Problem 1.

★★★★

Suppose you are given an $n \times m$ grid, and I then think of a rectangle with its corners on grid points. I then ask you to "black out" as many of the gridpoints as possible, in such a way that you can still guess my rectangle after I tell you all of the non-blacked out vertices that its corners lie on.

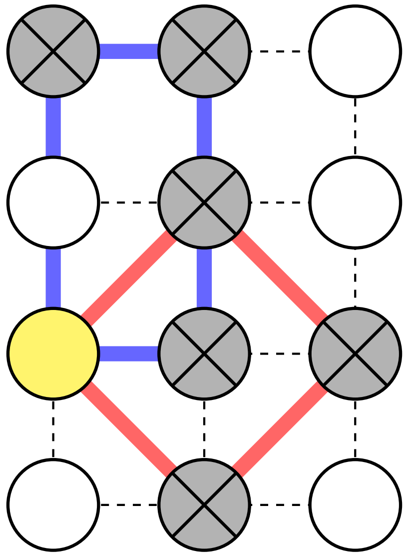


Figure 1: An example of an invalid "black out" for an 4×3 grid. The blue rectangle and the red rectangle have the same presentation, namely the gridpoint inside the yellow circle.

Question. How many vertices may be crossed out such that every rectangle can still be uniquely identified?

Related.

1. What if the interior of the rectangle is lit up instead?
2. What if all gridpoints that instersect the perimeter are lit up?
3. What if the rectangles must be square?
4. What if parallelograms are used instead of rectangles?
5. What if the rectangles must be horizontal, vertical, or 45° diagonal?
6. What if this is done on a triangular grid with equilateral triangles?
7. What if this is done in more dimensions (e.g. with a rectangular prism or tetrahedron?)

References.

`https://math.stackexchange.com/q/2465571/121988`

Problem 2.

Let G be some $n \times m$ grid as in Figure 1, where each cell has two opposite diagonals connected (uniformly at random). A cell is chosen (also uniformly at random), and the segment given by the path of diagonals that goes through the selected cell is is inspected.

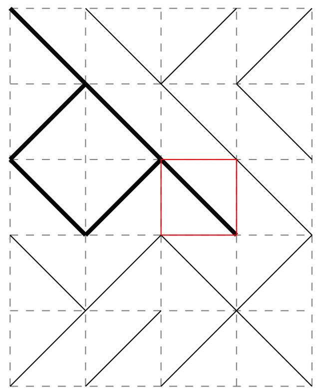


Figure 1: An example of a 4×5 grid, where a segment of size 6 has been selected.

Question. What is the expected length of the selected segment?

Related.

1. What is the expected number of segments in an $n \times m$ grid?
2. How long is the longest segment expected to be?
3. How does this change if the grid is toroidal, on a cylinder, on a Möbius strip, etc?

Problem 3.

Peter Winkler's Coins-in-a-Row game works as following:

> On a table is a row of fifty coins, of various denominations. Alice picks a coin from one of the ends and puts it in her pocket; then Bob chooses a coin from one of the (remaining) ends, and the alternation continues until Bob pockets the last coin.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed according to some probability distribution.

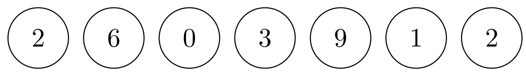


Figure 1: An instance of a seven coin game on a uniform distribution of $\{0, 1, \ldots, 9\}$. The first player has a strategy that allows her to win by one point.

Question. For some fixed ω, what is the expected first player's score of Peter Winkler's Coins-in-a-Row game when played with $X_1(\omega), X_2(\omega), \ldots, X_3(\omega)$ where both players are using a min-max strategy?

Note. Let

$$e = E[X_2 + X_4 + \ldots + X_{2n}] \text{ and } o = E[X_1 + X_2 + \ldots + X_{2n-1}]$$

When played with $2n$ coins, the first player's score is bounded below by $\max(e, o) - \min(e, o)$ by the strategy outlined by Peter Winkler.

Trivially the first player's score is bounded above by the expected value of the n largest coins minus the expected value of the n smallest coins.

Related.

1. If all possible n-coin games are played with coins marked 0 and 1, how many games exist where both players have a strategy to tie.
2. How does this change when played according to the (fair) Thue-Morse sequence?
3. What if the players are cooperating to help the first player make as much as possible (with perfect logic)?
4. What is both players are using the greedy algorithm?
5. What if one player uses the greedy algorithm and the other uses min-max? (i.e. What is the expected value of the score improvement when using the min-max strategy?)
6. What if one player selects a coin uniformly at random, and the other player uses one of the above strategies?

Problem 4.

Let a “popsicle stick weave” be a configuration of lines segments, called “sticks”, such that

(1) when you lift up any stick by the end, the structure supports itself (is in tension)

(2) the removal of any stick results in a configuration that no longer supports itself.

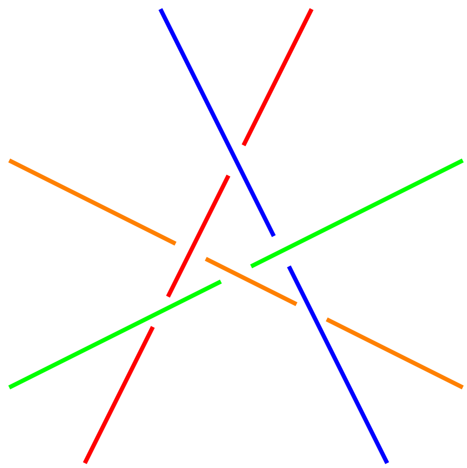


Figure 1: The unique example of a 4 stick crossing (up to reflection)

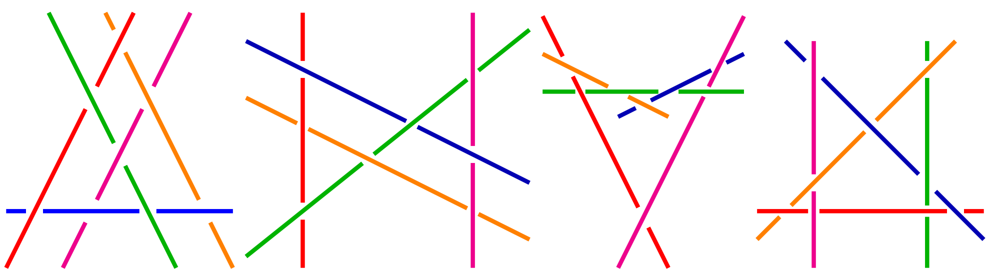


Figure 2: Four of five (?) known examples of five-stick crossings. Perhaps the fourth example shouldn’t count, because shortening the blue stick to avoid the blue-red crossing results in a valid configuration (the remaining known five-stick crossing).

Question. How many distinct popsicle stick weaves exist for n sticks?

Related.

1. What if the sticks are only allowed to touch three other sticks?
2. What if the sticks are another geometric object (e.g. semicircles)?

Problem 5.

Let

$$C_n = \{f : [n] \to \mathbb{N} \mid \text{the convex hull around } \{(1, f(1)), \dots, (n, f(n))\} \text{ forms an } n\text{-gon}\}$$

and then let $a(n)$ denote the least upper bound over all functions in C_n

$$a(n) = \min\{\max\{f(k) \mid k \in [n]\} \mid f \in C_n\}$$

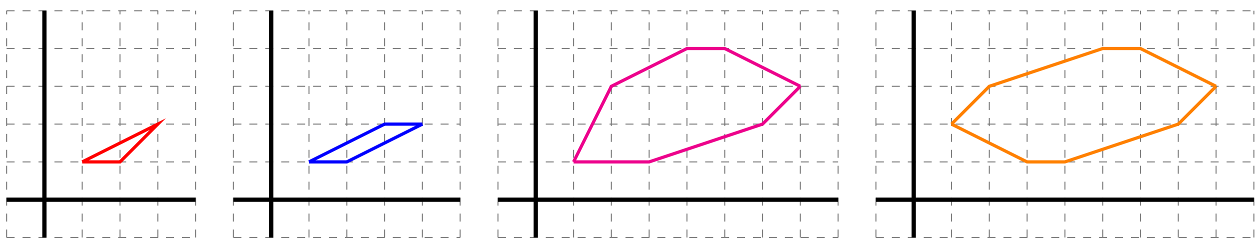


Figure 1: Examples of $a(3) = 2$, $a(4) = 2$, $a(7) = 4$, and $a(8) = 4$, where the polygons with an even number of vertices have rotational symmetry.

Question. Do these polygons converge to something asymptotically?

Related.

1. Does $a(2n) = a(2n - 1)$ for all n?
2. Do the minimal $2n$-gons always have a representative with rotational symmetry?
3. Are minimal $2n$-gons unique (up to vertical symmetry) with finitely many counterexamples?
4. What is the asymptotic growth of a(n)?

References.

A285521: “Table read by rows: the n-th row gives the lexicographically earliest sequence of length n such that the convex hull of (1, a(1)), ..., (n, a(n)) is an n-gon with minimum height.” (https://oeis.org/A285521)

Problem 6.

Let $f_{n,m} \colon [n] \to [m]$ be a uniformly random function. Consider the convex hull around $\{(1, f(1)), \ldots (n, f(n))\}$

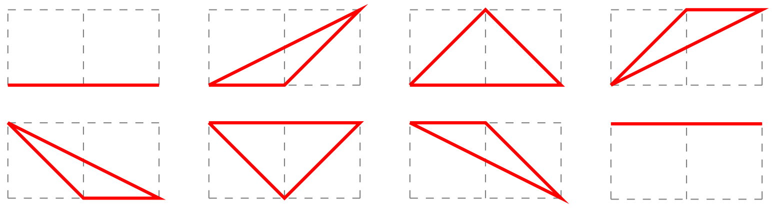


Figure 1: Examples of $f_{3,2}$. Here the expected number of sides on a convex hull is 2.75

Question. What is the probability of seeing a k-gon (for some fixed k), when given a uniformly random function $f_{n,m}$?

Related.

1. What value of k has the highest probability?
2. What is the expected value of the number of sides?
3. What if $f_{n,n}$ is restricted to be a permutation?
4. What if $f_{n,m}$ is injective?

Problem 7.

Given an $n \times n$ grid, consider all convex polygons with grid points as vertices. Let $m(n)$ be the greatest integer k such that there exists a convex k-gon on the $n \times n$ grid.

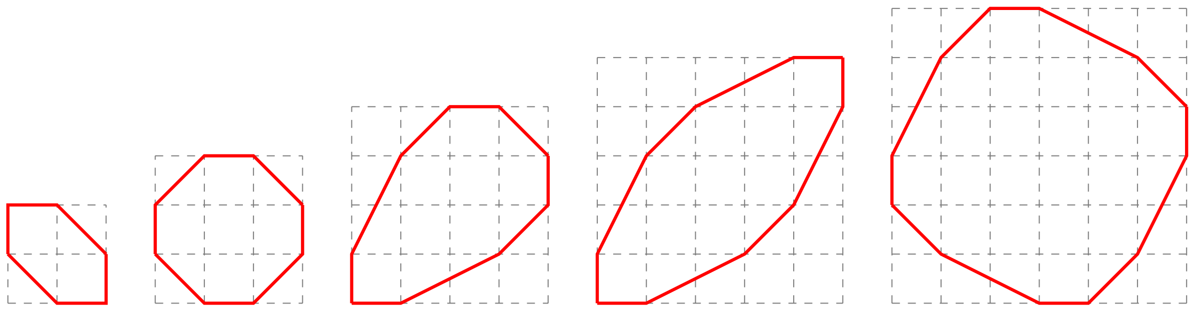


Figure 1: Examples that prove $m(3) = 6, m(4) = 8, m(5) \geq 9, m(6) \geq 10$, and $m(7) \geq (12)$

Question. What is $m(n)$?

Related.

1. What is a proof (or counterexample) that the examples shown are the best possible?
2. How does $m(n)$ grow asymptotically?
3. Do the shapes do anything interesting in the limit?
4. Are there finitely many maximal polygons without rotational symmetry (e.g. $m(5)$)?
5. How does this generalize to $m \times n$ grids?

References.

Problem 5.

Problem 6.

Problem 8.

Given an $n \times n$ grid, consider all the ways that convex polygons with grid points as vertices can be nested.

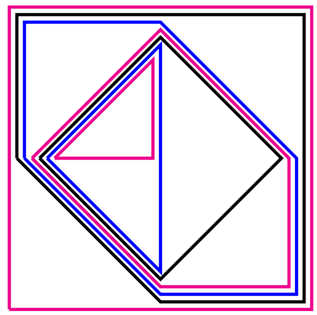


Figure 1: Seven nested convex polygons in the 3×3 grid.

Question. If we think of each polygon having the same height, what is the greatest volume that we can make by stacking the polygons this way?

Related.

1. What is the largest sum of the perimeters? The least?
2. What is the largest sum of the number of vertices? The least?
3. How many ways are there to stack $n^2 - 2$ polygons like this? Any number of polygons?
4. Does this generalize to polyhedra in the $n \times n \times n$ cube?
5. Does this generalize to polygons on a triangular grid?

Problem 9.

Consider all k-colorings of an $n \times n$ grid, where each row and column has $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ cells with each color.

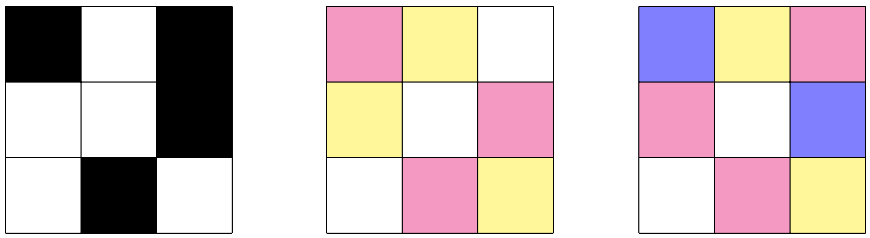


Figure 1: A valid 2-coloring, 3-coloring, and 4-coloring of an 3×3 grid.

Question. How many such k-colorings of the $n \times n$ grid?

Related.

1. What if there also must be a total of $\lfloor n^2/k \rfloor$ or $\lceil n^2/k \rceil$ cells of each color?
2. What if these are counted up to the dihedral action on the square D_4?
3. What if these are counted up to torus action?
4. What if these are counted up to permutation of the coloring?
5. Can this generalize to the cube? To a triangular tiling?

Problem 10.

Consider Ron Graham's sequence for LCM, that is, look at sequences such that

$$n = b_1 < b_2 < \ldots < b_t = k \text{ and } \mathrm{LCM}(b_1, \ldots, b_t) \text{ is square.}$$

Question. Let $A300516(n)$ be the least k (as a function of n) such that such a sequence exists?

$a(1) = 1$ via (1)	$a(11) = 121$ via $(11, 121)$	$a(21) = 49$ via $(21, 36, 49)$
$a(2) = 4$ via $(2, 4)$	$a(12) = 18$ via $(12, 18)$	$a(22) = 121$ via $(22, 64, 121)$
$a(3) = 3$ via $(3, 9)$	$a(13) = 169$ via $(13, 169)$	$a(23) = 529$ via $(23, 529)$
$a(4) = 4$ via (4)	$a(14) = 49$ via $(14, 16, 49)$	$a(24) = 48$ via $(24, 36, 48)$
$a(5) = 25$ via $(5, 25)$	$a(15) = 25$ via $(15, 16, 18, 25)$	$a(25) = 25$ via (25)
$a(6) = 12$ via $(6, 9, 12)$	$a(16) = 16$ via (16)	$a(26) = 169$ via $(26, 64, 169)$
$a(7) = 49$ via $(7, 49)$	$a(17) = 289$ via $(17, 289)$	$a(27) = 81$ via $(27, 81)$
$a(8) = 16$ via $(8, 16)$	$a(18) = 25$ via $(18, 20, 25)$	$a(28) = 49$ via $(28, 49)$
$a(9) = 9$ via (9)	$a(19) = 361$ via $(19, 361)$	$a(29) = 841$ via $(29, 841)$
$a(10) = 25$ via $(10, 16, 25)$	$a(20) = 25$ via $(20, 25)$	$a(30) = 50$ via (30)

Figure 1: Examples of $A300516(n)$ for $1 \leq n \leq 30$.

Related.

1. For what values n is $A300516(n)$ nonsquare?
2. For what values n does the corresponding sequence have three or more terms?
3. What is the analogous sequence for perfect cubes, etc?

References.

https://oeis.org/A300516

Problem 11.

Ron Graham's Sequence (A006255) is the least k for which there exists a strictly increasing sequence

$$n = b_1 < b_2 < \ldots < b_t = k \text{ where } b_1 \cdot \ldots \cdot b_t \text{ is square.}$$

There is a known way to efficiently compute analogous functions a_p where $a_p(n)$ is the least integer such that there exists a sequence

(a) $n = b_1 \leq b_2 \leq \ldots \leq b_t = a_p(n)$,

(b) any term appears at most $p - 1$ times, and

(c) $b_1 \cdot b_2 \cdot \ldots \cdot b_t$ is a p-th power.

Question. An efficient way to compute a_p is known when p is prime. What is an efficient way to compute a_c when c is composite?

$$\begin{aligned}
a_4(1) &= 1 \text{ via } 1 && = 1^4\\
a_4(2) &= 2 \text{ via } 2^2 \cdot 4 && = 2^4\\
a_4(3) &= 6 \text{ via } 3^2 \cdot 4 \cdot 6^2 && = 6^4\\
a_4(4) &= 4 \text{ via } 4^2 && = 2^4\\
a_4(5) &= 10 \text{ via } 5^2 \cdot 8^2 \cdot 10^2 && = 20^4\\
a_4(6) &= 9 \text{ via } 6^2 \cdot 8^2 \cdot 9 && = 12^4\\
a_4(7) &= 14 \text{ via } 7^2 \cdot 8^2 \cdot 14^2 && = 28^4\\
a_4(8) &\leq 15 \text{ via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 && = 60^4\\
a_4(9) &= 9 \text{ via } 9^2 && = 3^4\\
a_4(10) &\leq 18 \text{ via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 && = 180^4
\end{aligned}$$

Figure 1: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

Related.

1. For what values n is $a_4(n) < A006255(n)$?
2. Given some integers k, c, how many terms have $a_c(n) = k$? (e.g. $a_4(6) = a_4(9) = 9$.)
3. Does a_c contain arbitrarily many copies of the same value?
   (i.e. does there exist a sequence such that $a_4(n_1) = a_4(n_2) = \ldots = a_4(n_m)$ for arbitrarily large m?)
4. How many times does k appear in the image of a_c? (e.g. 9 appears twice, as $a(6)$ and $a(9)$.)
5. What integers are in the image of a_c?

References.

`https://oeis.org/A300516`

Problem 12.

Suppose you have a strip of toilet paper with n pieces, and you fold the paper evenly into d parts (divide by d) or fold the last k pieces in (subtract by k), until the length of the strip is less than k pieces.

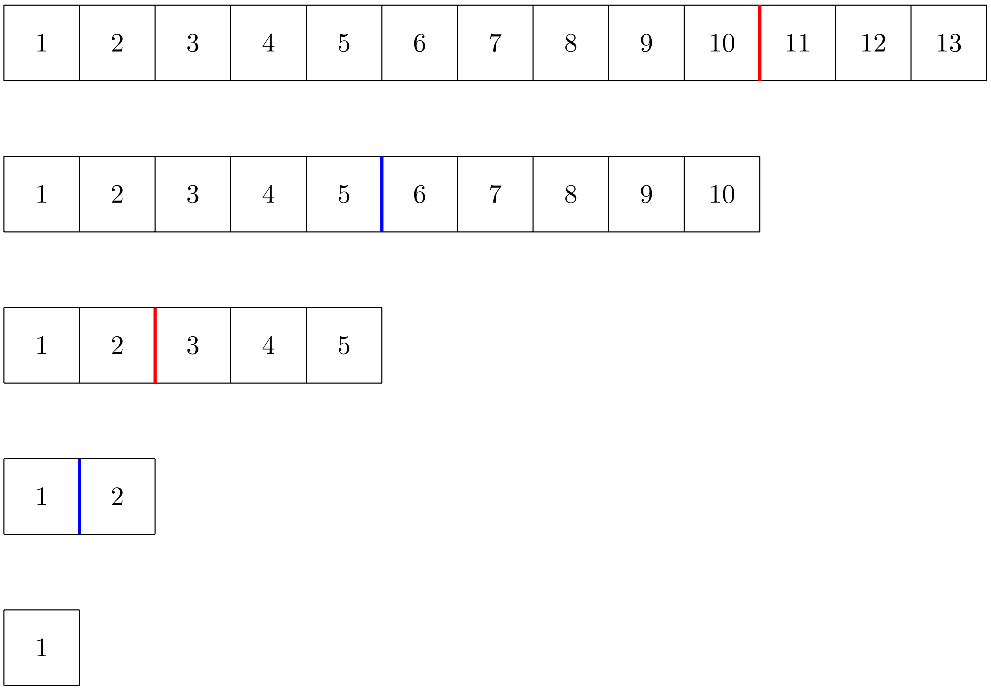


Figure 1: A folding of paper where $n = 13$, $d = 2$, and $k = 3$, showing that $a_{2,3}(13) \leq 4$. Where the red marks a subtraction by k and the blue marks a division by d.

Question. Is there an efficient way to compute $a_{d,k}(n)$?

Related.

1. What if you must keep folding until you cannot fold any longer?
2. What is the minimum number of terminal pieces? What is the minimum number of steps to this number?

References.

`https://oeis.org/A014701`

Problem 13.

OEIS sequence A261865 describes "$a(n)$ is the least $k \in \mathbb{N}$ such that some multiple of $\sqrt{k} \in (n, n+1)$." Clearly the asymptotic density of 2 in the sequence is $1/\sqrt{2}$.

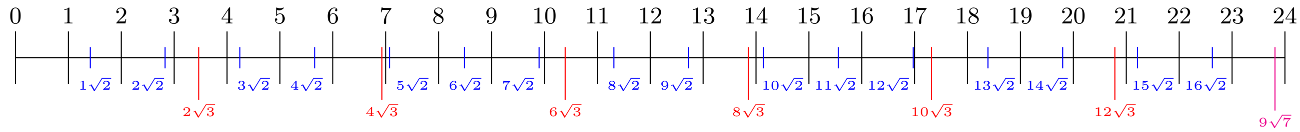


Figure 1: An illustration of $a(n)$ for $n \in \{1, 2, \ldots, 23\}$.

Question. Let $S_\alpha \subset \mathbb{N}$ denote the squarefree integers strictly less than α.
Is the asymptotic density of squarefree j given by

$$\frac{1}{\sqrt{j}} \prod_{s \in S_j} \left(1 - \frac{1}{\sqrt{s}}\right)?$$

Related.

1. Is there an algorithm to construct a value of n such that $a(n) > K$ for any specified K? (Perhaps using best Diophantine approximations or something?)
2. What is the asymptotic growth of the records?
3. Given some α what is the expected value of the smallest n such that $S_\alpha \subset \{a(1), \ldots, a(n)\}$?
4. This sequence uses the "base sequence" of $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\}$. On what other base sequences is this construction interesting?
5. What is the smallest $m \in \mathbb{N}$ such that $k2^{1/m} \in (n, n+1)$ for some $k \in \mathbb{N}$?
6. What is the smallest $k \in \mathbb{N}$ such that $k2^{1/m} \in (n, n+1)$ for some $m \in \mathbb{N}$?

References.

https://oeis.org/A261865

Problem 14.

Start with n piles with a single stone in each pile. If two piles have the same number of stones, then any number of stones can be moved between them.

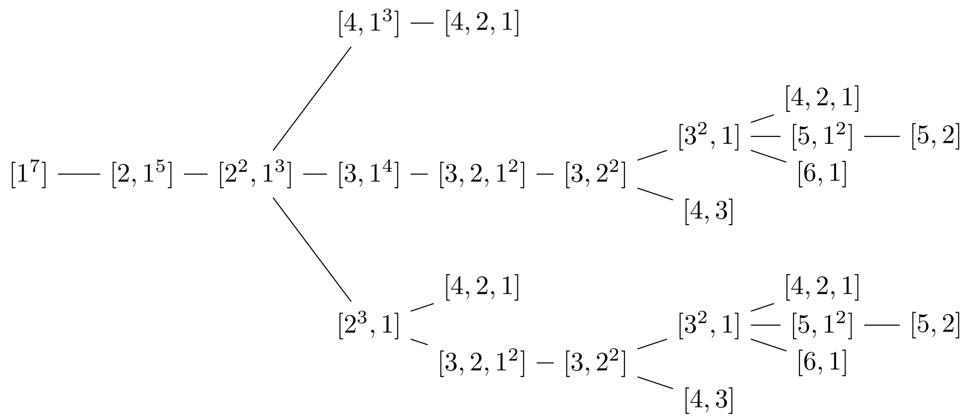


Figure 1: An illustration of all possible moves for $n = 7$.

Question. What is the greatest number of steps that can occur? Alternatively how many "levels" are in the tree of possible moves?

Related.

