Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way that no end is "orphaned".

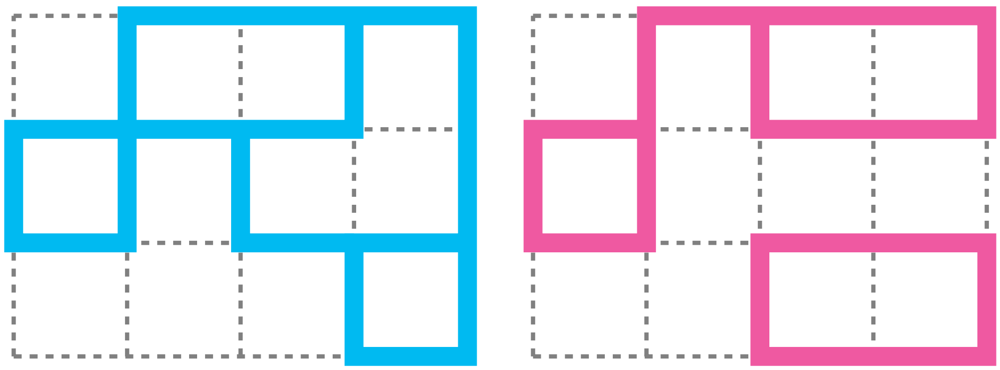

Figure 1: Two examples of a valid configurations on a 5×4 grid; the second is not connected.

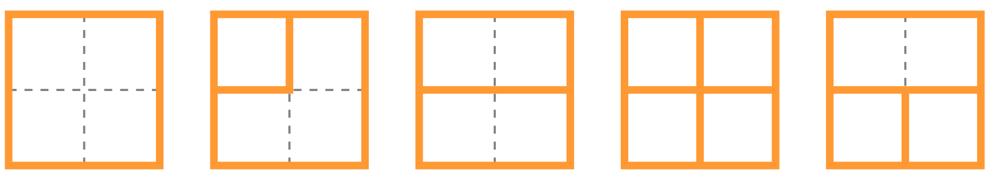

Figure 2: All(?) examples of valid configurations of 3×3 grids with border, up to dihedral action.

Question. Let $a_\ell(n)$ be the number of configurations on the $\ell \times n$ grid. What is a general formula for $a_\ell(n)$?

Related.

1. What if the matchsticks are of length k? Of $\{k_1, k_2, \dots, k_\ell\}$?
2. How does this generalize to a triangular/hexagonal lattice or to multiple dimensions? On the king graph? On the multipartite graph $K_{m,m,\dots,m}$?
3. What is the number of these configurations with rotational symmetry? Horizontal/vertical symmetry?
4. If such a configuration is chosen uniformly at random, what is the number of expected regions? (e.g. the first example has 4 interior regions.)
5. What if no gridpoint can have degree 0? Degree 2? 3? 4?
6. What if the entire border must be drawn?
7. What if the subgraph must be connected?
8. What if instead of horizontal/vertical lines, diagonals are allowed? All edges have integer slope? Edges don't intersect except at vertices?
9. How many k-ominoes fit in a "tube" of height m? Snuggly?

References.

A093129 ($2 \times n$), A301976 ($3 \times n$), A320097 ($4 \times n$), A320099 ($5 \times n$), A303930 ($2 \times n$ up to symmetry).

`http://mathworld.wolfram.com/KingGraph.html`