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Chapter 1

Generalized Spinning Switches

In this chapter, we explore puzzles about switches on the corners of a spinning table. Such puzzles have been written about and generalized since they were first popularized by Martin Gardner. In this chapter, we provide perhaps the fullest generalization yet, modeling both the switches and the spinning table as a wreath product of two arbitrary finite groups. We classify large families of wreath products depending on whether or not they correspond to a solvable puzzle, completely resolving the problem for p-groups, and providing novel examples for other families of groups. Lastly, we provide a number of open questions and conjectures, and provide other suggestions of how to generalize some of these ideas further.

1.1 Overview and Preliminaries

The paper is organized into six sections. This section, Section 1.1 provides a brief history of this genre of puzzles and introduces some of the first approaches to generalizing the puzzle further. Section 1.2 models these generalizations in the context of the wreath product, and formalizes the notation of puzzles being solvable. Section 1.3 explores situations where the puzzle does not have a winning strategy, and provides reductions that allow us to prove that entire families of puzzles are not solvable. Section 1.4 constructs a strategy for switches that behave like p-groups, and gives us ways of building strategies from smaller parts. Section 1.5 provides novel examples of puzzles that do not behave like p-groups, but still have winning strategies. Lastly, Section 1.6 provides further generalizations, and contains dozens of conjectures, open questions, and further directions.

1.1.1 History

Spinning Switches puzzles are a family of closely related puzzles that were first popularized by Martin Gardner in the form of pint glasses on a lazy susan. In the February 1979 edition of his column "Mathematical Games." [1] Gardner writes that he learned of the puzzle from Robert Tappay of Toronto who "believes it comes from the U.S.S.R.," a history that is not especially forthcoming.

My preferred version of the puzzle appears in Peter Winkler's 2004 book *Mathematical Puzzles A Connoisseur's Collection*

> Four identical, unlabeled switches are wired in series to a light bulb. The switches are simple buttons whose state cannot be directly observed, but can be changed by pushing; they are mounted on the corners of a rotatable square. At any point, you may push, simultaneously, any subset of the buttons, but then an adversary spins the square. Show that there is a deterministic algorithm that will enable you to turn on the bulb in at most some fixed number of steps. [2]

(Winkler's version will be a working example in many parts of the paper, so it is worth keeping in mind. An illustration can be found in Figure 1.1.)

Over the last three decades, various authors have consider generalizations of this puzzle. Here, we build on those results and go further. The first place authors looked to generalize was suggested by Gardner himself. In his March 1979 column, he provided the answer to the original puzzle and wrote

> The problem can also be generalized by replacing glasses with objects that have more than two positions. Hence the rotating table leads into deep combinatorial questions that as far as I know have not yet been explored. [3]

In 1993, Yehuda, Etzion, and Moran [4]. took on the challenge and developed a theory of the spinning switches puzzle where the switches behave like roulettes with a single “on” state. In this chapter we take Gardner’s charge to it’s logical conclusion and consider switches that behave like arbitrary “objects that have more than two positions”.

Another generalization of this puzzle could look at other ways of “spinning” the switches. In 1995, Ehrenborg and Skinner [5] did this in a puzzle they call ”Blind Bartender with Boxing Gloves”, that analyzed this puzzle while allowing the adversary to use an arbitrary, faithful group action to “scramble” the switches. We analyze our generalized switches within this same context.

This puzzle was re-popularized in 2019 when it appeared in “The Riddler” column from the publication FiveThirtyEight [6]. Shortly after this, in 2022, Yuri Rabinovich synthesized Yehuda and Ehrenborg’s results in a a paper that modeled the collection of switches as a vector space over a finite field, and modeled the “spinning” or “scrambling” as a faithful, linear group action.

Sidana [7] provides a detailed overview of the history of this and related problems.

1.1.2 A Solution to the Winkler’s Spinning Switches Puzzle

We will start by discuss the solution to Winkler’s version of the puzzle because the solution provides some insights and intuition for the techniques that we use later. Before solving the four-switch version of the puzzle, we will make Peter Winkler proud by beginning with a simpler, two-switch version.

Example 1.1.1. *Suppose that we have two identical unlabeled switches on opposite corners of a square table, as in Figure 1.1*

Then we have a three-step solution for solving the problem. We start by toggling both switches simultaneously. If this does not turn on the light, this means that the switches were (and still) are in different states.

Then, the adversary spins the table. Next, we toggle one of the two switches to ensure that the switches are both in the same state. If the light hasn’t turned on, both must be in the off state.

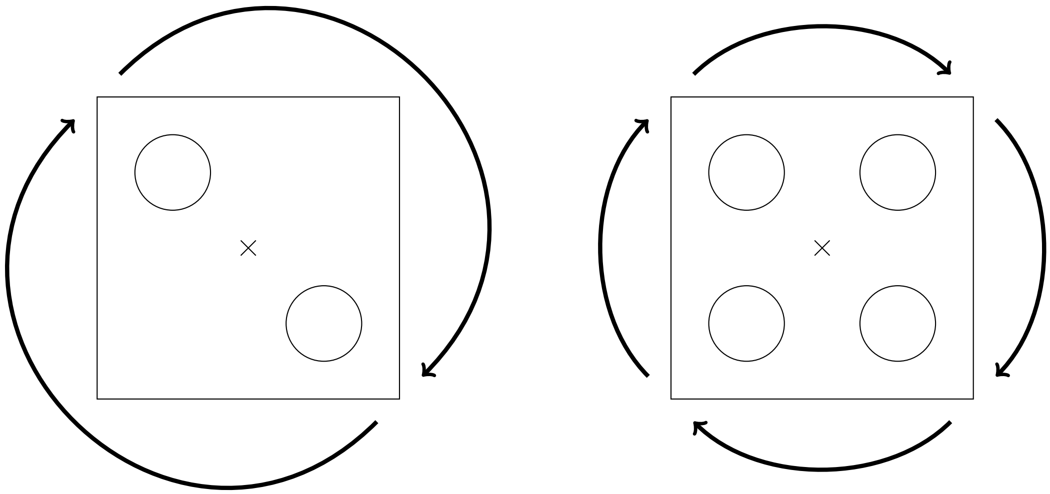

Figure 1.1: Illustration of both the two-switch and Winkler's original four-switch version of the puzzle, both on a spinning square table.

The adversary spins the table once more, but to no avail. We know both switches are in the off state, so we toggle them both simultaneously, turning on the lightbulb.

In order to bootstrap the two-switch solution into a four-switch solution, we must notice two things:

1. First, if we can get two switches along each diagonal into the same state respectively, then we can solve the puzzle by toggling both diagonals (all four switches), both switches in a single diagonal, and both diagonals again. In this (sub-)strategy, toggling both switches along a diagonal is equivalent to toggling a single single switch in the above example.

