Say have some empty space with a boundary. Inside this space there are 3 conductors, each with some charge. If we have two solutions to this problem V_1 and V_2 for the potential in this field given boundary conditions then we will have two fields $\vec{E}_1, \vec{E}_2$.

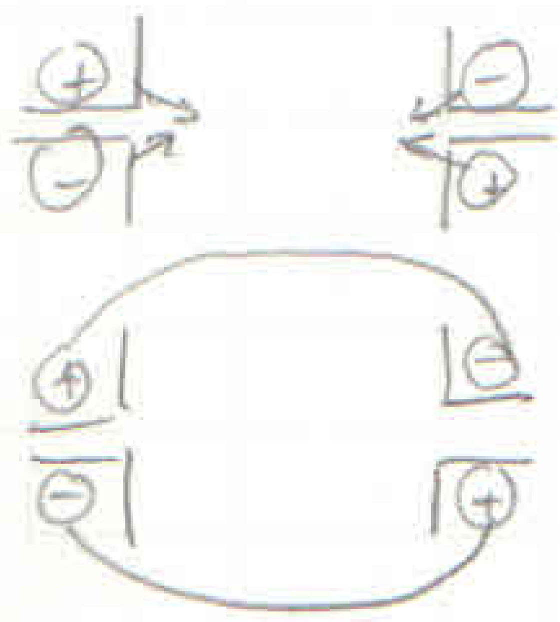
let $V_3 = V_1 - V_2$ and $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$ then on the boundary we know $V_3 = 0$

$$\Rightarrow \int_S \vec{E}_3 \cdot d\vec{a} = 0 \Rightarrow \int_S V_3 E_3 \, da = 0 = -\int_S V_3 \nabla V_3 \, da$$

$$= \int_V -\nabla (V_3 \nabla V_3) = -\int_V \nabla V_3 \nabla V_3 \, dV + \cancel{\int_V V_3 \nabla^2 V_3 \, dV}^{\,0}$$

$$= \int_V E_3^2 \, dV = 0 \Rightarrow E_3 = 0 \text{ so } E_1 = E_2$$ and the solutions are unique

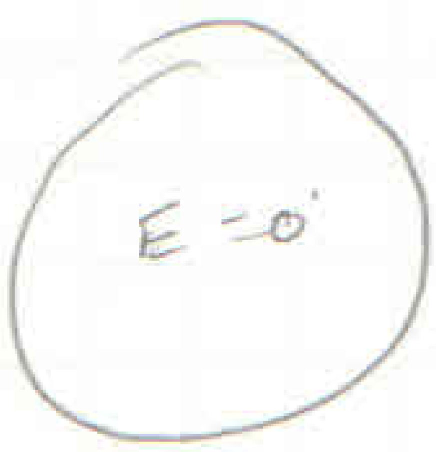
Now say we have 4 charges in boxes



well we know they will want to go towards each othere but the boxes prevent them

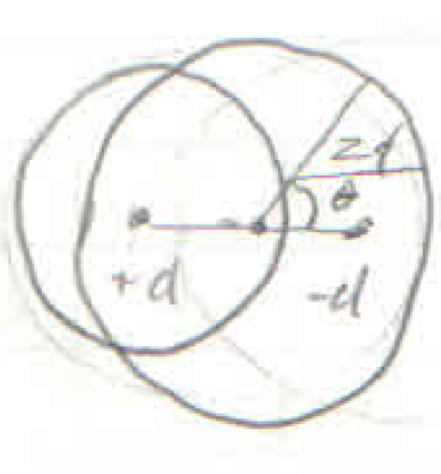
Now what if we connect them with a wire. Because of the wire, the potential should be constant inside the wire. Then this should all be modelled by two conductors of 0 charge! By uniqueness proved above, these are the same solutions so the wire must adjust it's charge distribution to cancel the effects.

Now consider a conducting sphere in a uniform $\vec{E}$



The charge will rearrange itself to cancel the $\vec{E}$ field. If we were to pause time and remove $\vec{E}$ we would observe a uniform $-\vec{E}$ in the conductor.

