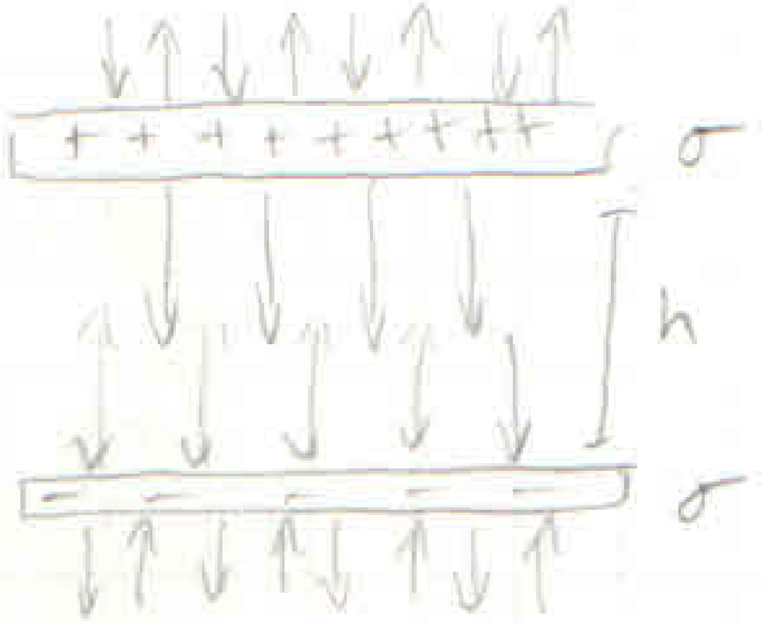
What about a flat plane?

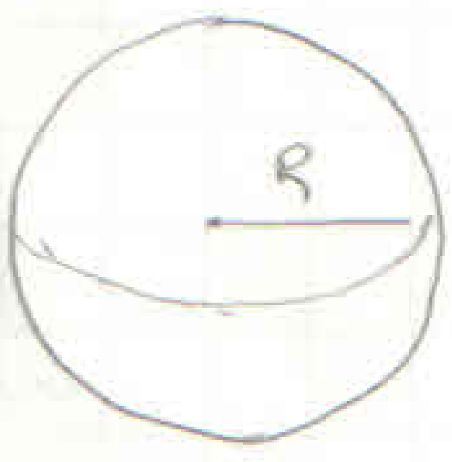
$2A\vec{E} = \frac{\sigma A}{\varepsilon_0}\hat{z}$ by Gauss's law so $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$

so $V = \frac{\sigma z}{2\varepsilon_0}$ now let's try two plates



The electric field is $E = \sigma/\varepsilon_0$ and 0 outside.
So $V = \sigma/\varepsilon_0 \cdot h$.

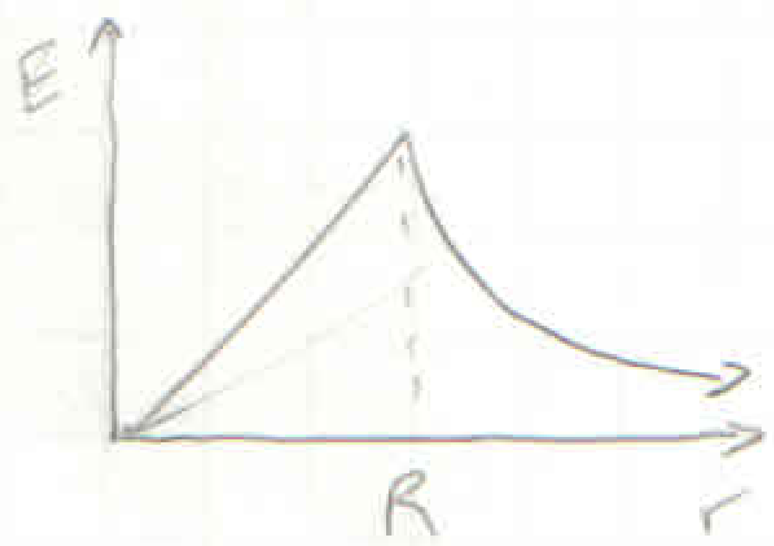
What about a spherical uniform ball of charge the



charge density ρ then $4\pi r^2 \vec{E} = \frac{4/3\pi R^3 \rho}{\varepsilon_0}$

if $r > R$ we have $\vec{E}(r) = \frac{Q_{in}}{4\pi r^2 \varepsilon_0}\hat{r}$ since Q_{in} goes only to R

if $r \leq R$ we have $\vec{E}(r) = \frac{r\rho}{3\varepsilon_0}$



So it is linear until $r = R$ and then drops off like a point charge at $r=0$ with value at Q_{in}

