Magnets

People in ancient Greece discovered these weird rocks that always point the same direction. They didn't really know how it worked but made it into compassed. People discovered that potting currents near the compasses made the magic move. How does this work? what is the magic?

Well we get magnetism, a force that appears as a relativistic correction to the electric force. Basically it happens if Gauss law holds for relativity. Then Let's denote the magnetic field as $\vec{B}$.

Experimentally we find $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$. Let's think of a particle in a constant $\vec{B}$.

This make a circle since $\vec{v} \perp \vec{B}$ so $\vec{B}$ here does no work

so $q\vec{v} \times \vec{B} = mv^2/R \Rightarrow R = \frac{mv}{qB}$

So then if I want a particle to go in a circle we only need to build a particle collider we only need to build the magnetic field in the ring shape.

@ The LHC the protons have 7 TeV and relativistically

$p = \gamma m v \overset{v \to c}{\approx} \gamma m c$ and $E = \gamma m c^2$ so $p \approx \frac{E}{c}$

thus above gives $qBR = p \Rightarrow qBRc = E$

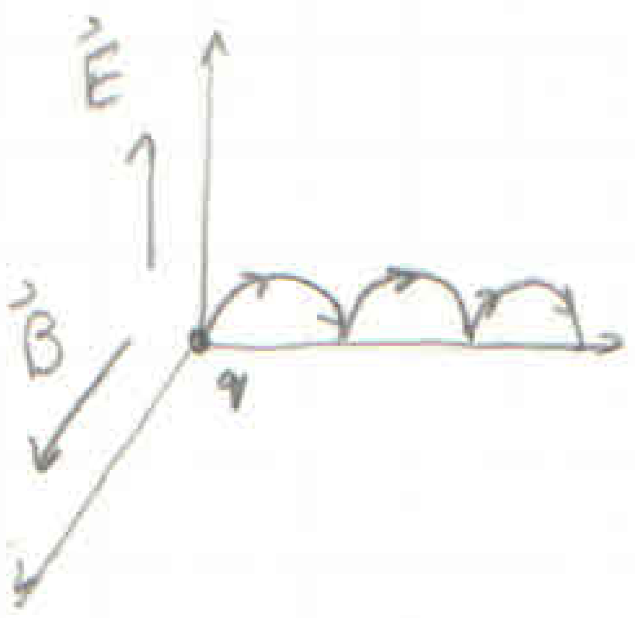
thus $B = \frac{7\,TeV}{qcR}$ and we note that accelerating charges enough makes the particles decay the energy. So let's see how much the LHC can generate in terms of magnetic field.

so $B = \frac{7 \cdot 10^{12}\, eV}{e \cdot 3 \cdot 10^8 \cdot 13000m} \approx 2$ Teslas which does not mean much but is huge.

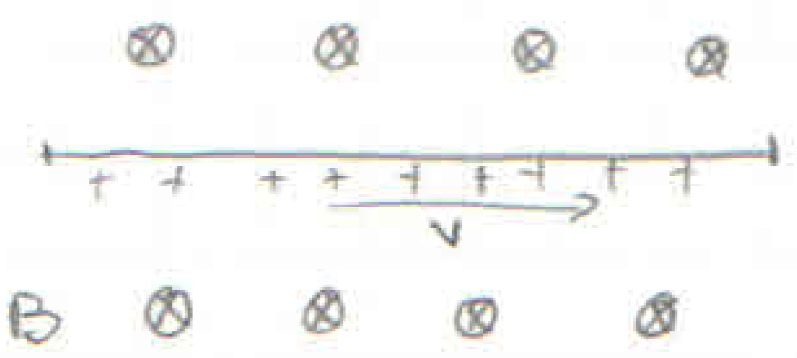
How long does it take to go around?

well $v = \omega R$ so $\omega = \frac{qB}{m}$ and so $\nu = \frac{1}{2\pi}\frac{qB}{m}$

This is a weird law, consider this particle at rest



Since $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$, the particle will start moving in small arcs because the the cross product with the magnetic field. Since the B field never does work when the particle returns to the axis it must stop. This creates cycloids! Note that one can get a similar effect with gravity.