1. Let $A292726$ be the total number of distinct states. What is $A292726$? (e.g. $A292726(7) = 14$.)
2. Let $c = A000041 - A292726$ be the total number of states that *cannot* be acheived. (e.g. $c(5) = 1$ via the state $[5]$.)
3. Is $c(p) = 1$ for all primes p? Is $c(n) = 0$ if and only if n is a power of 2?
4. Let $\ell = A292836$ be the least number of steps to a terminal state. (e.g. $\ell(7) = 4$ ending in $[4,2,1]$.)
5. Let $g = A292729$ be the greatest number of steps to a terminal state. (e.g. $g(7) = 8$ ending in $[5,2]$.)
6. Let p be the total number of paths, i.e. the number of leaves in the tree. (e.g. $p(7) = 10$.)
7. Let t be the number of distinct *terminal* states. (e.g, $t(7) = 4$ via $[4,2,1]$, $[4,2]$, $[6,1]$, and $[4,3]$.)
8. What if you can move stones between any sets of piles that share the same number of stones? (e.g. $[2^3] \to [6]$ or $[2^3] \to [4,1,1]$)

References.

`https://oeis.org/A292836`

Problem 15.

Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

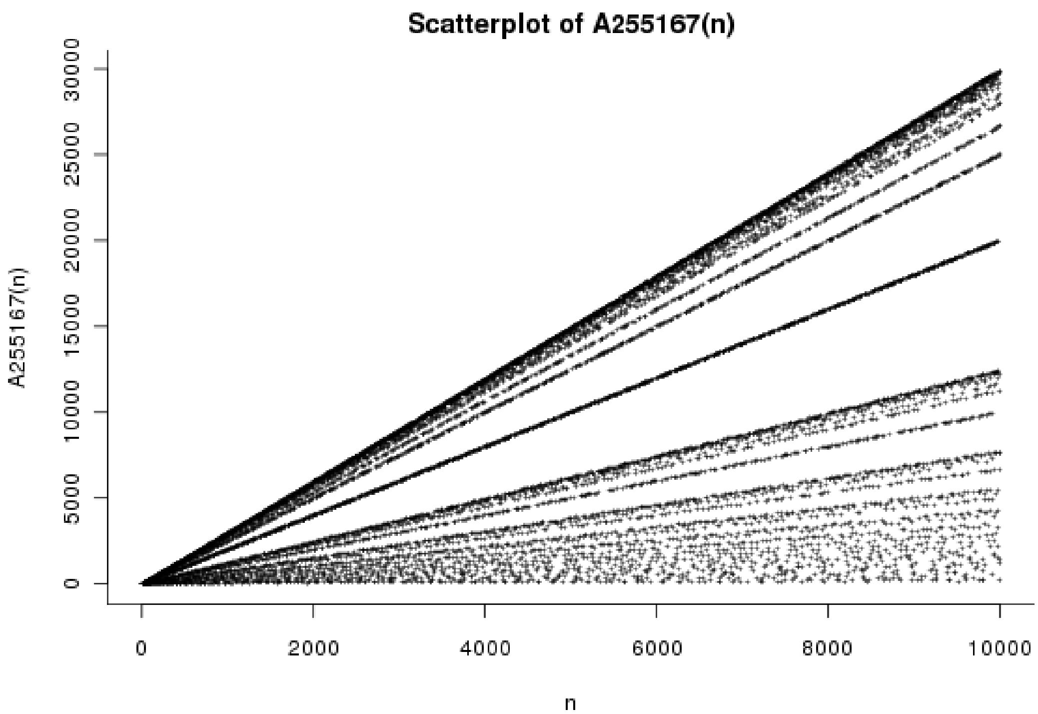
$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

A006255 is bounded above by A072905, the least $k > n$ such that $k \cdot n$ is square.

Question. Does there exist any n for which $A006255(n) = A072905(n)$. In other words, is there any non-square n for which $n \cdot A006255(n)$ is square?

Related.

1. Does the gap $A072905(n) - A006255(n)$ have a nonzero lower bound?



Note. This is equivalent to showing that for any $a < b$ with the same squarefree part, there is some subset of $\{a+1, a+2, \ldots, b-1\}$ such that the product of the elements of the subset has the same squarefree part as a (and b).

References.

`https://oeis.org/A006255`

`https://oeis.org/A072905`

`https://oeis.org/A255167`

Problem 16.

Let $a_3(n)$ be the least $k > n$ such that nk or nk^2 is a cube, and let $A299117$ be the image of $a_3(n)$.

$a_3(1) = 8$ via $1 \cdot 8 = 2^3$	$a_3(11) = 88$ via $11 \cdot 88 = 22^3$	$a_3(21) = 168$ via $21 \cdot 168^2 = 84^3$
$a_3(2) = 4$ via $2 \cdot 4 = 2^3$	$a_3(12) = 18$ via $12 \cdot 18 = 6^3$	$a_3(22) = 176$ via $22 \cdot 176^2 = 88^3$
$a_3(3) = 9$ via $3 \cdot 9 = 3^3$	$a_3(13) = 104$ via $13 \cdot 104^2 = 52^3$	$a_3(23) = 184$ via $23 \cdot 184^2 = 92^3$
$a_3(4) = 16$ via $4 \cdot 16 = 4^3$	$a_3(14) = 112$ via $14 \cdot 112^2 = 56^3$	$a_3(24) = 72$ via $24 \cdot 72 = 12^3$
$a_3(5) = 25$ via $5 \cdot 25 = 5^3$	$a_3(15) = 120$ via $15 \cdot 120^2 = 60^3$	$a_3(25) = 40$ via $25 \cdot 40 = 10^3$
$a_3(6) = 36$ via $6 \cdot 36 = 6^3$	$a_3(16) = 32$ via $16 \cdot 32 = 8^3$	$a_3(26) = 208$ via $26 \cdot 208^2 = 104^3$
$a_3(7) = 49$ via $7 \cdot 49 = 7^3$	$a_3(17) = 136$ via $17 \cdot 136^2 = 68^3$	$a_3(27) = 64$ via $27 \cdot 64 = 12^3$
$a_3(8) = 27$ via $8 \cdot 27 = 6^3$	$a_3(18) = 96$ via $18 \cdot 96 = 12^3$	$a_3(28) = 98$ via $28 \cdot 98 = 14^3$
$a_3(9) = 24$ via $9 \cdot 24 = 6^3$	$a_3(19) = 152$ via $19 \cdot 152^2 = 76^3$	$a_3(29) = 232$ via $29 \cdot 232^2 = 116^3$
$a_3(10) = 80$ via $10 \cdot 80 = 20^3$	$a_3(20) = 50$ via $20 \cdot 50 = 10^3$	$a_3(30) = 240$ via $30 \cdot 240^2 = 120^3$

Question. Is there another way to characterize what integers are in $A299117$?

Note. The function a_3 is an injection.
$A299117$ contains every cube, because $a(n^3) = (n+1)^3$.
$A299117$ contains the square of every prime, because $a(p) = p^2$.

Related.

1. Does $A299117$ contain every square?
2. Does $A299117$ contain any squarefree number?
3. What about the generalization: the image of a_β where $a_\beta(n)$ is the least $k > n$ such that $nk, nk^2, \ldots, nk^{\beta-2}$, or $nk^{\beta-1}$ is a β-th power? Prime β is an injection—is this well behaved?

References.

https://oeis.org/A277781

https://oeis.org/A299117

Problem 17.

Consider placing any number of queens (of the same color) on an $n \times n$ chessboard in such a way as to maximize the number of legal moves available.

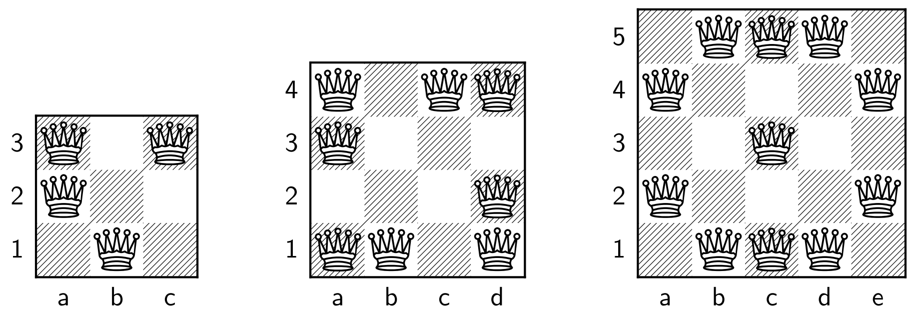


Figure 1: Examples of $a_q(3) = 17, a_q(4) = 40, a_q(5) = 76$.

Question. Is Alec Jones's conjecture true: $a_q(n) = 8(n-2)^2$ for $n \geq 6$, by placing the queens around the perimeter?

Related.

1. What about the analogous function for rooks (a_r) or bishops (a_b)?
2. What if the chessboard is a torus? Cylinder? Möbius strip?
3. What if the chessbaord is $n \times m$?
4. Is $a_b(n) = \lfloor a_q(n)/2 \rfloor$? for all n?
5. What if queens can attack?

References.

A278211: `http://oeis.org/A278211`

A278212: `http://oeis.org/A278212`

A275815: `http://oeis.org/A275815`

Problem 18.

Let U_n be the set of sequences of positive integers of length n such that no substring occurs twice.

$$(1,1,2,2,1,3,1) \in U_7$$
$$(1,2,1,2,3) \notin U_5 \text{ because } (1,2) \text{ occurs twice.}$$
$$(1,1,1) \notin U_3 \text{ because } (1,1) \text{ occurs twice.}$$

Figure 1: An example and two non-examples of sequences with no repeated substrings.

Question. What is the number of sequences in U_n where the sum of terms is minimized?

Related.

1. What is the minimum least common multiple of a sequence in U_n? How many such minimal sequences?
2. What is the minimum product of a sequence in U_n? How many such minimal sequences?
3. What if substrings are considered forward and backward?
4. What if only substrings of length greater k are considered?
5. What if any term can appear at most ℓ times?

References.

https://oeis.org/A259280

Problem 19.

Consider the fuction A300002(n) which is the lexicographically earliest sequence of positive integers such that no $k + 2$ points are on a polynomial of degree k. (i.e. no two points are equal, no three points are colinear, no four points are on a parabola, etc.)

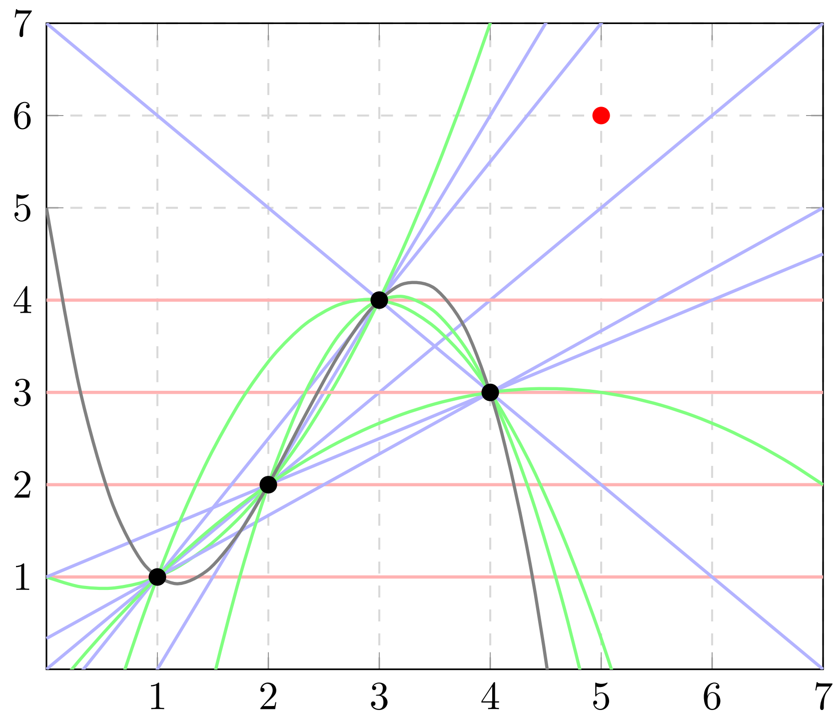


Figure 1: The first four points together with all interpolated polynomials. The red point marks the lowest integer coordinate $(5, k)$ that does not lie on an interpolated polynomial. (Degree 0 polynomials are plotted in red, degree 1 in blue, degree 2 in green and degree 3 in gray.)

Question. Do all positive integers occur in this sequence?

Related.

1. What is the asymptotic growth of this sequence?
2. Does *any* permutation of the natural numbers have the property that no $k+2$ points are on a polynomial of degree k?

References.

`https://oeis.org/A300002`

Problem 20.

Let h be the maximum number of penny-to-penny connections on the vertices of a hexagonal lattice, and let $t(n)$ be the analogous sequence on the vertices of a triangular lattice.

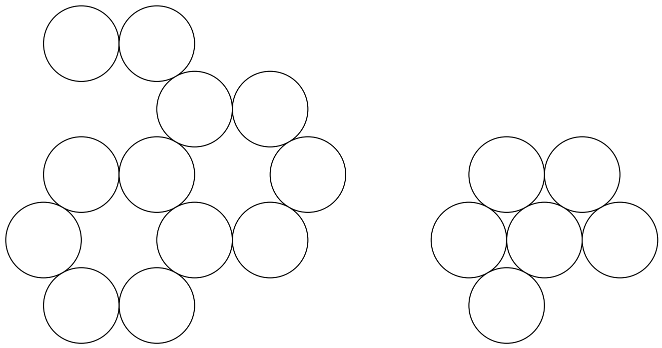


Figure 1: An example for $h(12) = 13$ and $t(6) = 9$

Question. What is a combinatorial proof that $h(2n) - t(n) = A216256(n)$.

Note. A216256 is

$$\underbrace{1}_{1}, \underbrace{2}_{1}, \underbrace{3, 3}_{2}, \underbrace{4, 4, 4}_{3}, \underbrace{5, 5, 5}_{3}, \underbrace{6, 6, 6, 6}_{4}, \underbrace{7, 7, 7, 7, 7}_{5}, \underbrace{8, 8, 8, 8, 8}_{5}, \underbrace{9, 9, 9, 9, 9, 9}_{6}, \ldots$$

Related.

1. `https://oeis.org/A216256`
2. $t(n)$: `https://oeis.org/A047932`
3. $h(n)$: `https://oeis.org/A263135`

Problem 21.

Consider all rectangles with all corners on gridpoints on an $n \times m$ grid.

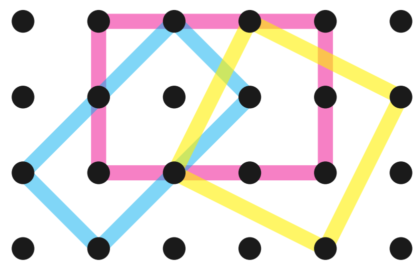


Figure 1: An example of three rectangles with all corners on gridpoints of a 4×6 grid.

Question. How many such rectangles exist?

Related.

1. How many squares exist? Rhombi? Parallelograms? Kites (A189417)? Trapezoids? Quadrilaterals? Convex quadrilaterals? n-gons?
2. What if we want to count only "primitive" squares, in the sense that the sides of the square only intersect gridpoints at the corners?
3. Number of rectangles on the cylinder? Torus? Möbius strip?
4. Number of "rotation classes", where two squares are equivalent if one can be transformed into the other by shifting and stretching?
5. Number of "orientation classes" where two squares are equivalent if one can be transformed into the other by shifting?
6. How many right triangles?
7. What if this is done on an $n \times m \times k$ grid?
8. What if the rectangles must be diagonal?
9. What if this is done on a triangular lattice with equilateral triangles? Primitive equilateral triangles?
10. How many tetrahedra are in an n-sided tetrahedra?

References.

Problem 1.

https://oeis.org/A000332

https://oeis.org/A085582

https://arxiv.org/pdf/1605.00180.pdf

http://people.missouristate.edu/lesreid/POW03_01.html

Problem 22.

The prime ant looks along the number line starting at 2. When she reaches a composite number, she divides by its least prime factor, and adds that factor to the previous term, and steps back.

2	3	4	5	6	7	8	9	10
2	5	2	5	6	7	8	9	10
2	5	2	7	3	7	8	9	10
2	5	2	7	3	9	4	9	10
2	5	2	7	6	3	4	9	10
2	5	2	9	2	3	4	9	10
2	5	5	3	2	3	4	9	10
2	5	5	3	2	3	7	3	10

Figure 1: An illustration of the prime ant's positions after the first 7 steps.

Question. Does the ant eventually stay to the right of any fixed position?

Related.

1. Are there any positions that stay permanently greater than 7? Than 11?
2. Does sequence of numbers converge in the long run? If so, what to? $(2, 5, 5, 3, 2, \ldots)$
3. Let S be a subset of $\mathbb{N}$ and let $f : S \times S^c \to \mathbb{N}^2$. For what "interesting" sets S and functions f can we answer the above questions?
   (In the example S is the prime numbers and f maps $(p, c) \mapsto (p + \operatorname{lpf}(c), \operatorname{gpf}(c))$.)

References.

`https://codegolf.stackexchange.com/q/144695/53884`

`https://math.stackexchange.com/q/2487116/121988`

`https://oeis.org/A293689`

Problem 23.

Consider polyominoes where each cell has one of n colors, and each distinct pair of colors is adjacent (horizontally or vertically) to each other somewhere in the polyomino. Let an n-minimum polyomino be one that has the minimum number of cells.

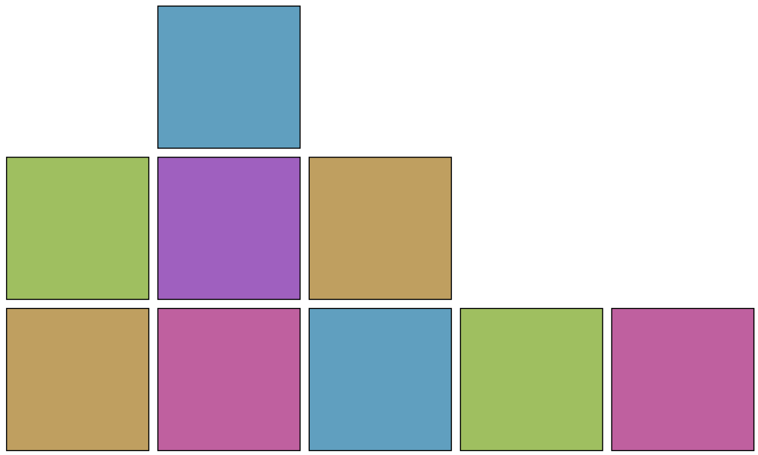


Figure 1: An example of a minimum polyomino for $n = 5$; $a(5) = 9$

Question. How many such n-minimum polyominoes exist?

Related.

1. What if the "distinct" restriction is lifted? (e.g. a blue label must somewhere be adjacent to another blue label.)
2. What is a way to determine the size of an n-minimum polyomino for large n?
3. What if this is done on a triangular or hexagonal grid?
4. What if this is done on a three dimensional cube lattice?

References.

https://oeis.org/A278299

Problem 24.

Consider partitions of the $n \times m$ grid in which every piece has $180°$ rotational symmetry.

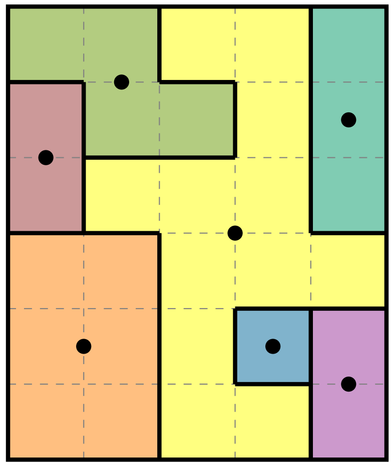


Figure 1: A partition of the 5×6 grid into 7 parts with rotational symmetry.

Question. How many such partitions of the $n \times n$ grid exist? Up to dihedral action?

Related.

1. How many partitions into exactly k parts?
2. How many partitions with other types of symmetry?
3. How many partitions of a torus? Cylinder? Möbius strip?
4. How many partitions of a triangular or hexagonal lattice?
5. How many partitions of an $n \times m \times p$ cuboid?
6. How many placements of centers results in a unique solution? Multiple solutions? No solutions?
7. What if there is the additional restriction that putting together any proper subset of adjacent parts must not exhibit symmetry? (e.g two adjacent unit squares cannot be colored differently.)
8. What partitions have parts with the greatest average number of sides? (e.g. in the example the average part has $(5(4) + 8 + 16)/7 = 44/7 \approx 6.29$ sides.)
9. What partitions have the smallest ratio of rectangular parts? (e.g. in the example, 2 out of 7 parts are non-rectangular.)
10. What partitions have the greatest number of non-rectangular parts, total? (e.g. in the example, two of the parts are non-rectangular.)

References.

`https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/galaxies.html`

Problem 25.

Consider all rectangles composed of n squares such that the greatest common divisor of all the sidelengths is 1.

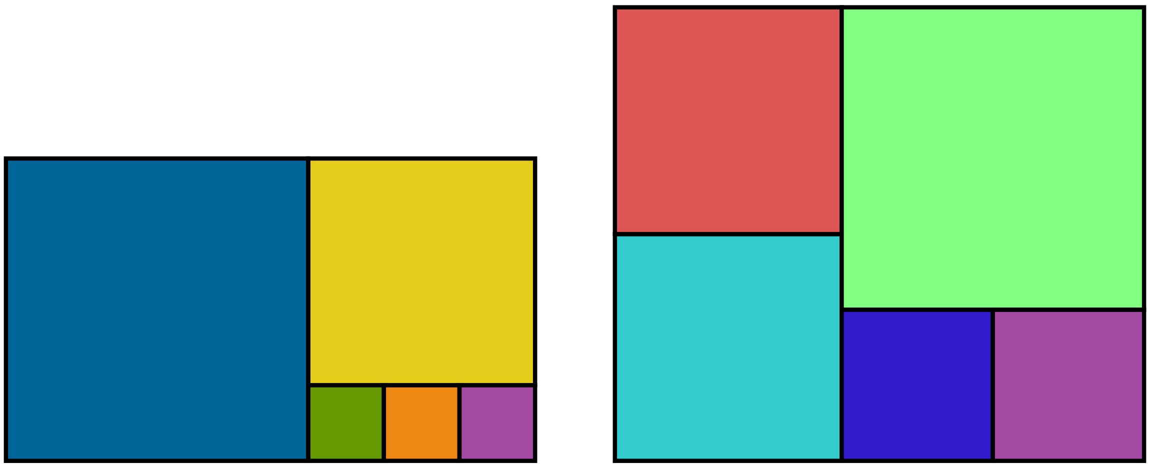


Figure 1: Two examples of rectangles made from $n = 5$ squares. In the first $\gcd(1, 1, 1, 3, 4) = 1$ and in the second $\gcd(2, 2, 3, 3, 4) = 1$.

Question. Given n squares, how many such rectangles exist?

Related.

1. How many ways are there to make convex polygons out of n equilateral triangles?
2. How many ways are there to make cuboids out of n cubes?

Problem 26.

Consider an n-coloring of a triangular grid such that no sub-triangle has corners all with the same color.

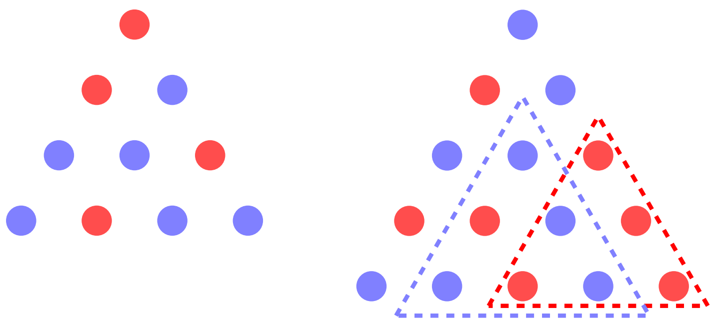


Figure 1: On the left is an example of a triangle on two labels that has no sub-triangles with equal corners. On the right is a non-example of such a triangle on two labels: it has two sub-triangles with equal corners.

Question. Given n labels, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

Related.

1. Given an n-coloring of a triangle of side length k, what number of sub-triangles with equal corners must exist?
2. How many such triangles exist?
3. What if diagonal equilateral triangles also are not allowed to have equal corners?
4. What if this is done with hexagons instead of triangles?
5. What if this is done on a square grid?
6. What if for $n \geq 3$ no *two* corners are allowed to be equal? (This is a bit like a peaceable queens problem on a hexagonal chessboard.)

References.

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

Problem 27.

A country has a strange legislative procedure. For each bill, the body is split up into k_1 committees of $\lfloor n/k_1 \rfloor$ or $\lfloor n/k_1 \rfloor$ legislators each, each of which picks a representative. These k_1 representatives are split up into k_2 sub-committees with $\lfloor k_1/k_2 \rfloor$ or $\lfloor k_1/k_2 \rfloor$ legislators each, which each elect a representative, and so on until $k_T = 1$ and the final committee votes on the bill.

There are a few rules:

1. Each committee (and subcommittee and so on) much have at least ℓ members.
2. Ties are settled by a coin toss.
3. The president does not get to vote, but she does get to choose the number of committees and who goes in each one.
4. There are α supporters who will always vote in the president's interests and $n - \alpha$ who will always vote against.

Let $a_\ell(n)$ be the minimum number of supporters (α) required for the president to be able to pass every bill.

