$$A \times (B \times C) = B(A \cdot C) + C(A \cdot B)$$

The Levi-Cevita Tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is even} \\ -1 & \text{if } ijk \text{ is odd} \\ 0 & \text{else} \end{cases} \qquad \text{and } (A \times B)_i = \varepsilon_{ijk} A_j B_k$$

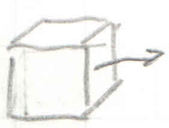
so $(A \times (B \times C))_i = \varepsilon_{ijk} A_j (B \times C)_k = \varepsilon_{ijk} A_j \varepsilon_{klm} B_l C_m$

note $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

so above becomes cool!

$$(A \times (B \times C))_i = A_j B_i C_j - A_j B_j C_i = B(A \cdot C) - C(A \cdot B)$$

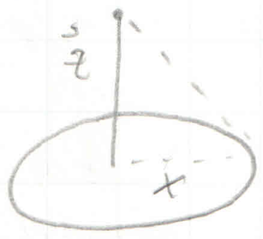
How do we derive divergence? say we have a box of infinitessimal volume. How much is in and out



$$\vec{F} \cdot \vec{n} - \vec{n}\,(\vec{F} + \vec{F}_x\,dx) \rightarrow dz\,dy\,(F(x) + dx\,F(x)_x - F(x))$$
$$= dz\,dy\,dx\,\frac{\partial F}{\partial x} \; ?$$

Insert the lecture history powerpoint here. Insert powerpoint 2

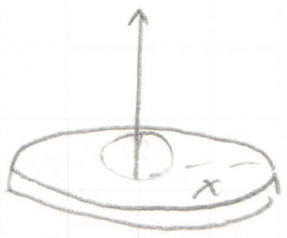
Field from a ring/disc



So the electric field from the disc is

$$\vec{E} = q/4\pi\varepsilon_0 \cdot \frac{\vec{z}}{\sqrt{z^2+x^2}^3}$$, we know the x and y components are 0 by symmetry

Now for a flat disc we have



We can think of a disc as a bunch of rings and add them with an integral

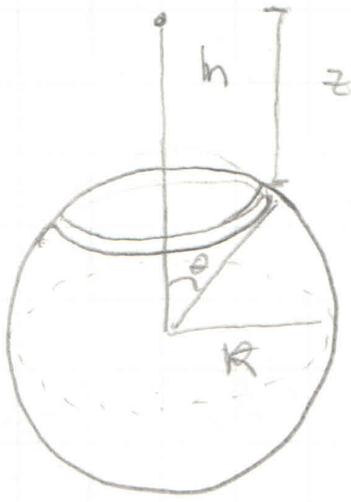
$$\vec{E} = \int_{x=0}^{x=R} \frac{q(x)}{4\pi\varepsilon_0} \frac{\vec{z}}{(z^2+x^2)^{3/2}}$$ where $q(x) = 2\pi x \sigma\,dx$ is the charge per ring for some density σ

$$= \int_{x=0}^{x=R} \frac{2\pi x \sigma\,dx}{4\pi\varepsilon_0} \frac{\vec{z}}{(z^2+x^2)^{3/2}}\,dx$$

$$= \frac{-2\pi\sigma\,\vec{z}}{4\pi\varepsilon_0\sqrt{x^2+z^2}}\Big|_0^R \quad \text{if } R \to \infty \text{ we get}$$

$$= 1/2\,\sigma/\varepsilon_0\,\hat{z}$$

Spherical shell



We start by sketching how to divide up the sphere into rings. The radius of each ring is $x = R\sin\theta$. The vertical height is $z = h - R\cos\theta$ and my charge density $Q(r,\theta) = 4\pi r^2 d\sigma$ for a charge density σ. $Q(r,\theta) = 4\pi r^2 \sin\theta\, d\theta$.

our integral is thus $\vec{E} = \int_0^{\pi} \frac{1}{4\pi\varepsilon_0} \frac{2\pi r^2 \sin\theta\, d\theta\,(h - R\cos\theta)\hat{z}}{((h - R\cos\theta)^2 + R^2\sin^2\theta)^{3/2}}$

$$= \int_0^{\pi} \frac{1}{4\pi\varepsilon_0} \frac{2\pi R^2 \sigma\,(h - R\cos\theta)\sin\theta\, d\theta\,\hat{z}}{(h^2 + R^2 - 2hR\cos\theta)^{3/2}} \qquad \text{let } u = \cos\theta,\ du = -\sin\theta$$

$$= \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0} \frac{2\pi R^2 \sigma\,(h - Ru)\hat{z}}{(h^2 + R^2 - 2hRu)^{3/2}} du$$

which we plug into a compoter and plot as a function of h.

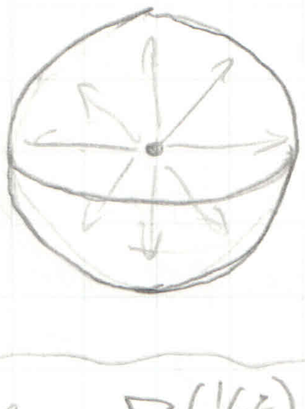
Wow! 0 inside the sphere and then if behaves like a point charge at the center of the sphere

This is a consequence of how the field falls off at r^2. Lets consider a ring and explore the consequences.

These vectors cancel out on opposite parts of the sphere. If we had $F \propto r^{-2+\varepsilon}$ the force wouldn't get weak as fast and the force would be too strong close to the field resulting in a vector pointing in.

Vector Calculus: Gauss's law

Let's think of a particle enclosed in a sphere!



since $\nabla(1/r^2) = 0$ we know that the sphere shape is not important

say the charge is in the center of the sphere, Then we know that it has a field outward $\propto 1/r^2$ then at the surface of the sphere we have flux

$\Phi = \nabla \cdot q/\varepsilon_0 V = q/\varepsilon_0 \Rightarrow \iint \vec{E}\cdot ds = q/\varepsilon_0$. we can apply the divergence theorem to get

$$\iiint \vec{\nabla} E\, dV \text{ and } q/\varepsilon_0 = \iiint \rho/\varepsilon_0\, dV$$

$$\Rightarrow \vec{\nabla} E = \rho/\varepsilon_0.$$ This is highly applicable.