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Chapter 1

Spinning Switches

TODO (Abstract)

1. Generalizes switches to arbitrary groups.
2. Proves a result when switches look like p-groups.
3. Finds the "correct" model: the wreath products.
4. Provides reductions for when strategies don't exist, which are easy to prove with the wreath product model.
5. Comes up with an example where something isn't a prime power. ($S_3 \wr C_2$ is nontrivial. Of course $G \wr \mathbf{1}$ also works.)

1.1 TODO

1. Fill out Example 1.4.1.
2. Provide example for first reduction. (Theorem 1.4.2)
3. Give other part of example for second reduction. (Theorem 1.4.4)
4. Example for third reduction. (Theorem 1.4.6)

5. Define/discuss minimal switching strategies?

6. Mention conjecture that most groups are 2-groups

7. Infinite switching strategy? $\mathbb{Z} \wr C_2$ or $\mathbb{Z}_2^\infty \wr C_2$.

   (By $\mathbb{Z}_2^\infty$, I mean something isomorphic to the integers under XOR: $(\mathbb{Z}, \oplus)$.)

1.2 Overview and Preliminaries

This section provides a brief history of the problem and provides the idea for the more general context. Section 1.3 models these generalizations in the context of the wreath product. Section **??** is where all of the references are. [TODO: put this section elsewhere] Section 1.4 allows us to prove when Player B does not have a winning strategy. Section 1.5 allows us to make a statement about games that have a prime number of possible moves. Section 1.6 gives us an example of new kinds of puzzles that have solutions. Section 1.7 gives us an example of new kinds of puzzles that have solutions.

1.2.1 History

A closely related puzzle was popularized by Martin Gardner in the February 1979 edition of his column “Mathematical Games.” [1] He wrote that he learned of the puzzle from Robert Tappay of Toronto who “believes it comes from the U.S.S.R.”

The version under consideration in this paper is first hinted at in 1993 by Yehuda and collaborators [2]. Ehrenborg and Skinner consider something very similar, which they call the ”Blind Bartender with Boxing Gloves” [3]. This was re-popularized in 2019 when it appeared in “The Riddler” from FiveThirtyEight [4]. Sidana [5] provides a detailed overview of the history of this and related problems.

My preferred version appears in Peter Winkler’s 2004 book *Mathematical Puzzles A Connoisseur’s Collection*

> Four identical, unlabeled switches are wired in series to a light bulb. The switches are simple buttons whose state cannot be directly observed, but can be changed by pushing; they are mounted on the corners of a rotatable square. At any point, you may push, simultaneously, any subset of the buttons, but then an adversary spins the square. Show that there is a deterministic algorithm that will enable you to turn on the bulb in at most some fixed number of steps. [6]

This version will be a working example in many parts of the paper, so it is worth keeping in mind.

1.2.2 A Solution to the Original Spinning Switches Puzzle

In this short section, we will discuss the solution to this puzzle because it will give some insights and intuition for the techniques that we use later. It is useful to start with even a simpler example, which would make Peter Winkler proud.

Example 1.2.1. *Suppose that instead of having four identical unlabeled switches, we instead only have two: on the diagonals of a rotatable square table.*

Then we have a three-step solution for solving the problem. We start by toggling both switches simultaneously. If this does not turn on the light, this means that the switches were (and still) are in different states.

Then, the adversary spins the table. Next, we toggle one of the two switches to ensure that the switches are both in the same state. If the light hasn't turned on, both must be in the off state.

The adversary spins the table once more, but to no avail. We know both switches are in the off state, so we toggle them both simultaneously, turning on the lightbulb.

In order to bootstrap this solution into a large solution, we must notice two things.

First, if we can get two switches along each diagonal into the same state respectively, then we can solve the puzzle by toggling both diagonals (all four switches), both switches in a single diagonal, and both diagonals again. In this (sub-)strategy, toggling both switches along a diagonal is equivalent to toggling a single single switch in the above example.

Second, we can indeed get both diagonals into the same state by toggling a switch from each diagonal (two switches on any side of square), then a single switch from one diagonal, followed by a switch from each switch.

We will interleave these strategies in a particular way, following the notation of [7].

Definition 1.2.2. *Given two sequences $A = \{a_i\}_{i=1}^{N}$ and $B = \{b_i\}_{i=1}^{M}$, we can define the **interleave** operation as*

$$A \circledast B = (\underbrace{a_1, a_2, \ldots, a_N}_{A}, b_1, \underbrace{a_1, a_2, \ldots, a_N}_{A}, b_2, \underbrace{a_1, a_2, \ldots, a_N}_{A}, \ldots, b_M, \underbrace{a_1, a_2, \ldots, a_N}_{A}).$$

which has length $(M+1)N + M = (M+1)(N+1) - 1$.

Typically it is useful to interleave two strategies when A solves the puzzle given that the switches are in a particular state, and B gets the switches into that particular state. Usually, we also need A not to "interrupt" what B is doing. In the problem of switches on a square table, B will ensure that the switches are in the same state within each diagonal, and A will turn on the light when that's the case. Moreover, A does not change the state within each diagonal.

