Table of Contents

Chapter 1

Spinning Switches

What this paper does:

1. Generalizes switches to arbitrary groups.
2. Proves a result when switches look like p-groups.
3. Finds the "correct" model: the wreath products.
4. Provides reductions for when strategies don't exist, which are easy to prove with the wreath product model.
5. Comes up with an example where something isn't a prime power. ($S_3 \wr C_2$ is nontrivial. Of course $G \wr \mathbf{1}$ also works.)

1.1 TODO

1. Provide solution to Winkler puzzle.
2. Fill out Example 1.5.1.
3. Incorporate Section 1.4 into earlier sections.
4. Provide example for first reduction. (Theorem 1.5.2)

5. Give other part of example for second reduction. (Theorem 1.5.4)

6. Example for third reduction. (Theorem 1.5.5)

7. Define/discuss minimal switching strategies?

8. Mention conjecture that most groups are 2-groups

1.2 Overview and Preliminaries

This section provides a brief history of the problem and provides the idea for the more general context. Section 1.3 models these generalizations in the context of the wreath product. Section 1.4 is where all of the references are. [TODO: put this section elsewhere] Section 1.5 allows us to prove when Player B does not have a winning strategy. Section 1.6 allows us to make a statement about games that have a prime number of possible moves. Section 1.7 gives us an example of new kinds of puzzles that have solutions. Section 1.8 gives us an example of new kinds of puzzles that have solutions.

1.2.1 History

A closely related puzzle was popularized by Martin Gardner in the February 1979 edition of his column "Mathematical Games." [1] He wrote that he learned of the puzzle from Robert Tappay of Toronto who "believes it comes from the U.S.S.R."

The version under consideration in this paper is first hinted at in 1993 by Yehuda and collaborators [2]. Ehrenborg and Skinner consider something very similar, which they call the "Blind Bartender with Boxing Gloves" [3].

This was re-popularized in 2019 when it appeared in "The Riddler" from FiveThirtyEight [4].

My preferred version appears in Peter Winkler's 2004 book *Mathematical Puzzles A Connoisseur's Collection*

> Four identical, unlabeled switches are wired in series to a light bulb. The switches are simple buttons whose state cannot be directly observed, but can be changed by pushing; they are mounted on the corners of a rotatable square. At any point, you may push, simultaneously, any subset of the buttons, but then an adversary spins the square. Show that there is a deterministic algorithm that will enable you to turn on the bulb in at most some fixed number of steps. [5]

TODO: Sidana paper has nice history.

1.2.2 Generalizing Switches

"The problem can also be generalized by replacing glasses with objects that have more than two positions. Hence the rotating table leads into deep combinatorial questions that as far as I know have not yet been explored." [6]

Switches that instead behave like n-state roulettes with a single on position are considered by Yehuda, Etzionn, and Moran in 1993 [2]. Yuri Rabinovich [7] goes further by considering collections of switches that behave like vector spaces over finite fields. I go further still by considering switches that behave like arbitrary finite groups—or more generally still, finite quasigroups with identity.

[A schematic for a switch that looks like D_4.]

1.2.3 Generalizing Spinning

We can also consider different ways of rearranging the switches. In a 1995 paper [3], Ehrenborg and Skinner provide a criterion for which permutations of ordinary, 2-way switches yield a winning strategy. Rabinovich [7] settles the problem for switches in a finite vector space.

For example, one could imagine a "switch" that behaves like the symmetric group S_3, consisting of three identical-looking parts that need to be arranged in a particular order in order for the switch to be on.