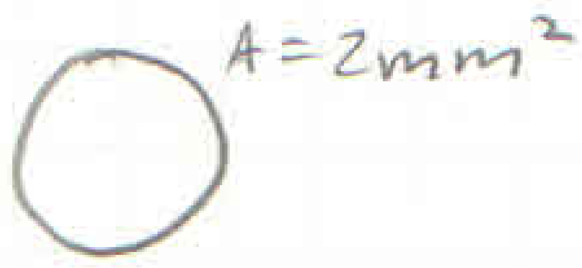


Let's thing of a current in a wire through a constant magnetic field. For each moving charge $|F| = qvB$ so total $|F| = nvBq$ where n is the charges per unit length. Denote $I = qvn$ then $|F| = IB$ and for the whole wire $|F| = \ell IB \Rightarrow \vec{F} = I\vec{\ell} \times \vec{B}$ and so for a loop

$$\vec{F} = \oint I d\vec{\ell} \times \vec{B} = I \oint d\vec{\ell} \times \vec{B}$$

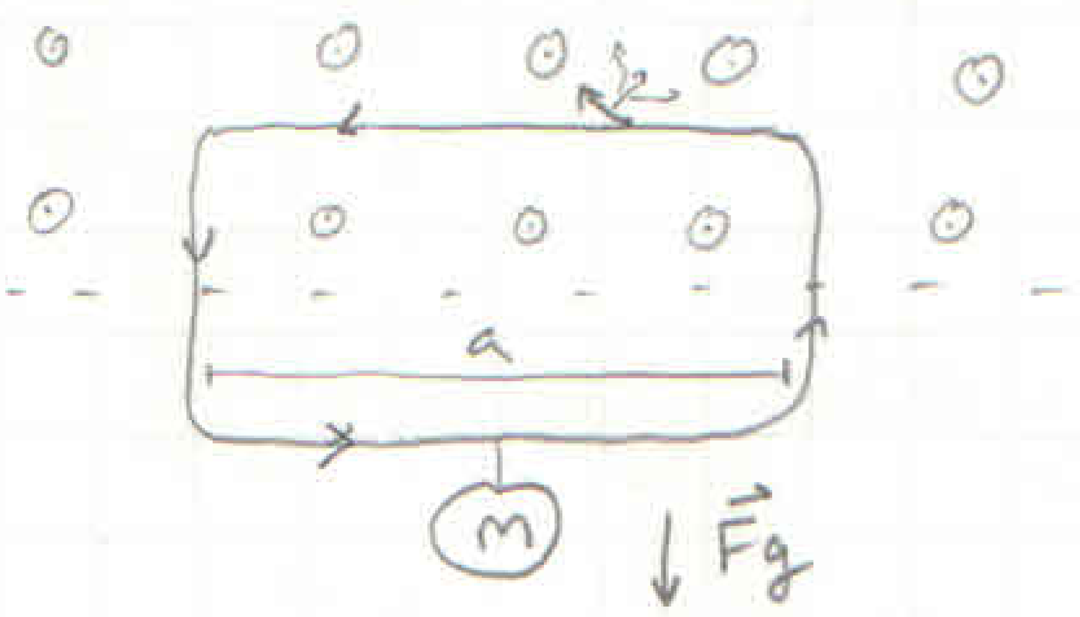
so if we pass 10 amps through a long extension chord what is v?

$I = qvn \Leftrightarrow 10\,A = evn$ well for n we say in copper we have one free flowing electron per atom. Copper has $\sim 10^{-25}$ kg/atom



and with a density of $\sim 9\,g/cm^3$ or $9\,ton/m^3$ $\Rightarrow \sim 10^{29}$ atom/kg so using $\rho = 9\,g/cm^3$ and the area of the wire we get $\sim 2 \cdot 10^{23}$ electrons/surface thus $10\,A / (1.6 \cdot 10^{-19} \cdot 2 \cdot 10^{23}) \sim 3 \cdot 10^{-4}\,m/s = 0.3\,mm/s$.

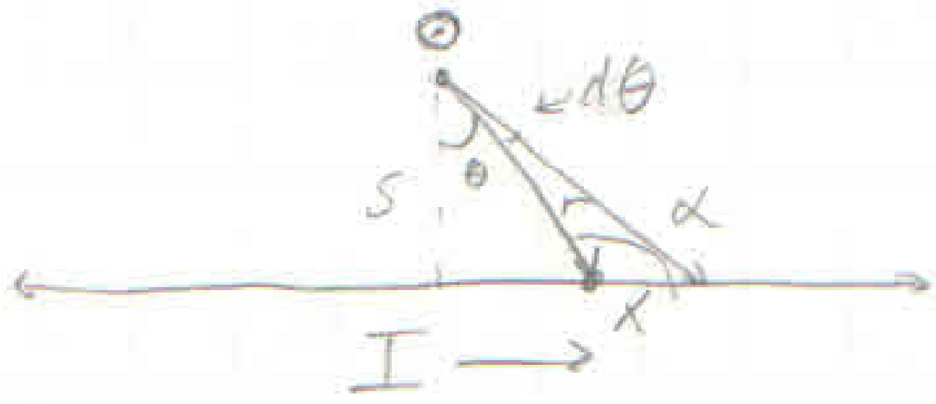
This is tiny movement, in AC we can see how far an electron moves by dividing be 2x frequency. Thus $\Delta x = 0.3\,mm/s / (120\,Hz) \Rightarrow \Delta x = 0.0025\,mm$. Nice.



