According to Fáry's theorem, any planar graph can be drawn as a planar straight-line graph. This problem studies which of these straight-line graphs are the smallest.

Given a planar graph G let S_G be the set of all straight-line embeddings of G where the shortest edge has length 1. Given an embedding E, let $\ell(E)$ be the average edge length, and let $f(G) = \inf\{\ell(E) : E \in S_G\}$.

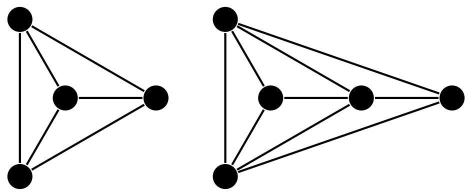

Figure 1: Conjectured minimal embeddings showing $f(K_4) \leq \frac{3\sqrt{3}+3}{6}$ and $f(K_5 \setminus \{e\}) \leq \frac{3\sqrt{3}+4+\sqrt{28}}{9}$.

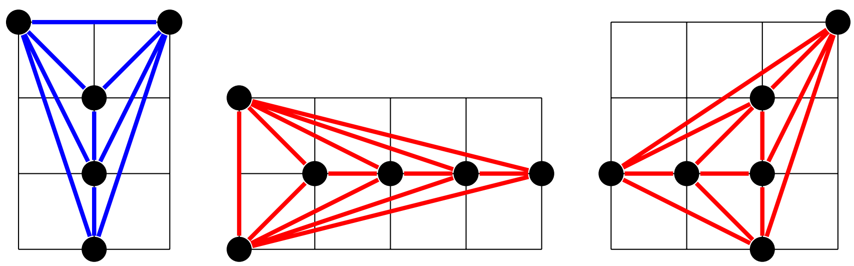

Figure 2: Smallest (?) grids that contain all planar simple graphs on n vertices are $[3] \times [4]$ for $n = 5$ and $[5] \times [4]$ for $n = 6$.

Question. Given some graph G, what is $f(G)$?

Related.

1. How does $\max\{f(G) : G \text{ is planar with } n \text{ vertices}\}$ grow with respect to n?
2. What if the vertices must be on $\mathbb{Z}^2$? What is the smallest square that can contain all planar graphs with n vertices?
3. What is the smallest nmk such that K_n can be drawn in $[n] \times [m] \times k$ with straight-line edges and no edges intersecting?
4. Is it always possible to write a planar graph as a straight-line graph with integer edge lengths? If not, when is this possible? If so, what's the minimal edge sum?
5. What is the longest non-self-intersecting polygonal chain that can fit in an $n \times m$ grid?
6. What is the supremum of $\ell(E)$ over all straight-line embeddings with longest edge *at most* 1? Are these just rescalings of the original case?

References.

Problem 74.

Problem 85.

`https://oeis.org/A000109`