Difficulty: 3/4 Interest: 2/4

Given two vector valued functions $u, v\colon \mathbb{R}^n \to \mathbb{R}^n$, that are linearly independent at every point, let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x_0, x_1) = |\alpha| + |\beta| \text{ where } x_1 - x_0 = \alpha \cdot u(x_0) + \beta \cdot v(x_0).$$

Next let the length of a curve $\Gamma\colon [0,1] \to \mathbb{R}^n$ be given by

$$\mathcal{L}(\Gamma) = \lim_{N\to\infty} \sum_{k=0}^{N-1} f\left(\Gamma\left(\frac{j}{N}\right), \Gamma\left(\frac{j+1}{N}\right)\right).$$

Let the distance $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ from x_0 to x_1 be given by the infimum of the length over all curves from x_0 to x_1:

$$d(x_0, x_1) = \inf\{\, \mathcal{L}(\Gamma) : \Gamma(0) = x_0 \text{ and } \Gamma(1) = x_1 \,\}.$$

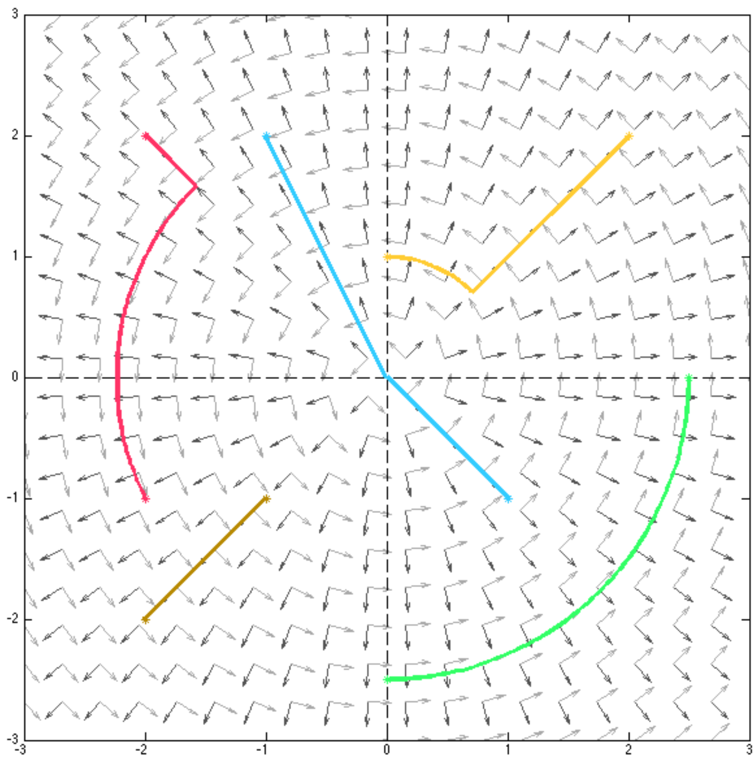

Figure 1: Five examples of shortest curves when $u(x_1, x_2) = (x_1, x_2)/||(x_1, x_2)||$ and $v(x_1, x_2) = (-x_2, x_1)/||(x_1, x_2)||$.

Question. What are the necessary conditions on u and v for this to be a well-defined metric space?

Related.

1. If $|u(x)| = |v(x)| = 1$ for all $x \in \mathbb{R}^n$, what is greatest possible (Euclidean) length of the circumference of a unit circle?

2. If u and v are well-behaved and selected at random according to some distribution, what is the expected length of the circumference of a unit circle?