Let's consider this loop of wire with a magnetic field going through it's top half with a mass along the bottom. For the field to hold up the mass, the current must go around counter clockwise. The magnitude of the force is $F = IBa$ and so $I = mg/Ba$. However, we are opposing gravity which requires work. Magnetic fields don't do work classically so where is this coming from? Well when the electrons are accelerated up, the magnetic field pushes back on the electrons making the electric field in the conductor do the work (Battery does work). Note the sides cancel out.

Let's take a look at the Biot-Savart laws

$\vec{B} = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \hat{I} \times \hat{r}/r^2$, on a wire $\vec{B} = \frac{\mu_0}{4\pi} I \int d\vec{\ell} \times \hat{r}/r^2$. Let's think of an infinite straight wire.



$\hat{r} \times d\vec{\ell} = d\ell \sin(\alpha) = d\ell \sin(\pi - \theta) = d\ell \cos(\theta)$

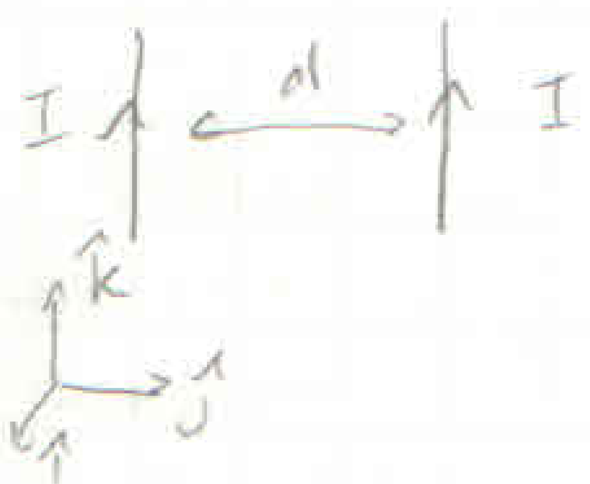
and $\cos\theta = s/r \Rightarrow 1/r^2 = \frac{\cos^2\theta}{s^2}$

note $\tan\theta = x/s \Rightarrow x = \tan\theta\, s \Rightarrow dx = \frac{s}{\cos^2\theta} d\theta$

so $\frac{\mu_0}{4\pi} I \int 1/r^2\, d\bar{x} \cdot \hat{r} = \frac{\mu_0}{4\pi} I \int \frac{\cos^2\theta}{s^2} \cdot \frac{s \cos\theta\, d\theta}{\cos^2\theta}$

Now we reach the question of "what are our bounds?" Well we have that any thing less than $-\pi/2, \pi/2$ would lead to a finite wire, this would require creation of charge and deletion of charge. Thus we get

$$\frac{\mu_0 I}{4\pi s}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{\mu_0 I}{2\pi s}$$ which is like the field of a line of charge but now with a different direction Now out of the paper.

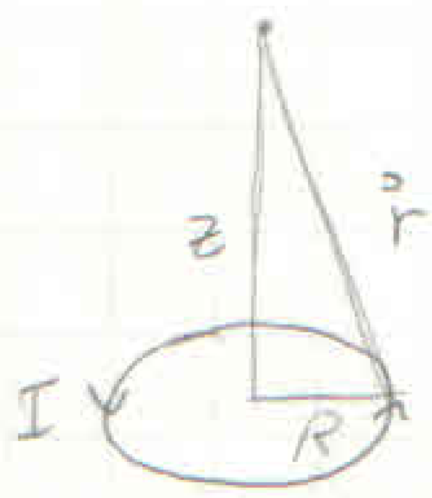


What then happens with two parallel wires

$$B = \frac{\mu_0 I}{2\pi d} \Rightarrow F/\ell = I\times B = \frac{\mu_0 I^2}{2\pi d}$$

