The number of ways to draw a triangle on a triangular grid is given by

$$\sum_{k=1}^{n-1} k \cdot t(n-k) = \binom{n+2}{4}$$

where $t(m)$ is the m-th triangular number.

The number of ways to draw a square on a square grid is given by

$$\sum_{k=1}^{n-1} k \cdot (n-k)^2 = n^2 \left(\frac{n^2-1}{12}\right)$$

the 4-dimensional pyramidal number.

The number of ways to draw a hexagon on a hexagonal grid is given by

$$\sum_{k=1}^{n-1} k \cdot h(n-k) = \sum_{k=1}^{n-1} k^3 = t(n-1)^2$$

where $h(m)$ is the m-th hexagonal number.

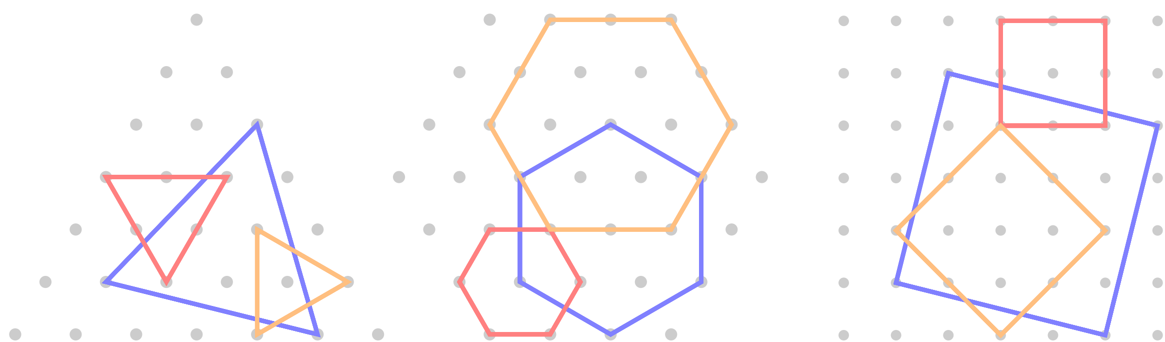

Figure 1: All connected configurations of 4 coins. Six out of the seven possible polyhexes are present.

Question. Is there a combinatorial explanation for why these numbers have nice closed forms?

Related.

1. Can this be generalized to arbitrary regular n-gons in hyperbolic space?
2. How many triangles are on the “centered triangular number” grid?

References.

Problem 25.

`https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling`