perimeter are lit up?
3. What if the rectangles must be square?
4. What if parallelograms are used instead of rectangles?
5. What if the rectangles must be horizontal, vertical, or 45° diagonal?
6. What if this is done on a triangular grid with equilateral triangles?
7. What if this is done in more dimensions (e.g. with a rectangular prism or tetrahedron?)

**References.**

`https://math.stackexchange.com/q/2465571/121988`

# Problem 2.

Let $G$ be some $n \times m$ grid as in Figure 1, where each cell has two opposite diagonals connected (uniformly at random). Choose a cell (also uniformly at random), and consider the component that goes through this cell.

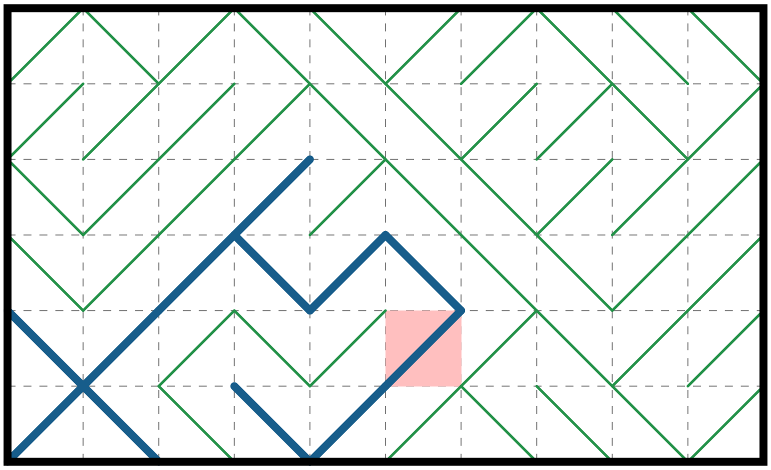


Figure 1: An example of a $6 \times 10$ grid, where a component of size 12 has been selected.

**Question.** What is the expected size of the selected component?

**Related.**

1. What is the expected number of components in an $n \times m$ grid?
2. How long is the longest component expected to be?
3. How does this change if the grid on a torus/cylinder/Möbius strip/etc?

# Problem 3.

Peter Winkler's Coins-in-a-Row game works as following:

> On a table is a row of fifty coins, of various denominations. Alice picks a coin from one of the ends and puts it in her pocket; then Bob chooses a coin from one of the (remaining) ends, and the alternation continues until Bob pockets the last coin.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed according to some probability distribution.

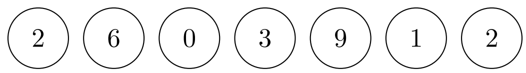



Figure 1: An instance of a seven coin game on a uniform distribution of $\{0, 1, \ldots, 9\}$. The first player has a strategy that allows her to win by one point.

**Question.** For some fixed $\omega$, what is the expected first player's score of Peter Winkler's Coins-in-a-Row game when played with $X_1(\omega), X_2(\omega), \ldots, X_3(\omega)$ where both players are using a min-max strategy?

**Note.** Let

$$e = E[X_2 + X_4 + \ldots + X_{2n}] \text{ and } o = E[X_1 + X_2 + \ldots + X_{2n-1}]$$

When played with $2n$ coins, the first player's score is bounded below by $\max(e, o) - \min(e, o)$ by the strategy outlined by Peter Winkler.

Trivially the first player's score is bounded above by the expected value of the $n$ largest coins minus the expected value of the $n$ smallest coins.

**Related.**

1. If all possible $n$-coin games are played with coins marked 0 and 1, how many games exist where both players have a strategy to tie.
2. How does this change when played according to the (fair) Thue-Morse sequence?
3. What if the players are cooperating to help the first player make as much as possible (with perfect logic)?
4. What is both players are using the greedy algorithm?
5. What if one player uses the greedy algorithm and the other uses min-max? (i.e. What is the expected value of the score improvement when using the min-max strategy?)
6. What if one player selects a coin uniformly at random, and the other player uses one of the above strategies?

# Problem 4.

Let a "popsicle stick weave" be a configuration of lines segments, called "sticks", such that

(1) when you lift up any stick by the end, the structure supports itself (is in tension)

(2) the removal of any stick results in a configuration that no longer supports itself.

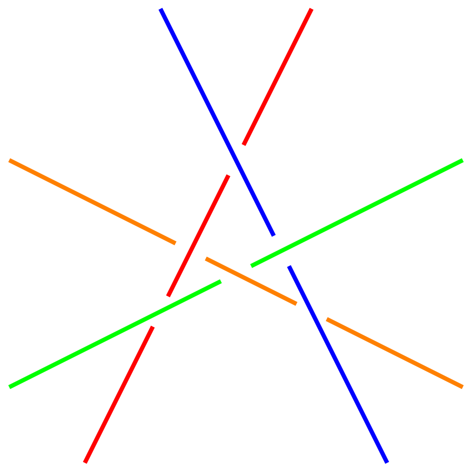


Figure 1: The unique example of a 4 stick crossing (up to reflection)

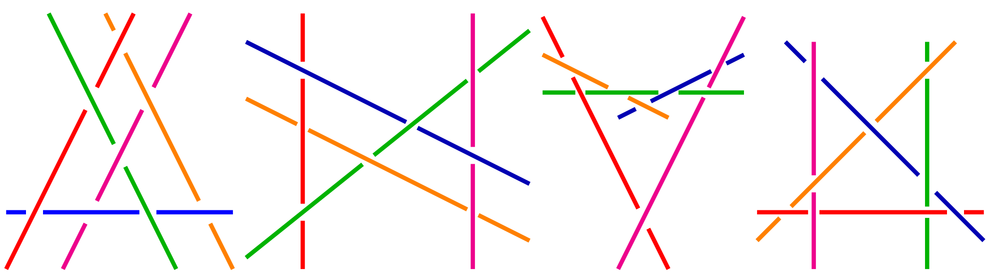


Figure 2: Four of five (?) known examples of five-stick crossings. Perhaps the fourth example shouldn't count, because shortening the blue stick to avoid the blue-red crossing results in a valid configuration (the remaining known five-stick crossing).

**Question.** How many distinct popsicle stick weaves exist for $n$ sticks?

**Related.**

1. What if the sticks are only allowed to touch three other sticks?
2. What if the sticks are another geometric object (e.g. semicircles)?

# Problem 5.

Let

$$C_n = \{f : [n] \to \mathbb{N} \mid \text{the convex hull around } \{(1, f(1)), \ldots, (n, f(n))\} \text{ forms an } n\text{-gon}\}$$

and then let $a(n)$ denote the least upper bound over all functions in $C_n$

$$a(n) = \min\{\max\{f(k) \mid k \in [n]\} \mid f \in C_n\}$$

Figure 1: Examples of $a(3) = 2$, $a(4) = 2$, $a(7) = 4$, and $a(8) = 4$, where the polygons with an even number of vertices have rotational symmetry.

**Question.** Do these polygons converge to something asymptotically?

**Related.**

1. Does $a(2n) = a(2n - 1)$ for all $n$?
2. Do the minimal $2n$-gons always have a representative with rotational symmetry?
3. Are minimal $2n$-gons unique (up to vertical symmetry) with finitely many counterexamples?
4. What is the asymptotic growth of a(n)?

**References.**

A285521: "Table read by rows: the n-th row gives the lexicographically earliest sequence of length n such that the convex hull of (1, a(1)), ..., (n, a(n)) is an n-gon with minimum height." (https://oeis.org/A285521)

# Problem 6.

Let $f_{n,m} \colon [n] \to [m]$ be a uniformly random function, and consider the convex hull around the points $\{(1, f_{n,m}(1)), \ldots (n, f_{n,m}(n))\}$ in $\mathbb{Z}^2$.

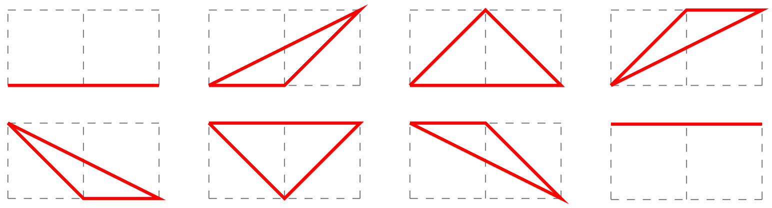


Figure 1: Examples of $f_{3,2}$. Here the expected number of sides on a convex hull is 2.75

**Question.** What is the probability of seeing a $k$-gon (for some fixed $k$), when given a uniformly random function $f_{n,m}$?

**Related.**

1. What value of $k$ has the highest probability?
2. What is the expected value of the number of sides?
3. What if $f_{n,n}$ is restricted to be a permutation?
4. What if $f_{n,m}$ is injective?

# Problem 7.

Given an $n \times n$ grid, consider all convex polygons with grid points as vertices. Let $m(n)$ be the greatest integer $k$ such that there exists a convex $k$-gon on the $n \times n$ grid.

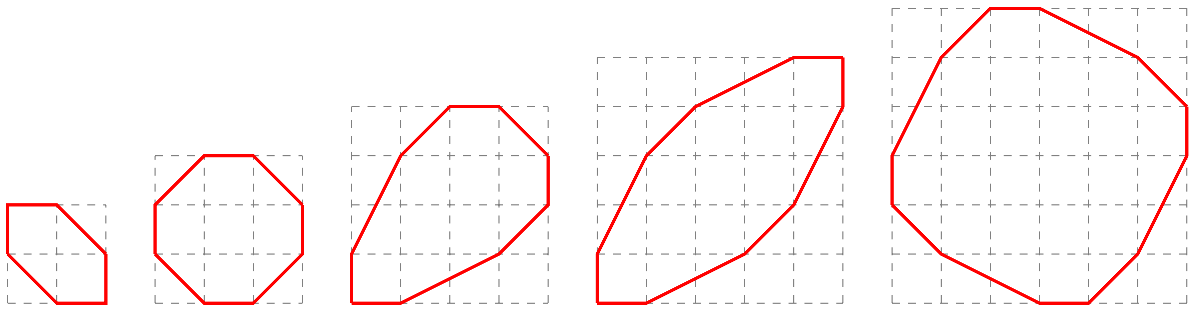


Figure 1: Examples that prove $m(3) = 6, m(4) = 8, m(5) \geq 9, m(6) \geq 10$, and $m(7) \geq (12)$

**Question.** What is $m(n)$?

**Related.**

1. What is a proof (or counterexample) that the examples shown are the best possible?
2. How does $m(n)$ grow asymptotically?
3. Do the shapes do anything interesting in the limit?
4. Are there finitely many maximal polygons without rotational symmetry (e.g. $m(5)$)?
5. How does this generalize to $m \times n$ grids?

**References.**

Problem 5.

Problem 6.

# Problem 8.

Given an $n \times n$ grid, consider all the ways that convex polygons with grid points as vertices can be nested.

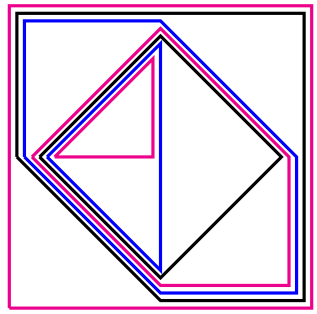


Figure 1: Seven nested convex polygons in the $3 \times 3$ grid.

**Question.** If we think of each polygon having the same height, what is the greatest volume that we can make by stacking the polygons this way?

**Related.**

1. What is the largest sum of the perimeters? The least?
2. What is the largest sum of the number of vertices? The least?
3. How many ways are there to stack $n^2 - 2$ polygons like this? Any number of polygons?
4. Does this generalize to polyhedra in the $n \times n \times n$ cube?
5. Does this generalize to polygons on a triangular grid?

# Problem 9.

Consider all $k$-colorings of an $n \times n$ grid, where each row and column has $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ cells with each color.

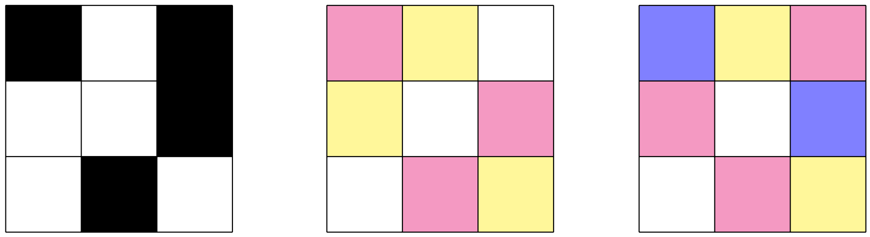


Figure 1: A valid 2-coloring, 3-coloring, and 4-coloring of an $3 \times 3$ grid.

**Question.** How many such $k$-colorings of the $n \times n$ grid?

**Related.**

1. What if there also must be a total of $\lfloor n^2/k \rfloor$ or $\lceil n^2/k \rceil$ cells of each color?
2. What if these are counted up to the dihedral action on the square $D_4$?
3. What if these are counted up to torus action?
4. What if these are counted up to permutation of the coloring?
5. Can this generalize to the cube? To a triangular tiling?

# Problem 10.

Consider Ron Graham's sequence for LCM, that is, look at sequences such that

$$n = b_1 < b_2 < \ldots < b_t = k \text{ and } \mathrm{LCM}(b_1, \ldots, b_t) \text{ is square.}$$

**Question.** Let $A300516(n)$ be the least k (as a function of n) such that such a sequence exists?

| | | |
|---|---|---|
| $a(1) = 1$ via $(1)$ | $a(11) = 121$ via $(11, 121)$ | $a(21) = 49$ via $(21, 36, 49)$ |
| $a(2) = 4$ via $(2, 4)$ | $a(12) = 18$ via $(12, 18)$ | $a(22) = 121$ via $(22, 64, 121)$ |
| $a(3) = 3$ via $(3, 9)$ | $a(13) = 169$ via $(13, 169)$ | $a(23) = 529$ via $(23, 529)$ |
| $a(4) = 4$ via $(4)$ | $a(14) = 49$ via $(14, 16, 49)$ | $a(24) = 48$ via $(24, 36, 48)$ |
| $a(5) = 25$ via $(5, 25)$ | $a(15) = 25$ via $(15, 16, 18, 25)$ | $a(25) = 25$ via $(25)$ |
| $a(6) = 12$ via $(6, 9, 12)$ | $a(16) = 16$ via $(16)$ | $a(26) = 169$ via $(26, 64, 169)$ |
| $a(7) = 49$ via $(7, 49)$ | $a(17) = 289$ via $(17, 289)$ | $a(27) = 81$ via $(27, 81)$ |
| $a(8) = 16$ via $(8, 16)$ | $a(18) = 25$ via $(18, 20, 25)$ | $a(28) = 49$ via $(28, 49)$ |
| $a(9) = 9$ via $(9)$ | $a(19) = 361$ via $(19, 361)$ | $a(29) = 841$ via $(29, 841)$ |
| $a(10) = 25$ via $(10, 16, 25)$ | $a(20) = 25$ via $(20, 25)$ | $a(30) = 50$ via $(30)$ |

Figure 1: Examples of $A300516(n)$ for $1 \leq n \leq 30$.

**Related.**

1. For what values $n$ is $A300516(n)$ nonsquare?
2. For what values $n$ does the corresponding sequence have three or more terms?
3. What is the analogous sequence for perfect cubes, etc?

**References.**

`https://oeis.org/A300516`

# Problem 11.

Ron Graham's Sequence (A006255) is the least k for which there exists a strictly increasing sequence

$$n = b_1 < b_2 < \ldots < b_t = k \text{ where } b_1 \cdot \ldots \cdot b_t \text{ is square.}$$

There is a known way to efficiently compute analogous functions $a_p$ where $a_p(n)$ is the least integer such that there exists a sequence

(a) $n = b_1 \leq b_2 \leq \ldots \leq b_t = a_p(n)$,

(b) any term appears at most $p - 1$ times, and

(c) $b_1 \cdot b_2 \cdot \ldots \cdot b_t$ is a $p$-th power.

**Question.** An efficient way to compute $a_p$ is known when $p$ is prime. What is an efficient way to compute $a_c$ when $c$ is composite?

$$\begin{array}{lll}
a_4(1) = 1 & \text{via } 1 & = 1^4 \\
a_4(2) = 2 & \text{via } 2^2 \cdot 4 & = 2^4 \\
a_4(3) = 6 & \text{via } 3^2 \cdot 4 \cdot 6^2 & = 6^4 \\
a_4(4) = 4 & \text{via } 4^2 & = 2^4 \\
a_4(5) = 10 & \text{via } 5^2 \cdot 8^2 \cdot 10^2 & = 20^4 \\
a_4(6) = 9 & \text{via } 6^2 \cdot 8^2 \cdot 9 & = 12^4 \\
a_4(7) = 14 & \text{via } 7^2 \cdot 8^2 \cdot 14^2 & = 28^4 \\
a_4(8) \leq 15 & \text{via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 & = 60^4 \\
a_4(9) = 9 & \text{via } 9^2 & = 3^4 \\
a_4(10) \leq 18 & \text{via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 & = 180^4
\end{array}$$

Figure 1: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

**Related.**

1. For what values $n$ is $a_4(n) < A006255(n)$?
2. Given some integers $k, c$, how many terms have $a_c(n) = k$? (e.g. $a_4(6) = a_4(9) = 9$.)
3. Does $a_c$ contain arbitrarily many copies of the same value?
   (i.e. does there exist a sequence such that $a_4(n_1) = a_4(n_2) = \ldots = a_4(n_m)$ for arbitrarily large $m$?)
4. How many times does $k$ appear in the image of $a_c$? (e.g. 9 appears twice, as $a(6)$ and $a(9)$.)
5. What integers are in the image of $a_c$?

**References.**

`https://oeis.org/A277494`

# Problem 12.

Suppose you have a strip of toilet paper with $n$ pieces, and you fold the paper evenly into $d$ parts (divide by $d$) or fold the last $k$ pieces in (subtract by $k$), until the length of the strip is less than $k$ pieces.

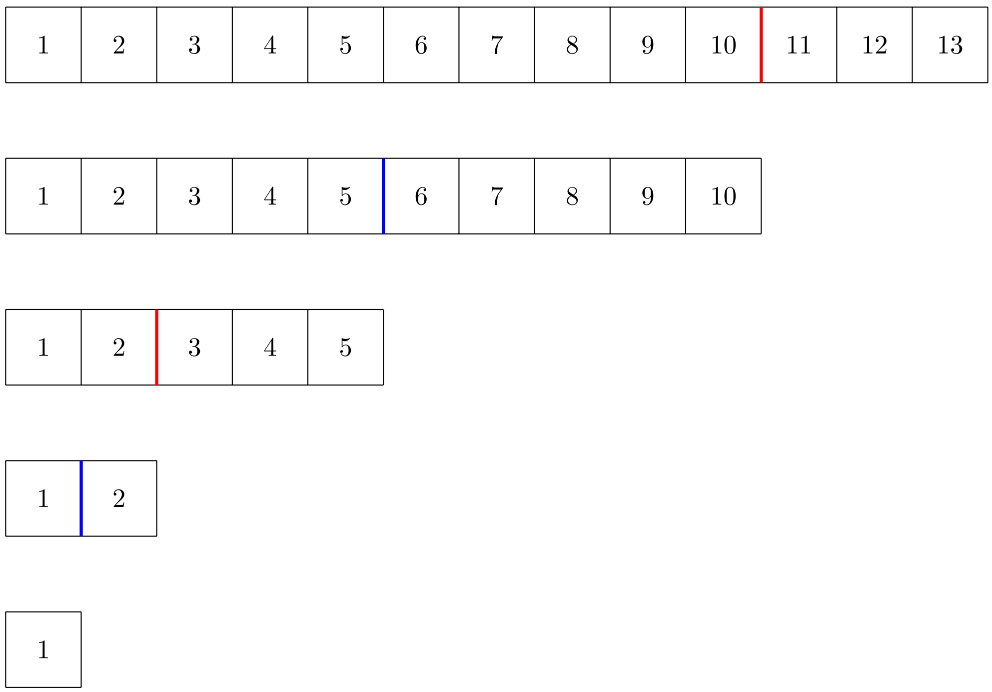



Figure 1: A folding of paper where $n = 13$, $d = 2$, and $k = 3$, showing that $a_{2,3}(13) \leq 4$. Where the red marks a subtraction by $k$ and the blue marks a division by $d$.

**Question.** Is there an efficient way to compute $a_{d,k}(n)$?

**Related.**

1. What if you must keep folding until you cannot fold any longer?
2. What is the minimum number of terminal pieces? What is the minimum number of steps to this number?

**References.**

https://oeis.org/A014701

# Problem 13.

OEIS sequence A261865 describes "$a(n)$ is the least $k \in \mathbb{N}$ such that some multiple of $\sqrt{k} \in (n, n+1)$." Clearly the asymptotic density of 2 in the sequence is $1/\sqrt{2}$.

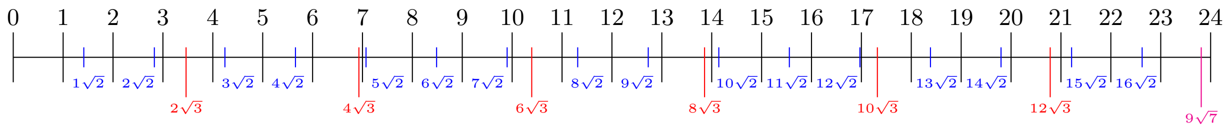



Figure 1: An illustration of $a(n)$ for $n \in \{1, 2, \ldots, 23\}$.

**Question.** Let $S_\alpha \subset \mathbb{N}$ denote the squarefree integers strictly less than $\alpha$.
Is the asymptotic density of squarefree $j$ given by

$$\frac{1}{\sqrt{j}} \prod_{s \in S_j} \left(1 - \frac{1}{\sqrt{s}}\right)?$$

**Related.**

1. Is there an algorithm to construct a value of $n$ such that $a(n) > K$ for any specified $K$? (Perhaps using best Diophantine approximations or something?)
2. What is the asymptotic growth of the records?
3. Given some $\alpha$ what is the expected value of the smallest $n$ such that $S_\alpha \subset \{a(1), \ldots, a(n)\}$?
4. This sequence uses the "base sequence" of $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\}$. On what other base sequences is this construction interesting?
5. What is the smallest $m \in \mathbb{N}$ such that $k2^{1/m} \in (n, n+1)$ for some $k \in \mathbb{N}$?
6. What is the smallest $k \in \mathbb{N}$ such that $k2^{1/m} \in (n, n+1)$ for some $m \in \mathbb{N}$?

**References.**

https://oeis.org/A261865

# Problem 14.

Start with $n$ piles with a single stone in each pile. If two piles have the same number of stones, then any number of stones can be moved between them.

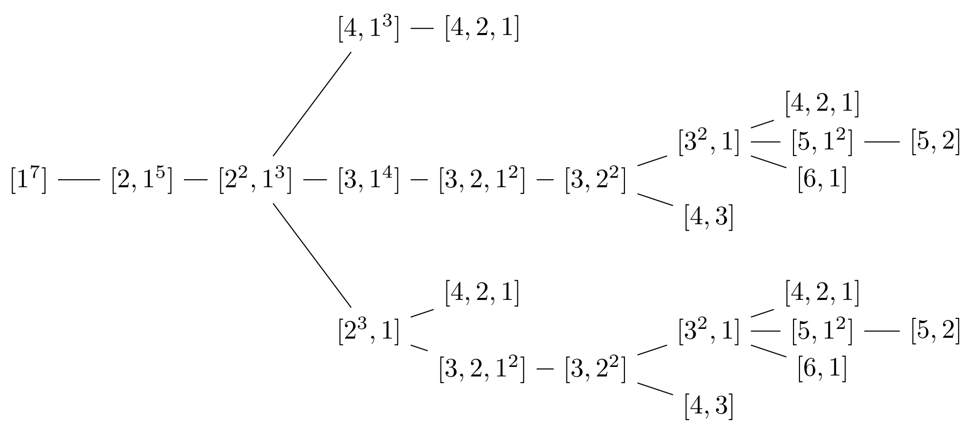



Figure 1: An illustration of all possible moves for $n = 7$.

**Question.** What is the greatest number of steps that can occur? Alternatively how many "levels" are in the tree of possible moves?

**Related.**

1. Let $A292726$ be the total number of distinct states. What is $A292726$? (e.g. $A292726(7) = 14$.)
2. Let $c = A000041 - A292726$ be the total number of states that *cannot* be acheived. (e.g. $c(5) = 1$ via the state [5].)
3. Is $c(p) = 1$ for all primes $p$? Is $c(n) = 0$ if and only if $n$ is a power of 2?
4. Let $\ell = A292836$ be the least number of steps to a terminal state. (e.g. $\ell(7) = 4$ ending in $[4, 2, 1]$.)
5. Let $g = A292729$ be the greatest number of steps to a terminal state. (e.g. $g(7) = 8$ ending in $[5, 2]$.)
6. Let $p$ be the total number of paths, i.e. the number of leaves in the tree. (e.g. $p(7) = 10$.)
7. Let $t$ be the number of distinct *terminal* states. (e.g, $t(7) = 4$ via $[4, 2, 1]$, $[4, 2]$, $[6, 1]$, and $[4, 3]$.)
8. What if you can move stones between any sets of piles that share the same number of stones? (e.g. $[2^3] \to [6]$ or $[2^3] \to [4, 1, 1]$)

**References.**

https://oeis.org/A292836

# Problem 15.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

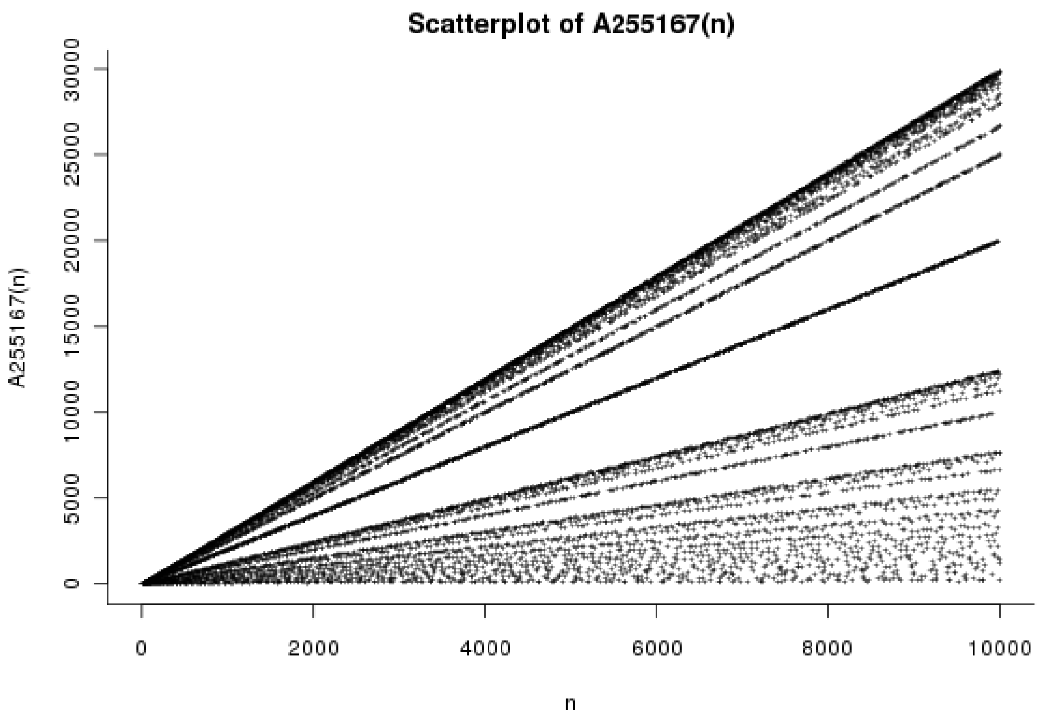
$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

A006255 is bounded above by A072905, the least $k > n$ such that $k \cdot n$ is square.

**Question.** Does there exist any $n$ for which $A006255(n) = A072905(n)$. In other words, is there any non-square $n$ for which $n \cdot A006255(n)$ is square?

**Related.**

1. Does the gap $A072905(n) - A006255(n)$ have a nonzero lower bound?




**Note.** This is equivalent to showing that for any $a < b$ with the same squarefree part, there is some subset of $\{a+1, a+2, \ldots, b-1\}$ such that the product of the elements of the subset has the same squarefree part as $a$ (and $b$).

**References.**

https://oeis.org/A006255

https://oeis.org/A072905

https://oeis.org/A255167

# Problem 16.

Let $a_3(n)$ be the least $k > n$ such that $nk$ or $nk^2$ is a cube, and let $A299117$ be the image of $a_3(n)$.

| | | |
|---|---|---|
| $a_3(1) = 8$ via $1 \cdot 8 = 2^3$ | $a_3(11) = 88$ via $11 \cdot 88 = 22^3$ | $a_3(21) = 168$ via $21 \cdot 168^2 = 84^3$ |
| $a_3(2) = 4$ via $2 \cdot 4 = 2^3$ | $a_3(12) = 18$ via $12 \cdot 18 = 6^3$ | $a_3(22) = 176$ via $22 \cdot 176^2 = 88^3$ |
| $a_3(3) = 9$ via $3 \cdot 9 = 3^3$ | $a_3(13) = 104$ via $13 \cdot 104^2 = 52^3$ | $a_3(23) = 184$ via $23 \cdot 184^2 = 92^3$ |
| $a_3(4) = 16$ via $4 \cdot 16 = 4^3$ | $a_3(14) = 112$ via $14 \cdot 112^2 = 56^3$ | $a_3(24) = 72$ via $24 \cdot 72 = 12^3$ |
| $a_3(5) = 25$ via $5 \cdot 25 = 5^3$ | $a_3(15) = 120$ via $15 \cdot 120^2 = 60^3$ | $a_3(25) = 40$ via $25 \cdot 40 = 10^3$ |
| $a_3(6) = 36$ via $6 \cdot 36 = 6^3$ | $a_3(16) = 32$ via $16 \cdot 32 = 8^3$ | $a_3(26) = 208$ via $26 \cdot 208^2 = 104^3$ |
| $a_3(7) = 49$ via $7 \cdot 49 = 7^3$ | $a_3(17) = 136$ via $17 \cdot 136^2 = 68^3$ | $a_3(27) = 64$ via $27 \cdot 64 = 12^3$ |
| $a_3(8) = 27$ via $8 \cdot 27 = 6^3$ | $a_3(18) = 96$ via $18 \cdot 96 = 12^3$ | $a_3(28) = 98$ via $28 \cdot 98 = 14^3$ |
| $a_3(9) = 24$ via $9 \cdot 24 = 6^3$ | $a_3(19) = 152$ via $19 \cdot 152^2 = 76^3$ | $a_3(29) = 232$ via $29 \cdot 232^2 = 116^3$ |
| $a_3(10) = 80$ via $10 \cdot 80 = 20^3$ | $a_3(20) = 50$ via $20 \cdot 50 = 10^3$ | $a_3(30) = 240$ via $30 \cdot 240^2 = 120^3$ |

**Question.** Is there another way to characterize what integers are in $A299117$?

**Note.** The function $a_3$ is an injection.
$A299117$ contains every cube, because $a(n^3) = (n+1)^3$.

**Related.**

1. Does $A299117$ contain every square?
2. Does $A299117$ contain any squarefree number?
3. What about the generalization: the image of $a_m$ where $a_m(n)$ is the least $k > n$ such that $n^\alpha k^\beta$ is a $m$-th power, with $\alpha, \beta \in \{1, 2, \ldots, m-1\}$.

**References.**

OEIS sequences A277781, A299117, and A343881.

# Problem 17.

Consider placing any number of queens (of the same color) on an $n \times n$ chessboard in such a way as to maximize the number of legal moves available.

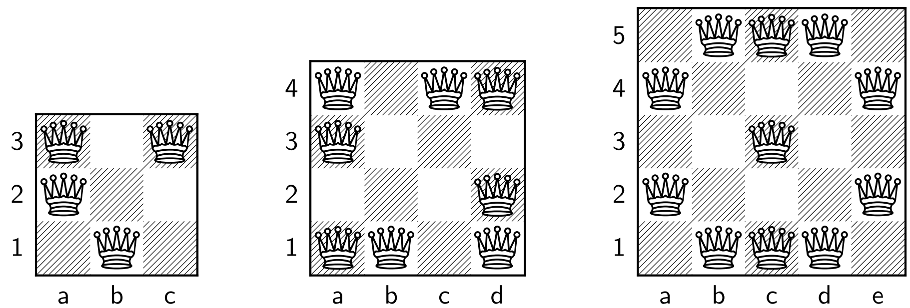



Figure 1: Examples of $a_q(3) = 17, a_q(4) = 40, a_q(5) = 76$.

**Question.** Is Alec Jones's conjecture true: $a_q(n) = 8(n-2)^2$ for $n \geq 6$, by placing the queens around the perimeter?

**Related.**

1. What about the analogous function for rooks ($a_r$) or bishops ($a_b$)?
2. What if the chessboard is a torus? Cylinder? Möbius strip?
3. What if the chessboard is $n \times m$?
4. Is $a_b(n) = \lfloor a_q(n)/2 \rfloor$? for all $n$?
5. What if queens can attack?

**References.**

A278211: `http://oeis.org/A278211`

A278212: `http://oeis.org/A278212`

A275815: `http://oeis.org/A275815`

# Problem 18.

Let $U_n$ be the set of sequences of positive integers of length $n$ such that no substring occurs twice.

$$\begin{aligned}
(1,1,2,2,1,3,1) &\in U_7 \\
(1,2,1,2,3) &\notin U_5 \text{ because } (1,2) \text{ occurs twice.} \\
(1,1,1) &\notin U_3 \text{ because } (1,1) \text{ occurs twice.}
\end{aligned}$$

Figure 1: An example and two non-examples of sequences with no repeated substrings.

**Question.** What is the number of sequences in $U_n$ where the sum of terms is minimized?

**Related.**

1. What is the minimum least common multiple of a sequence in $U_n$? How many such minimal sequences?
2. What is the minimum product of a sequence in $U_n$? How many such minimal sequences?
3. What if substrings are considered forward and backward?
4. What if only substrings of length greater $k$ are considered?
5. What if any term can appear at most $\ell$ times?

**References.**

https://oeis.org/A259280

# Problem 19.

Consider the fuction A300002(n) which is the lexicographically earliest sequence of positive integers such that no $k+2$ points are on a polynomial of degree $k$. (i.e. no two points are equal, no three points are colinear, no four points are on a parabola, etc.)

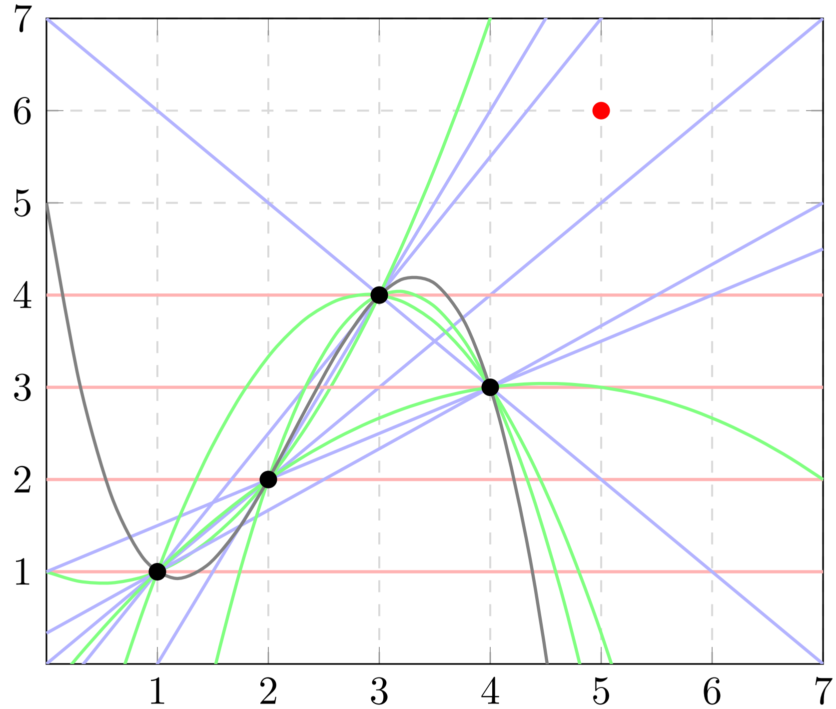



Figure 1: The first four points together with all interpolated polynomials. The red point marks the lowest integer coordinate $(5, k)$ that does not lie on an interpolated polynomial. (Degree 0 polynomials are plotted in red, degree 1 in blue, degree 2 in green and degree 3 in gray.)