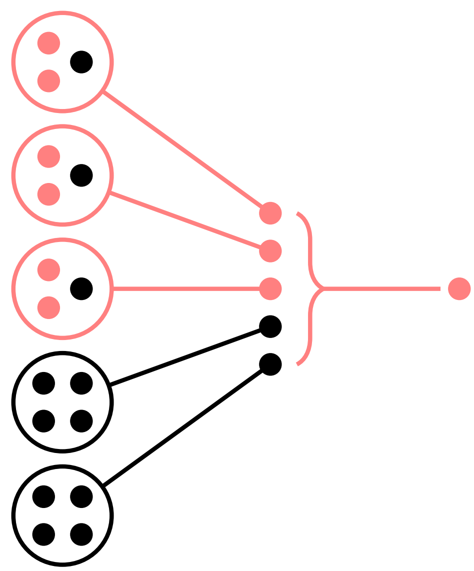


Figure 1: An example of $n = 17$ legislators with a minimum comittee size of $\ell = 3$, which demonstrates that $a_3(17) \leq 6$.

Question. What is an efficient way to compute $a_\ell(n)$ for general ℓ and n?

Related.

1. What if the president gets to choose who is on each committee but the opposition party gets to choose the committee size? Vice versa?
2. What if $k_1 \leq k_2 \leq \ldots \leq k_T$? Or $k_1 \geq k_2 \geq \ldots \geq k_T$?
3. What if ties go to the president? To the opposition?

References.

https://oeis.org/A290323

https://math.stackexchange.com/q/2395044/121988

Problem 28.

Consider tilings of the $n \times n$ grid up to D_8 action where the tiles are diagonals.

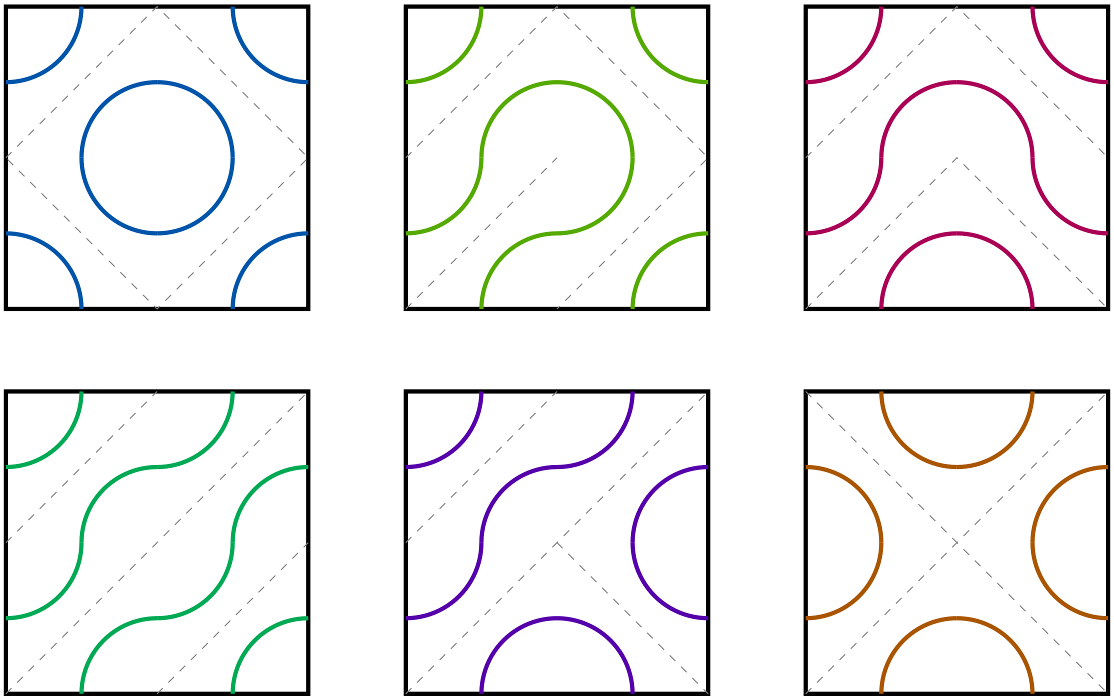


Figure 1: An example of the $a(2) = 6$ different ways to fill the 2×2 grid with diagonal tiles (up to dihedral action).

Question. How many such tilings exist?

Related.

1. What if grids are only counted up to C_4 (rotation) action?
2. What if this is counted on the torus/cylinder/Möbius strip?
3. What if each tile can have no diagonals or both diagonals?
4. What if tiles are black or white?
5. Is there an obvious bijection between the results on the $2n \times 2n$ grid for black/white versus diagonal tile types?

References.

`https://oeis.org/A295229`

Problem 29.

Consider the rectangles from Problem 25: those composed of n squares such that the greatest common divisor of all the sidelengths is 1. If rectangles are measured by the longest side, the smallest rectangles are given by $A295753$.

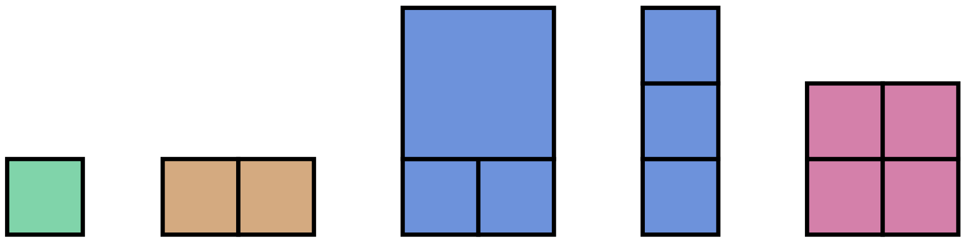


Figure 1: Examples of $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, and $a(4) = 1$.

Question. How many distinct rectangles composed of n squares have a longest side of $A295753(n)$?

Related.

1. Is the largest rectangle (as measured by smallest side) unique for large n?
2. What if smallest rectangle is measured by perimeter?

Note. Largest rectangles might be Fibonacci spirals, or they might be similar to the second example or the examples in the References.

References.

https://en.wikipedia.org/wiki/Squaring_the_square

https://oeis.org/A295753

Problem 30.

Consider all configurations of n nonattacking rooks on an $n \times n$ board up to dihedral action.

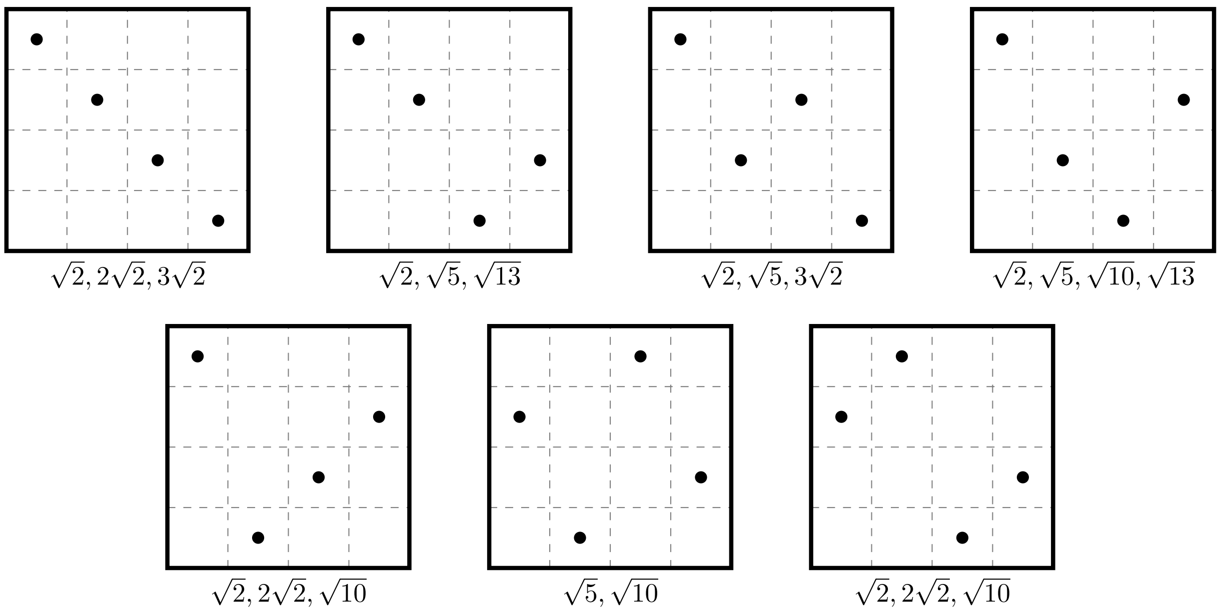


Figure 1: Each figure is marked with the distinct distances between pieces.

Question. What is the minimum number of distinct distances on such a figure?

Related.

1. What is the minimum number of distinct directions on such a figure? (Directions up to dihedral action?)
2. What if this is done with n queens instead of rooks?
3. What if this is done with $0 \le k \le n^2$ pieces, any of which are allowed to be in attacking positions?
4. What if distance is measured by the taxicab metric? d_3? d_∞? Number of knight-moves away? Number of king moves away?
5. How many configurations of nonattacking rooks on the torus, rectangle, triangluar grid, and other geometries?
6. Are any configurations of nonattacking rooks on the torus that can be meaningfully called a "generalized Costas array"?

References.

`https://en.wikipedia.org/wiki/Costas_array`

The maximum and minimum number of distinct distances is given by A320448 and A319476 respectively.

The number of extremal boards is given by A320573 and A320575 (A320574 and A320576, up to symmetry).

A193838: smallest square pegboard from which n points with distinct mutual distances can be chosen.

Problem 31.

Consider square, triangular, and hexagonal grids that are filled in with with tiles of different patterns.

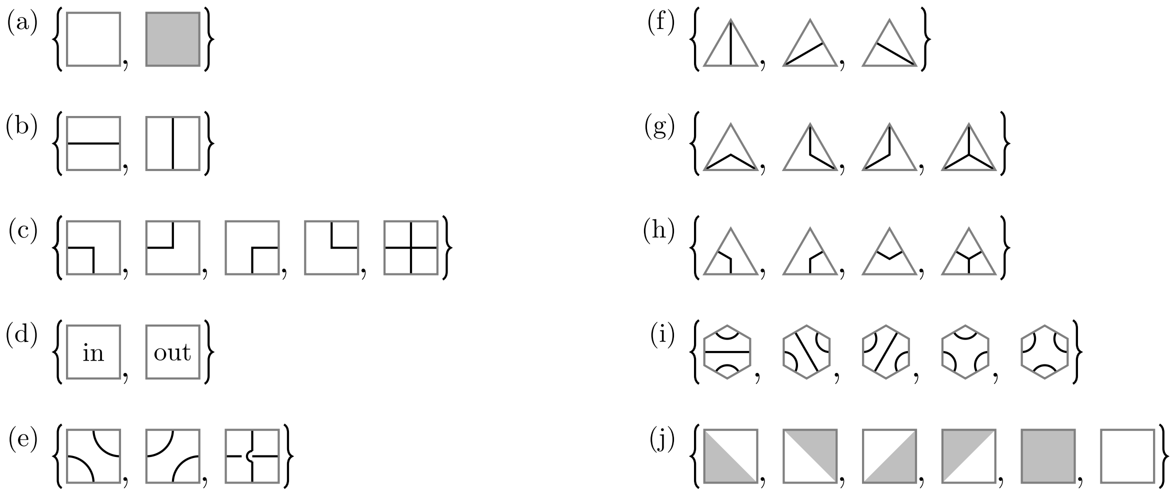


Figure 1: Ten examples of different tiles.

Question. How many essentially different grids of size n exist with these tiles? (Up to dihedral action? Up to cyclic action?)

Related.

1. The square grid can be $n \times n$ or $n \times m$.
2. The hexagonal grid can have triangles with side length n or hexagons with side length n.
3. The triangular grid can have triangles with side length n or hexagons with side length n.
4. The square grid can be quotiented to be a cylinder, torus, or Möbius strip.
5. What if shapes have to "match-up" (e.g. the lines in the third example or colors in the last example have to be "smooth".)
6. How many distinct regions, as in Problem 2?

References.

Problem 2.

Problem 28.

https://en.wikipedia.org/wiki/Burnside%27s_lemma

Problem 32.

Starting with an $n \times m$ grid, remove one corner at a time (uniformly at random) until the grid is gone.

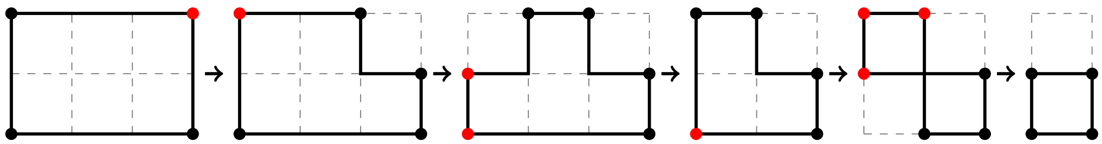


Figure 1: An example of a process starting with a 2×3 grid.

Question. If a stopping point is chosen randomly, how many corners are expected?

Related.

1. What if the deletion is uniform with respect to faces instead of vertices?
2. How many sides are expected?
3. If all polygons in the process are considered, what is the expected number of corners on the polygon with the greatest number of corners?
4. What figure produces the greatest number of corners?
5. How many possible processes exist (up to, say, dihedral action)?
6. What if each figure must stay path connected?
7. What if paths cannot travel through corners? (e.g. the second-to-last figure is illegal.)

Problem 33.

Consider all of the ways to stack n "blocks" of different shapes on a platform of length k.

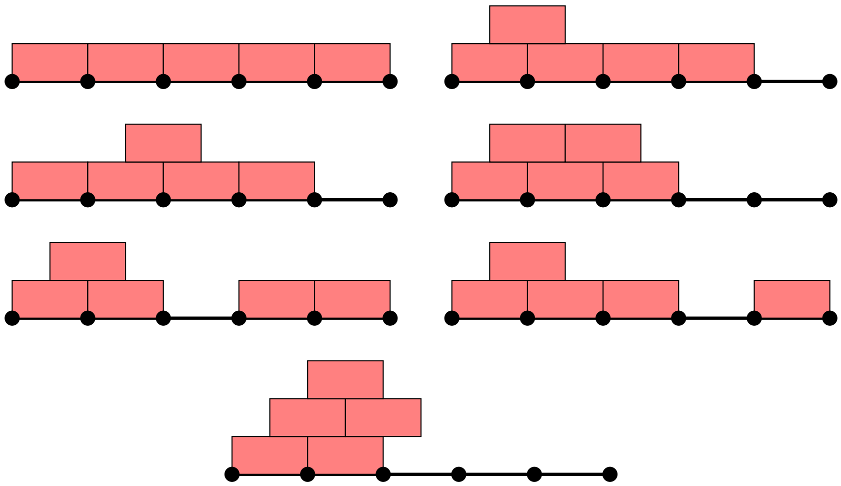
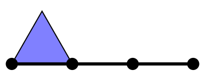
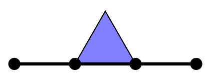


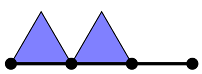
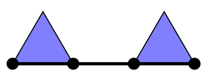
Figure 1: Seven examples of five length 2 bricks on a length 5 platform.

Question. How many different stacks exist for these shapes?

Related.

1. What if we use triangular blocks?

   (a) What if  and  are considered to be distinct?

   (b) What if  and  are considered to be the same (because one turns into the other by "sliding".)

   (c) What if "upside-down" triangles can be placed in the gaps?

   (d) What if "upside-down" triangles *must* be placed in the gaps in order to stack on top?

2. What about bricks of length 3?

3. What about tetrahedra and cuboids?

4. What if bricks can be stacked directly on top of each other?

5. What if the stack must be connected?

6. What if reflections are considered to be the same?

Note. The triangle stacking problem appears to be counted by Catalan numbers. If cantilevers are not allowed, the brick stacking problem reduces to the triangle stacking problem.

References.

`https://oeis.org/A005169` (Connected triangles on arbitrarily long platform)

`https://oeis.org/A168368`

`https://math.stackexchange.com/q/2731692/121988`

Problem 34.

Consider ways to partition the $n \times m$ grid so that no three tiles of the same partition fall on a line.

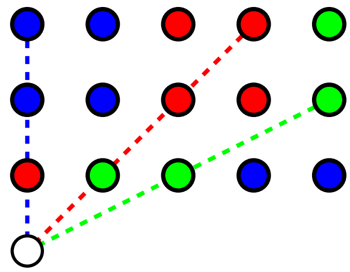


Figure 1: A 3 partition of the 5×3 grid. The white circle cannot be in any of the existing partitions, otherwise three circles of the same color would fall on the same line.

Question. How many colors are required to satisfy the "no three in a row" criterion?

Related.

1. What if this is generalized to k in a row?
2. What if this is generalized to a triangular or hexagonal grid?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if this only queen moves or rook moves are considered?
5. How many distinct configurations exist with a minimal number of partitions?
6. How many distinct configurations exist with k partitions?

References.

Problem 26.

Problem 35.

Say that a *minimally interpolable permutation* f is a permutation of $\{1, 2, \ldots, n\}$ such that no $k+2$ of the points $\{(1, f(1)), \ldots, (n, f(n))\}$ fall on a degree k polynomial.

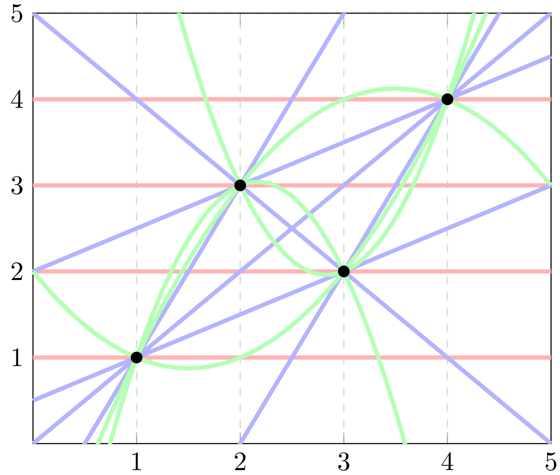


Figure 1: A minimally interpolable permutation of $\{1, 2, 3, 4\}$. (Degree 0 polynomials are plotted in red, degree 1 in blue, and degree 2 in green.)

Question. Do such permutations always exist? If not, what is the least N such that there is a minimally interpolable function from $[n]$ into $[N]$?

Related.

1. How many minimally interpolable permutations exist?
2. Does the number of minimally interpolable permutations increase as a function of n?
3. Is there a method to explicitly construct a minimally interpolable permuation?
4. If such permutations do not always exist, what is the least M such that there exists a subset $S \subset [M]$ and a surjection $g\colon S \to [N]$ with the aforementioned property?

References.

Problem 19.

`https://oeis.org/A301802`

`https://codegolf.stackexchange.com/q/160382/53884`

Problem 36.

Consider an n-coloring of a triangular grid such that no upright sub-triangle has the same coloring as any other (up to rotation).

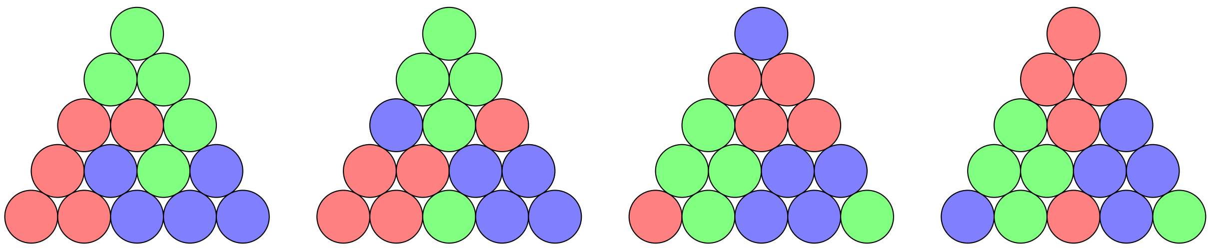


Figure 1: Four examples of 3-colorings of the length 5 triangle. In all cases, 10 different colorings appear exactly once. In the first example, starting from the top: (1) GGG, (2) RRG, (3) RGG, (4) RRB, (5) RGB, (6) GGB, (7) RRR, (8) RBB, (9) GBB, and (10) BBB. (Incidentally, this is *all* of the colorings, so $a(3) = 5$.)

Question. Given n colors, what is the biggest triangle that can be constructed? Call the side length of such a triangle $a(n)$.

Related.

1. What if inverted triangles are counted too?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. How many patterns exist for a triangle of length k with the minimum number of labels?
4. What if diagonal equilateral triangles are also considered? (For example, take the second circle on every side as measured clockwise from each corner.)
5. What if this is done on a square grid?
6. What if this is done on hexagonal shapes?
7. What if this is done on tetrahedra or cuboids?
8. Consider the lexicographically earliest infinite case. Does every triangle eventually appear?

References.

`https://math.stackexchange.com/a/2416790/121988`

`https://math.stackexchange.com/a/2636168/121988`

Problem 37.

Start with an $n \times n$ grid of boxes and place lines through gridpoints at the border. A box is considered "on" if a line travels through its interior.

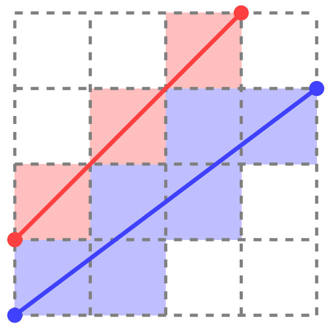


Figure 1: An example of two lines drawn on a grid. The seven white squares still require a line to be drawn through them.

Question. What is the minimum number of lines required to turn on all of the squares in and $n \times n$ grid?

Related.

1. What if touching the corner of a square also turns it on?
2. What if two triangles with the same coloring but different rotations are counted as different?
3. What if no two lines can be parallel? If no two line segments can be congruent?
4. What if no two lines can intersect?
5. How many fully "on" grids exist? How many such minimal grids? (A grid is minimal if removing any line results in a square turning off.)
6. Suppose a grid is on if an even number of lines pass through it and off if an odd number of lines pass through it. How many such grids?
7. How about on an $n \times m$ grid?
8. What if this is done on a triangular grid?
9. What if this is done on a cuboid? On a cuboid with planes passing through the cubes?

Note. In the case where "grid is on if an even number of lines pass through it and off if an odd number of lines pass through it", there exist 2^k configurations. If we further restrict to dihedral-symmetric grids, there are 2^j configurations.
It appears that $k = 5n^2 - 14n + 9$, the 12-gonal numbers. Is there a bijection between the basis elements and the 12-gonal numbers?

Problem 38.

There is a well known magic trick called "Communicating the Card" in which a spectator draws k cards from an n-card deck and shows them to the magician's assistant. He then shows $k-1$ of them to the magician in a particular order, after which she (the magician) can deduce the remaining card. In this variation, the largest possible deck is $k! + k - 1$ cards.

$f(1,2) = \{1,2,3\}$	$f(4,8) = \{1,4,8\}$	$f(7,2) = \{2,4,7\}$	$f(8,3) = \{3,5,8\}$
$f(2,1) = \{1,2,4\}$	$f(5,1) = \{1,5,6\}$	$f(8,2) = \{2,4,8\}$	$f(3,6) = \{3,6,7\}$
$f(1,5) = \{1,2,5\}$	$f(5,7) = \{1,5,7\}$	$f(2,5) = \{2,5,6\}$	$f(6,3) = \{3,6,8\}$
$f(1,6) = \{1,2,6\}$	$f(5,8) = \{1,5,8\}$	$f(5,2) = \{2,5,7\}$	$f(7,3) = \{3,7,8\}$
$f(1,7) = \{1,2,7\}$	$f(6,7) = \{1,6,7\}$	$f(8,5) = \{2,5,8\}$	$f(4,5) = \{4,5,6\}$
$f(1,8) = \{1,2,8\}$	$f(6,8) = \{1,6,8\}$	$f(6,2) = \{2,6,7\}$	$f(5,4) = \{4,5,7\}$
$f(1,3) = \{1,3,4\}$	$f(7,8) = \{1,7,8\}$	$f(8,6) = \{2,6,8\}$	$f(8,4) = \{4,5,8\}$
$f(3,1) = \{1,3,5\}$	$f(2,3) = \{2,3,4\}$	$f(8,7) = \{2,7,8\}$	$f(4,6) = \{4,6,7\}$
$f(6,1) = \{1,3,6\}$	$f(3,2) = \{2,3,5\}$	$f(3,4) = \{3,4,5\}$	$f(6,4) = \{4,6,8\}$
$f(7,1) = \{1,3,7\}$	$f(2,6) = \{2,3,6\}$	$f(4,3) = \{3,4,6\}$	$f(7,4) = \{4,7,8\}$
$f(8,1) = \{1,3,8\}$	$f(2,7) = \{2,3,7\}$	$f(3,7) = \{3,4,7\}$	$f(5,6) = \{5,6,7\}$
$f(1,4) = \{1,4,5\}$	$f(2,8) = \{2,3,8\}$	$f(3,8) = \{3,4,8\}$	$f(6,5) = \{5,6,8\}$
$f(4,1) = \{1,4,6\}$	$f(2,4) = \{2,4,5\}$	$f(3,5) = \{3,5,6\}$	$f(7,5) = \{5,7,8\}$
$f(4,7) = \{1,4,7\}$	$f(4,2) = \{2,4,6\}$	$f(5,3) = \{3,5,7\}$	$f(7,6) = \{6,7,8\}$

Figure 1: An example of an encoding where $k = 3$ and $n = k! + k - 1 = 8$.