Well the definition of an amp is that 2 1meter wires seperated by 1 m at 1 amp generates $2\cdot 10^{-7}$ N

$$\Rightarrow \frac{\mu_0 \cdot 1A^2}{2\pi\cdot 1m} = 2\cdot 10^{-7}N \Rightarrow \mu_0 = 4\pi\cdot 10^{-7}\frac{Nm}{A^2}$$



What is B at this point? well since we are rotationally symmetric so no radial forces.
For a small chunk of line

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\begin{vmatrix}\hat{x} & \hat{\theta} & \hat{z}\\ 0 & Rd\theta & 0\\ \frac{x}{r^3} & 0 & -\frac{z}{r^3}\end{vmatrix}, B = \frac{\mu_0 I}{4\pi}\int \overbrace{\cancel{-\frac{z}{r^3}Rd\theta\hat{x}}}^{0} - \frac{Rd\theta\times\hat{e}}{r^3}$$

$$\text{so}\quad \vec{B} = 2\pi R^2/4\pi(z^2+R^2)^{3/2} = \frac{\mu_0 R^2 I}{2(z^2+R^2)^{3/2}}\hat{z}\quad \text{let } x = R$$

Notice that we get something that scales with $1/r^3$! A dipole!

Let's try get the divergence and curl of B? Let's return to the line wire

what is $\oint_\ell \vec{B}\cdot d\vec{\ell}$? well along ℓ we have $\vec{B} \parallel \hat{\ell}$

By the RHR so $\oint_\ell \vec{B}\cdot d\vec{\ell} = 2\pi r B = \mu_0 I$ hmmm.

Well what about $\oint_{\ell'} \vec{B}\cdot d\vec{\ell} = 0$ outside of the wire since the sides are 0 and the in and outside cancel since $\oint_\ell \vec{B}\cdot d\vec{\ell}$ is radius independent.

This suggests that $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$ and using Stoke's

$\iint \vec{\nabla}\times\vec{B}\, da = \mu_0 I$, for a small enough region this suggests

$\vec{\nabla}\times\vec{B} = \mu_0 I/A$ Let's try make this more rigorous.

What is $\nabla\cdot(\vec{A}\times\vec{B}) = \partial_i \varepsilon_{ijk} A_j B_k = \varepsilon_{ijk}\partial_i A_j B_k$
$= \varepsilon_{ijk} B_k \partial_i A_j + \varepsilon_{ijk} A_j \partial_i B_k$
$= B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

now $$B(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{\|\vec{r}-r'\|^3}\,d^3r'$$

thus $$\nabla\cdot B(\vec{r}) = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(J(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{\|r-r'\|^3}\right)d^3r'$$ we move divergence in since it's w.r.t r not r'

$$= \frac{\mu_0}{4\pi}\int\left[J\cdot\underbrace{\cancel{\nabla\times\frac{(\vec{r}-\vec{r}')}{\|r-r'\|^3}}}_{1/r^2!}^{\to 0} - \frac{(\vec{r}-\vec{r}')}{\|r-r'\|^3}\cdot\underbrace{\cancel{\nabla\times\vec{J}}}_{\partial r'/\partial(r)=0}^{\to 0}\right]d^3r'$$
$$= \int 0\, d^3r = 0$$ nice.

similarly $$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int\nabla\times\left[\vec{J}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{\|r-r'\|^3}\right]d^3r$$

now note $\vec{\nabla}\times(\vec{A}\times\vec{B}) = \varepsilon_{ijk}\partial_j \varepsilon_{kem} A_e B_m$
$* = \varepsilon_{ijk}\varepsilon_{kem}\left[A_e\partial_j B_m + B_m\partial_j A_e\right]$

using $\varepsilon_{ijk}\varepsilon_{kem} = \delta_{ie}\delta_{jm} - \delta_{im}\delta_{je}$

$* = A_i\partial_j B_j - A_j\partial_j B_i - B_i\partial_j A_j + B_j\partial_j A_i$
$= (\nabla\cdot B)\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - (\nabla\cdot\vec{A})\vec{B} + (\vec{B}\cdot\vec{\nabla})\cdot A$

so above becomes

$$\nabla\times B = \frac{\mu_0}{4\pi}\int J\overbrace{\left(\nabla\cdot\frac{(\vec{r}-\vec{r}')}{\|r-r'\|^3}\right)}^{4\pi\delta(r-r')} - (\vec{J}\cdot\vec{\nabla})\frac{(\vec{r}-\vec{r}')}{\|\vec{r}-\vec{r}'\|^3}\,d^3r$$

$$= \mu_0\vec{J} - \frac{\mu_0}{4\pi}\int(\vec{J}\cdot\underset{\text{Tensor}}{\vec{\nabla}})\frac{(\vec{r}-\vec{r}')}{\|\vec{r}-\vec{r}'\|^3}\,d^3r$$ Let's see the x-component