**Question.** Do all positive integers occur in this sequence?

**Related.**

1. What is the asymptotic growth of this sequence?
2. Does *any* permutation of the natural numbers have the property that no $k+2$ points are on a polynomial of degree $k$?

**References.**

https://oeis.org/A300002

# Problem 20.

Let $h$ be the maximum number of penny-to-penny connections on the vertices of a hexagonal lattice, and let $t(n)$ be the analogous sequence on the vertices of a triangular lattice.

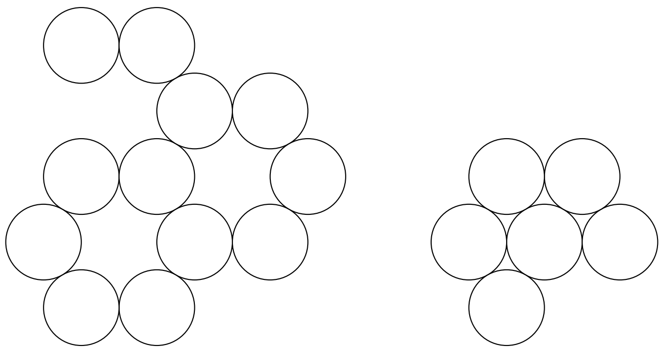


Figure 1: An example for $h(12) = 13$ and $t(6) = 9$

**Question.** What is a combinatorial proof that $h(2n) - t(n) = A216256(n)$.

**Note.** A216256 is

$$\underbrace{1}_{1}, \underbrace{2}_{1}, \underbrace{3, 3}_{2}, \underbrace{4, 4, 4}_{3}, \underbrace{5, 5, 5}_{3}, \underbrace{6, 6, 6, 6}_{4}, \underbrace{7, 7, 7, 7, 7}_{5}, \underbrace{8, 8, 8, 8, 8}_{5}, \underbrace{9, 9, 9, 9, 9, 9}_{6}, \ldots$$

**Related.**

1. `https://oeis.org/A216256`
2. $t(n)$: `https://oeis.org/A047932`
3. $h(n)$: `https://oeis.org/A263135`

# Problem 21.

Consider all rectangles with all corners on gridpoints on an $n \times m$ grid.

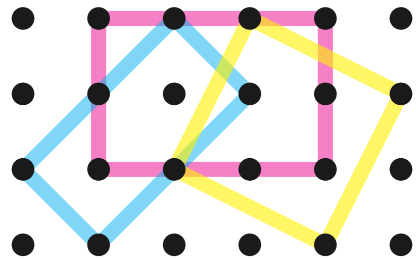


Figure 1: An example of three rectangles with all corners on gridpoints of a $4 \times 6$ grid.

**Question.** Given some shape, how many of these shapes can be constrained to the $n \times m$ grid?

**Related.**

1. What if we want to count only "primitive" squares, in the sense that the sides of the square only intersect grid points at the corners?
2. Number of rectangles on the cylinder? Torus? Möbius strip?
3. Number of "rotation classes", where two squares are equivalent if one can be transformed into the other by shifting and stretching?
4. Number of "orientation classes" where two squares are equivalent if one can be transformed into the other by shifting?
5. What if this is done on an $n \times m \times k$ grid?
6. What if the rectangles must be diagonal?
7. What if this is done on a triangular lattice with primitive equilateral triangles?

**Note.** Equilateral triangles in triangle is A000332; tetrahedra in a tetrahedron is A269747; triangles in a tetrahedron is A334581; tetrahedra in cube is 2*A103158; equilateral triangles in cube is A102698; rectangles in square is A085582; rectangles in rectangle is A289832; isosceles triangles in rectangle is A271910; right isosceles triangles in square is A187452; right triangles in square is A077435; convex quadrilaterals in square A189413; quadrilaterals in square A189414; trapezoids in square A189415; parallelograms in square is A189416; kites in square A189417; rhombi in square A189418.

**References.**

Problem 1.

https://arxiv.org/pdf/1605.00180.pdf

http://people.missouristate.edu/lesreid/POW03_01.html

# Problem 22.

The prime ant looks along the number line starting at 2. When she reaches a composite number, she divides by its least prime factor, and adds that factor to the previous term, and steps back.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 5 | 2 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 5 | 2 | 7 | 3 | 7 | 8 | 9 | 10 |
| 2 | 5 | 2 | 7 | 3 | 9 | 4 | 9 | 10 |
| 2 | 5 | 2 | 7 | 6 | 3 | 4 | 9 | 10 |
| 2 | 5 | 2 | 9 | 2 | 3 | 4 | 9 | 10 |
| 2 | 5 | 5 | 3 | 2 | 3 | 4 | 9 | 10 |
| 2 | 5 | 5 | 3 | 2 | 3 | 7 | 3 | 10 |

Figure 1: An illustration of the prime ant's positions after the first 7 steps.

**Question.** Does the ant eventually stay to the right of any fixed position?

**Related.**

1. Are there any positions that stay permanently greater than 7? Than 11?
2. Does sequence of numbers converge in the long run? If so, what to? $(2, 5, 5, 3, 2, \ldots)$
3. Let $S$ be a subset of $\mathbb{N}$ and let $f : S \times S^c \to \mathbb{N}^2$. For what "interesting" sets $S$ and functions $f$ can we answer the above questions?
   (In the example $S$ is the prime numbers and $f$ maps $(p, c) \mapsto (p + \operatorname{lpf}(c), \operatorname{gpf}(c))$.)

**References.**

`https://codegolf.stackexchange.com/q/144695/53884`

`https://math.stackexchange.com/q/2487116/121988`

`https://oeis.org/A293689`

# Problem 23.

Consider polyominoes where each cell has one of $n$ colors, and each distinct pair of colors is adjacent (horizontally or vertically) to each other somewhere in the polyomino. Let an $n$-minimum polyomino be one that has the minimum number of cells.

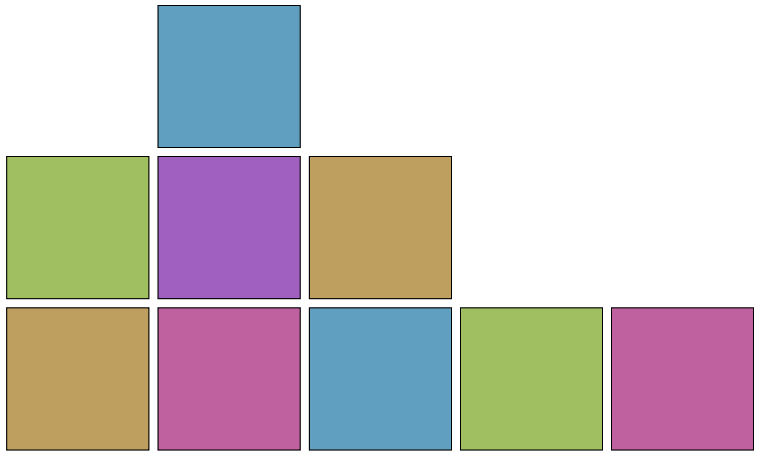


Figure 1: An example of a minimum polyomino for $n = 5$; $a(5) = 9$

**Question.** How many such $n$-minimum polyominoes exist?

**Related.**

1. What if the "distinct" restriction is lifted? (e.g. a blue label must somewhere be adjacent to another blue label.)
2. What is a way to determine the size of an $n$-minimum polyomino for large $n$?
3. What if this is done on a triangular or hexagonal grid?
4. What if this is done on a three dimensional cube lattice?

**References.**

`https://oeis.org/A278299`

`http://www.peterkagey.com/square_games`

# Problem 24.

Consider partitions of the $n \times m$ grid in which every piece has $180°$ rotational symmetry.

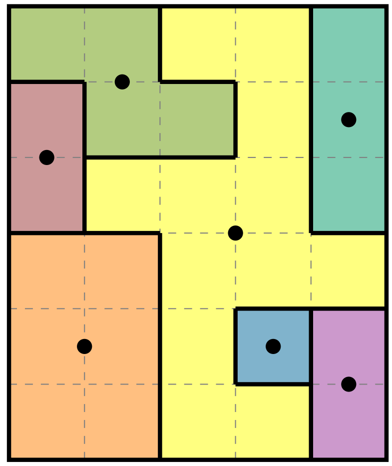


Figure 1: A partition of the $5 \times 6$ grid into 7 parts with rotational symmetry.

**Question.** How many such partitions of the $n \times n$ grid exist? Up to dihedral action?

**Related.**

1. How many partitions into exactly $k$ parts?
2. How many partitions with other types of symmetry?
3. How many partitions of a torus? Cylinder? Möbius strip?
4. How many partitions of a triangular or hexagonal lattice?
5. How many partitions of an $n \times m \times p$ cuboid?
6. How many placements of centers results in a unique solution? Multiple solutions? No solutions?
7. What if there is the additional restriction that putting together any proper subset of adjacent parts must not exhibit symmetry? (e.g two adjacent unit squares cannot be colored differently.)
8. What partitions have parts with the greatest average number of sides? (e.g. in the example the average part has $(5(4) + 8 + 16)/7 = 44/7 \approx 6.29$ sides.)
9. What partitions have the smallest ratio of rectangular parts? (e.g. in the example, 2 out of 7 parts are non-rectangular.)
10. What partitions have the greatest number of non-rectangular parts, total? (e.g. in the example, two of the parts are non-rectangular.)

**References.**

https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/galaxies.html

# Problem 25.

Consider all rectangles composed of $n$ squares such that the greatest common divisor of all the sidelengths is 1.

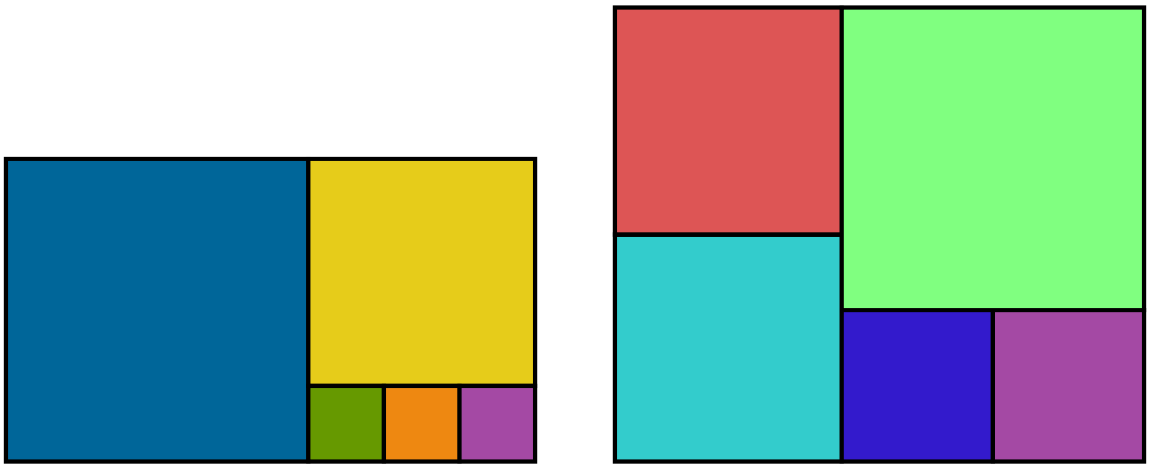


Figure 1: Two examples of rectangles made from $n = 5$ squares. In the first $\gcd(1, 1, 1, 3, 4) = 1$ and in the second $\gcd(2, 2, 3, 3, 4) = 1$.

**Question.** Given $n$ squares, how many such rectangles exist?

**Related.**

1. How many ways are there to make convex polygons out of $n$ equilateral triangles?
2. How many ways are there to make cuboids out of $n$ cubes?

**References.**

http://mathworld.wolfram.com/PerfectSquareDissection.html

# Problem 26.

Consider an $n$-coloring of a triangular grid such that no sub-triangle has corners all with the same color.

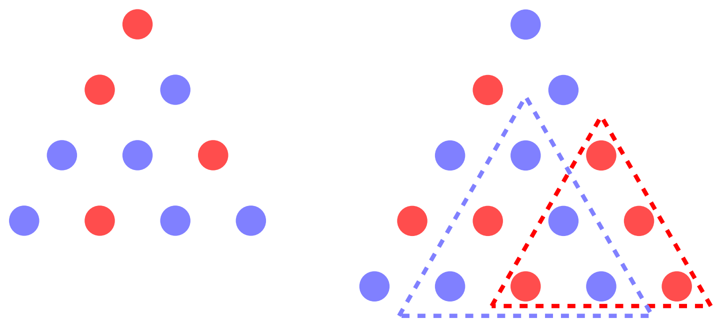


Figure 1: On the left is an example of a triangle on two labels that has no sub-triangles with equal corners. On the right is a non-example of such a triangle on two labels: it has two sub-triangles with equal corners.

**Question.** Given $n$ labels, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

**Related.**

1. Given an $n$-coloring of a triangle of side length $k$, what number of sub-triangles with equal corners must exist?
2. How many such triangles exist?
3. What if diagonal equilateral triangles also are not allowed to have equal corners?
4. What if this is done with hexagons instead of triangles?
5. What if this is done on a square grid?
6. What if for $n \geq 3$ no *two* corners are allowed to be equal? (This is a bit like a peaceable queens problem on a hexagonal chessboard.)

**References.**

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

# Problem 27.

A country has a strange legislative procedure. For each bill, the body is split up into $k_1$ committees of $\lfloor n/k_1 \rfloor$ or $\lfloor n/k_1 \rfloor$ legislators each, each of which picks a representative. These $k_1$ representatives are split up into $k_2$ sub-committees with $\lfloor k_1/k_2 \rfloor$ or $\lfloor k_1/k_2 \rfloor$ legislators each, which each elect a representative, and so on until $k_T = 1$ and the final committee votes on the bill.

There are a few rules:

1. Each committee (and subcommittee and so on) much have at least $\ell$ members.
2. Ties are settled by a coin toss.
3. The president does not get to vote, but she does get to choose the number of committees and who goes in each one.
4. There are $\alpha$ supporters who will always vote in the president's interests and $n - \alpha$ who will always vote against.

Let $a_\ell(n)$ be the minimum number of supporters ($\alpha$) required for the president to be able to pass every bill.

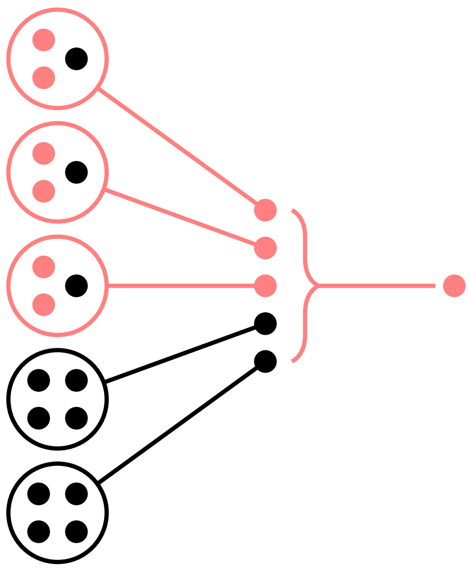


Figure 1: An example of $n = 17$ legislators with a minimum comittee size of $\ell = 3$, which demonstrates that $a_3(17) \leq 6$.

**Question.** What is an efficient way to compute $a_\ell(n)$ for general $\ell$ and $n$?

**Related.**

1. What if the president gets to choose who is on each committee but the opposition party gets to choose the committee size? Vice versa?
2. What if $k_1 \leq k_2 \leq \ldots \leq k_T$? Or $k_1 \geq k_2 \geq \ldots \geq k_T$?
3. What if ties go to the president? To the opposition?

**References.**

https://oeis.org/A290323

https://math.stackexchange.com/q/2395044/121988

# Problem 28.

Consider tilings of the $n \times n$ grid up to $D_8$ action where the tiles are diagonals.

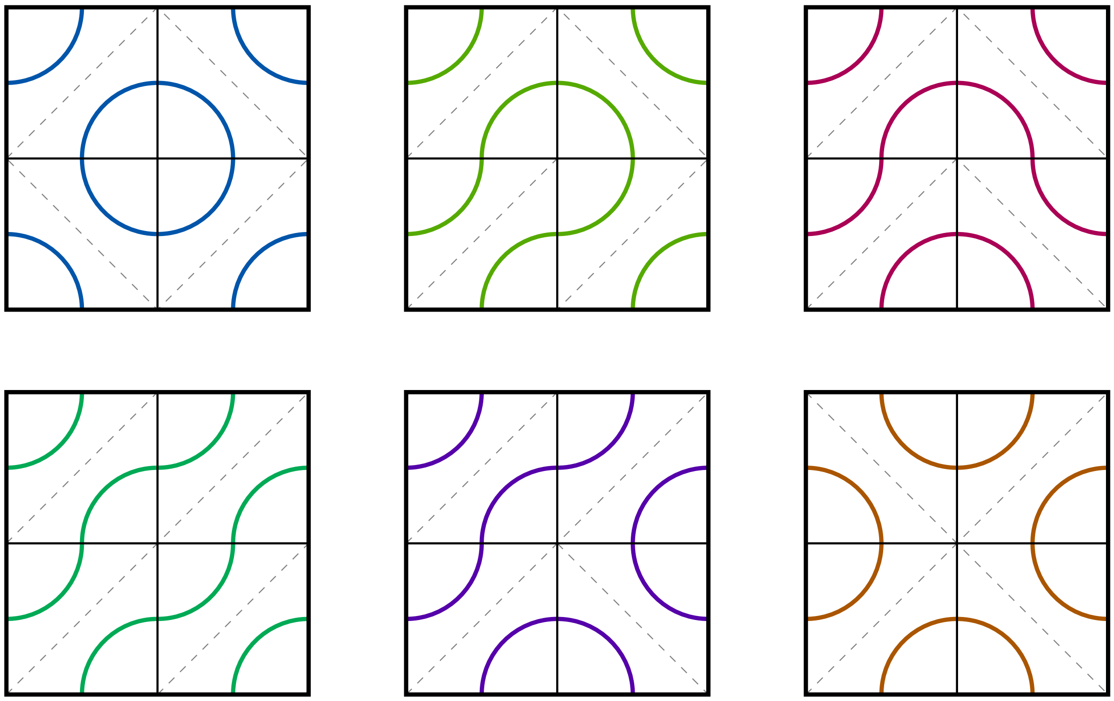


Figure 1: An example of the $a(2) = 6$ different ways to fill the $2 \times 2$ grid with diagonal tiles (up to dihedral action).

**Question.** How many such tilings exist?

**Related.**

1. What if grids are only counted up to $C_4$ (rotation) action?
2. What if this is counted on the torus/cylinder/Möbius strip?
3. What if each tile can have no diagonals or both diagonals?
4. What if tiles are black or white?
5. Is there an obvious bijection between the results on the $2n \times 2n$ grid for black/white versus diagonal tile types?

**References.**

https://oeis.org/A295229

# Problem 29.

Consider the rectangles from Problem 25: those composed of $n$ squares such that the greatest common divisor of all the sidelengths is 1. If rectangles are measured by the longest side, the smallest rectangles are given by $A295753$.

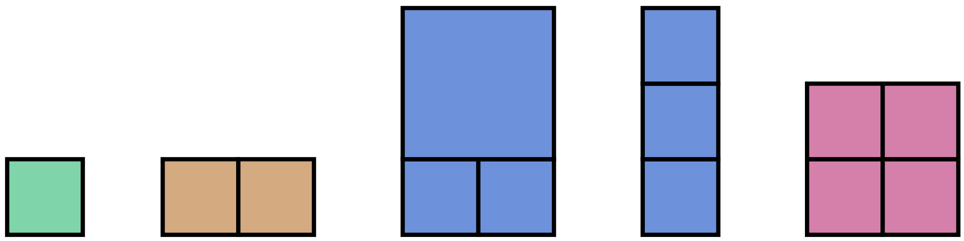


Figure 1: Examples of $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, and $a(4) = 1$.

**Question.** How many distinct rectangles composed of $n$ squares have a longest side of $A295753(n)$?

**Related.**

1. Is the largest rectangle (as measured by smallest side) unique for large $n$?
2. What if smallest rectangle is measured by perimeter?

**Note.** Largest rectangles might be Fibonacci spirals, or they might be similar to the second example or the examples in the References.

**References.**

https://en.wikipedia.org/wiki/Squaring_the_square

https://oeis.org/A295753

# Problem 30.

Consider all configurations of $n$ nonattacking rooks on an $n \times n$ board up to dihedral action.

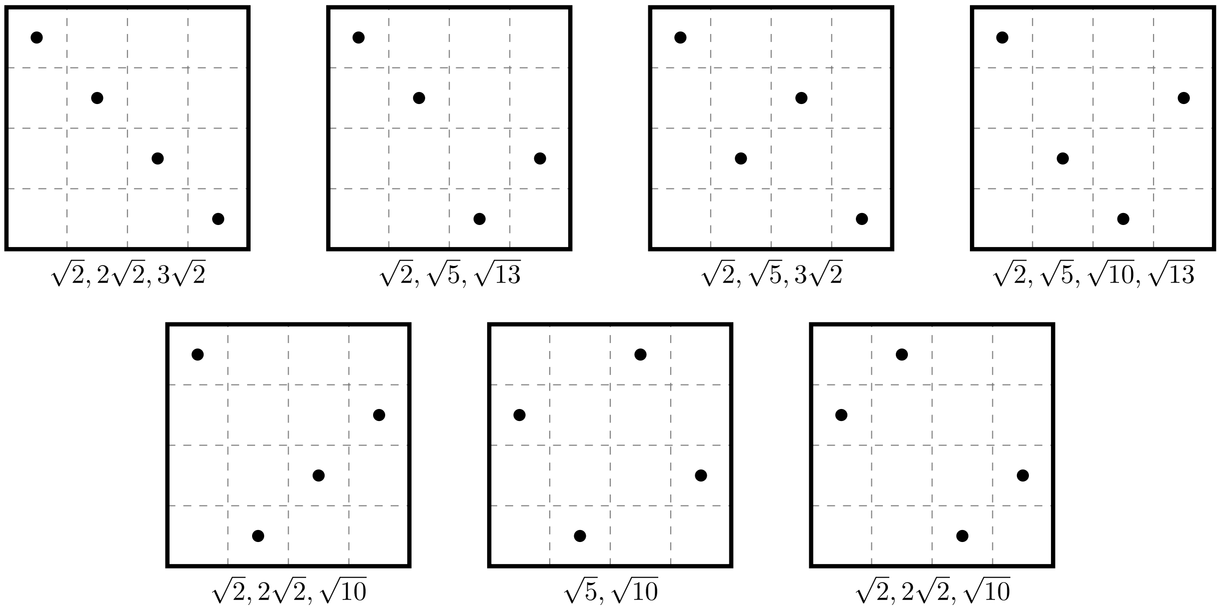



Figure 1: Each figure is marked with the distinct distances between pieces.

**Question.** What is the minimum number of distinct distances on such a figure?

**Related.**

1. What is the minimum number of distinct directions on such a figure? (Directions up to dihedral action?)
2. What if this is done with $n$ queens instead of rooks?
3. What if this is done with $0 \leq k \leq n^2$ pieces, any of which are allowed to be in attacking positions?
4. What if distance is measured by the taxicab metric? $d_3$? $d_\infty$? Number of knight-moves away? Number of king moves away?
5. How many configurations of nonattacking rooks on the torus, rectangle, triangluar grid, and other geometries?
6. Are any configurations of nonattacking rooks on the torus that can be meaningfully called a "generalized Costas array"?

**References.**

`https://en.wikipedia.org/wiki/Costas_array`

The maximum and minimum number of distinct distances is given by A320448 and A319476 respectively.

The number of extremal boards is given by A320573 and A320575 (A320574 and A320576, up to symmetry).

A193838: smallest square pegboard from which $n$ points with distinct mutual distances can be chosen.

# Problem 31.

Consider square, triangular, and hexagonal grids that are filled in with with tiles of different patterns.

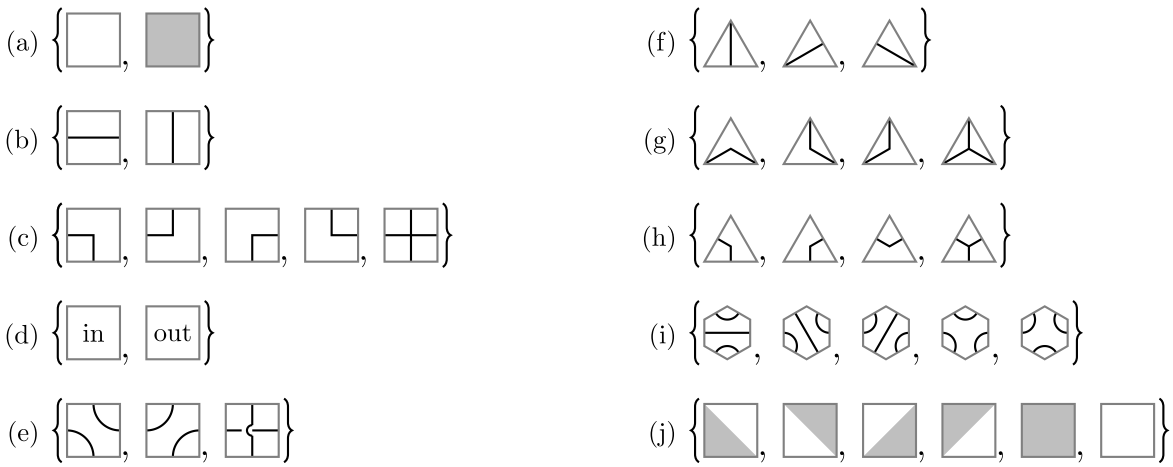



Figure 1: Ten examples of different tiles.

**Question.** How many essentially different grids of size $n$ exist with these tiles? (Up to dihedral action? Up to cyclic action?)

**Related.**

1. The square grid can be $n \times n$ or $n \times m$.
2. The hexagonal grid can have triangles with side length $n$ or hexagons with side length $n$.
3. The triangular grid can have triangles with side length $n$ or hexagons with side length $n$.
4. The square grid can be quotiented to be a cylinder, torus, or Möbius strip.
5. What if shapes have to "match-up" (e.g. the lines in the third example or colors in the last example have to be "smooth".)
6. How many distinct regions, as in Problem 2?

**References.**

Problem 2.

Problem 28.

https://en.wikipedia.org/wiki/Burnside%27s_lemma

https://en.wikipedia.org/wiki/Palago

# Problem 32.

Starting with an $n \times m$ grid, remove one corner at a time (uniformly at random) until the grid is gone.

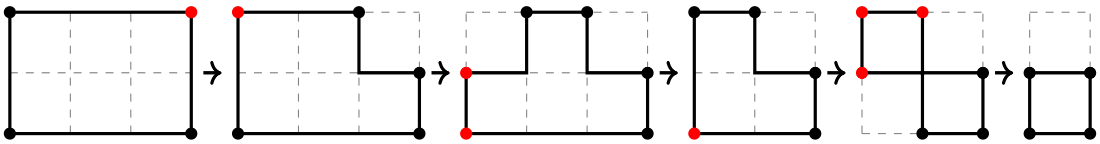


Figure 1: An example of a process starting with a $2 \times 3$ grid.

**Question.** If a stopping point is chosen randomly, how many corners are expected?

**Related.**

1. What if the deletion is uniform with respect to faces instead of vertices?
2. How many sides are expected?
3. If all polygons in the process are considered, what is the expected number of corners on the polygon with the greatest number of corners?
4. What figure produces the greatest number of corners?
5. How many possible processes exist (up to, say, dihedral action)?
6. What if each figure must stay path connected?
7. What if paths cannot travel through corners? (e.g. the second-to-last figure is illegal.)

**References.**

Problem 8.

# Problem 33.

Consider all of the ways to stack $n$ "blocks" of different shapes on a platform of length $k$.

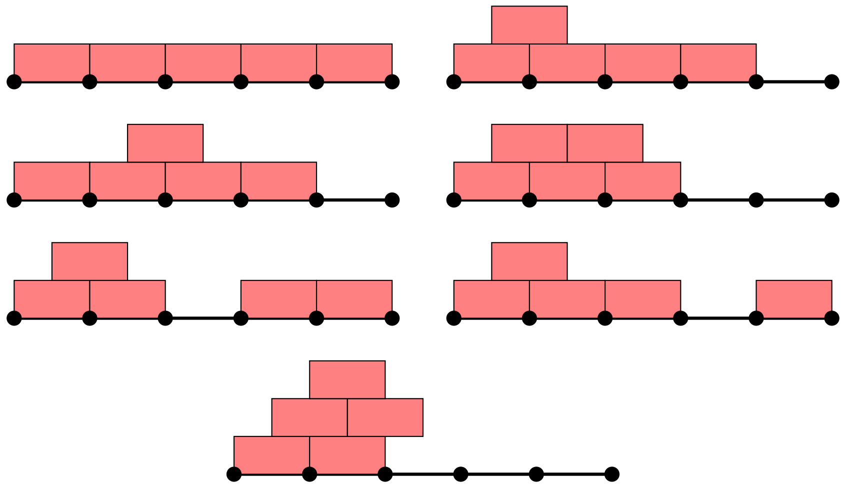
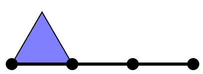
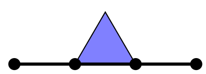
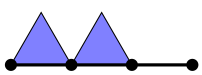
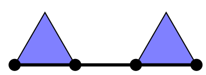


Figure 1: Seven examples of five length 2 bricks on a length 5 platform.

**Question.** How many different stacks exist for these shapes?

**Related.**

1. What if we use triangular blocks?
   (a) What if  and  are considered to be distinct?
   (b) What if  and  are considered to be the same (because one turns into the other by "sliding".)
   (c) What if "upside-down" triangles can be placed in the gaps?
   (d) What if "upside-down" triangles *must* be placed in the gaps in order to stack on top?
2. What about bricks of length 3?
3. What about tetrahedra and cuboids?
4. What if bricks can be stacked directly on top of each other?
5. What if the stack must be connected?
6. What if reflections are considered to be the same?

**Note.** The triangle stacking problem appears to be counted by Catalan numbers. If cantilevers are not allowed, the brick stacking problem reduces to the triangle stacking problem.

**References.**

`https://oeis.org/A005169` (Connected triangles on arbitrarily long platform)

`https://oeis.org/A168368`

`https://math.stackexchange.com/q/2731692/121988`

# Problem 34.

Consider ways to partition the $n \times m$ grid so that no three tiles of the same partition fall on a line.

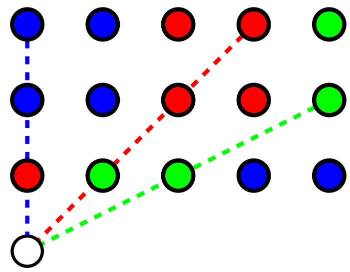


Figure 1: A 3 partition of the $5 \times 3$ grid. The white circle cannot be in any of the existing partitions, otherwise three circles of the same color would fall on the same line.

**Question.** How many colors are required to satisfy the "no three in a row" criterion?

**Related.**

1. What if this is generalized to $k$ in a row?
2. What if this is generalized to a triangular or hexagonal grid?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if this only queen moves or rook moves are considered?
5. How many distinct configurations exist with a minimal number of partitions?
6. How many distinct configurations exist with $k$ partitions?

**References.**

Problem 26.

# Problem 35.

Say that a *minimally interpolable permutation* $f$ is a permutation of $\{1, 2, \ldots, n\}$ such that no $k+2$ of the points $\{(1, f(1)), \ldots, (n, f(n))\}$ fall on a degree $k$ polynomial.

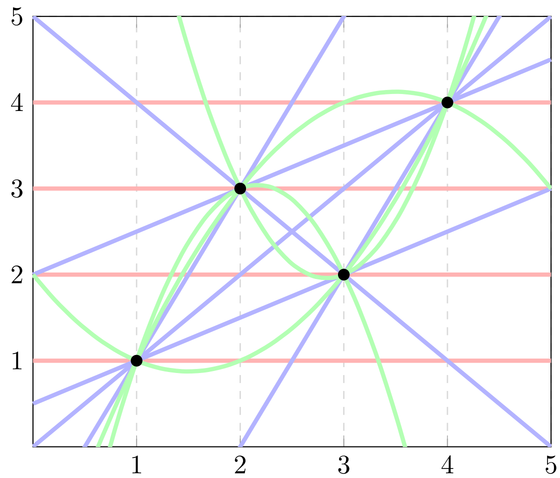



Figure 1: A minimally interpolable permutation of $\{1, 2, 3, 4\}$. (Degree 0 polynomials are plotted in red, degree 1 in blue, and degree 2 in green.)

**Question.** Do such permutations always exist? If not, what is the least $N$ such that there is a minimally interpolable function from $[n]$ into $[N]$?

**Related.**

1. How many minimally interpolable permutations exist?
2. Does the number of minimally interpolable permutations increase as a function of $n$?
3. Is there a method to explicitly construct a minimally interpolable permuation?
4. If such permutations do not always exist, what is the least $M$ such that there exists a subset $S \subset [M]$ and a surjection $g \colon S \to [N]$ with the aforementioned property?

**References.**

Problem 19.

`https://oeis.org/A301802`

`https://codegolf.stackexchange.com/q/160382/53884`

# Problem 36.

Consider an $n$-coloring of a triangular grid such that no upright sub-triangle has the same coloring as any other (up to rotation).

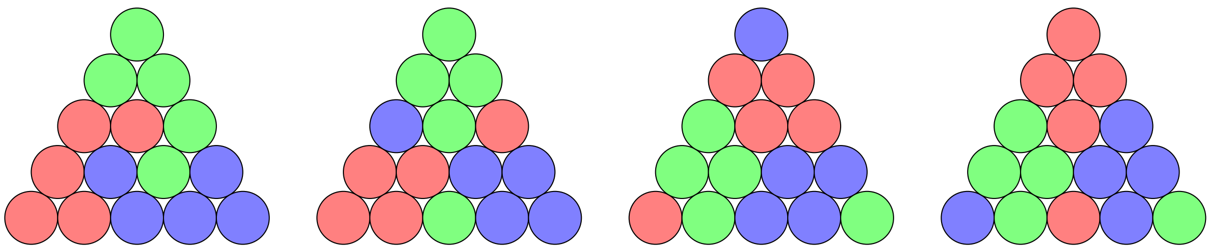


Figure 1: Four examples of 3-colorings of the length 5 triangle. In all cases, 10 different colorings appear exactly once. In the first example, starting from the top: (1) GGG, (2) RRG, (3) RGG, (4) RRB, (5) RGB, (6) GGB, (7) RRR, (8) RBB, (9) GBB, and (10) BBB. (Incidentally, this is *all* of the colorings, so $a(3) = 5$.)

**Question.** Given $n$ colors, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

**Related.**

1. What if inverted triangles are counted too?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. How many patterns exist for a triangle of length $k$ with the minimum number of labels?
4. What if diagonal equilateral triangles are also considered? (For example, take the second circle on every side as measured clockwise from each corner.)
5. What if this is done on a square grid?
6. What if this is done on hexagonal shapes?
7. What if this is done on tetrahedra or cuboids?
8. Consider the lexicographically earliest infinite case. Does every triangle eventually appear?

**References.**

Problem 26.

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

# Problem 37.

Start with an $n \times n$ grid of boxes and place lines through gridpoints at the border. A box is considered "on" if a line travels through its interior.

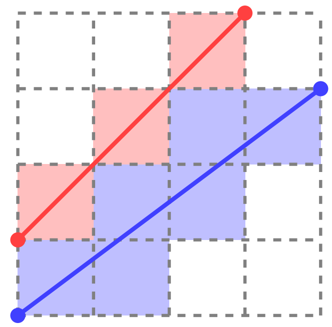


Figure 1: An example of two lines drawn on a grid. The seven white squares still require a line to be drawn through them.