Proposition 1.2.3. *There exists a fifteen-move strategy that guarantees that the light in Winkler's puzzle turns on.*

Proof. We begin by formalizing the two strategies. We will say that the first strategy S_1 where we toggle the two switches in a diagonal together will consist of the following three moves:

1. Switch **a**ll of the bulbs (A).

2. Switch the **d**iagonal consisting of the upper-left and lower-right bulbs (D).

3. Switch **a**ll of the bulbs (A).

We will say that the second strategy S_2 where we get the two switches within each diagonal into the same state consists of the following three moves:

1. Switch both switches on the left **s**ide (S).

2. Switch **one** switch (1).

3. Switch both switches on the left **s**ide (S).

Then the 15 move strategy is

$$S_1 \circledast S_2 = (A,S,A,D,A,S,A,1,A,S,A,D,A,S,A)$$

$\square$

We will generalize this construct in Theorem 1.5.1, which offers a formal proof that this strategy works.

It is worth briefly noting that $S_1 \circledast S_2$ is the fourth *Zimin word* (also called a *sequipower*), an idea that comes up in the study of combinatorics on words.

1.2.3 Generalizing Switches

"The problem can also be generalized by replacing glasses with objects that have more than two positions. Hence the rotating table leads into deep combinatorial questions that as far as I know have not yet been explored." [8]

Two kinds of switches are considered by Yehuda, Etzionn, and Moran in 1993 [2]: switches with a single "on" position that behave like n-state roulettes ($\mathbb{Z}_n$) and switches that behave like the finite field $\mathbb{F}_q$. Yuri Rabinovich [7] goes further by considering collections of switches that behave like arbitrary finite dimensional vector spaces over finite fields. We can generalize this notion further by considering switches that behave like arbitrary finite groups.

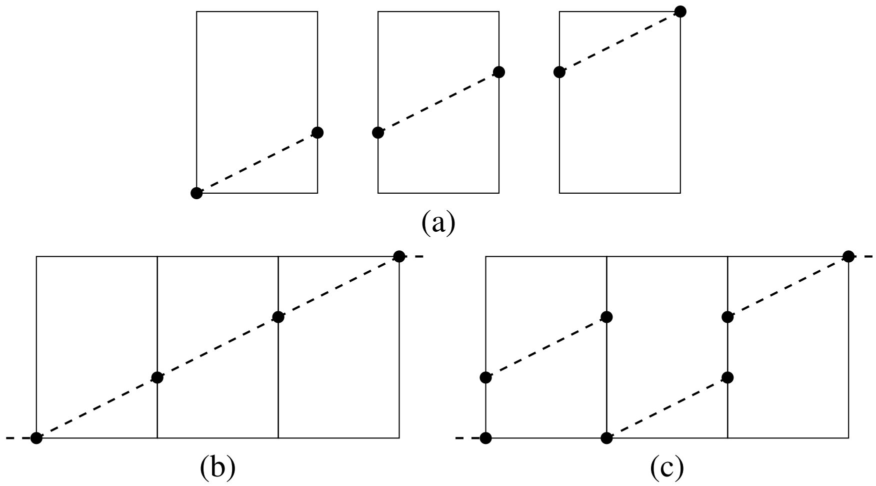

Figure 1.1: Part (a) shows a simple schematic for a switch that behaves like S_3, the symmetric group on three letters. The three rectangles can be permuted arbitrarily, but only configuration (b) completes the circuit. All other configurations fail to complete the circuit (e.g. (c)).

Example 1.2.4. *One could imagine a switch that behaves like the symmetric group S_3, consisting of three identical-looking parts that need to be arranged in a particular order in order for the switch to be on, as illustrated in Figure 1.1.*

One subtlety is that we will use a group G to model both the "internal state" of a switch itself and the set of "moves" or changes that can be made to the switch. We can think of the moves changing the state via a right group action of G on itself.

The reason that using a group to model a switch is because groups have many of the properties we would expect in a desirable switch.

Note 1.2.5. *The group axioms for a group $(G,\cdot)$ are*

1. (Closure) The group $(G,\cdot)$ is equipped with a binary operation, $\cdot\colon G\times G\to G$. That is, for all pairs of elements $g_1, g_2 \in G$ their product is in G

$$g_1 \cdot g_2 \in G.$$

   In the context of switches, this means that if the switch is in some state $g_1 \in G$ and player B moves it with action $g_2 \in G$, then $g_1 \cdot g_2 \in G$ is a valid switch for the state.

2. (Identity) There exists an element $\mathrm{id}_G \in G$ such that for all $g \in G$,

$$\mathrm{id}_G \cdot g = g \cdot \mathrm{id}_G = g.$$

This axiom is useful because it means that Player B can "do nothing" to a switch and leave it in whatever state it is in. Because the identity is a distinguished element in G, we will also use the convention that id_G is the "on" or "winning" state for a given switch. (It is worth noting that all of the arguments work with small modification regardless of which element is designated as the on state.)

3. (Inverses) For each element $g \in G$ there exists an inverse element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = \mathrm{id}_G.$$

This axiom states that no matter what state a switch is in, there is a move that will transition it into the on state.

4. (Associativity) Given three elements $g_1, g_2, g_3 \in G$,

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

This is axiom is not strictly necessary for modeling switches, but as we will see in a later definition, it gives us a convenient way to describe the conditions for a winning strategy. (In Subsection 1.7.2, we briefly discuss dropping the associativity axiom by considering switches that behave like quasigroups with identity.)

1.2.4 Generalizing Spinning

We can also consider generalizations of "spinning" the switches. In particular, we will adopt the generalization from Ehrenborg and Skinner's [3] 1995 paper, which use arbitrary finite group actions to permute the switches. In particular, they provide a criterion that determines which group actions yield a winning strategy in the case of a given number of "ordinary" switches (those that behave like $\mathbb{Z}_2$). Rabinovich [7] stretches these results a bit further and looks at certain group actions on collections of switches that behave like a finite dimensional vector space over a finite field. We build on this result in the context of more general switches.

