Consider all r-colorings of the $n \times m$ grid where no two colors are adjacent (horizontally/vertically) more than once.

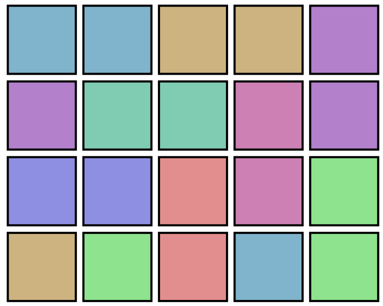

Figure 1: An 8-coloring of the 4×5 grid where no two colors are adjancent more than once. There is no 7-coloring.

Question. Let $r_{n\times m}$ be the smallest integer such that there exists an $r_{n\times m}$-coloring of the $n \times m$ grid. What is $r_{n\times m}$?

Related.

1. What if colors are not allowed to be self-adjacent?
2. How many $a(n, m)$-colorings exist up to permutation of the colors?
3. What if this is done on a triangular or hexagonal grid?
4. What if orientation matters? (A horizontal adjacency is distinct from a vertical adjacency.)
5. What if order matters? (red-green is distinct from green-red.)
6. What if diagonal adjacencies are considered?

Note.

$$
\begin{array}{lllll}
r_{1\times 1} = 1 & & & & \\
r_{1\times 2} = 1 & r_{2\times 2} = 3 & & & \\
r_{1\times 3} = 2 & r_{2\times 3} = 4 & r_{3\times 3} = 5 & & \\
r_{1\times 4} = 2 & r_{2\times 4} = 5 & r_{3\times 4} = 6 & r_{4\times 4} = 7 & \\
r_{1\times 5} = 3 & r_{2\times 5} = 5 & r_{3\times 5} = 7 & r_{4\times 5} = 8 & r_{5\times 5} = 9
\end{array}
$$

References.

Problem 23.

Problem 36.

Problem 49.