**Question.** What is the minimum number of lines required to turn on all of the squares in and $n \times n$ grid?

**Related.**

1. What if touching the corner of a square also turns it on?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. What if no two lines can be parallel? If no two line segments can be congruent?
4. What if no two lines can intersect?
5. How many fully "on" grids exist? How many such minimal grids? (A grid is minimal if removing any line results in a square turning off.)
6. Suppose a grid is on if an even number of lines pass through it and off if an odd number of lines pass through it. How many such grids?
7. How about on an $n \times m$ grid?
8. What if this is done on a triangular grid?
9. What if this is done on a cuboid? On a cuboid with planes passing through the cubes?

**Note.** In the case where "grid is on if an even number of lines pass through it and off if an odd number of lines pass through it", there exist $2^k$ configurations. If we further restrict to dihedral-symmetric grids, there are $2^j$ configurations.
It appears that $k = 5n^2 - 14n + 9$, the 12-gonal numbers. Is there a bijection between the basis elements and the 12-gonal numbers?

# Problem 38.

There is a well known magic trick called “Communicating the Card” in which a spectator draws $k$ cards from an $n$-card deck and shows them to the magician’s assistant. He then shows $k-1$ of them to the magician in a particular order, after which she (the magician) can deduce the remaining card. In this variation, the largest possible deck is $k! + k - 1$ cards.

| | | | |
|---|---|---|---|
| $f(1,2) = \{1,2,3\}$ | $f(4,8) = \{1,4,8\}$ | $f(7,2) = \{2,4,7\}$ | $f(8,3) = \{3,5,8\}$ |
| $f(2,1) = \{1,2,4\}$ | $f(5,1) = \{1,5,6\}$ | $f(8,2) = \{2,4,8\}$ | $f(3,6) = \{3,6,7\}$ |
| $f(1,5) = \{1,2,5\}$ | $f(5,7) = \{1,5,7\}$ | $f(2,5) = \{2,5,6\}$ | $f(6,3) = \{3,6,8\}$ |
| $f(1,6) = \{1,2,6\}$ | $f(5,8) = \{1,5,8\}$ | $f(5,2) = \{2,5,7\}$ | $f(7,3) = \{3,7,8\}$ |
| $f(1,7) = \{1,2,7\}$ | $f(6,7) = \{1,6,7\}$ | $f(8,5) = \{2,5,8\}$ | $f(4,5) = \{4,5,6\}$ |
| $f(1,8) = \{1,2,8\}$ | $f(6,8) = \{1,6,8\}$ | $f(6,2) = \{2,6,7\}$ | $f(5,4) = \{4,5,7\}$ |
| $f(1,3) = \{1,3,4\}$ | $f(7,8) = \{1,7,8\}$ | $f(8,6) = \{2,6,8\}$ | $f(8,4) = \{4,5,8\}$ |
| $f(3,1) = \{1,3,5\}$ | $f(2,3) = \{2,3,4\}$ | $f(8,7) = \{2,7,8\}$ | $f(4,6) = \{4,6,7\}$ |
| $f(6,1) = \{1,3,6\}$ | $f(3,2) = \{2,3,5\}$ | $f(3,4) = \{3,4,5\}$ | $f(6,4) = \{4,6,8\}$ |
| $f(7,1) = \{1,3,7\}$ | $f(2,6) = \{2,3,6\}$ | $f(4,3) = \{3,4,6\}$ | $f(7,4) = \{4,7,8\}$ |
| $f(8,1) = \{1,3,8\}$ | $f(2,7) = \{2,3,7\}$ | $f(3,7) = \{3,4,7\}$ | $f(5,6) = \{5,6,7\}$ |
| $f(1,4) = \{1,4,5\}$ | $f(2,8) = \{2,3,8\}$ | $f(3,8) = \{3,4,8\}$ | $f(6,5) = \{5,6,8\}$ |
| $f(4,1) = \{1,4,6\}$ | $f(2,4) = \{2,4,5\}$ | $f(3,5) = \{3,5,6\}$ | $f(7,5) = \{5,7,8\}$ |
| $f(4,7) = \{1,4,7\}$ | $f(4,2) = \{2,4,6\}$ | $f(5,3) = \{3,5,7\}$ | $f(7,6) = \{6,7,8\}$ |

Figure 1: An example of an encoding where $k = 3$ and $n = k! + k - 1 = 8$.

**Question.** What if the assistant can show any number of cards less than $k$, and the magician must guess all of the remaining cards?

**Related.**

1. How many different encodings exist (up to relabeling)?
2. What if the magician just needs to guess one of the remaining cards?
3. What if there are $\ell$ identical copies of a deck, how many cards can the original trick support?
4. If the assistant shows $k-2$ cards to the magician, what is the biggest deck that this trick can be done with? $k-j$?

**References.**

`http://oeis.org/A030495`

`https://www.reddit.com/r/math/comments/7l1t84/a_combinatorists_card_trick/`

`https://web.northeastern.edu/seigen/11Magic/Articles/Best%20Card%20Trick.pdf`

# Problem 39.

This one is based on correspondence from Alec Jones: Consider all of the ways of partitioning the complete graph on $n$ vertices into smaller complete graphs.

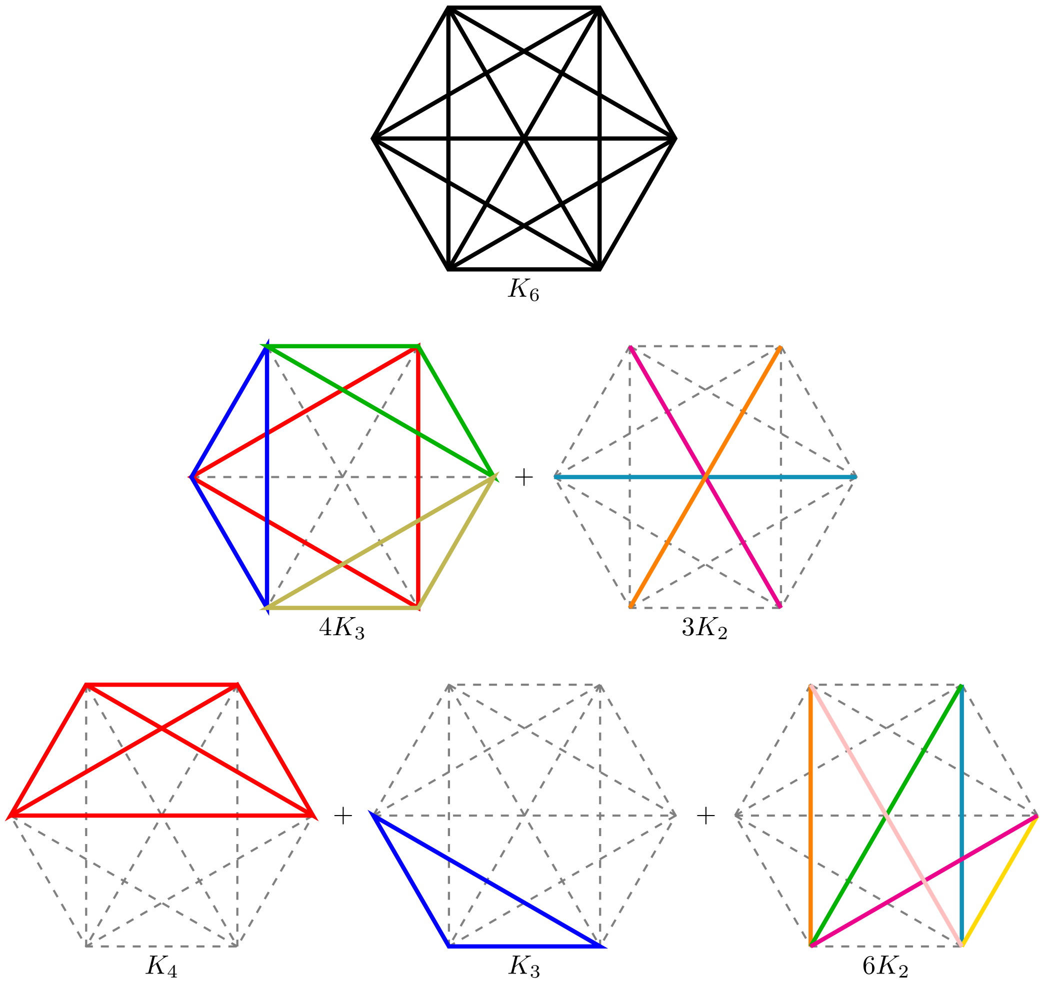



Figure 1: An example three ways to partition $K_6$ into complete graphs: the trivial partition, a partition into 4 copies of $K_3$ and 3 copies of $K_2$, and a partition into 1 copy of $K_4$, 1 copy of $K_3$, and 6 copies of $K_2$.

**Question.** How many such partitions exist, up to graph isomorphism?

**Related.**

1. What if the union of $K_j$ graphs cannot contain a $K_{j-1}$ subgraph?
2. What if the partition can only consist of two "sizes" of complete graphs, as in the second example?
3. How many such partitions exist up to dihedral action?

# Problem 40.

From correspondence with Alec Jones: Consider a game played on the $m \times n$ rectangular grid, where players take turns placing their pieces onto the board. Each player gets a point for each 3-in-a-row that they make.

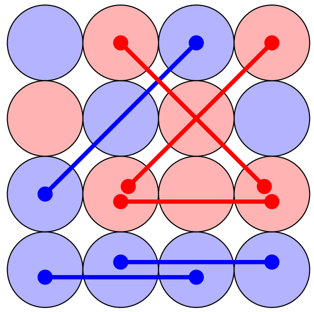


Figure 1: In this game on a $4 \times 4$ board, the red player and blue player tie with three points each.

**Question.** Which player has a winning strategy?

**Related.**

1. What is the score differential under perfect play?
2. If players cooperate, what is the greatest score differential?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if the game is played with $k$ players or requires $\ell$-in-a-row?
5. What if the game is played on a triangular grid?
6. What if the game is played in $d$ dimensions?

**References.**
Problem 34.

# Problem 41.

A problem inspired by a Project Euler problem: suppose an $n$-robot takes steps that are $1/n$ of a circle, and turns right or left after every step.

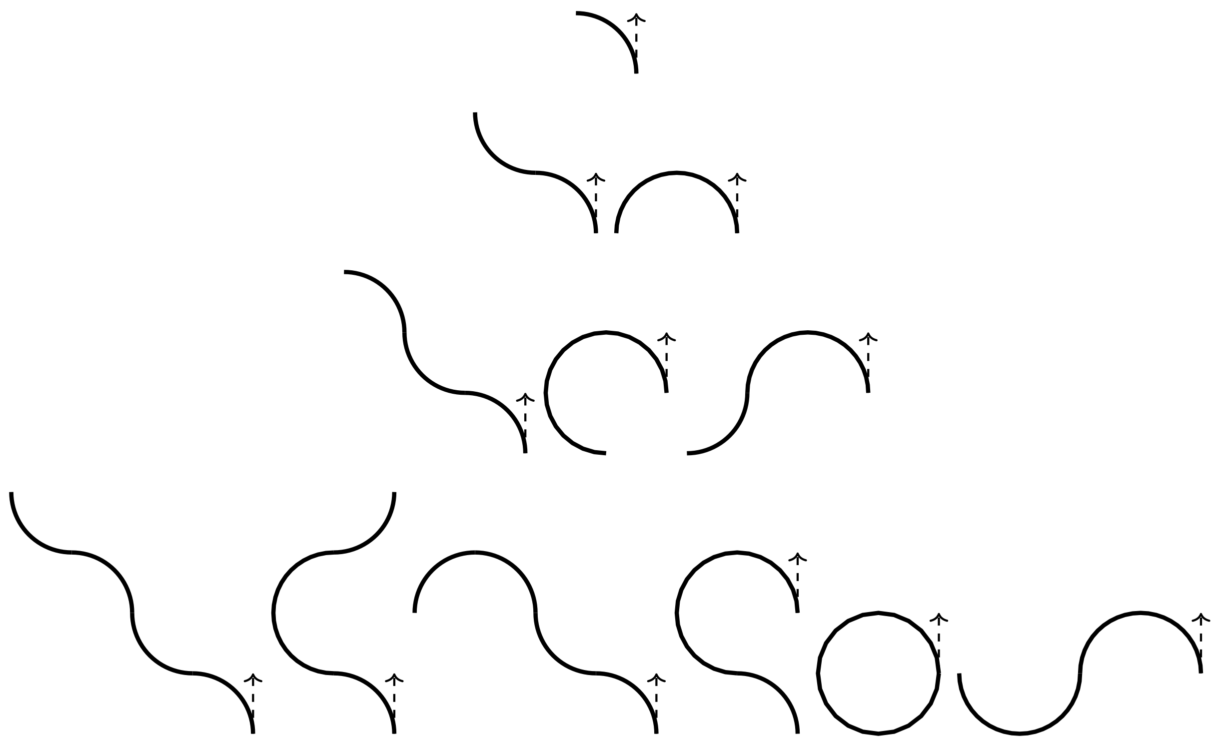


Figure 1: An example of distinct paths of $k$ steps (up to dihedral action) for a 4-robot. $a(1) = 1$, $a(2) = 2$, $a(3) = 3$, and $a(4) = 6$.

**Question.**

How many walks exist such that the robot ends up at the original position and orientation after $k$ steps?

**Related.**

1. How many distinct paths exist for an $n$-robot, where the robot never retraces its steps?
2. What if the robot is allowed to retrace its steps?
3. What is the smallest radius that can contain a $k$-step walk if the robot cannot retrace its steps? (The robot returns to where it started in the same direction that it started.)
4. Can smooth loop paths occur when the number of steps is not a multiple of $n$?
5. What if the orientation of the path matters (i.e. *not* counted up to dihedral action)?
6. What if this is done on a torus, cylinder, or Möbius strip?
7. What if the robot cannot cross its own path?

**References.**

https://projecteuler.net/index.php?section=problems&id=208

# Problem 42.

Consider walks in a city, starting mid-block, where (1) at each intersection the walker goes left right or straight, (2) at each mid-block, the walker decides whether or not to turn around, and (3) she ends up back at her apartment.

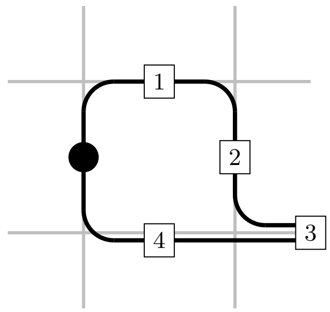



Figure 1: An example of a 5-step walk returning to the apartment.

**Question.** How many $n$-block walks can the walker take?

**Related.**

1. What if the walker does not want to walk along the same strip of road twice?
2. What if the walker does not want to walk along the same *side* of the same strip of road? (Suppose she always walks on the right side of the street.)
3. What if the walker never wants to revisit the same intersection?
4. How many walks up to dihedral action?
5. What if the walker does not turn around?
6. What if the walker never goes straight? Never goes right?

**References.**

Problem 41.

# Problem 43.

Consider a puzzle on a (blank) $n \times m$ board, where each column and row has a number denoting the number of markers that should go in that column or row. The player's goal is to fill in the grid in such a way that the row/column "histograms" are satisfied.

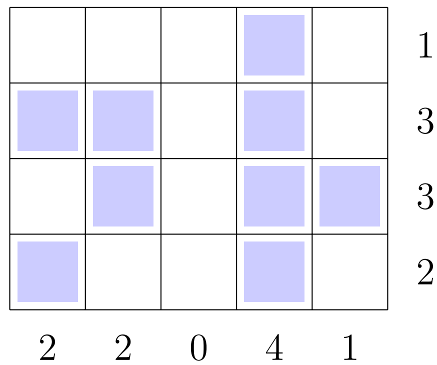



Figure 1: Example of a solution to the puzzle $(2, 2, 0, 4, 1) \times (1, 3, 3, 2)$. Is the solution unique?

**Question.** What is a procedure for determining if a grid has a solution? If it has a unique solution?

**Related.**

1. What if the game is played on a $d$-dimensional hypercube?
2. What if the game is played on a triangle? Tetrahedron?
3. What is the greatest amount of ambiguity a non-unique board can have? (i.e. what is the greatest number of solutions?)
4. How many maximally ambiguous boards exist?
5. How many distinct boards exist up to dihedral action? Up to torus action?
6. What if multiple markers can be put in each cell?

**References.**

https://oeis.org/A297077

# Problem 44.

From Alec Jones. Let $a_k(n)$ count the number of $k$-gons with vertices on the $n \times n$ grid.

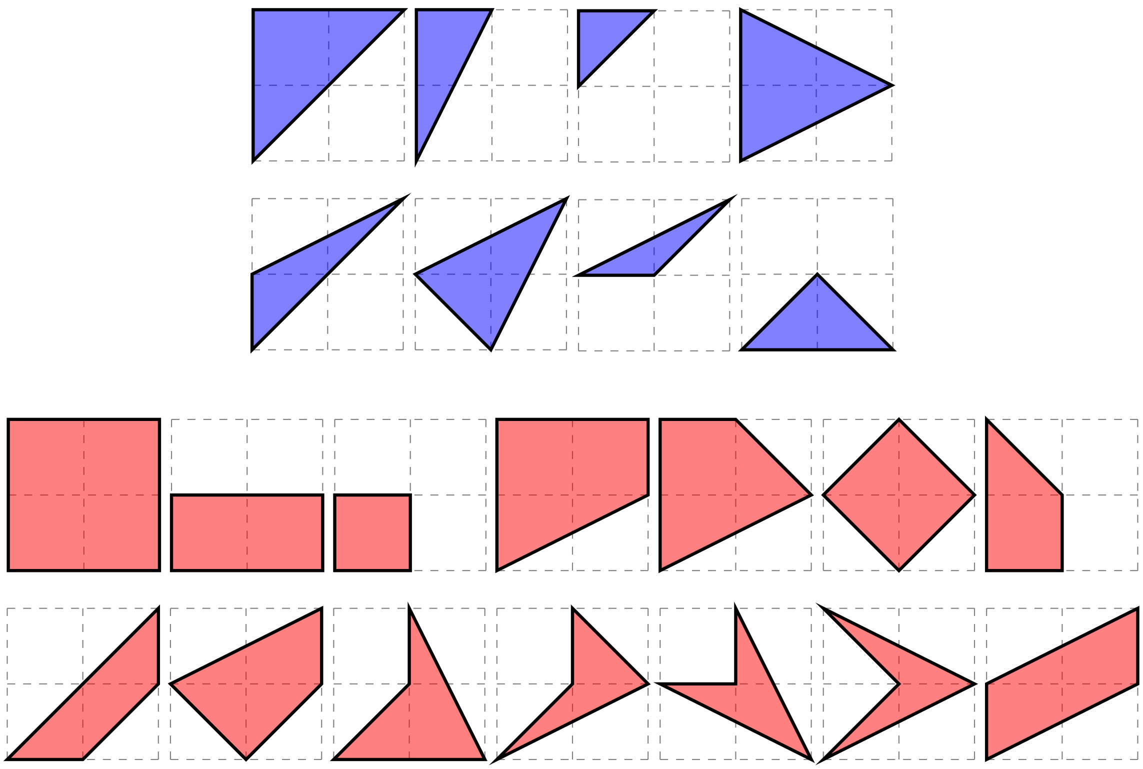


Figure 1: An example showing that $a_3(2) \geq 8$ and $a_4(2) \geq 14$.

**Question.** What is $a_k(n)$?

**Related.**

1. For a fixed $n$, what is the value of $k$ such that $a_k(n)$ is maximized?
2. Here two polygons are considered equivalent if they are congruent. What if two polygons are considered equivalent if they are similar? If they are the same under dihedral action? If they are the same over linear transformation? (e.g. stretching/skewing)
3. What if concave polygons are excluded?
4. What if this is done on an $n \times m$ grid?
5. What if we don't deduplicate based on congruence?
6. What if this is done on a hypercube or a triangular grid?

# Problem 45.

A polyform counting problem from Alec Jones: let $a_k(n)$ count the number of polyabolos with $n$ faces and $k$ exposed edges.

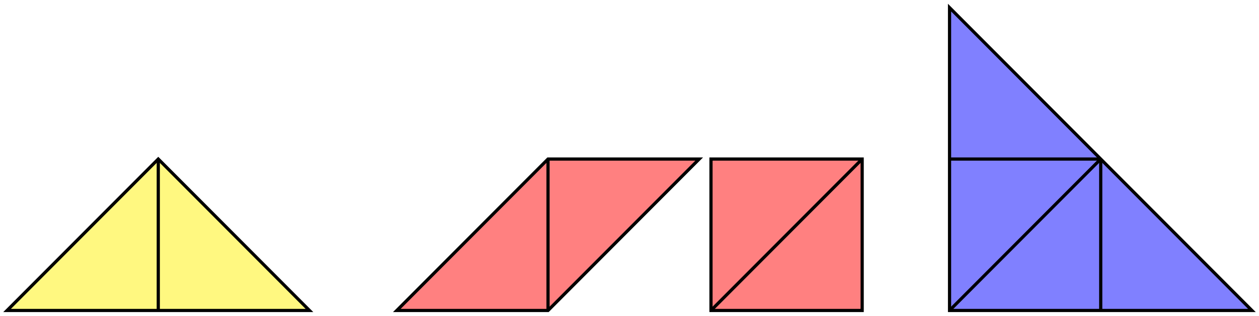


Figure 1: An example in yellow showing that $a_3(2) \geq 1$, two examples in red showing that $a_4(2) \geq 2$, and an example in blue showing that $a_3(4) \geq 1$.

**Question.** What is the smallest $k$ such that for some fixed $n$, $a_k(n) > 0$?

**Related.**

1. What is the largest $k$ such that for some fixed $n$, $a_k(n) > 0$?
2. What if $\hat{a}_k(n)$ counts polyiamonds instead?
3. What if concave polygons are excluded?
4. Is the following function well-defined?

$$b(k) = \max\{\, a_k(n) : n \in \mathbb{N} \,\}$$

5. Is the following function interesting?

$$c(n) = \max\{\, a_k(n) : k \in \mathbb{N} \,\}$$

**References.**

https://en.wikipedia.org/wiki/Polyiamond

https://en.wikipedia.org/wiki/Polyabolo

# Problem 46.

Define an $n$-triangle to be a triangle with integer coordinates and perimeter in $[n, n+1)$.

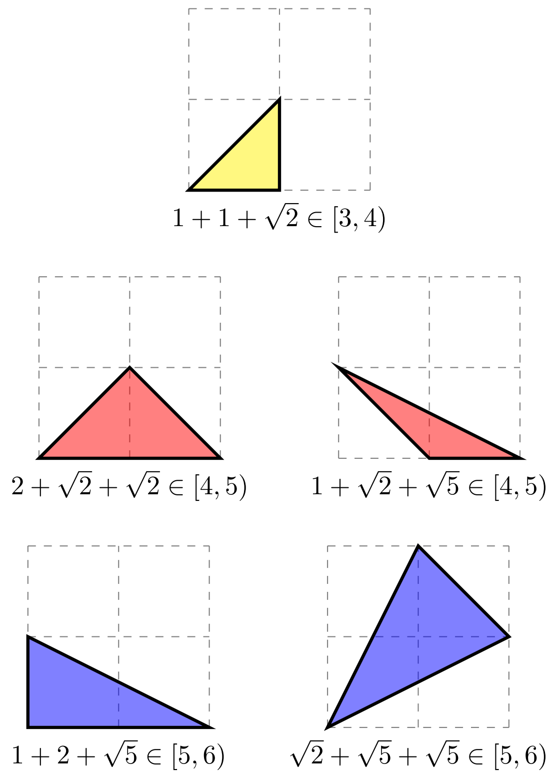



Figure 1: An example in yellow showing that $a(3) = 1$, and example in red showing that $a(4) = 2$, and an example in blue showing that $a(5) = 3$.

**Question.** Let $a(n)$ count $n$-triangles up to dihedral action. What is the asymptotic growth of $a(n)$?

**Related.**

1. How many tetrahedra?
2. How many quadrilaterals?

**References.**

`https://oeis.org/A298079` counts the number up to congruence.

Problem 44

# Problem 47.

Let a palindromic partition be a partition of a string into palindromes.

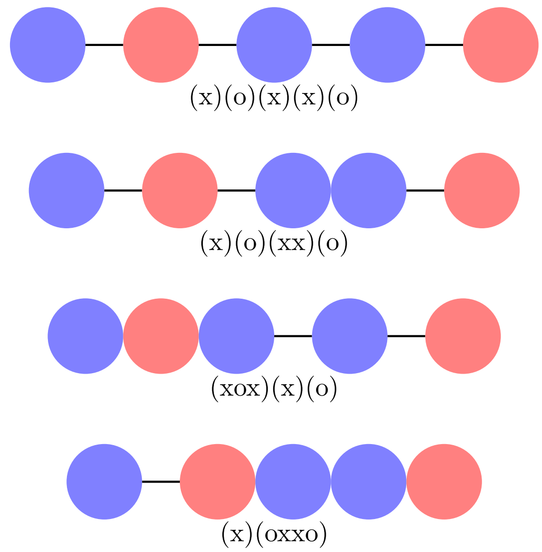



Figure 1: An example of four palindromic partitions of the string “xoxxo”.

**Question.** Given some string, how many palindromic partitions does it have?

**Related.**

1. What is the least number of parts $p$ such that an arbitrary string of length $\ell$ over a $k$-letter alphabet can be partitioned into $p$ or fewer parts?
2. What is the length of the shortest string that cannot be partitioned into fewer than $p$ parts?
3. How many length $\ell$ strings require the “worst-case” number of parts?
4. Which length $\ell$ strings have the greatest number of distinct partitions? The least?
5. What is the smallest number of parts that any string with $m$ o’s and an arbitrary number of x’s can be partitioned into?

**References.**

`https://oeis.org/A298481` the number of ways to partition the binary representation of n into the minimal number of palindromic parts.

# Problem 48.

Consider folding a strip of $n$ equilateral triangles down to 1 triangle in as few moves as possible.

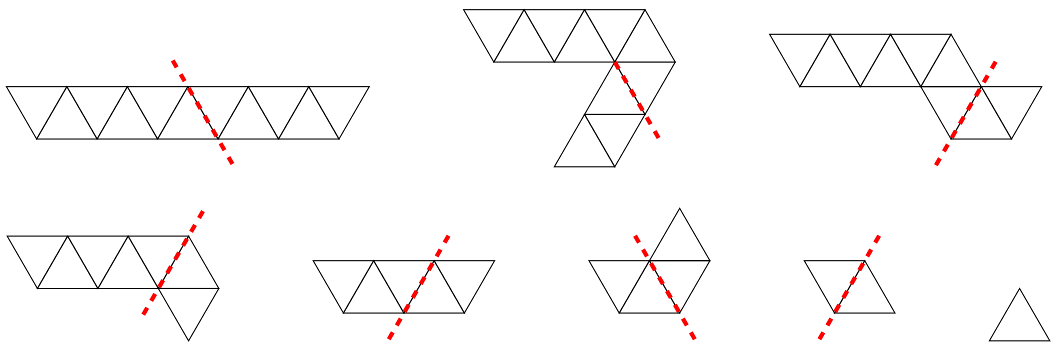


Figure 1: An example demonstrating that $a(11) \leq 7$.

**Question.** How many folds are required to fold a strip of $n$ triangles down to one?

**Related.**

1. What if other $n$-iamonds are considered? Which $n$-iamond takes the greatest number of folds?
2. Does there exist a family of $n$-iamonds that require more than $\mathcal{O}(\log_2(n))$ folds?
3. Given an $n$-iamond uniformly at random, what's the expected value of the number of folds required?
4. What if you must fold a single cell versus across a line?
5. Consider the graded poset of polyiamonds given by the covering relation $x \lessdot y$ if $y$ is one fold away from $x$. How many polyiamonds have rank $n$?
   - There is at least one $2^k$-iamond with rank $k$. How many $2^k$-iamonds are there with rank $k$?
   - What's the second largest polyiamond of rank $k$?
   - What's the smallest polyiamond of rank $k$?
   - Is this poset Sperner?
6. Is there a sensible way to generalize this construction to other polyforms?

# Problem 49.

Consider a 2-coloring of a triangular grid of length $\ell$. Then label each cell with its greatest number of neighbors of one color.

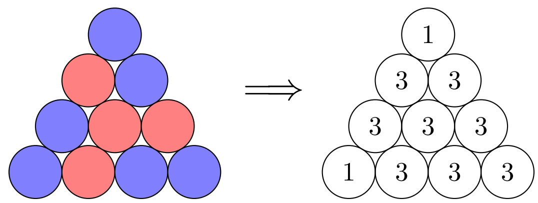



Figure 1: The second cell (reading top to bottom and left to right) is labeled with a $\max(3, 1) = 3$ because it has 3 blue neighbors and 1 red neighbor.

**Question.** How many colorings exist of a length $\ell$ triangle such that the maximum label is 3?

**Related.**

1. If the "number triangle" is summed for each coloring, which coloring has the smallest sum?
2. How does this generalize for a $k$-coloring?
3. How does this generalize to a $n \times m$ square grid where horizontal-vertical connections are counted? Diagonal connections? Both?
4. How does this generalize to a tetrahedron, torus, Möbius strip, cylinder, or cube?
5. How many colorings exist of a length $\ell$ triangle such that the maximum label is 4 or 5?

# Problem 50.

Call $s$ an "initial permutable" string if for every initial substring of odd length, the first half of the string is a permutation of the second half.

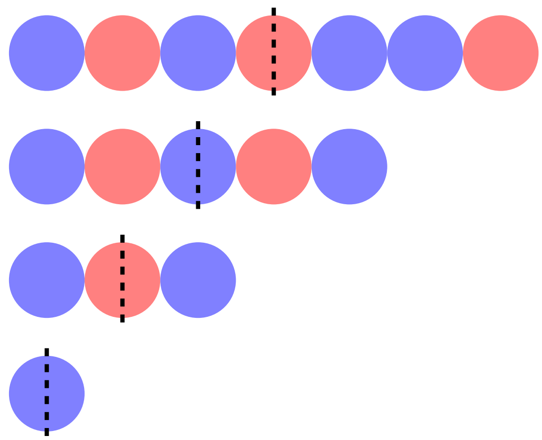


Figure 1: "BRBRBBR" is an example of an initial permutable string. Because each initial odd substring (the string itself, "BRBRB", "BRB", and "B") has the property that the first half of the string is a rearrangement of second half.

**Question.** What is the growth of $a_2(n)$, the number of initial permutable strings of length $2n-1$ over a 2-letter alphabet?

**Related.**

1. Can this be generalized to a $k$-letter alphabet?

**References.**

`http://oeis.org/A297789`

# Problem 51.

Given two vector valued functions $u, v\colon \mathbb{R}^n \to \mathbb{R}^n$, that are linearly independent at every point, let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x_0, x_1) = |\alpha| + |\beta| \text{ where } x_1 - x_0 = \alpha \cdot u(x_0) + \beta \cdot v(x_0).$$

Next let the length of a curve $\Gamma\colon [0,1] \to \mathbb{R}^n$ be given by

$$\mathcal{L}(\Gamma) = \lim_{N\to\infty} \sum_{k=0}^{N-1} f\left(\Gamma\left(\frac{j}{N}\right), \Gamma\left(\frac{j+1}{N}\right)\right).$$

Let the distance $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ from $x_0$ to $x_1$ be given by the infimum of the length over all curves from $x_0$ to $x_1$:

$$d(x_0, x_1) = \inf\{\, \mathcal{L}(\Gamma) : \Gamma(0) = x_0 \text{ and } \Gamma(1) = x_1 \,\}.$$

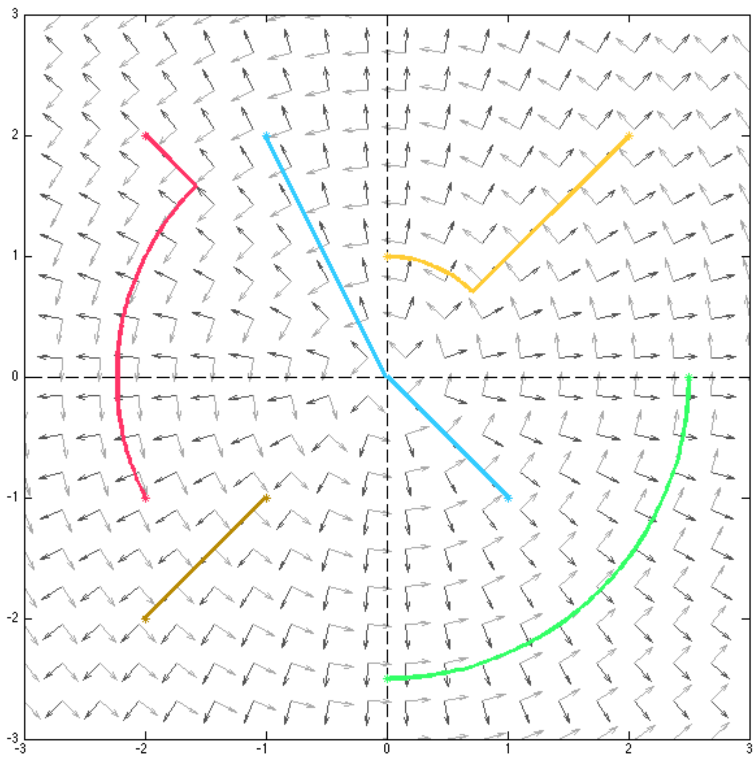



Figure 1: Five examples of shortest curves when $u(x_1, x_2) = (x_1, x_2)/||(x_1, x_2)||$ and $v(x_1, x_2) = (-x_2, x_1)/||(x_1, x_2)||$.

**Question.** What are the necessary conditions on $u$ and $v$ for this to be a well-defined metric space?

**Related.**

1. If $|u(x)| = |v(x)| = 1$ for all $x \in \mathbb{R}^n$, what is greatest possible (Euclidean) length of the circumference of a unit circle?

2. If $u$ and $v$ are well-behaved and selected at random according to some distribution, what is the expected length of the circumference of a unit circle?

# Problem 52.

Suppose that there is a “cop” and a “robber” on an infinite grid, where each starts at some given position with some given orientation on the grid, and each can move according to some rule set.

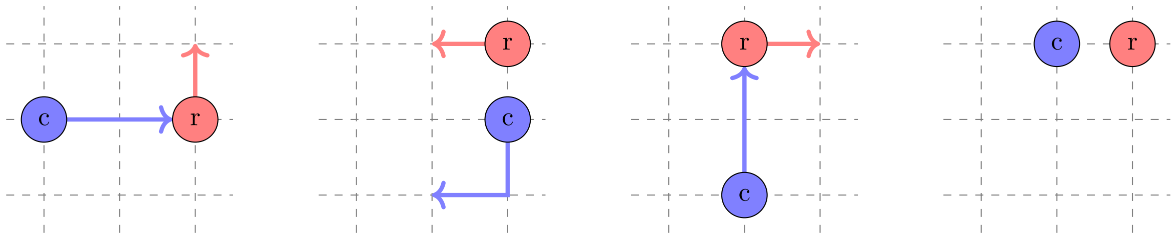



Figure 1: In this example, the cop can perform any of the following moves $C = \{$ 2 units straight, 1 unit right + 1 unit right, 1 unit right + 1 unit straight $\}$ and the robber can move one unit in any direction along the grid. After three steps, the cop has not caught the robber, but if the robber moves forward, backward, or right, then she will be caught.

**Question.** Is there a procedure for determining in general whether the cop can catch the robber?

**Related.**

1. Is there a procedure that can put a bound on the number of steps it will take for the cop to catch the robber?
2. If the cop/robber perform moves in their rule set according to some distribution, what is the probability that the cop will eventually catch the robber?
3. How does this generalize to a torus, Möbius strip, cylinder, multiple dimensions, a triangular grid, or a hexagonal grid?