1.3 The Wreath Product Model

Peter Winkler's version of the puzzle consists of four two-way switches on the corners of a rotating square table. The behavior of the switches are naturally modeled as $\mathbb{Z}_2$, and the rotating table is modeled as the cyclic group C_4.

If we put these together as a wreath product of $\mathbb{Z}_2$ by C_4, it will result in the expected structure.

1.3.1 Modeling Generalized Spinning Switches Puzzles

Definition 1.3.1 ([9]). *Let G and H be groups, let Ω be a finite H-set, and let $K = \prod_{\omega\in\Omega} G_\omega$, where $G_\omega \cong G$ for all $\omega \in \Omega$. Then the* ***wreath product*** *of G by H denoted by $G \wr H$, is the semidirect product of K by H, where H acts on K by $h \cdot (d_\omega) = d_{h^{-1}\omega}$ for $g \in H$ and $(g_\omega) \in \prod_{\omega\in\Omega} G_\omega$. The normal subgroup K of $G \wr H$ is called the* ***base*** *of the wreath product.*

The group operation is $(k,h)\cdot(k',h') = (k(h\cdot k'),hh')j$

The reason this definition is used is because it models the game well: G models the behavior of the switches, Ω models the positions of the switches, and the action of H on Ω models the ways the adversary can "spin" or scramble the switches.

An element of $(k,h) \in G \wr H$ represents a turn of the game: Player B chooses an element of the base $k \in K$ to indicate how they want to modify each of their switches and then Player A chooses $h \in H$ to indicate how they want to permute the switches.

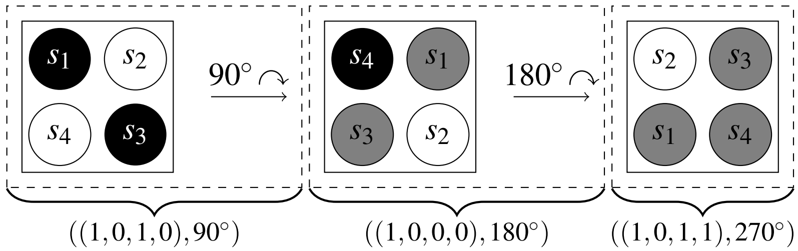

Figure 1.2: An illustration of two turns each in the Spinning Switches puzzle, modeled as elements of a wreath product.

Example 1.3.2. *Consider the setup in the original version of the problem consisting of two-way switches ($\mathbb{Z}_2$) on the corners of a rotating square ($C_4 \cong \langle 0^\circ, 90^\circ, 180^\circ, 270^\circ \rangle$). This can be modeled as a game on the wreath product $\mathbb{Z}_2 \wr C_4$. We will use the convention that the base of the wreath product, K is ordered upper-left, upper-right, lower-right, lower-left, and the group action is specified by degrees in the clockwise direction.*

Consider the following two turns:

1. *During the first turn, Player B toggles the upper-left and lower-right switches, and Player A rotates the table 90° clockwise. This is represented by the element $((1,0,1,0),90^\circ) \in \mathbb{Z}_2 \wr C_4$.*
2. *During the second turn, Player B toggles the upper-left switch, and Player A rotates the table 90° clockwise. This is represented by the element $((1,0,0,0),180^\circ) \in \mathbb{Z}_2 \wr C_4$.*

As illustrated in Figure 1.2, the net result of these two turns is the same as a single turn where Player B toggles the upper-left, upper-right, and lower-left switches and Player A rotates the board 270° clockwise.

The multiplication under the wreath product agrees with this:

$$
\begin{aligned}
((1,0,1,0),90^\circ) \cdot ((1,0,0,0),180^\circ) &= ((1,0,1,0) + \underbrace{90^\circ \cdot (1,0,0,0)}_{(0,0,0,1)}, 90^\circ + 180^\circ) \\
&= ((1,0,1,1),270^\circ)
\end{aligned}
$$

Occasionally it is useful to designate a particular state of the switches as the on state or the winning state, and ordinarily the identity state is the choice given for this. However, the existence of a winning strategy does not depend on a particular choice in the winning state; instead, a winning strategy is equivalent to a choice of moves that will walk over all of the possible configuration states, regardless of the choice of the adversaries spin.

1.3.2 Switching Strategy

Definition 1.3.3. *A* ***switching strategy*** *is a finite sequence, $\{k_i \in K\}_{i=1}^N$, such that for every sequence $\{h_i \in H\}_{i=1}^N$,*

$$p(\{\underbrace{e_{G\wr H}}_{m_0},\underbrace{(k_1,h_1)}_{m_1},\underbrace{(k_1,h_1)\cdot(k_2,h_2)}_{m_2},\cdots,\underbrace{(k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)}_{m_N}\}) = K.$$

where $p\colon G\wr H \to K$ is the projection map from the wreath product onto its base.

This definition is useful because it puts the problem into purely algebraic terms. It is also useful because it abstracts away the initial state of the switches: regardless of the initial state $k \in K$, a existence of a switching strategy means that its inverse $k^{-1} \in K$ appears in the sequence.

