Phys 350 Honours Electricity and Magnetism

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Contents

Lecture 1

Kronecker Delta Function

The Kronecker delta function is a function of two variables $\delta(i,j) \coloneqq \delta_{ij}$

Which is defined as:

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Levi-Civita Tensor

A tensor is a generalization of a matrix for higher orders.

For three dimensions, the tensor is useful for representing the cross product.

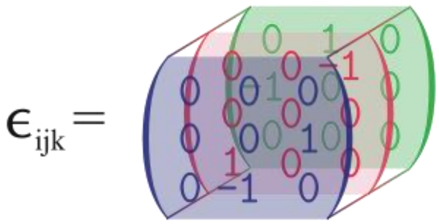

Figure 1: The Levi-Civita of 3 indices Tensor is a 3-dimensional array

Where i, j, k are indexes between 1 and 3.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

This is based on properties of permutations. A **permutation** can represent any a bijective function of a finite set (as well as other cases which don't matter here).

Thus, a permutation of the elements $(1,2,3)$ can be written as the function

$\sigma: \{1,2,3\} \to \{1,2,3\}$

When thinking of a permutation σ of $(1,2,3)$

The permutation can be represented by the ordered list $(\sigma(1), \sigma(2), \sigma(3))$

$(\sigma(1), \sigma(2), \sigma(3))$ will either be a part of the cycle $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$ (forwards) Or the cycle $3 \to 2 \to 1 \to 3 \to 2 \to \cdots$ (backwards). For a forward cycle, the permutation has a **positive sign** ($\text{sign}(\sigma) = +1$). For a backwards cycle, the permutation is a **negative sign** ($\text{sign}(\sigma) = -1$).

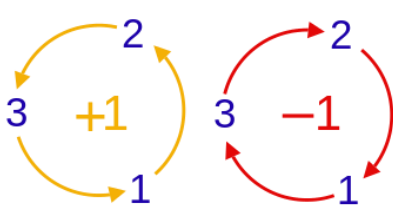

Figure 2 given a permutation of (1,2,3) its order from one of the two possible cycles, which also determines the sign of the permutation.

The definition of the sign of a permutation becomes a bit more complicated for permutations of more than 3 elements, but since we are doing physics here we usually only care about concepts that apply to 3 dimensions.

Let $i, j, k \in \{1,2,3\}$ be distinct.

Let $\sigma: (1,2,3) \mapsto (i, j, k)$ be a permutation

Consider a permutation τ that maps $(1,2,3)$ to (i, j, k) but where only two of i, j, k are swapped

(such as $\tau(1,2,3) = (j, i, k)$ or $\tau(1,2,3) = (i, k, j)$ or $\tau(1,2,3) = (k, i, j)$) then the resulting permutation τ has an opposite sign to $\sigma(1,2,3) = (i, j, k)$.

Now with the concept of sign of a permutation

$\epsilon_{ijk} = \text{sign}\big((i, j, k)\big)$ for distinct indices i, j, k

But what if $\boldsymbol{i, j, k}$ are not distinct? (at least two of them are identical)

If there are repeated indexes (such as ϵ_{iik}), then swapping the repeated index is supposed to change the sign, but in that case, we still have ϵ_{iik}

Thus $\epsilon_{iik} = -\epsilon_{iik} \Rightarrow \epsilon_{iik} = 0$ thus same case applies to any repeated indices

$\Rightarrow$if at least two of i, j, k in ϵ_{ijk} are identical, then $\epsilon_{ijk} = 0$

In summary

$$\epsilon_{ijk} = \{\text{sign}\big((i, j, k)\big)$$

Einstein Summation Notation

When there are repeated indices in **the same term** with **no defined value**, then it so happens that we often need so sum terms over that repeated index as it goes from 1 to 3.