**References.**

http://demonstrations.wolfram.com/TheHomicidalChauffeurProblem/

https://en.wikipedia.org/wiki/Homicidal_chauffeur_problem

# Problem 53.

A problem based on a conversation with Alec Jones. Consider a variation on the "concavity classes" of polygons as described by OEIS sequence A227910. Say that two $n$-gons are in the same concavity class if one can be continuously deformed into the other (or a mirror image of the other) while (1) remaining an $n$-gon the entire time, and (2) preserving the number of sides of the convex hull.

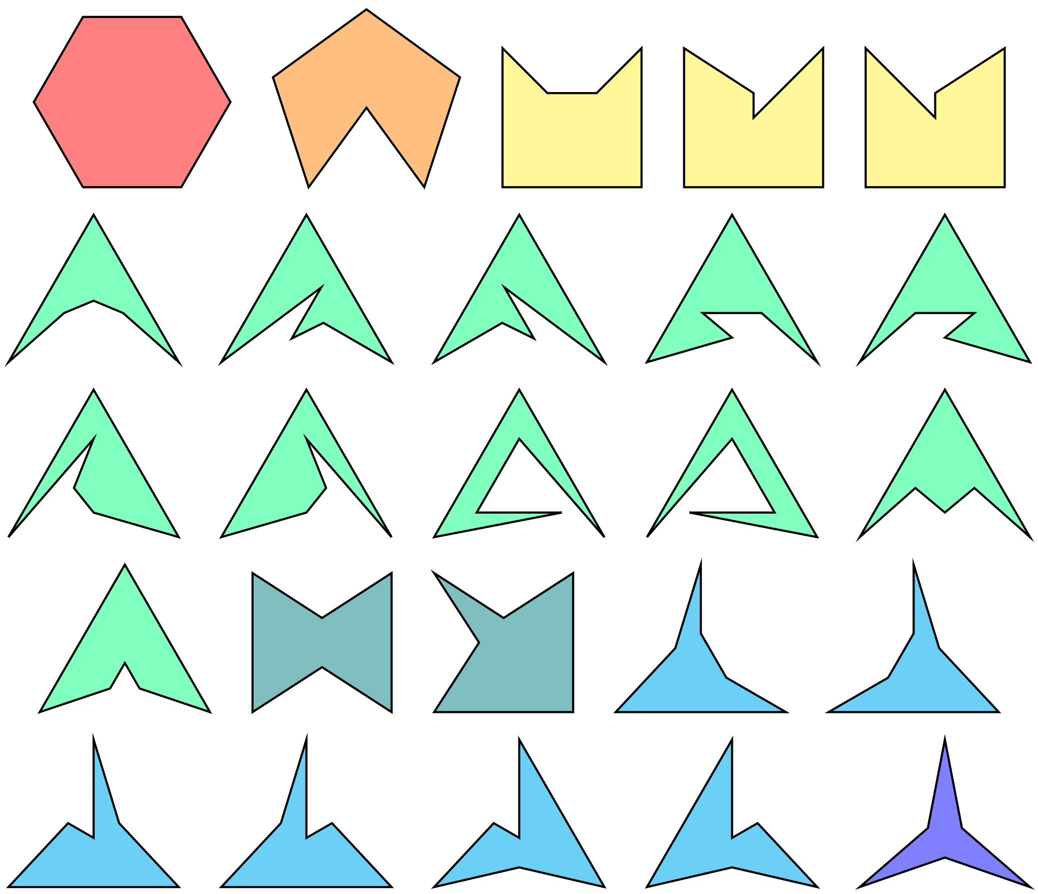


Figure 1: The $a(6) = 25$ concavity classes on the hexagons. There are $a(3) = 1$ triangles, $a(4) = 2$ quadrilaterals, and $a(5) = 6$ pentagons.

**Question.** How many convexity classes are there of an arbitrary $n$-gon?

**Related.**

1. What is the smallest square lattice that contains at least one representative of each concavity class of the $n$-gon for some fixed $n$? (That is, the polygons must have integer coordinates.)
2. (Is this the correct definition?)

**References.**

`https://oeis.org/A227910`

# Problem 54.

Consider convex polygons with integer coordinates. The notion of a best Diophantine approximation can be generalized to equilateral triangles by saying that a triangle is a better diophantine approximation if the ratio of the largest side to the smallest side is less than the ratio of any other triangle with smaller perimeter.

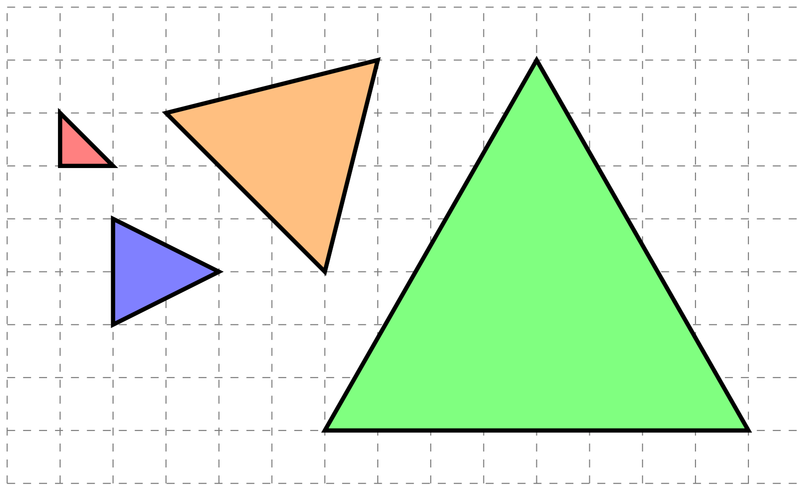


Figure 1: Four best (?) Diophantine approximations of an equilateral triangle. The red triangle has a ratio of $\sqrt{2/1} \approx 1.41$, the blue has a ratio of $\sqrt{5/4} \approx 1.118$, the orange has a ratio of $\sqrt{18/17} \approx 1.029$, and the green has a ratio of $\sqrt{64/63} \approx 1.008$.

**Question.** What is the growth of the perimeter of the $k$-th best Diophantine approximation of an equilateral triangle as a function of $k$?

**Related.**

1. How can this be generalized in a reasonable way to regular $n$-gons? (Just looking at side lengths isn't enough—angles can behave badly.)
2. What if this is done on tetrahedra?

**References.**

`https://math.stackexchange.com/q/2251555/121988`

`https://en.wikipedia.org/wiki/Near-miss_Johnson_solid`

# Problem 55.

For each $2n$-gon there exists some number $\ell_n$ such that there is a equilateral convex polygon with integer coordinates such that all sides have length $\ell_n$.

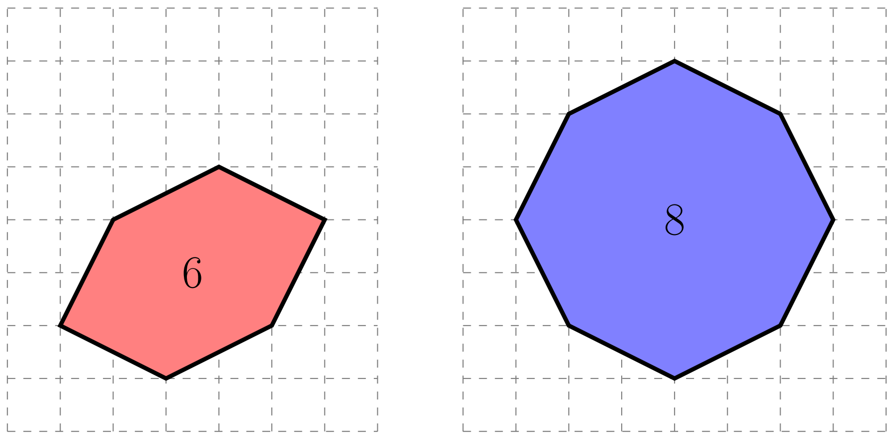



Figure 1: Example demonstrating that $\ell_6 = \ell_8 = \sqrt{5}$.

**Question.** For what values of $m$ do there exist equilateral $(2m-1)$-gons with integer coordinates?

**Related.**

1. For $m$ such that there are no equilateral $(2m-1)$-gons, what is do the best Diophantine approximations look like (in the sense of Problem 54)?
2. Does this generalize to polyhedra?

**References.**

https://oeis.org/A071383

Problem 54

# Problem 56.

☆☆☆☆

Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way that no end is "orphaned".

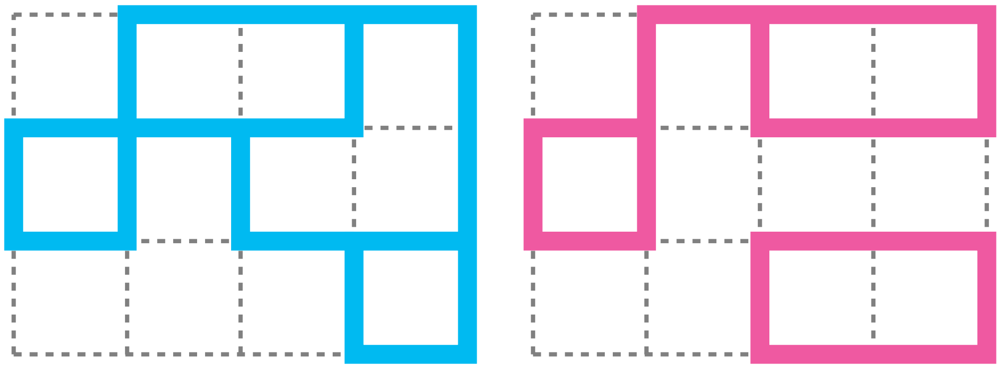


Figure 1: Two examples of a valid configurations on a $5 \times 4$ grid; the second is not connected.

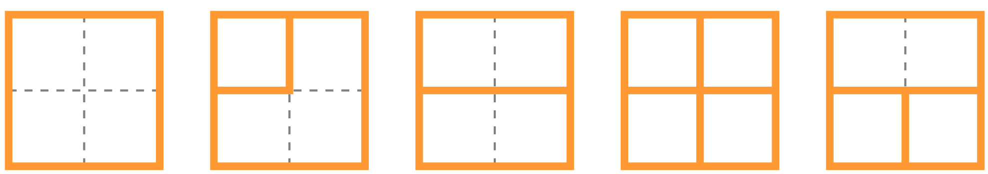


Figure 2: All(?) examples of valid configurations of $3 \times 3$ grids with border, up to dihedral action.

**Question.** Let $a_\ell(n)$ be the number of configurations on the $\ell \times n$ grid. What is a general formula for $a_\ell(n)$?

**Related.**

1. What if the matchsticks are of length $k$? Of $\{k_1, k_2, \ldots, k_\ell\}$?
2. How does this generalize to a triangular/hexagonal lattice or to multiple dimensions? On the king graph? On the multipartite graph $K_{m,m,\ldots,m}$?
3. What is the number of these configurations with rotational symmetry? Horizontal/vertical symmetry?
4. If such a configuration is chosen uniformly at random, what is the number of expected regions? (e.g. the first example has 4 interior regions.)
5. What if no gridpoint can have degree 0? Degree 2? 3? 4?
6. What if the entire border must be drawn?
7. What if the subgraph must be connected?
8. What if instead of horizontal/vertical lines, diagonals are allowed? All edges have integer slope? Edges don't intersect except at vertices?
9. How many $k$-ominoes fit in a "tube" of height $m$? Snuggly?

**References.**

A093129 ($2 \times n$), A301976 ($3 \times n$), A320097 ($4 \times n$), A320099 ($5 \times n$), A303930 ($2 \times n$ up to symmetry).

`http://mathworld.wolfram.com/KingGraph.html`

# Problem 57.

Consider equivalence classes of polygonal chains on $n$ segments where

(1) Edges can cross, but no segment can have a vertex on another segment's edge.

(2) Two chains are equivalent if one can move to the other without an edge crossing over a vertex, or a crossing being otherwise changed.

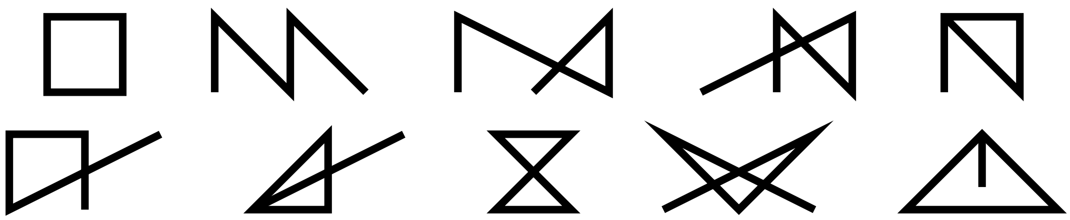


Figure 1: Examples of all known classes of polygonal chains of length 4.

**Question.** How many classes of polygonal chains exist on $n$ segments?

**Related.**

1. What if all segments are of unit length, so the final example is not allowed?
2. What if the fifth and seventh example are considered the same because they are isomorphic as graphs? (Even if vertices are added at each intersection)
3. What is the smallest grid that can contain the figures if vertices must be placed on gridpoints?

**References.**

Problem 53.

# Problem 58.

Consider a peaceable queens problem in an $n \times n \times n$ chess "cube", where a queen can move in any diagonal direction.

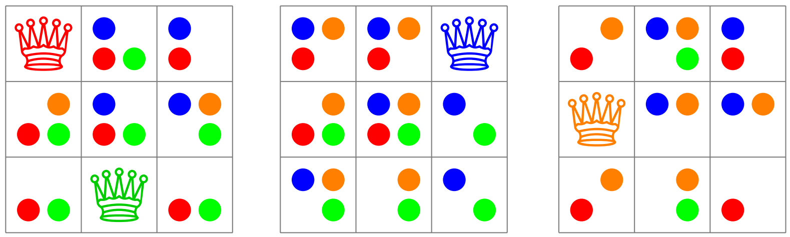


Figure 1: At least four hyper-queens can be placed peaceably on a $3 \times 3 \times 3$ board.

**Question.** What is the greatest number of queens that can be placed on an $n \times n \times n$ board?

**Related.**

1. If $n^{k-1}$ queens can be placed on a $\underbrace{n \times n \times \ldots \times n}_{k}$ board for sufficently large $n$, how large must $n$ be?

**References.**

https://math.stackexchange.com/q/2232287/121988

# Problem 59.

A snail travels along the grid in unit steps—but it hates crossing its trail, so if a step is going to cross is trail, it will only go half way.

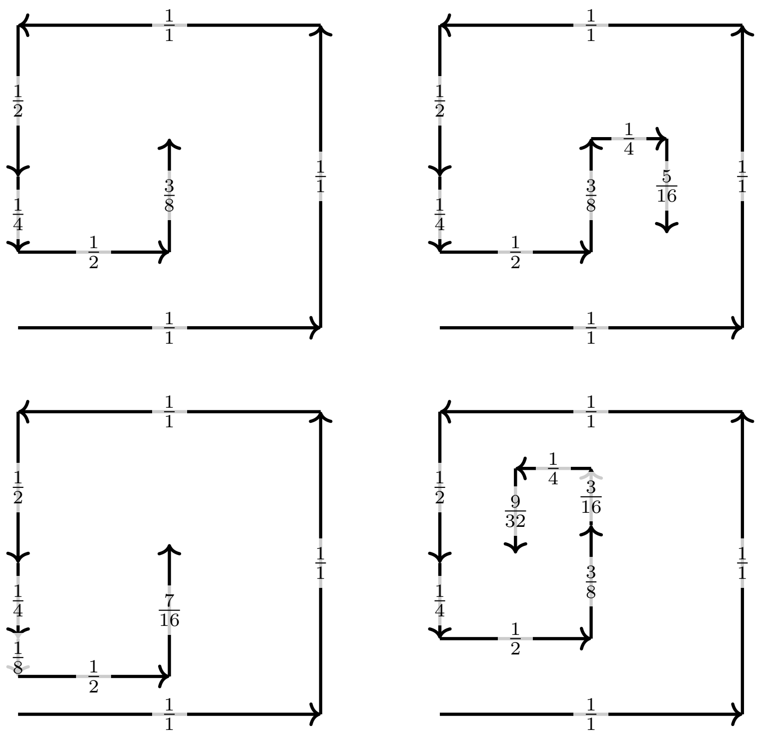



Figure 1: Left-to-right and top-to-bottom: examples of walks that end in step sizes that have numberators of 3, 5, 7, and 9.

**Question.** Let $a(n)$ be the minimum number of steps the snail must take before it can take a step of size $(2n-1)/2^k$. What is $a(n)$?

**Related.**

1. What if the snail must always turn left or right?
2. What if the snail is walking on a triangular or hexagonal grid?
3. What is the set of points the snail can step on after finitely many steps?
4. How many distinct points can the snail reach after $m$ steps?

**References.**

https://math.stackexchange.com/q/2678852/121988

https://oeis.org/A300444

# Problem 60.

Say that two sequences with distinct elements are in the same equivalence class if their first differences have the same signs. (e.g. $(1, 3, 2, 3)$ and $(7, 8, -1, 0)$ are equivalent because their first differences are both $(+, -, +)$.)

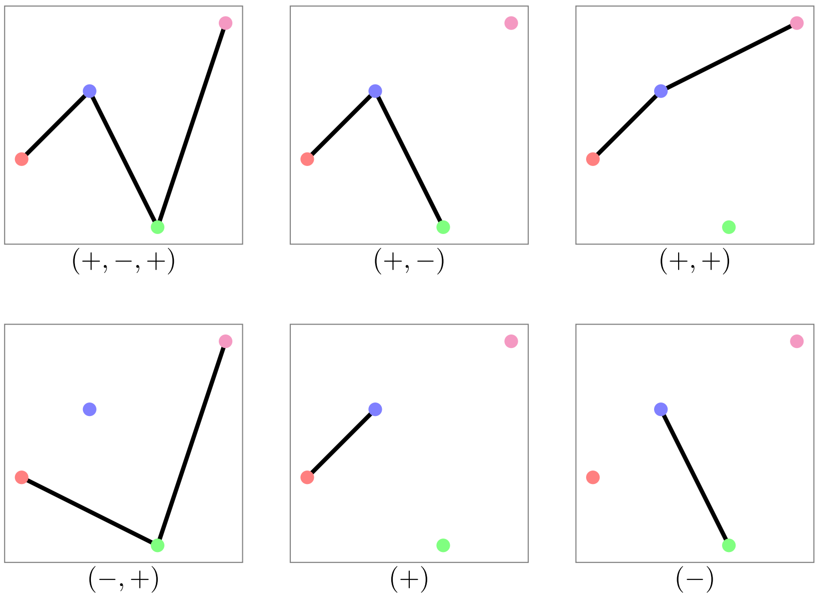



Figure 1: $(0, 1, -1, 2)$ has subsequences in the following six equivalence classes: $(+, -, +)$, $(+, +)$, $(+, -)$, $(-, +)$, $(+)$, $(-)$. No length 4 sequence has its subsequences in more equivalence classes, so $a(4) = 6$.

**Question.** What is the general formula for $a(n)$?

**Related.**

1. What if the sequences do not necessarily consist of distinct elements?
2. What if two sequences are considered to be equivalent if they are in the same "sort order"; that is, if both sequences have their biggest element in the same position, their second biggest in the same position, and so on.
3. What if $(+, +) \sim (+)$?
4. Is the number of equivalence classes for the subsequences determined by the number of local minima and maxima?

**Note.** A quick attempt finds that $a(2) = 1$, $a(3) = 3$, $a(4) = 6$, and $a(5) = 11$. (Fibonacci minus 2?)
For related question 3, conjecture the answer is $a'(n) = 2n - 3$ for $n \geq 2$.
For related question 3 without distinct elements (as in related question 1), the inital terms are

$$a(2) = 1 \text{ via } (+)$$
$$a(3) = 4 \text{ via } (+, -), (+), (-), (=)$$
$$a(4) = 8 \text{ via } (+, -, +), (+, -), (+, =), (=, +), (-,_)\, (+), (-), (=)$$
$$a(5) = 15; a(6) = 25; a(7) = 40$$
$$a(8) = 62; a(9) = 94; a(10) = 141; a(11) = 210; a(12) = 311 \text{ (conjectured*)}$$

* Assumes sequence is $(1, 2, 1, 2, 1, 2, \ldots)$.

# Problem 61.

Consider all $r$-colorings of the $n \times m$ grid where no two colors are adjancent (horizontally/vertically) more than once.

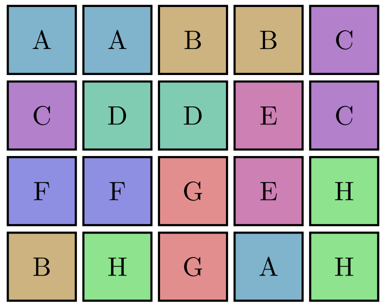



Figure 1: An 8-coloring of the $4 \times 5$ grid where no two colors are adjancent more than once. There is no 7-coloring.

**Question.** Let $r_{n\times m}$ be the smallest integer such that there exists an $r_{n\times m}$-coloring of the $n \times m$ grid. What is $r_{n\times m}$?

**Related.**

1. What if colors are not allowed to be self-adjacent?
2. How many $a(n, m)$-colorings exist up to permutation of the colors?
3. What if this is done on a triangular or hexagonal grid?
4. What if orientation matters? (A horizontal adjacency is distinct from a vertical adjacency.)
5. What if order matters? (red-green is distinct from green-red.)
6. What if diagonal adjacencies are considered?

**Note.**

$$\begin{array}{lllll}
r_{1\times 1} = 1 & & & & \\
r_{1\times 2} = 1 & r_{2\times 2} = 3 & & & \\
r_{1\times 3} = 2 & r_{2\times 3} = 4 & r_{3\times 3} = 5 & & \\
r_{1\times 4} = 2 & r_{2\times 4} = 5 & r_{3\times 4} = 6 & r_{4\times 4} = 7 & \\
r_{1\times 5} = 3 & r_{2\times 5} = 5 & r_{3\times 5} = 7 & r_{4\times 5} = 8 & r_{5\times 5} = 9
\end{array}$$

**References.**

Problem 23.

Problem 36.

Problem 49.

# Problem 62.

Consider ways to place colored markers on an $n \times m$ grid so that no two pairs of markers of the same color have the same distance between them.

| | | | | |
|---|---|---|---|---|
| A | B | B | C | A |
| A | D | E | B | C |
| D | A | E | D | C |
| C | E | F | D | A |

Figure 1: This arrangement has 6 different colors of markers. There are 5 red (A) markers and no valid way to place 6 red markers.

**Question.** What is $c_{n\times m}$ the greatest number of markers of a given color can be placed on the $n \times m$ grid?

**Related.**

1. How many colors of markers are required to fill the grid?
2. What if this is done on the $d_1$, $d_\infty$, or $d_3$ metric?
3. What if this is done on a triangular or hexagonal grid?
4. What is the smallest board that can contain $k$ markers?

**Note.** $c_{n\times m}(c_{n\times m} - 1)/2 \leq A301853(n, m) - 1$.

**References.**

Problem 30.

`https://oeis.org/A301853`

# Problem 63.

Consider all of the shapes that can be made with a rubber band and a rubber hand.

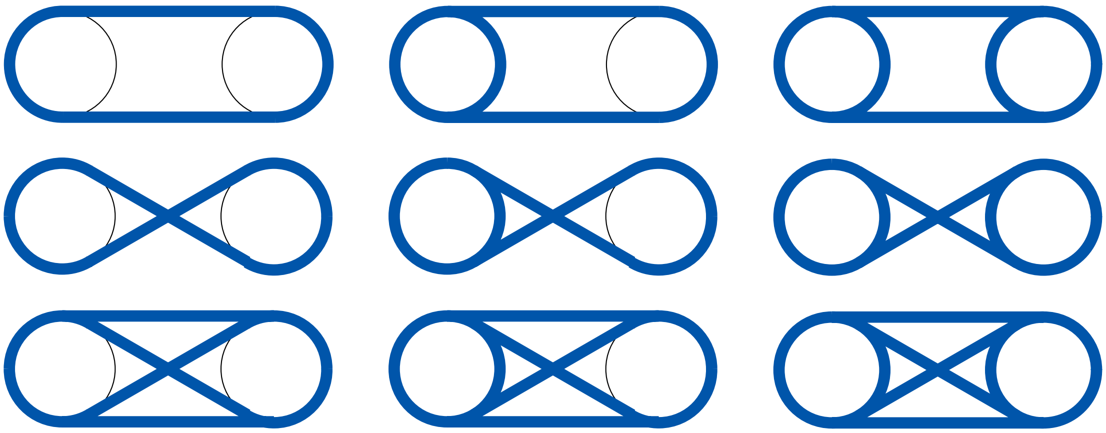


Figure 1: There are (at least) 9 ways to weave a rubber band between two fingers up to reflection/rotation.

**Question.** How many figures can be made with $n$ fingers and a rubber band?

**Related.**

1. Is there an analog in higher dimensions?
2. What if all fingers must be aligned?
3. What if all fingers must be on the corners of an $n$-gon?

# Problem 64.

Consider the art gallery problem on all "museum"-equivalence classes of polygons.

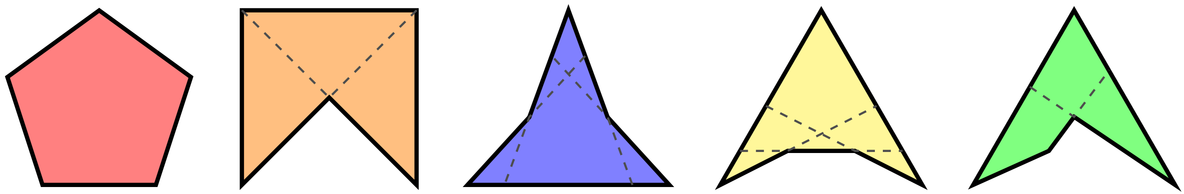
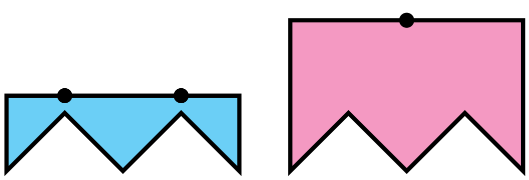


Figure 1: The concavity classes from Problem 53. It appears that the second and fifth polygons are museum-equivalent. Are the third and fourth polygons equivalent?

**Question.** If each polygon is a museum, how many guards are required to patrol the museum?

**Related.**

1. What if guards are stationed at a corner in the polygon?
2. What if guards are allowed to patrol along a wall?
3. What if the polygons are on a torus or cylinder?
4. What if the polygons are orthogonal (i.e. each wall meets at a right angle)?
5. What if the guards must patrol the outside of the polygon?
6. How many equivalence classes of museums exist? For example, the following museums are distinct, because the first requires two guards, and the second requires only one.



**References.**

Problem 53.

https://en.wikipedia.org/wiki/Art_gallery_problem

# Problem 65.

Starting with a configuration of coins, slide one coin at a time such that the coin ends up in a position where it is tangent to two other coins.

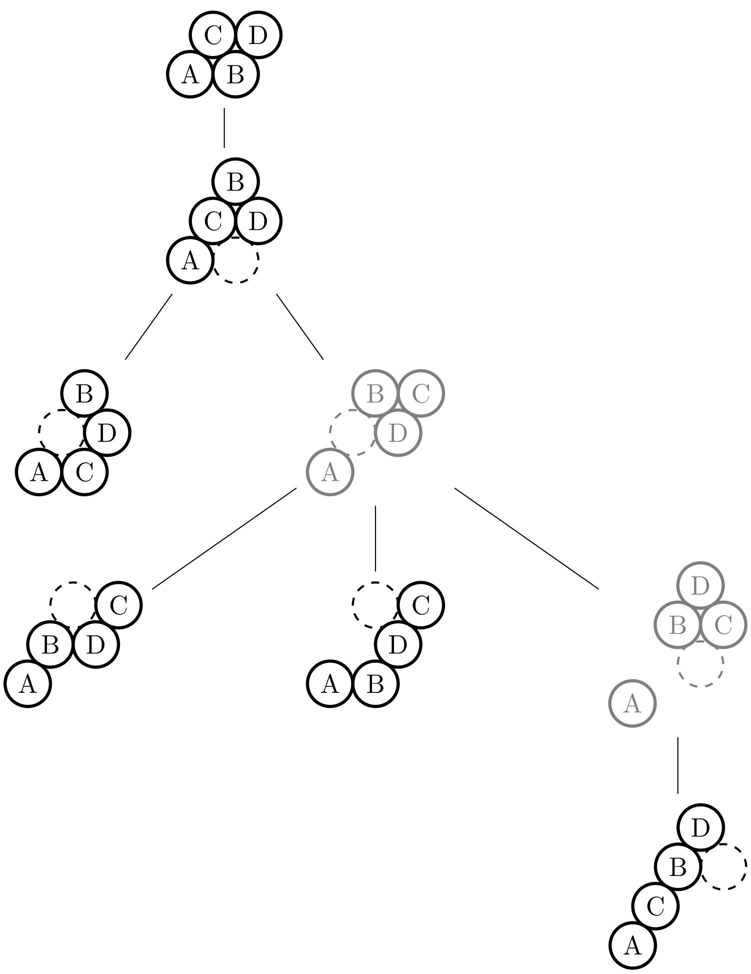



Figure 1: All connected configurations of 4 coins. Six out of the seven possible polyhexes are present.

**Question.** In general, given $n$ coins starting in a "spiral" configuration, how many polyhexes can be reached by the above procedure?

**Related.**

1. What if this is done with hyperspheres in $\mathbb{R}^d$?
2. Is there a sensible way to categorize non-connected configurations?
3. Which polyhexes require the greatest amount of moves?

**References.**

`https://en.wikipedia.org/wiki/Polyhex_(mathematics)`

`https://www.youtube.com/watch?v=_pP_C7HEy3g`

Martin Gardner, SciAm, Feb 1966

# Problem 66.

The number of ways to draw a triangle on a triangular grid is given by

$$\sum_{k=1}^{n-1} k \cdot t(n-k) = \binom{n+2}{4} = A000332(n-2)$$

where $t(m)$ is the $m$-th triangular number, and A000332 is a figurate number based on the 4-simplex.

The number of ways to draw a square on a square grid is given by

$$\sum_{k=1}^{n-1} k \cdot (n-k)^2 = \frac{1}{6}\binom{n^2}{2} = n^2\left(\frac{n^2-1}{12}\right) = A002415(n)$$

where A002415 is a figurate number based on the 4-dimensional pyramid.

The number of ways to draw a hexagon on a hexagonal grid is given by

$$\sum_{k=1}^{n-1} k \cdot h(n-k) = \left(\frac{n(n+1)}{2}\right)^2 = A000537(n-1).$$

where $h(m) = \text{A003215}(m)$ is the $m$-th centered hexagonal number.

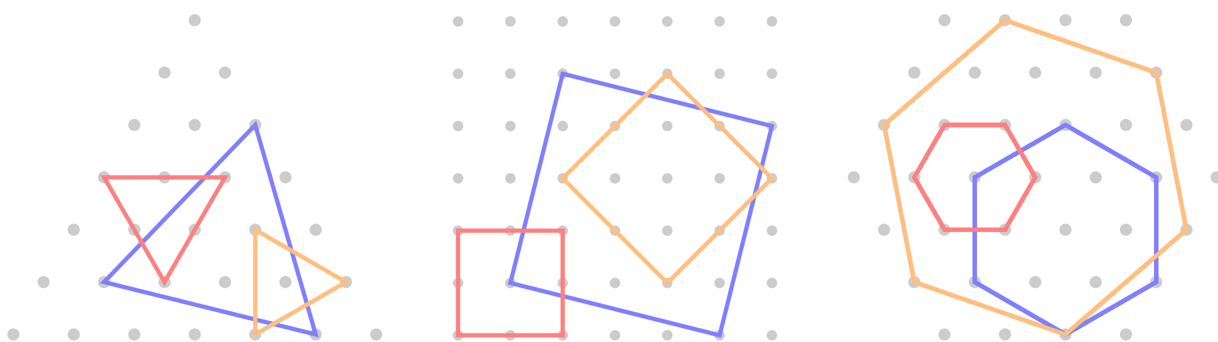


Figure 1: An illustration of three triangles on a triangular grid, three squares on a square grid, and three hexagons on a centered hexagonal grid.

**Question.** Is there a combinatorial explanation for why these numbers relate to 4-dimensional polytopes?

**Related.**

1. Can this be generalized to arbitrary regular $n$-gons in hyperbolic space?
2. How many triangles are on the "centered triangular number" grid?

**References.**

Problem 21.

`https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling`

# Problem 67.

Consider all of the ways to take a square piece of paper and make two “precise” creases—that is, we can make a perpendicular bisector of the line segment connecting two distinguished points, and we can take any two creases and bisect the angle between them.

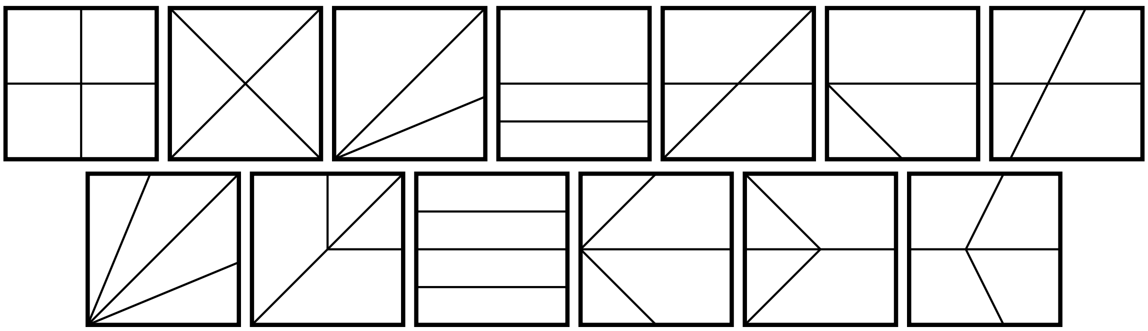


Figure 1: Thirteen (all?) two-fold crease patterns.

**Question.** How many such crease patterns exist on $n$ creases?

**Related.**

1. If we “overlap” all diagrams, how many distinct lines?
2. What if we start with a rectangle? Equilateral triangle?
3. What if $n$ is the number of folds, and unfolding counts as a fold?
4. What if we restrict the possible folds—for example, disallow folding a crease between two distinguished points?

**References.**

`https://en.wikipedia.org/wiki/Crease_pattern`

Andreas Aronsson: Divide into equal parts.

# Problem 68.

Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way as to form a maze.

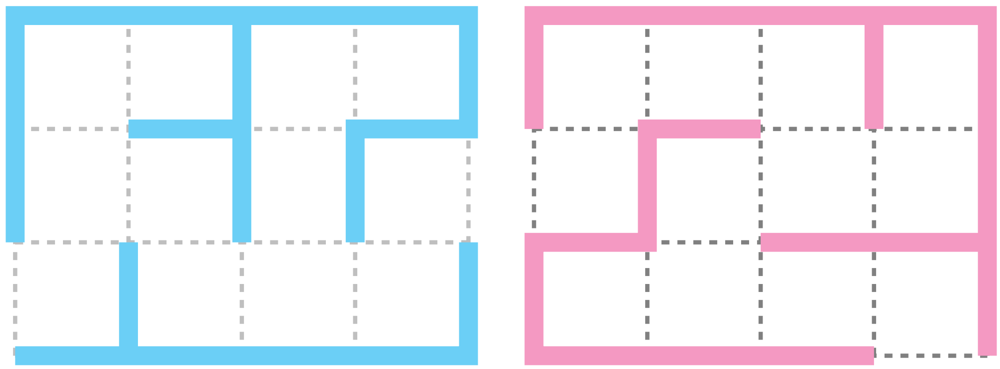


Figure 1: Two mazes on a $(5 \times 4)$-cell grid.

**Question.** How many distinct mazes can be drawn on the grid?

**Related.**

1. What if every $1 \times 1$ cell must be reachable?
2. What if there are no dead ends?
3. What if there are to be identically $k$ dead ends?
4. What if paths that loop are not allowed?
5. What if the entrance and exit have prescribed positions?
6. What if this is done on a hexagonal or triangular grid? On a torus?
7. Is there a meaningful way to assign "difficulty" to a maze?

**Note.** This appears to be the number of spanning trees on the $n \times m$ grid graph such that the start and end are leaves.