Lemma 1.3.4. *A sequence of moves is guaranteed to "turn on the lightbulb" if and only if it is a switching strategy.*

Proof. Without loss of generality, say that the "on" setting for the switches is id_K. In the puzzle, we have an initial (hidden) state, k. Thus, after the i-th move, the wreath product element that represents the state of the switches is

$$p\left((k,\mathrm{id}_H)\cdot(k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right) = k\cdot p\left((k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right),$$

by associativity, where we can factor out the first term because the "spin" is id_H, which acts trivially, that is $(k,\mathrm{id}_H)\cdot(k',h') = (kk',h')$.

Thus, by considering different initial states, we see that in order to reach the on state, there must exist some i, such that $p\left((k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right) = k^{-1}$ for all k and adversarial sequences $\{h_i\}_{i=1}^N$. □

It's also worth noting that this model can be thought of as a random model or an adversarial model: the sequence $\{h_i \in H\}$ can be chosen after the sequence $\{k_i \in K\}$ in a deterministic way or randomly.

One useful consequence of this definition is that it quite straightforward to prove certain lemmas. For example, the minimum length for a switching strategy has a simple lower bound.

Lemma 1.3.5. *Every switching strategy $\{k_i \in K\}_{i=1}^N$ is a sequence of length at least $|K|-1$.*

Proof. By the pigeonhole principle, because the set

$$\{e_{G\wr H}, (k_1,h_1), (k_1,h_1)\cdot(k_2,h_2), \cdots, (k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)\}$$

has at most $N+1$ elements. In order for the projection to be equal to K,

$$p(\{e_{G\wr H}, (k_1,h_1), (k_1,h_1)\cdot(k_2,h_2), \cdots, (k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)\}) = K,$$

$N+1 \geq |K|$, so $N \geq |K|-1$. $\square$

Definition 1.3.6. *A **minimal switching strategy** for $G \wr H$ is a switching strategy of length $N = |K|-1$.*

In practice, every wreath product known by the author to have a switching strategy also has a known minimal switching strategy. In Section 1.7, we ask whether this property always holds.

1.4 Reductions

There are essentially three ways to show that $G \wr H$ does not have a solution: directly, or via one of two *reductions* (or a combination thereof).

1.4.1 Puzzles Without Switching Strategies

Using results from Rabinovich [7], we can give examples of puzzles that don't have solutions. This section allows us to take those examples and stretch them into wider families of examples.

Example 1.4.1. *The game $\mathbb{Z}_2 \wr C_3$ does not have a switching strategy. Here's how to see it...*

1.4.2 Reductions on Switches

Theorem 1.4.2. *If $G \wr H$ does not have a switching strategy and G' is a group with a quotient $G'/N \cong G$, then $G' \wr H$ does not have a switching strategy.*

Proof. I will prove the contrapositive, and suppose that $G' \wr H$ has a switching strategy $\{k'_i \in K'\}_{i=1}^N$. The quotient map $\varphi \colon G' \mapsto G$ extends coordinatewise to $\hat{\varphi} \colon K' \mapsto K$.

The sequence $\{\hat{\varphi}(k'_i) \in K\}_{i=1}^N$ is a switching strategy on $G \wr H$. [Say something about how the projection map is ?linear? wrt $\hat{\varphi}$? Say ϕ induces a homomorphism from $G' \wr H$?]

Want to prove

$$p((\hat{\varphi}(k'_1), h_1) \dots (\hat{\varphi}(k'_i), h_i)) = \hat{\varphi}(p'((k'_1, h_1) \dots (k'_i, h_i)))$$

where $p \colon G \wr H \to K$ and $p' \colon G' \wr H \to K'$ □

Example 1.4.3. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_6 \wr C_3$ does not have a switching strategy either. This is illustrated in Figure 1.3*

1.4.3 Reductions on Spinning

Theorem 1.4.4. *If $G \wr H$ does not have a switching strategy and H' is a group with a subgroup $A \leq H'$ such that $A \cong H$, then $G \wr H'$ does not have a switching strategy.*

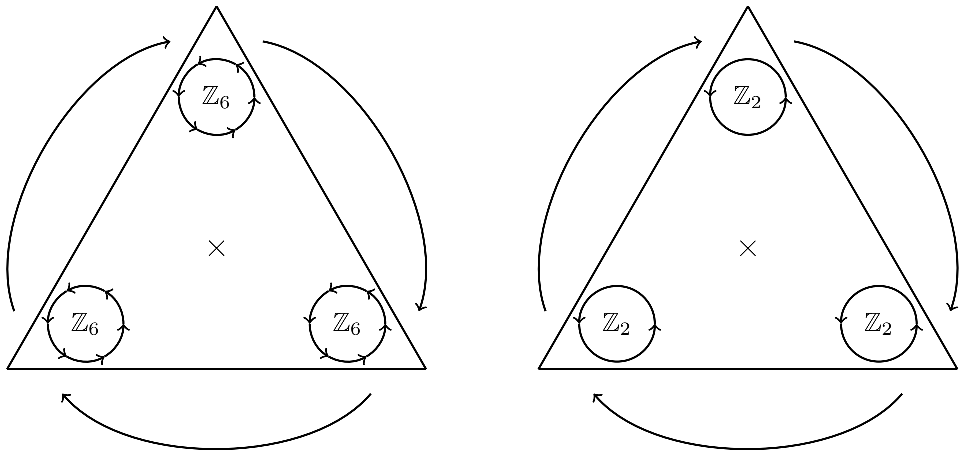

Figure 1.3: A reduction on switches: $\mathbb{Z}_6 \wr C_3$ reduces to $\mathbb{Z}_2 \wr C_3$, which is known not to have a switching strategy.