Question. What if the assistant can show any number of cards less than k, and the magician must guess all of the remaining cards?

Related.

1. How many different encodings exist (up to relabeling)?
2. What if the magician just needs to guess one of the remaining cards?
3. What if there are ℓ identical copies of a deck, how many cards can the original trick support?
4. If the assistant shows $k-2$ cards to the magician, what is the biggest deck that this trick can be done with? $k-j$?

References.

http://oeis.org/A030495

https://www.reddit.com/r/math/comments/7l1t84/a_combinatorists_card_trick/

https://web.northeastern.edu/seigen/11Magic/Articles/Best%20Card%20Trick.pdf

Problem 39.

This one is based on correspondence from Alec Jones: Consider all of the ways of partitioning the complete graph on n vertices into smaller complete graphs.

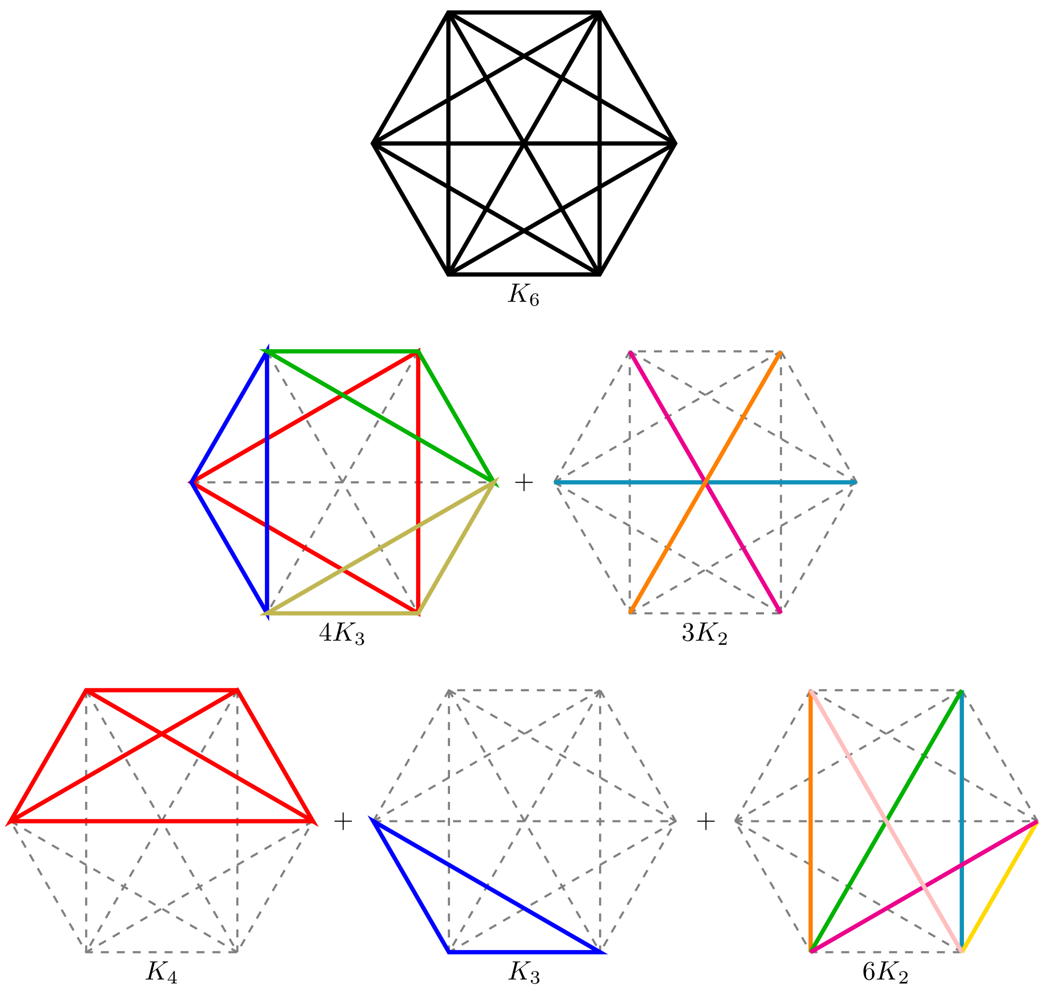


Figure 1: An example three ways to partition K_6 into complete graphs: the trivial partition, a partition into 4 copies of K_3 and 3 copies of K_2, and a partition into 1 copy of K_4, 1 copy of K_3, and 6 copies of K_2.

Question. How many such partitions exist, up to graph isomorphism?

Related.

1. What if the union of K_j graphs cannot contain a K_{j-1} subgraph?
2. What if the partition can only consist of two "sizes" of complete graphs, as in the second example?
3. How many such partitions exist up to dihedral action?

Problem 40.

From correspondence with Alec Jones: Consider a game played on the $m \times n$ rectangular grid, where players take turns placing their pieces onto the board. Each player gets a point for each 3-in-a-row that they make.

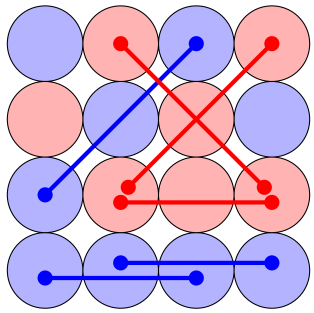


Figure 1: In this game on a 4×4 board, the red player and blue player tie with three points each.

Question. Which player has a winning strategy?

Related.

1. What is the score differential under perfect play?
2. If players cooperate, what is the greatest score differential?
3. What if this is generalized to a torus or cylinder or Möbius strip?
4. What if the game is played with k players or requires ℓ-in-a-row?
5. What if the game is played on a triangular grid?
6. What if the game is played in d dimensions?

References.

Problem 34.

Problem 41.

A problem inspired by a Project Euler problem: suppose an n-robot takes steps that are $1/n$ of a circle, and turns right or left after every step.

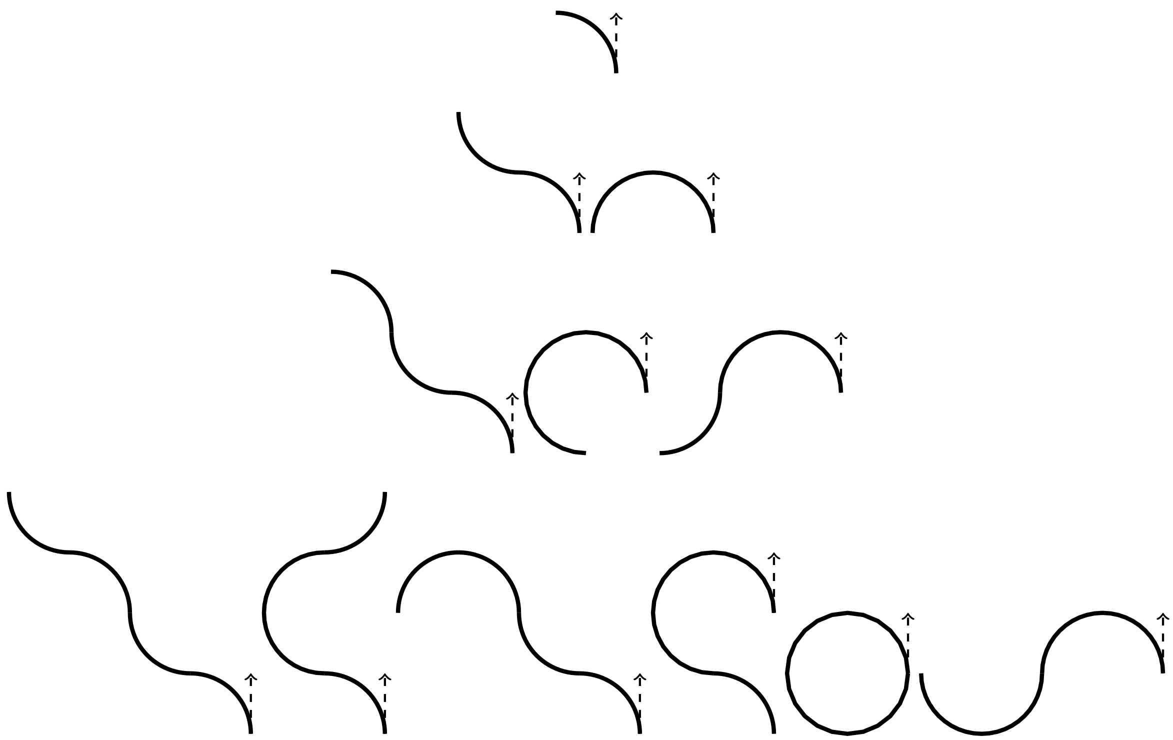


Figure 1: An example of distinct paths of k steps (up to dihedral action) for a 4-robot. $a(1) = 1$, $a(2) = 2$, $a(3) = 3$, and $a(4) = 6$.

Question.

How many walks exist such that the robot ends up at the original position and orientation after k steps?

Related.

1. How many distinct paths exist for an n-robot, where the robot never retraces its steps?
2. What if the robot is allowed to retrace its steps?
3. What is the smallest radius that can contain a k-step walk if the robot cannot retrace its steps? (The robot returns to where it started in the same direction that it started.)
4. Can smooth loop paths occur when the number of steps is not a multiple of n?
5. What if the orientation of the path matters (i.e. *not* counted up to dihedral action)?
6. What if this is done on a torus, cylinder, or Möbius strip?
7. What if the robot cannot cross its own path?

References.

`https://projecteuler.net/index.php?section=problems&id=208`

Problem 42.

Consider walks in a city, starting mid-block, where (1) at each intersection the walker goes left right or straight, (2) at each mid-block, the walker decides whether or not to turn around, and (3) she ends up back at her apartment.

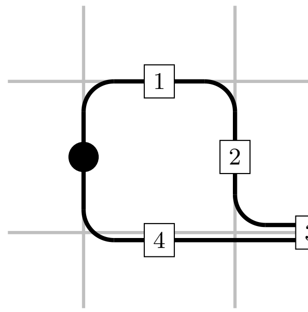


Figure 1: An example of a 5-step walk returning to the apartment.

Question. How many n-block walks can the walker take?

Related.

1. What if the walker does not want to walk along the same strip of road twice?
2. What if the walker does not want to walk along the same *side* of the same strip of road? (Suppose she always walks on the right side of the street.)
3. What if the walker never wants to revisit the same intersection?
4. How many walks up to dihedral action?
5. What if the walker does not turn around?
6. What if the walker never goes straight? Never goes right?

References.

Problem 41.

Problem 43.

Consider a puzzle on a (blank) $n \times m$ board, where each column and row has a number denoting the number of markers that should go in that column or row. The player's goal is to fill in the grid in such a way that the row/column "histograms" are satisfied.

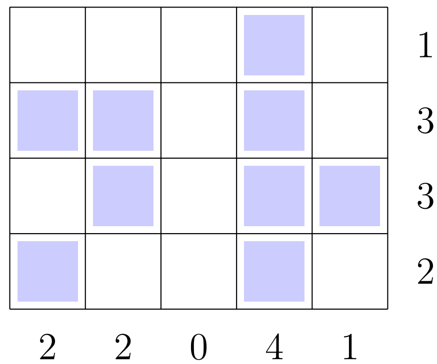


Figure 1: Example of a solution to the puzzle $(2,2,0,4,1) \times (1,3,3,2)$. Is the solution unique?

Question. What is a procedure for determining if a grid has a solution? If it has a unique solution?

Related.

1. What if the game is played on a d-dimensional hypercube?
2. What if the game is played on a triangle? Tetrahedron?
3. What is the greatest amount of ambiguity a non-unique board can have? (i.e. what is the greatest number of solutions?)
4. How many maximally ambiguous boards exist?
5. How many distinct boards exist up to dihedral action? Up to torus action?
6. What if multiple markers can be put in each cell?

References.

`https://oeis.org/A297077`

Problem 44.

From Alec Jones. Let $a_k(n)$ count the number of k-gons with vertices on the $n \times n$ grid.

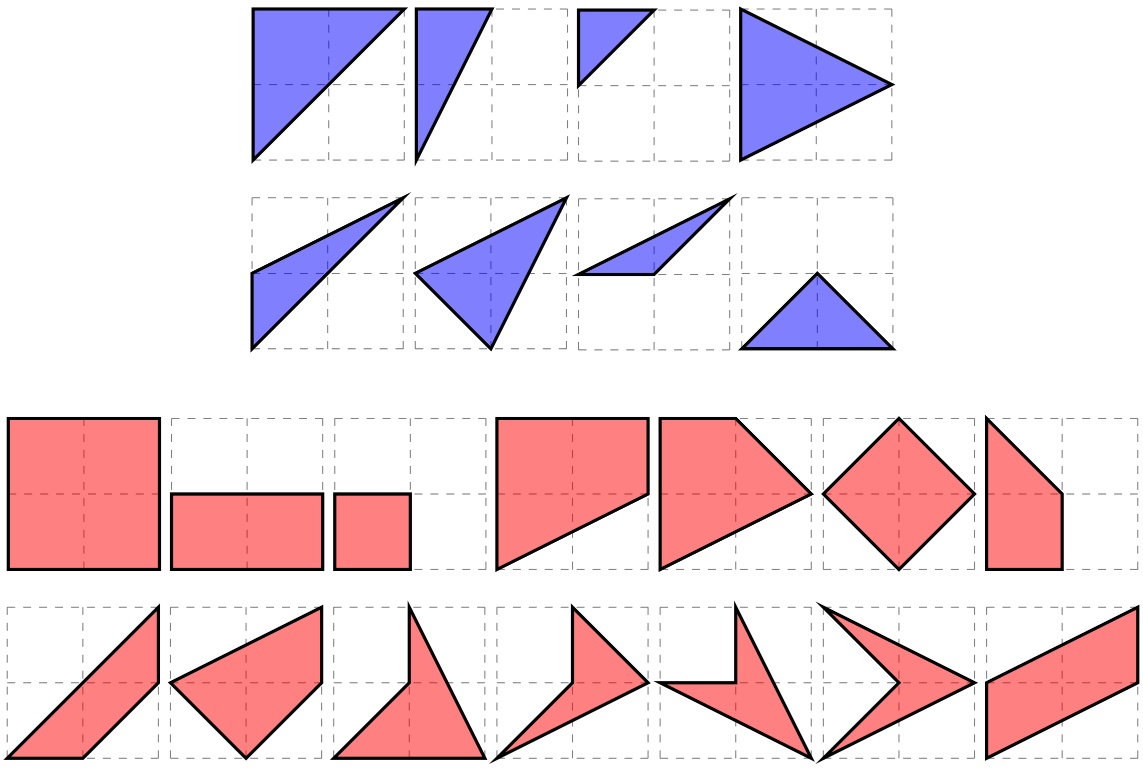


Figure 1: An example showing that $a_3(2) \geq 8$ and $a_4(2) \geq 14$.

Question. What is $a_k(n)$?

Related.

1. For a fixed n, what is the value of k such that $a_k(n)$ is maximized?
2. Here two polygons are considered equivalent if they are congruent. What if two polygons are considered equivalent if they are similar? If they are the same under dihedral action? If they are the same over linear transformation? (e.g. stretching/skewing)
3. What if concave polygons are excluded?
4. What if this is done on an $n \times m$ grid?
5. What if we don't deduplicate based on congruence?
6. What if this is done on a hypercube or a triangular grid?

Problem 45.

A polyform counting problem from Alec Jones: let $a_k(n)$ count the number of polyabolos with n faces and k exposed edges.

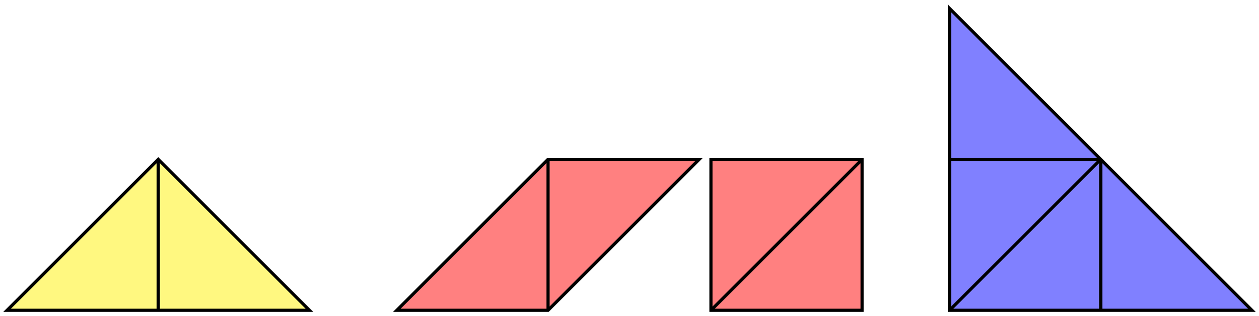


Figure 1: An example in yellow showing that $a_3(2) \geq 1$, two examples in red showing that $a_4(2) \geq 2$, and an example in blue showing that $a_3(4) \geq 1$.

Question. What is the smallest k such that for some fixed n, $a_k(n) > 0$?

Related.

1. What is the largest k such that for some fixed n, $a_k(n) > 0$?
2. What if $\hat{a}_k(n)$ counts polyiamonds instead?
3. What if concave polygons are excluded?
4. Is the following function well-defined?
$$b(k) = \max\{\, a_k(n) : n \in \mathbb{N} \,\}$$
5. Is the following function interesting?
$$c(n) = \max\{\, a_k(n) : k \in \mathbb{N} \,\}$$

References.

https://en.wikipedia.org/wiki/Polyiamond

https://en.wikipedia.org/wiki/Polyabolo

Problem 46.

Define an n-triangle to be a triangle with integer coordinates and perimeter in $[n, n+1)$.

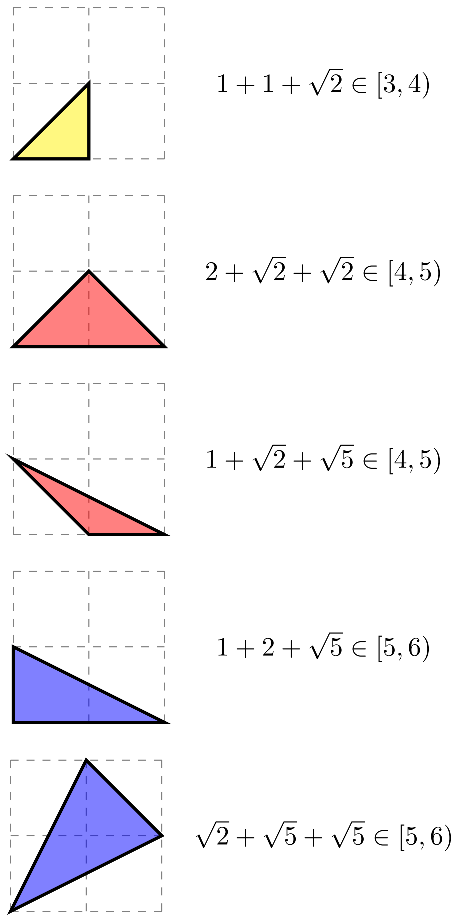


Figure 1: An example in yellow showing that $a(3) = 1$, and example in red showing that $a(4) = 2$, and an example in blue showing that $a(5) = 3$.

Question. Let $a(n)$ count n-triangles up to dihedral action. What is the asymptotic growth of $a(n)$?

Related.

1. How many tetrahedra?
2. How many quadrilaterals?

References.

`https://oeis.org/A298079` counts the number up to congruence.

Problem 44

Problem 47.

Let a palindromic partition be a partition of a string into palindromes.

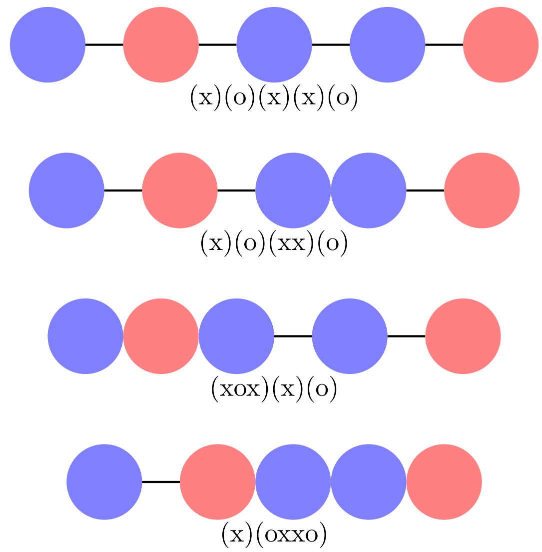


Figure 1: An example of four palindromic partitions of the string "xoxxo".

Question. Given some string, how many palindromic partitions does it have?

Related.

1. What is the least number of parts p such that an arbitrary string of length ℓ over a k-letter alphabet can be partitioned into p or fewer parts?
2. What is the length of the shortest string that cannot be partitioned into fewer than p parts?
3. How many length ℓ strings require the "worst-case" number of parts?
4. Which length ℓ strings have the greatest number of distinct partitions? The least?
5. What is the smallest number of parts that any string with m o's and an arbitrary number of x's can be partitioned into?

References.

`https://oeis.org/A298481` the number of ways to partition the binary representation of n into the minimal number of palindromic parts.

Problem 48.

Consider folding a strip of n equilateral triangles down to 1 triangle in as few moves as possible.

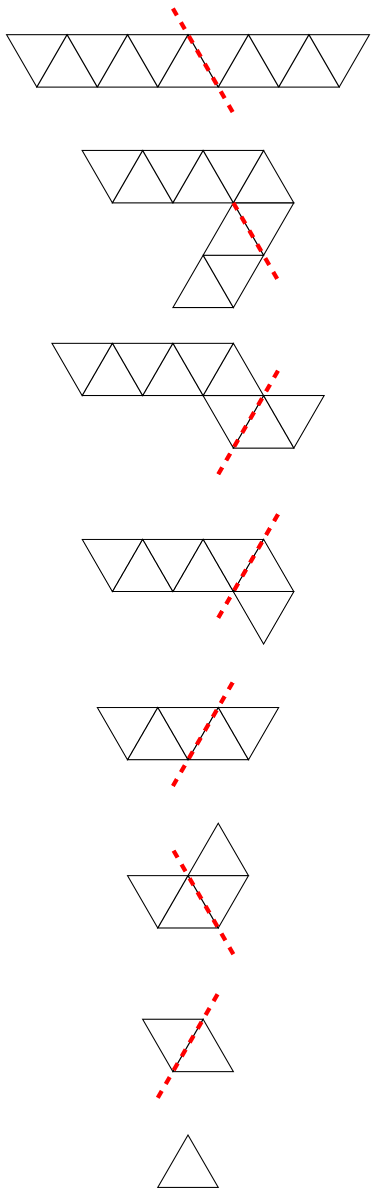


Figure 1: An example demonstrating that $a(11) \leq 7$.

Question. How many folds are required to fold a strip of n triangles down to one?

Related.

1. What if you can choose the starting configuration of the triangles? (e.g for $n = 10$, you can start from the second example)
2. What if you can fold along an entire line? (Not just a single cell.)

Problem 49.

Consider a 2-coloring of a triangular grid of length ℓ. Then label each cell with its greatest number of neighbors of one color.

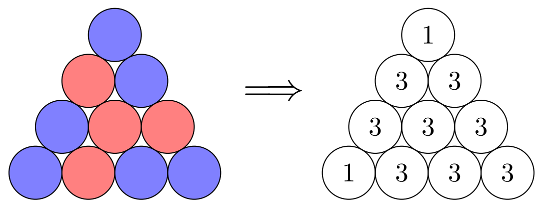


Figure 1: The second cell (reading top to bottom and left to right) is labeled with a $\max(3, 1) = 3$ because it has 3 blue neighbors and 1 red neighbor.

Question. How many colorings exist of a length ℓ triangle such that the maximum label is 3?

Related.

1. If the "number triangle" is summed for each coloring, which coloring has the smallest sum?
2. How does this generalize for a k-coloring?
3. How does this generalize to a $n \times m$ square grid where horizontal-vertical connections are counted? Diagonal connections? Both?
4. How does this generalize to a tetrahedron, torus, Möbius strip, cylinder, or cube?
5. How many colorings exist of a length ℓ triangle such that the maximum label is 4 or 5?

Problem 50.

Call s an "initial permutable" string if for every initial substring of odd length, the first half of the string is a permutation of the second half.

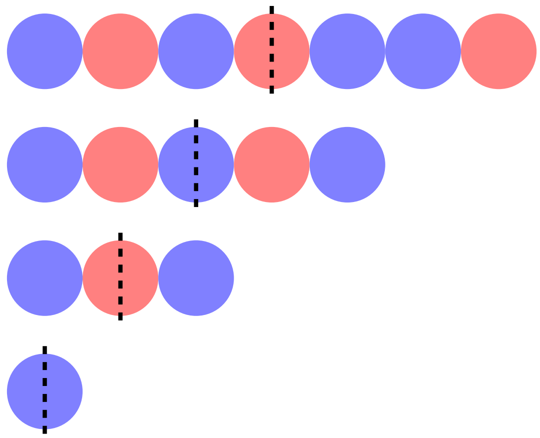


Figure 1: "BRBRBBR" is an example of an initial permutable string. Because each initial odd substring (the string itself, "BRBRB", "BRB", and "B") has the property that the first half of the string is a rearrangement of second half.

