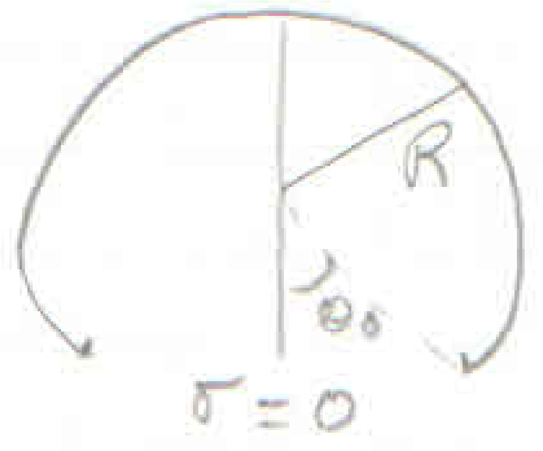
Now let's look at a Van der Graph generator.

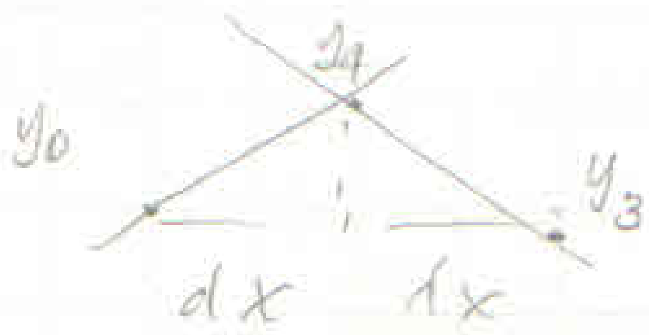


$V = V_0$ So for $\theta > \theta_0$ $\sum P_\ell C_\ell = V_0$ if $R = r = 1$

Similary we can alway normalise r/R and get $P_\ell C_\ell (r/R)^{-(\ell+1)}$ or $P_\ell C_\ell (r/R)^\ell$ these imply

$$\vec{E} = 1/R \, (P_\ell C_\ell \, (-(\ell+1)) \ \ldots$$

Say we have V and we want to find ρ. How can we compute $\nabla^2 V$?



slope 1 : $\frac{y_1 - y_0}{dx}$, slope 2 : $\frac{y_2 - y_1}{dx}$ then $\frac{\Delta slope}{\Delta x}$ is what we want $\frac{\Delta slope}{\Delta x} = \frac{y_1 - y_0 - y_2 + y_1}{\Delta x^2} = 0 = y_1 = \frac{y_0 + y_2}{2}$

which is the mean of points around it. This generalises fairly easily to $\mathbb{R}^n$

Then with repeated averages will actually get us to where we need to be and will give us V given the boundary conditions. Then one can take the second derivatrie approximation to get ρ.

Let's head back to the spherical solution of seperation of variables We remember that

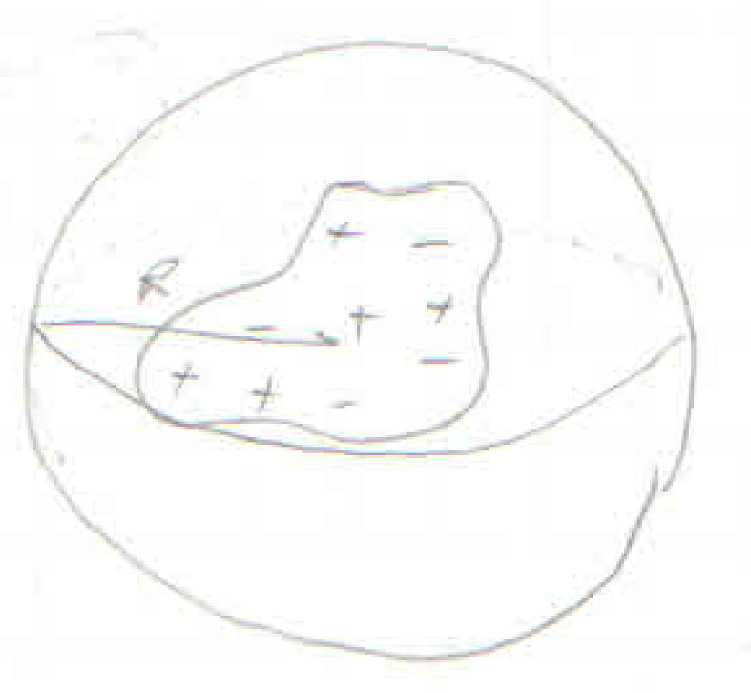
$$V(r, \theta) = \sum_\ell P_\ell^m (\cos\theta) \, r^{\{-(\ell+1), \ell\}} \, C_\ell \, e^{im\phi} \quad \ell \text{ int} \geq 0, \ |m| \leq \ell$$