2. Second, we can indeed get both diagonals into the same state by toggling a switch from each diagonal (two switches on any side of square), then a single switch from one diagonal, followed by a switch from each switch.

We will interleave these strategies in a particular way, following the notation of Rabinovich [8].

Definition 1.1.2. *Given two sequences $A = \{a_i\}_{i=1}^N$ and $B = \{b_i\}_{i=1}^M$, we can define the **interleave** operation as*

$$\begin{aligned} A \circledast B &= (A, b_1, A, b_2, A, \ldots, b_M, A) && (1.1) \\ &= (\underbrace{a_1, a_2, \ldots, a_N}_{A}, b_1, \underbrace{a_1, a_2, \ldots, a_N}_{A}, b_2, \underbrace{a_1, a_2, \ldots, a_N}_{A}, \ldots, b_M, \underbrace{a_1, a_2, \ldots, a_N}_{A}). && (1.2) \end{aligned}$$

which has length $(M+1)N + M = MN + M + N$.

Typically it is useful to interleave two strategies when A solves the puzzle given that the switches are in a particular state, and B gets the switches into that particular state. Usually, we also need A not to "interrupt" what B is doing. In the problem of four switches on a square table, B will ensure that the switches are in the same state within each diagonal, and A will turn on the light when that's the case. Moreover, A does not change the state within each diagonal.

Proposition 1.1.3. *There exists a fifteen-move strategy that guarantees that the light in Winkler's puzzle turns on.*

Proof. We begin by formalizing the two strategies. We will say that the first strategy S_1 where we toggle the two switches in a diagonal together will consist of the following three moves:

1. Switch **a**ll of the bulbs (A).

2. Switch the **d**iagonal consisting of the upper-left and lower-right bulbs (D).

3. Switch **a**ll of the bulbs (A).

We will say that the second strategy S_2 where we get the two switches within each diagonal into the same state consists of the following three moves:

1. Switch both switches on the left **s**ide (S).

2. Switch **one** switch (1).

3. Switch both switches on the left **s**ide (S).

Then the 15 move strategy is

$$S_1 \circledast S_2 = (A,S,A,D,A,S,A,1,A,S,A,D,A,S,A)$$

$\square$

We will generalize this construct in Theorem 1.4.1, which offers a formal proof that this strategy works.

It is worth briefly noting that $S_1 \circledast S_2$ is the fourth *Zimin word* (also called a *sequipower*), an idea that comes up in the study of combinatorics on words.

1.1.3 Generalizing Switches

Two kinds of switches are considered by Bar Yehuda, Etzion, and Moran in 1993 [4]: switches with a single "on" position that behave like n-state roulettes ($\mathbb{Z}_n$) and switches that behave like the finite field $\mathbb{F}_q$, both on a rotating k-gonal table. Yuri Rabinovich [8] goes further by considering collections of switches that behave like arbitrary finite dimensional vector spaces over finite fields that are acted on by a linear, faithful group action. We generalize this notion further by considering switches that behave like arbitrary finite groups.

Example 1.1.4. *In Figure 1.2, we provide a schematic for a switch that behaves like the symmetric group S_3. It consists of three identical-looking parts that need to be arranged in a particular order in order for the switch to be on.*

We could also construct a switch that behaves like the dihedral group of the square, D_8. This switch a flat, square prism that can slot into a square hole, and only one of the $|D_8| = 8$ rotations of the prism completes the circuit.

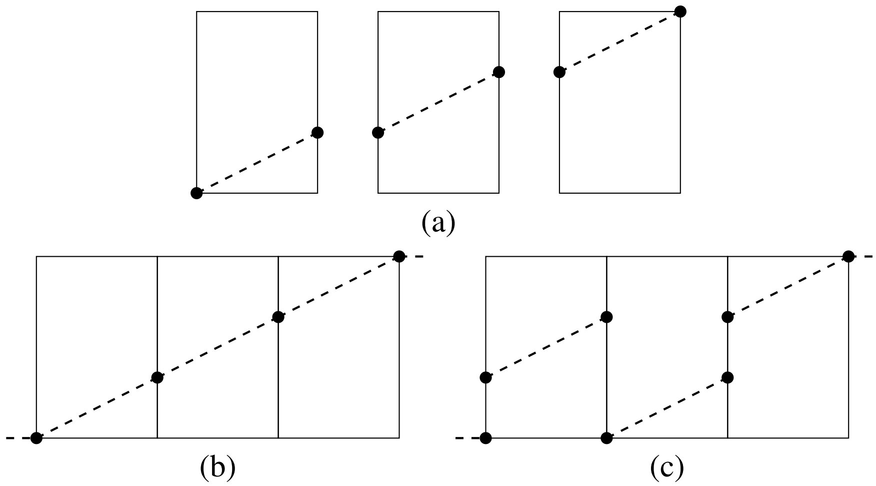

Figure 1.2: Part (a) shows a simple schematic for a switch that behaves like S_3, the symmetric group on three letters. The three rectangles can be permuted arbitrarily, but only configuration (b) completes the circuit. All other configurations fail to complete the circuit (e.g. (c)).

One subtlety of using a group G to model a switch is that both the "internal state" of a switch itself and the set of "moves" or changes are modeled by G. Perhaps we think of the state as the underlying set of G and the moves act via right group action of G on itself.

The reason that using a group to model a switch is because groups have many of the properties we would expect in a desirable switch.

Note 1.1.5. *The axioms for a group $(G,\cdot)$ closely follow what we would expect from a switch.*

1. (Closure) The group $(G,\cdot)$ is equipped with a binary operation, $\cdot\colon G\times G\to G$. That is, for all pairs of elements $g_1,g_2\in G$ their product is in G

$$g_1\cdot g_2\in G.$$

   In the context of switches, this means that if the switch is in some state $g_1\in G$ and player B moves it with action $g_2\in G$, then $g_1\cdot g_2\in G$ is a valid switch for the state.

2. (Identity) There exists an element $\mathrm{id}_G\in G$ such that for all $g\in G$,

$$\mathrm{id}_G\cdot g = g\cdot \mathrm{id}_G = g.$$

This axiom is useful because it means that Player B can "do nothing" to a switch and leave it in whatever state it is in. Because the identity is a distinguished element in G, we will also use the convention that id_G is the "on" or "winning" state for a given switch. (It is worth noting that all of the arguments work with small modification regardless of which element is designated as the on state.)

3. (Inverses) For each element $g \in G$ there exists an inverse element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = \mathrm{id}_G .$$

This axiom states that no matter what state a switch is in, there is a move that will transition it into the on state.