Proof. Again we will prove the contrapositive. Assume that $G \wr H'$ does have a switching strategy, $\{k_i\}_{i=1}^N$. Then by definition, for any sequence $\{h'_i\}_{i=1}^N$, the projection of the sequence

$$p(\{(k_1, h'_1) \cdot (k_2, h'_2) \cdots (k_i, h'_i)\}_{i=1}^N) = K,$$

and in particular this is true when h'_i is restricted to be in the subgroup H. Thus a switching strategy for $G \wr H'$ is also a valid switching strategy for $G \wr H$. $\square$

Example 1.4.5. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_2 \wr C_6$ does not have a switching strategy either.*

(TODO: commentary: What we're more interested in though is the reduction of $\mathbb{Z}_2 \wr C_6$ to $\mathbb{Z}_2 \wr C_3$. (Closely related to Theorem 1.4.4))

Theorem 1.4.6. *Suppose that H' is a group with a subgroup $A \leq H'$ such that $A \cong H$, and let*

$$\mathrm{Orb}(\omega) = \{\omega \cdot a : a \in A\} \subseteq \Omega$$

be the (right) orbit of $\omega \in \Omega$ under A. If $G \wr_{\mathrm{Orb}(\omega)} H$ does not have a switching strategy, then $G \wr_\Omega H'$ does not have a switching strategy.

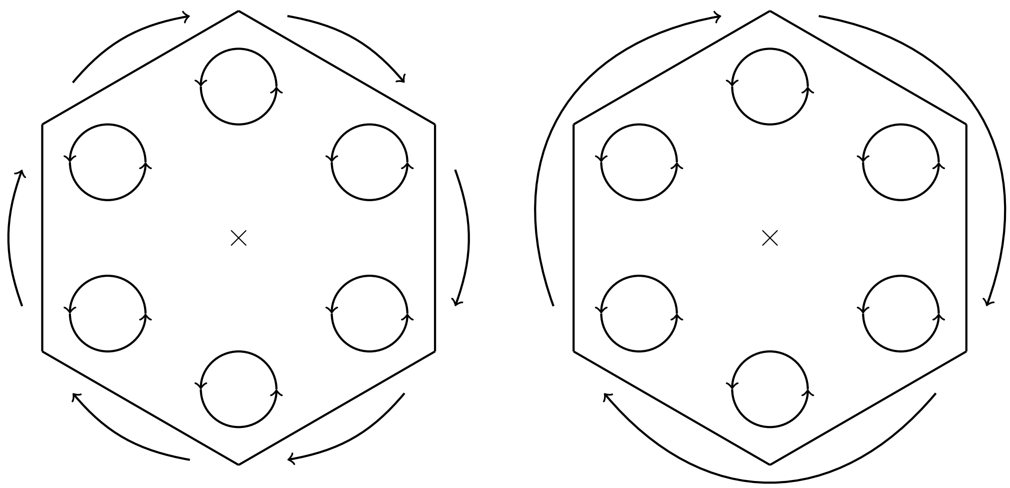

Figure 1.4: If there were a solution to $\mathbb{Z}_2 \wr C_6$, then there would be a solution to ...

Proof. We start by making the contrapositive assumption that $G \wr_\Omega H'$ has a switching strategy $\{k_i \in K\}_{i=1}^N$, and we consider the projection $p_\omega : K \to K_\omega$ where $K = \prod_{\omega \in \Omega} G_\omega$ and $K_\omega = \prod_{\omega' \in \mathrm{Orb}(\omega)} G_{\omega'}$.

Then $\{p_\omega(k_i) \in K_\omega\}_{i=1}^N$ is a switching strategy for $G \wr_{\mathrm{Orb}(\omega)} H$, since a projection is a surjective map. $\square$

Example 1.4.7. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_2 \wr C_6$ does not have a switching strategy either.*

1.5 Switching Strategies on p-Groups

In this section, we'll develop a broad family of switching strategies, namely those on p-groups.

1.5.1 Switching Strategy Decomposition

Our first constructive theorem provides a technique that can be used to construct switching strategies for switches that behave like a group G in terms of a normal group and its corresponding quotient group.

Theorem 1.5.1. *The wreath product $G \wr H$ has a switching strategy if there exists a normal subgroup $N \trianglelefteq G$ such that both $N \wr H$ and $G/N \wr H$ have switching strategies.*

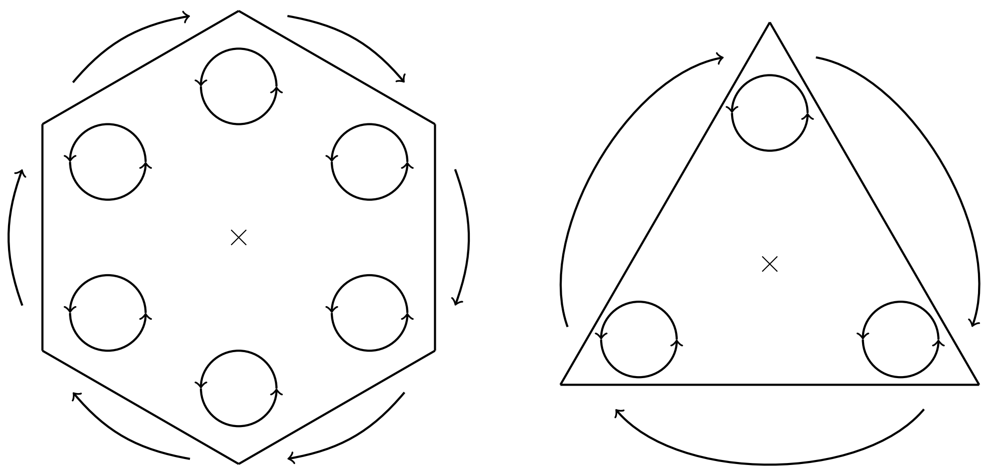

Figure 1.5: We know that $\mathbb{Z}_2 \wr_\Omega C_6$ cannot have a switching strategy, because that would imply a switching strategy for $\mathbb{Z}_2 \wr_{\Omega'} C_3$, where Ω' is the orbit of the top switch rotations of multiples of $120°$.

