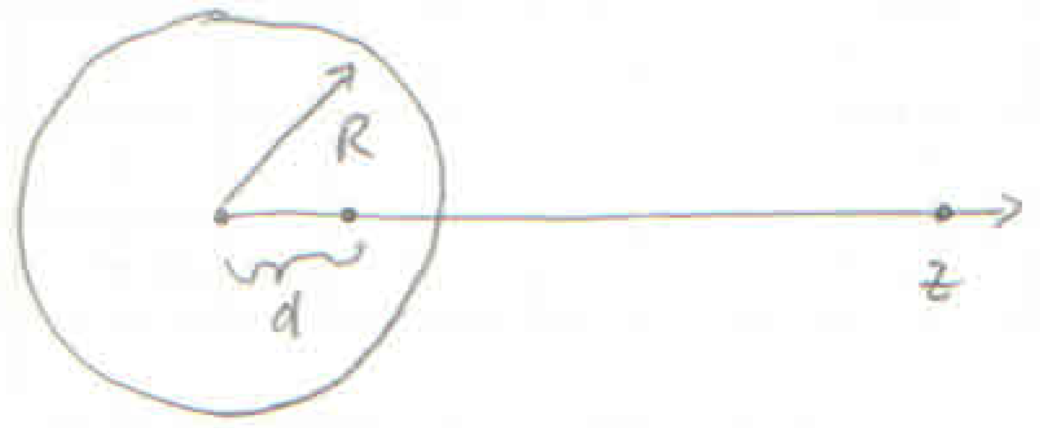


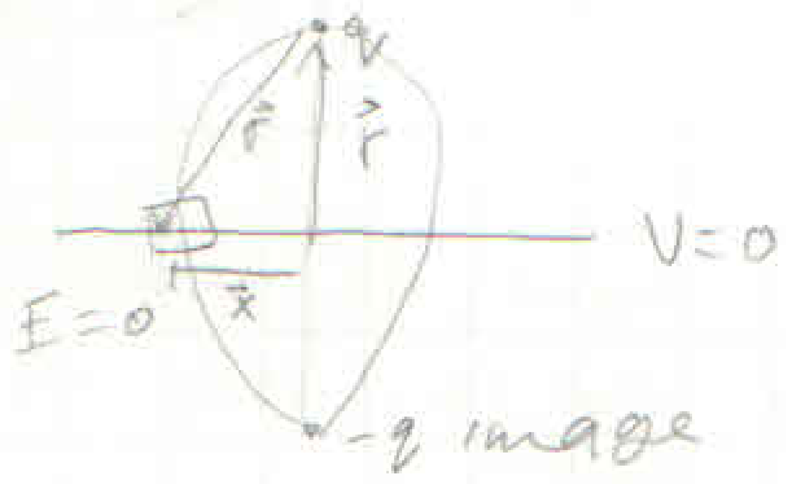
say this is our solution, we know this is a solution by uniqueness of the boundary cond. then $\frac{q}{z-R} = \frac{Q}{R-d} = 0$ on the boundary and