4. (Associativity) Given three elements $g_1, g_2, g_3 \in G$,

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

This is axiom is not strictly necessary for modeling switches, but as we will see in a later definition, it gives us a convenient way to describe the conditions for a winning strategy. (In Subsection 1.6.3, we briefly discuss dropping the associativity axiom by considering switches that behave like quasigroups with identity.)

1.1.4 Generalizing Spinning

We can also consider generalizations of "spinning" the switches. In particular, we will adopt the generalization from Ehrenborg and Skinner's [5] 1995 paper, which use arbitrary faithful group actions to permute the switches. In particular, they provide a criterion that determines which group actions yield a winning strategy in the case of a given number of "ordinary" switches (those that behave like $\mathbb{Z}_2$). Rabinovich [8] stretches these results a bit further and looks at faithful linear group actions on collections of switches that behave are modeled as a finite dimensional vector space over a finite field. We build on this result in the context of more general switches.

1.2 The Wreath Product Model

Peter Winkler's version of the puzzle consists of four two-way switches on the corners of a rotating square table. The behavior of the switches are naturally modeled as $\mathbb{Z}_2$, and the rotating table is modeled as the cyclic group C_4. We will take the wreath product of $\mathbb{Z}_2$ by C_4 in order to get a mathematical model of the generalized spinning switches puzzle.

1.2.1 Modeling Generalized Spinning Switches Puzzles

We don't evoke wreath products arbitrarily: we use them because they are the right abstraction to model a generalized spinning switches puzzle where G describes the behavior of the switches, Ω describes the positions of the switches, and the action of H on Ω models the ways the adversary can permute the switches.

Definition 1.2.1 ([9])**.** *Let G and H be groups, let Ω be a finite H-set, and let $K = \prod_{\omega\in\Omega} G_\omega$, where $G_\omega \cong G$ for all $\omega \in \Omega$. Then the* **wreath product** *of G by H denoted by $G \wr H$, is the semidirect product of K by H, where H acts on K by $h \cdot (d_\omega) = d_{h^{-1}\omega}$ for $g \in H$ and $(g_\omega) \in \prod_{\omega\in\Omega} G_\omega$. The normal subgroup K of $G \wr H$ is called the* **base** *of the wreath product.*

The group operation is $(k,h)\cdot(k',h') = (k(h\cdot k'),hh')$

An element of $(k,h) \in G \wr H$ represents a turn of the game: The puzzle-solver chooses an element of the base $k \in K$ to indicate how they want to modify each of their switches and then their adversary chooses $h \in H$ and acts with h on Ω to permute the switches.

Example 1.2.2. *Consider the setup in the Winkler's version of the puzzle that consists of two-way switches ($\mathbb{Z}_2$) on the corners of a rotating square ($C_4 \cong \langle 0^\circ, 90^\circ, 180^\circ, 270^\circ\rangle$). The game itself corresponds to the wreath product $\mathbb{Z}_2 \wr C_4$. We will use the convention that the base of the wreath*

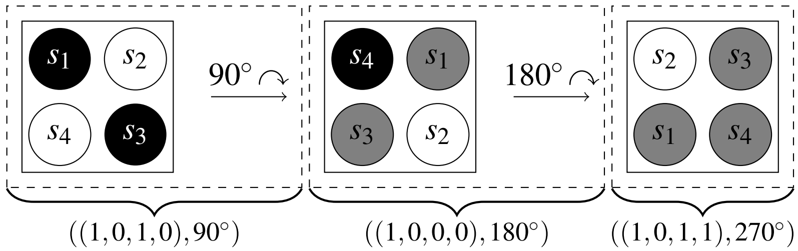

Figure 1.3: An illustration of two turns each in the Spinning Switches puzzle, modeled as elements of a wreath product.

product, K is ordered upper-left, upper-right, lower-right, lower-left; the group action is specified by degrees in the clockwise direction.

Consider the following two turns:

1. *During the first turn, the puzzle-solver toggles the upper-left and lower-right switches, and the adversary rotates the table 90° clockwise. This is represented by the element $((1,0,1,0),90^\circ) \in \mathbb{Z}_2 \wr C_4$.*

2. *During the second turn, the puzzle-solver toggles the upper-left switch, and the adversary rotates the table 90° clockwise. This is represented by the element $((1,0,0,0),180^\circ) \in \mathbb{Z}_2 \wr C_4$.*

As illustrated in Figure 1.3, the net result of these two turns is the same as a single turn where the puzzle-solver toggles the upper-left, upper-right, and lower-left switches and the adversary rotates the board 270° clockwise.

The multiplication under the wreath product agrees with this:

$$
\begin{aligned}
((1,0,1,0),90^\circ)\cdot((1,0,0,0),180^\circ) &= ((1,0,1,0)+\underbrace{90^\circ\cdot(1,0,0,0)}_{(0,0,0,1)},90^\circ+180^\circ) \\
&= ((1,0,1,1),270^\circ)
\end{aligned}
$$

Occasionally it is useful to designate a particular state of the switches as the "on" state or the winning state. We will use the convention that the lightbulb turns on when all of the switches are equal to the identity, that is $\mathrm{id}_K \in K$. It is worth noting, however, that the existence of a winning strategy does not depend on a particular choice in the winning state. Instead, a winning strategy is equivalent to a choice of moves that will walk over all of the possible configuration states, regardless of the choice of the adversaries spin.

1.2.2 Switching Strategy

We will begin by formalizing the notation of a winning strategy in a generalized spinning switches puzzle. Informally, a switching strategy is a sequence of moves that the puzzle-solver can make that will put the switches into every possible state, which ensures the the "on" state is reached regardless of the initial (hidden) state of the switches.

Definition 1.2.3. *A* ***switching strategy*** *for $G \wr H$ is a finite sequence of elements in the base K, $\{k_i \in K\}_{i=1}^N$, such that for every sequence of elements in H, $\{h_i \in H\}_{i=1}^N$,*

$$p(\{\underbrace{e_{G\wr H}}_{m_0}, \underbrace{(k_1,h_1)}_{m_1}, \underbrace{(k_1,h_1)\cdot(k_2,h_2)}_{m_2}, \cdots, \underbrace{(k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)}_{m_N}\}) = K.$$

where $p\colon G \wr H \to K$ is the projection map from the wreath product onto its base.

This definition is useful because it puts the problem into purely algebraic terms. It is also useful because it abstracts away the initial state of the switches: regardless of the initial state $k \in K$, a existence of a switching strategy means that its inverse $k^{-1} \in K$ appears in the sequence. (This follows the convention that $\mathrm{id}_K \in K$ is designated as the "on" state. If k' is chosen to the "on" state, then the sequence must contain $k^{-1}k'$.)