Proof. Let $S_{G/N} = \{k_i^{G/N} \in K_{G/N}\}$ denote the switching strategy for $G/N \wr H$, and let $S_N = \{k_i^N \in K_N\}$ denote the switching strategy for $N \wr H$.

First, we partition G into $|G|/|N| = m$ cosets of N:

$$G = g_1 N \sqcup g_2 N \sqcup \cdots \sqcup g_m N.$$

From the switching strategy $\{k_i^{G/N} \in K_{G/N}\}$, we can get a sequence $S_{G'} = \{k'_i \in K_G\}$ by picking the coset representatives coordinatewise.

This sequence is not itself a switching strategy, but it does "hit" all combinations of cosets. That is, for every "spinning sequence" $\{h_i \in H\}$, and sequence of cosets $(g_{i_1}H, g_{i_2}H, \ldots, g_{i_m}H)$, there exists an index n such that

$$p((k'_1, h_1)\ldots(k'_n, h_n)) \in g_{i_1}H \times g_{i_2}H \times \cdots \times g_{i_m}H$$

Now if we interleave $S'_G \circledast S_N$, this forms a switching strategy because ...

$$S'_G \circledast S_N = (\underbrace{k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_0}, \underbrace{k'_1, k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_1}, \ldots, \underbrace{k'_{n'}, k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_{n'}})$$

The partial products that end in block B_i all have switches in the same cosets of N, and the S_N strategy then hits all elements of K_G that belong to that combination of cosets.

□

Theorem 1.5.2. *[7] Assume that a finite group H acts linearly and faithfully on a vector space V over a finite field $\mathbb{F}_q$ of characteristic p. Then (G,V) is friendly if and only if G is a $p-$group.*

Corollary 1.5.3. *If H is a finite group that acts faithfully on Ω, then the wreath product $G \wr H$ has a switching strategy whenever $|G| = p^n$ for some n.*

Proof. If $|G| = p^n$, then either $G \cong \mathbb{Z}_p$ or G is not simple. If $G \cong \mathbb{Z}_p$, then there exists a strategy. Otherwise, G is not simple, so choose a normal subgroup N of order $|N| = p^t$ which gives a quotient G/N with order $|G/N| = p^{n-t}$. Then the result follows by induction. □

Corollary 1.5.4. *Based on the above construction, if $|G| = p^n$, then $G \wr C_{p^\ell}$ has a palindromic strategy of length $p^{np^\ell} - 1$.*

Proof. Is this true for general $H \neq C_{p^\ell}$? □

1.6 Switching Strategies on Other Wreath Products

So far, the literature has only contained examples of spinning strategies on wreath products that are themselves p-groups: $|G \wr_\Omega H| = |G|^{|\Omega|} \cdot |H|$, where H acts faithfully.

1.6.1 $G \wr \mathbf{1}$

Of course, if $H = \mathbf{1}$ is the trivial group, then $G \wr \mathbf{1} \cong G$ has a switching strategy even if G is not a p-group. (In fact, it has $(|G|-1)!$ switching strategies!)

1.6.2 Two interchangeable copies of a symmetric group $S_n \wr C_2$

Theorem 1.6.1. *$S_n \wr C_2$ has a switching strategy.*

We'll construct a switching strategy. Now our switching strategy has two parts: the first is that we ensure that the two switches have every possible difference. The second is that we show that we can get either of the switches to take on every possible value without disturbing the difference.

Proof. We start with the observation that the symmetric group can be generated by transpositions: $S_n = \langle t_1, t_2, \ldots, t_N \rangle$. This means that there is a sequence of transpositions $t_{i_1}, t_{i_2}, \ldots, t_{i_M}$ such that $\{\mathrm{id}, t_{i_1}, t_{i_1}t_{i_2}, \ldots, t_{i_1}t_{i_2}\cdots t_{i_M}\} = S_n$.

Strategy A: We first assume that we have a sequence of moves that can run through all values of the first coordinate. Of course we do, just let $t' = (t,t) \in K$ and use the sequence $\{t'_{i_1}, t'_{i_2}, \ldots, t'_{i_M}\}$.

Strategy B: Let the i-th move be denoted by $m_i = (k_i, k'_i)$. We want $t_1 t_2^{-1}$ to be whatever we choose. We can do this too by applying the sequence $\{(t_{i_j}, \mathrm{id}_G)\}_{j=1}$ this is because

$$t_1(t_2 t)^{-1} = t_1 t^{-1} t_2^{-1} = (t_1 t) t_2^{-1}$$

because t is a transposition, and thus $t = t^{-1}$. Moreover, when we apply the first strategy, it doesn't change this difference: $(t_1 t)(t_2 t)^{-1} = t_1 t t^{-1} t_2^{-1} = t_1 t_2^{-1}$

Combined: $S_2 \circledast S_1$ $\square$

Example 1.6.2. *If $a \in S_3$, let a_1 mean multiplying one of the two copies by a and a_2 mean multiplying both of the copies by a. Then the following is a strategy:*

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$
$$(12)_1$$
$$(12)_2(13)_2(12)_2(13)_2(12)_2$$
$$(13)_1$$
$$(12)_2(13)_2(12)_2(13)_2(12)_2$$
$$(12)_1$$
$$(12)_2(13)_2(12)_2(13)_2(12)_2$$
$$(13)_1$$
$$(12)_2(13)_2(12)_2(13)_2(12)_2$$
$$(12)_1$$
$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

In general, if you can walk through G with elements of order 2, then there is a strategy.

