Ron Graham's (A006255) sequence is the least k for which there exists a strictly increasing sequence

$$n = a_1 \leq a_2 \leq \ldots \leq a_T = k \text{ where } a_1 \cdot \ldots \cdot a_T \text{ is square.}$$

There is a known way to efficiently compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a p-th power, where p is a prime and where any term appears at most $p-1$ times.

Question. What is an efficient way to compute analogous sequences wherein $a_1 \cdot \ldots \cdot a_T$ is a c-th power, where c is composite?

$$\begin{aligned}
a_4(1) &= 1 \text{ via } 1 &&= 1^4\\
a_4(2) &= 2 \text{ via } 2^2 \cdot 4 &&= 2^4\\
a_4(3) &= 6 \text{ via } 3^2 \cdot 4 \cdot 6^2 &&= 6^4\\
a_4(4) &= 4 \text{ via } 4^2 &&= 2^4\\
a_4(5) &= 10 \text{ via } 5^2 \cdot 8^2 \cdot 10^2 &&= 20^4\\
a_4(6) &= 9 \text{ via } 6^2 \cdot 8^2 \cdot 9 &&= 12^4\\
a_4(7) &= 14 \text{ via } 7^2 \cdot 8^2 \cdot 14^2 &&= 28^4\\
a_4(8) &\leq 15 \text{ via } 8^2 \cdot 9 \cdot 10^2 \cdot 15^2 &&= 60^4\\
a_4(9) &= 9 \text{ via } 9^2 &&= 3^4\\
a_4(10) &\leq 18 \text{ via } 10^2 \cdot 12^2 \cdot 15^2 \cdot 18^2 &&= 180^4
\end{aligned}$$

Figure 1: Examples of $a_4(n)$ for $n \in \{1, 2, \ldots, 10\}$.

Related.

1. For what values n is $a(n) < A006255(n)$?
2. How many c-th power sequences have $a_T = a_c(n)$?
3. Do any such c-th power sequences exactly two distinct terms?

References.

`https://oeis.org/A300516`