Proposition 1.2.4. *A finite sequence of moves is guaranteed to reach the "on" state if and only if it is a switching strategy.*

Proof. Without loss of generality, say that the "on" state for the switches is id_K. In the puzzle, we have an initial (hidden) state, k. Thus, after the i-th move, the wreath product element that represents the state of the switches is

$$p\left((k,\mathrm{id}_H)\cdot(k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right) = k\cdot p\left((k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right),$$

by associativity. We can factor out the first term because the "spin" is id_H, which acts trivially: $(k,\mathrm{id}_H)\cdot(k',h') = (kk',h')$

The initial state can be any $k \in K \setminus \{\mathrm{id}_K\}$, and this isn't known to the puzzle-solver. In order to reach the "on" state, there must exist some i, such that $p\left((k_1,h_1)\cdot(k_2,h_2)\cdots(k_i,h_i)\right) = k^{-1}$. k and adversarial sequences $\{h_i\}_{i=1}^N$. $\square$

It's also worth noting that this model can be thought of as a random model or an adversarial model: the sequence $\{h_i \in H\}$ can be chosen after the sequence $\{k_i \in K\}$ in a deterministic way or randomly.

1.2.3 Bounds on the length of switching strategies

One useful consequence of this definition is that it quite straightforward to prove certain propositions. For example, the minimum length for a switching strategy has a simple lower bound.

Proposition 1.2.5. *Every switching strategy $\{k_i \in K\}_{i=1}^N$ is a sequence of length at least $|K|-1$.*

Proof. This follows from an application of the Pigeonhole Principle. Because the set

$$\{e_{G\wr H},(k_1,h_1),(k_1,h_1)\cdot(k_2,h_2),\cdots,(k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)\}$$

has at most $N+1$ elements. In order for the projection to be equal to K,

$$p(\{e_{G\wr H},(k_1,h_1),(k_1,h_1)\cdot(k_2,h_2),\cdots,(k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)\}) = K,$$

it must be the case that $N+1 \geq |K|$. Therefore $N \geq |K|-1$. $\square$

Minimal length switching strategies are common, so we give them a name.

Definition 1.2.6. *A* ***minimal switching strategy*** *for $G \wr H$ is a switching strategy of length $N = |K|-1$.*

In practice, every wreath product known by the author to have a switching strategy also has a known minimal switching strategy. In Section 1.6, we ask whether this property always holds.

1.3 Reductions

In this section, we develop examples of generalized spinning switches puzzles that do not have switching strategies using three techniques: directly, by a reduction on switches, or by a reduction on spinning.

1.3.1 Puzzles Known to Have No Switching Strategies

Our richest collection of known puzzles without switching strategies comes from a theorem of Rabinovich, which models switches as a vector space over a finite field.

Theorem 1.3.1. *[8] Assume that a finite "spinning" group H acts linearly and faithfully on a collection of switches that behave like a vector space V over a finite field $\mathbb{F}_q$ of characteristic p. Then the resulting puzzle has a switching strategy if and only if H is a p-group.*

It's worth noting that Rabinovich's switches are less general than arbitrary finite groups, but the "spinning" is more general: in addition to permuting the switches, the group action might add linear combinations of them as well.

Example 1.3.2. *By the theorem of Rabinovich [8], the game $\mathbb{Z}_2 \wr C_3$ does not have a switching strategy. In Rabinovich's notation, the vector space of switches is $\mathbb{Z}_2^3$ over the field $\mathbb{F}_2 = \mathbb{Z}_2$.*

The wreath product $\mathbb{Z}_2 \wr C_3$ is perhaps the simplest example of a generalized spinning switches puzzle without a switching strategy, so we will continue to use it as a basis of future examples.

1.3.2 Reductions on Switches

With Theorem 1.3.1 providing a family of wreath products without spinning switches to reduce to, we now introduce a theorem that allows us to prove that large families of wreath products also do not have a switching strategy.

Theorem 1.3.3. *If $G \wr H$ does not have a switching strategy and G' is a group with a quotient $G'/N \cong G$, then $G' \wr H$ does not have a switching strategy.*

Proof. We will prove the contrapositive, and suppose that $G' \wr H$ has base K' and a switching strategy $\{k'_i \in K'\}_{i=1}^{N}$.

The quotient map $\varphi\colon G' \mapsto G$ extends coordinatewise to $\varphi\colon K' \mapsto K$, which further extends in the first coordinate to $G \wr H$: $\varphi(k,h) := (\varphi(k),h)$.

It is necessary to verify that $\varphi\colon G' \wr H \to G \wr H$ is indeed a homomorphism.

$$\begin{aligned}
\varphi((k'_\alpha,h_\alpha)) \cdot \varphi((k'_\beta,h_\beta)) &= (\varphi(k'_\alpha),h_\alpha) \cdot (\varphi(k'_\beta),h_\beta) \\
&= (\varphi(k'_\alpha)(h_\alpha \cdot \varphi(k'_\beta)),h_\alpha h_\beta) \\
&= (\varphi(k'_\alpha)\varphi(h_\alpha \cdot k'_\beta),h_\alpha h_\beta) \\
&= (\varphi(k'_\alpha(h_\alpha \cdot k'_\beta)),h_\alpha h_\beta) \\
&= \varphi((k'_\alpha(h_\alpha \cdot k'_\beta),h_\alpha h_\beta)) \\
&= \varphi((k'_\alpha,h_\alpha) \cdot (k'_\beta,h_\beta))
\end{aligned}$$

Therefore the sequence $\{\varphi(k'_i) \in K\}_{i=1}^{N}$ is a switching strategy on $G \wr H$, because the quotient map $\varphi\colon G' \to G$ (and thus $\varphi\colon K' \to K$) is injective. $\square$

Example 1.3.4. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_6 \wr C_3$ does not have a switching strategy either, as illustrated in Figure 1.4.*

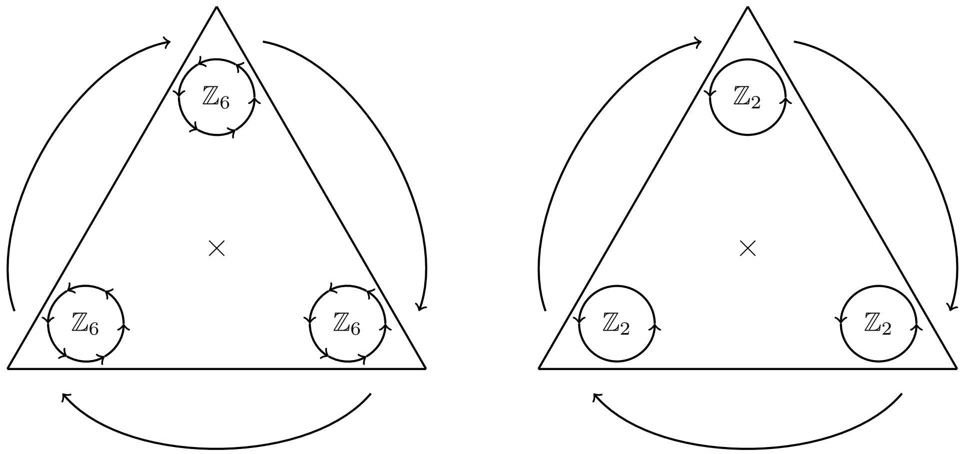

Figure 1.4: A reduction on switches: $\mathbb{Z}_6 \wr C_3$ reduces to $\mathbb{Z}_2 \wr C_3$, which is known not to have a switching strategy.