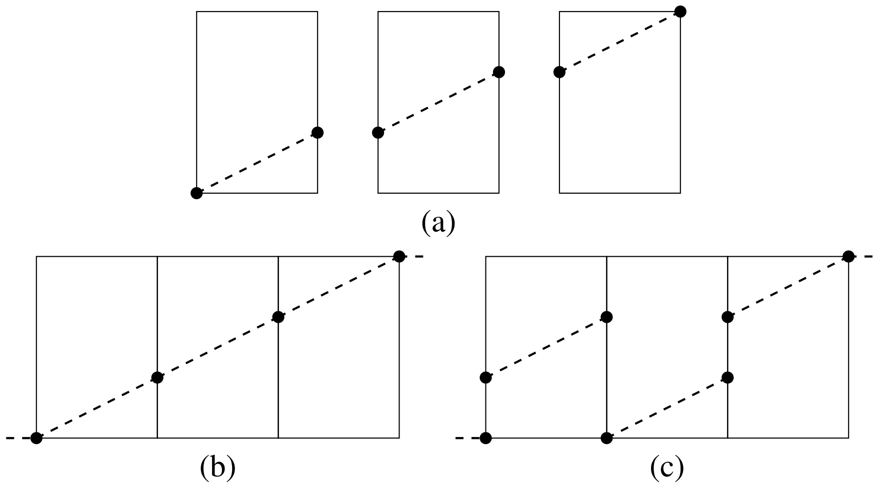

Figure 1.1: Part (a) shows a simple schematic for a switch that behaves like S_3, the symmetric group on three letters. The three rectangles can be permuted arbitrarily, but only configuration (b) completes the circuit. All other configurations fail to complete the circuit (e.g. (c)).

Or one could imagine a switch that behaves like the dihedral group of the square, D_8 where the square has a single, unique orientation that completes the circuit. Or abstractly, one could think of each switch as an abstract group element, where Player B can multiply by anything they like.

1.3 The Wreath Product Model

Remind you of the definition of a wreath product, and give examples of how it models the spinning switches puzzle.

1.3.1 Modeling Generalized Spinning Switches Puzzles

Definition 1.3.1 (Literally copied from Rotman)**.** *Let D and Q be groups, let Ω be a finite Q-set, and let $K = \prod_{\omega\in\Omega} D_\omega$, where $D_\omega \cong D$ for all $\omega \in \Omega$. Then the* ***wreath product*** *of D by Q denoted by $D \wr Q$, is the semidirect product of K by Q, where Q acts on K by $q \cdot (d_\omega) = d_{q\omega}$ for $q \in Q$ and $(d_\omega) \in \prod_{\omega\in\Omega} D_\omega$. The normal subgroup K of $D \wr Q$ is called the* ***base*** *of the wreath product.*

TODO: I prefer the Wikipedia version where $q \cdot (d_\omega) = d_{q^{-1}\omega}$

The reason this definition is used is because it models the game well, where G models the behavior of the switches, Ω models the switches themselves, and the way H acts on Ω models the ways the adversary can "spin" the board.

An element of $(k,h) \in G \wr H$ represents a turn of the game: Player B chooses k to indicate how they want to modify each of their switches and then Player A chooses k to indicate how they want to permute the switches.

Example 1.3.2. *Consider the setup in the original version of the problem consisting of two-way switches ($\mathbb{Z}_2$) on the corners of a rotating square ($C_4 \cong \langle 0^\circ, 90^\circ, 180^\circ, 270^\circ \rangle$). This can be modeled as a game on the wreath product $\mathbb{Z}_2 \wr C_4$. We will use the convention that the base of the wreath product, K is ordered upper-left, upper-right, lower-right, lower-left, and the group action is specified by degrees in the clockwise direction.*

Consider the following two turns:

1. *Turn 1:* $((1,0,1,0),90^\circ) \in \mathbb{Z}_2 \wr C_4$.
   (a) *Player B toggles the upper-left and lower-right switches.*
   (b) *Player A rotates the table* 90° *clockwise.*
2. *Turn 2:* $((1,0,0,0),180^\circ) \in \mathbb{Z}_2 \wr C_4$.
   (a) *Player B toggles the upper-left switch.*
   (b) *Player A rotates the table* 90° *clockwise.*

As illustrated in Figure 1.2, the net result of these two turns is the same as a single turn where Player B toggles the upper-left, upper-right, and lower-left switches and Player A rotates the board 270° *clockwise.*

The multiplication under the wreath product agrees with this:

$$((1,0,1,0),90^\circ) \cdot ((1,0,0,0),180^\circ) = ((1,0,1,1),270^\circ)$$

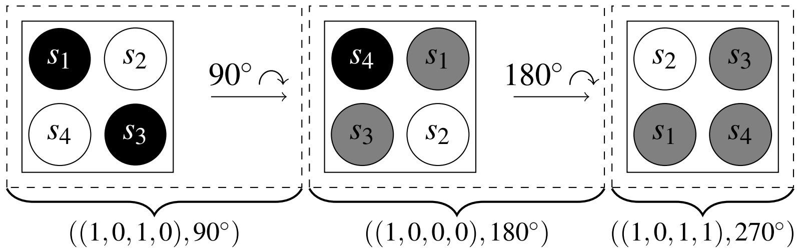

Figure 1.2: An illustration of two turns each in the Spinning Switches puzzle, modeled as elements of a wreath product.