$J\cdot\nabla\frac{x-x'}{\|r-r'\|^3}$ but note $\nabla(J\cdot \underset{\text{scalar field}}{a}) = \vec{J}\vec{\nabla}a + a\cancel{\vec{\nabla}\cdot J}^{\,0}$
now switch to x'

$= \nabla\cdot\frac{\vec{J}(x-x')}{\|r-r'\|^3} = -\nabla' J\frac{(x-x')}{\|r-r'\|^3} \Rightarrow \iiint\nabla\cdot J\frac{(x-x')}{\|r-r'\|}d^3r = \cancel{\iint\frac{J(x-x')}{\|r-r'\|^3}d^2a}^{\to 0}$
compact J.

Thus $\nabla\times B = \mu_0\vec{J}$. This is called Ampere's law.
Note that
$$\underbrace{\nabla\cdot(\vec{\nabla}\times\vec{B})}_{0!} = \mu_0\nabla\cdot\vec{J} \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$$

So built into this law is that the divergence of J is 0. But this is not always the case. But notice that if the charge density changes, the amount of flow in and out should sum to this change $\Rightarrow \frac{\partial\rho}{\partial t} = \nabla\cdot\vec{J}$ so then
$$\frac{\partial\rho}{\partial t} = \varepsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot E = \varepsilon_0\cdot\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\cdot\left(J+\varepsilon_0\frac{\partial E}{\partial t}\right) = 0$$

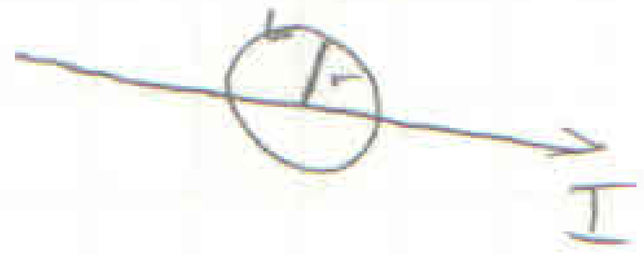
But this doesn't make sense! We may not get a curl being two values so we will make a guess that instead

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t} \right)$$ which turns out to be correct. There is an electric correction too!

Then the integral version is $\displaystyle\int_{\partial S} \vec{B} \cdot d\vec{l} = \iint_S \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \mu_0 \iint_S \vec{J} \cdot d\vec{a}$

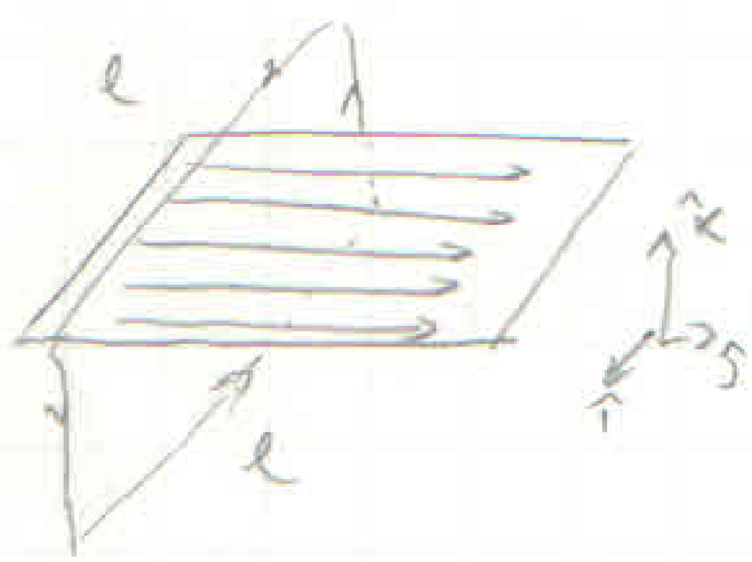
$\Rightarrow \displaystyle\int_{\partial S} \vec{B} \cdot dl = I_{enc} \cdot \mu_0$ This is only for magnetostatics.