1.3.3 Reductions on Spinning

We can do two similar reductions on the "spinning" group of a wreath product. These theorems say that if a given wreath product $G \wr H$ does not have a switching strategy, then a similar wreath product $G \wr H'$ with a "more complicated" spinning group H' will not have a switching strategy either.

Theorem 1.3.5. *If $G \wr H$ does not have a switching strategy and H' is a group with a subgroup $A \leq H'$ such that $A \cong H$, then $G \wr H'$ does not have a switching strategy.*

Proof. Again we will prove the contrapositive. Assume that $G \wr H'$ does have a switching strategy, $\{k_i\}_{i=1}^N$. Then by definition, for any sequence $\{h'_i\}_{i=1}^N$, the projection of the sequence

$$p(\{(k_1,h'_1)\cdot(k_2,h'_2)\cdots(k_i,h'_i)\}_{i=1}^N) = K,$$

and in particular this is true when h'_i is restricted to be in the subgroup H. Thus a switching strategy for $G \wr H'$ is also a valid switching strategy for $G \wr H$. $\square$

Example 1.3.6. *Consider the wreath product $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ where Ω' consists of six switches on the corners of a hexagon as illustrated in Figure 1.3.6. While the group action of C_3 on Ω' is not*

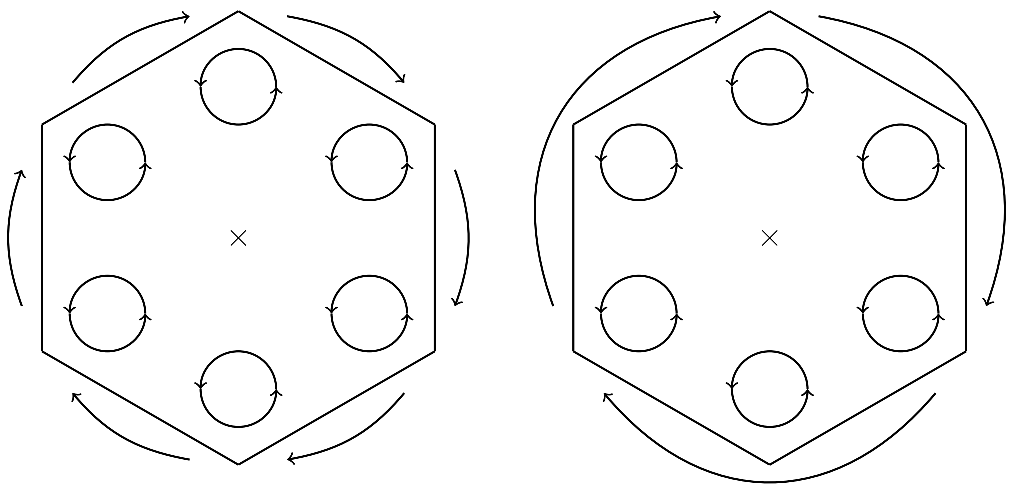

Figure 1.5: If there were a solution to $\mathbb{Z}_2 \wr_{\Omega_6} C_6$, then there would be a solution to $\mathbb{Z}_2 \wr_{\Omega_6} C_3$.

transitive, we know that $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ *does not have a switching strategy, because in particular there is no way to ensure that the top, bottom-right, and bottom-left switches hit every state.*

Since $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ *doesn't have a switching strategy,* $\mathbb{Z}_2 \wr_{\Omega_6} C_6$ *cannot not have a switching strategy either.*

In the above example, we noted that $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ does not have a switching strategy by focusing on a triangle of switches and using the knowledge that $\mathbb{Z}_2 \wr C_3$ (two-way switches on a rotating triangular board) does not have a switching strategy. The following theorem allows us to take that very shortcut.

Theorem 1.3.7. *Suppose that* H' *is a group with a subgroup* $A \leq H'$ *such that* $A \cong H$*, and let*

$$\mathrm{Orb}(\omega) = \{\omega \cdot a : a \in A\} \subseteq \Omega$$

be the (right) orbit of $\omega \in \Omega$ *under* A*. If* $G \wr_{\mathrm{Orb}(\omega)} H$ *does not have a switching strategy, then* $G \wr_\Omega H'$ *does not have a switching strategy.*

Proof. We start by making the contrapositive assumption that $G \wr_\Omega H'$ has a switching strategy $\{k_i \in K\}_{i=1}^N$, and we consider the projection $p_\omega \colon K \to K_\omega$ where $K = \prod_{\omega' \in \Omega} G_{\omega'}$ and $K_\omega = \prod_{\omega' \in \mathrm{Orb}(\omega)} G_{\omega'}$.

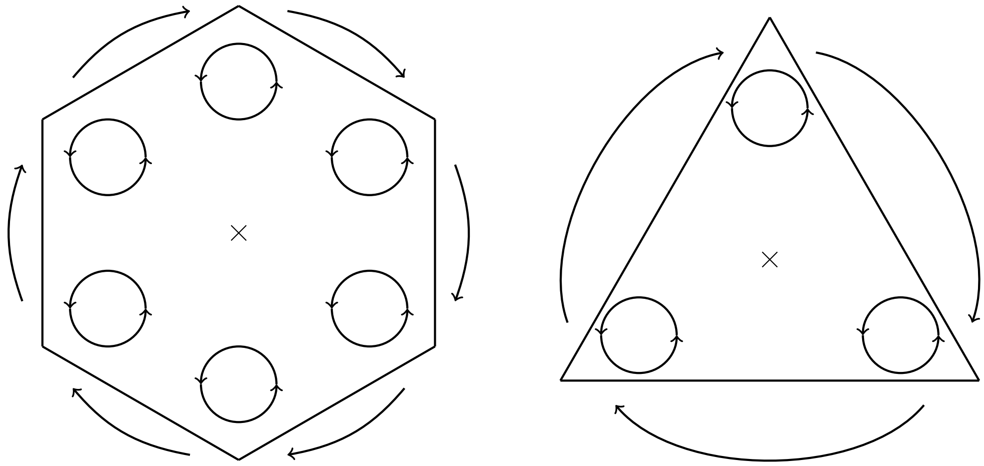

Figure 1.6: We know that $\mathbb{Z}_2 \wr_\Omega C_6$ cannot have a switching strategy, because that would imply a switching strategy for $\mathbb{Z}_2 \wr_{\Omega'} C_3$, where Ω' is the orbit of the top switch rotations of multiples of 120°.