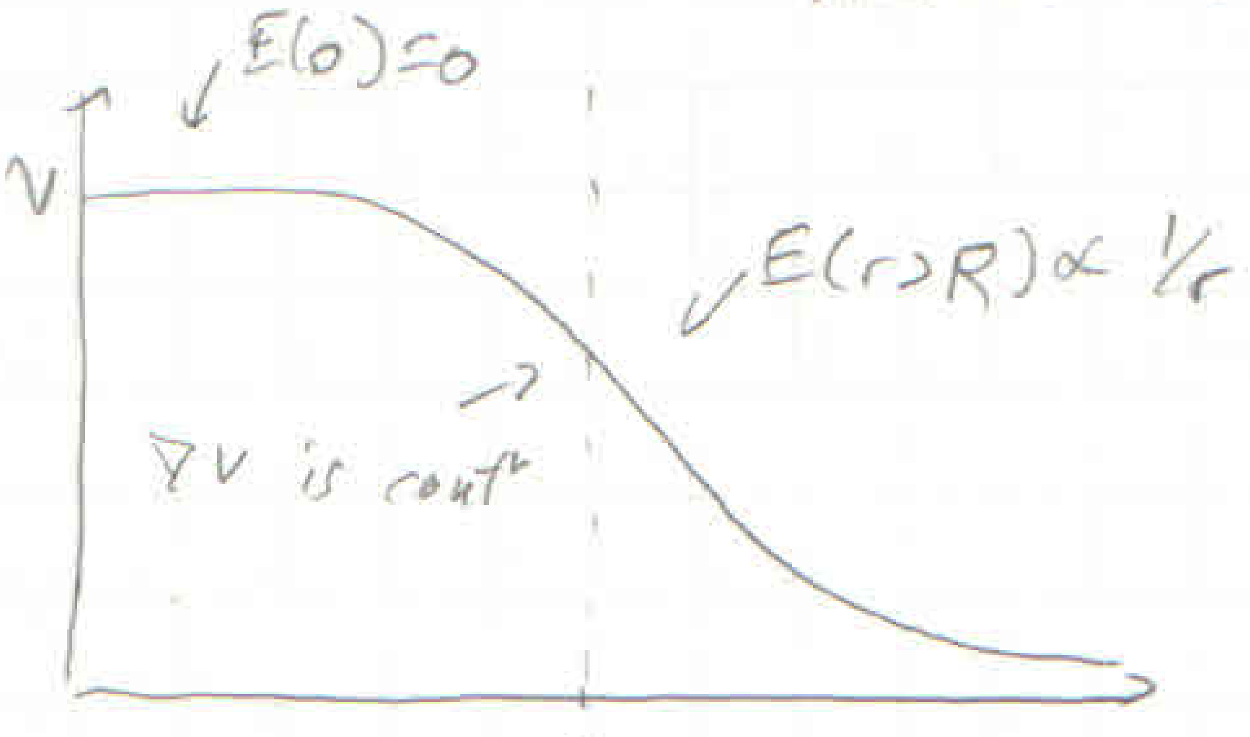
The work needed to move two charges together is $W = qV = q_1 q_2/4\pi\varepsilon_0 r$

For a group of charges, the work done to move many particles together is $\frac{1}{2}\sum_{i \neq j} q_i q_j/4\pi\varepsilon_0 r_{ij}$. 1/2 since we don't want to double count (i,j) and (j,i)

Imagine this hollow sphere the potential inside it is $q/4\pi\varepsilon_0|r|$ for $r > R$ and 0 otherwise. To build the sphere, we imagine moving a charge in to the sphere to get $dW = \int_0^q \frac{q\,dq}{4\pi\varepsilon_0}$ and get $W = \frac{q^2}{8\pi\varepsilon_0}$ to build the ring.

What about a solid sphere? This sphere has finite charge density ρ.

How do we know this field is continuous? One can imagine shrinking a pill box around the surface. The flux of the box is $E_{top} \cdot A_{top} - (E_{bot} - A_{bot}) = A \cdot \Delta E$ and by Gauss's law $A \cdot \Delta E = q/\varepsilon_0$ and since ρ is finite $q_{inside} \to 0$ as we shrink the box $\Rightarrow \Delta E \to 0$.



Since E is continuous, we know there are no kinks in $V(r)$. We also know $E(0) = 0$ so the potential is flat at $r=0$ and $E(r>R) \propto 1/r^2 \Rightarrow V(r>R) \propto 1/r$.