Occasionally it is useful to designate a particular state of the switches as the on state or the winning state, and ordinarily the identity state is the choice given for this. However, the existence of a winning strategy does not depend on a particular choice in the winning state; instead, a winning strategy is equivalent to a choice of moves that will walk over all of the possible configuration states, regardless of the choice of the adversaries spin.

1.3.2 Switching Strategy

Definition 1.3.3. *A* ***switching strategy*** *is a finite sequence, $\{k_i \in K\}_{i=1}^N$, such that for every sequence $\{h_i \in H\}_{i=1}^N$,*

$$p(\{e_{G \wr H}, (k_1,h_1), (k_1,h_1) \cdot (k_2,h_2), \cdots, (k_1,h_1) \cdot (k_2,h_2) \cdots (k_N,h_N)\}) = K.$$

where $p \colon G \wr_\Omega H \to K$ is the projection map from the wreath product $G \wr_\Omega H$ onto its base K.

This definition is useful because it puts the problem into purely algebraic terms. It is also useful because it abstracts away the initial state of the switches: regardless of the initial state $k \in K$, a existence of a switching strategy means that its inverse $k^{-1} \in K$ appears in the sequence.

Lemma 1.3.4. *A sequence of moves is guaranteed to "turn on the lightbulb" if and only if it is a switching strategy.*

Proof. TODO: The initial state (k) is hidden, so we only turn on the lightbulb if our projection contains k^{-1}.

Use associativity and show that if the initial state is (k, id), then it bumps everything like left multiplication by k. $\square$

It's also worth noting that this model can be thought of as a random model or an adversarial model: the sequence $\{h_i \in H\}$ can be chosen after the sequence $\{k_i \in K\}$ in a deterministic way or randomly.

1. Random model to adversarial model

2. Retroactively changes initial conditions

3. Spin

4. Proves lower bound of number of moves

1.4 Historical Progress

1.4.1 Yehuda (1993) [Roulette wheel]

Theorem 1.4.1. *The game on $\mathbb{Z}_n \wr C_m$ has a switching strategy if and only if $n = p^\alpha$ and $m = p^\beta$ where p is prime and α and β are nonnegative integers. They also deal with words with q^β letters over $\mathbb{F}_q$.*

1.4.2 Ehrenborg/Skinner (1995) [Scrambling]

Theorem 1.4.2. *Interested in fixing $G = \mathbb{Z}_2$ and looking at permutation representations of H, and determining when switching strategies exist. They look at a particular "sub-poset" of the set partitions of H partially ordered by refinement, and give a condition which is equivalent to a switching strategy.*

1.4.3 Sharma/Sidana (2021) [Other related games]

1.4.4 Yuri Rabinovich (2022)

[Switches are V over $\mathbb{F}_q$, arbitrary scrambling]

Theorem 1.4.3. *Let V be a vector space over a finite field $\mathbb{F}_q$ of characteristic p, and let V^+ be the group under addition. Let G be a group that acts linearly and faithfully on V. Then $G \wr V^+$ has a switching strategy if and only if G is a p-group.*

1.5 Reductions

There are essentially three ways to show that $G \wr H$ does not have a solution: directly, or via one of two *reductions* (or a combination thereof).

1.5.1 Puzzles Without Switching Strategies

Using results from Rabinovich [7], we can give examples of puzzles that don't have solutions. This section allows us to take those examples and stretch them into wider families of examples.

Example 1.5.1. *The game $\mathbb{Z}_2 \wr C_3$ does not have a switching strategy. Here's how to see it...*

1.5.2 Reductions on Switches

Theorem 1.5.2. *If $G \wr H$ does not have a switching strategy and G' is a group with a quotient $G'/N \cong G$, then $G' \wr H$ does not have a switching strategy.*

Proof. I will prove the contrapositive, and suppose that $G' \wr H$ has a switching strategy $\{k_i' \in K'\}_{i=1}^N$. The quotient map $\varphi\colon G' \mapsto G$ extends coordinatewise to $\hat{\varphi}\colon K' \mapsto K$.