Then $\{p_\omega(k_i) \in K_\omega\}_{i=1}^N$ is a switching strategy for $G \wr_{\mathrm{Orb}(\omega)} H$, since the projection is a surjective map. $\square$

Example 1.3.8. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_2 \wr C_6$ does not have a switching strategy either, as illustrated in Figure 1.6.*

Now that we've proven that large families of wreath products do not have switching strategies, it's worthwhile to construct families of wreath products that do have switching strategies.

1.4 Switching Strategies on p-Groups

In this section, we'll develop a broad family of switching strategies, namely those where G and H (and thus $G \wr H$) are p-groups.

1.4.1 Switching Strategy Decomposition

Our first constructive theorem provides a technique that can be used to construct switching strategies for switches that behave like a group G in terms of a normal group and its corresponding quotient group.

Theorem 1.4.1. *The wreath product $G \wr H$ has a switching strategy if there exists a normal subgroup $N \trianglelefteq G$ such that both $N \wr H$ and $G/N \wr H$ have switching strategies.*

Proof. Let $S_{G/N} = \{k_i^{G/N} \in K_{G/N}\}$ denote the switching strategy for $G/N \wr H$, and let $S_N = \{k_i^N \in K_N\}$ denote the switching strategy for $N \wr H$.

We ultimately would like to interleave these two strategies, but $k_i^{G/N} \notin K_G$. To find the appropriate analog, we partition G into $[G : N] = m$ (TODO: left?) cosets of N,

$$G = g_1 N \sqcup g_2 N \sqcup \cdots \sqcup g_m N,$$

each with a chosen representative in G. Now we define a map $r\colon G/N \to G$ that chooses the chosen representative of the coset, and extends coordinatewise. We use this map to define a sequence $S = \{r(k_i^{G/N}) \in K_G\}$.

We claim that these two sequences interleaved, $S_N \circledast S$, is a switching strategy for $G \wr H$. To prove this claim, we observe two facts:

1. Multiplying by elements of S_N will not change cosets and will walk through every element of its coset.

2. Multiplying by elements of S will walk through all cosets.

Therefore the interleaved sequence will walk through all elements of each coset, and thus is surjective onto K. □

1.4.2 Construction of switching strategies on p groups

We start with a corollary of Theorem 1.3.1.

Corollary 1.4.2. *If H is a finite p-group that acts faithfully on Ω, then the wreath product $G \wr H$ has a switching strategy whenever $|G| = p^n$ for some n.*

Proof. If $|G| = p^n$, then either $G \cong \mathbb{Z}_p$ or G is not simple.

If $n = 1$, $G \cong \mathbb{Z}_p \cong \mathbb{F}_p$, then there exists a switching strategy by Theorem 1.3.1. This is because H permutes the coordinates of $V = \mathbb{F}_p^{|\Omega|}$, and so it is a linear action on the vector space.

Otherwise, G is not simple. This means that G has a normal subgroup N of order $|N| = p^t$ (with $t \geq 1$) and a quotient G/N with order $|G/N| = p^{n-t}$. Because $t < n$ and $n - t < n$, we eventually end up at the $n = 1$ case by induction. □

This means that whenever G and H (and thus $G \wr H$) are p-groups, then $G \wr H$ has a switching strategy.

1.4.3 A folklore conjecture

Here we note a conjecture from folklore, which—if true—implies that we have *almost* solved the problem in its full generality.

Conjecture 1.4.3 (Folklore)**.** *Almost all groups are* 2*-groups.*

One reason for this conjecture is computational. According to the On-Line Encyclopedia of Integer Sequences [10], there are $A000001(2^{10}) = 49487367289$ groups of order 2^{10} and there are $A063756(2^{11} - 1) = 49910536613$ groups of order less than 2^{11}. This means that more than 99.15% of the groups of order less than 2^{11} are of order 2^{10}.

If this conjecture is true, then most types of switches have switching strategies on most kinds of faithful finite group actions. Of course, while most finite groups may be 2-groups, most mathematicians are more interested in groups that *aren't*. This next section develops two families of examples of switching strategies where the switches do not behave like p-groups.

1.5 Switching Strategies on Other Wreath Products

So far, the literature has only contained examples of spinning strategies on wreath products that are themselves p-groups: $|G \wr_\Omega H| = |G|^{|\Omega|} \cdot |H|$, where H acts faithfully.

1.5.1 $G \wr \mathbf{1}$

The first example of a wreath product $G \wr H$ that has switching strategy but where G is not a p-group occurs when $H = \mathbf{1}$ is the trivial group. In this case, player B cannot "spin" the switches at all, so player A has perfect information the entire time. Thus, $G \wr \mathbf{1} \cong G$ has a switching strategy for all finite groups G. In fact, $G \wr \mathbf{1}$ has many switching strategies.

Proposition 1.5.1. *The wreath product $G \wr \mathbf{1}$ has $(|G|-1)!$ minimal switching strategies.*

Proof. There are $(|G|-1)!$ permutations of $G \setminus \{\mathrm{id}_G\}$, and each one corresponds to a minimal switching strategy.

Suppose that $(k_1, k_2, \ldots, k_{|G|-1})$ is such a permutation, then define a switching strategy as $\{k'_i\}_{i=1}^{|G|-1}$ where $k'_1 = k_1$ and $k'_i = k_{i-1}^{-1} k_i$.

Then we claim by induction that $(k'_1, \mathrm{id}) \cdot (k'_2, \mathrm{id}) \cdots (k'_j, \mathrm{id}) = (k_j, \mathrm{id})$. By construction, the base case is true when $j = 1$. If the claim holds up to $j-1$, then

$$\underbrace{(k'_1, \mathrm{id}) \cdot (k'_2, \mathrm{id}) \cdots (k'_{j-1}, \mathrm{id})}_{(k_{j-1}, \mathrm{id})}(k'_j, \mathrm{id}) = (k_{j-1}, \mathrm{id})(k_{j-1}^{-1} k_j, \mathrm{id}) = (k_j, \mathrm{id}),$$

as desired. Thus the projection of the partial products is

$$p(\{\underbrace{e_{G\wr\mathbf{1}}}_{m_0}, \underbrace{(k'_1, h_1)}_{m_1}, \underbrace{(k'_1, h_1) \cdot (k'_2, h_2)}_{m_2}, \cdots, \underbrace{(k'_1, h_1) \cdot (k'_2, h_2) \cdots (k'_N, h_N)}_{m_N}\}) \tag{1.3}$$

$$= p(\{e_{G\wr\mathbf{1}}, (k_1, \mathrm{id}), (k_2, \mathrm{id}), \cdots, (k'_N, \mathrm{id})\}) \tag{1.4}$$

$$= \{e_{G\wr\mathbf{1}}, k_1, k_2, \cdots, k_{|G|-1}\} = K, \tag{1.5}$$

where $\{k_1, k_2, \ldots, k_{|G|-1}\}$ spans $G \setminus \{\mathrm{id}_G\} \cong K \setminus \{\mathrm{id}_K\}$ by assumption. $\square$

While the trivial wreath product is an important example to keep in mind for generating counterexamples, we're generally more interested in the situation where Player A permutes the switches to create uncertainty for Player B.