1.7 Open questions

1.7.1 Palindromic switching strategies

In all known examples, when there exists a switching strategy S, there exists a *palindromic* switching strategy $S' = \{k'_i \in K\}_{i=0}^{N}$ such that $k'_i = k'_{N-i}$ for all i.

Conjecture 1.7.1. *Whenever $G \wr H$ has a switching strategy, it also has a palindromic switching strategy.*

I'm interested in the answer even in the case of $G \wr 1 \equiv G$.

(MSE)

1.7.2 Quasigroup switches

In the paper we modeled switches as groups. This is because groups have desirable properties:

1. Closure. Regardless of which state a switch is in, modifying the state is the set of states.

2. Identity. We don't have to toggle a switch on a given turn.

3. Inverses. If a switch is off, we can always turn it on.

It turns out that we don't need the axiom of associativity, because the sequencing is naturally what computer scientists call "left associative". Thus, we can model switches a bit more generally as *loops* (i.e. quasigroups with identity.)

1.7.3 Expected number of turns

Recall that the original conception of a generalized spinning switches puzzle is to turn on all of the switches at once. That if the original state of the puzzle is $k \in K$, we "win" on move i if $k^{-1} = p((k_1,h_1)(k_2,h_2)\dots(k_i,h_i))$. It is natural to ask about the expected value of the number of turns given various sequences of moves. Notice that this is a question we can ask even about generalized spinninng switches puzzles that do not have a switching strategy.

Indeed, Winkler [6] (TODO, this is probably the wrong citation!) notes in the solution of his puzzle:

> This puzzle reached me via Sasha Barg of the University of Maryland, but seems to be known in many places. Although no fixed number of steps can guarantee turning the bulb on in the three-switch version [with two-way switches], a smart randomized algorithm can get the bulb on in at most $5\frac{5}{7}$ steps on average, against any strategy by an adversary who sets the initial configuration and turns the platform. [10]

In all cases, when computing the expected number of turns, we will assume that the initial hidden state $k \in K$ is not the winninng state id_K, and that the adversaries "spins" are indepedent and identically distributed uniformly random elements $h_j \in H$.

Proposition 1.7.2. *If Player B chooses $k_j \in K \setminus \{\mathrm{id}_K\}$ uniformly at random (that is, never choosing the "do nothing" move) then the distribution of the resulting state will be uniformly distrubuted among the $|K|-1$ different states, the probability of the resulting state being the winning state is*

$$\mathbb{P}(p((k_1,h_1)(k_2,h_2)\dots(k_j,h_j)) = k^{-1} \mid p((k_1,h_1)(k_2,h_2)\dots(k_{j-1},h_{j-1}) \neq k^{-1})) = \frac{1}{|K|-1},$$

and the expected number of moves is $|K|-1$.

Proof. Because the new states are in 1-to-1 correspondence with the elements of $K \setminus \{\mathrm{id}_K\}$, since $k_j \in \setminus\{\mathrm{id}_K\}$ is chosen uniformly at random, $p((k_1,h_1)(k_2,h_2)\dots(k_j,h_j)$ is uniformly distributed among all elements of K besides the projection of the first $j-1$ elements. The expected value is $|K|-1$ because the number of turns follows a geometric distribution with parameter $(|K|-1)^{-1}$. □

Unsuprisingly, when a generalized spinning switches puzzle has a *minimal* switching strategy, then we can do better than this, on average.

Proposition 1.7.3. *If the generalized spinning switches puzzle, $G \wr H$, has a minimal switching strategy, then the expected number of moves is $|K|/2$. TODO: And no other strategy can do better than this.*

Proof. TODO: Every move is equally likely to be our winner. (clean up proof)

If there's a switching strategy of length $|K|-1$, then the sequence walks through every state exactly once, and so every move is equally likely we're equally likely to win on any turn, so the expected number is $(N+1)/2$ with an N move strategy. □

Proposition 1.7.4. *Whether or not the generalized spinning switches puzzle $G \wr H$ has a switching strategy, there always exists a (perhaps infinite) strategy whose expected value of moves is strictly less than $|K|-1$.*

Proof. We can always do a bit better than the naive play by saying never do $(g,g,...,g) \in K$ followed by $(g^{-1},g^{-1},...,g^{-1}) \in K$. $\square$

Conjecture 1.7.5. *There exists a constant $c < 1$ such that for all generalized spinning switches puzzles, the expected number of moves is less than $c|K|$.*

1.7.4 Minimal switching strategies

Proposition 1.7.6. *When $G \wr H$ has a switching strategy, it always has a switching strategy of length $N < 2^{|K|-1}$.*

Proof. TODO: We can keep track of the possible states. Initially, the possible states are $K \setminus \{\mathrm{id}_K$, but in subsequent steps, the state can be anything in $2^{K \setminus \{\mathrm{id}_K}$. $\square$

In fact we can do better, because we can look at element of K up to actions of H. (To do: I'm sure this has a useful name. Look up Burnside to see what it's called there.)