The sequence $\{\hat{\varphi}(k_i') \in K\}_{i=1}^N$ is a switching strategy on $G \wr H$. [Say something about how the projection map is ?linear? wrt $\hat{\varphi}$? Say ϕ induces a homomorphism from $G' \wr H$?]

Want to prove

$$p((\hat{\varphi}(k_1'),h_1)\dots(\hat{\varphi}(k_i'),h_i)) = \hat{\varphi}(p'((k_1',h_1)\dots(k_i',h_i)))$$

where $p\colon G \wr H \to K$ and $p'\colon G' \wr H \to K'$ $\square$

Example 1.5.3. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_6 \wr C_3$ does not have a switching strategy either.*

1.5.3 Reductions on Spinning

Theorem 1.5.4. *If $G \wr H$ does not have a switching strategy and H' is a group with a subgroup $A \leq H'$ such that $A \cong H$, then $G \wr H'$ does not have a switching strategy.*

Theorem 1.5.5. *(Closely related to Theorem 1.5.4) If H' is a group with a subgroup $A \leq H'$ such that $A \cong H$, Ω' is an orbit of $\omega \in \Omega$ under A, and $G \wr_{\Omega'} H$ does not have a switching strategy, then $G \wr H'$ does not have a switching strategy.*

Proof. I will also prove the contrapositive. Assume that $G \wr H'$ does have a switching strategy, $\{k_i\}_{i=1}^N$. Then by definition, for any sequence $\{h_i'\}_{i=1}^N$, the projection of the sequence

$$p(\{(k_1,h_1')\cdot(k_2,h_2')\cdots(k_i,h_i')\}_{i=1}^N) = K,$$

and in particular this is true when h_i' is restricted to be in the subgroup H. Thus a switching strategy for $G \wr H'$ is also a valid switching strategy for $G \wr H$. $\square$

TODO: we have to be careful here, because the simple proof doesn't change the number of switches, it just makes the set of "rotations" smaller. In the case of the example of $\mathbb{Z}_2 \wr C_6$, our group action is no longer transitive, but instead we have two triangular orbits.

TODO: (Something is wrong about this question, but the spirit is right) Is it true that if $G \wr_\Omega H$ doesn't work then $G \wr_{\Omega'} H'$ doesn't work where Ω' is any orbit under N?

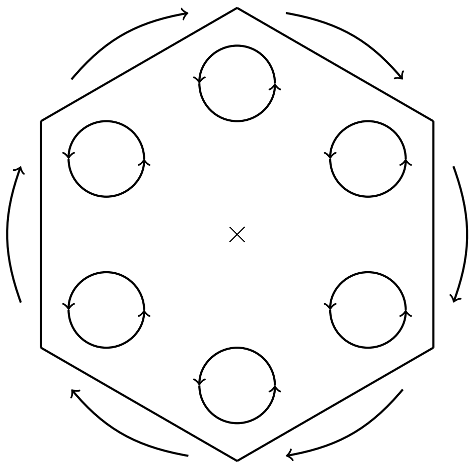

Figure 1.3: If there were a solution to $\mathbb{Z}_2 \wr C_6$, then there would be a solution to ...

Example 1.5.6. *We know that $\mathbb{Z}_2 \wr C_3$ doesn't have a switching strategy. This means that $\mathbb{Z}_2 \wr C_6$ does not have a switching strategy either.*

1.6 Switching Strategies on p-Groups

In this section, we'll develop a broad family of switching strategies, namely those on p-groups.

Theorem 1.6.1. *The wreath product $G \wr H$ has a switching strategy if there exists a normal subgroup $N \trianglelefteq G$ such that both $N \wr H$ and $G/N \wr H$ have switching strategies.*

Proof. Let $S_{G/N} = \{k_i^{G/N} \in K_{G/N}\}$ denote the switching strategy for $G/N \wr H$, and let $S_N = \{k_i^N \in K_N\}$ denote the switching strategy for $N \wr H$.

First, we partition G into $|G|/|N| = m$ cosets of N:

$$G = g_1 N \sqcup g_2 N \sqcup \cdots \sqcup g_m N.$$

From the switching strategy $\{k_i^{G/N} \in K_{G/N}\}$, we can get a sequence $S_{G'} = \{k_i' \in K_G\}$ by picking the coset representatives coordinatewise.