1.5.2 Two copies of the symmetric groups on a rectangular table ($S_n \wr C_2$)

In this section, we will exploit the fact that the symmetric group S_n can be generated by self-inverse elements to construct a switching strategy for $S_n \wr C_2$. This switching strategy has two parts. The first part ensures that the two switches have every possible difference. The second part show that we can get either of the switches to take on every possible value without disturbing the difference.

Theorem 1.5.2. *$S_n \wr C_2$ has a switching strategy.*

Proof. We start with the observation that the symmetric group can be generated by transpositions: $S_n = \langle t_1, t_2, \ldots, t_N \rangle$. This means that there is a sequence of transpositions $t_{i_1}, t_{i_2}, \ldots, t_{i_M}$ such that $\{\mathrm{id}, t_{i_1}, t_{i_1}t_{i_2}, \ldots, t_{i_1}t_{i_2}\cdots t_{i_M}\} = S_n$.

Strategy A: We first assume that we have a sequence of moves that can run through all values of the first coordinate. Of course we do, just let $t' = (t,t) \in K$ and use the sequence $\{t'_{i_1}, t'_{i_2}, \ldots, t'_{i_M}\}$.

Strategy B: Let the i-th move be denoted by $m_i = (k_i, k'_i)$. We want $t_1 t_2^{-1}$ to be whatever we choose. We can do this too by applying the sequence $\{(t_{i_j}, \mathrm{id}_G)\}_{j=1}$ this is because

$$t_1(t_2 t)^{-1} = t_1 t^{-1} t_2^{-1} = (t_1 t) t_2^{-1}$$

because t is a transposition, and thus $t = t^{-1}$. Moreover, when we apply the first strategy, it doesn't change this difference: $(t_1 t)(t_2 t)^{-1} = t_1 t t^{-1} t_2^{-1} = t_1 t_2^{-1}$

Combined: $S_2 \circledast S_1$ $\square$

Example 1.5.3. *If $a \in S_3$, let a_1 mean multiplying one of the two copies by a and a_2 mean multiplying both of the copies by a. Then the following is a strategy:*

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(13)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(13)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

In general, if you can walk through G with elements of order 2, then there is a strategy.

Proposition 1.5.4. *For $n > 1$, $S_n \wr C_m$ does not have a switching strategy whenever m is not a power of 2.*

Proof. The alternating group A_n is an index 2 subgroup of S_n, so A_n is normal, and $S_n/A_n \cong \mathbb{Z}_2$. Since we know that $\mathbb{Z}_2 \wr C_m$ has no switching strategy when m is not a power of 2, by the reduction in Theorem 1.3.3, $S_n \wr C_m$ does not have a switching strategy. □

1.6 Open questions

1.6.1 Generalizations of $S_3 \wr C_2$

In Example 1.5.3, we constructed a strategy for $S_n \wr C_2$, by exploiting the fact that S_n can be generated by elements of order 2.

Conjecture 1.6.1. *There exists a switching strategy for $S_n \wr C_4$.*

Conjecture 1.6.2. *There exists a switching strategy for $A_n \wr C_3$.*

The generalization of this conjecture, which is as likely to be false as it is to be true doesn't have evidence to support it.

Conjecture 1.6.3. *If G can be generated by elements of order p^n, and H is a p-group acting faithfully on the set of switches Ω, then $G \wr_\Omega H$ has a switching strategy.*

1.6.2 Palindromic switching strategies

In all known examples, when there exists a switching strategy S, there exists a *palindromic* switching strategy $S' = \{k'_i \in K\}_{i=0}^N$ such that $k'_i = k'_{N-i}$ for all i.

Conjecture 1.6.4. *Whenever $G \wr H$ has a switching strategy, it also has a palindromic switching strategy.*

I'm interested in the answer even in the case of $G \wr 1 \equiv G$.

(MSE)

1.6.3 Quasigroup switches

In the paper we modeled switches as groups. This is because groups have desirable properties:

1. Closure. Regardless of which state a switch is in, modifying the state is the set of states.

2. Identity. We don't have to toggle a switch on a given turn.

3. Inverses. If a switch is off, we can always turn it on.

It turns out that we don't need the axiom of associativity, because the sequencing is naturally what computer scientists call "left associative". Thus, we can model switches a bit more generally as *loops* (i.e. quasigroups with identity.)

1.6.4 Expected number of turns

Recall that the original conception of a generalized spinning switches puzzle is to turn on all of the switches at once. That if the original state of the puzzle is $k \in K$, we "win" on move i if $k^{-1} = p((k_1,h_1)(k_2,h_2)\dots(k_i,h_i))$. It is natural to ask about the expected value of the number of turns given various sequences of moves. Notice that this is a question we can ask even about generalized spinninng switches puzzles that do not have a switching strategy.

Indeed, Winkler [2] (TODO, this is probably the wrong citation!) notes in the solution of his puzzle:

> This puzzle reached me via Sasha Barg of the University of Maryland, but seems to be known in many places. Although no fixed number of steps can guarantee turning the bulb on in the three-switch version [with two-way switches], a smart randomized algorithm can get the bulb on in at most $5\frac{5}{7}$ steps on average, against any strategy by an adversary who sets the initial configuration and turns the platform. [11]

In all cases, when computing the expected number of turns, we will assume that the initial hidden state $k \in K$ is not the winninng state id_K, and that the adversaries "spins" are indepedent and identically distributed uniformly random elements $h_j \in H$.

Proposition 1.6.5. *If Player B chooses $k_j \in K \setminus \{\mathrm{id}_K\}$ uniformly at random (that is, never choosing the "do nothing" move) then the distribution of the resulting state will be uniformly distrubuted among the $|K|-1$ different states, the probability of the resulting state being the winning state is*

$$\mathbb{P}(p((k_1,h_1)(k_2,h_2)\dots(k_j,h_j)) = k^{-1} \mid p((k_1,h_1)(k_2,h_2)\dots(k_{j-1},h_{j-1}) \neq k^{-1})) = \frac{1}{|K|-1},$$

and the expected number of moves is $|K|-1$.

