Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way that no end is "orphaned".

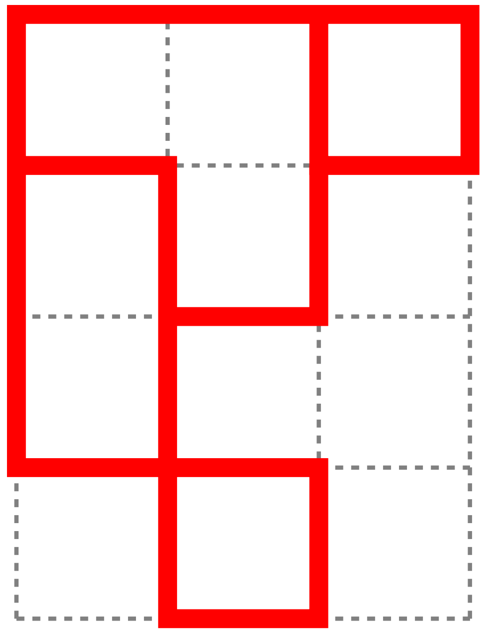

Figure 1: An example of a valid configuration on a 3×4 grid.

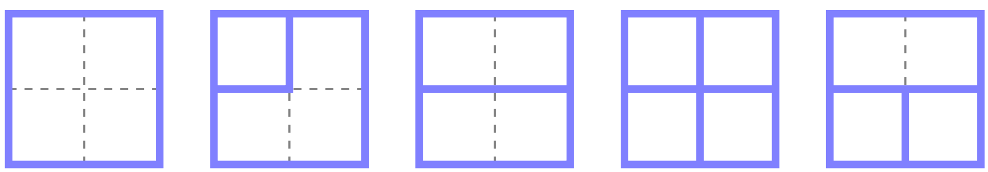

Figure 2: All(?) examples of valid configurations of 2×2 grids with border, up to dihedral action.

Question. Let $a_\ell(n)$ be the number of configurations on the $\ell \times n$ grid. What is a general formula for $a_\ell(n)$?

Related.

1. What if the matchsticks are of length k?
2. How does this generalize to a triangular/hexagonal lattice or to multiple dimensions?
3. What is the number of these configurations with rotational symmetry? Horizontal/vertical symmetry?
4. If such a configuration is chosen uniformly at random, what is the number of expected regions? (e.g. the first example has 4 interior regions.)