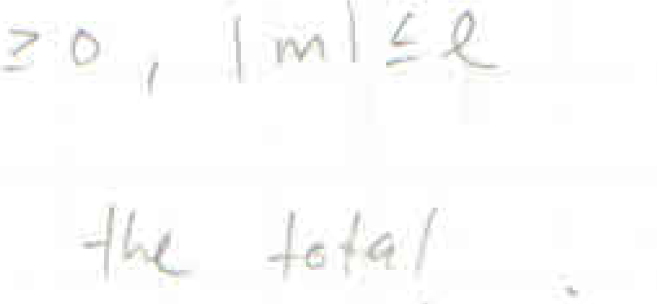
m describes the ripples on the azimoth and ℓ is the total amount of ripples. Usually $P_\ell^m(\cos\theta)e^{im\phi}$ is denoted Y_ℓ^m. I suggest seing the powerpoints to see some figures. How can we find C_ℓ?

$$\int Y_{\ell m}^* Y_{\ell' m'} = \mu_{\ell m} \delta_{\ell\ell'} \delta_{mm'} \quad \text{and} \quad \int Y_{\ell m}^* V = \int \sum Y_{\ell m} Y_{\ell m} C_\ell$$

$$\Rightarrow \int Y_{\ell m}^* V = C_{\ell m} \text{ if we rescale so } \mu_{\ell m} = 1 \qquad = \mathcal{C}_{\ell m} \mu_{\ell m}$$

Say we have some blob of charge. Let us wrap it in a sphere, then getting the potential on the surface provides a boundary. So long as all the charge is inside, If the sphere is on the origin we can use spherical harmonics. Now let's think of a shell of charge, we can get a measure of the charge density with Gauss' law. Also recall σ is continuous.





let $V_{out} = \sum Y_{\ell m} (r/R)^{-(\ell+1)} a_{\ell m}$, $V_{in} = \sum Y_{\ell m} (r/R)^\ell b_{\ell m}$, $a_{\ell m} = b_{\ell m}$ for continuity

and $\vec{E} = -\nabla_r V_{out} = -\sum Y_{\ell m} a_{\ell m} \, 1/R \, (-(\ell+1)) \, r/R^{-(\ell+2)}$
$-\nabla_R V_{in} = -\sum Y_{\ell m} a_{\ell m} \, 1/R \ \ell (r/R)^{\ell-1}$

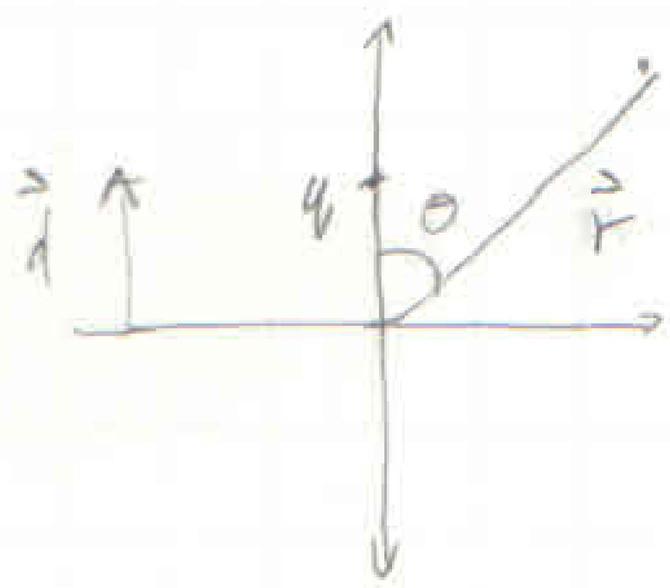
since $\vec{E}$ is radial then

$$\Delta E = \sum Y_{\ell m} a_{\ell m} \, 1/R \, ((\ell+1)(r/R)^{-(\ell+2)} + \ell (r/R)^{\ell-1}) \quad \text{let } r = R \text{ we get}$$