Question. What is the growth of $a_2(n)$, the number of initial permutable strings of length $2n-1$ over a 2-letter alphabet?

Related.

1. Can this be generalized to a k-letter alphabet?

References.

`http://oeis.org/A297789`

Problem 51.

Given two vector valued functions $u, v\colon \mathbb{R}^n \to \mathbb{R}^n$, that are linearly independent at every point, let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x_0, x_1) = |\alpha| + |\beta| \text{ where } x_1 - x_0 = \alpha \cdot u(x_0) + \beta \cdot v(x_0).$$

Next let the length of a curve $\Gamma\colon [0,1] \to \mathbb{R}^n$ be given by

$$\mathcal{L}(\Gamma) = \lim_{N\to\infty} \sum_{k=0}^{N-1} f\left(\Gamma\left(\frac{j}{N}\right), \Gamma\left(\frac{j+1}{N}\right)\right).$$

Let the distance $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ from x_0 to x_1 be given by the infimum of the length over all curves from x_0 to x_1:

$$d(x_0, x_1) = \inf\{\, \mathcal{L}(\Gamma) : \Gamma(0) = x_0 \text{ and } \Gamma(1) = x_1 \,\}.$$

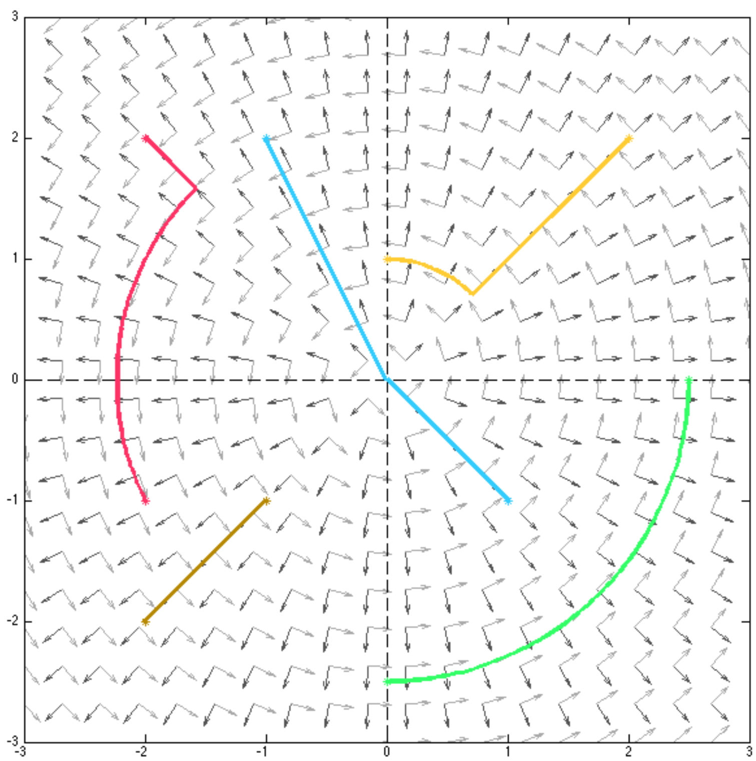


Figure 1: Five examples of shortest curves when $u(x_1, x_2) = (x_1, x_2)/||(x_1, x_2)||$ and $v(x_1, x_2) = (-x_2, x_1)/||(x_1, x_2)||$.

Question. What are the necessary conditions on u and v for this to be a well-defined metric space?

Related.

1. If $|u(x)| = |v(x)| = 1$ for all $x \in \mathbb{R}^n$, what is greatest possible (Euclidean) length of the circumference of a unit circle?

2. If u and v are well-behaved and selected at random according to some distribution, what is the expected length of the circumference of a unit circle?

Problem 52.

★★★★

Suppose that there is a "cop" and a "robber" on an infinite grid, where each starts at some given position with some given orientation on the grid, and each can move according to some rule set.

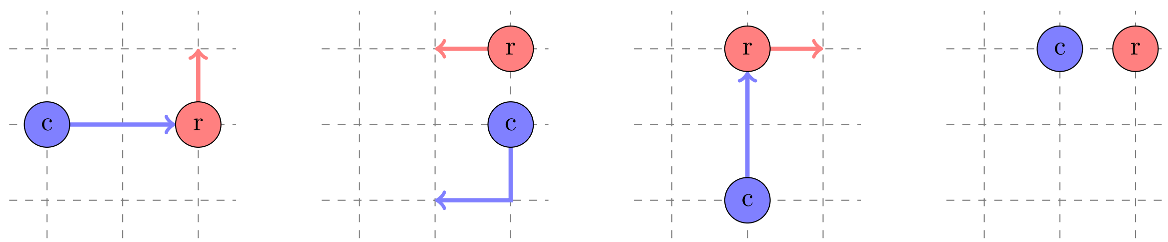


Figure 1: In this example, the cop can perform any of the following moves $C = \{$ 2 units straight, 1 unit right + 1 unit right, 1 unit right + 1 unit straight $\}$ and the robber can move one unit in any direction along the grid. After three steps, the cop has not caught the robber, but if the robber moves forward, backward, or right, then she will be caught.

Question. Is there a procedure for determining in general whether the cop can catch the robber?

Related.

1. Is there a procedure that can put a bound on the number of steps it will take for the cop to catch the robber?
2. If the cop/robber perform moves in their rule set according to some distribution, what is the probability that the cop will eventually catch the robber?
3. How does this generalize to a torus, Möbius strip, cylinder, multiple dimensions or a triangular/hexagonal grid?

References.

http://demonstrations.wolfram.com/TheHomicidalChauffeurProblem/

https://en.wikipedia.org/wiki/Homicidal_chauffeur_problem

Problem 53.

A problem based on a conversation with Alec Jones. Consider a variation on the "concavity classes" of polygons as described by OEIS sequence A227910. Say that two n-gons are in the same concavity class if one can be continuously deformed into the other (or a mirror image of the other) while (1) remaining an n-gon the entire time, and (2) preserving the number of sides of the convex hull.

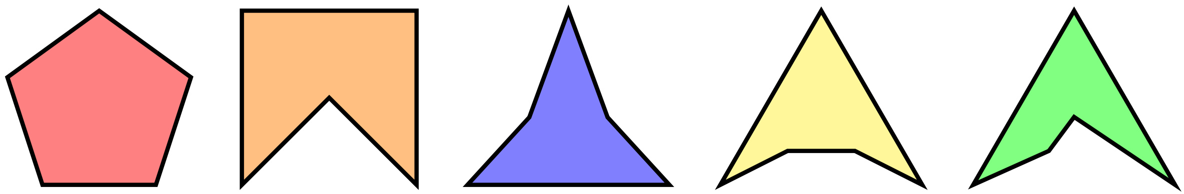


Figure 1: The $a(5) = 5$ concavity classes on the pentagon.

Question. How many convexity classes are there of an arbitrary n-gon?

Related.

1. What is the smallest square lattice that contains at least one representative of each concavity class of the n-gon for some fixed n? (That is, the polygons must have integer coordinates.)
2. (Is this the correct defintion?)

References.

https://oeis.org/A227910

Problem 54.

Consider convex polygons with integer coordinates. The notion of a best Diophantine approximation can be generalized to equilateral triangles by saying that a triangle is a better diophantine approximation if the ratio of the largest side to the smallest side is less than the ratio of any other triangle with smaller perimeter.

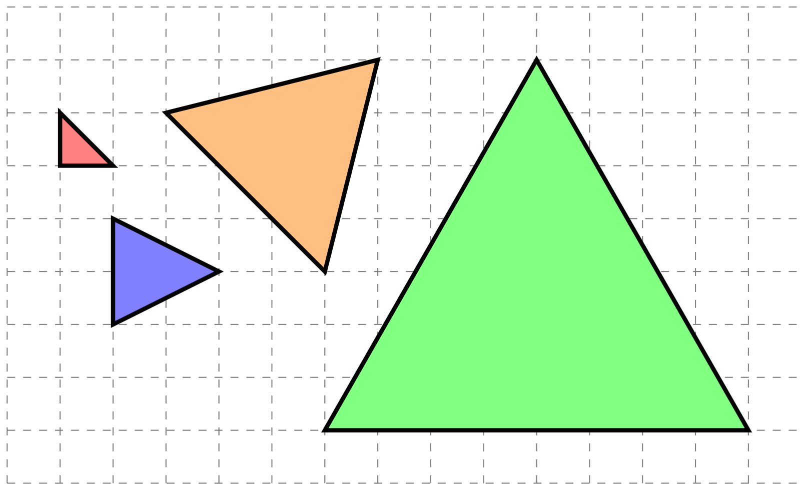


Figure 1: Four best (?) Diophantine approximations of an equilateral triangle. The red triangle has a ratio of $\sqrt{2/1} \approx 1.41$, the blue has a ratio of $\sqrt{5/4} \approx 1.118$, the orange has a ratio of $\sqrt{18/17} \approx 1.029$, and the green has a ratio of $\sqrt{64/63} \approx 1.008$.

Question. What is the growth of the perimeter of the k-th best Diophantine approximation of an equilateral triange as a function of k?

Related.

1. How can this be generalized in a reasonable way to regular n-gons? (Just looking at side lengths isn't enough—angles can behave badly.)
2. What if this is done on tetrahedra?

References.

`https://math.stackexchange.com/q/2251555/121988`

Problem 55.

For each $2n$-gon there exists some number ℓ_n such that there is a equilateral convex polygon with integer coordinates such that all sides have length ℓ_n.

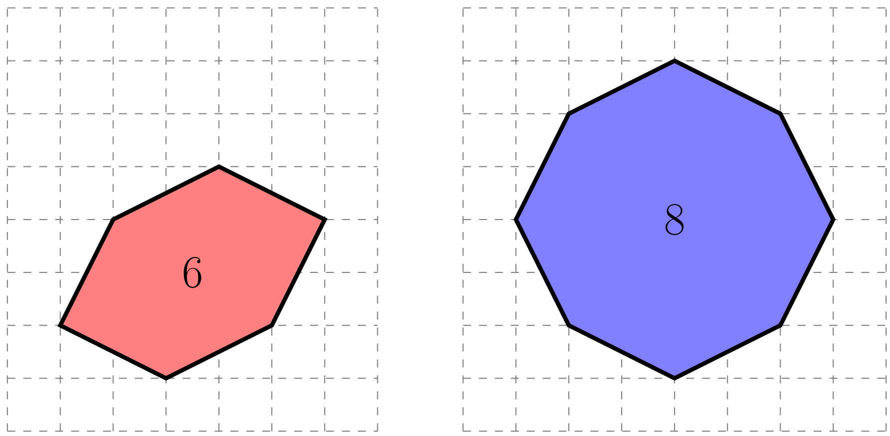


Figure 1: Example demonstrating that $\ell_6 = \ell_8 = \sqrt{5}$.

Question. For what values of m do there exist equilateral $(2m-1)$-gons with integer coordinates?

Related.

1. For m such that there are no equilateral $(2m-1)$-gons, what is do the best Diophantine approximations look like (in the sense of Problem 54)?
2. Does this generalize to polyhedra?

References.

https://oeis.org/A071383

Problem 54

Problem 56.

Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way that no end is "orphaned".

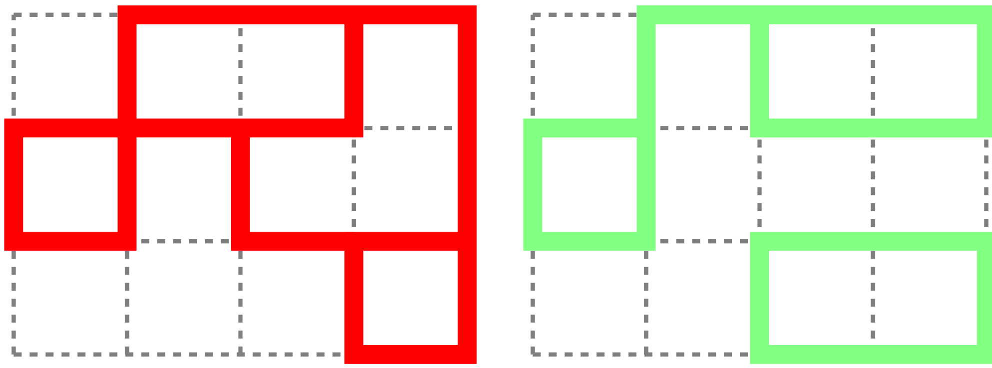


Figure 1: Two examples of a valid configurations on a 5×4 grid; the second has an "island" in the lower right corner.

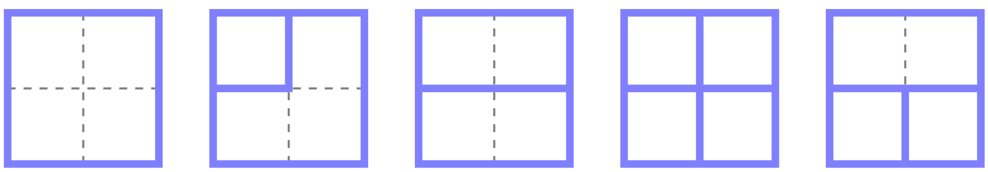


Figure 2: All(?) examples of valid configurations of 3×3 grids with border, up to dihedral action.

Question. Let $a_\ell(n)$ be the number of configurations on the $\ell \times n$ grid. What is a general formula for $a_\ell(n)$?

Related.

1. What if the matchsticks are of length k?
2. How does this generalize to a triangular/hexagonal lattice or to multiple dimensions?
3. What is the number of these configurations with rotational symmetry? Horizontal/vertical symmetry?
4. If such a configuration is chosen uniformly at random, what is the number of expected regions? (e.g. the first example has 4 interior regions.)
5. What if no gridpoint can have degree 0? Degree 2? 3? 4?
6. What if the entire border must be drawn?
7. What if the the graph must be connected (i.e. cannot have an "island".)
8. What if instead of horizontal/vertical lines, diagonals are allowed? All edges have integer slope? Edges don't intersect except at vertices?
9. How many k-ominoes fit in a "tube" of height m? Snuggly?

References.

The number of $2 \times n$ grids appears to be given by A093129.

The number of $3 \times n$ grids is given by A301976.

Problem 57.

Consider equivalence classes of polygonal chains on n segments where

(1) Edges can cross, but no segment can have a vertex on another segment's edge.

(2) Two chains are equivalent if one can move to the other without an edge crossing over a vertex, or a crossing being otherwise changed.

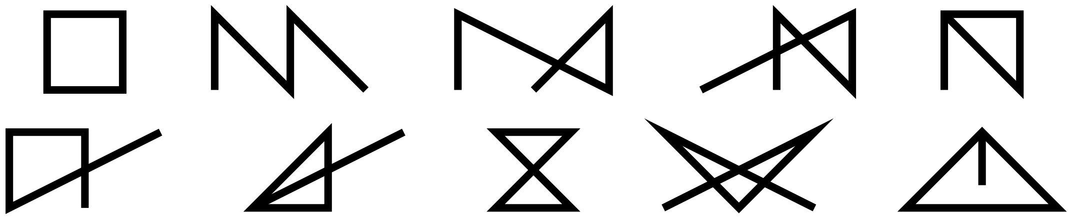


Figure 1: Examples of all known classes of polygonal chains of length 4.

Question. How many classes of polygonal chains exist on n segments?

Related.

1. What if all segments are of unit length, so the final example is not allowed?
2. What if the fifth and seventh example are considered the same because they are isomorphic as graphs? (Even if vertices are added at each intersection)
3. What is the smallest grid that can contain the figures if vertices must be placed on gridpoints?

References.

Problem 53.

Problem 58.

Consider a peaceable queens problem in an $n \times n \times n$ chess "cube", where a queen can move in any diagonal direction.

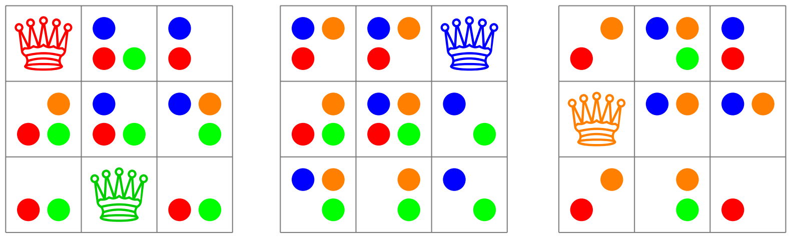


Figure 1: At least four hyper-queens can be placed peaceably on a $3 \times 3 \times 3$ board.

Question. What is the greatest number of queens that can be placed on an $n \times n \times n$ board?

Related.

1. If n^{k-1} queens can be placed on a $\underbrace{n \times n \times \ldots \times n}_{k}$ board for sufficently large n, how large must n be?

References.

`https://math.stackexchange.com/q/2232287/121988`

Problem 59.

A snail travels along the grid in unit steps—but it hates crossing its trail, so if a step is going to cross is trail, it will only go half way.

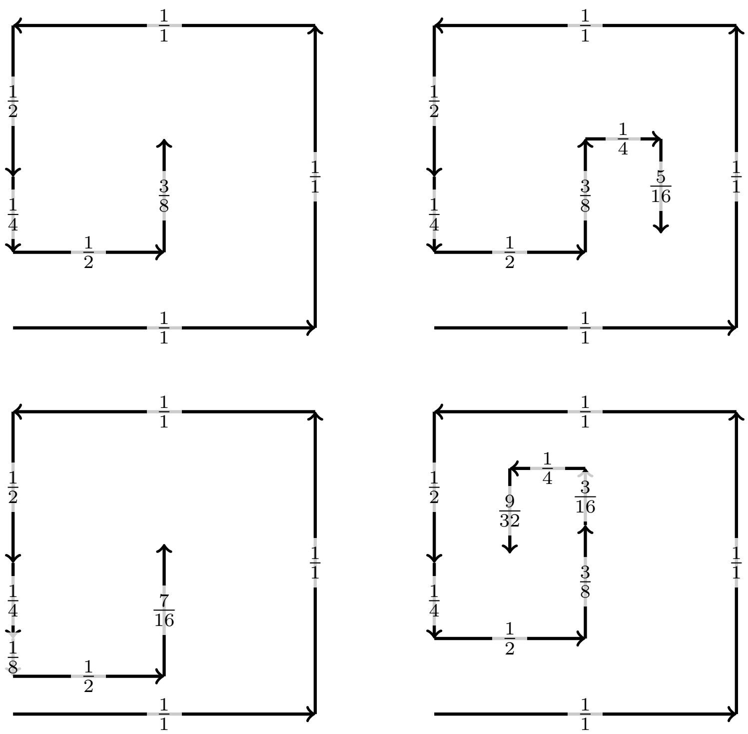


Figure 1: Left-to-right and top-to-bottom: examples of walks that end in step sizes that have numberators of 3, 5, 7, and 9.

Question. Let $a(n)$ be the minimum number of steps the snail must take before it can take a step of size $(2n-1)/2^k$. What is $a(n)$?

Related.

1. What if the snail must always turn left or right?
2. What if the snail is walking on a triangular or hexagonal grid?
3. What is the set of points the snail can step on after finitely many steps?
4. How many distinct points can the snail reach after m steps?

References.

https://math.stackexchange.com/q/2678852/121988

https://oeis.org/A300444

Problem 60.

Say that two sequences with distinct elements are in the same equivalence class if their first differences have the same signs. (e.g. $(1, 3, 2, 3)$ and $(7, 8, -1, 0)$ are equivalent because their first differences are both $(+, -, +)$.)

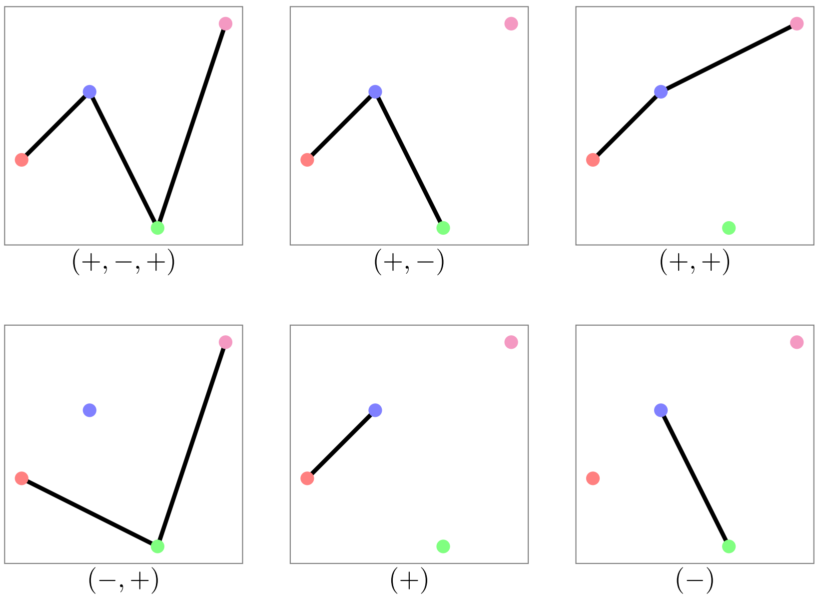


Figure 1: $(0, 1, -1, 2)$ has subsequences in the following six equivalence classes: $(+, -, +)$, $(+, +)$, $(+, -)$, $(-, +)$, $(+)$, $(-)$. No length 4 sequence has its subsequences in more equivalence classes, so $a(4) = 6$.

Question. What is the general formula for $a(n)$?

Related.

1. What if the sequences do not necessarily consist of distinct elements?
2. What if two sequences are considered to be equivalent if they are in the same "sort order"; that is, if both sequences have their biggest element in the same position, their second biggest in the same position, and so on.
3. What if $(+, +) \sim (+)$?
4. Is the number of equivalence classes for the subsequences determined by the number of local minima and maxima?

Note. A quick attempt finds that $a(2) = 1$, $a(3) = 3$, $a(4) = 6$, and $a(5) = 11$. (Fibonacci minus 2?)
For related question 3, conjecture the answer is $a'(n) = 2n - 3$ for $n \geq 2$.
For related question 3 without distinct elements (as in related question 1), the inital terms are

$$a(2) = 1 \text{ via } (+)$$
$$a(3) = 4 \text{ via } (+,-),(+),(-),(=)$$
$$a(4) = 8 \text{ via } (+,-,+),(+,-),(+,=),(=,+),(-,)\,(+),(-),(=)$$
$$a(5) = 15; a(6) = 25; a(7) = 40$$
$$a(8) = 62; a(9) = 94; a(10) = 141; a(11) = 210; a(12) = 311 \text{ (conjectured*)}$$

* Assumes sequence is $(1, 2, 1, 2, 1, 2, \ldots)$.

Problem 61.

Consider all r-colorings of the $n \times m$ grid where no two colors are adjancent (horizontally/vertically) more than once.

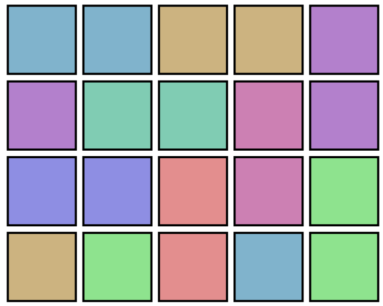


Figure 1: An 8-coloring of the 4×5 grid where no two colors are adjancent more than once. There is no 7-coloring.

Question. Let $r_{n\times m}$ be the smallest integer such that there exists an $r_{n\times m}$-coloring of the $n \times m$ grid. What is $r_{n\times m}$?

Related.

1. What if colors are not allowed to be self-adjacent?
2. How many $a(n, m)$-colorings exist up to permutation of the colors?
3. What if this is done on a triangular or hexagonal grid?
4. What if orientation matters? (A horizontal adjacency is distinct from a vertical adjacency.)
5. What if order matters? (red-green is distinct from green-red.)
6. What if diagonal adjacencies are considered?

Note.

$r_{1\times1} = 1$				
$r_{1\times2} = 1$	$r_{2\times2} = 3$			
$r_{1\times3} = 2$	$r_{2\times3} = 4$	$r_{3\times3} = 5$		
$r_{1\times4} = 2$	$r_{2\times4} = 5$	$r_{3\times4} = 6$	$r_{4\times4} = 7$	
$r_{1\times5} = 3$	$r_{2\times5} = 5$	$r_{3\times5} = 7$	$r_{4\times5} = 8$	$r_{5\times5} = 9$

References.

Problem 23.

Problem 36.

Problem 49.

Problem 62.

Consider ways to place colored markers on an $n \times m$ grid so that no two pairs of markers of the same color have the same distance between them.

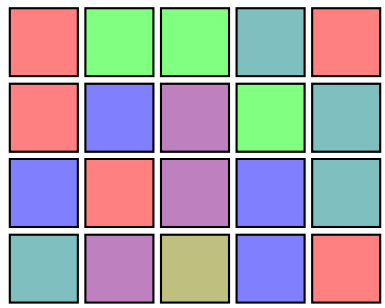


Figure 1: There are 5 markers and $5(5-1)/2 = 10$ distinct distances between red tiles. (There is no way to place 6 tiles.) Placing another red marker would result in two pairs of red markers being the same distance apart.

