Call s an "initial permutable" string if for every initial substring of odd length, the first half of the string is a permutation of the second half.

$$\begin{gathered} 1\,0\,1\,0\,1\,1\,0 \rightarrow 1\,0\,1\,0 \;\sim 0\,1\,1\,0 \\ 1\,0\,1\,0\,1 \rightarrow 1\,0\,1 \sim 0\,1\,0 \\ 1\,0\,1 \rightarrow 1\,0 \sim 0\,1 \\ 1 \rightarrow 1 \sim 1 \end{gathered}$$

Figure 1: "1010110" is an example of an initial permutable string. Because each initial odd substring (the string itself, "10101", "101", and "1") has the property that the first half of the string is a rearrangement of second half.

Question. What is the growth of $a_2(n)$, the number of initial permutable strings of length $2n-1$ over a 2-letter alphabet?

Related.

1. Can this be generalized to a k-letter alphabet?
2. How does this generalize for a k-coloring?
3. How does this generalize to a $n \times m$ square grid where horizontal-vertical connections are counted? Diagonal connections? Both?
4. How does this generalize to a tetrahedron, torus, Möbius strip, cylinder, or cube?
5. How many colorings exist of a length ℓ triangle such that the maximum label is 4 or 5?