Consider ways to draw diagonals on the cells of $n \times n$ toriodal grid such that no two diagonals touch.

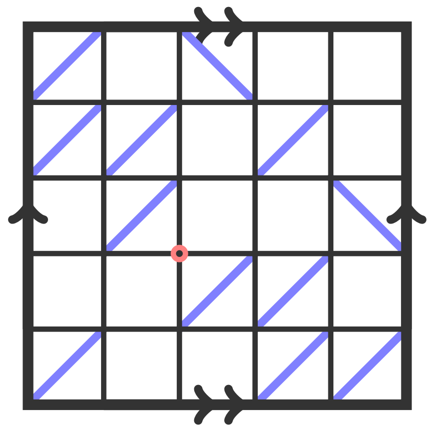

Figure 1: A maximal configuration of a 5×5 toroidal grid: 12 diagonal lines can be drawn. The unused vertex is marked with a circle.

Question. What is the greatest number of diagonals that can be drawn on a $n \times n$ toroidal grid?

Note. Let $m(n)$ be the maximum number of diagonals on an $n \times n$, grid.
Then

$$\begin{aligned} m(2n) &= 2n^2 \text{ and} \\ 2n^2 + n \leq m(2n+1) &\leq 2n^2 + 2n. \end{aligned}$$

Related.

1. How many configurations exist for a given grid size up to group action?
2. What if this is done on an $n \times m$ grid?
3. What if this is done on a cylinder, Klein bottle, projective space, etc?
4. What is the maximum number of diagonals that can go SW to NE?
5. Does this generalize to three or more dimensions?
6. Can something similar be done on a hexagonal or triangular grid?
7. How many configurations exist if touching is allowed, but cycles aren't?
8. What if we color edges of the grid rather than diagonals on the faces?

References.

`https://oeis.org/A264041`

`https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/slant.html`