Difficulty: 3/4   Interest: 1/4

Call s an "initial permutable" string if for every initial substring of odd length, the first half of the string is a permutation of the second half.

$$1\ 0\ 1\ 0\ 1\ 1\ 0 \rightarrow 1\ 0\ 1\ 0\ \sim 0\ 1\ 1\ 0$$
$$1\ 0\ 1\ 0\ 1 \rightarrow 1\ 0\ 1 \sim 0\ 1\ 0$$
$$1\ 0\ 1 \rightarrow 1\ 0 \sim 0\ 1$$
$$1 \rightarrow 1 \sim 1$$

Figure 1: "1010110" is an example of an initial permutable string. Because each initial odd substring (the string itself, "10101", "101", and "1") has the property that the first half of the string is a rearrangement of second half.

Question. What is the growth of $a_2(n)$, the number of initial permutable strings of length $2n - 1$ over a 2-letter alphabet?

Related.

1. Can this be generalized to a k-letter alphabet?

References.

http://oeis.org/A297789