Proof. Because the new states are in 1-to-1 correspondence with the elements of $K \setminus \{\mathrm{id}_K\}$, since $k_j \in \setminus\{\mathrm{id}_K\}$ is chosen uniformly at random, $p((k_1,h_1)(k_2,h_2)\ldots(k_j,h_j)$ is uniformly distributed among all elements of K besides the projection of the first $j-1$ elements. The expected value is $|K|-1$ because the number of turns follows a geometric distribution with parameter $(|K|-1)^{-1}$. □

Unsuprisingly, when a generalized spinning switches puzzle has a *minimal* switching strategy, then we can do better than this, on average.

Proposition 1.6.6. *If the generalized spinning switches puzzle, $G \wr H$, has a minimal switching strategy, then the expected number of moves is $|K|/2$. TODO: And no other strategy can do better than this.*

Proof. TODO: Every move is equally likely to be our winner. (clean up proof)

If there's a switching strategy of length $|K|-1$, then the sequence walks through every state exactly once, and so every move is equally likely we're equally likely to win on any turn, so the expected number is $(N+1)/2$ with an N move strategy. □

Proposition 1.6.7. *Whether or not the generalized spinning switches puzzle $G \wr H$ has a switching strategy, there always exists a (perhaps infinite) strategy whose expected value of moves is strictly less than $|K|-1$.*

Proof. We can always do a bit better than the naive play by saying never do $(g,g,...,g) \in K$ followed by $(g^{-1},g^{-1},...,g^{-1}) \in K$. □

Conjecture 1.6.8. *There exists a constant $c < 1$ such that for all generalized spinning switches puzzles, the expected number of moves is less than $c|K|$.*

1.6.5 Minimal switching strategies

Proposition 1.6.9. *When $G \wr H$ has a switching strategy, it always has a switching strategy of length $N < 2^{|K|-1}$.*

Proof. TODO: We can keep track of the possible states. Initially, the possible states are $K \setminus \{\mathrm{id}_K$, but in subsequent steps, the state can be anything in $2^{K\setminus\{\mathrm{id}_K}$. □

In fact we can do better, because we can look at element of K up to actions of H. (To do: I'm sure this has a useful name. Look up Burnside to see what it's called there.)

I conjecture, however, that we can do much better still.

Conjecture 1.6.10. *Whenever $G \wr H$ has a switching strategy, it also has a minimal switching strategy. (That is, a switching strategy of length $|K|-1$.)*

1.6.6 Multiple moves between each turn

We could modify the puzzle so that the adversary's spinning sequence $\{h_i \in H\}$ is constained so that $h_i = e_H$ whenever $i \not\cong 0 \bmod k$; that is, the adversary can only spin every k turns. For any finite setup $G \wr H$, there exists k such that Player B can win. (For example, take $k > |K|$ so that Player B can just do a walk of K.)

How can you compute the minimum k such that Player B has a strategy for each choice of $G \wr H$? This is an interesting statistic.

1.6.7 Nonhomogeneous switches

We could imagine a square board with different sorts of switches—for instance one of the corners has an ordinary 2-way switch and another has a 3-way switch and so on.

Example 1.6.11. *Act using $\mathbb{Z} \wr H$. Then we have a collection of "projection-like" maps for each coordinate:*

$p' \colon \underbrace{\mathbb{Z} \times \mathbb{Z}}_{K} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ *which sends* $p'(x,y) = (x \bmod 2, x \bmod 3)$.

Definition 1.6.12. *A* ***nonhomogeneous generalized spinning switches puzzle*** *is a a wreath product of the free group on k generators by a "rotation" group H,* $\mathbb{F}_k \wr_\Omega H$*, together with a product of finite groups indexed by* Ω*,* $K' = \prod_{\omega\in\Omega} G_\omega$ *each with group presentation*

$$G_\omega = \langle g_1^\omega, g_2^\omega, \ldots, g_k^\omega \mid R_\omega \rangle,$$

and a corresponding sequence of evaluation maps $e_\omega \colon \mathbb{F}_\omega \to G_\omega$*.*

When all of the groups are isomorphic, this essentially simplifies to the original definition.

Definition 1.6.13. *TODO: This definition is incomplete and unintelligible.*

Let X be a nonhomogeneous generalized spinning switches puzzle with wreath product $\mathbb{F}_k \wr_\Omega H$ *which has base K.*

Then let $e \colon K \to K$ *be the evaluation map evaluated coordinatewise on K.*

Then a ***nonhomogeneous switching strategy*** *is a sequence in K such that the evaluation/projection map* $e \circ p \colon \mathbb{F}_k \wr_\Omega H \to K'$

Proposition 1.6.14. *In the specific case that* $k = 1$*,* $\Omega = [n]$*,* $H = C_n$*, and* $\{G_\omega\}_{\omega\in\Omega}$ *behave is a sequence of cyclic groups of pairwise coprime order. Then the nonhomogeneous generalized spinning switches puzzle has a switching strategy, namely* $\{(1,1,\ldots,1)\}_{i=1}^{|K'|-1}$*.*

Proof. TODO Chinese remainder theorem, right? □

When do such setups have a switching strategy? (We can also put this problem into purely algebraic terms.) Of course, if one is a switch and another is like S_3 then it's not clear how to keep them indistinguishable to Player B. In the case of the switches, we can have $\mathbb{Z}$ act on either of them.

1.6.8 Counting switching strategies

Is there a good way to count the number of switching strategies? How about up to the action of H?

In the case of $S_3 \wr \mathbf{1}$, I counted the palindromic switching strategies, which can give a lower bound on the number of palindromic switching strategies of $S_3 \wr C_2$. (MSE)

1.6.9 Infinite Switching Strategies

In Definition 1.2.3, a switching strategy was defined as a finite sequence on finite wreath products. However, it might be interesting to extend the definitions to switches with a countably infinite number of states, to a countably infinite number of switches, or both. To keep K countable in the latter cases, we may need to restrict to the restricted wreath product, where $K \cong \bigoplus_{\omega\in\Omega} G_\omega$ is defined to be a direct sum instead of a direct product.

Because there are an infinite number of states, any switching strategy must also be an infinite sequence.

Definition 1.6.15. *A **infinite switching strategy** on an infinite wreath product $G \wr H$ is a sequence $\{k_i \in K\}_{i=1}^{\infty}$ such that for all $k \in K$ and all infinite sequences $\{h_i \in H\}_{i=1}^{\infty}$, there exists some $N \geq 0$ such that the projection*

$$p((k_1,h_1)\cdot(k_2,h_2)\cdots(k_N,h_N)) = k^{-1}.$$