This sequence is not itself a switching strategy, but it does "hit" all combinations of cosets. That is, for every "spinning sequence" $\{h_i \in H\}$, and sequence of cosets $(g_{i_1}H, g_{i_2}H, \ldots, g_{i_m}H)$, there exists an index n such that

$$p((k'_1, h_1)\ldots(k'_n, h_n)) \in g_{i_1}H \times g_{i_2}H \times \cdots \times g_{i_m}H$$

Now if we intersperse $S'_G \circledast S_N$, this forms a switching strategy because ...

$$S'_G \circledast S_N = (\underbrace{k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_0}, \underbrace{k'_1, k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_1}, \ldots, \underbrace{k'_{n'}, k_1^N, k_2^N, \ldots, k_{n_N}^N}_{B_{n'}})$$

The partial products that end in block B_i all have switches in the same cosets of N, and the S_N strategy then hits all elements of K_G that belong to that combination of cosets.

□

Theorem 1.6.2. *[7] Assume that a finite group H acts linearly and faithfully on a vector space V over a finite field $\mathbb{F}_q$ of characteristic p. Then (G,V) is friendly if and only if G is a $p-$group.*

Corollary 1.6.3. *If H is a finite group that acts faithfully on Ω, then the wreath product $G \wr H$ has a switching strategy whenever $|G| = p^n$ for some n.*

Proof. If $|G| = p^n$, then either $G \cong \mathbb{Z}_p$ or G is not simple. If $G \cong \mathbb{Z}_p$, then there exists a strategy. Otherwise, G is not simple, so choose a normal subgroup N of order $|N| = p^t$ which gives a quotient G/N with order $|G/N| = p^{n-t}$. Then the result follows by induction. □

Corollary 1.6.4. *Based on the above construction, if $|G| = p^n$, then $G \wr C_{p^\ell}$ has a palindromic strategy of length $p^{np^\ell} - 1$.*

Proof. Is this true for general $H \neq C_{p^\ell}$? □

1.7 Switching Strategies on Other Wreath Products

So far, the literature has only contained examples of spinning strategies on wreath products that are themselves p-groups: $|G \wr_\Omega H| = |G|^{|\Omega|} \cdot |H|$, where H acts faithfully.

Of course, if $H = \mathbf{1}$ is the trivial group, then $G \wr \mathbf{1} \cong G$ has a switching strategy even if G is not a p-group. (In fact, it has $(|G|-1)!$ switching strategies!)

1.7.1 $S_n \wr C_2$

Theorem 1.7.1. *$S_n \wr C_2$ has a switching strategy.*

We'll construct a switching strategy. Now our switching strategy has two parts: the first is that we ensure that the two switches have every possible difference. The second is that we show that we can get either of the switches to take on every possible value without disturbing the difference.

Proof. We start with the observation that the symmetric group can be generated by transpositions: $S_n = \langle t_1, t_2, \ldots, t_N \rangle$. This means that there is a sequence of transpositions $t'_1, t'_2, \ldots, t'_M$ such that $\{\mathrm{id}, t'_1, t'_1 t'_2, \ldots, t'_1 t'_2 \ldots t'_M\} = S_n$.

Strategy A: (S_2)

Strategy B: (S_1)

Combined: $S_2 \circledast S_1$ $\square$

Example 1.7.2. *If $a \in S_3$, let a_1 mean multiplying one of the two copies by a and a_2 mean multiplying both of the copies by a. Then the following is a strategy:*

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(13)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(13)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(12)_1$$

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

In general, if you can walk through G with elements of order 2, then there is a strategy.

1.8 Open questions

1.8.1 Palindromic switching strategies

In all known examples, when there exists a switching strategy S, there exists a *palindromic* switching strategy $S' = \{k'_i \in K\}_{i=0}^{N}$ such that $k'_i = k'_{N-i}$ for all i.

Conjecture 1.8.1. *Whenever $G \wr H$ has a switching strategy, it also has a palindromic switching strategy.*

I'm interested in the answer even in the case of $G \wr 1 \equiv G$.

(MSE)

1.8.2 Quasigroup switches

In the paper we modeled switches as groups. This is because groups have desirable properties:

1. Closure. Regardless of which state a switch is in, modifying the state is the set of states.

2. Identity. We don't have to toggle a switch on a given turn.

3. Inverses. If a switch is off, we can always turn it on.

It turns out that we don't need the axiom of associativity, because the sequencing is naturally what computer scientists call "left associative". Thus, we can model switches a bit more generally as *loops* (i.e. quasigroups with identity.)

