Consider a system of first-order finite difference equations (linear recurrences) $a_1, a_2, \ldots, a_N$ where

$$a_i(n) = \alpha_{i1}a_1(n-1) + \alpha_{i2}a_2(n-1) + \ldots + \alpha_{iN}a_N(n-1).$$

is the ith such equation.

$$\begin{array}{ll}
a_1(1) = 1 & a_1(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + 2a_5(n-1) + a_7(n-1) \\
a_2(1) = 0 & a_2(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_3(1) = 0 & a_3(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_4(1) = 0 & a_4(n) = a_2(n-1) + a_3(n-1) + a_5(n-1) \\
a_5(1) = 0 & a_5(n) = a_1(n-1) + a_2(n-1) + a_4(n-1) + a_5(n-1) \\
a_6(1) = 0 & a_6(n) = a_4(n-1) + a_5(n-1) + a_6(n-1) \\
a_7(1) = 1 & a_7(n) = a_5(n-1) + a_7(n-1)
\end{array}$$

Figure 1: In this system of equations, the function $a(n) = a_5(n) + a_7(n)$ (which counts no-leaf subgraphs of the $2 \times n$ grid.) satisfies the recursion $a(n) = 5a(n-1) - 5a(n-2)$ for $n > 2$.

Question. Given some linear combination $a(n) = k_1 a_{i_1}(n) + k_2 a_{i_2}(n) + \ldots + k_m a_{i_m}(n)$ of these finite difference equations, what is the smallest order finite difference equation that $a(n)$ satisfies?

Related.

1. How does the order of such a recurrence depend on the initial conditions of the system?
2. What if the initial recurrences have order greater than 1? For example:

$$\begin{aligned}
a_i(n) = \; & \alpha_{i,1,1}a_1(n-1) + \alpha_{i,1,2}a_1(n-2) + \ldots + \alpha_{i,1,k_1}a_1(n-k_1) \; + \\
& \alpha_{i,2,1}a_2(n-1) + \alpha_{i,2,2}a_2(n-2) + \ldots + \alpha_{i,2,k_2}a_2(n-k_2) \; + \ldots \\
& \alpha_{i,t,1}a_t(n-1) \; + \alpha_{i,t,2}a_t(n-2) \; + \ldots + \alpha_{i,t,k_t}a_t(n-k_t)
\end{aligned}$$

Note. Let α_{ij} be the coefficient of $a_j(n-1)$ in the finite difference equation for $a_i(n)$, and denote the minimum polynomial of the matrix $A = [\alpha_{ij}]_{i,j=1}^{N}$ by

$$\det(xI - A) = x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_1 x + \beta_0$$

then

$$a(n) = -\beta_{m-1}a(n-1) - \beta_{m-2}a(n-2) - \ldots - \beta_1 a(n-m+1) - \beta_0 a(n-m),$$

but there may be a lower-order recurrence.

References.

A special case of Problem 56 is counted in the example.