Choose a point p in $\mathbb{R}^2$, and consider all balls $\mathcal{B}_p(r)$ of radius r centered at p. Let $f(\mathcal{B}_p(r))$ be the number of "boxes" of $\mathbb{Z}^2$ that are partly contained in the interior of a ball.

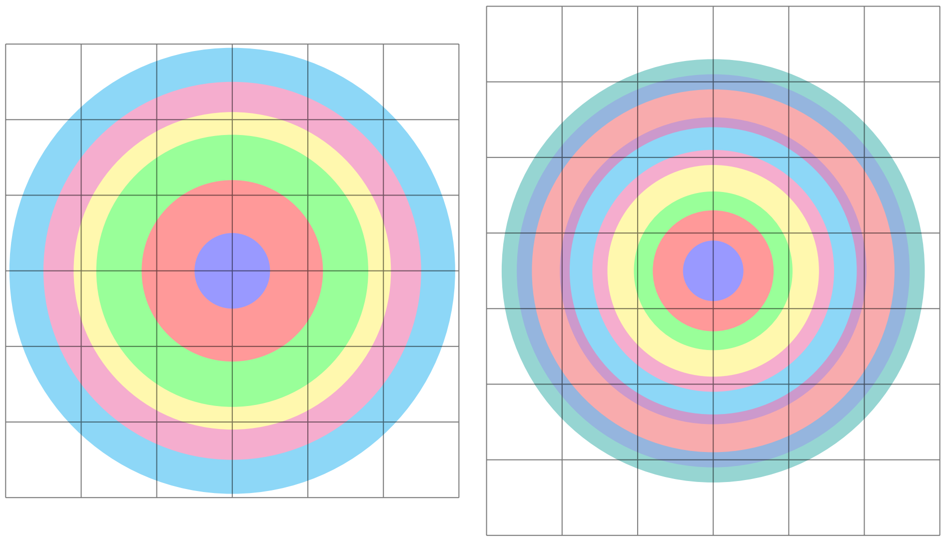

Figure 1: When $p \in \mathbb{Z}^2$, the range of f is $\{0, 4, 12, 16, 24, 32, 36, \dots\}$; when p is in the middle of an edge, the range of f is $\{0, 2, 6, 8, 12, 16, 20, 22, 26, 34, 38, ..\}$; when p has irrational coordinates, the range of f is $\mathbb{N}$.

Question. What are all possible sequences for varying p?

Related.

1. What if f' counts the number of vertices in a circle? Or if f'' counts the number of boxes that are *entirely* inside of a circle?
2. What happens on other lattices, tilings, or higher dimensional analogs?
3. How does this vary when the "circles" are generated by other metrics?

References.

Problem 30.