We draw a circle around and get



$$\int_{\partial S} \vec{B} \cdot d\vec{l} = B\, 2\pi r = \mu_0 I \Rightarrow \vec{B} = \mu_0 I / 2\pi r\, \hat{\theta}$$

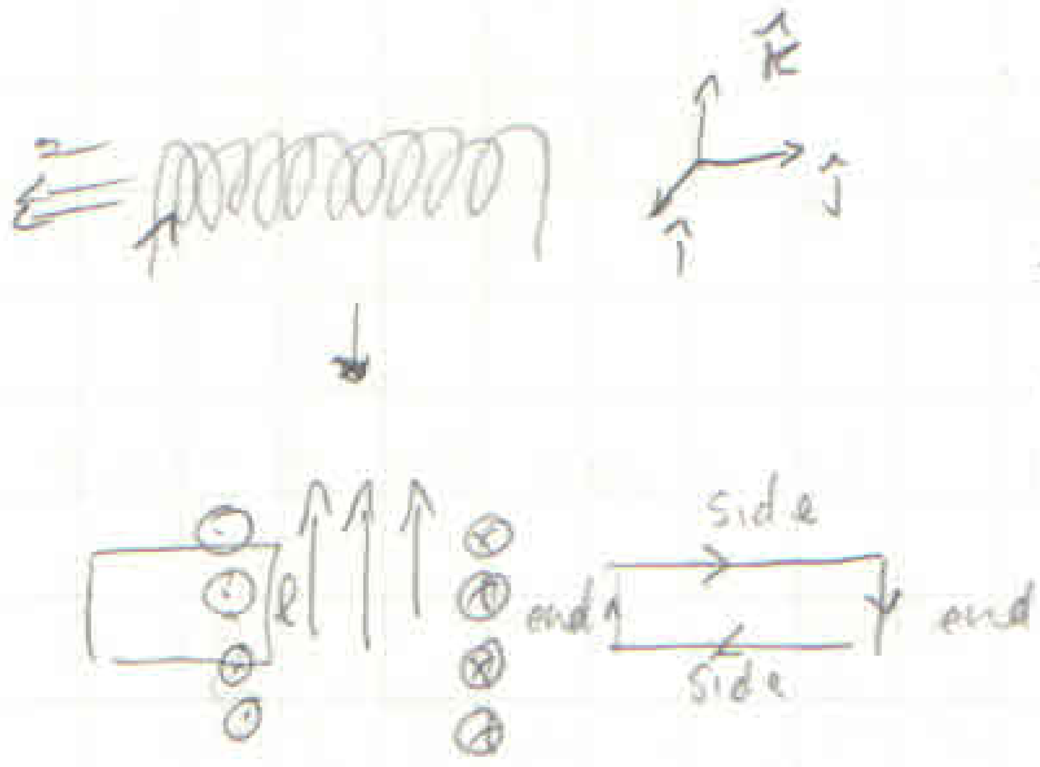
Nice and easy! What about an infinite sheet of charge with a current density



We can tell already that that up and down components must be 0 since if we rotate the plane the current is being multiplied by -1, but the radial may not change sides with rotation, $B\hat{k} = -B\hat{k}$ $\Rightarrow B\hat{k} = 0$

Thus $\displaystyle\int B\, dl = 2lB = \mu_0 \underbrace{K}_{\text{current density (linear)}} l \Rightarrow \vec{B} = \frac{\mu_0 K}{2} \cdot \begin{cases} -\hat{i} & \text{below} \\ \hat{i} & \text{above} \end{cases}$

Let's take a look at a solenoid



At infinity, the field should drop to 0 since this should scale with $1/R^3$. So the drawing a box outside and go around it then the sides are 0 and the ends are the same value since $I_{enc} = 0$. On the inside there can't be a radial component by rotational symmetry nor may there be a tangential component since drawing a ring inside gives $B_{\hat{\theta}}\, 2\pi r = \mu_0 \cdot 0 \Rightarrow B_{\hat{\theta}} = 0$

So this only leaves the direction of the solenoid. Drawing a loop half in half out gives $B \cdot l = \mu_0 I \underbrace{n}_{\text{coil density}} \cdot l$

$\Rightarrow B = \mu_0 I n$

Since the divergence of B is 0 we may write $\vec{B}$ as the curl of another field A

Thus if $\nabla \times A = B$ and since the curl of a divergence is zero we may say that $A + \nabla V$ is valid for any potential. Thus we may make the choice that $\nabla \cdot A = 0$ by tuning the potential field. Ex: Let $\nabla \cdot A = \Phi$ then let V be s.t. $\Delta V = -\Phi$ thus $\nabla \cdot (A - \nabla V) = \nabla \cdot A + \Delta V = 0$ works!