I conjecture, however, that we can do much better still.

Conjecture 1.7.7. *Whenever $G \wr H$ has a switching strategy, it also has a minimal switching strategy. (That is, a switching strategy of length $|K|-1$.)*

1.7.5 Multiple moves between each turn

We could modify the puzzle so that the adversary's spinning sequence $\{h_i \in H\}$ is constained so that $h_i = e_H$ whenever $i \not\cong 0 \bmod k$; that is, the adversary can only spin every k turns. For any finite setup $G \wr H$, there exists k such that Player B can win. (For example, take $k > |K|$ so that Player B can just do a walk of K.)

How can you compute the minimum k such that Player B has a strategy for each choice of $G \wr H$? This is an interesting statistic.

1.7.6 Nonhomogeneous switches

We could imagine a square board with different sorts of switches—for instance one of the corners has an ordinary 2-way switch and another has a 3-way switch and so on.

Example 1.7.8. *Act using $\mathbb{Z} \wr H$. Then we have a collection of "projection-like" maps for each coordinate:*

$p' \colon \underbrace{\mathbb{Z} \times \mathbb{Z}}_{K} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ *which sends* $p'(x,y) = (x \bmod 2, x \bmod 3)$.

Definition 1.7.9. *A **nonhomogeneous generalized spinning switches puzzle** is a a wreath product of the free group on k generators by a "rotation" group H, $\mathbb{F}_k \wr_\Omega H$, together with a product of finite groups indexed by Ω, $K' = \prod_{\omega \in \Omega} G_\omega$ each with group presentation*

$$G_\omega = \langle g_1^\omega, g_2^\omega, \ldots, g_k^\omega \mid R_\omega \rangle,$$

and a corresponding sequence of evaluation maps $e_\omega \colon \mathbb{F}_\omega \to G_\omega$.

When all of the groups are isomorphic, this essentially simplifies to the original definition.

Definition 1.7.10. *TODO: This definition is incomplete and unintelligible.*

Let X be a nonhomogeneous generalized spinning switches puzzle with wreath product $\mathbb{F}_k \wr_\Omega H$ which has base K.

Then let $e \colon K \to K$ be the evaluation map evaluated coordinatewise on K.

*Then a **nonhomogeneous switching strategy** is a sequence in K such that the evaluation/projection map $e \circ p \colon \mathbb{F}_k \wr_\Omega H \to K'$*

Proposition 1.7.11. *In the specific case that $k = 1$, $\Omega = [n]$, $H = C_n$, and $\{G_\omega\}_{\omega \in \Omega}$ behave is a sequence of cyclic groups of pairwise coprime order. Then the nonhomogeneous generalized spinning switches puzzle has a switching strategy, namely $\{(1, 1, \ldots, 1)\}_{i=1}^{|K'|-1}$.*

Proof. TODO Chinese remainder theorem, right? $\square$

When do such setups have a switching strategy? (We can also put this problem into purely algebraic terms.) Of course, if one is a switch and another is like S_3 then it's not clear how to keep them indistinguishable to Player B. In the case of the switches, we can have $\mathbb{Z}$ act on either of them.

1.7.7 Counting switching strategies

Is there a good way to count the number of switching strategies? How about up to the action of H?

In the case of $S_3 \wr \mathbf{1}$, I counted the palindromic switching strategies, which can give a lower bound on the number of palindromic switching strategies of $S_3 \wr C_2$. (MSE)

1.7.8 Yehuda's "open game"

Yehuda has an "open game" version of the puzzle that goes like this: Everyone can see the state of the board. Player B says what moves (positionally) they want to make. Player A rotates the board however they see fit *then* applies Player B's move.

Conjecture 1.7.12. *If G is abelian, Player B can always win by repeatedly choosing the inverse of the board.*

Note that the conjecture fails for certain nonabelian groups, namely $S_3 \wr C_2$ if the initial state is $((123),(132))$ because $(123)(123) = (132)$ and $(132)(132) = (123)$.

Example 1.7.13. *This strategy works for $\mathbb{Z}_2 \wr C_4$. Here's one example.*

- *The initial state of the board is one switch off and all of the others on.*
- *Player B says to turn on the off switch.*
- *Player A turns the board in order to turn off an adjacent switch.*
- *Player B says to turn on those two adjacent switches.*

- *Player A turns the board in order to toggle one of the switches, leaving one diagonal on and one off.*
- *Player B says to turn on those two diagonal swtiches.*
- *Player A rotates the board to instead turn off the two on switches so that all switches are off.*
- *Player B says to turn on all of the switches.*
- *Player A knows that rotating the board does not do anything, so they turn on all of the switches.*
- *Player B wins.*

This strategy also works for $\mathbb{Z}_3 \wr C_3$.

1.7.9 Generalizations of $S_3 \wr C_2$

In Example 1.6.2, we constructed a strategy for $S_n \wr C_2$, by exploiting the fact that S_n can be generated by elements of order 2.

Conjecture 1.7.14. *There exists a switching strategy for $S_n \wr C_4$.*

Conjecture 1.7.15. *There exists a switching strategy for $A_n \wr C_3$.*

The generalization of this conjecture, which is as likely to be false as it is to be true doesn't have evidence to support it.

Conjecture 1.7.16. *If G can be generated by elements of order p^n, and H is a p-group acting faithfully on the set of switches Ω, then $G \wr_\Omega H$ has a switching strategy.*