**References.**

Problem 56.

`https://oeis.org/A116469`

# Problem 69.

Say that an $n$-robot takes steps that are $1/n$ of a circle ($2\pi/n$ radians). Call a $(k, j)$-step pattern a walk that starts with $k$ right turns, followed by $j$ left turns, followed by $k$ right turns, and so on until the robot reaches its original position in the original orientation.

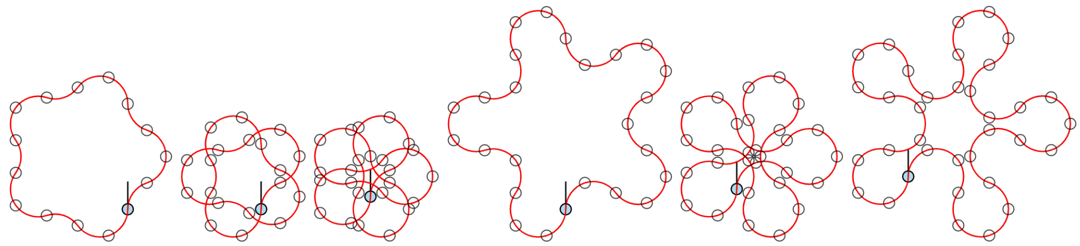


Figure 1: A 5-robot walks in $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, and $(3, 4)$-step patterns, respectively.

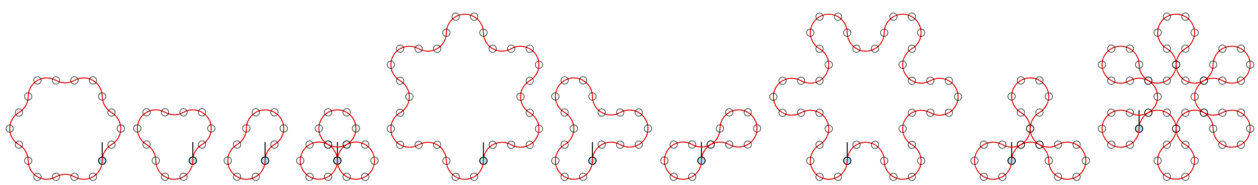


Figure 2: A 6-robot walks in $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, and $(4, 5)$-step patterns, respectively.

**Question.** For an $n$-robot, which of these paths encloses the most area? The least area?

**Related.**

1. Which of these figures has the largest convex hull? Smallest convex hull?
2. Is there a way to tell at a glance whether or not these walks will self-intersect? How many times?
3. Is there a way to tell at a glance if a $(k, j)$-step pattern will "go off to infinity"?
4. Are the areas enclosed by these figures "nice" numbers?
5. How does this generalize to $(a_1, a_2, \ldots, a_k)$-step patterns?
6. How many steps are taken before the figure "reconnects"?
7. For what step patterns are the "footprints" (the small grey circles in the figure) closest together (the $(2, 4)$-step pattern for the 5-robot)? How many steps are required to get two footprints within $\varepsilon$?
8. What if the robot turns $1/n$ of a circle when it turns right, but $1/m$ of a circle when it turns left?
9. What if the robot turns with some other rational number $a/b$ of a circle?
10. What if the robot only needs to reach the original position, but not original orientation?

**Note.** It is likely that 3, 4, and 6-robots are special cases because the footprints appear at lattice points.

**References.**

Problem 41.

`https://cemulate.github.io/project-euler-208/`

`https://codegolf.stackexchange.com/q/196399/53884`

# Problem 70.

Starting with a pair of integers $(a, b)$, there exists an algorithm for making the two integers equal by repeated applications of the map $(x, y) \overset{\alpha}{\mapsto} (2x, y+1)$ or $(x, y) \overset{\beta}{\mapsto} (x+1, y)$.

$$\begin{array}{rllllllll}
(4,0) & \overset{\beta}{\mapsto} (5,0) & \overset{\beta}{\mapsto} (6,0) & \overset{\alpha}{\mapsto} (12,1) & \overset{\beta}{\mapsto} (13,2) & \overset{\beta}{\mapsto} (14,4) & \overset{\beta}{\mapsto} (15,8) & \overset{\beta}{\mapsto} (16,16) \\
(5,4) & \overset{\beta}{\mapsto} (6,8) & \overset{\beta}{\mapsto} (7,16) & \overset{\beta}{\mapsto} (8,32) & \overset{\alpha}{\mapsto} (16,33) & \overset{\beta}{\mapsto} (17,66) & \overset{\alpha}{\mapsto} (34,67) & \overset{\alpha}{\mapsto} (68,68) \\
(8,1) & \overset{\beta}{\mapsto} (9,2) & \overset{\beta}{\mapsto} (10,4) & \overset{\alpha}{\mapsto} (20,5) & \overset{\beta}{\mapsto} (21,10) & \overset{\alpha}{\mapsto} (42,11) & \overset{\beta}{\mapsto} (43,22) & \overset{\beta}{\mapsto} (44,44) \\
(9,6) & \overset{\beta}{\mapsto} (10,12) & \overset{\beta}{\mapsto} (11,24) & \overset{\beta}{\mapsto} (12,48) & \overset{\alpha}{\mapsto} (24,49) & \overset{\beta}{\mapsto} (25,98) & \overset{\alpha}{\mapsto} (50,99) & \overset{\alpha}{\mapsto} (100,100) \\
(11,7) & \overset{\beta}{\mapsto} (12,14) & \overset{\beta}{\mapsto} (13,28) & \overset{\beta}{\mapsto} (14,56) & \overset{\alpha}{\mapsto} (28,57) & \overset{\beta}{\mapsto} (29,114) & \overset{\alpha}{\mapsto} (58,115) & \overset{\alpha}{\mapsto} (116,116)
\end{array}$$

Figure 1: Five examples of (shortest) seven-step paths to equality, starting from $(4, 0)$, $(5, 4)$, $(8, 1)$, $(9, 6)$, and $(11, 7)$.

**Question.** What is an algorithm for the shortest path to equality?

**Related.**

1. What are some good upper bounds for the shortest path length?
2. Can this be generalized to other maps (e.g. $(x, y) \mapsto (3x, y + 2)$)?
3. What is the least $k$ such that there is a path from $(a, b)$ to $(k, k)$? Is there a way to characterize all such values for $k$?

**References.**

`https://oeis.org/A304027`

`https://codegolf.stackexchange.com/q/164085/53884`

# Problem 71.

Let a "polyarc" be a path composed of quarter circular arcs, and a polyarc-configuration be a placement of polyarcs on an $n \times m$ grid such that no part of a polyarc is inside another polyarc.

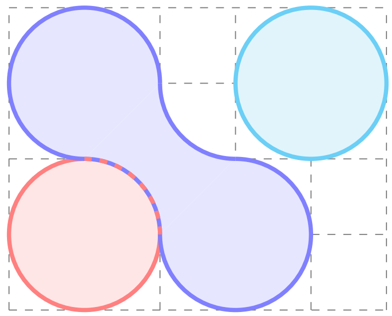


Figure 1: An example of a polyarc-configuration

**Question.** How many such polyarc-configurations exist on the $n \times m$ grid?

**Related.**

1. What if polyarcs can be inside other polyarcs?
2. What if all polyarcs must be tangent to another polyarc?
3. With if the polyarc-configuration must be connected?
4. What configuration gives maximum area for the entire polyarc-configuration?
5. Can this be done on a triangular/hexagonal grid with 1/3 or 1/6 arcs?
6. How many different non-self-intersecting 4-robot walks can fit inside of an $n \times m$ grid? Self-intersecting?
7. What if instead of quarter circles, boxes were made with diagonal line segments?
8. Is there a nice multi-dimensional analog?

**References.**

Problem 69.

Roundominoes by Kate Jones

# Problem 72.

Let a $k$-tile *multipolyform* be a generalized polyform on a tiling, that is, a choice of $k$ tiles in the tiling that are edge-adjacent, up to isometries of the tiling.

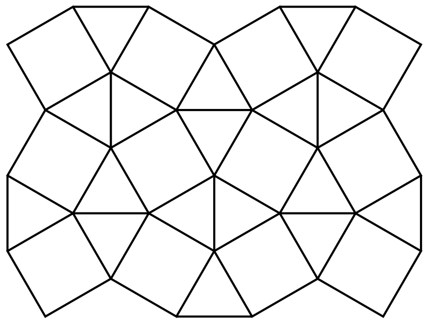


Figure 1: The snub square tiling.

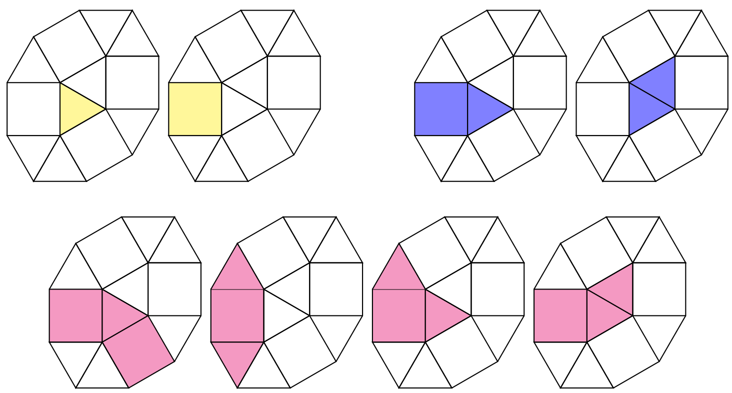


Figure 2: All 1-tile, 2-tile, and 3-tile multipolyforms on a snub square tiling.

**Note.** It is computationally hard to count polyominos, polyiamonds, polyhexes, etc.

**Question.** What are the number of multipolyforms on the eleven uniform tilings?

**Related.**

1. For a given tiling, what is the smallest region that can contain all $k$-polyforms? (See Problem 77.)
2. What are the number of polysticks on a given tiling? For a given graph?
3. Do the multipolyforms described in the example grow significantly faster than polyominos? How does the tiling affect asymptotic growth?
4. What about other tilings, such as the 15 pentagonal tilings? Penrose tilings?

**References.**

`https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons`

`https://en.wikipedia.org/wiki/Polyform`

Square tiling, triangular tiling, hexagonal tiling, snub square tiling.

# Problem 73.

A Heronian 2-simplex (triangle) is a triangle with both integer sides and integer area. A Heronian $n$-simplex is an $n$-simplex with integer volume and where all sides are Heronian $(n-1)$-simplices.

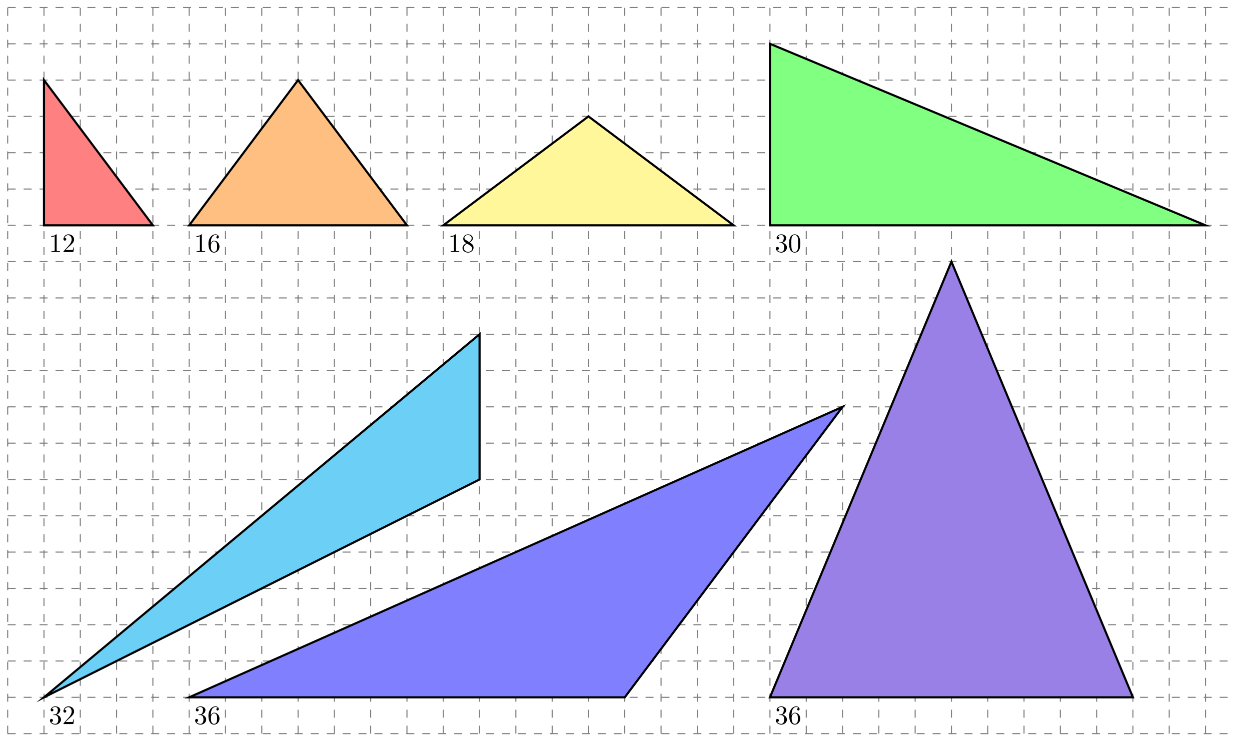



Figure 1: The seven smallest primitive Heronian triangles as measured by perimeter. The areas are 6, 12, 12, 30, 24, 60, and 36, respectively.

**Question.** Do Heronian $n$-simplices exist for all integers $n$?

**Related.**

1. Do infinitely many primitive Heronian $n$-simplices exist for each $n$?
2. What is the smallest Heronian $n$-simplex as measured $k$-dimensional volume of the largest $k$-face? as measured by sum volume of $k$-faces? (These agree when $k = n$.)
3. Are all Heronian $n$-simplices lattice simplices, as is the case for $n \leq 3$?
4. What if the definition is relaxed so that only, say, the volume and the edge-lengths must be integers?
5. Are other "Heronian polytopes" lattice polytopes, where a Heronian polytope is polytope where the $k$-dimensional volume of every $k$-face is an integer.

**References.**

https://www.jstor.org/stable/2695390

https://oeis.org/A272388

https://en.wikipedia.org/wiki/Heronian_tetrahedron

https://en.wikipedia.org/wiki/Simplex

# Problem 74.

In the “nine dots puzzle” or “thinking outside the box puzzle”, a player is asked to connect dots arranged in a $3 \times 3$ grid using four lines. This can be generalized to connecting the dots of a $n \times n$ grid with $2n - 2$ lines.

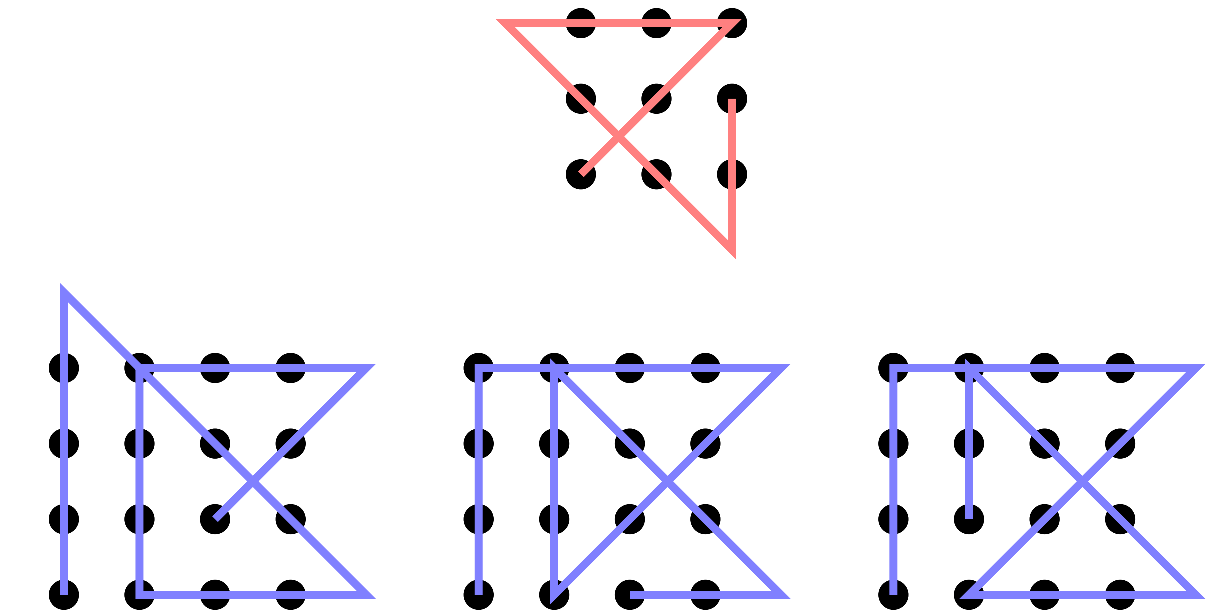


Figure 1: The unique (?) way of completing the $3 \times 3$ grid, and three distinct ways of completing the $4 \times 4$ grid.

**Question.** How many distinct solutions exist on the $n \times n$ grid?

**Related.**

1. What if you want to minimize the area “outside” of the grid?
2. What if you must start and end from the same point?
3. What if you want to minimize the path length?
4. Do any of these have lines that aren’t horizontal, vertical, or 45° diagonal?
5. What if this is done on other figures? (Triangles, Diamonds, Octagons, Stars, etc.)
6. Can this be generalized into higher dimensions with lines? Hyperplanes?
7. What if the “pencil” can be lifted $k \geq 1$ times?
8. What if this is done on a torus or cylinder?

**References.**


`https://math.stackexchange.com/q/21851/121988`

`https://en.wikipedia.org/wiki/Thinking_outside_the_box#Nine_dots_puzzle`

Thinking outside outside the box by Rob Eastaway

# Problem 75.

Ezgi showed me a puzzle where dominoes are placed to form a rectangle such that there is no line that separates the dominoes into two rectangles.

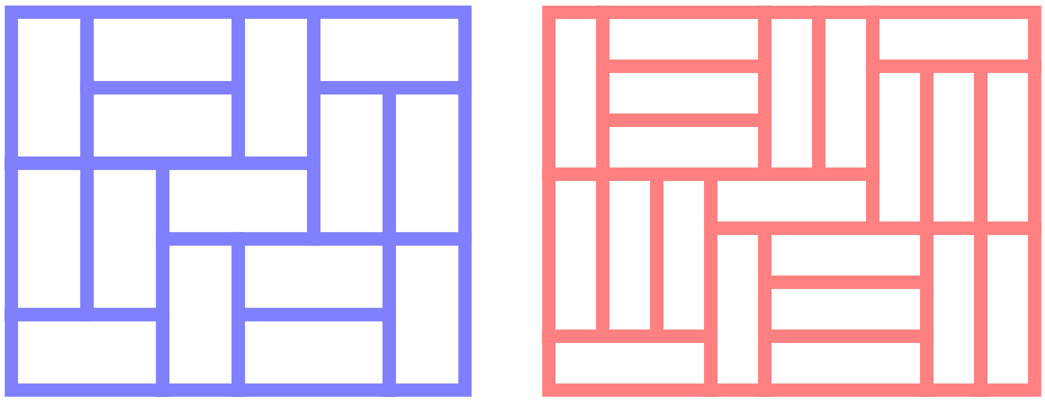


Figure 1: On the right is the smallest way to place dominoes into a rectangle such that there is no way to partition the dominoes into two rectangles. Is the left a minimal configuration with $1 \times 3$ triominoes?

**Question.** What size grids have such configurations?

**Related.**

1. How many configurations exist for a given grid size?
2. What if other rectangular polyominoes are used? (e.g. $3 \times 2$ hexominoes)
3. Are there analogous problems for triangular grids? Higher dimensions?

**Note.** These are sometimes called "Fault-free domomino tilings".

**References.**

Project Euler, Problem 215.

http://mathworld.wolfram.com/Fault-FreeRectangle.html

# Problem 76.

Consider ways to draw diagonals on the cells of $n \times n$ toriodal grid such that no two diagonals touch.

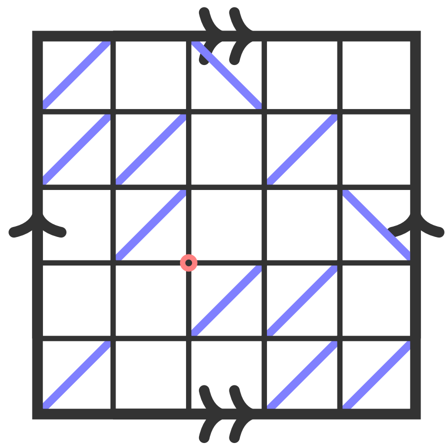


Figure 1: A maximal configuration of a $5 \times 5$ toroidal grid: 12 diagonal lines can be drawn. The unused vertex is marked with a circle.

**Question.** What is the greatest number of diagonals that can be drawn on a $n \times n$ toroidal grid?

**Note.** Let $m(n)$ be the maximum number of diagonals on an $n \times n$ grid.
Then

$$\begin{aligned} m(2n) &= 2n^2 \text{ and} \\ 2n^2 + n \leq m(2n+1) &\leq 2n^2 + 2n. \end{aligned}$$

**Related.**

1. How many configurations exist for a given grid size up to group action?
2. What if this is done on an $n \times m$ grid?
3. What if this is done on a cylinder, Klein bottle, projective space, etc?
4. What is the maximum number of diagonals that can go SW to NE?
5. Does this generalize to three or more dimensions?
6. Can something similar be done on a hexagonal or triangular grid?
7. How many configurations exist if touching is allowed, but cycles aren't?
8. What if we color edges of the grid rather than diagonals on the faces?

**References.**

https://oeis.org/A264041

https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/slant.html

# Problem 77.

Consider regions of the plane that can contain all free $n$-ominoes.

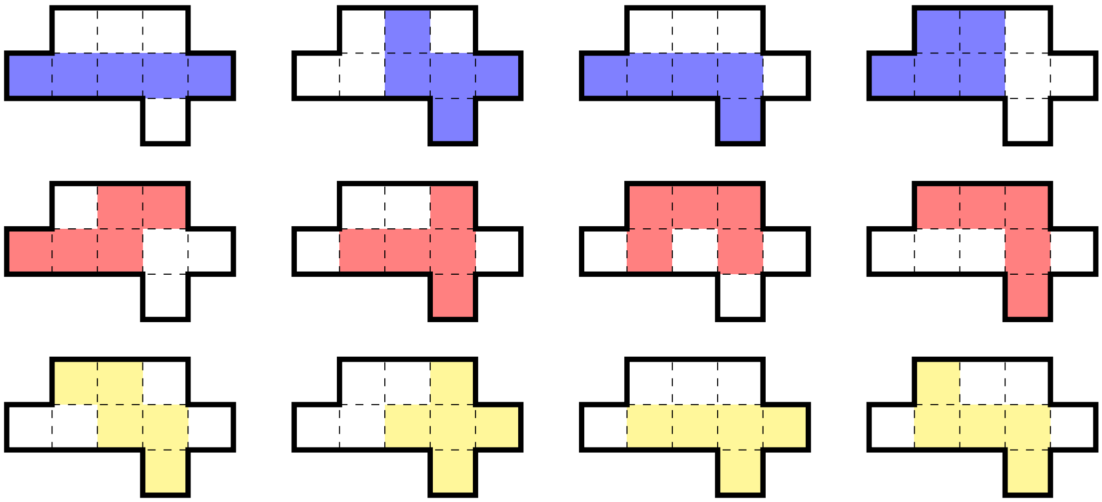


Figure 1: A computer search has proven that a nine-cell region of the plane is the smallest possible region that contains all 5-ominoes.

**Question.** What is the smallest region of the plane (with respect to area) that can contain all free $n$-ominoes?

**Related.**

1. What about fixed polyominoes? One-sided polyominoes (those that can be rotated but not flipped)?
2. What about other polyforms such as polyhexes or polycubes?
3. What if the region must be convex?
4. What is the smallest convex region that contains all length $n$ polysticks (along grid lines)?
5. How many distinct minimal covering sets (call this $c(n)$)?
6. What is the asymptotic growth in area of such a region? (Somewhere between linear and quadratic.)
7. Is there a limiting shape?
8. Alec Jones wonders if there always exists a covering set such that a single cell is used by all polyominoes.

**Note.** If $c(n)$ counts the number of distinct minimal covering sets of $n$-ominoes, then $c(1) = c(2) = c(3) = 1$, $c(4) = c(5) = 2$, and $c(6) = 14$.

**References.**

Problem 75

`https://en.wikipedia.org/wiki/Moser%27s_worm_problem`

`https://en.wikipedia.org/wiki/Polystick`

`https://math.stackexchange.com/q/2831675/121988`

# Problem 78.

It is known that trapezoids consisting of 1, 3, and 5 equilateral triangles in a line can tile an equilateral triangle.

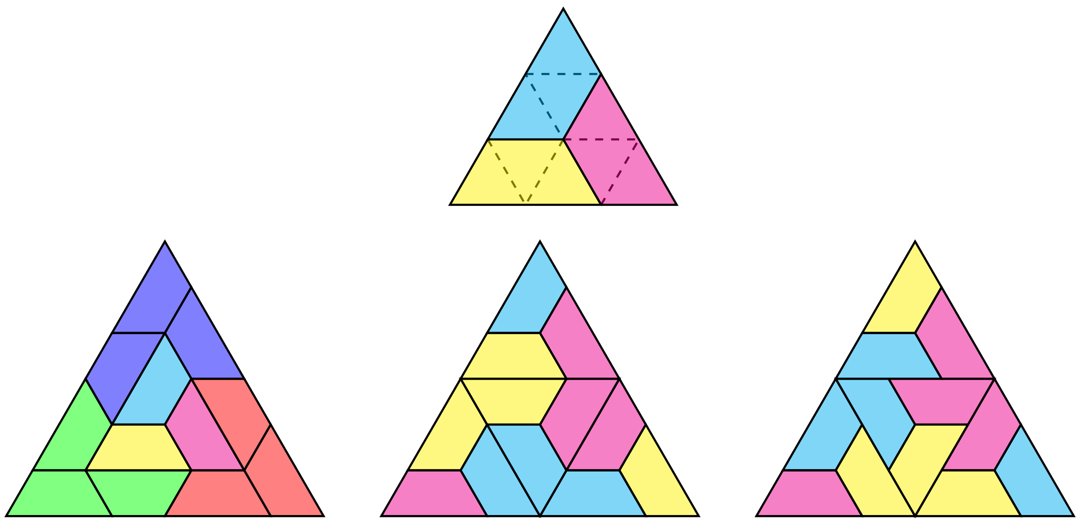


Figure 1: A equilateral triangle made of 3-trapezoids.

**Question.** Can all $(2n-1)$-trapezoids be arranged to form an equilateral triangle?

**Related.**

1. What is the smallest triangle that can be formed this way?
2. Is there a construction that makes such triangles given some $k$-trapezoid?
3. How many such tilings exist for a given size trapezoid and triangle?
4. Can other shapes be tiled (e.g. hexagon, arbitrary trapezoid)?
5. Does this generalize to square/hexagonal tilings? Multiple dimensions?

**Note.** This problem appears to have been around since at least July 2007.

**References.**

Problem 75

`https://math.stackexchange.com/q/2215781/121988`

`https://mathoverflow.net/a/267763/104733`

# Problem 79.

★★★★

Suppose you have two finite groups $G$ and $H$, and a $H$-set $\Omega$. Denote the base of the wreath product of $G$ by $H$ as $K = \prod_{\omega \in \Omega} g_\omega$.

We're interested in determining when it is possible to construct a finite sequence, $\{k_i \in K\}_{i=1}^N$, such that for any choice of $\alpha \in G \wr_\Omega H$ and any sequence $\{h_i \in H\}_{i=1}^N$, there exists $n \leq N$ satisfying

$$\alpha \cdot (k_1, h_1) \cdot (k_2, h_2) \cdots (k_n, h_n) = (e_K, h)$$

for some $h \in H$.

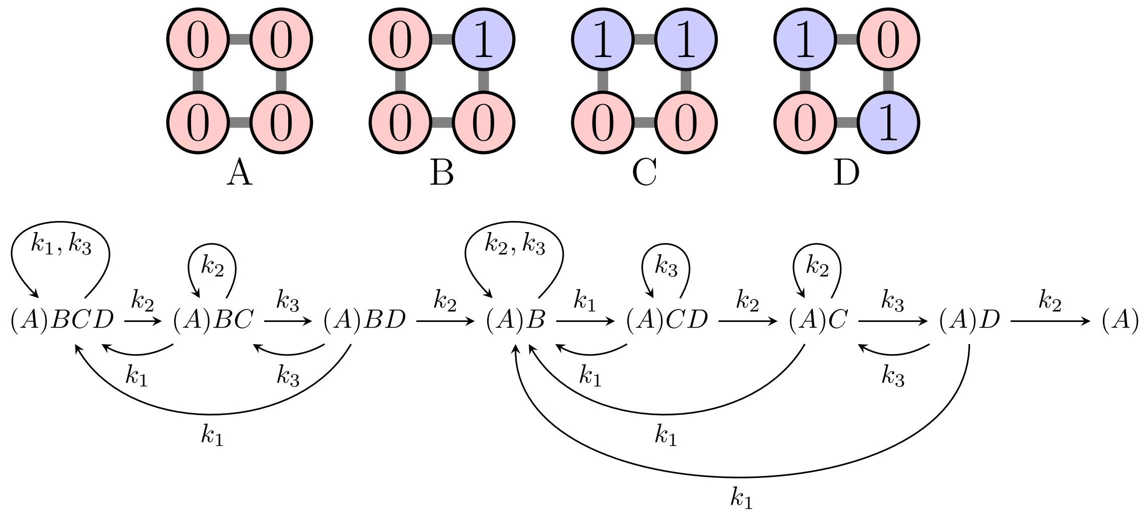



Figure 1: Suppose $G = \mathbb{Z}_2$, $H = C_4$, and $\Omega$ is the rotations of the square, with $H$ acting in the ordinary way. Moreover, suppose that $k_0 = (1,1,1,1)$, $k_1 = (1,0,0,0)$, $k_2 = (1,0,1,0)$, and $k_3 = (1,1,0,0)$. Then the sequence $(k_0, k_2, k_0, k_3, k_0, k_2, k_0, k_1, k_0, k_2, k_0, k_3, k_0, k_2, k_0)$ satisfies the above condition.

**Question.** What conditions on $G \wr_\Omega H$ guarantee such a finite sequence of elements in $K$?

**Related.**

1. Given $G \wr_\Omega H$, what is $N$, the minimum length of the sequence?
2. If both $\alpha$ and the sequence $\{h_i \in H\}_{i=1}^N$ are chosen uniformly at random, what is the expected value of the minimal $n$ such that $\alpha \cdot (k_1, h_1) \cdot (k_2, h_2) \cdot \cdots \cdot (k_n, h_n) = (e, k)$?
3. What if there is a set of elements in the base, any one of which is valid. (e.g. all sets satisfying the condition that the number of heads is congruent to 2 (mod 3).)
4. What if something about $s_i$ is told to you after the $i$th move for each $i$ (and the strategy can depend on this information)?

**Note.** This problem appears in Peter Winkler's book under the name "Spinning Switches". The can be generalized to $H = C_{2^n}$ and $\Omega$ a $2^n$-gon, but no other $m$-gon. The idea is that (1) you can't solve any $m$-gon with $m$ odd, and (2) you can solve a $m$-gon if and only if you can solve a $d$-gon for every proper divisor $d \mid m$. If you allow "coins" to have three states, then you can solve this on a $3^n$-gon.

**References.**

http://mathriddles.williams.edu/?p=77

https://mathoverflow.net/q/332600/104733

https://en.wikipedia.org/wiki/Four_glasses_puzzle

# Problem 80.

Consider figures created out of "blocks" starting from some base state and with the rule that each new block needs to touch as many old blocks as possible.

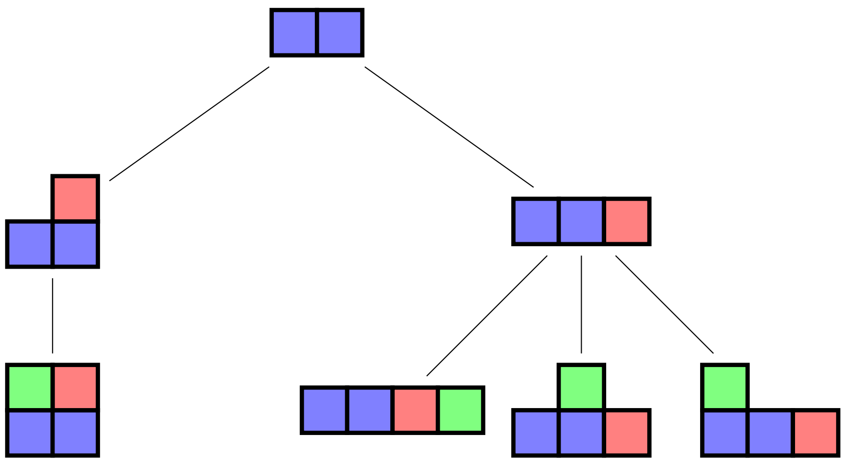


Figure 1: On the leftmost path, the final transition is from an "L" to a square, because the maximum number of faces that can touch is two, so the block must be added in the upper left corner. Counting the number of vertices gives $a(1) = 1$, $a(2) = 2$, and $a(3) = 4$.

**Question.** How many distinct figures (up to group action) can be made with $n$ blocks, always following a greedy algorithm (with respect to number of faces touching)?

**Related.**

1. What if this is done with circles on a hexagonal grid? (Polyiamonds, etc.)
2. What if this is done in more than 2 dimensions?
3. What if the starting shape is different? (e.g. the "T" tetromino)
4. What if the blocks are different? (e.g. dominoes)
5. What if the constraint is changed? (e.g. each block must touch exactly two sides)

**References.**

Problem 65

# Problem 81.

Consider pairs subsets of $[n] = \{1, 2, 3, \ldots, n\}$ such that the arithmetic mean of the subsets is equal. How many different pairs of subsets can we find, up to some sort of depedence, where two pairs are equivalent if there exists a linear transformation that takes one pair to the other, or if there exists a "chain" of subsets that implies equality.

$$\left(\frac{1+5}{2} = \frac{1+3+5}{3}\right) \cong \left(\frac{3+5}{2} = \frac{3+4+5}{3}\right) \tag{1}$$

$$\left(\frac{1+5}{2} = \frac{3}{1}\right) \cong \left(\frac{3}{1} = \frac{2+4}{2}\right) \Longrightarrow \left(\frac{1+5}{2} = \frac{2+4}{2}\right) \tag{2}$$

Figure 1: The first equalities are considered equivalent under the linear transformation $x \mapsto \frac{1}{2}(x+5)$. The equality $\frac{1}{2}(1+5) = \frac{1}{2}(2+4)$ is a combination of equations (1) and (2), and so is not an independent equation.

**Question.** Is there some notion of a "basis" for these pairs of subsets, from which we can work out all pairs with equal means?

**Related.**

1. What is the minimal "basis" that can describe all pairs of subsets with equal means?
2. What if the subsets in the pair need to be disjoint?
3. Is there a way to combine two pairs of subsets into another pair?
4. Can this generalize to multisets?

# Problem 82.

How many functions $f_{n,k} \colon P([k]) - \emptyset \to \{0, 1, 2, \ldots, n\}$ exist between nonempty subsets of $[k]$ and nonnegative integers less than or equal to $n$ such that there exists a sequence of finite sets $(A_1, A_2, \ldots, A_k)$ satisfying

$$f(S) = \# \bigcap_{i \in S} A_i$$

for all $S \in P([k]) - \emptyset$?

| | $\#A$ | $\#B$ | $\#C$ | $\#(A \cap B)$ | $\#(A \cap C)$ | $\#(B \cap C)$ | $\#(A \cap B \cap C)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 6 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 7 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 9 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 10 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 11 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 12 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 13 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| 14 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Figure 1: For $n = 1$, $k = 3$, there are fifteen such functions.

