radial sol: then $\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \ell(\ell+1)$ say $R = Cr^k$

then $\frac{C}{Cr^k}\frac{\partial}{\partial r}\left(r^2 \, (k) r^{k-1}\right) = k(k+1) = \ell(\ell+1)$

$\Rightarrow k = \ell$ or $k = -(\ell+1)$ $\ell > 0$ but this gives two solutions.

If $k = \ell$, $V \to \infty$ as $r \to \infty$ no good!

$k = -(\ell+1)$, $V \to \infty$ as $r \to 0$ also no good. Both need to be used together.

And angular Θ is relabeled as T for less confusion

$1/T/\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial r}{\partial\theta}\right) = -\ell(\ell+1)$ guess that $T(\theta) = \sum c_k \cos^k(\theta)$

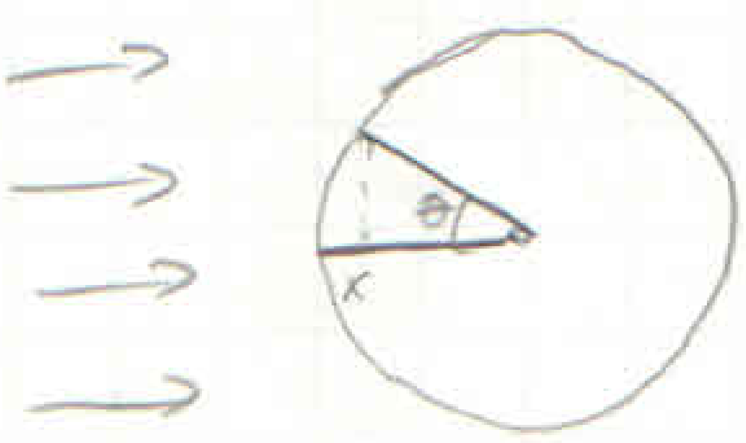
then some math gives

$$c_k = \frac{(k+2)(k+1)}{\ell(\ell+1) - k(k+1)} c_{k+2}, \quad c_{k+2} = -c_k \frac{\ell(\ell+1) - k(k+1)}{(k+1)(k+2)}$$

note that when $k = \ell$, this series ends since the multiplier is 0. Note that if k and ℓ are not both odd or even we have a problem!

let $\ell = 5$, say $c_1 = 1$, $c_3 = -14/3$, $c_5 = 21/5$ But this sucks!

Define The legendre polynomial as P_ℓ, $(\ell+1)P_{\ell+1} = (2\ell+1)P_\ell \cdot x - \ell P_{\ell-1}$. These are orthogonal on $[-1, 1]$ and can provide a solution to the PDE. Let's return to the phere in uniform field

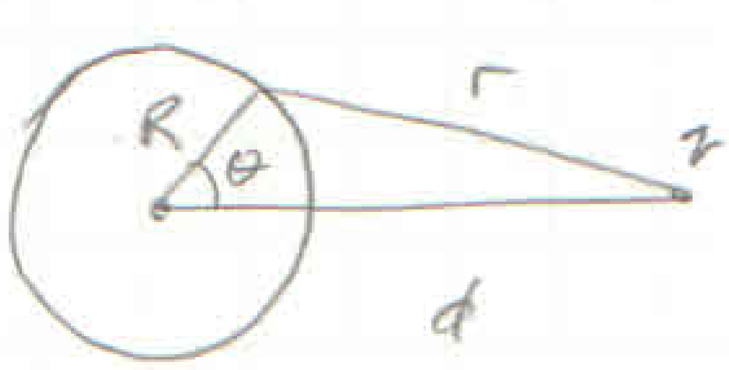


well we know $V \propto x$ in the uniform field thus we can say $V = C_1 \cos\theta$ on the sphere so $V(\theta=0) = RE$, $V(\theta=\pi) = -RE \Rightarrow C_1 = RE$

Note that $V(r,\theta) = R(r)T(\theta)$ and to match the legendre polynomial which has correct boundary conditions at $\ell = 1$ and to not get a diverging V in the center we use $R(r) = Cr^{-(\ell+1)}$ and to normalise this $C = R^2$

$\Rightarrow V(r,\theta) = RE\cos\theta\,(R^2/r^2) = R^3E/r^2 \cos\theta$

Return to the case of the point charge outside of the conducting sphere.



Well the potential near the surface of the sphere is

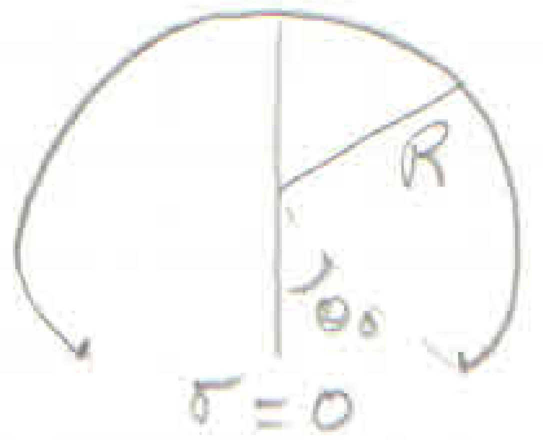
$$V(R,\theta) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{d^2 + R^2 - 2dR\cos\theta}}$$

Well using spherical harmonics $V_s = \sum c_\ell P_\ell(\cos\theta) R^{-(\ell+1)}$ and say the potential on the surface is 0. $V_s + V(R,\theta) = 0$

$$\Rightarrow \sum c_\ell P_\ell(\cos\theta) R^{-(\ell+1)} = -kq/\sqrt{d^2 + R^2 - 2dR\cos\theta}$$

We can numerically integrate this. Then we get a solution that agrees well with images.

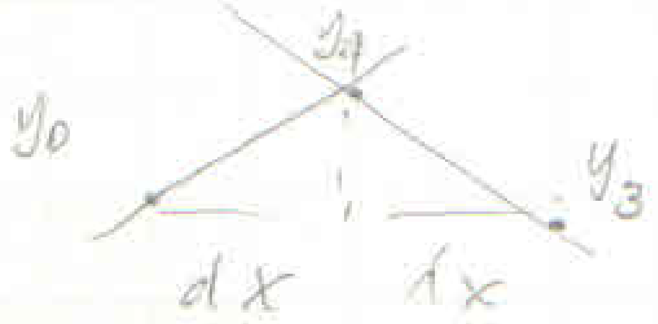
Now let's look at a Van der Graph generator.



So for $\theta > \theta_0$ $\sum P_\ell C_\ell = V_0$ if $R = r = 1$ similary we can alway normalise r/R and get $P_\ell C_\ell (r/R)^{-(\ell+1)}$ or $P_\ell C_\ell (r/R)^\ell$ these imply

$$\vec{E} = 1/R \left(P_\ell C_\ell \left(-(\ell+1)\right) \ldots\right.$$

Say we have V and we want to find ρ how can we compute $\nabla^2 V$?



slope 1: $\frac{y_1 - y_0}{dx}$, slope 2: $\frac{y_2 - y_1}{dx}$ then $\frac{\Delta \text{Slope}}{\Delta x}$ is what we want $\frac{\Delta \text{Slope}}{\Delta x} = \frac{y_1 - y_0 - y_2 + y_1}{\Delta x^2} = 0 = y_1 = \frac{y_0 + y_2}{2}$

which is the mean of points around it.
This generalises fairly easily to $\mathbb{R}^n$

Then with repeated averages will actually get us to where we need to be and will give us V given the boundary conditions. Then one can take the second derivative approximation to get ρ.