Question. What is $c_{n\times m}$ the greatest number of markers of a given color can be placed on the $n \times m$ grid?

Related.

1. How many colors of markers are required to fill the grid?
2. What if this is done on the d_1, d_∞, or d_3 metric?
3. What if this is done on a triangular or hexagonal grid?
4. What is the smallest board that can contain k markers?

Note. $c_{n\times m}(c_{n\times m} - 1)/2 \leq A301853(n, m) - 1$.

References.

Problem 30.

`https://oeis.org/A301853`

Problem 63.

Consider all of the shapes that can be made with a rubber band and a rubber hand.

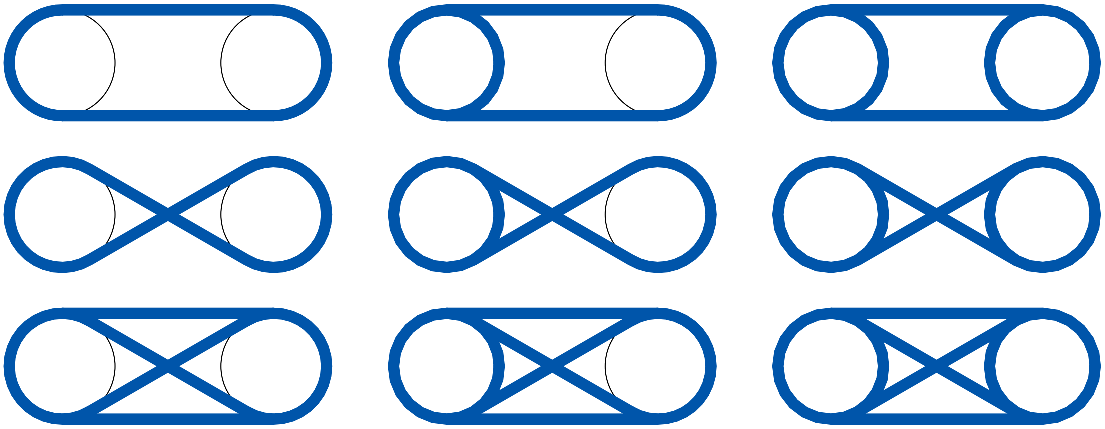


Figure 1: There are (at least) 9 ways to weave a rubber band between two fingers up to reflection/rotation.

Question. How many figures can be made with n fingers and a rubber band?

Related.

1. Is there an analog in higher dimensions?
2. What if all fingers must be aligned?
3. What if all fingers must be on the corners of an n-gon?

Problem 64.

Consider the art gallery problem on all "museum"-equivalence classes of polygons.

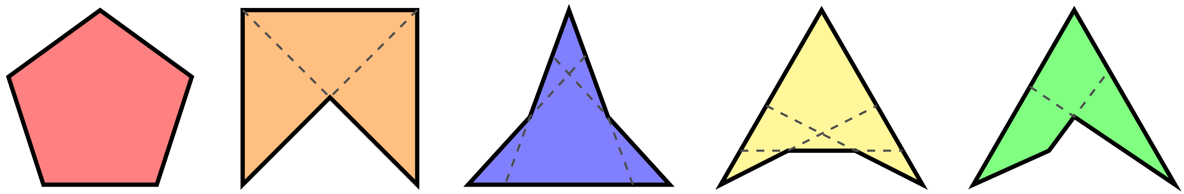
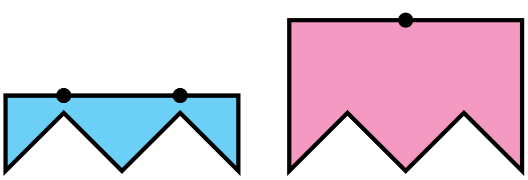


Figure 1: The concavity classes from Problem 53. It appears that the second and fifth polygons are museum-equivalent. Are the third and fourth polygons equivalent?

Question. If each polygon is a museum, how many guards are required to patrol the museum?

Related.

1. What if guards are stationed at a corner in the polygon?
2. What if guards are allowed to patrol along a wall?
3. What if the polygons are on a torus or cylinder?
4. What if the polygons are orthogonal (i.e. each wall meets at a right angle)?
5. What if the guards must patrol the outside of the polygon?
6. How many equivalence classes of museums exist? For example, the following museums are distinct, because the first requires two guards, and the second requires only one.



References.

Problem 53.

`https://en.wikipedia.org/wiki/Art_gallery_problem`

Problem 65.

Starting with a configuration of coins, slide one coin at a time such that the coin ends up in a position where it is tangent to two other coins.

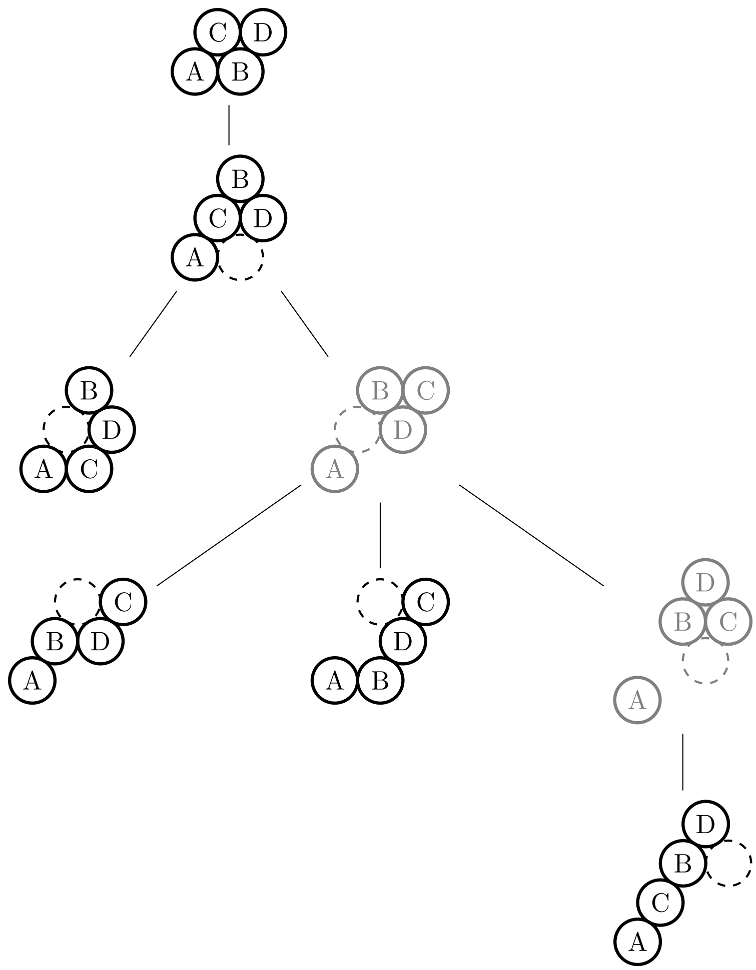


Figure 1: All connected configurations of 4 coins. Six out of the seven possible polyhexes are present.

Question. In general, given n coins starting in a “spiral” configuration, how many polyhexes can be reached by the above procedure?

Related.

1. What if this is done with hyperspheres in $\mathbb{R}^d$?
2. Is there a sensible way to categorize non-connected configurations?
3. Which polyhexes require the greatest amount of moves?

References.

`https://en.wikipedia.org/wiki/Polyhex_(mathematics)`

`https://www.youtube.com/watch?v=_pP_C7HEy3g`

Problem 66.

The number of ways to draw a triangle on a triangular grid is given by

$$\sum_{k=1}^{n-1} k \cdot t(n-k) = \binom{n+2}{4}$$

where $t(m)$ is the m-th triangular number.

The number of ways to draw a square on a square grid is given by

$$\sum_{k=1}^{n-1} k \cdot (n-k)^2 = n^2 \left(\frac{n^2-1}{12}\right)$$

the 4-dimensional pyramidal number.

The number of ways to draw a hexagon on a hexagonal grid is given by

$$\sum_{k=1}^{n-1} k \cdot h(n-k) = \sum_{k=1}^{n-1} k^3 = t(n-1)^2$$

where $h(m)$ is the m-th hexagonal number.

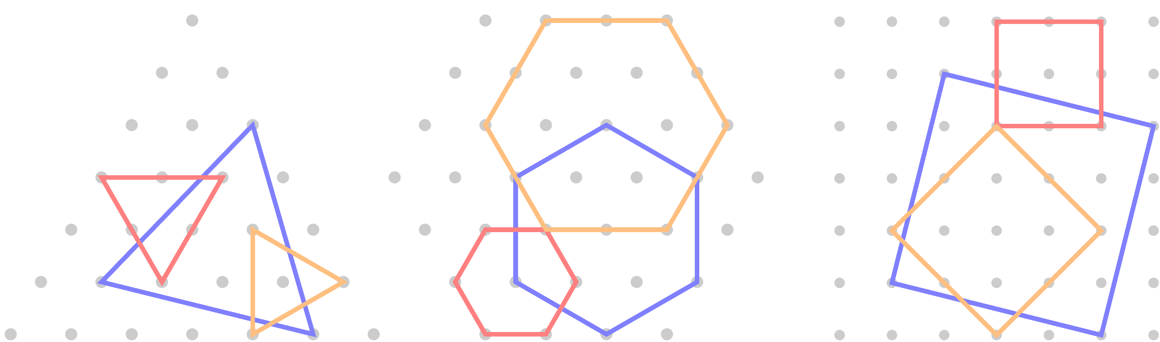


Figure 1: All connected configurations of 4 coins. Six out of the seven possible polyhexes are present.

Question. Is there a combinatorial explanation for why these numbers have nice closed forms?

Related.

1. Can this be generalized to arbitrary regular n-gons in hyperbolic space?
2. How many triangles are on the “centered triangular number” grid?

References.

Problem 21.

`https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling`

Problem 67.

Consider all of the ways to take a square piece of paper and make two “precise” creases—that is, we can make a crease between two distinguished points, we can crease the paper such that any two distinguished points touch, and we can take an angle and bisect it.

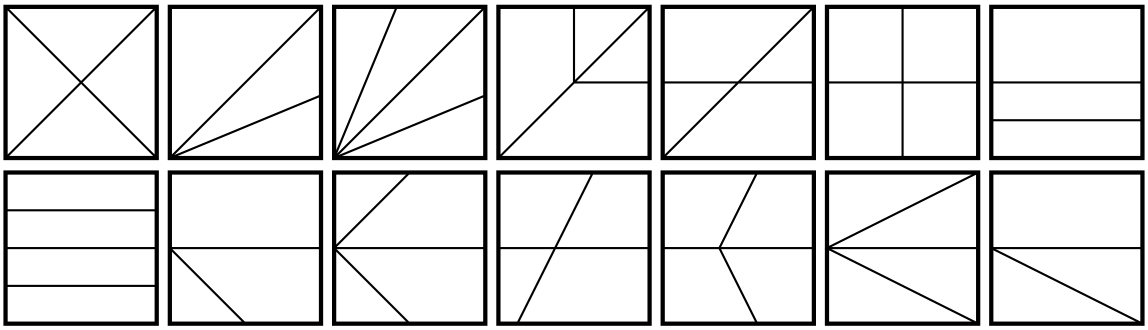


Figure 1: Fourteen (all?) two-crease patterns.

Question. How many such crease patterns exist on n creases?

Related.

1. If we “overlap” all diagrams, how many distinct lines?
2. What if we start with a rectangle? Equilateral triangle?
3. What if n is the number of folds, and unfolding counts as a fold?
4. What if we restrict the possible folds—for example, disallow folding a crease between two distinguished points?

References.

`https://en.wikipedia.org/wiki/Crease_pattern`

Problem 68.

★★★★

Consider ways to lay matchsticks (of unit length) on the $n \times m$ grid in such a way as to form a maze.

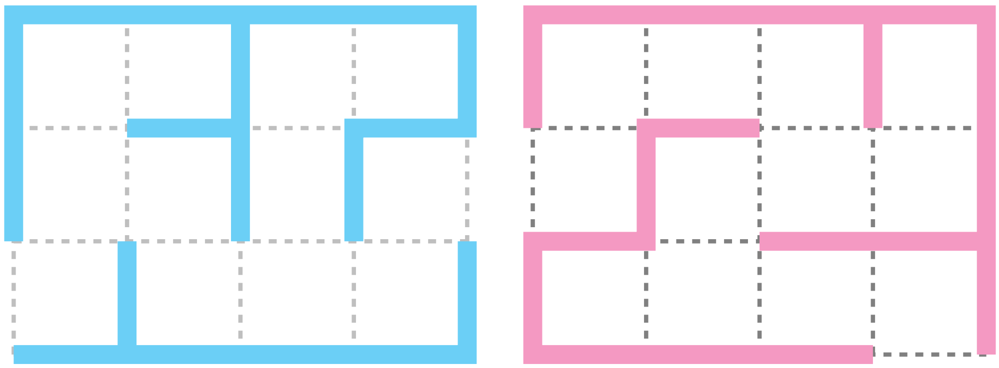


Figure 1: Two mazes on a (5×4)-cell grid.

Question. How many distinct mazes can be drawn on the grid?

Related.

1. What if every 1×1 cell must be reachable?
2. What if there are no dead ends?
3. What if there are to be identically k dead ends?
4. What if paths that loop are not allowed?
5. What if the entrance and exit have prescribed positions?
6. What if this is done on a hexagonal or triangular grid? On a torus?
7. Is there a meaningful way to assign "difficulty" to a maze?

Note. This appears to be the number of spanning trees on the $n \times m$ grid graph such that the start and end are leaves.

References.

Problem 56.

https://oeis.org/A116469

Problem 69.

Say that an n-robot takes steps that are $1/n$ of a circle ($2\pi/n$ radians). Call a (k, j)-step pattern a walk that starts with k right turns, followed by j left turns, followed by k right turns, and so on until the robot reaches its original position in the original orientation.

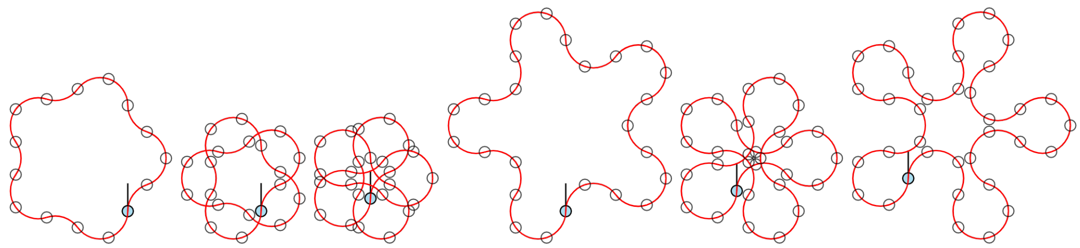


Figure 1: A 5-robot walks in $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, and $(3, 4)$-step patterns, respectively.

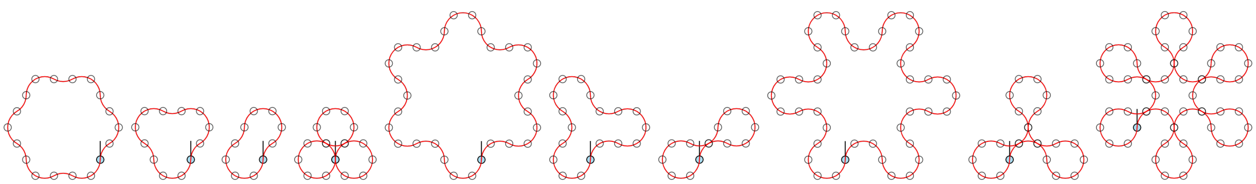


Figure 2: A 6-robot walks in $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, and $(4, 5)$-step patterns, respectively.

Question. For an n-robot, which of these paths encloses the most area? The least area?

Related.

1. Which of these figures has the largest convex hull? Smallest convex hull?
2. Is there a way to tell at a glance whether or not these walks will self-intersect?
3. Is there a way to tell at a glance if a (k, j)-step pattern will "go off to infinity"?
4. Are the areas enclosed by these figures "nice" numbers?
5. How does this generalize to $(a_1, a_2, \ldots, a_k)$-step patterns?
6. How many steps are taken before the figure "reconnects"?
7. For what step patterns are the "footprints" (the small grey circles in the figure) closest together (the $(2, 4)$-step pattern for the 5-robot)? How many steps are required to get two footprints within ε?
8. What if the robot turns $1/n$ of a circle when it turns right, but $1/m$ of a circle when it turns left?
9. What if the robot turns with some other rational number a/b of a circle?
10. What if the robot only needs to reach the original position, but not original orientation?

Note. It is likely that 3, 4, and 6-robots are special cases because the footprints appear at lattice points.

References.

Problem 41.

https://cemulate.github.io/project-euler-208/

Problem 70.

Starting with a pair of integers (a, b), there exists an algorithm for making the two integers equal by repeated applications of the map $(x, y) \overset{\alpha}{\mapsto} (2x, y+1)$ or $(x, y) \overset{\beta}{\mapsto} (x+1, y)$.

$$
\begin{array}{llllllll}
(4,0) & \overset{\beta}{\mapsto} (5,0) & \overset{\beta}{\mapsto} (6,0) & \overset{\alpha}{\mapsto} (12,1) & \overset{\beta}{\mapsto} (13,2) & \overset{\beta}{\mapsto} (14,4) & \overset{\beta}{\mapsto} (15,8) & \overset{\beta}{\mapsto} (16,16) \\
(5,4) & \overset{\beta}{\mapsto} (6,8) & \overset{\beta}{\mapsto} (7,16) & \overset{\beta}{\mapsto} (8,32) & \overset{\alpha}{\mapsto} (16,33) & \overset{\beta}{\mapsto} (17,66) & \overset{\alpha}{\mapsto} (34,67) & \overset{\alpha}{\mapsto} (68,68) \\
(8,1) & \overset{\beta}{\mapsto} (9,2) & \overset{\beta}{\mapsto} (10,4) & \overset{\alpha}{\mapsto} (20,5) & \overset{\beta}{\mapsto} (21,10) & \overset{\alpha}{\mapsto} (42,11) & \overset{\beta}{\mapsto} (43,22) & \overset{\beta}{\mapsto} (44,44) \\
(9,6) & \overset{\beta}{\mapsto} (10,12) & \overset{\beta}{\mapsto} (11,24) & \overset{\beta}{\mapsto} (12,48) & \overset{\alpha}{\mapsto} (24,49) & \overset{\beta}{\mapsto} (25,98) & \overset{\alpha}{\mapsto} (50,99) & \overset{\alpha}{\mapsto} (100,100) \\
(11,7) & \overset{\beta}{\mapsto} (12,14) & \overset{\beta}{\mapsto} (13,28) & \overset{\beta}{\mapsto} (14,56) & \overset{\alpha}{\mapsto} (28,57) & \overset{\beta}{\mapsto} (29,114) & \overset{\alpha}{\mapsto} (58,115) & \overset{\alpha}{\mapsto} (116,116)
\end{array}
$$

Figure 1: Five examples of (shortest) seven-step paths to equality, starting from $(4, 0)$, $(5, 4)$, $(8, 1)$, $(9, 6)$, and $(11, 7)$.

Question. What is an algorithm for the shortest path to equality?

Related.

1. What are some good upper bounds for the shortest path length?
2. Can this be generalized to other maps (e.g. $(x, y) \mapsto (3x, y+2)$)?
3. What is the least k such that there is a path from (a, b) to (k, k)? Is there a way to characterize all such values for k?

References.

`https://oeis.org/A304027`

`https://codegolf.stackexchange.com/q/164085/53884`

Problem 71.

Let a "polyarc" be a path composed of quarter circular arcs, and a polyarc-configuration be a placement of polyarcs on an $n \times m$ grid such that no part of a polyarc is inside another polyarc.

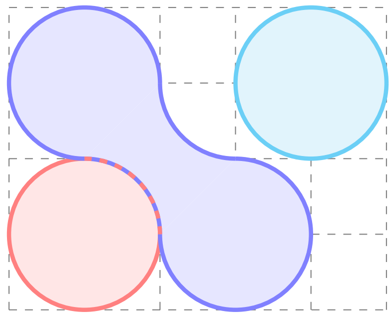


Figure 1: An example of a polyarc-configuration

Question. How many such polyarc-configurations exist on the $n \times m$ grid?

Related.

1. What if polyarcs can be inside other polyarcs?
2. What if all polyarcs must be tangent to another polyarc?
3. With if the polyarc-configuration must be connected?
4. What configuration gives maximum area for the entire polyarc-configuration?
5. Can this be done on a triangular/hexagonal grid with 1/3 or 1/6 arcs?
6. How many different non-self-intersecting 4-robot walks can fit inside of an $n \times m$ grid? Self-intersecting?
7. What if instead of quarter circles, boxes were made with diagonal line segments?
8. Is there a nice multi-dimensional analog?

References.

Problem 69.

Problem 72.

Let a k-tile *multipolyform* be a generalized polyform on a tiling, that is, a choice of k tiles in the tiling that are edge-adjacent.

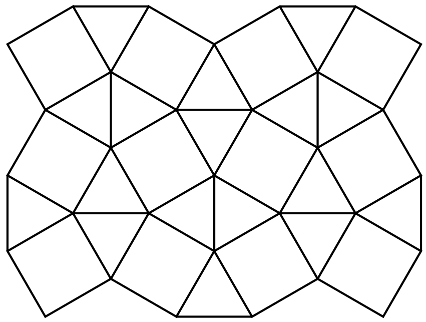


Figure 1: The snub square tiling.

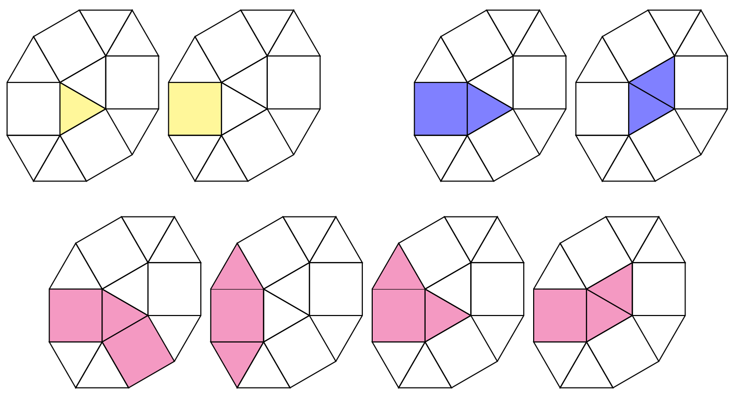


Figure 2: All 1-tile, 2-tile, and 3-tile multipolyforms on a snub square tiling.

Note. It is hard to count polyominos and polyforms more generally.

Question. Is there a unified method for counting multipolyforms on an arbitrary tiling—that is, a method that is not *ad hoc* for each tiling?

Related.

1. What is the smallest region of the plane that can contain all k-polyforms? (As in Moser's worm problem.)
2. Do the multipolyforms described in the example grow significantly faster than polyominos? What aspects of the tiling does the asymptotic growth depend on?

References.

`https://en.wikipedia.org/wiki/Snub_square_tiling#/media/File:1-uniform_n9.svg`

`https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons`

`https://en.wikipedia.org/wiki/Polyform`

Problem 73.

A Heronian 2-simplex (triangle) is a triangle with both integer sides and integer area. A Heronian n-simplex is an n-simplex with integer volume and where all sides are Heronian $(n-1)$-simplices.

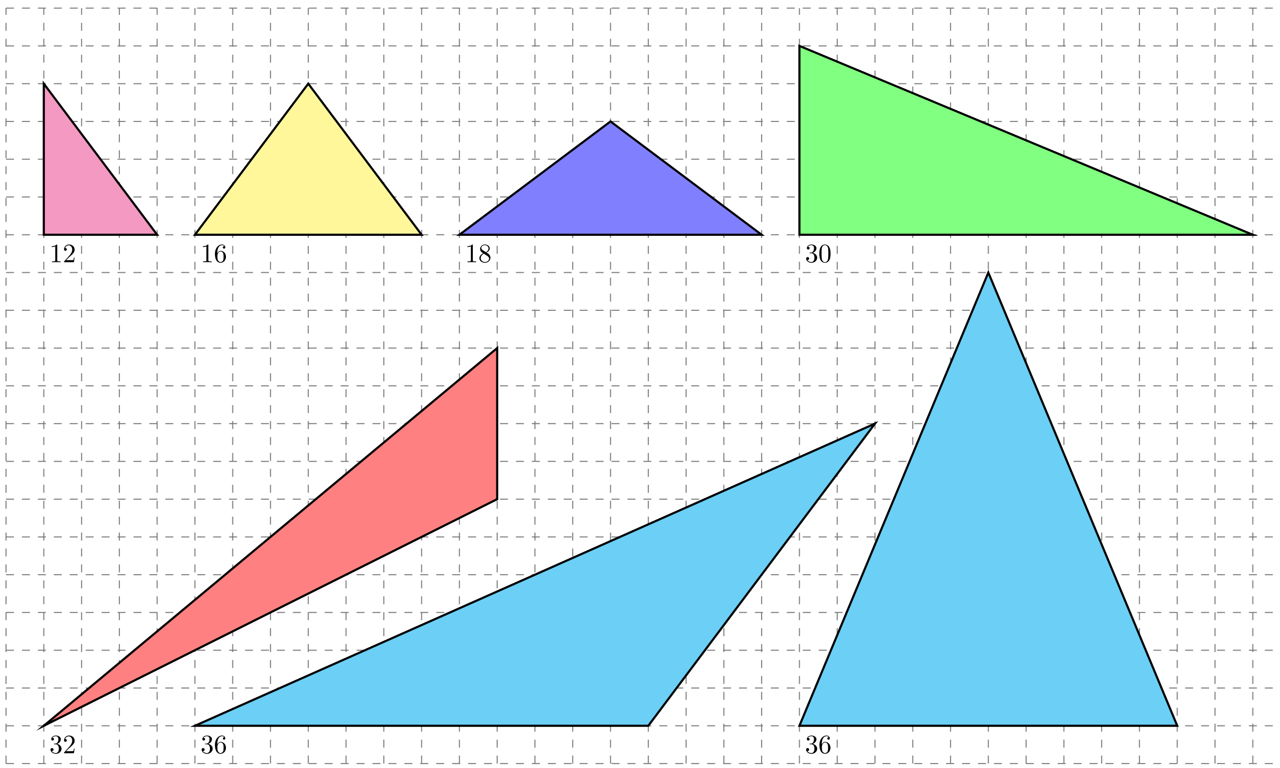


Figure 1: The seven smallest primitive Heronian triangles as measured by perimeter.