Recall now the ball problem from the homework where the field inside was constant. If we take the limit as the small ball goes to being the big ball we should get a similar case



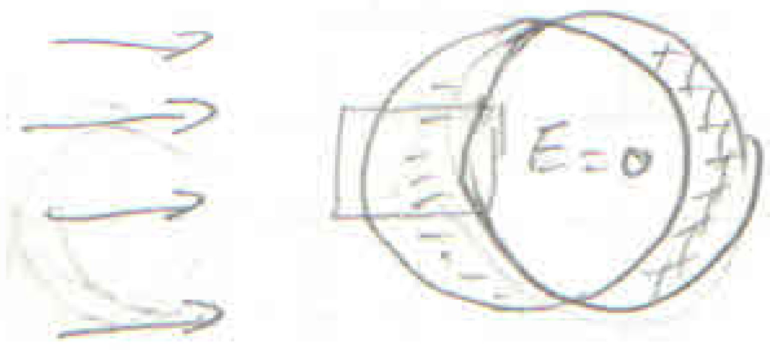
The field here is $\dfrac{(r-d)e}{3\varepsilon_0} - \dfrac{(r+d)e}{3\varepsilon_0} = \dfrac{-2ed}{3\varepsilon_0}$

but e needs to preserve charge so as $d \to 0$, $e \to \infty$ in a way that de must remain constant $\sigma(\theta)$

$$\xrightarrow[e \to \infty]{d \to 0} \frac{\sigma(\theta)}{3\varepsilon_0}$$ but we need to cancel the field so

$\sigma(\theta) = E(\theta)\, 3\varepsilon_0$.

let's continue observing the spheres cutting into each other.



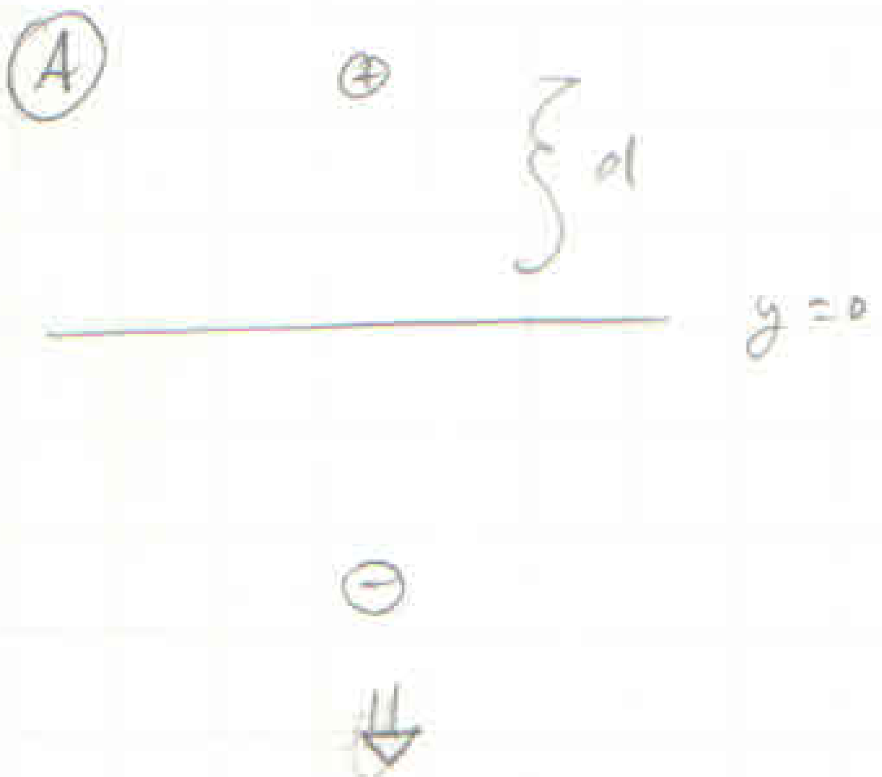
The field outside the sphere will get stronger. We can say $\sigma = \sigma_0 \cos\theta$ since it will be at a maximum from yesterday's analysis. Inside a pill box we can observe this as a pill box with two parallel plates. we get $E = \frac{\sigma_0}{\varepsilon_0}$, $\sigma_0 = E\varepsilon_0$. The field of a sphere is $E = \frac{e(\vec{r}-\vec{d})}{3\varepsilon_0}$ and in the home work we saw the net for us was

$$\vec{E} = \frac{e\vec{r} - e\vec{r} + ed}{3\varepsilon_0} = \frac{ed}{3\varepsilon_0} \Rightarrow \sigma_0 = 3\varepsilon_0 E \quad \text{so} \quad \sigma = 3\varepsilon_0 \vec{E}\cos\theta .$$

Because the field is constant it will not move, the left and right fields will cancel. Furthermore, letting the particles free should do work in the field, moving to a lower energy state, however if the field is at infinity, it was never out of a field so they were already in a low energy state. IRL moving the sphere into the field does the work required to allow the particles to move.