$\Delta E = \sum Y_{\ell m} a_{\ell m} (2\ell+1) 1/R = \sigma/\epsilon_0$, we examine the R independent V

$$\sum b_{\ell m} Y_{\ell m} r^{-(\ell+1)} = \sum a_{\ell m} Y_{\ell m} (r/R)^{-(\ell+1)}$$ what is the relationship between $a_{\ell m}$ and $b_{\ell m}$.

Ok! I guess not, we will move to dipoles. Imagine the following situation with a charge and a point in 3-D defining a plane.



Then $V = \dfrac{q}{4\pi\epsilon_0 |\vec{r} - \vec{d}|}$ $\quad r >> d$.

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\sqrt{1+d^2/r^2 - 2d/r\cos\theta}}$$

$\uparrow$ small

$$\approx \frac{q}{4\pi\epsilon_0 r} \frac{1}{1-d/R\cos\theta} \approx \frac{q}{4\pi\epsilon_0 r}(1+d/R\cos\theta + \dots)$$

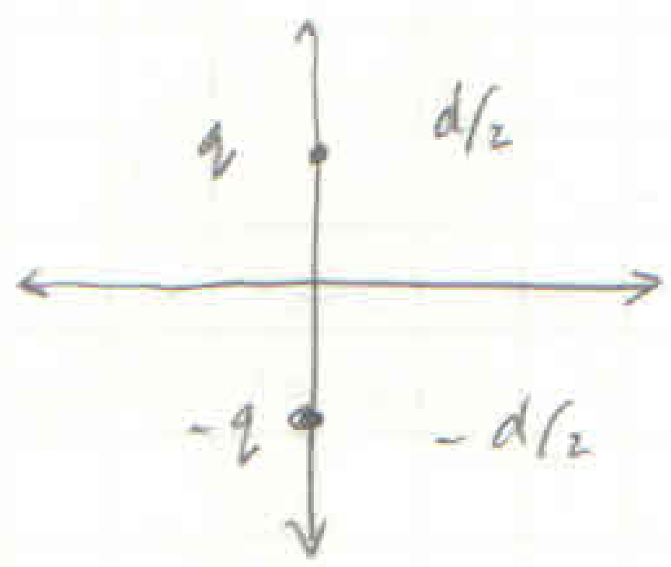
we cut off the rest. What if I have a charge density e then I can rewrite

$$V \approx \int \frac{e\, d^3v}{4\pi\epsilon_0 r}(1+d/r\cos\theta) = \int \frac{e\, d^3v}{4\pi\epsilon_0 r} + \int \frac{e d\cos\theta\, d^3v}{4\pi\epsilon_0 r^2}$$

$$= \frac{Q}{4\pi\epsilon_0 r} + \int \frac{e\, \hat{r}\cdot\vec{d}}{4\pi\epsilon_0 r^2} d^3v = \frac{Q}{4\pi\epsilon_0} + \frac{1}{4\pi\epsilon_0} \int e\, \vec{d}\cdot\vec{r}\, d^3v$$

$\uparrow$ This is the dipole moment.