**Question.** How many such functions exist? Equivalently, how many ways to fill in a $k$-"base set" Venn diagram with integers such that no base set has more than $n$ elements?

**Related.**

1. What if $\#A_i = \#A_j$ for all $i, j < n$?
2. What if $A_i \not\subset A_j$ for all $i \neq j$?
3. What if this is done with unordered sets? (e.g. the second, third, and fourth functions in the example are all considered equivalent.)
4. What if the correpsonding diagrams need to be realizable as grid rectangles with areas corresponding to the values in the table?
5. What if this is done with set union instead of set intersection?

**References.**

OEIS Sequence A000330 handles the case where $k = 2$.

OEIS Sequence A319777 handles the case where $k = 3$.

# Problem 83.

Consider a system of first-order finite difference equations (linear recurrences) $a_1, a_2, \ldots, a_N$ where

$$a_i(n) = \alpha_{i1}a_1(n-1) + \alpha_{i2}a_2(n-1) + \ldots + \alpha_{iN}a_N(n-1).$$

is the $i$th such equation.

$$\begin{array}{ll}
a_1(1) = 1 & a_1(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + 2a_5(n-1) + a_7(n-1) \\
a_2(1) = 0 & a_2(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_3(1) = 0 & a_3(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_4(1) = 0 & a_4(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_5(1) = 0 & a_5(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + a_5(n-1) \\
a_6(1) = 0 & a_6(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_7(1) = 1 & a_7(n) = a_5(n-1) + a_7(n-1)
\end{array}$$

Figure 1: In this system of equations, the function $a(n) = a_5(n) + a_7(n)$ (which counts no-leaf subgraphs of the $2 \times n$ grid.) satisfies the recursion $a(n) = 5a(n-1) - 5a(n-2)$ for $n > 2$.

**Question.** Given some linear combination $a(n) = k_1a_{i_1}(n)+k_2a_{i_2}(n)+\ldots+k_ma_{i_m}(n)$ of these finite difference equations, what is the smallest order finite difference equation that $a(n)$ satisfies?

**Related.**

1. How does the order of such a recurrence depend on the initial conditions of the system?
2. What if the initial recurrences have order greater than 1? For example:

$$\begin{aligned}
a_i(n) = {} & \alpha_{i,1,1}a_1(n-1) + \alpha_{i,1,2}a_1(n-2) + \ldots + \alpha_{i,1,k_1}a_1(n-k_1) \; + \\
& \alpha_{i,2,1}a_2(n-1) + \alpha_{i,2,2}a_2(n-2) + \ldots + \alpha_{i,2,k_2}a_2(n-k_2) \; + \ldots \\
& \alpha_{i,t,1}a_t(n-1) \; + \alpha_{i,t,2}a_t(n-2) \; + \ldots + \alpha_{i,t,k_t}a_t(n-k_t)
\end{aligned}$$

**Note.** Let $\alpha_{ij}$ be the coefficient of $a_j(n-1)$ in the finite difference equation for $a_i(n)$, and denote the minimum polynomial of the matrix $A = [\alpha_{ij}]_{i,j=1}^N$ by

$$\det(xI - A) = x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_1x + \beta_0$$

then

$$a(n) = -\beta_{m-1}a(n-1) - \beta_{m-2}a(n-2) - \ldots - \beta_1a(n-m+1) - \beta_0a(n-m),$$

but there may be a lower-order recurrence.

**References.**

A special case of Problem 56 is counted in the example.

# Problem 84.

Starting with a row of $n$ coins all heads up, repeatedly flip over a coin which is heads and its neighbor to the right. If the chosen coin is the rightmost coin, there is no neighbor to flip.

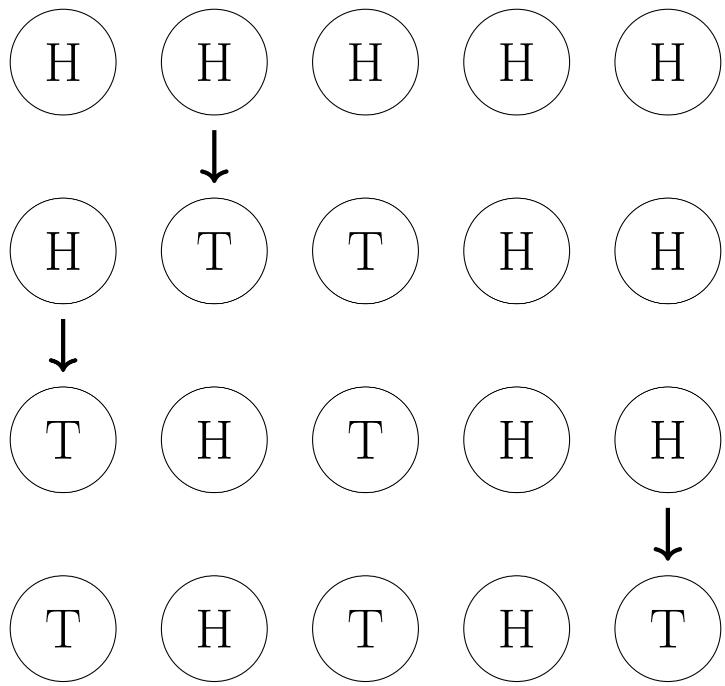



Figure 1: Since the sequence of coin flips strictly increases lexicographically (with $T > H$), the process must eventually halt.

**Question.** If the puzzle is modified so that when a coin is chosen, either the right or left neighbor is chosen (with probability $p$ and $1 - p$ respectively), what is the optimum strategy for maximizing the total number of flips?

**Related.**

1. What is the strategy for minimizing the number of flips?
2. What is the expected number of total flips under optimal play?
3. What if the direction is randomly chosen, and then you choose which coin to flip? (i.e. you know the direction before you make your choice.)
4. What if the (infinite) sequence of choices have to all be made ahead of time?
5. What if this is done on a different geometry, such as a circle or grid?
6. What if one, neither, or both neighbors have some probability of being flipped?
7. What if coins have more than two states? (e.g dice instead of coins)
8. What if you can flip over a contiguous section of heads?

**References.**

Problem 79.

# Problem 85.

The Erdős distinct distance problem asks for the number of distinct distances determined by $n$ points in the Euclidean plane.

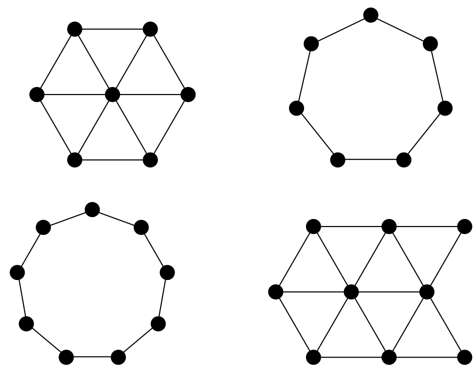


Figure 1: Sets with 3 and 4 distinct distances on 7 and 9 vertices respectively. There are no larger sets with an equal or smaller number of vertices.

**Question.** If $n$ points are constrained to the grid $\mathbb{Z}^2$, what is the minimal number of distinct distances?

**Related.**

1. How many such figures?
2. What if the figures are constrained to some subset of $\mathbb{Z}^2$, e.g. $[n] \times [n]$?
3. What about on other grids (triangular, hexagonal, etc)?
4. Can this be meaningfully done on more exotic topologies, e.g. $Z_n^2$?
5. What about $\mathbb{Z}^n$ for $n > 2$? What is the asymptotic behavior?
6. What if distance is measured via $d_1$, $d_3$, or $d_\infty$?

**References.**

Problem 30.

`https://oeis.org/A186704`: Erdős distinct distance problem.

# Problem 86.

Euler's well is a labeling of the $n \times k$ grid with a permutation in $S_{n\times k}$ such that the upper left corner is labeled with 1.

> Water is poured into the well from a point above the section marked 1, at the rate of 1 cubic foot per minute. Assume that water entering a region of constant depth immediately disperses to all orthogonally adjacent lower-depth regions evenly along that region's exposed perimeter (an assumption that Euler insisted on).
>
> After how many minutes will the water begin to accumulate in [the lower right corner]?

| | | | | |
|---|---|---|---|---|
| 1 | 14 | 9 | 20 | 3 |
| 5 | 13 | 24 | 17 | 18 |
| 25 | 16 | 4 | 21 | 6 |
| 10 | 2 | 15 | 19 | 23 |
| 7 | 22 | 8 | 12 | 11 |

Figure 1: A labeling of the $5 \times 5$ grid where the labels are a permutation of the integers from 1 to 25.

**Question.** For a random permtation in $S_{n\times k}$, what is the expected amount of time it takes for water to reach the lower right hand corner of the grid?

**Related.**

1. What if water can flow diagonally?
2. What if the source or sink are in different places? What if there are multiple sources/sinks?
3. What if this is done on a torus? Triangular/hexagonal grid? Three dimensions?
4. What if the numbers are not neccesarily a permutation?
5. What if the well is a Latin square?
6. What is an efficient algorithm for computing this for an arbitrary permutation?
7. What is the expected value of number of "wet" squares at the end?
8. How many wells have minimal filling times?
9. How many wells up to "fill level"? (e.g. two wells are equivalent if each square has the same height after the water flows all the way)

**References.**

`http://chalkdustmagazine.com/blog/well-well-well/`

`https://oeis.org/A321853`

# Problem 87.

Consider partitions of the $n \times m$ grid into triangles with vertices on gridpoints.

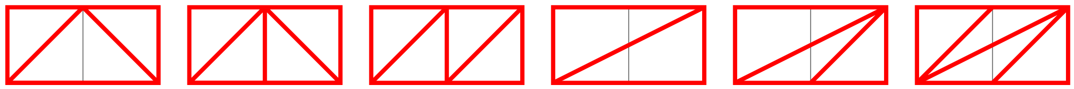


Figure 1: All six partitions of the $2 \times 1$ grid into triangles with gridpoint vertices, up to dihedral action.

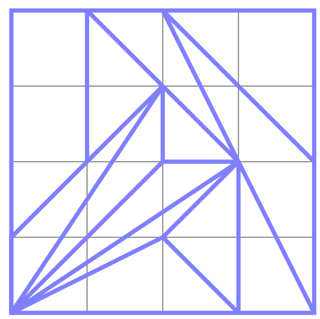


Figure 2: An example of a partitions of the $4 \times 4$ grid into triangles with no “empty” gridpoints.

**Question.** How many such partitions exist?

**Related.**

1. What if these are counted up to rotation/reflection?
2. What if this is done on a triangular/hexagonal grid?
3. How many partitions with the maximal number of triangles? With $k$ triangles?
4. What if all triangles must be right triangles? Acute? Obtuse?
5. What if each gridpoint must touch a triangle? What is the minimum number of faces?
6. What if each gridpoint must touch as many triangles as possible? What is the minimum number of faces? What’s the expected number of faces? (i.e. there’s no way to draw a new edge?)
7. What if this is done on a grid in hyperbolic space?

**Note.** $a_1(n) = A051708(n)$

**References.**

`https://oeis.org/A051708`

`https://codegolf.stackexchange.com/q/176646/53884`

# Problem 88.

Consider ways of partitioning nonattacking rooks in such a way that no rook lies in the convex hull of its partition. Let $a(\sigma)$ be the minimum number of parts of such a partition.

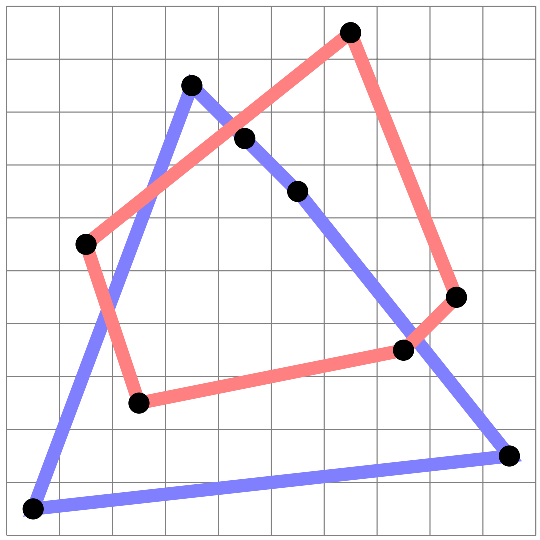


Figure 1: An illustration showing that $a(\sigma) = 2$ for $\sigma = 16398710452 \in S_{10}$.

**Question.** What is the expected value of $a(\sigma)$ for a uniformly random $\sigma \in S_n$?

**Related.**

1. What if each point must be on the corner of the convex hull?
2. What is the maximum number of convex hulls required?
3. What is the expected number of convex hulls? (i.e. how many different ways can a $\sigma$ be partitioned into $a(\sigma)$ convex hulls?
4. What if the convex hulls are not allowed to overlap?
5. What is the expected value of the largest subset of $((1, \sigma(1)), \ldots, (n, \sigma(n)))$ such that no points are in the interior of the convex hull?
6. What if this is done for non-attacking queens?
7. What if this is done for an arbitrary configuration of $k$ pieces on an $n \times m$ board?
8. What if the convex hull of the permutation is taken, and then the convex hull of the interior, and the convex hull of that interior and so on?
9. What if a no three-on-a-line rule is used instead? No $k + 2$ on a degree $k$ polynomial?

**Note.**

$A156831(n) = \{\sigma \in S_n : a(\sigma) = 1\}$.

**References.**

`https://oeis.org/A156831`

Problem 5, 6, 7.

# Problem 89.

According to Peter Winkler's problem "Red and Blue Dice" in *Mathematical Mind-Benders*, given two sequences of length $n$ with letters in $[n]$, there must always exist a (nonempty) subsequence in each such that the sum of each of the subsequences are equal. Furthermore, there must exist a *substring* (i.e. contiguous subset) in both such that the sum of each substring is equal.

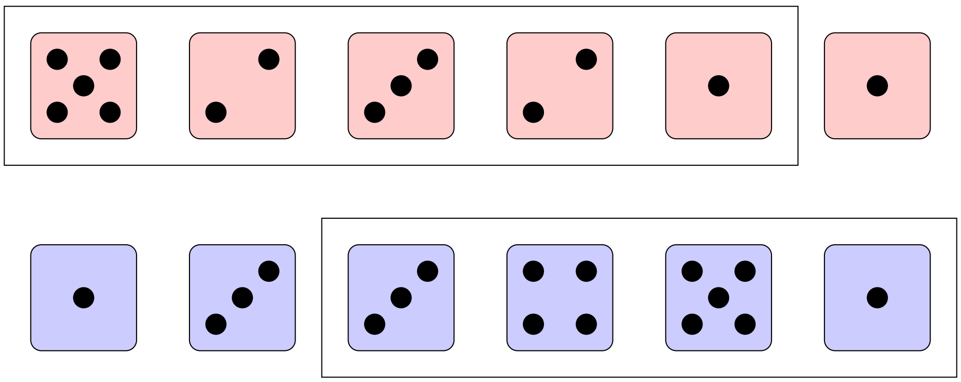


Figure 1: The first five dice in the top row have the same sum as the last four dice in the bottom row.

**Question.** How many equal-sum subsequences (respectively substrings) are guaranteed to exist?

**Related.**

1. What if the subsequence or substring must have length greater than or equal to $k$?
2. What if the subsequences or substrings must be of equal length?
3. What if the substring is allowed to wrap around?
4. What if this is done over permutations $S_n$ instead of subsets $[n]^n$?
5. What if the sequences are of length $\ell < n$, and must be injections from $\mathbb{N}$ to $[n]$?
6. What if there are three sequences? $m$ sequences?
7. Given two random sequences, what is the expected number of equal-sum subsequences?

**References.**

https://math.stackexchange.com/q/3035452/121988

# Problem 90.

The puzzle Figure/Ground by Ian Gilman features a grid with two colors. In the grid any (horizontal/vertical) connected component can be moved exposing the other color beneath.

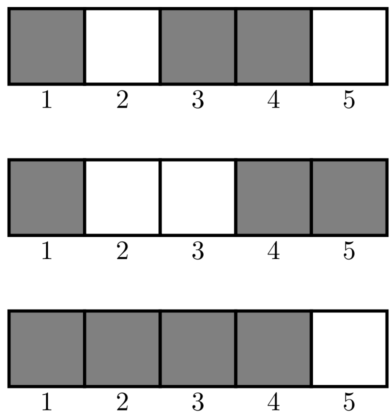



Figure 1: It is possible to get from the first configuration to the second configuration by moving the $(3, 4)$-block to position $(4, 5)$ or by moving the 5-block to position 3. It is possible to get from the first configuration to the third by moving the block in position 2 to position 5.

**Question.** Is there an efficient algorithm to determine whether it's possible to get from one configuration to another?

**Related.**

1. On a $1 \times n$ grid, what is the greatest number of steps between two configurations?
2. Starting with the $1 \times n$ grid where even squares are black and odd squares are white, is is possible to get to any configuration with both colors present? Do other starting configurations have this property?
3. What if this is done on a $n \times m$ grid? A $n_1 \times \ldots \times n_k$ grid? A triangular/hexagonal grid? Torus?
4. What if more colors were used?

**References.**

http://www.clockworkgoldfish.com/figureground/list/sky/

# Problem 91.

Consider a puzzle that consists of an $n \times n$ grid with $n$ marked cells. The goal of the puzzle is to partition the grid into $n$-cell regions of size $n$, each containing exactly one marked cell.

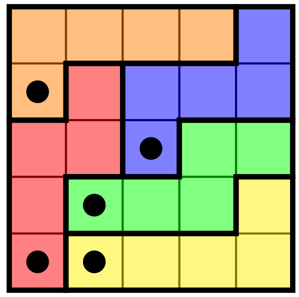


Figure 1: An example of a $5 \times 5$ grid with a unique solution.

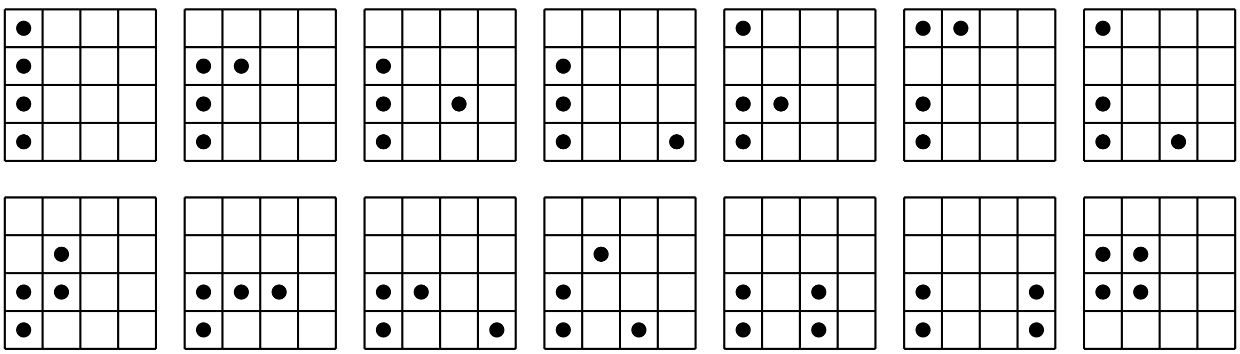


Figure 2: Fourteen (all, up to dihedral action?) markings with exactly one solution.

**Question.** How many $n \times n$ boards exist with a unique solution? Up to dihedral action?

**Related.**

1. How many $n \times n$ boards exist with no solution? Multiple solutions?
2. What board has the most solutions?
3. What if this is done on an $n \times m$ board with $k$ marked cells where $k|nm$ and each region has $nm/k$ cells?
4. What if the board is a torus? Triangular/hexagonal grid? Multiple dimensions?
5. What if instead of marked cells there are marked regions?
6. What if cells must must be rectangular? Symmetric?
7. What if every region must be a walk starting at a marked cell? (As in the example.)

**References.**

Problem 24

`https://math.stackexchange.com/q/3072735/121988`

`https://codegolf.stackexchange.com/q/179074/53884`

`https://en.wikipedia.org/wiki/Flow_Free`

# Problem 92.

Consider walks on an $n \times m$ grid, where the walk can only self-intersect at a perpendicular step.

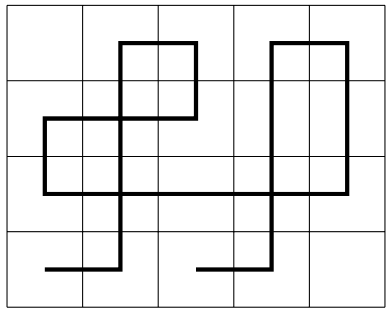


Figure 1: An example of a walk on a $5 \times 5$ grid.

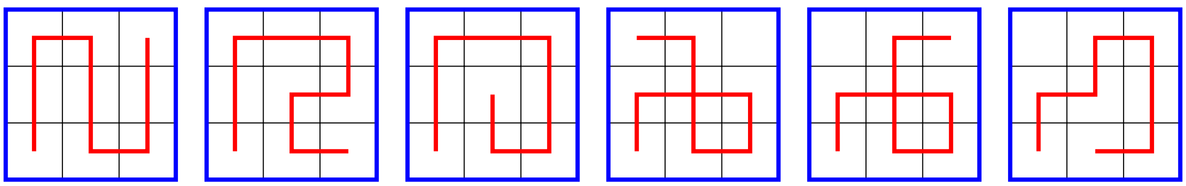


Figure 2: Six (all?) "as-long-as-possible" paths starting in the lower left corner, up to dihedral action on the $3 \times 3$ grid.

**Question.** How many $n \times m$ boards exist with a unique maximal path?

**Related.**

1. What if paths must be "as long as possible", in the sense that they can't be extended at either end? Exactly $k$ steps?
2. What if this is done on a torus, triangular grid, cube, etc?
3. What if paths much touch every square at least once?
4. How many up to dihedral action? How many with dihedral symmetry?
5. What if paths must start at, say, the upper right corner?
6. What if the path must have at least one self-intersection?
7. What if paths are allowed to be loops (i.e., end on same square as they began on?) What if they must be loops?
8. What is the greatest number of king steps? Fewest on an "as-long-as-possible" path? Rook steps?
9. What if diagonal moves are allowed? Only diagonal moves?
10. What if multiple paths can be drawn on the same grid, only intersecting perpendicularly?

**References.**

Special case of Problem 31. Problem 42 and 56.

Pipe Mania puzzle game. `https://en.wikipedia.org/wiki/Pipe_Mania`

# Problem 93.

Consider the intersection of a regular $n$-gon with a regular $m$-gon, both with sides of unit length.

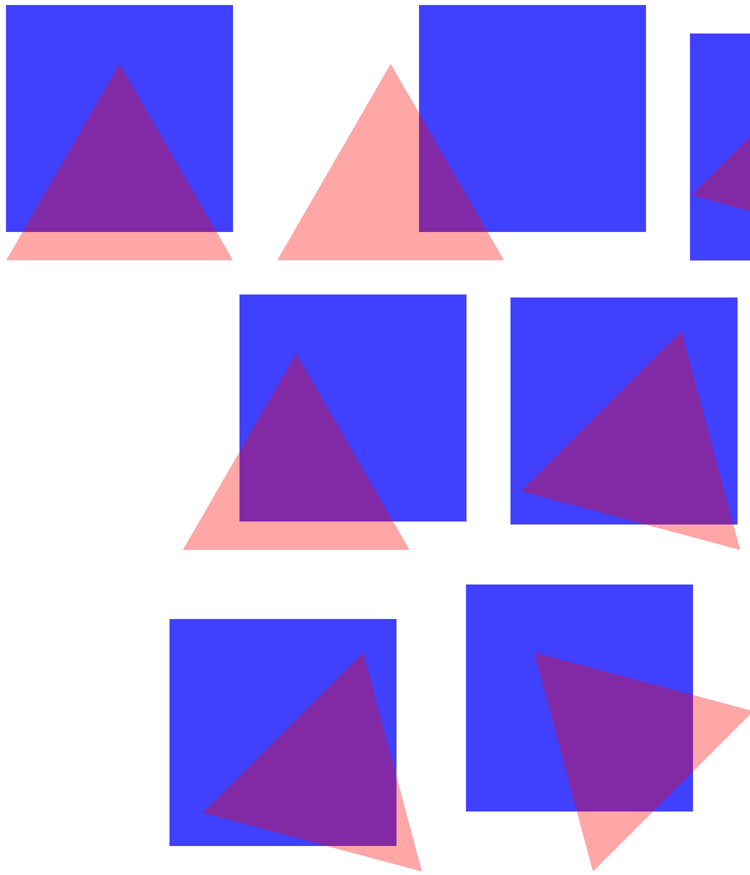


Figure 1: Seven (all?) classes of intersections between a 4-gon and a 3-gon. The three triangular intersections may be considered distinct because one has all three of its sides contributed from the triangle, one has two sides from the triangle, and one has two sides from the square.

**Question.** What are the possible classes of polygons that can be realized as the intersection of an $n$-gon and an $m$-gon?

**Related.**

1. What is the largest $k$ for which a regular unit $n$-gon and $m$-gon can intersect in a $k$-gon?
2. What if the polygons have unit area instead of unit length?
3. What if the regular polygons can be any size at all?
4. What if the polygons do not need to be regular? If the intersection does not need to be connected?
5. What if the polygons have integer vertices and minimal area?

# Problem 94.

Consider convex polygons with integer vertices and minimal area.

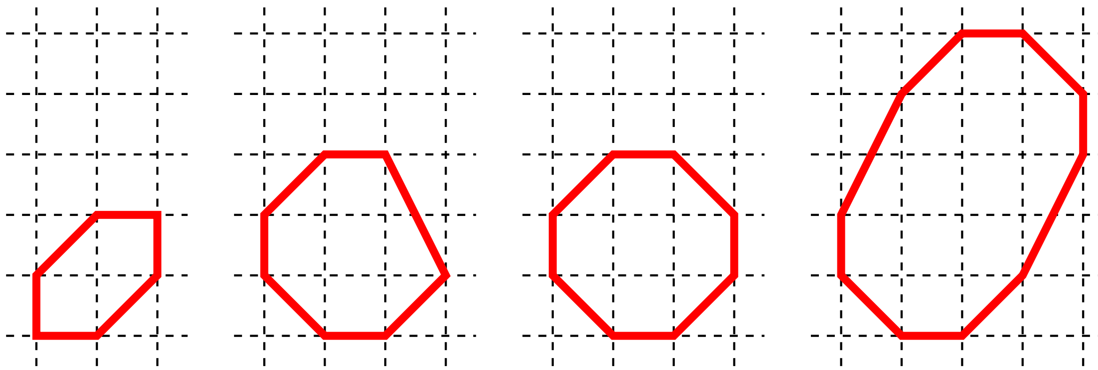


Figure 1: Candidates for minimal area $6, 7, 8$ and 10-gons.

**Question.** What is the minimal area of a convex lattice $n$-gon.

**Related.**

1. What if the sum of side lengths is minimized instead? Measured via the taxicab metric?
2. What if the polygons are minimized with respect to the height/width of the smallest grid? Or the number of complete cells they contain? (e.g. the examples contain 0, 4, 5, and 10 cells respectively) The number of partial cells they contain? (e.g. 4, 9, 9, 18) respectively?
3. What if the polygons can be concave?
4. What if the concave polygons cannot have any acute angles?
5. What if the concave polygons can be decomposed into polyabolos?
6. What if the polygons must have 180° or horizontal symmetry?

**References.**

https://en.wikipedia.org/wiki/Polyabolo

Problem 5, Problem 7, and Problem 88.

# Problem 95.

★★★☆

How many non-intersecting walks from $(1,1)$ to $(n,m)$ with steps up and to the right exist on the $n \times m$ torus?

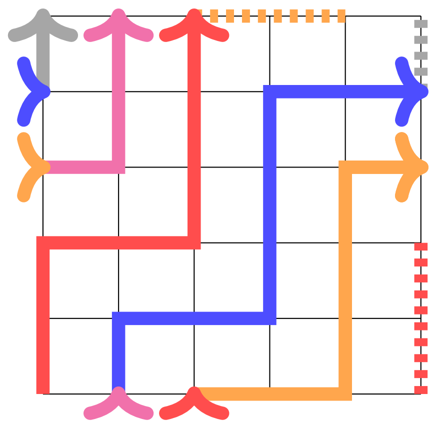


Figure 1: An example of a walk on a $5 \times 5$ torus that touches every lattice point.

**Question.** How many such walks exist?

**Related.**

1. What if the walks must touch every lattice point?
2. What if the walks must wrap around the torus exactly $k$ times? (For $k = 1$ and $m = n$, this is the number of walks along the edges of an $n \times n$ non-toroidal grid.)
3. What if there always must be weakly more "up" steps than "right" steps? (generalization of staying above the diagonal) Strongly more?
4. What if this is done on a cylinder? Möbius strip? More dimensions?
5. What if walks can intersect at a right angle? What if there must be exactly $k$ intersections? Only at $(0,0)$?
6. What if more general loops were counted? (i.e. any walk from $(0,0)$ to $(0,m)$, $(n,0)$ or $(n,m)$.)

**References.**

Problem 92.

`https://oeis.org/A324603`

# Problem 96.

Consider ways of nesting centered squares in the $n \times n$ square lattice.

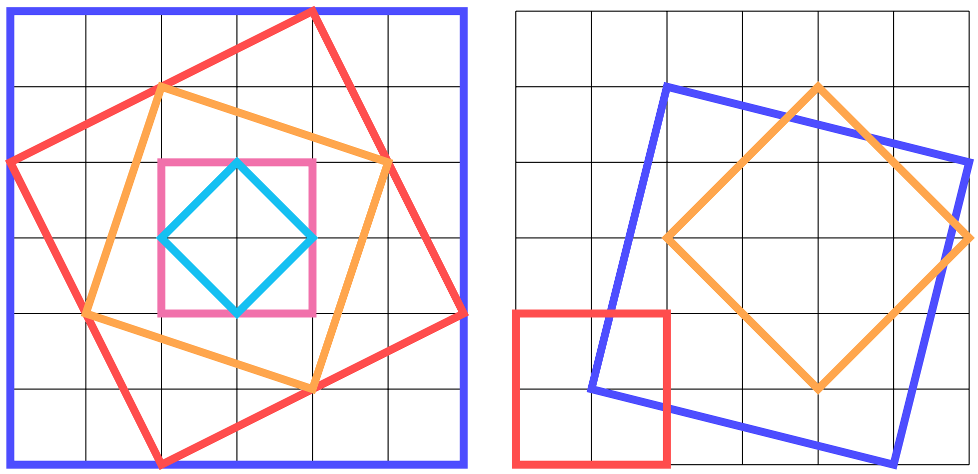


Figure 1: An example of five non-parallel centered squares in the size 6 square, and an example of three non-parallel non-centered squares that do not share any lattice points.

**Question.** What is the largest number of squares that can be nested?

**Related.**

1. What is the maximum sum of the areas? Perimeters?
2. How many maximal configurations exist?
3. What if the nested squares must be "snug", that is, all four of their corners must be on their outer neighbor.
4. What if the nested squares cannot be parallel (i.e every square must be a scaling *and* a rotation of another square)?
5. What if they don't need to be centered, but instead no two squares can share a lattice point? If no two squares can be parallel?
6. What if this is done on a triangular or hexagonal lattice?

**References.**

Problems 21 and 66.

# Problem 97.

Consider a $n \times m$ grid of ones and zeroes, which represent the heights of the cells. It rains, and the grid fills up with the rain moving horizontally and vertically.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |

Figure 1: An example of five non-parallel centered squares in the size 6 square, and an example of three non-parallel non-centered squares that do not share any lattice points.

**Question.** What is the expected area of a lake? Of the sum of all lakes?

**Related.**

1. What is the expected number of lakes? Of islands? Of lakes on islands?
2. What if water can flow diagonally too?
3. What if the heights can take on arbitrary values?
4. What if there is a border around the grid of height $k$?
5. What if the cell is height 0 with probability $p$ and height 1 with probability $(1 - p)$?
6. How does this generalize to triangular/hexagonal grids? More dimensions?
7. How does this generalize to a cylinder?

**References.**

Problem 86.
https://codegolf.stackexchange.com/q/2638/53884

# Problem 98.

Suppose you choose two points on a line segment uniformly at random, defining three smaller line segments. The probability that these three line segments satisfy the triangle inequality is $\frac{1}{4}$.
Similarly, if you choose five points on a line segment, it is conjectured that they can form a tetrahedron with probability $\frac{1}{79}$.

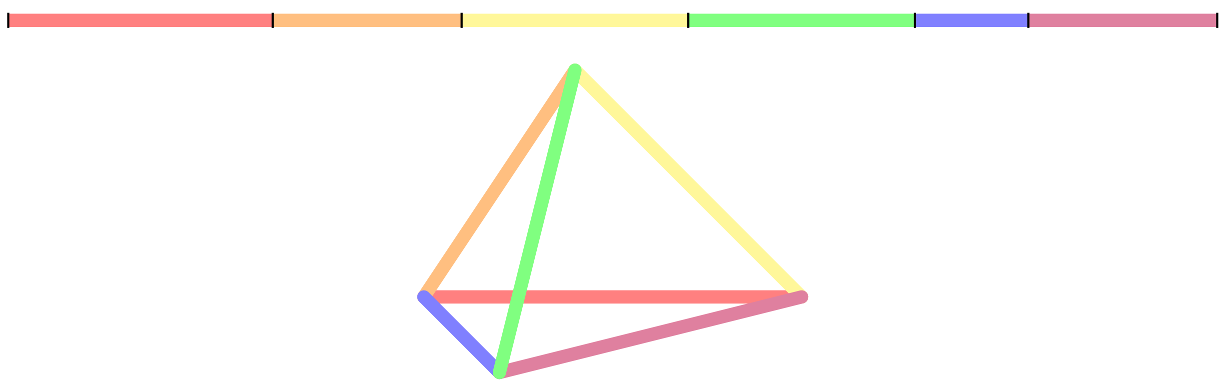


Figure 1: An example of a partition of a line segment and the resulting tetrahedron.

**Question.** If the line segment is split into $\frac{k(k+1)}{2}$ pieces using this prescription, do the resulting segments form a $k$-simplex with rational probability?

**Related.**

1. What if the segments form prescribed sides of the $k$-simplex? What if any configuration is valid?
2. What if there are more than $\frac{k(k+1)}{2}$ pieces, what is the probability that there exists some subset of $\frac{k(k+1)}{2}$ of them that forms a $k$-simplex?
3. What is the expected hyper-volume of such a tetrahedron?
4. What is the probability that all $A006473(k+1)$ arrangements of edges form a $k$-simplex?

**References.**

Problem 73.

Problem 78 also deals with a generalization of a Stack Exchange question.

`https://mathoverflow.net/q/142983/104733`

`https://oeis.org/A006473`

# Problem 99.

There are two popular, essentially identical, iPhone games in the app store: *AMAZE!!!* and *Roller Splat!*. The goal of the puzzle is to reach every (white) square in the board—the catch is that you can only move in as-long-as-possible rook moves.

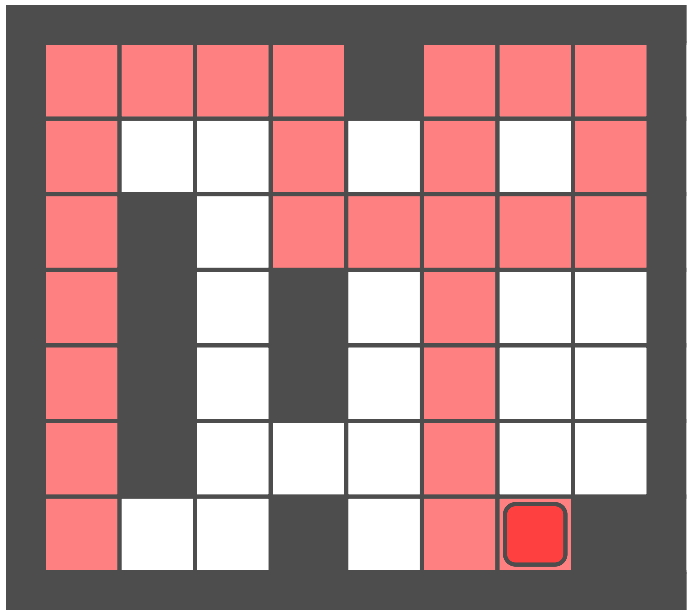


Figure 1: Starting from the lower left corner, the board can be filled using the following 25 moves: $\underbrace{\uparrow\rightarrow\downarrow\rightarrow\uparrow\leftarrow\downarrow\rightarrow}_{\text{illustrated above}}\uparrow\downarrow\leftarrow\uparrow\leftarrow\rightarrow\downarrow\leftarrow\uparrow\downarrow\leftarrow$

**Question.** How many solvable puzzles exist on an $n \times m$ board?

**Related.**

1. What if we only want to count "primitive" puzzles—those that cannot exist on a smaller board?
2. What if we count up to symmetries of the rectangle?
3. Which puzzle requires the greatest number of moves?
4. What if we do this on a torus? Möbius strip? More dimensions?
5. Given some configuration, what is an algorithm to figure out how to solve it?