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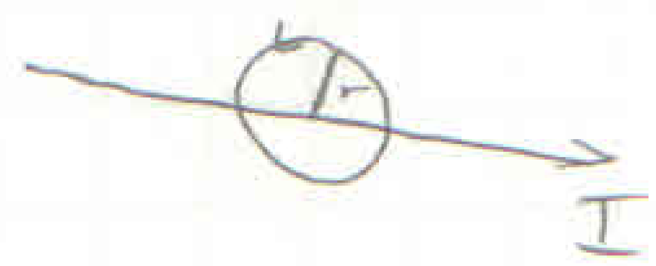

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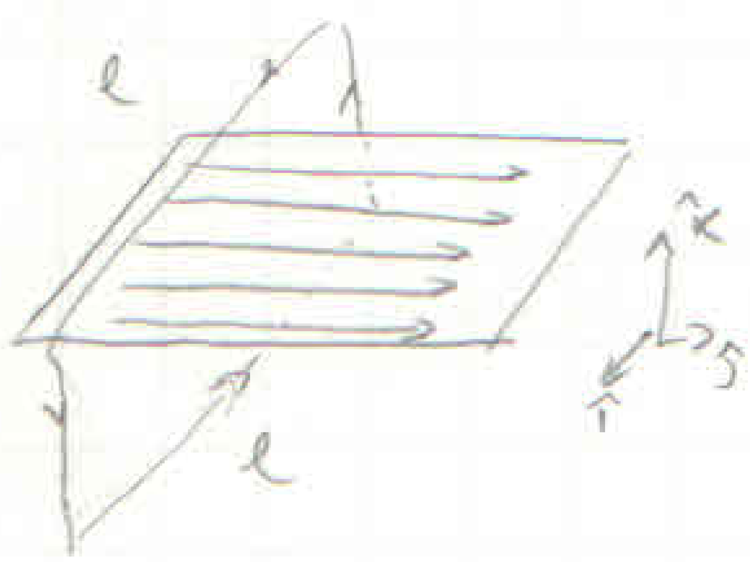
Nice and easy! What about an infinite sheet of charge with a current density



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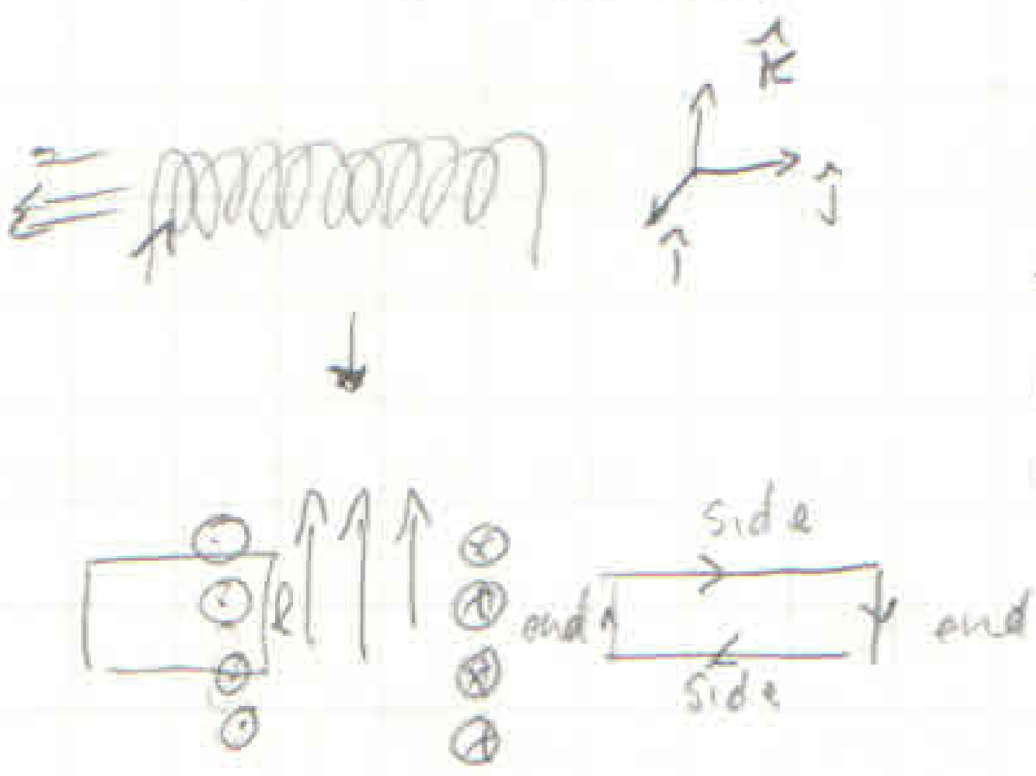

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thus $\nabla\cdot(A - \nabla V) = \nabla\cdot A + \Delta V = 0$ works!