By Einstein summation convention, when given 3-vectors ($\vec{A} = (A_1, A_2, A_3), \vec{B} = (B_1, B_2, B_3)$, etc…)

Then $\epsilon_{ijk}A_k$ is a shorthand to $\sum_{k=1}^{3} \epsilon_{ijk}A_k$ since the k index is repeated in the term

$\epsilon_{ijk}B_jA_k$ is a shorthand to $\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}B_jA_k$ since j and k is repeated

$\epsilon_{ijk} + A_jB_k$ is not a shorthand for anything since there are no repeated indices in the same term

Dot Product

For vectors $\vec{A} = (A_1, A_2, A_3), \vec{B} = (B_1, B_2, B_3)$

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = A_i B_i \quad \text{(Einstein shorthand summation for repeated indices)}$$

Cross Product

For vectors $\vec{A} = (A_1, A_2, A_3), \vec{B} = (B_1, B_2, B_3)$

Let $i \in \{1,2,3\}$

Then the i^{th} component of $\vec{A} \times \vec{B}$ is

$$\left(\vec{A} \times \vec{B}\right)_i = \sum_{k=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk} A_j B_k = \epsilon_{ijk} A_j B_k$$

For example, the first component (x component) of $\vec{A} \times \vec{B}$

$i = 1$

$$\left(\vec{A} \times \vec{B}\right)_1 = \sum_{k=1}^{3} \left(\underbrace{\epsilon_{11k} A_1 B_k}_{0} + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k \right) = \sum_{k=1}^{3} (\epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k)$$

$$= (\underbrace{\epsilon_{121}}_{0} A_2 B_1 + \underbrace{\epsilon_{131}}_{0} A_3 B_1) + (\underbrace{\epsilon_{122}}_{0} A_2 B_2 + \underbrace{\epsilon_{132}}_{-1} A_3 B_2) + (\underbrace{\epsilon_{123}}_{1} A_2 B_3 + \underbrace{\epsilon_{133}}_{0} A_3 B_3)$$

$$= A_2 B_3 - A_3 B_2$$

While the Levi-Civita method is inefficient for manual computation of cross product (many terms become 0), it is useful for proving identities

Vector Product Identities

Levi-Civita and Kronecker Delta Identity

There are a few identities that need to be proven from scratch but are then useful in proving other identities.

We can prove the following identity by looping over all possible indices' values:

Identity: $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Note that $\epsilon_{ijk}\epsilon_{klm} \coloneqq \sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{klm}$ by Einstein summation. Then k is a dummy variable and the expressions on each side are functions of i, j, l and m (but not k).

Proof:

Case $i = l$ and $j = m$:

Case $i = l = j = m$:

$\epsilon_{ijk}\epsilon_{klm} = 0$ since all have repeated indexes

$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 1 \cdot 1 - 1 \cdot 1 = 0$

$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = \delta_{jm} - \delta_{im}\delta_{jl} = 0$

Case $i = l \neq j = m$:

$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 1 \cdot 1 - 0 \cdot 0 = 1$

$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{kij}$

The permutations (i, j, k) and (k, i, j) are 2 transpositions from each other

$(i, j, k) \rightarrow (j, i, k) \rightarrow (k, i, j)$

Thus ϵ_{ijk} and ϵ_{kij} have the same sign which means $\epsilon_{ijk}\epsilon_{kij} \geq 0$.

When summing up $\epsilon_{ijk}\epsilon_{kij}$, only one of the 3 possible k values will be distinct from i, j and thus be nonzero

$\Rightarrow \epsilon_{ijk}\epsilon_{kij} = 1 + 0 + 0 = 1$

$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{kij} = 1 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Case $i = m$ and $j = l$:

Case $i = m = j = l$:

Already did it

$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = \delta_{jm} - \delta_{im}\delta_{jl} = 0$

Case $i = m \neq j = l$:

$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0 \cdot 0 - 1 \cdot 1 = -1$

$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{kji}$

The permutations (i, j, k) and (k, j, i) are a transposition from each other (swapping i and k).

Thus ϵ_{ijk} and ϵ_{kij} have opposite signs which means $\epsilon_{ijk}\epsilon_{kji} \leq 0$.