# Problem 100.

Consider attempting to fill in a $n \times n$ Latin square with the numbers 1 through $n$, one number at a time. A clumsy filling is an incomplete filling in which no more valid moves are possible.

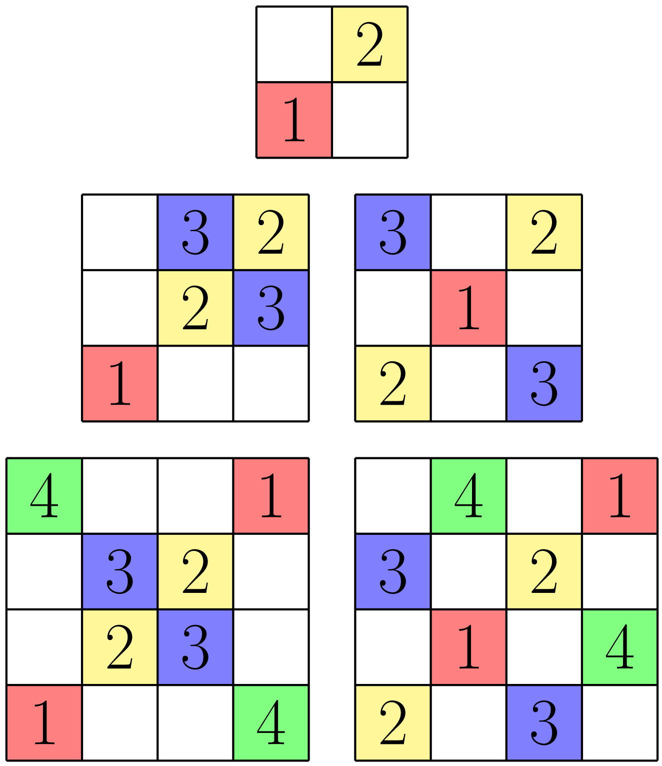


Figure 1: Clumsy filling for $n = 2, 3, 4$ can be achieved with 2, 5, and 8 or fewer entries respectively; thus $a(2) = 2, a(4) \leq 5$, and $a(4) \leq 8$.

**Question.** Let $a(n)$ be the fewest number of entries required for a clumsy filling. What is $a(n)$?

**Related.**

1. How many "essentially different" fillings are there?
   (Two fillings are the same if related by permuting the symbols or dihedral action of the board.)
2. Can minimal clumsy fillings be built iteratively, as suggested by the leftmost diagrams in the example?
3. What if this is done on a group table instead of a Latin square (quasigroup table)?

**References.**

`https://www.youtube.com/watch?v=U5NLgivoKDQ`

# Problem 101.

In a letter, Alec Jones asks notes that there are eleven distinct nets of the cube, considered as free polyominoes.

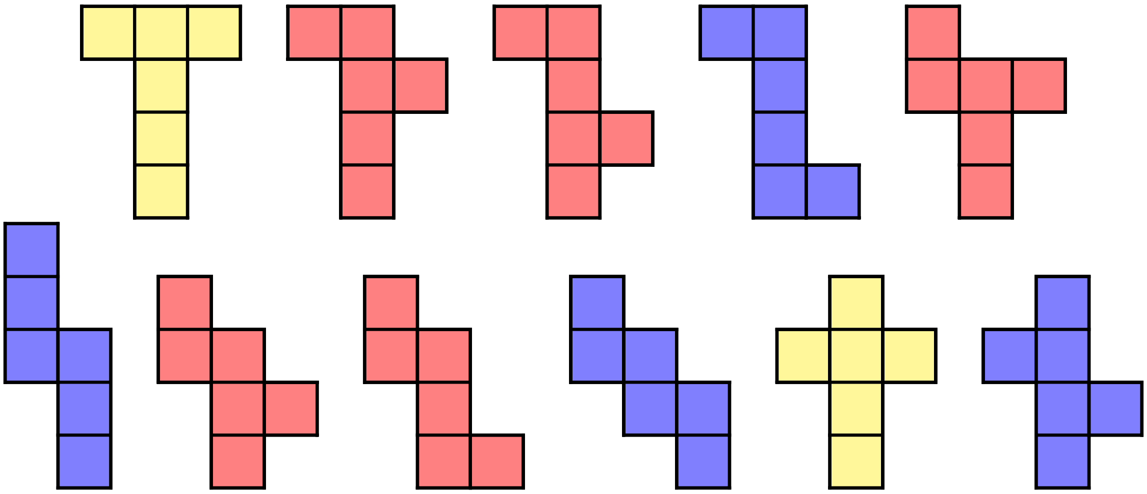


Figure 1: Eleven distinct nets of the (3-hyper)cube. Two, marked in yellow, have reflection symmetry, that is they are achiral; four, marked in blue, have rotational symmetry.

**Question.** Alec asks, how many nets are there of the $n$-hypercube?

**Related.**

1. How many of the nets exhibit some sort of symmetry, as shown in the example.
2. How many nets are there of the $n$-simplex? Other polyhedra?
3. How many nets for a rectangular analog? That is for some $a_1, a_2, \ldots, a_n$,

   $$R = \{(x_1, x_2, \ldots, x_n) : x_i = 0 \text{ or } x_i = a_i\}.$$

   Note that in the case of the $n$-hypercube, $a_1 = a_2 = \ldots = a_n = 1$.
4. Must all of these nets "use up" $n$ dimensions. (In the case of $n = 3$ in the example, yes, because the "straight line" 6-omino cannot be folded into a cube.)
5. What is the net with the smallest convex hull by hypervolume? Largest?
6. How many of the nets tile $(n-1)$-space?

**Note.** There is 1 net for the 2-hypercube, 11 nets for the 3-hypercube, and 261 nets for the 4-hypercube. The answer is unknown for the 5-hypercube. The corresponding sequence is tagged "hard" on OEIS.

**References.**

https://oeis.org/A091159

https://mathoverflow.net/q/198722/104733

# Problem 102.

In geometry, a deltahedron is a polyhedron whose faces are all equilateral triangles.

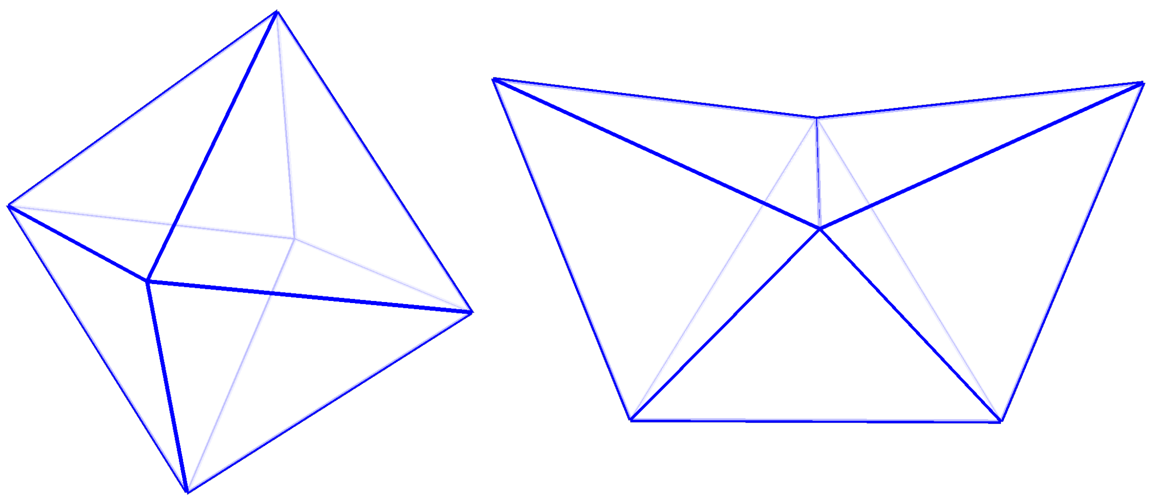


Figure 1: The two deltahedra consisting of eight triangles: the octahedron and the biaugmented tetrahedron.

**Question.** How many polytopes can you make out of $n$ equilateral triangles?

**Related.**

1. What if none of the adjacent faces can be coplanar? (e.g. the "scaled-up" tetrahedron is not allowed)
2. How many with some sort of symmetry?
3. What if squares or pentagons are used instead?
4. What if pentagons and triangles are used?
5. How does this generalize to higher dimensions?
6. Which $n$-cell nets can produce the greatest number of distinct polytopes?
7. How many $n$-cell nets can form at least one polytope?
8. There is one augmented tetrahedron, one bi-augmented tetrahedron, and three tri-augmented tetrahedra. How many $k$-augmented tetrahedra?

**References.**

`https://en.wikipedia.org/wiki/Deltahedron`

**Note.**
The tetrahedron is the only deltahedron with 4 faces.
The triangular bipyramid is the only deltahedron with 6 faces.
There are at least five examples of deltahedra with 10 faces: the pentagonal dipyramid, the augmented octahedron, and the three ways to augment the biaugmented tetrahedron.

# Problem 103.

The Snake Cube is

> a mechanical puzzle, a chain of 27 or 64 cubelets, connected by an elastic band running through them. The cubelets can rotate freely. The aim of the puzzle is to arrange the chain in such a way that they will form $3 \times 3 \times 3$ or $4 \times 4 \times 4$ cube.

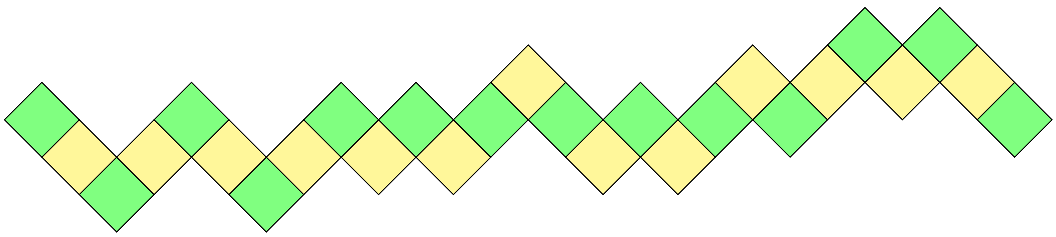


Figure 1: The most common instance of the Snake Cube.

**Question.** How many chains exists for an $n \times n \times n$ cube?

**Related.**

1. How about for an $n \times m \times p$ rectangular prism?
2. Can the "hardest" puzzles be quantified? Perhaps those with the greatest or fewest number of solutions?
3. Can this be generalized to higher dimensions?
4. Can this be generalized to other polytopes?

**References.**

Problem 48.

Problem 101.

`https://en.wikipedia.org/wiki/Snake_cube`

`https://github.com/scholtes/snek` (Counts solutions to Rubik's Snake)

# Problem 104.

According to Fáry's theorem, any planar graph can be drawn as a planar straight-line graph. This problem studies which of these straight-line graphs are the smallest.

Given a planar graph $G$ let $S_G$ be the set of all straight-line embeddings of $G$ where the shortest edge has length 1. Given an embedding $E$, let $\ell(E)$ be the average edge length, and let $f(G) = \inf\{\ell(E) : E \in S_G\}$.

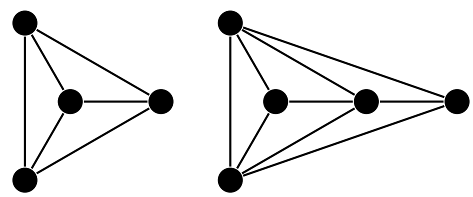
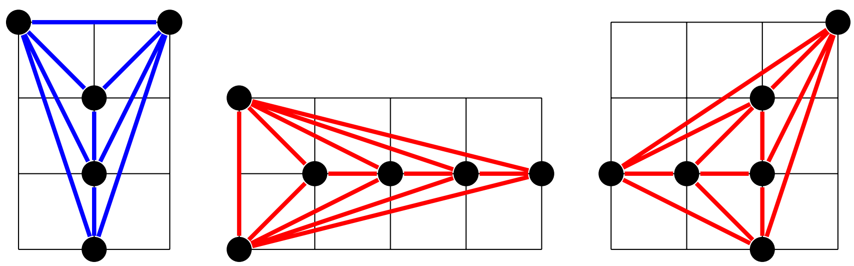


Figure 1: Conjectured minimal embeddings showing $f(K_4) \leq \dfrac{3\sqrt{3}+3}{6}$ and $f(K_5 \setminus \{e\}) \leq \dfrac{3\sqrt{3}+4+\sqrt{28}}{9}$.



$n = 5$ $n = 6$ $n = 6$

Figure 2: Smallest (?) grids that contain all planar simple graphs on $n$ vertices are $[3] \times [4]$ for $n = 5$ and $[5] \times [4]$ for $n = 6$.

**Question.** Given some graph $G$, what is $f(G)$?

**Related.**

1. How does $\max\{f(G) : G \text{ is planar with } n \text{ vertices}\}$ grow with respect to $n$?
2. What if the vertices must be on $\mathbb{Z}^2$? What is the smallest square that can contain all planar graphs with $n$ vertices?
3. What is the smallest $nmk$ such that $K_n$ can be drawn in $[n] \times [m] \times k$ with straight-line edges and no edges intersecting?
4. Is it always possible to write a planar graph as a straight-line graph with integer edge lengths? If not, when is this possible? If so, what's the minimal edge sum?
5. What is the longest non-self-intersecting polygonal chain that can fit in an $n \times m$ grid?
6. What is the supremum of $\ell(E)$ over all straight-line embeddings with longest edge *at most* 1? Are these just rescalings of the original case?

**References.**

Problem 74.

Problem 85.

`https://oeis.org/A000109`

# Problem 105.

Consider maximal non-self-intersecting polygonal chains on $[n] \times [m]$ stable under $180°$ rotation.

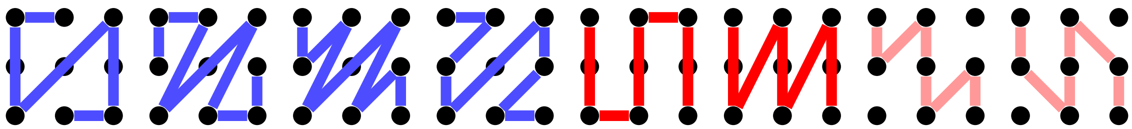


Figure 1: The 6 (or 8) maximal polygonal chains with vertices in $[3] \times [3]$.

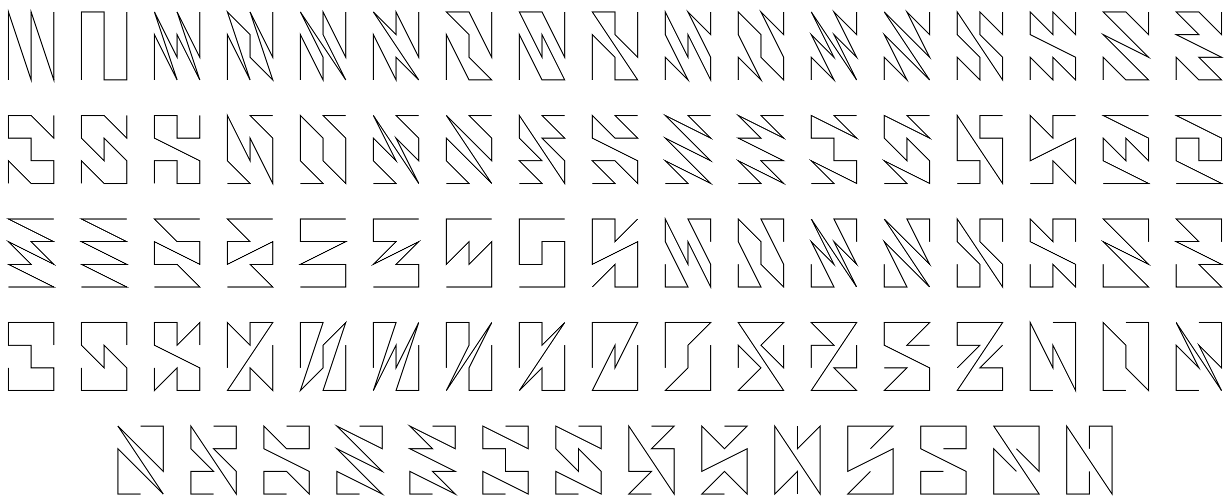


Figure 2: $f(4 \times 3) = 82$

**Question.** How many non-self-intersecting polygonal chains with vertex set equal to $[n] \times [m]$ are stable under $180°$ rotation?

**Related.**

1. What if this is done with other kinds of symmetry? (e.g. horizontal or vertical reflection)
2. What if this is done for polygons instead of polygonal chains?
3. What if maximal means that the polygonal chain cannot be extended, a weaker condition than that the vertex set is $[n] \times [m]$. (This includes the last two chains in the example.)
4. What is the maximal length of such a chain with respect to $\ell_1, \ell_2$, and $\ell_\infty$? What if the symmetry restriction is dropped?
5. What if the only allowed moves are king moves? Rook moves?
6. What if this is done with vertex set $[n_1] \times [n_2] \times \cdots \times [n_k]$?

**References.**

Problems 5, 44, 46, 55, 68, 74, 87, and 104.

# Problem 106.

Consider a frog hopping on a circular collection of $n$ lily pads. The frog hops to any lily pad, and then hops with increasing steps. At the $k$-th step, the frog looks $k$ steps in the clockwise direction and $k$ steps in the counterclockwise direction and hops to whatever lily pad she has visited less. If there is a tie, she hops in the clockwise direction.

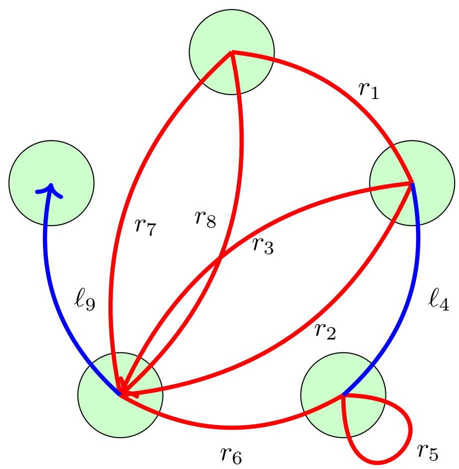



Figure 1: For $n = 5$, all lily pads will have been reached after nine hops.

**Question.** How many hops does it take to reach all lily pads?

**Related.**

1. What if ties are broken by hopping in the same direction instead of hopping clockwise?
2. What if instead of hopping with steps $1, 2, 3, \ldots,$ a different sequence is used?
3. How many positions are reached exactly once?
4. If the hops are in a random direction, what's the expected time to reach every lily pad? What's the expected value of the most-reached lily pad?
5. If you get to choose clockwise or counterclockwise each hop, how many ways are there to reach every lily pad in exactly $n$ hops?

**References.**

https://math.stackexchange.com/q/3418970/121988

https://oeis.org/A282442

https://oeis.org/A329230

# Problem 107.

Suppose that Arthur chooses an arbitrary subset $A \subseteq [n]$, and Bri attempts to discover it by repeatedly asking questions of the form, “How many elements does $A$ have in common with $B_i \subseteq [n]$?”

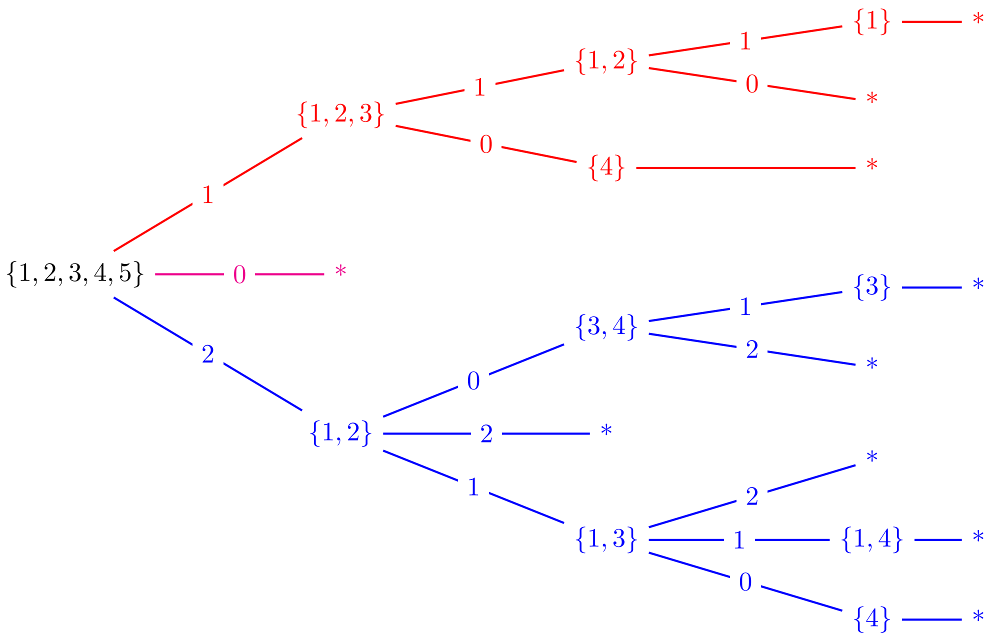



Figure 1: A strategy that Bri can use to discover $A$ in four guesses or fewer, note that the cases where $A$ is size $3, 4$, or $5$ follow by symmetry.

**Question.** Let $a(n) = k$ be the least integer $k$ such that there exists a strategy where Bri can always determine $A$ in $k$ guesses or fewer. What is $a(n)$?

**Related.**

1. What if instead of giving the size of the intersection, Arthur gives the size of the symmetric difference?
2. What if $A$ is a multiset? Where $i$ can occur with multiplicity at most $a_i$?
3. What are some upper and lower bounds?
4. How many (essentially different) optimal strategies exist? (e.g., do you always have to start by guessing the entire set?)
5. What is the best *average case* strategy?
6. What if there are restrictions on Bri's subsets? For example, if the size of Bri's subsets must be weakly decreasing, or if Bri's subsets cannot simultaneously contain both $i$ and $i + 1$?
7. What if Arthur instead picks a column from a given matrix, how many questions of the form “what is the $i$th entry” does Bri have to ask in order to determine the column?

**References.**

https://math.stackexchange.com/a/25297/121988

# Problem 108.

One way to form minimum perimeter polyominoes is to arrange the tiles in a square spiral, however, there are often minimal-perimeter configurations that are not formed by a square spiral.

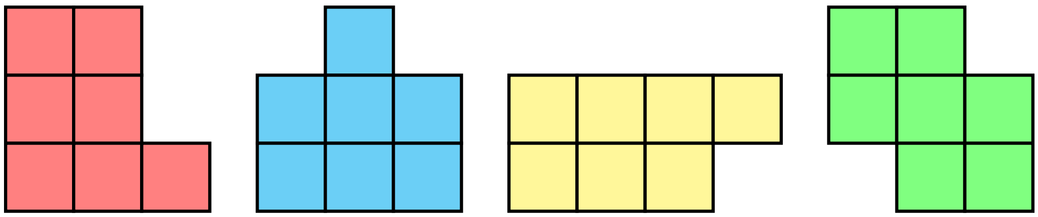


Figure 1: All A100092(7) = 4 minimal-perimeter 7-ominoes, the first of which is the beginning of a square spiral.

**Question.** Given (pseudo-)polyforms on some plane tiling, what is the minimum perimeter of a region containing $n$ cells?

**Note.** In the case of the pseudo-polyform on a snub square tiling, a spiral does not appear to be the way to minimize the perimeter of an $n$-form.

**Related.**

1. How many such minimal-perimeter regions exist? How many regions have some sort of symmetry?
2. What is the minimum perimeter if the region must be symmetric under mirror image? Under 180° rotation?
3. What do these look like on the "ordinary" polyforms: polyominoes, polyiamonds, polyhexes, etc.
4. Next, what about the pseudo-polyforms that specifically live on the snub square tiling, truncated hexagonal tiling, and all of the fifteen pentagonal tilings?
5. What about irregular tilings like the Penrose tiling?
6. What about higher dimenional tilings, and minimizing side lengths or surface area or both?
7. What about (pseudo-)polyforms that can't cover the plane or don't correspond to a tiling (e.g. polypents)
8. What about minimizing (or maximizing) via other metrics? The perimeter of the convex hull? The sum of the angles? The number of sides touching internally?
9. What is the minimum perimeter region that can contain all free (pseudo-)$n$-forms? Fixed forms? How many fillings does minimal-perimeter region have?

**References.**

Problems 72 and 77.

`https://oeis.org/A027709`

`https://oeis.org/A100092`

# Problem 109.

Consider polyforms formed by facets of an $n$-dimensional hypercube. If such a polyform has $k$ cells, call it a $k$-polyfacet. Count these up to symmetries of the cube.

Figure 1: On the left, the two 3-polyfacets on the cube, and on the right, the two 4-polyfacets on the cube. The 0-, 1-, 2-, 5-, and 6-polyfacets are unique on the cube.

**Question.** How many $k$-polyfacets live on the $n$-cube?

**Note.** The following table gives the number of $k$-polyfacets on an $n$-cube:

```
n\k| 0  1  2                  ...                 2n-1 2n|
---+------------------------------------------------------+
 2 | 1, 1, 1,                                      1, 1 |
 3 | 1, 1, 1, 2,                                2, 1, 1 |
 4 | 1, 1, 1, 2, 3,                          2, 2, 1, 1 |
 5 | 1, 1, 1, 2, 3, 3,                    3, 2, 2, 1, 1 |
 6 | 1, 1, 1, 2, 3, 3, 4,              3, 3, 2, 2, 1, 1 |
 7 | 1, 1, 1, 2, 3, 3, 4, 4,        4, 3, 3, 2, 2, 1, 1 |
 8 | 1, 1, 1, 2, 3, 3, 4, 4, 5,  4, 4, 3, 3, 2, 2, 1, 1 |
 9 | 1, 1, 1, 2, 3, 3, 4, 4, 5, 5,  5, 4, 4, 3, 3, 2, 2, 1, 1 |
10 | 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1 |
```

Notice that $T(n,k) = T(n,n-k)$ for all $k \notin \{2, n-2\}$. In this case, $T(n,2) = 1$ and $T(n,n-2) = 2$.

**Related.**

1. What if (not necessarily connected) 2-colorings are considered instead of polyforms? $k$-colorings?
2. How many $d$-dimensional $k$-poly-$d$-faces live on the $n$-cube? $n$-simplex? $n$-orthoplex? $n$-demicube?
3. How many fixed polyforms? One-sided polyforms?
4. If we chop up the hypercube into an $\ell \times \cdots \times \ell$ "Rubik's" hypercube, how many polyfacets live on this subdivision?
5. Let $T(n,k)$ denote the $k$-polyfacets on an $n$-cube. Which of the $T(n,k)$ polyfacets has the most symmetry? The least?

**References.**

Problems 72, 101.

`https://codegolf.stackexchange.com/q/201054/53884`

OEIS Sequences A333333, A333362, and A333418.

# Problem 110.

Consider the graded poset on $\mathbb{N}_{>0}$ given by covering relations $n - \frac{n}{p} \lessdot n$ for all primes $p \mid n$.

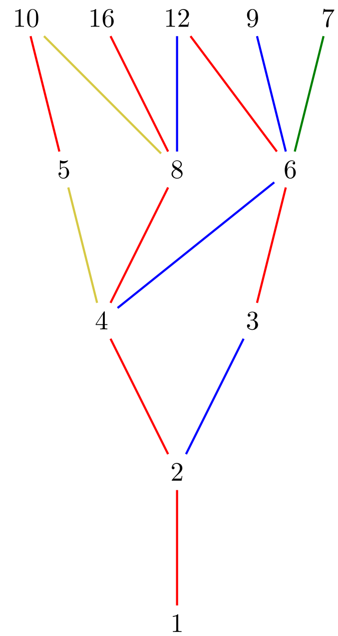



Figure 1: Ranks 0 through 4 of the poset. Prime divisors $2, 3, 5$, and $7$ are displayed as red, blue, gold, and green respectively. For example, $12 \gtrdot 12 - \frac{12}{3} = 8$.

**Question.** Is this poset a lattice?

**Related.**

1. If not, is this poset a join- or meet-semilattice?
2. If so, is this poset a distributive lattice? A modular lattice?
3. It appears that (at least for small values of $n$ and $k$) $(n \vee k) \mid \operatorname{lcm}(n, k)$. What is $\frac{\operatorname{lcm}(n,k)}{n \vee k}$?
4. Is there a good way to construct the least integer (in the usual sense) for a given rank number?

**References.**

https://math.stackexchange.com/q/3632156/121988

https://oeis.org/A334230

# Problem 111.

The chromatic polynomial of a graph $G$, $\chi_G(n)$ gives the number of ways to color the vertices of the graph such that no two vertices of the same color are connected by an edge.

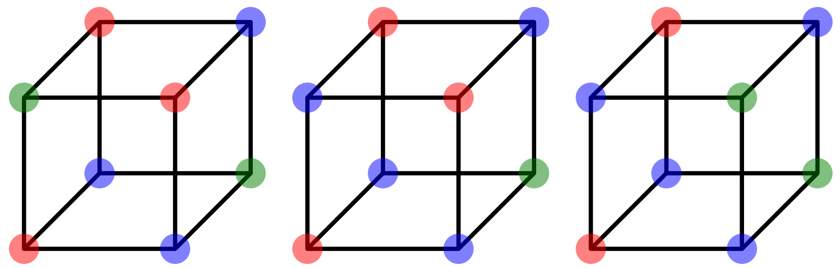


Figure 1: Three examples of 3-colorings of the cube. The chromatic polynomial of the cubic graph is $\chi_{Q_3}(n) = a(n) = n^8 - 12n^7 + 66n^6 - 214n^5 + 441n^4 - 572n^3 + 423n^2 - 133n$.

**Question.** Is there a way to generate the chromatic polynomial of an $n$-cube in polynomial time with respect to $n$?

**Related.**

1. What about up to permutations of the colors and/or isometries of the cube?
2. What about simplices, cross-polytopes, and demicubes?
3. What about other polytopes such as associahedra, permutahedra, and the 24-, 120-, and 600-cells?

**References.**

Problem 61.

`https://math.stackexchange.com/q/3632156/121988`

`https://oeis.org/A334278`

# Problem 112.

Suppose that you're on an $n \times m$ grid, and you'd like to place rectangles to fill up as many gridlines as possible—the catch is that if there are an even number of boxes on a gridline, then they cancel out.

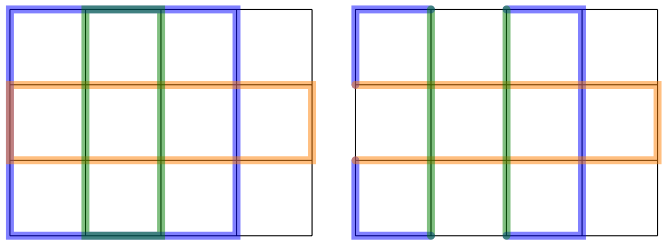


Figure 1: This arrangement with three boxes on the $4 \times 3$ grid misses only seven edges.

**Question.** What is the greatest number of edges that can be covered?

**Related.**

1. How many minimal arrangements of rectangles are there? (Right example)
2. How many maximal edge covers are there? (Left example)
3. What if the rectangles must be square?
4. What if a maximum of $k$ rectangles is allowed?
5. What if it takes 3 (or $c$) rectangles to cancel out?
6. Is this equivalent to finding disjoint collections of "almost" Eulerian cycles on a grid graph?
7. How does this generalize to higher dimensions, the triangular grid, etc?
8. What if we want to cover vertices instead of edges? Facets?
9. How many edges are covered if we use every possible rectangle?
10. Placement of rectangles generates an abelian group. What is the group's structure? Is the size of the group the number of possible edge covers?

**References.**

`https://math.stackexchange.com/q/1579862/121988`

Problems 37 and 74.

# Problem 113.

Consider arrangements of $n$ lines in the plane.

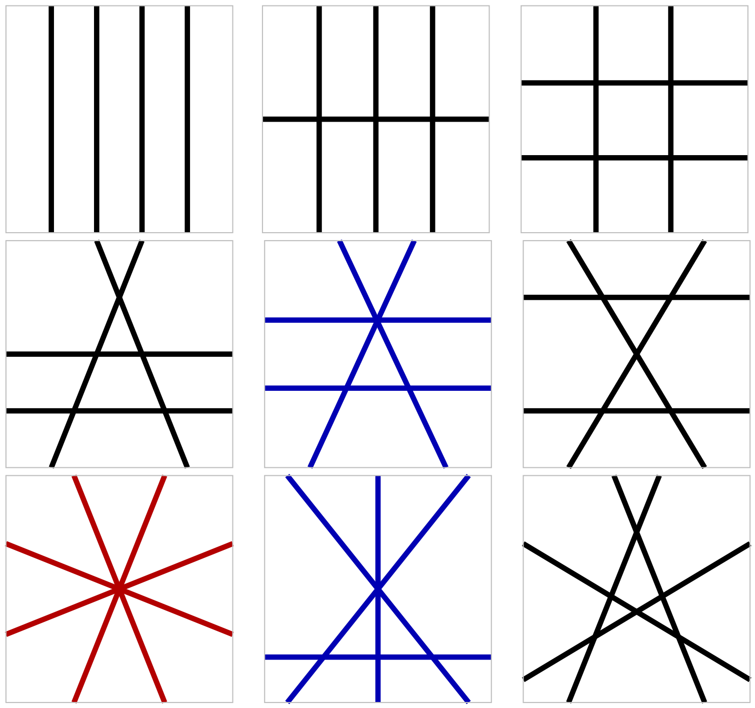


Figure 1: There are $A241600(4) = 9$ arrangements of 4 lines in the plane, which split the plane into $5, 8, 9, 10, 9, 10, 8, 10$, and 11 parts respectively.

**Question.** How many nonisomorphic ways can $n$ lines split the plane into $k$ parts?

**Related.**

1. What if only two lines can go through a single point?
2. What if circles are used instead of lines? Circles on a sphere? Lines on a torus?
3. Hyperplanes in higher dimensional space?
4. How many arrangements are there if the bounded regions must have equal area?
5. How many different polygons can be embedded such that every side is on a line? Convex polygons?

**References.**

OEIS sequences A241600, A177862, and A250001.

# Problem 114.

According to Wikipedia

> A Johnson solid is a strictly convex polyhedron for which each face is a regular polygon.

Let Hull($P$) denote the convex hull of a polygon $P$. Say that a 1-concave Johnson solid is a polyhedra $P$ with regular polygonal faces (such that no two faces lay the same plane) and with the property that the "concave" part $\text{Hull}(P) - P$ is a connected convex polyhedron.