we can use other side of the sphere to get $\frac{q}{z+R} = \frac{Q}{R+d} = 0$

$$q(R-d) = Q(z-R), \quad q(R+d) = Q(z+R) \quad \ldots \text{ etc.}$$

kill me papi

Return to the infinite conducting grounded plane

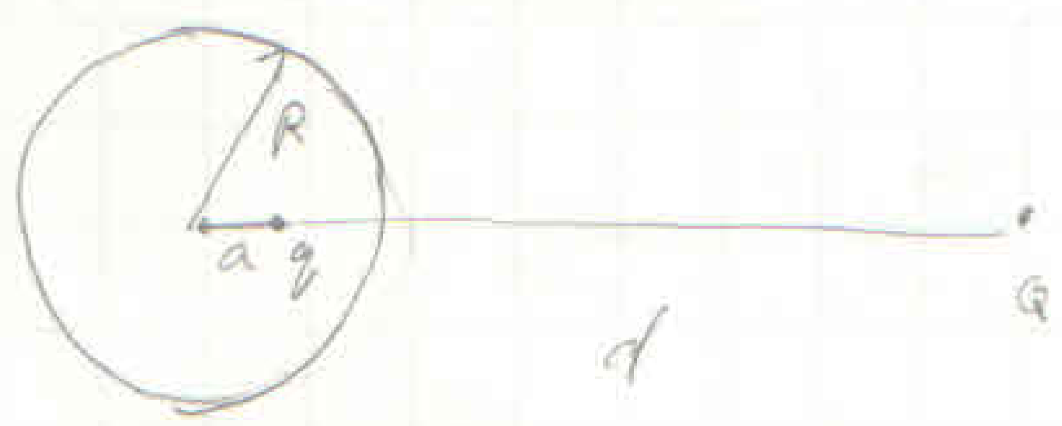


We create the image as before, we know $\vec{E}$ below the conductor is 0, and by uniqueness of B.C. That $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}+\vec{x}}{(r^2+x^2)^{3/2}} - \frac{q}{4\pi\epsilon_0}\frac{(\vec{x}-\vec{r})}{(r^2+x^2)^{3/2}}$

$$\Rightarrow A \cdot -\frac{2q}{4\pi\epsilon_0}\frac{\vec{r}}{(r^2+x^2)^{3/2}} = \frac{\sigma A}{\epsilon_0} \quad \text{by Gauss' law}$$

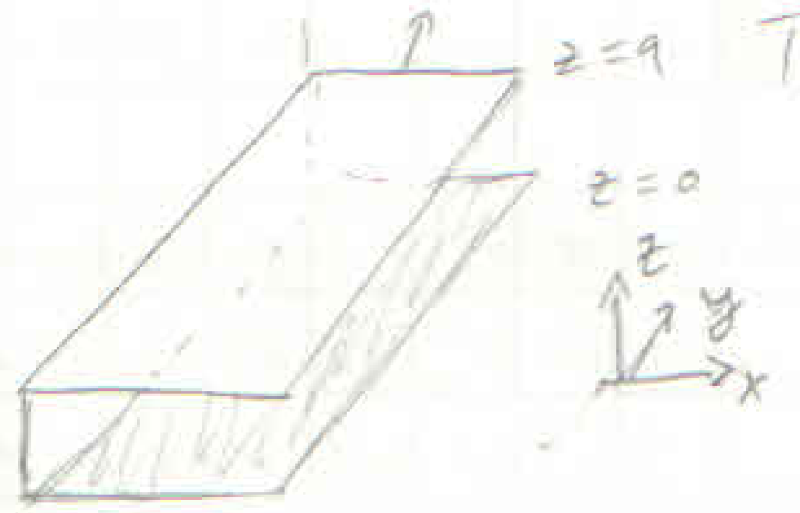
so $\sigma = \frac{-qr}{2\pi(x^2+r^2)^{3/2}}$ then we can compute $q_{\text{induced}} = \int_0^\infty \int_0^{2\pi} r\sigma \, d\theta \, dr$

we see that the 'images' charge appears on the surface! $= q$



let $q = -\frac{R}{d}Q$ and $a = R^2/d$ then the sphere is an equipotential then the potential at 0 is QR/d. Then since the potential at the center is average on the sphere we can write the potential outside to be $V = \frac{Q}{4\pi\epsilon_0 d}$?

Consider the following shape with the top and bottom grounded.



That gives the following boundary conditions

$$V(x, a) = 0$$
$$V(x, 0) = 0$$
$$V(0, z) = f(z)$$
$$V(\infty, z) = 0$$

let's assume that V is seperable $V(x,z) = X(x)\,Z(z)$ then if $-\nabla^2 V = 0$

$$\Rightarrow X''(x)Z(z) + Z''(z)X(x) = 0$$
$$\Rightarrow X''(x)/X(x) + Z''(z)/Z(z) = 0 \Rightarrow \frac{X''}{X} = \lambda^2 \text{ and } \frac{Z''}{Z} = -\lambda^2$$

$$\Rightarrow X(x) = Ce^{\pm\lambda x}, \quad Z(z) = A\cos(\lambda z) + B\sin(\lambda z)$$

from the boundaries, for $X(\infty) = 0 \Rightarrow X(x) = e^{-\lambda x}$

so $V(x,z) = e^{-\lambda x}[\alpha\cos(\lambda z) + \beta\sin(\lambda z)]$

but $V(x,0) = e^{-\lambda x}[\alpha + 0]$ but this must be $0 \Rightarrow \alpha = 0$