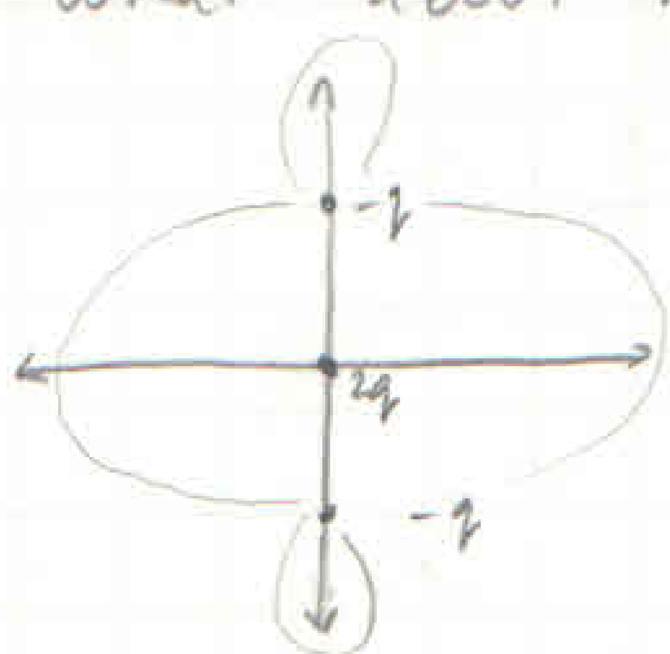
Let's test this and see what it gives!



$$V \approx q/4\pi\epsilon_0 r\left(\cancel{1} + \frac{d}{2r}\cos\theta\right) - q/4\pi\epsilon_0 r\left(\cancel{1} + \frac{-d}{2r}\cos\theta\right)$$
$$= q/4\pi\epsilon_0 r\,(d/r\cos\theta)$$ so where r is very far this looks like a dipole.

very cool.

What about this?



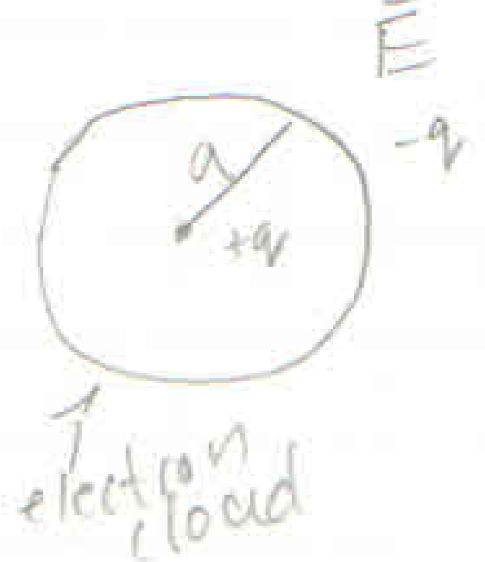
Well $V \approx 0$ since each component cancels in the dipole moment, however looking more at the field lines we see there is clearly a field. The dipole is not high enough order to capture this effect.

Now let's look at the force between particles

E(r) in back

$$F_{+q} = q\, E(\vec{r}+\vec{d}) \approx q\,(\vec{E}(r) + \vec{d}\cdot\underbrace{\nabla\vec{E}}_{\text{tensor}})$$

$$F_{-q} \approx -q\,(\vec{E}(r) +$$



This is an atom in an electric field. We will assume that $\vec{E}$ is constant in the electric field. The electrons will move with the field and the nucleus against.
The force between the nucleus and the electrons is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \overset{q}{ze} \left(\frac{r}{a}\right)^3 \frac{1}{r^2} = \frac{qr}{4\pi\epsilon_0 a^3} \quad \text{so since} \quad V = \tfrac{4}{3}\pi r^3$$

from shell geometry

dipole moment => $\vec{P} = \vec{E} \cdot 4\pi\epsilon_0 a^3 = 3V\epsilon_0$ where V is volume!

Then can we rip an atom apart? $E_{max} = 3\,kV/mm = 3e^6\,V/m$

=> $r = E_{max}\, 4\pi\epsilon_0 a^3/q = 3 \cdot 10^{-16}\,m$ but an atom is $\sim 10^{-11}\,m$
so there is no field strong enough.

What actually happens? Well what is air compared to water? It has $\sim 1/1000$ the density => the mean free path in air is roughly $\sim 1000x$ more. If in water the mean free path is 1 atom diameter then for air it is $1000 \cdot 1\text{atom} \approx 2 \cdot 10^{-7}\,m$