Question. Do Heronian n-simplices exist for all integers n?

Related.

1. Do infinitely many primitive Heronian n-simplices exist for each n?
2. What is the smallest Heronian n-simplex for each n as measured by volume? As measured by largest side? As measured by sum of sides? As measured by "surface area" (sum of volume of $(n-1)$-degree facets)?
3. Are all Heronian n-simplices lattice simplices?
4. What if the definition is relaxed so that only the volume and the edges must be integers?

References.

https://www.jstor.org/stable/2695390

https://oeis.org/A272388

https://en.wikipedia.org/wiki/Heronian_tetrahedron

https://en.wikipedia.org/wiki/Simplex

Problem 74.

In the "nine dots puzzle" or "thinking outside the box puzzle", a player is asked to connect dots arranged in a 3×3 grid using four lines. This can be generalized to connecting the dots of a $n \times n$ grid with $2n - 2$ lines.

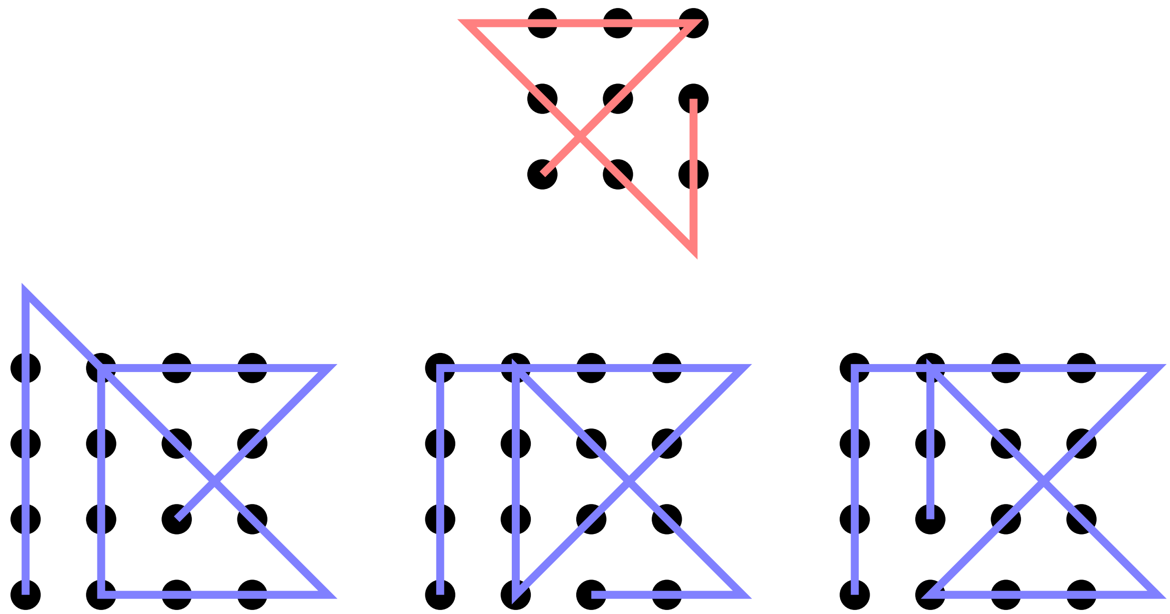


Figure 1: The unique (?) way of completing the 3×3 grid, and three distinct ways of completing the 4×4 grid.

Question. How many distinct solutions exist on the $n \times n$ grid?

Related.

1. What if you want to minimize the area "outside" of the grid?
2. What if you must start and end from the same point?
3. What if you want to minimize the path length?
4. Do any of these have lines that aren't horizontal, vertical, or 45° diagonal?
5. What if this is done on other figures? (Triangles, Diamonds, Octagons, Stars, etc.)
6. Can this be generalized into higher dimensions with lines? Planes?
7. What if the "pencil" can be lifted $k \geq 1$ times?
8. What if this is done on a torus or cylinder?

References.

https://math.stackexchange.com/q/21851/121988

https://en.wikipedia.org/wiki/Thinking_outside_the_box#Nine_dots_puzzle

https://en.wikipedia.org/wiki/Figurate_number

Problem 75.

Ezgi showed me a puzzle where dominoes are placed to form a rectangle such that there is no line that separates the dominoes into two rectangles.

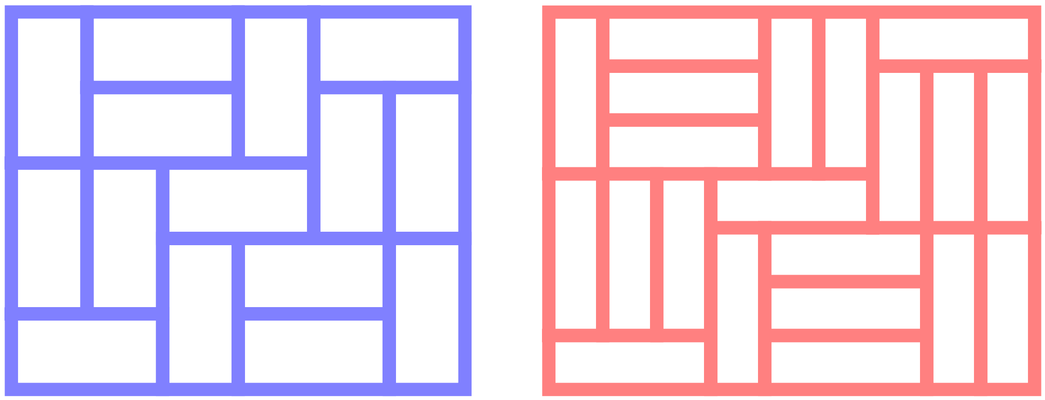


Figure 1: On the right is the smallest way to place dominoes into a rectangle such that there is no way to partition the dominoes into two rectangles. Is the left a minimal configuration with 1×3 triominoes?

Question. What size grids have such configurations?

Related.

1. How many configurations exist for a given grid size?
2. What if other rectangular polyominoes are used? (e.g. 3×2 hexominoes)
3. Are there analogous problems for triangular grids? Higher dimensions?

Note. These are sometimes called "Fault-free domomino tilings".

References.

Project Euler, Problem 215.

Problem 76.

Consider ways to draw diagonals on the cells of $n \times n$ toriodal grid such that no two diagonals touch.

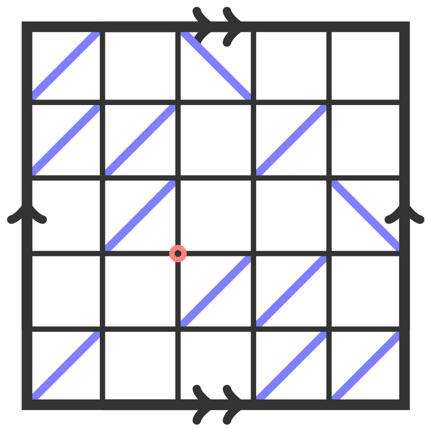


Figure 1: A maximal configuration of a 5×5 toroidal grid: 12 diagonal lines can be drawn. The unused vertex is marked with a circle.

Question. What is the greatest number of diagonals that can be drawn on a $n \times n$ toroidal grid?

Note. Let $m(n)$ be the maximum number of diagonals on an $n \times n$, grid.
Then

$$\begin{aligned} m(2n) &= 2n^2 \text{ and} \\ 2n^2 + n \le m(2n+1) &\le 2n^2 + 2n. \end{aligned}$$

Related.

1. How many configurations exist for a given grid size up to group action?
2. What if this is done on an $n \times m$ grid?
3. What if this is done on a cylinder, Klein bottle, projective space, etc?
4. What is the maximum number of diagonals that can go SW to NE?
5. Does this generalize to three or more dimensions?
6. Can something similar be done on a hexagonal or triangular grid?
7. How many configurations exist if touching is allowed, but cycles aren't?
8. What if we color edges of the grid rather than diagonals on the faces?

References.

`https://oeis.org/A264041`

`https://www.chiark.greenend.org.uk/~sgtatham/puzzles/js/slant.html`

Problem 77.

Consider regions of the plane that can contain all free n-ominoes.

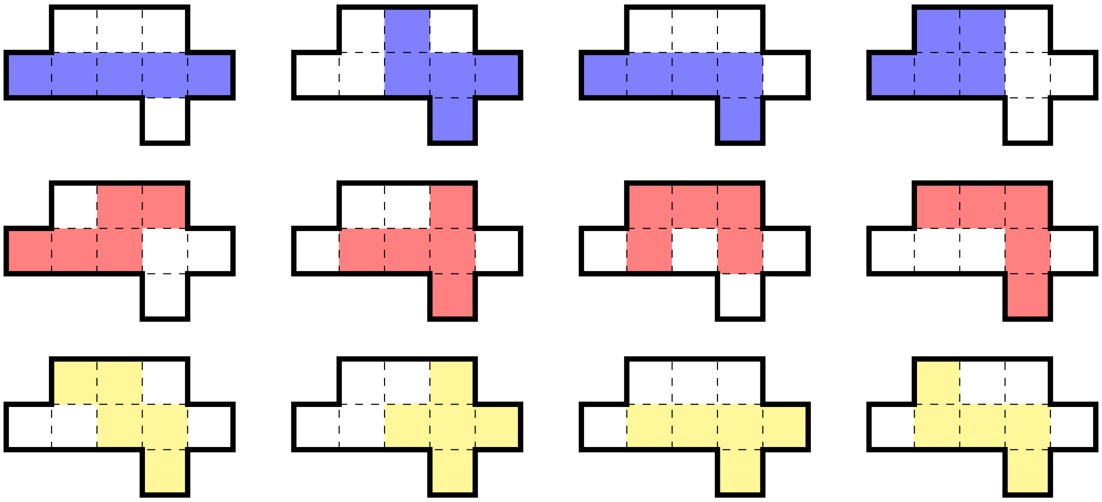


Figure 1: A computer search has proven that a nine-cell region of the plane is the smallest possible region that contains all 5-ominoes.

Question. What is the smallest region of the plane (with respect to area) that can contain all free n-ominoes?

Related.

1. What about fixed polyominoes? One-sided polyominoes (those that can be rotated but not flipped)?
2. What about other polyforms such as polyhexes or polycubes?
3. What if the region must be convex?
4. What is the smallest convex region that contains all length n polysticks (along grid lines)?
5. How many distinct minimal covering sets (call this $c(n)$)?
6. What is the asymptotic growth in area of such a region? (Somewhere between linear and quadratic.)
7. Is there a limiting shape?
8. Alec Jones wonders if there always exists a covering set such that a single cell is used by all polyominoes.

Note. If $c(n)$ counts the number of distinct minimal covering sets of n-ominoes, then $c(1) = c(2) = c(3) = 1$, $c(4) = c(5) = 2$, and $c(6) = 14$.

References.

Problem 75

`https://en.wikipedia.org/wiki/Moser%27s_worm_problem`

`https://en.wikipedia.org/wiki/Polystick`

`https://math.stackexchange.com/q/2831675/121988`

Problem 78.

It is known that trapezoids consisting of 1, 3, and 5 equilateral triangles in a line can tile an equilateral triangle.

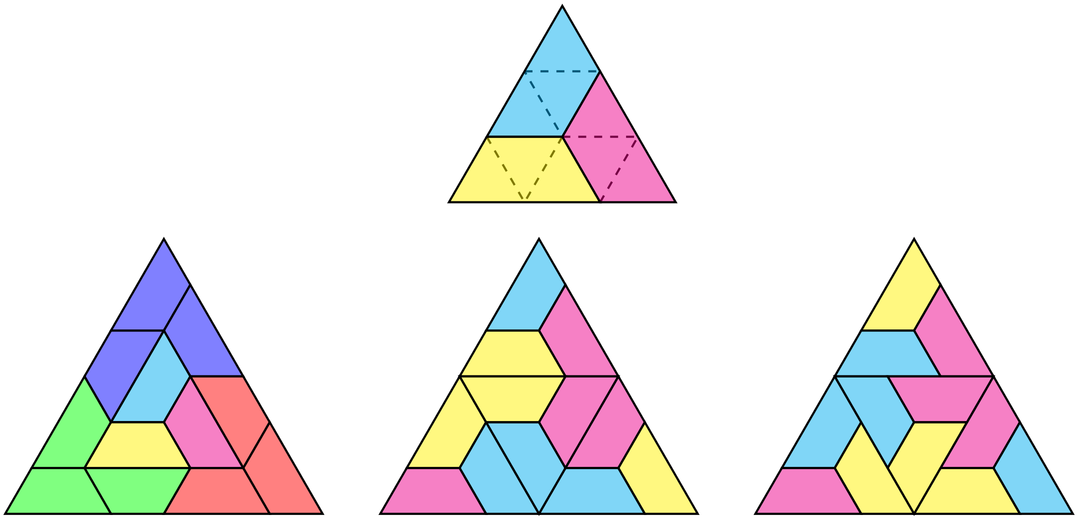


Figure 1: A equilateral triangle made of 3-trapezoids.

Question. Can all $(2n-1)$-trapezoids be arranged to form an equilateral triangle?

Related.

1. What is the smallest triangle that can be formed this way?
2. Is there a construction that makes such triangles given some k-trapezoid?
3. How many such tilings exist for a given size trapezoid and triangle?
4. Can other shapes be tiled (e.g. hexagon, arbitrary trapezoid)?
5. Does this generalize to square/hexagonal tilings? Multiple dimensions?

Note. If $c(n)$ counts the number of distinct minimal covering sets of n-ominoes, then $c(1) = c(2) = c(3) = 1$, $c(4) = c(5) = 2$, and $c(6) = 14$.

References.

Problem 75

`https://math.stackexchange.com/q/2215781/121988`

Problem 79.

Steven Miller poses a riddle:

> A square has a quarter in each corner. You are blindfolded and must get all quarters to be heads up or all to be tails up. You will be told when you have done this. You may flip however many you want, then ask if you are done (this constitutes a turn). The square is then rotated/spun an undisclosed number of times. You then get another turn and so on
> Is there a strategy that is guaranteed to work in a finite number of moves, and if so, what is that smallest number of moves you need to be 100% you'll be able to have all heads up or all tails up?

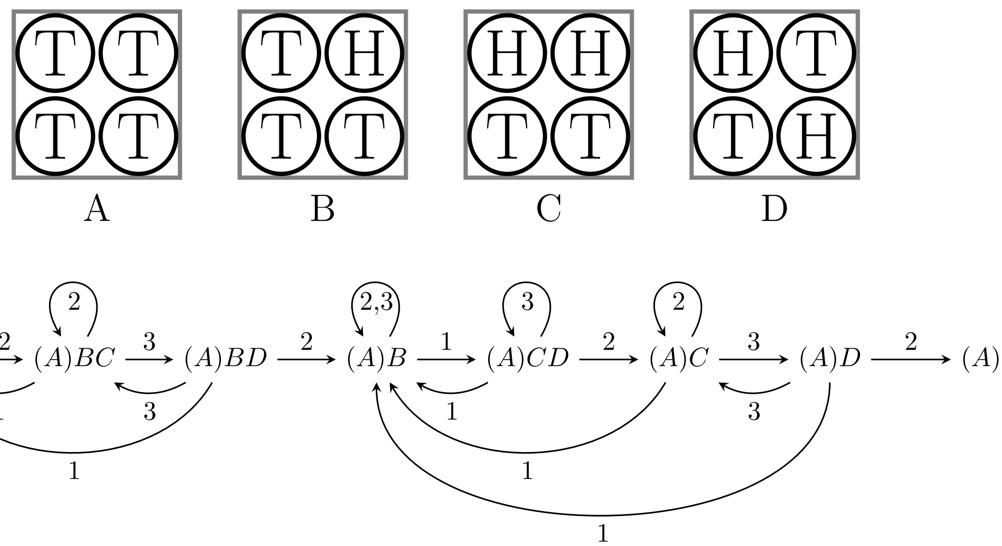


Figure 1: A equilateral triangle made of 3-trapezoids.

Question. If the three moves are (1) flipping a corner, (2) flipping two opposite corners, or (3) flipping over two adjacent corners, then the graph shows that all heads or all tails can be guaranteed in a minimum of seven moves.

Related.

1. Can this be generalized to an arbitrary n-gon?
2. Can this be generalized to multiple dimensions (e.g. a tetrahedron)?
3. What if the coins need to end up all heads?
4. What if the coins have k sides that are changed randomly? Changed sequentially?
5. What if the operator scrambles (but does not flip) the coins after each move?
6. What if the operator tells you how many coins are up after a given move?

References.

http://mathriddles.williams.edu/?p=77

Problem 80.

Consider figures created out of “blocks” starting from some base state and with the rule that each new block needs to touch as many old blocks as possible.

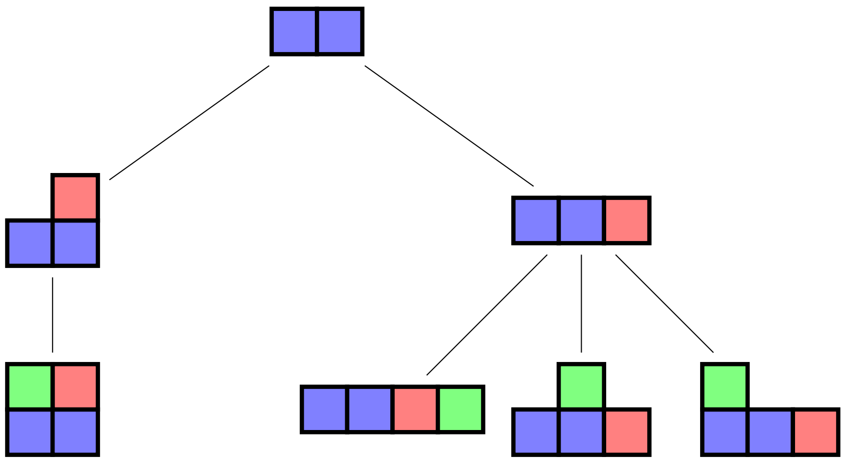


Figure 1: On the leftmost path, the final transition is from an “L” to a square, because the maximum number of faces that can touch is two, so the block must be added in the upper left corner. Counting the number of vertices gives $a(1) = 1$, $a(2) = 2$, and $a(3) = 4$.

Question. How many distinct figures (up to group action) can be made with n blocks, always following a greedy algorithm (with respect to number of faces touching)?

Related.

1. What if this is done with circles on a hexagonal grid? (Polyiamonds, etc.)
2. What if this is done in more than 2 dimensions?
3. What if the starting shape is different? (e.g. the “T” tetromino)
4. What if the blocks are different? (e.g. dominoes)
5. What if the constraint is changed? (e.g. each block must touch exactly two sides)

References.

Problem 65

Problem 81.

Consider pairs subsets of $[n] = \{1, 2, 3, \ldots, n\}$ such that the arithmetic mean of the subsets is equal. How many different pairs of subsets can we find, up to some sort of depedence, where two pairs are equivalent if there exists a linear transformation that takes one pair to the other, or if there exists a "chain" of subsets that implies equality.

$$\left(\frac{1+5}{2} = \frac{1+3+5}{3}\right) \cong \left(\frac{3+5}{2} = \frac{3+4+5}{3}\right) \tag{1}$$

$$\left(\frac{1+5}{2} = \frac{3}{1}\right) \cong \left(\frac{3}{1} = \frac{2+4}{2}\right) \Longrightarrow \left(\frac{1+5}{2} = \frac{2+4}{2}\right) \tag{2}$$

Figure 1: The first equalities are considered equivalent under the linear transformation $x \mapsto \frac{1}{2}(x+5)$. The equality $\frac{1}{2}(1+5) = \frac{1}{2}(2+4)$ is a combination of equations (1) and (2), and so is not an independent equation.

Question. Is there some notion of a "basis" for these pairs of subsets, from which we can work out all pairs with equal means?

Related.

1. What is the minimal "basis" that can describe all pairs of subsets with equal means?
2. What if the subsets in the pair need to be disjoint?
3. Is there a way to combine two pairs of subsets into another pair?
4. Can this generalize to multisets?

Problem 82.

How many functions $f_{n,k}\colon P([k]) - \emptyset \to \{0, 1, 2, \ldots, n\}$ exist between nonempty subsets of $[k]$ and nonnegative integers less than or equal to n such that there exists a sequence of finite sets $(A_1, A_2, \ldots, A_k)$ satisfying

$$f(S) = \# \bigcap_{i \in S} A_i$$

for all $S \in P([k]) - \emptyset$?

	$\#A$	$\#B$	$\#C$	$\#(A \cap B)$	$\#(A \cap C)$	$\#(B \cap C)$	$\#(A \cap B \cap C)$
1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0
3	0	1	0	0	0	0	0
4	1	0	0	0	0	0	0
5	0	1	1	0	0	0	0
6	0	1	1	0	0	1	0
7	1	0	1	0	0	0	0
8	1	0	1	0	1	0	0
9	1	0	1	0	0	0	0
10	1	1	0	1	0	0	0
11	1	1	1	0	0	0	0
12	1	1	1	1	0	0	0
13	1	1	1	0	1	0	0
14	1	1	1	0	0	1	0
15	1	1	1	1	1	1	1

Figure 1: For $n = 1$, $k = 3$, there are fifteen such functions.

Question. How many such functions exist? Equivalently, how many ways to fill in a k-"base set" Venn diagram with integers such that no base set has more than n elements?

Related.

1. What if $\#A_i = \#A_j$ for all $i, j < n$?
2. What if $A_i \not\subset A_j$ for all $i \neq j$?
3. What if this is done with unordered sets? (e.g. the second, third, and fourth functions in the example are all considered equivalent.)
4. What if the correpsonding diagrams need to be realizable as grid rectangles with areas corresponding to the values in the table?
5. What if this is done with set union instead of set intersection?

References.

OEIS Sequence A000330 handles the case where $k = 2$.

OEIS Sequence A319777 handles the case where $k = 3$.

Problem 83.

★★★★

Consider a system of linear recurrences $a_1, a_2, \ldots, a_t$ where

$$a_i(n) = b_{i,1}a_1(n-1) + b_{i,2}a_2(n-1) + \ldots + b_{i,t}a_t(n-1).$$

$$\begin{array}{ll}
a_1(1) = 0 & a_1(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + 2a_5(n-1) + a_7(n-1) \\
a_2(1) = 0 & a_2(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_3(1) = 0 & a_3(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_4(1) = 0 & a_4(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_5(1) = 0 & a_5(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + a_5(n-1) \\
a_6(1) = 0 & a_6(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_7(1) = 1 & a_7(n) = a_5(n-1) + a_7(n-1)
\end{array}$$

Figure 1: In this system of equations, the function $f(n) = a_5(n) + a_7(n)$ (which counts no-leaf subgraphs of the $2 \times n$ grid.) satisfies the recursion $f(n) = 5f(n-1) - 5f(n-2)$ for $n > 2$.

Question. Can any linear combination of these recurrences be turned into a single linear recurrence? If not, how far can it be "simplified"?

Related.

1. What are the number of terms in such a linear recurrence? (i.e. how "deep" does it go? In the example, it has a depth of 2.)
2. What if the initial recurrences have depth greater than 1? For example:

$$\begin{aligned} a_i(n) = {} & b_{i,1,1}a_1(n-1) + b_{i,1,2}a_1(n-2) + \ldots + b_{i,1,k_1}a_1(n-k_1) + \\ & b_{i,2,1}a_2(n-1) + b_{i,2,2}a_2(n-2) + \ldots + b_{i,2,k_2}a_2(n-k_2) + \ldots \\ & b_{i,t,1}a_t(n-1) + b_{i,t,2}a_t(n-2) + \ldots + b_{i,t,k_t}a_t(n-k_t) \end{aligned}$$

References.

A special case of Problem 56 is counted in the example.

Problem 84.

Starting with a row of n coins all heads up, repeatedly flip over a coin which is heads and its neighbor to the right. If the chosen coin is the rightmost coin, there is no neighbor to flip.