$V(x,a) = e^{-\lambda x}[\beta\sin(\lambda a)] \Rightarrow \lambda = n\pi/a$ for $n \in \mathbb{N}$

by linearity

$$V = \sum_\lambda e^{-\lambda x}\beta_\lambda \sin(\lambda z) = \sum_n e^{-\frac{n\pi}{a}x}\beta_n \sin\left(\frac{n\pi}{a}z\right)$$

but now how do we get the β_λ? say $f(z) = V_0$ @ $x=0$

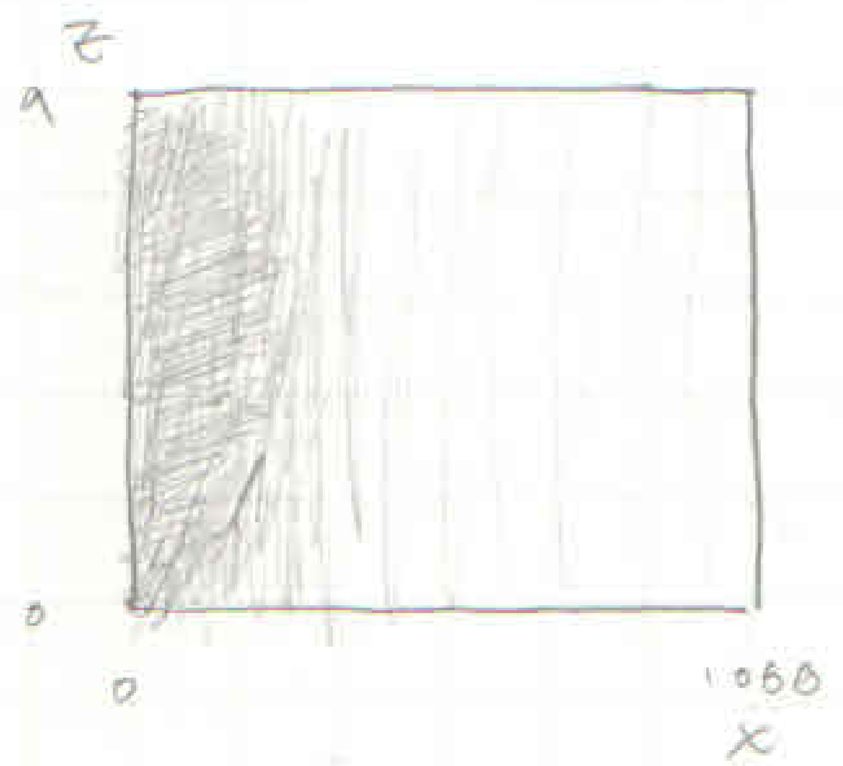
so $V(0,z) = \sum_n \beta_n \sin\left(\frac{n\pi z}{a}\right) = V_0 \Rightarrow \int_0^a V_0 \sin\left(\frac{m\pi z}{a}\right)dz = \int_0^a \sin\left(\frac{m\pi z}{a}\right)V(0,z)\,dz$

LHS.
$\Rightarrow -V_0 \cos\left(\frac{m\pi z}{a}\right) \cdot a/m\pi$ so $m=0 \Rightarrow 0$, $m=1 \Rightarrow 2aV_0/\pi$, $m=2 \Rightarrow 0$, $m=3 \Rightarrow \frac{2aV_0}{3\pi}$
$= \frac{V_0 2a}{m\pi}(1-(-1)^m)$ but the RHS becomes

$$\int_0^a C_m \sin^2(m\pi z/a)\,dz$$ since if $n \neq m$ this is 0

$$= C_m\, a/2 \Rightarrow C_m = \frac{2}{\cancel{a}} \frac{V_0 2a}{m\pi} = \frac{4V_0}{m\pi}.$$ m odd

