How many functions $f_{n,k}\colon P([k])-\emptyset \to \{0,1,2,\ldots,n\}$ exist between nonempty subsets of $[k]$ and nonnegative integers less than or equal to n such that there exists a sequence of finite sets $(A_1, A_2, \ldots, A_k)$ satisfying

$$f(S) = \# \bigcap_{i \in S} A_i$$

for all $S \in P([k]) - \emptyset$?

	$\#A$	$\#B$	$\#C$	$\#(A \cap B)$	$\#(A \cap C)$	$\#(B \cap C)$	$\#(A \cap B \cap C)$
1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0
3	0	1	0	0	0	0	0
4	1	0	0	0	0	0	0
5	0	1	1	0	0	0	0
6	0	1	1	0	0	1	0
7	1	0	1	0	0	0	0
8	1	0	1	0	1	0	0
9	1	0	1	0	0	0	0
10	1	1	0	1	0	0	0
11	1	1	1	0	0	0	0
12	1	1	1	1	0	0	0
13	1	1	1	0	1	0	0
14	1	1	1	0	0	1	0
15	1	1	1	1	1	1	1

Figure 1: For $n = 1$, $k = 3$, there are fifteen such functions.

Question. How many such functions exist? Equivalently, how many ways to fill in a k-"base set" Venn diagram with integers such that no base set has more than n elements?

Related.

1. What if $\#A_i = \#A_j$ for all $i, j < n$?
2. What if $A_i \not\subset A_j$ for all $i \neq j$?
3. What if this is done with unordered sets? (e.g. the second, third, and fourth functions in the example are all considered equivalent.)
4. What if the correpsonding diagrams need to be realizable as grid rectangles with areas corresponding to the values in the table?
5. What if this is done with set union instead of set intersection?

References.

OEIS Sequence A000330 handles the case where $k = 2$.

OEIS Sequence A319777 handles the case where $k = 3$.