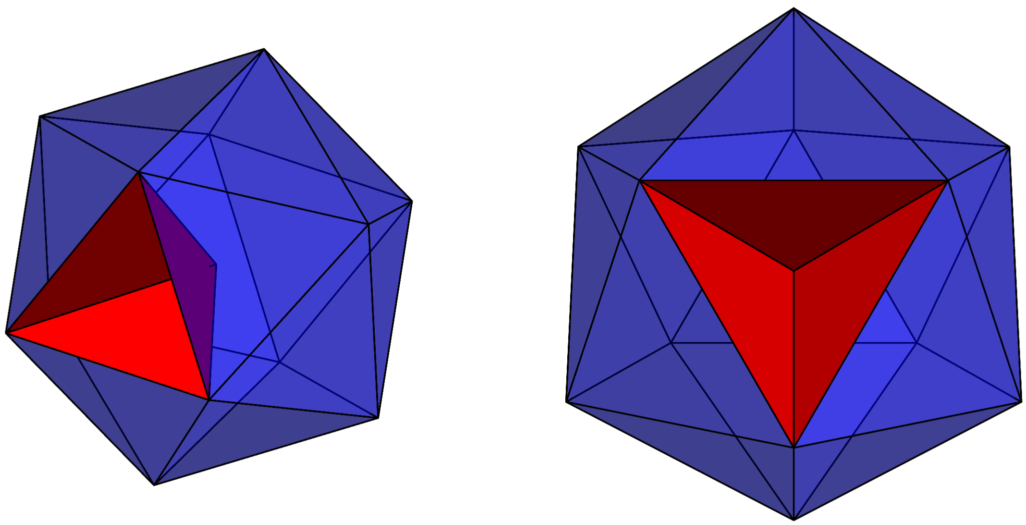


Figure 1: An example of a "1-concave" Johnson solid: an icosahedron with a tetrahedron removed.

**Question.** Excluding infinite families of 1-concave Johnson solids (e.g. prisms with a square pyramid removed), are there a finite number of 1-concave Johnson solids?

**Related.**

1. What if the "co-planar" restriction is relaxes so that no two *adjacent* faces may lie in the same plane?
2. How many 1-concave Johnson solids have a convex hull that is a Johnson solid?
3. Say that a 2-concave Johnson solid $P$ is a polyhedra with regular polygonal faces such that $P' = \text{Hull}(P) - P$ is a connected polyhedron, and $\text{Hull}(P') - P'$ is a connected convex polyhedron. Outside of infinite families, are there a finite number of 2-concave Johnson solids? $n$-concave Johnson solids?
4. Are all $n$-concave Johnson solids homeomorphic to a sphere?
5. Do there exist $k$-concave Johnson solids for all $k$?

**References.**

Problem 53.

`https://en.wikipedia.org/wiki/Johnson_solid`

# Problem 115.

Given some set of functions $\{p_i \colon [n] \to [k]\}_{i=1}^m$ consider the semigroup generated by $\langle p_1, p_2, \ldots, p_m \rangle$.

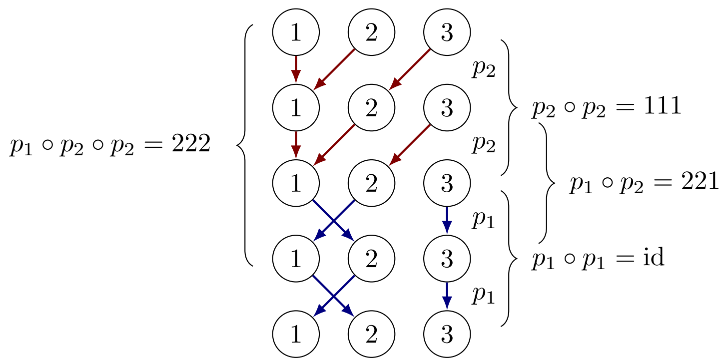



Figure 1: Given $p_1 = 213$ and $p_2 = 112$, there are six distinct functions that can be made from nonempty compositions of these functions: $p_1$, $p_1 \circ p_1$, $p_1 \circ p_2$, $p_1 \circ p_2 \circ p_2$, $p_2$, and $p_2 \circ p_2$.

**Question.** If the $p_i$s are chosen uniformly at random, what is the expected size of the semigroup?

**Related.**

1. What's the expected number of functions $m$ such that $|\langle p_1, p_2, \ldots, p_m \rangle| = k^n$? The minimum number of functions?
2. What is the largest semigroup as a function of $n$, $k$, and $m$?
3. Can you make any size semigroup with the right parameters? If not, what sizes can you make?

**References.**

Problem 6.

# Problem 116.

In the grid $\mathbb{Z}^2$, the only regular polygons that you can draw are squares. In $\mathbb{Z}^3$, you can draw equilateral triangles, squares, and regular hexagons, but no other regular polygons.

In $\mathbb{Z}^3$, you can also draw regular tetrahedra, cubes, and octahedra(?), but not dodecahedra or icosahedra. (Otherwise you could draw pentagons too!)

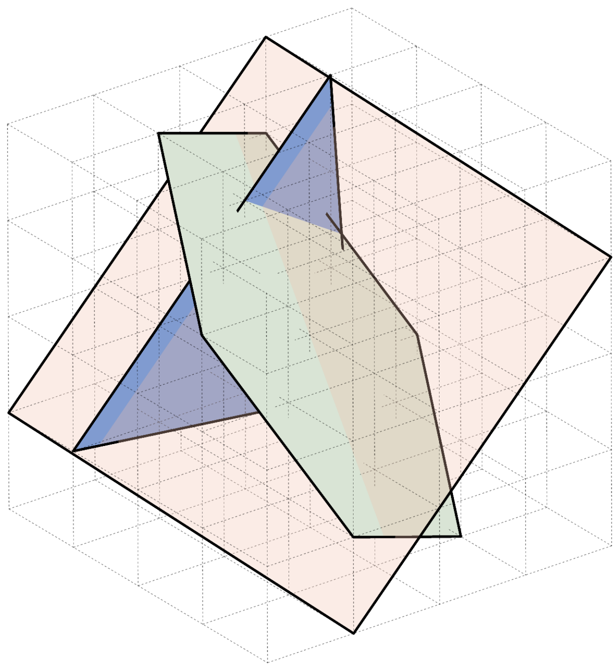


Figure 1: An example of an equilateral triangle, a square, and a regular hexagon drawn with integer coordinates in $[5]^3$.

**Question.** How many regular $k$-dimensional polytopes can be drawn with vertices in $[n]^\ell$?

**Related.**

1. What is the asymptotic growth of the number of $k$-dimensional polytopes?
2. What if other sorts of polytopes are considered? (E.g. Archimedean solids.)

**References.**

Problems 21, 54, 66, 94, 104.

A338323

# Problem 117.

Choose a point $p$ in $\mathbb{R}^2$, and consider all balls $\mathcal{B}_p(r)$ of radius $r$ centered at $p$. Let $f(\mathcal{B}_p(r))$ be the number of "boxes" of $\mathbb{Z}^2$ that are partly contained in the interior of a ball.

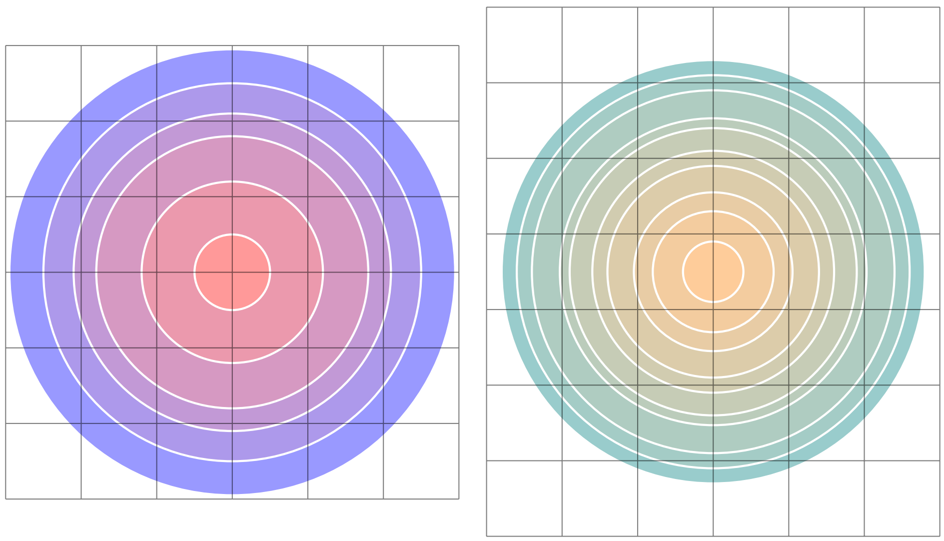


Figure 1: When $p \in \mathbb{Z}^2$, the range of $f$ is $\{0, 4, 12, 16, 24, 32, 36, \dots\}$; when $p$ is in the middle of an edge, the range of $f$ is $\{0, 2, 6, 8, 12, 16, 20, 22, 26, 34, 38, ..\}$; when $p$ has irrational coordinates, the range of $f$ is $\mathbb{N}$.

**Question.** What are all possible sequences for varying $p$?

**Related.**

1. What if $f'$ counts the number of vertices in a circle? Or if $f''$ counts the number of boxes that are *entirely* inside of a circle?
2. What happens on other lattices, tilings, or higher dimensional analogs?
3. How does this vary when the "circles" are generated by other metrics?

**References.**

Problem 30.

`https://codegolf.stackexchange.com/q/217444/53884`

# Problem 118.

Figure 1: For $(n, k) = (4, 2)$, it appears that there are $C(4) = 14$ valid patterns in six equivalence classes.

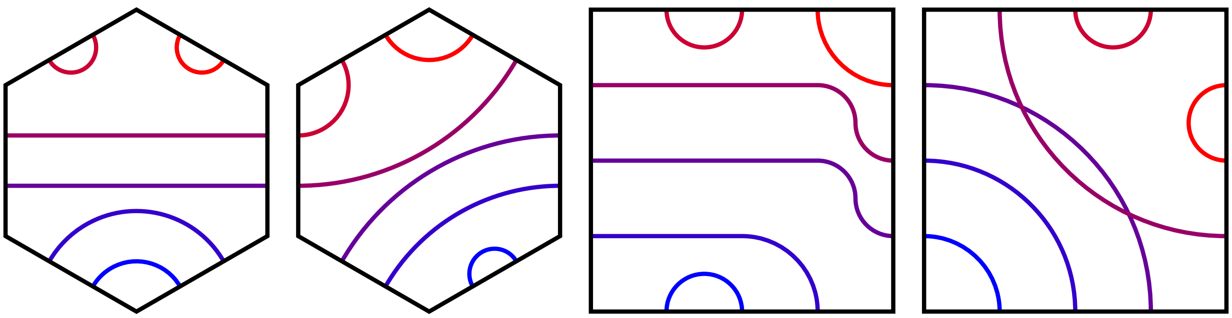


Figure 2: A valid $(6, 2)$-pattern does not always have a corresponding valid $(4, 3)$-pattern: the first square's pattern has curves that are not line segments or circular arcs, and the second square's pattern is self-overlapping.

**Question.** Given an $n$-gon with $k$ markings on each side, how much such patterns can be made using circular arcs and line segments such that each curve meets the boundary at a right angle?

**Note.** An $(n, k)$-pattern has or $C(nk/2)$ fewer realizations, where $C(m)$ is the $m$-th Catalan number.

**Related.**

1. How many $(n, k)$-patterns up to dihedral action of the $n$-gon?
2. For some fixed $k$, which values of $N$ allow for $C(Nk/2)$ $(N, k)$-patterns? If none, what are the obstructions?
3. How does this generalize to non-regular polygons or to higher dimensional polytopes?
4. What if curves other than circular arcs and line segments are allowed?

**References.**

Problems 28, 31, and 92.

# Problem 119.

A relation on a group $X$ is a subset $S \subseteq X \times X$. For a given relation if $(x, y) \in S$, then we say that $x$ is related to $y$ and denote it by $xRy$.
A relation is called "antitransitive" if $(x, y), (y, z) \in S$ implies $(x, z) \notin S$.

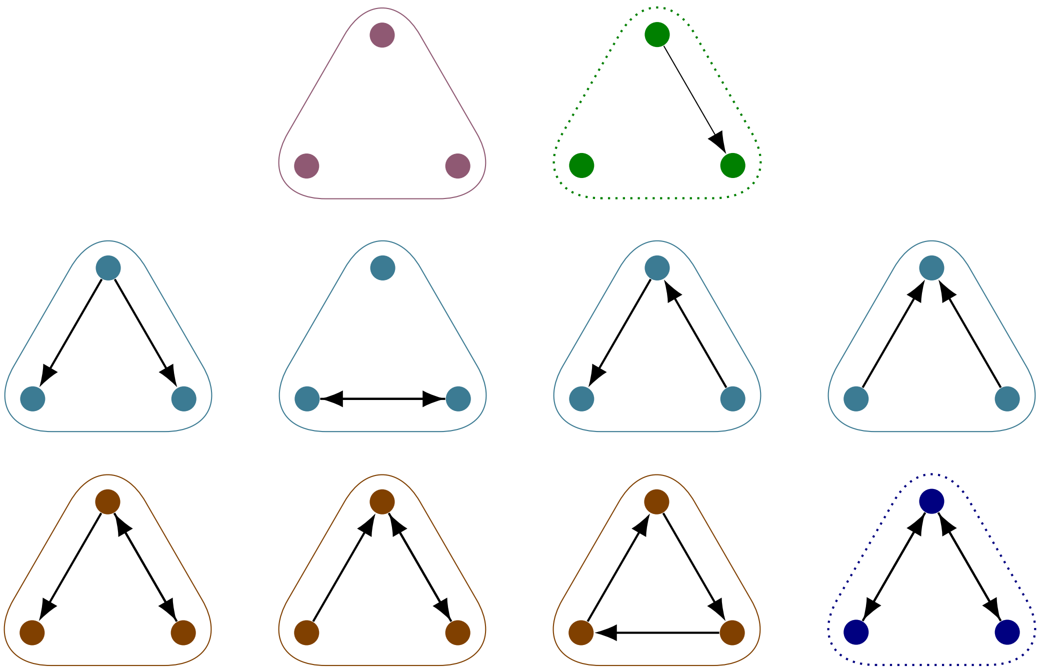


Figure 1: The ten antitransitive relations on 3 unlabeled nodes. There are 1, 1, 4, 3, and 1 relations with 0, 1, 2, 3, and 4 pairs respectively.

**Question.** What is the asymptotic growth of the number of antitransitive relations as a function of the number of (unlabeled) nodes?

**Related.**

1. On $n$ labeled nodes?
2. Given some subset of conditions (e.g. reflexive, asymmetric, antitransitive, connex, etc.), what is the asymptotic growth?
3. What's the ratio of the number of, say, transitive relations to antitransitive relations as $n \to \infty$.
4. How many relations with exactly $k$ pairs?
5. What's the greatest number of pairs?
6. With $\ell$ (strongly) connected components?

**References.**

Problem 39.

OEIS sequences A341471 and A341473.

# Problem 120.

A hyperbolic poly$_{\{p,k\}}$-form is a polyform on the hyperbolic plane that consists of $p$-gons on the tiling of the hyperbolic plane with Schläfli symbol $\{p, k\}$. A hyperbolic polyform with $n$ cells is called a hyperbolic $n_{\{p,k\}}$-form.

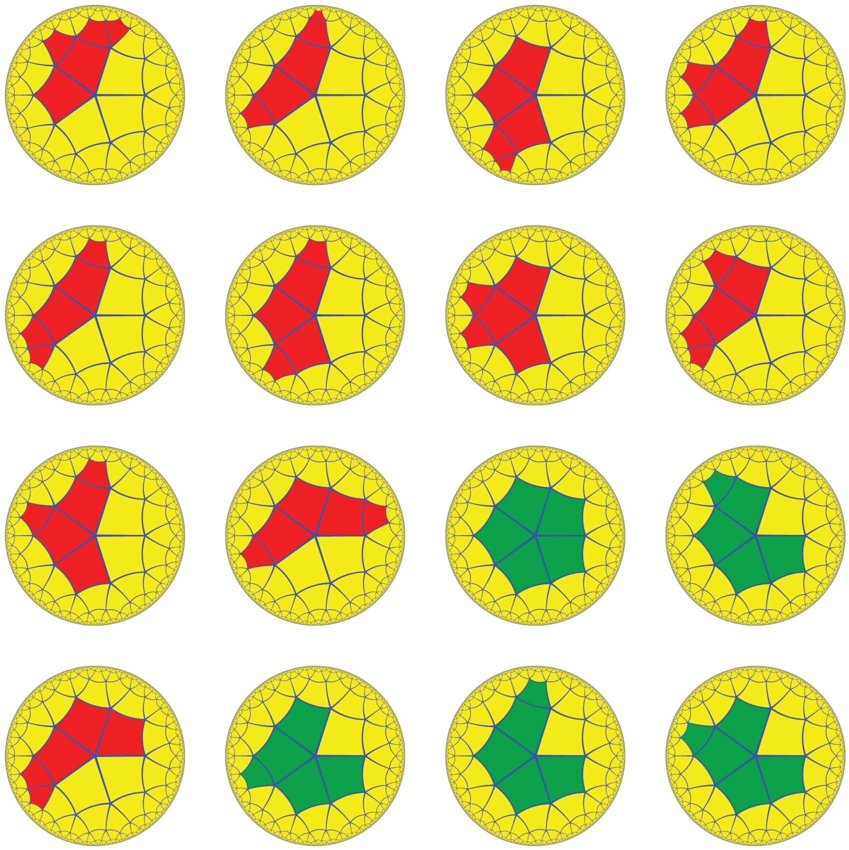


Figure 1: The $A119611(5)$ free pentominoes in the tessellation of the hyperbolic plane with Schläfli symbol $\{5, 4\}$.

**Question.** What is the asymptotic growth of $n_{\{p,k\}}$-form as a function of $n$, the number of cells?

**Related.**

1. Can this generalize to three or more dimensions the way polyominoes generalize to polycubes?
2. Is there a meaningful notion of fixed vs 1-sided poly$_{\{p,k\}}$-forms?

**References.**

Problems 71, 72, 77, 101, 103, 108, and 109.

Code Golf Stack Exchange: Counting polyominoes on the hyperbolic plane.

arXiv: Extremal $\{p, q\}$-Animals

# Problem 121.

In the game (tic-tac-toe)$^2$, each square is itself a smaller tic-tac-toe game; of course, one could imagine (tic-tac-toe)$^3$, where each of the squares in the smaller tic-tac-toe boards are themselves tic-tac-toe boards and so on. We're interested in counting (an abstraction of) the number of boards in this game, where two boards are considered the same if one is a rotation/reflection of another, or if any of the iterated games is a rotation/reflection of another, and so on.

In the simplest version, the boards are $2 \times 2$ and every square is filled in with an "X" or an "O", in perhaps unequal numbers.

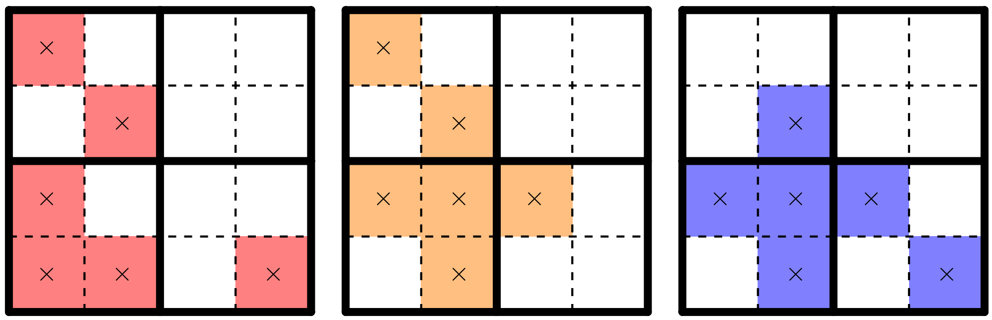


Figure 1: Three boards of depth 2 that are equivalent up to dihedral actions of both the smaller and larger boards.

**Question.** How many different boards of depth $n$ are there?

**Related.**

1. What if instead of just "X" and "O", there are more $k$ kinds of markers?
2. What if these are counted up to interchanging all "Xs" and "Os"?
3. What if done on nested $m \times m$ boards?
4. What if the biggest board is $m_1 \times m_1$, the second level of boards are $m_2 \times m_2$, the $i$-th level of boards are $m_i \times m_i$ and so on?
5. How does this work in higher dimensions? On other grids? On polygons on the vertices of polygons (on the vertices of polygons)?

**Note.** It appears that the group that acts on an ordinary board is $D_8$. The group that acts on a depth-2 board is $D_8 \wr D_8$, a depth-3 board is $(D_8 \wr D_8) \wr D_8$ and so on.

**References.**

Problems 31, 61, 79, 86, and 91.

Wikipedia, Ultimate tic-tac-toe.

# Problem 122.

OEIS sequence A169950 counts 0-1 polynomials, $f(x)$, by their *thickness*: the magnitude of the largest coefficient in the expansion of $f(x)^2$.

> Consider the $2^n$ monic polynomials $f(x)$ with coefficients 0 or 1 and degree $n$. Sequence gives triangle read by rows, in which $T(n,k)$ $(n \geq 0)$ is the number of such polynomials of thickness $k$ $(1 \leq k \leq n+1)$.

On April 19, 2021 my Twitter bot @OeisTriangles tweeted an image in which that the parity of this triangle resembled the The Sierpiński triangle, suggesting that there is a recursive structure in terms of the above rows.

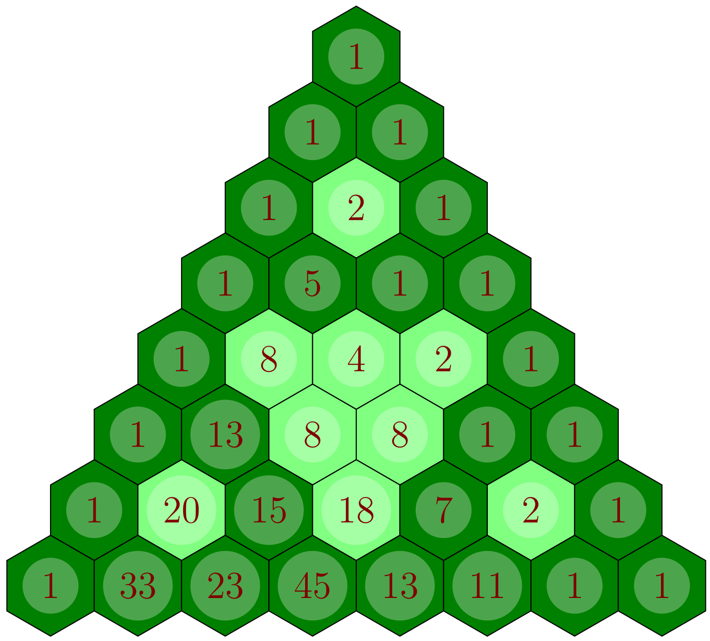



Figure 1: First eight rows of OEIS sequence A169950, where odd-valued cells are dark and even-valued cells are light.

**Question.** What is a recurrence for the values in this triangle?

**Related.**

1. What if $\{-1, 0, 1\}$-polynomials are considered?
2. What if the "$m$-thickness" is the largest coefficient when taken to the power $m$.
3. What if the sum of coefficients is considered?
4. How many 0-1 polynomials $f(x)$ have $\text{thickness}(f(x))+1 = \text{thickness}(xf(x)+1)$? $\text{thickness}(f(x))+2 = \text{thickness}(xf(x)+1)$ What is the asymptotic density of such 0-1 polynomials as a function of degree?

**References.**

David Speyer, Math Stack Exchange answer.

# Problem 123.

Polysticks can be used to model nets of a (not necessarily convex) polyhedron with square faces, by thinking of the vertices of the polystick as faces of a polycube and the edges of the polystick as Scotch tape connecting two faces at an edge.

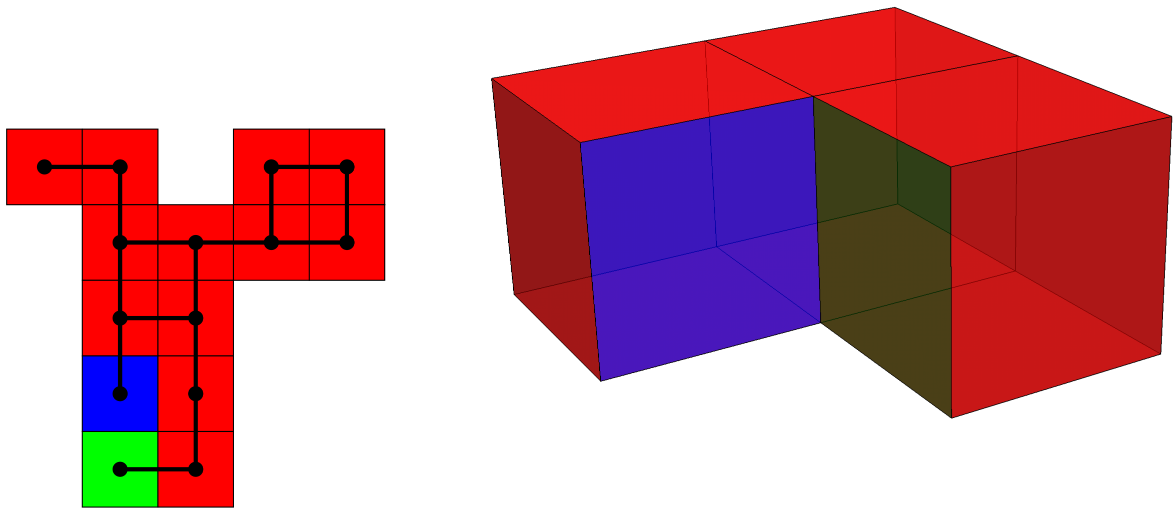


Figure 1: A polystick with 15 edges that models a net of a polycube.

**Question.** Which polysticks can be used to realize a polyhedron with square faces?

**Related.**

1. Is there a computationally efficient algorithm of determining whether a given polystick can be folded into a polyhedron?
2. Is there a computationally efficient algorithm of determining the number of ways a given polystick can be folded into a polyhedron?
3. How many polysticks can be folded into a rigid structure with no degrees of freedom?
4. What if we model polyhedra with triangular faces instead? Pentagonal?

**References.**

https://en.wikipedia.org/wiki/Polystick

# Problem 124.

A "mod-$n$ XOR-triangle" is a Pascal-like inverted triangle, where the first row consists of numbers in $\{0, 1, \ldots, n\}$, and the values in the cells of the subsequent rows are computed as the sum (mod $n$) of the numbers in the cells above them.

In the case of mod-3 XOR-triangles, if we additionally impose a constraint that the boundary must be rotationally symmetric, some of the resulting triangles appears to have emergent "central circles" reminiscent of the "Arctic circles" that appear in lozenge and domino tilings.

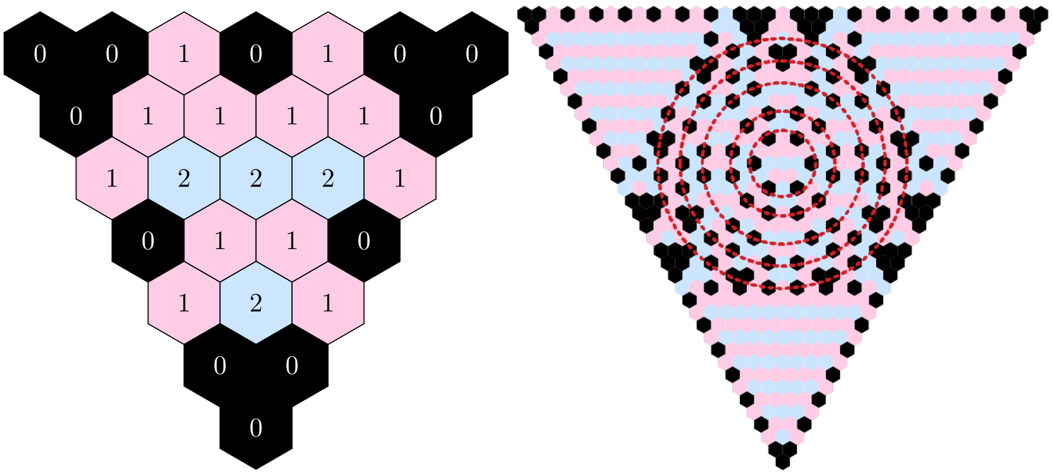



Figure 1: An illustration of the construction of a mod 3 sum triangle, and some candidates for central circles for an example of a triangle with 37 cells per side.

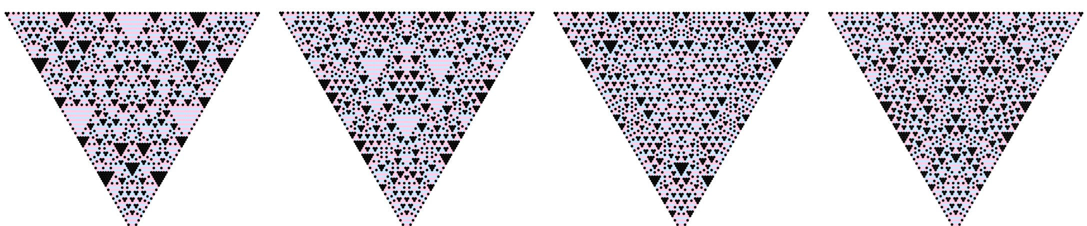


Figure 2: Four more examples of triangles that appear to have central circles

**Question.** Are these central circles in mod-3 XOR-triangles optical illusions or coincidences? If not, what's a mathematical explanation for why they exist?

**Related.**

1. What about other moduli?
2. Why does the boundary condition cause the black (0) cells to be symmetric with respect to the dihedral group of the triangle?
3. Can we create meaningful analogs where the shape is a square, tetrahedron, or another shape instead of a triangle?

**References.**

https://math.stackexchange.com/q/4088671/121988

# Problem 125.

Let $G$ be a finite group, and call a sequence $\{g_i \in G\}_{i=1}^{|G|-1}$ a *Hamiltonian walk* if for each $g \in G$ there exists some $i \geq 0$ such that the $n$-th partial product $p_n = g$ where $p_n = g_1 \cdot g_2 \cdots g_n$.

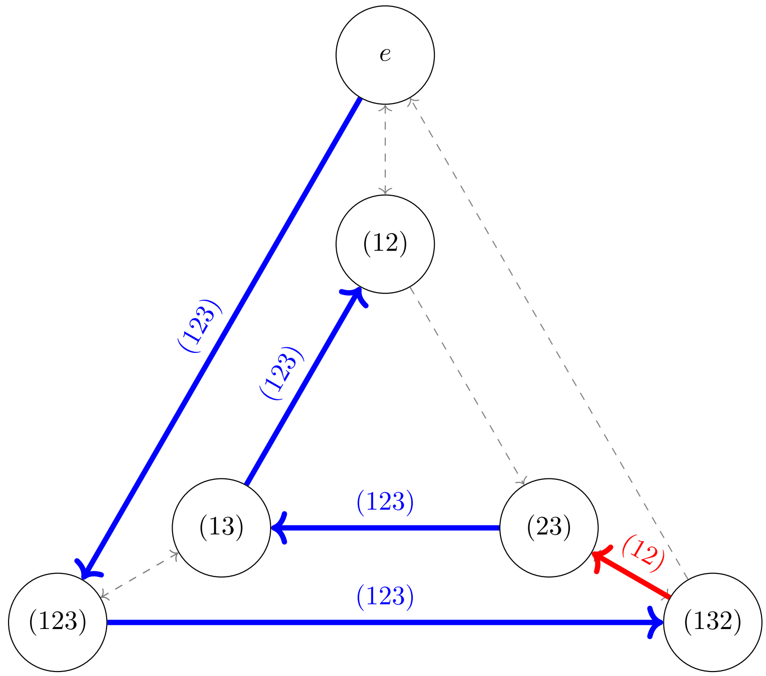



Figure 1: An example showing that the sequence $((123),(123),(12),(123),(123))$ is a palindromic Hamiltonian walk for $S_3$, where $p_0 = e$, $p_1 = (123)$, $p_2 = (132)$, $p_3 = (23)$, $p_4 = (13)$, and $p_5 = (12)$.

**Question.** Does every finite group have a Hamiltonian walk that is a palindrome?

**Related.**

1. If not, does every finite group have a Hamiltonian walk whose reversal is also a Hamiltonian walk?
2. Is there an efficient way to compute how many *essentially different* Hamiltonian walks $G$ has?
3. For a group $G$, what proportion of Hamiltonian walks are reversible?
4. Does every finite semigroup have a reversible Hamiltonian walk?

**References.**

Problem 79.

`https://math.stackexchange.com/q/3706654/121988`

# Problem 126.

Consider a spanning tree on the following graph chosen uniformly at random. If three "terminal" vertices are chosen from the tree, there exists a unique "critical" vertex such that every path between any two of the terminal vertices goes through the critical vertex.

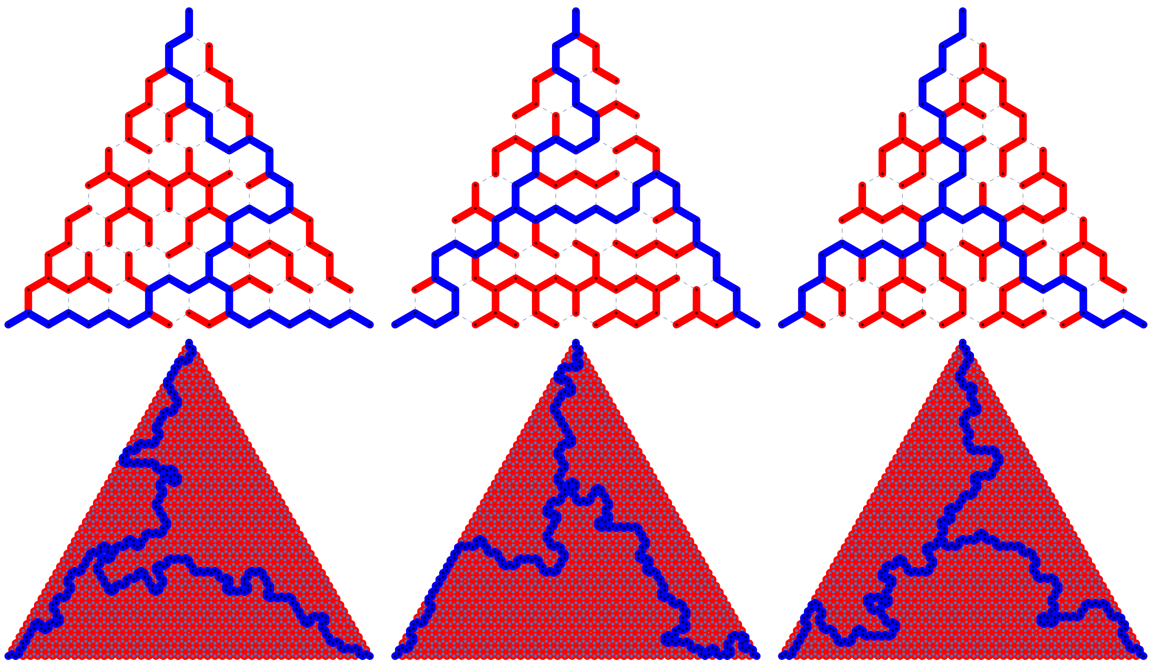


Figure 1: First six figures are for $n = 10$, next six are for $n = 50$.

**Question.** What is the distribution of the terminal vertices as a function of the triangle size?

**Related.**

1. If we draw the graphs in the way shown above, scaled so that the bottom is unit length, does the limit have a well-defined distribution?
2. What is the expected value of the total length of the blue path? The distribution?
3. What is the expected number of regions?
4. What is the expected ratio of the largest red region to smallest?

**References.**

https://oeis.org/A351888

# Problem 127.

Consider the two-player game on a $n$-letter string on the alphabet $\{a, b\}$ where players take turns removing palindromic substrings of their choosing.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | | | | |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | | | | | | |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | | | | | | | | | |
| 0 | 1 | 0 | 0 | 1 | 1 | | | | | | | | | | | | | | |
| 0 | 1 | | | | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | | | | | |

Figure 1: A seven move game: Player 1 erases red strings and Player 2 erases cyan strings. In this game, Player 1 won.

**Question.** How many $n$-letter games does Player 1 have a winning strategy?

**Related.**

1. What is a winning strategy?
2. If the game is chosen uniformly at random, what is the probability that the first player has
3. What if players take turns according to the Thue-Morse sequence?
4. What if players collect points based on the number of 1s they erase?
5. What if this is played on a larger alphabet?
6. What if instead of palindromes, players remove $AA$ subwords, $ABA$ subwords, or other patterns?
7. In a single-player version of the game, where the goal is to finish in as few moves a possible, which $n$-letter games require the most moves?

**References.**

Problem 3.

https://oeis.org/A298475

https://oeis.org/A298481