1.8.3 Expected number of turns

If we're to play the game uniformly at random, we're equally likely to win on any turn (of course, we never choose the "do nothing" move), so the expected number of moves is $|K|-1$.

If there's a switching strategy, we're equally likely to win on any turn, so the expected number is $(N+1)/2$ with an N move strategy. In all cases, when a strategy is known, a *minimal* strategy is known, so this is reduced to $|K|/2$.

Conjecture 1.8.2. *Whenever $G \wr H$ has a switching strategy, it also has a minimal switching strategy.*

For setups that don't have a switching strategy, what is an (infinite) strategy that minimizes the expected number of turns? We can always do a bit better than the naive play by saying never do $(g, g, \ldots, g) \in K$ followed by $(g^{-1}, g^{-1}, \ldots, g^{-1}) \in K$.

This puzzle reached me via Sasha Barg of the University of Mary- land, but seems to be known in many places. Although no fixed number of steps can guarantee turning the bulb on in the three-switch version, a smart randomized algorithm can get the bulb on in at most $5\frac{5}{7}$ steps on average, against any strategy by an adversary who sets the initial configuration and turns the platform. [8]

1.8.4 Multiple moves between each turn

We could modify the puzzle so that the adversary's spinning sequence $\{h_i \in H\}$ is constained so that $h_i = e_H$ whenever $i \not\cong 0 \bmod k$; that is, the adversary can only spin every k turns. For any finite setup $G \wr H$, there exists k such that Player B can win. (For example, take $k > |K|$ so that Player B can just do a walk of K.)

How can you compute the minimum k such that Player B has a strategy for each choice of $G \wr H$? This is an interesting statistic.

1.8.5 Different sorts of buttons

We could imagine a square board with different sorts of buttons—for instance one of the corners has an ordinary 2-way button and another has a 3-way button and so on. When do such setups have a switching strategy. (We can also put this problem into purely algebraic terms.) Of course, if one is a button and another is like S_3 then it's not clear how to keep them indistinguishable to Player B. In the case of the buttons, we can have $\mathbb{Z}$ act on either of them.

1.8.6 Counting switching strategies

Is there a good way to count the number of switching strategies? How about up to the action of H?

In the case of $S_3 \wr \mathbf{1}$, I counted the palindromic switching strategies, which can give a lower bound on the number of palindromic switching strategies of $S_3 \wr C_2$. (MSE)

1.8.7 Yehuda's "open game"

Yehuda has an "open game" version of the puzzle that goes like this: Everyone can see the state of the board. Player B says what moves (positionally) they want to make. Player A rotates the board however they see fit *then* applies Player B's move.

Conjecture 1.8.3. *If G is abelian, Player B can always win by repeatedly choosing the inverse of the board.*

Example 1.8.4. *This strategy works for* $\mathbb{Z}_2 \wr C_4$*. Here's one example.*

- *The initial state of the board is one switch off and all of the others on.*
- *Player B says to turn on the off switch.*
- *Player A turns the board in order to turn off an adjacent switch.*
- *Player B says to turn on those two adjacent switches.*
- *Player A turns the board in order to toggle one of the switches, leaving one diagonal on and one off.*
- *Player B says to turn on those two diagonal swtiches.*
- *Player A rotates the board to instead turn off the two on switches so that all switches are off.*
- *Player B says to turn on all of the switches.*
- *Player A knows that rotating the board does not do anything, so they turn on all of the switches.*
- *Player B wins.*

This strategy also works for $\mathbb{Z}_3 \wr C_3$.

1.8.8 Generalizations of $S_3 \wr C_2$

In Example 1.7.2, we constructed a strategy for $S_n \wr C_2$, by exploiting the fact that S_n can be generated by elements of order 2.

Conjecture 1.8.5. *There exists a switching strategy for* $S_n \wr C_4$.

Conjecture 1.8.6. *There exists a switching strategy for* $A_n \wr C_3$.

The generalization of this conjecture, which is as likely to be false as it is to be true doesn't have evidence to support it.

Conjecture 1.8.7. *If G can be generated by elements of order p^n, and H is a p-group acting faithfully on the set of switches Ω, then $G \wr_\Omega H$ has a switching strategy.*