When summing up, $\epsilon_{ijk}\epsilon_{kji}$ only one term will be nonzero

$\Rightarrow \epsilon_{ijk}\epsilon_{kij} = -1 + 0 + 0 = -1$

$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = -1 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

The unexamined cases left (logic notation ahead):

$\big((i \neq l) \vee (j \neq m)\big) \wedge \big((i \neq m) \vee (j \neq l)\big)$ $\qquad \wedge$:="and" $\vee$:="or"

By distributive property of logic

$$\Big((i \neq l) \wedge \big((i \neq m) \vee (j \neq l)\big)\Big) \vee \Big((j \neq m) \wedge \big((i \neq m) \vee (j \neq l)\big)\Big)$$

$$= \Big(\big((i \neq l) \wedge (i \neq m)\big) \vee \big((i \neq l) \wedge (j \neq l)\big)\Big) \vee \Big(\big((j \neq m) \wedge (i \neq m)\big) \vee \big((j \neq m) \wedge (j \neq l)\big)\Big)$$

$$= \underbrace{\big((i \neq l) \land (i \neq m)\big)}_{\text{case 1}} \text{ or } \underbrace{\big((i \neq l) \land (j \neq l)\big)}_{\text{case 2}} \text{ or } \underbrace{\big((j \neq m) \land (i \neq m)\big)}_{\text{case 3}} \text{ or } \underbrace{\big((j \neq m) \land (j \neq l)\big)}_{\text{case 4}}$$

For each of the 4 cases, the inequalities make both terms in $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ contain a 0 factor,

Thus $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0 - 0 = 0$ in all those remaining cases.

Case 1 $(i \neq l)$ and $(i \neq m)$:

Case $l = m$: then $\epsilon_{klm} = 0$

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case $l \neq m$:

Then $\epsilon_{klm} \neq 0$ only when $k = i$ but in this case $\epsilon_{ijk} = 0$,

Which means $\epsilon_{ijk}\epsilon_{klm} = 0$ since it’s a summation of zero terms

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case 2 $(i \neq l)$ and $(j \neq l)$:

Case $i = j$: then $\epsilon_{ikj} = 0$

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case $i \neq j$:

Then $\epsilon_{ijk} \neq 0$ only when $k = l$ but in this case $\epsilon_{klm} = 0$,

Which means $\epsilon_{ijk}\epsilon_{klm} = 0$ since it’s a summation of zero terms

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case 3 $(j \neq m)$ and $(i \neq m)$:

Case $i = j$: then $\epsilon_{ikj} = 0$

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case $i \neq j$:

Then $\epsilon_{ijk} \neq 0$ only when $k = m$ but in this case $\epsilon_{klm} = 0$,

Which means $\epsilon_{ijk}\epsilon_{klm} = 0$ since it’s a summation of zero terms

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case 4 $(j \neq m)$ and $(j \neq l)$:

Case $m = l$: then $\epsilon_{klm} = 0$

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Case $m \neq l$:

Then $\epsilon_{klm} \neq 0$ only when $k = j$ but in this case $\epsilon_{ijk} = 0$,

Which means $\epsilon_{ijk}\epsilon_{klm} = 0$ since it's a summation of zero terms

$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = 0 = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

So in those 4 cases

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 0$$

In all possible cases, $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ ■

BAC CAB Rule

Let $\vec{A}, \vec{B}$ and $\vec{C}$ be vectors in 3 dimensional space.

Then

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Proof:

for any $i \in \{1,2,3\}$

$$\left(\vec{A} \times (\vec{B} \times \vec{C})\right)_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k = \epsilon_{ijk} A_j (\epsilon_{klm} B_l C_m) = \epsilon_{ijk}\epsilon_{klm}(A_j B_l C_m)$$

Note that the second ϵ has different indices since it is in a different and unrelated cross product

Using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$\Rightarrow \left(\vec{A} \times (\vec{B} \times \vec{C})\right)_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_j B_l C_m)$$

$$\left(\vec{A} \times (\vec{B} \times \vec{C})\right) = \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m$$

Which sums over i, j, l, m

The only nonzero terms are when $i = l$ and $j = m$ for the first term or $i = m$ and $j = l$ for the second term

$$\Rightarrow \left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right) = A_j B_i C_j - A_l B_l C_i = B_i\left(A_j C_j\right) - C_i(A_l B_l) = \left(\vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)\right)_i$$