Consider polyforms formed by facets of an n-dimensional hypercube. If such a polyform has k cells, call it a k-polyfacet. Count these up to symmetries of the cube.

Figure 1: On the left, the two 3-polyfacets on the cube, and on the right, the two 4-polyfacets on the cube. The 0-, 1-, 2-, 5-, and 6-polyfacets are unique on the cube.

Question. How many k-polyfacets live on the n-cube?

Note. The following table gives the number of k-polyfacets on an n-cube:

```
n\k| 0  1  2                        ...                         2n-1 2n|
---+-------------------------------------------------------------------+
 2 | 1, 1, 1,                                                     1, 1 |
 3 | 1, 1, 1, 2,                                               2, 1, 1 |
 4 | 1, 1, 1, 2, 3,                                         2, 2, 1, 1 |
 5 | 1, 1, 1, 2, 3, 3,                                   3, 2, 2, 1, 1 |
 6 | 1, 1, 1, 2, 3, 3, 4,                             3, 3, 2, 2, 1, 1 |
 7 | 1, 1, 1, 2, 3, 3, 4, 4,                       4, 3, 3, 2, 2, 1, 1 |
 8 | 1, 1, 1, 2, 3, 3, 4, 4, 5,                 4, 4, 3, 3, 2, 2, 1, 1 |
 9 | 1, 1, 1, 2, 3, 3, 4, 4, 5, 5,           5, 4, 4, 3, 3, 2, 2, 1, 1 |
10 | 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6,     5, 5, 4, 4, 3, 3, 2, 2, 1, 1 |
11 | 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1 |
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Notice that $T(n,k) = T(n, n-k)$ for all $k \notin \{2, n-2\}$. In this case, $T(n,2) = 1$ and $T(n, n-2) = 2$.

Related.

1. How many d-dimensional k-poly-d-faces live on the n-cube?
2. How many d-dimensional k-poly-d-faces live on the n-simplex?
3. How many d-dimensional k-poly-d-faces live on the n-orthoplex?
4. How many k-polyfacets live on the n-demicube?
5. How many fixed polyforms? One-sided polyforms?
6. If we chop up the hypercube into an $\ell \times \cdots \times \ell$ "Rubik's" hypercube, how many polyfacets live on this subdivision?
7. Let $T(n,k)$ denote the k-polyfacets on an n-cube. Which of the $T(n,k)$ polyfacets has the most symmetry? The least?

References.

Problems 72, 101.