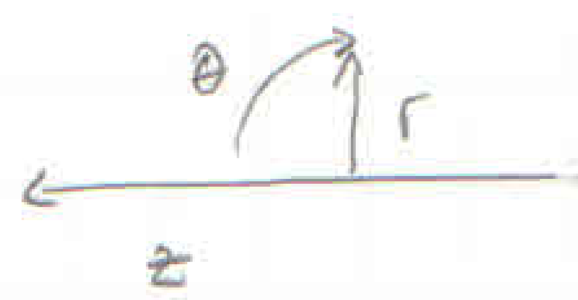
so then $\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$ using Levi Cevita we get

$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\Delta}\cdot A = \mu_0 J$ by our choice $\vec{\nabla}\cdot\vec{A} = 0$ so

$-\vec{\Delta}\cdot A = \mu_0 J \Rightarrow \begin{cases} -\Delta A_x = \mu_0 J_x \\ -\Delta A_y = \mu_0 J_y \\ -\Delta A_z = \mu_0 J_z \end{cases}$ 3 Laplace's eq.

Thus we get 3 coupled PDEs for a J s.t. $\nabla \cdot J = 0$ otherwise we would need $\frac{\partial E}{\partial t}$.

For a line of current in the z direction we can reduce the above equations to $\Delta A_z = -\mu_0 I$ which is similar to $\Delta V = -e/\varepsilon_0$ thus we know the solution is



$$A_z = -\frac{\mu_0 I}{2\pi}\ln(r) \quad \text{and} \quad A = -\frac{\mu_0 I}{2\pi}\ln(r)\,\hat{z}$$

and $\vec{\nabla}\times\vec{A} = \frac{-\partial A_z}{\partial r}\hat{\theta} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$ which is correct. Hurray!

Now Let's return to the solenoid.

So inside the solenoid we get

$$A\cdot 2\pi r = \int \vec{A}\cdot d\vec{l} = \iint (\nabla\times A)\cdot dA = \iint \underbrace{\vec{B}}_{B_0 \text{ inside}}\cdot d\vec{A} = B_0 \cdot a_{solenoid}$$

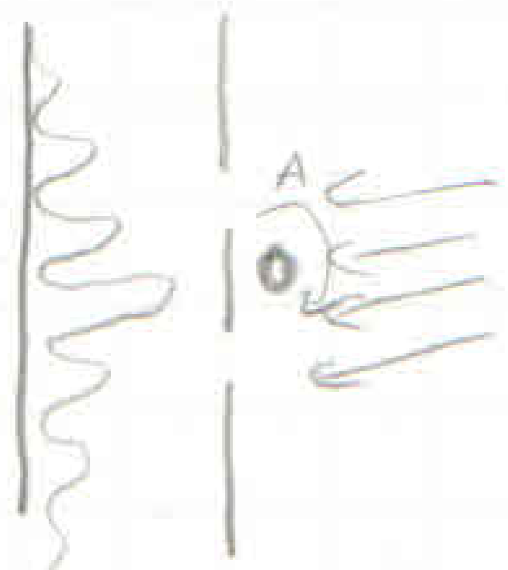
$\Rightarrow A = \frac{B_0 \cdot a_{solenoid}}{2\pi r}$ notice there is a strange drop off to the vector potential even when $B = 0$.

Let's also note that $[A] = kg\,m/s/q = [p]/[q]$

$\Rightarrow$ qA looks like momentum so let's use some QM.

$p = \hbar k$ then $k = p/\hbar = \frac{qA}{\hbar}$ so $\Delta\phi$ is a change in phase then

$\Delta\phi = \frac{q}{\hbar}\int \vec{A}\cdot d\vec{l}$ so let's think about the double slit experiment



here we have an interference pattern from the double slit, but placing a solenoid between the slits has no field but does have a vector potential. It is experimentally verified that despite having no field to act with the interference pattern is phase shifted. cool.