Say we have two particles, what is the potential each point

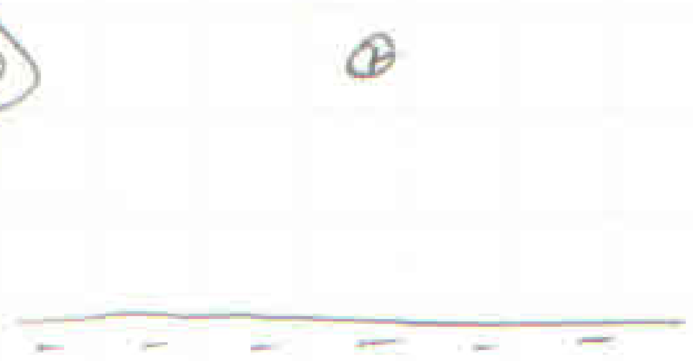
(A)



$$V = \frac{kq}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{kq}{\sqrt{x^2+y^2+(z+d)^2}}$$

Then there is a potential on the middle line, thus if we create a boundary that matches that potential then we have the same solution. So for above the plane

(B)



This makes the same situation by uniqueness. this is the method of images. What is the energy of these situations? In A we know it to be $q^2/4\pi\varepsilon_0 r$ but in B the field outside the boundary is 0, cutting the field in half. We expect $E_B = \frac{1}{2}E_A$. Not everything is the same!

$$V = -\frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-R)^2}} + \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z)^2}}$$

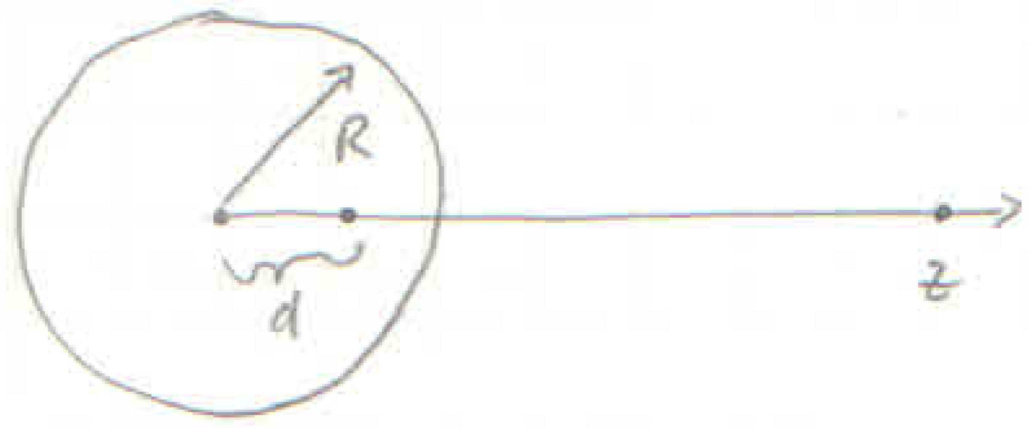
let's look for equipotentials for images let $V=0$ then

$$\frac{Q}{\cancel{4\pi\varepsilon_0}\sqrt{x^2+y^2+(z-R)^2}} = \frac{q}{\cancel{4\pi\varepsilon_0}\sqrt{x^2+y^2+z^2}}$$

$$Q^2/(x^2+y^2+(z-R)^2) = q^2/(x^2+y^2+z^2) \rightarrow Q^2(x^2+y^2+z^2) = q^2(x^2+y^2+(z-R)^2)$$

$$(Q^2-q^2)x^2 + (Q^2-q^2)y^2 + (Q^2-q^2)z^2 + 2q^2Rz - q^2R^2 = 0$$

$$x^2+y^2+z^2 - \frac{2q^2Rz}{q^2-Q^2} = \frac{-R^2q^2}{q^2-Q^2}$$ is a sphere then there is a $V=0$ surface



say this is our solution, we know this is a solution by uniqueness of the boundary cond.
then $\frac{q}{z-R} = \frac{Q}{R-d} = 0$ on the boundary and

we can use other side of the sphere to get $\frac{q}{z+R} = \frac{Q}{R+d} = 0$

$q(R-d) = Q(z-R)$, $q(R+d) = Q(z+R)$ ... etc.