



Suppose you have two finite groups G and H , and a H -set Ω . Denote the base of the wreath product of G by H as $K = \prod_{\omega \in \Omega} g_{\omega}$.

We're interested in determining when it is possible to construct a finite sequence, $\{k_i \in K\}_{i=1}^N$, such that for any choice of $\alpha \in G \wr_{\Omega} H$ and any sequence $\{h_i \in H\}_{i=1}^N$, there exists $n \leq N$ satisfying

$$\alpha \cdot (k_1, h_1) \cdot (k_2, h_2) \cdots (k_n, h_n) = (e_K, h)$$

for some $h \in H$.

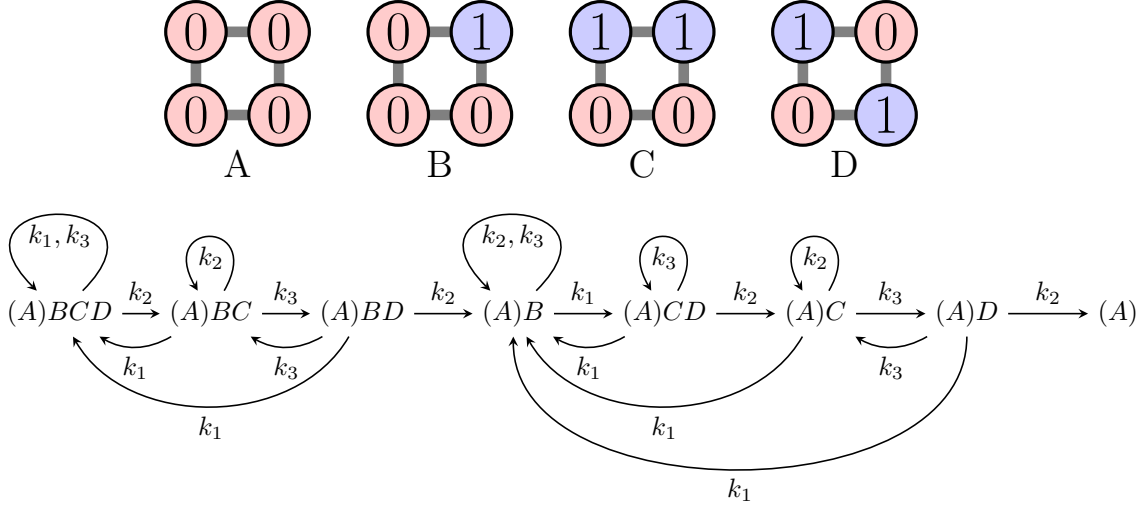


Figure 1: Suppose $G = \mathbb{Z}_2$, $H = C_4$, and Ω is the rotations of the square, with H acting in the ordinary way. Moreover, suppose that $k_0 = (1, 1, 1, 1)$, $k_1 = (1, 0, 0, 0)$, $k_2 = (1, 0, 1, 0)$, and $k_3 = (1, 1, 0, 0)$. Then the sequence $(k_0, k_2, k_0, k_3, k_0, k_2, k_0, k_1, k_0, k_2, k_0, k_3, k_0, k_2, k_0)$ satisfies the above condition.

Question. What conditions on $G \wr_{\Omega} H$ guarantee such a finite sequence of elements in K ?

Related.

1. Given $G \wr_{\Omega} H$, what is N , the minimum length of the sequence?
2. If both α and the sequence $\{h_i \in H\}_{i=1}^N$ are chosen uniformly at random, what is the expected value of the minimal n such that $\alpha \cdot (k_1, h_1) \cdot (k_2, h_2) \cdots (k_n, h_n) = (e, k)$?
3. What if there is a set of elements in the base, any one of which is valid. (e.g. all sets satisfying the condition that the number of heads is congruent to 2 (mod 3).)
4. What if something about s_i is told to you after the i th move for each i (and the strategy can depend on this information)?

Note. This problem appears in Peter Winkler's book under the name "Spinning Switches". The can be generalized to $H = C_{2^n}$ and Ω a 2^n -gon, but no other m -gon. The idea is that (1) you can't solve any m -gon with m odd, and (2) you can solve a m -gon if and only if you can solve a d -gon for every proper divisor $d \mid m$. If you allow "coins" to have three states, then you can solve this on a 3^n -gon.

References.

<http://mathriddles.williams.edu/?p=77>

<https://mathoverflow.net/q/332600/104733>

https://en.wikipedia.org/wiki/Four_glasses_puzzle