So using Gauss's law $E_{sphere}(r) = r\rho/3\varepsilon_0$ ($r<R$), and such the potential is

$V = -\int E(r)\,dr = \frac{-r^2\rho}{6\varepsilon_0}$ for $r \le R$ and $V(r) = kq/r$ but this has a problem since V must be continuous. $V(r\le R)$ has a reference at $r=0$ and $V(r>R)$ is referenced at $r\to\infty$. We can fix this by adding a constant and shifting the reference point. A bit of math gives:

$$V = \frac{Q}{4\pi\varepsilon_0}\begin{pmatrix} 1/r & r>R \\ \frac{r^2}{2R^3}+\frac{1}{2R} & r\le R \end{pmatrix}$$

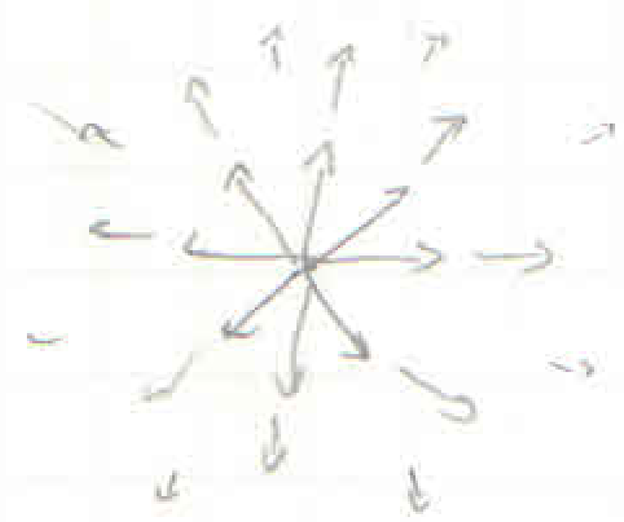
How about building it? We can guess since we know each charge will repel one another so Q^2 and we know the potential $\frac{1}{4\pi\varepsilon_0 r}$ so we can guess $W \propto \frac{Q^2}{4\pi\varepsilon_0 R}$. Let's do the math to try justify that. Let's imagine constructing a thin shell of thickness dr

leads to $\int_0^R \frac{\overbrace{4/3\,\pi r^3\rho}^{q_{in}}}{4\pi\varepsilon_0 r}\cdot\overbrace{\rho 4\pi r^2 dr}^{q_{shell}} = \int_0^R \frac{4\pi r^4\rho^2 dr}{3\varepsilon_0}$

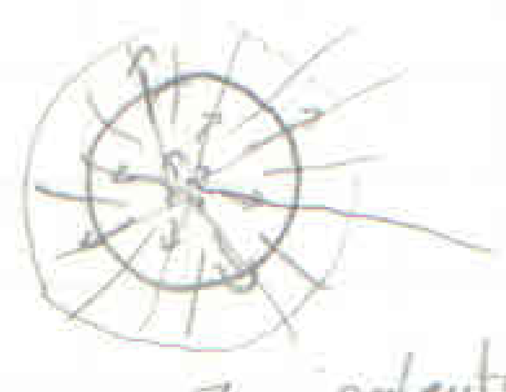
$$= \frac{4\pi}{15}\frac{R^5\rho^2}{\varepsilon_0} = \frac{4\pi}{15}\frac{1}{R}R^6\rho^2 = \frac{3}{5}\frac{1}{4\varepsilon_0 R\pi}\Big(\underbrace{\tfrac{4\pi}{3}R^3}_{Q}\Big)^2$$

$$= \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$ cool the factor is $3/5$.

<u>Visualisation</u>



We can visualise a field by summing the directions of field to create field lines. The density of the lines is the strength of the field. Lines can only stop and start on charges. Let's try a solid ball



The equipotential lines are where $\Delta V = 0$. These are perpendicular to the field as the field is the -gradient of the potential.

equipotential

Let's try build a general form of the work to build things

$W = \frac{1}{2}\sum\sum \frac{q_i q_j}{4\pi\varepsilon_0 R_{ij}}$ convert to integral form $W = \frac{1}{2}\iiint \rho V\, d^3x$.

We note that we change from the double sum of individual charges to the triple integral of charge over space. We know from Gauss' law that $\nabla\cdot E = \rho/\varepsilon_0$ so the integral becomes

$$W = \tfrac{1}{2}\varepsilon_0\iiint \nabla\cdot E\, V\, d^3x \overset{\text{product rule}}{=} \frac{\varepsilon_0}{2}\iiint \nabla(\vec{E}V) - \vec{E}\,\underbrace{\nabla V}_{E!}\, d^3x$$

$$\overset{IBP}{=} \tfrac{1}{2}\varepsilon_0\left(\iiint E\cdot E\, d^3x + \oint VE\cdot dn\right)$$