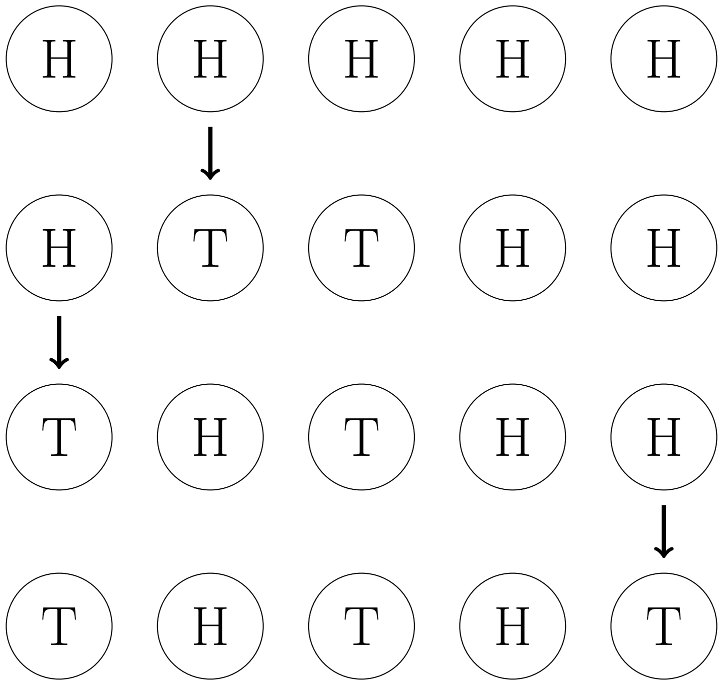


Figure 1: Since the sequence of coin flips strictly increases lexicographically (with $T > H$), the process must eventually halt.

Question. If the puzzle is modified so that when a coin is chosen, either the right or left neighbor is chosen (with probability p and $1-p$ respectively), what is the optimum strategy for maximizing the total number of flips?

Related.

1. What is the strategy for minimizing the number of flips?
2. What is the expected number of total flips under optimal play?
3. What if the direction is randomly chosen, and then you choose which coin to flip? (i.e. you know the direction before you make your choice.)
4. What if the (infinite) sequence of choices have to all be made ahead of time?
5. What if this is done on a different geometry, such as a circle or grid?
6. What if one, neither, or both neighbors have some probability of being flipped?
7. What if coins have more than two states? (e.g dice instead of coins)
8. What if you can flip over a contiguous section of heads?

References.

Problem 79.

Problem 85.

The Erdős distinct distance problem asks for the number of distinct distances determined by n points in the Euclidean plane.

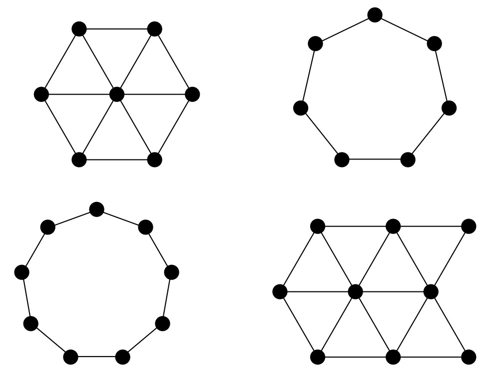


Figure 1: Sets with 3 and 4 distinct distances on 7 and 9 vertices respectively. There are no larger sets with an equal or smaller number of vertices.

Question. If n points are constrained to the grid $\mathbb{Z}^2$, what is the minimal number of distinct distances?

Related.

1. How many such figures?
2. What if the figures are constrained to some subset of $\mathbb{Z}^2$, e.g. $[n] \times [n]$?
3. What about on other grids (triangular, hexagonal, etc)?
4. Can this be meaningfully done on more exotic topologies, e.g. Z_n^2?
5. What about $\mathbb{Z}^n$ for $n > 2$? What is the asymptotic behavior?
6. What if distance is measured via d_1, d_3, or d_∞?

References.

Problem 30.

`https://oeis.org/A186704`: Erdős distinct distance problem.

Problem 86.

Euler's well is a labeling of the $n \times k$ grid with a permutation in $S_{n \times k}$ such that the upper left corner is labeled with 1.

> Water is poured into the well from a point above the section marked 1, at the rate of 1 cubic foot per minute. Assume that water entering a region of constant depth immediately disperses to all orthogonally adjacent lower-depth regions evenly along that regions exposed perimeter (an assumption that Euler insisted on).
>
> After how many minutes will the water begin to accumulate in [the lower right corner]?

1	14	9	20	3
5	13	24	17	18
25	16	4	21	6
10	2	15	19	23
7	22	8	12	11

Figure 1: A labeling of the 5×5 grid where the labels are a permutation of the integers from 1 to 25.

Question. For a random permtation in $S_{n \times k}$, what is the expected amount of time it takes for water to reach the lower right hand corner of the grid?

Related.

1. What if water can flow diagonally?
2. What if the source or sink are in different places? What if there are multiple sources/sinks?
3. What if this is done on a torus? Triangular/hexagonal grid? Three dimensions?
4. What if the numbers are not neccesarily a permutation?
5. What if the well is a Latin square?
6. What is an efficient algorithm for computing this for an arbitrary permutation?
7. What is the expected value of number of "wet" squares at the end?
8. How many wells have minimal filling times?
9. How many wells up to "fill level"? (e.g. two wells are equivalent if each square has the same height after the water flows all the way)

References.

http://chalkdustmagazine.com/blog/well-well-well/

https://oeis.org/A321853

Problem 87.

Consider partitions of the $n \times m$ grid into triangles with vertices on gridpoints.

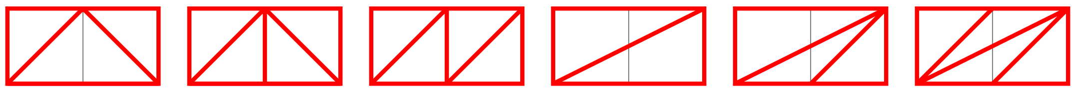


Figure 1: All six partitions of the 2×1 grid into triangles with gridpoint vertices, up to dihedral action.

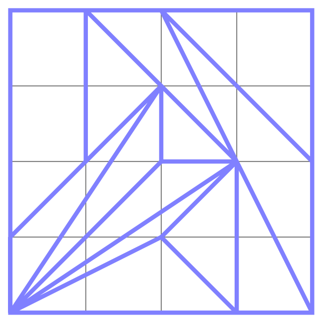


Figure 2: An example of a partitions of the 4×4 grid into triangles with no "empty" gridpoints.

Question. How many such partitions exist?

Related.

1. What if these are counted up to rotation/reflection?
2. What if this is done on a triangular/hexagonal grid?
3. How many partitions with the maximal number of triangles? With k triangles?
4. What if all triangles must be right triangles? Acute? Obtuse?
5. What if each gridpoint must touch a triangle? What is the minimum number of faces?
6. What if each gridpoint must touch as many triangles as possible? What is the minimum number of faces? What's the expected number of faces? (i.e. there's no way to draw a new edge?)
7. What if this is done on a grid in hyperbolic space?

Note. $a_1(n) = A051708(n)$

References.

`https://oeis.org/A051708`

`https://codegolf.stackexchange.com/q/176646/53884`

Problem 88.

Consider ways of partitioning nonattacking rooks in such a way that no rook lies in the convex hull of its partition. Let $a(\sigma)$ be the minimum number of parts of such a partition.

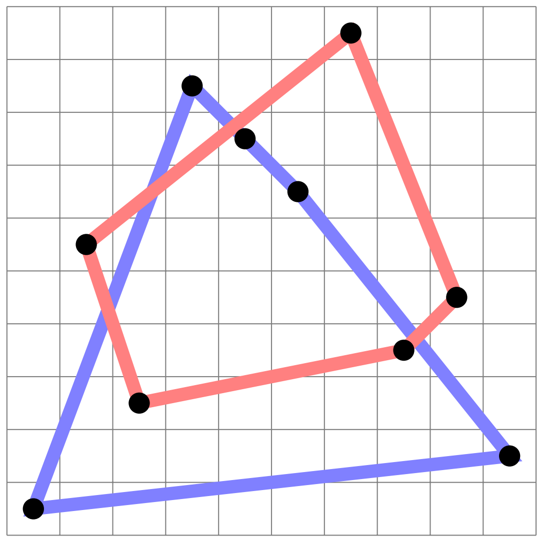


Figure 1: An illustration showing that $a(\sigma) = 2$ for $\sigma = 16398710452 \in S_{10}$.

Question. What is the expected value of $a(\sigma)$ for a uniformly random $\sigma \in S_n$?

Related.

1. What if each point must be on the corner of the convex hull?
2. What is the maximum number of convex hulls required?
3. What is the expected number of convex hulls? (i.e. how many different ways can a σ be partitioned into $a(\sigma)$ convex hulls?
4. What if the convex hulls are not allowed to overlap?
5. What is the expected value of the largest subset of $((1, \sigma(1)), \ldots, (n, \sigma(n)))$ such that no points are in the interior of the convex hull?
6. What if this is done for non-attacking queens?
7. What if this is done for an arbitrary configuration of k pieces on an $n \times m$ board?
8. What if the convex hull of the permutation is taken, and then the convex hull of the interior, and the convex hull of that interior and so on?
9. What if a no three-on-a-line rule is used instead? No $k + 2$ on a degree k polynomial?

Note.

$A156831(n) = \{\sigma \in S_n : a(\sigma) = 1\}$.

References.

`https://oeis.org/A156831`

Problem 5, 6, 7.

Problem 89.

According to Peter Winkler's problem "Red and Blue Dice" in *Mathematical Mind-Benders*, given two sequences of length n with letters in $[n]$, there must always exist a (nonempty) subsequence in each such that the sum of each of the subsequences are equal. Furthermore, there must exist a *substring* (i.e. contiguous subset) in both such that the sum of each substring is equal.

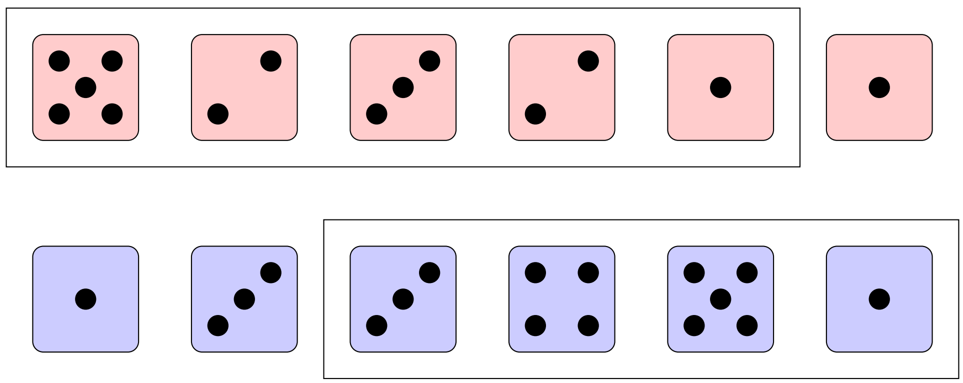


Figure 1: The first five dice in the top row have the same sum as the last four dice in the bottom row.

Question. How many equal-sum subsequences (respectively substrings) are guaranteed to exist?

Related.

1. What if the subsequence or substring must have length greater than or equal to k?
2. What if the subsequences or substrings must be of equal length?
3. What if the substring is allowed to wrap around?
4. What if this is done over permutations S_n instead of subsets $[n]^n$?
5. What if the sequences are of length $\ell < n$, and must be injections from $\mathbb{N}$ to $[n]$?
6. What if there are three sequences? m sequences?
7. Given two random sequences, what is the expected number of equal-sum subsequences?

References.

`https://math.stackexchange.com/q/3035452/121988`

Problem 90.

The puzzle Figure/Ground by Ian Gilman features a grid with two colors. In the grid any (horizontal/vertical) connected component can be moved exposing the other color beneath.

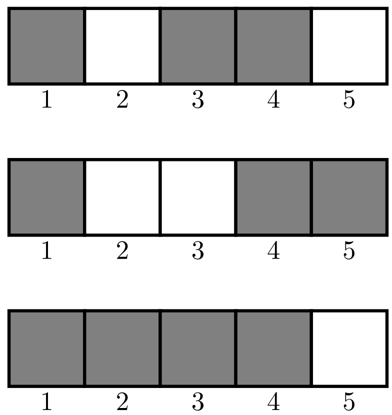


Figure 1: It is possible to get from the first configuration to the second configuration by moving the $(3, 4)$-block to position $(4, 5)$ or by moving the 5-block to position 3. It is possible to get from the first configuration to the third by moving the block in position 2 to position 5.

Question. Is there an efficient algorithm to determine whether it's possible to get from one configuration to another?

Related.

1. On a $1 \times n$ grid, what is the greatest number of steps between two configurations?
2. Starting with the $1 \times n$ grid where even squares are black and odd squares are white, is is possible to get to any configuration with both colors present? Do other starting configurations have this property?
3. What if this is done on a $n \times m$ grid? A $n_1 \times \ldots \times n_k$ grid? A triangular/hexagonal grid? Torus?
4. What if more colors were used?

References.

`http://www.clockworkgoldfish.com/figureground/list/sky/`

Problem 91.

Consider a puzzle that consists of an $n \times n$ grid with n marked cells. The goal of the puzzle is to partition the grid into n-cell regions of size n, each containing exactly one marked cell.

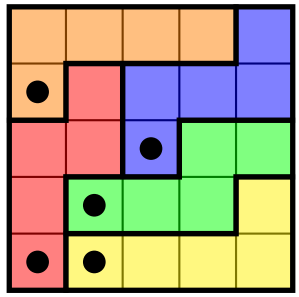


Figure 1: An example of a 5×5 grid with a unique solution.

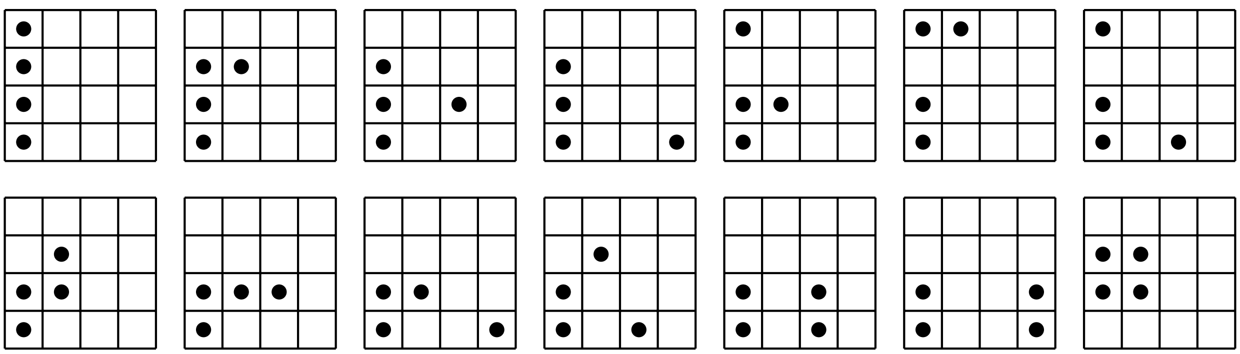


Figure 2: Fourteen (all, up to dihedral action?) markings with exactly one solution.

Question. How many $n \times n$ boards exist with a unique solution?

Related.

1. How many $n \times n$ boards exist with no solution? Multiple solutions?
2. What board has the most solutions?
3. What if this is counted up to dihedral action?
4. What if this is done on an $n \times m$ board with k marked cells where $k|nm$ and each region has nm/k cells?
5. What if the board is a torus? Triangular/hexagonal grid? Multiple dimensions?
6. What if instead of marked cells there are marked regions? (e.g. two adjacent cells are both marked "1" and must be captured by the same region)
7. What if cells must must be rectangular? Symmetric?
8. What if every region must be a walk starting at a marked cell? (As in the example.)

References.

Problem 24

`https://math.stackexchange.com/q/3072735/121988`

Problem 92.

Consider walks on an $n \times m$ grid, where the walk can only self-intersect at a perpendicular step.

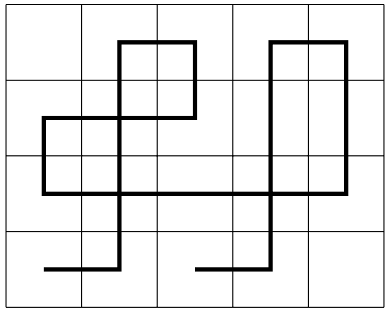


Figure 1: An example of a walk on a 5×5 grid.

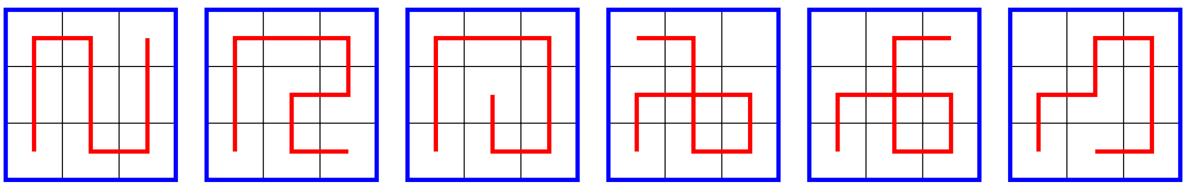


Figure 2: Six (all?) "as-long-as-possible" paths starting in the lower left corner, up to dihedral action on the 3×3 grid.

Question. How many $n \times n$ boards exist with a unique solution?

Related.

1. What if paths must be "as long as possible", in the sense that they can't be extended at either end? Exactly k steps?
2. What if this is done on a torus, triangular grid, cube, etc?
3. What if paths much touch every square at least once?
4. How many up to dihedral action? How many with dihedral symmetry?
5. What if paths must start at, say, the upper right corner?
6. What if the path must have at least one self-intersection?
7. What if paths are allowed to be loops (i.e., end on same square as they began on?) What if they must be loops?
8. What is the greatest number of king steps? Fewest on an "as-long-as-possible" path? Rook steps?
9. What if diagonal moves are allowed? Only diagonal moves?
10. What if multiple paths can be drawn on the same grid, only intersecting perpendicularly?

References.

Special case of Problem 31. Problem 42 and 56.

Pipe Mania puzzle game. `https://en.wikipedia.org/wiki/Pipe_Mania`

Problem 93.

Consider the intersection of a regular n-gon with a regular m-gon, both with sides of unit length.

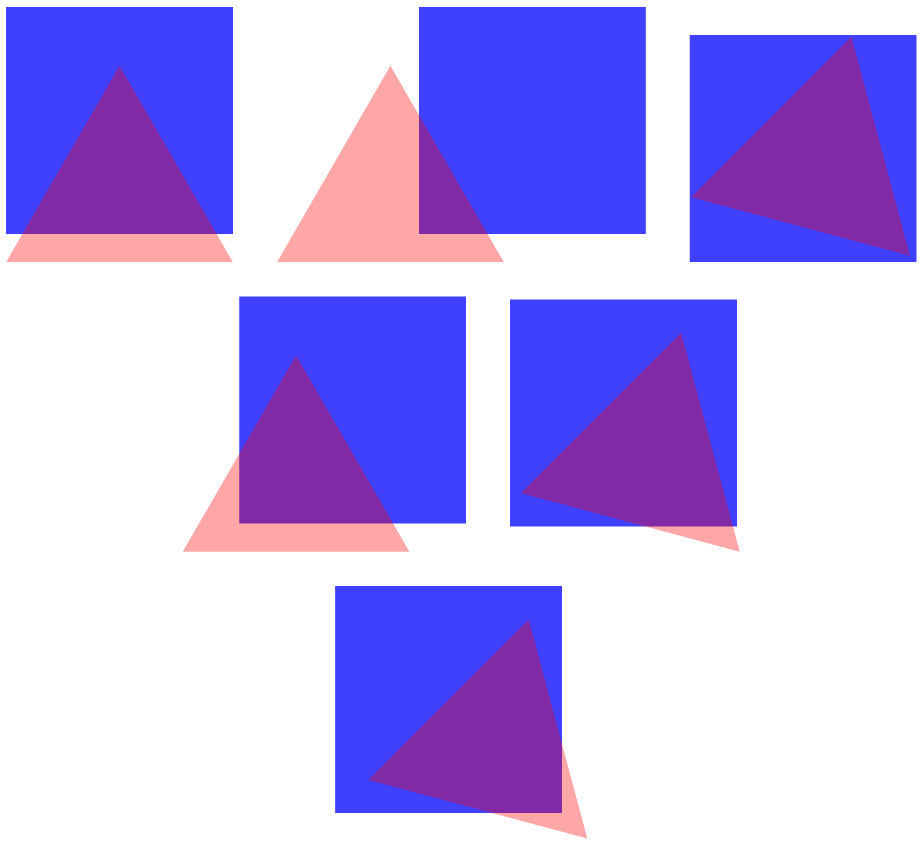


Figure 1: Six (all?) classes of intersections between a 4-gon and a 3-gon. The three triangular intersections may be considered distinct because one has all three of its sides contributed from the triangle, one has two sides from the triangle, and one has two sides from the square.

Question. What are the possible classes of polygons that can be realized as the intersection of an n-gon and an m-gon?

Related.

1. What is the largest k for which a regular unit n-gon and m-gon can intersect in a k-gon?
2. What if the polygons have unit area instead of unit length?
3. What if the regular polygons can be any size at all?
4. What if the polygons do not need to be regular? If the intersection does not need to be connected?
5. What if the polygons have integer vertices and minimal area?

Problem 94.

Consider convex polygons with integer vertices and minimal area.

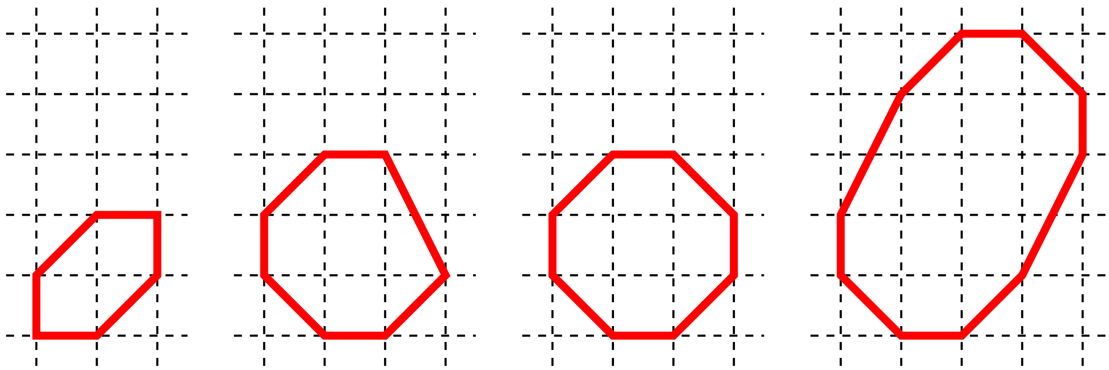


Figure 1: Candidates for minimal area $6, 7, 8$ and 10-gons.

Question. What is the minimal area of a convex lattice n-gon.

Related.

1. What if the sum of side lengths is minimized instead? Measured via the taxicab metric?
2. What if the polygons are minimized with respect to the height/width of the smallest grid? Or the number of complete cells they contain? (e.g. the examples contain 0, 4, 5, and 10 cells respectively) The number of partial cells they contain? (e.g. 4, 9, 9, 18) respectively?
3. What if the polygons can be concave?
4. What if the concave polygons cannot have any acute angles?
5. What if the concave polygons can be decomposed into polyabolos?
6. What if the polygons must have 180° or horizontal symmetry?

References.

`https://en.wikipedia.org/wiki/Polyabolo`

Problem 5, Problem 7, and Problem 88.

Problem 95.

How many non-intersecting walks from $(1, 1)$ to (n, m) with steps up and to the right exist on the $n \times m$ torus?

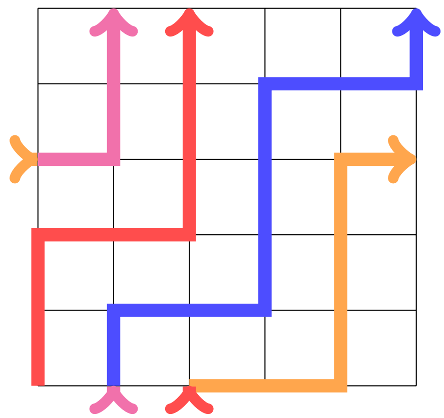


Figure 1: An example of a walk on a 5×5 torus that touches every lattice point.

Question. How many such walks exist?

Related.

1. What if the walks must touch every lattice point?
2. What if the walks must wrap around the torus exactly k times? (For $k = 1$ and $m = n$, this is the number of walks along the edges of an $n \times n$ non-toroidal grid.)
3. What if there always must be weakly more "up" steps than "right" steps? (generalization of staying above the diagonal) Strongly more?
4. What if this is done on a cylinder? Möbius strip? More dimensions?
5. What if walks can intersect at a right angle? What if there must be exactly k intersections?
6. What if more general loops were counted? (i.e. any walk from $(0, 0)$ to $(0, m)$, $(n, 0)$ or (n, m).)

References.

Problem 92.