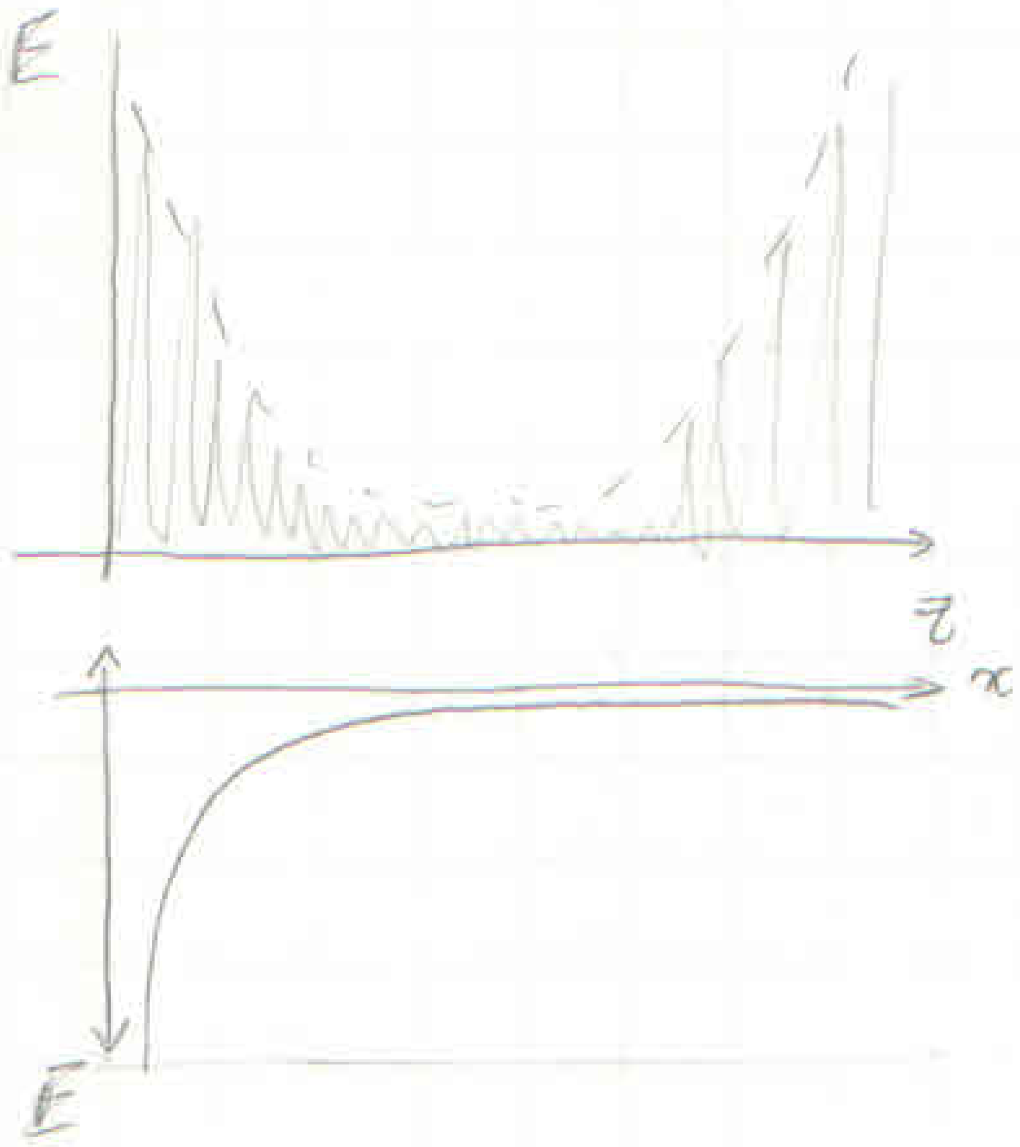
Thus, plotting $\sum_n e^{\frac{2\pi}{a} n \cdot x} \cdot 4V_0/n\pi\,(1-(-1)^n)\sin(n\pi/a\, z)$ get's us

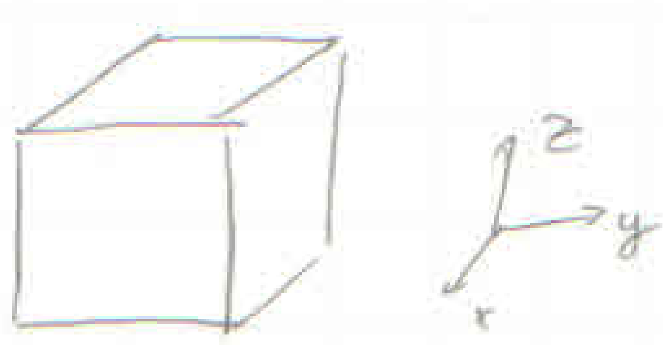


where dark is stronger than light. The rounded edges are the Gibbs effect (lovingly called batman ears) but quickly decays out to give a physical solution. How do we get the charge distribution on each plate?

Using Gauss' law and how $\vec{E}$ is $\perp$ to equipotentials we have that
$$EA = q/\varepsilon_0 = \sigma A/\varepsilon_0 \Rightarrow \sigma = \vec{E}\varepsilon_0$$

Since V is fairly constant near the center of left surface, we can predict and compute that σ will look like the following figure with oscillatory artifacts from the transform. Charges tend to ∞ on the corners since near the edges E jumps to 0 instantly. Similarly along the top and bottom are as in the second plot. The infinite negative charge aims to cancel out the infinite positive charge. This can be made better. Let's try another problem



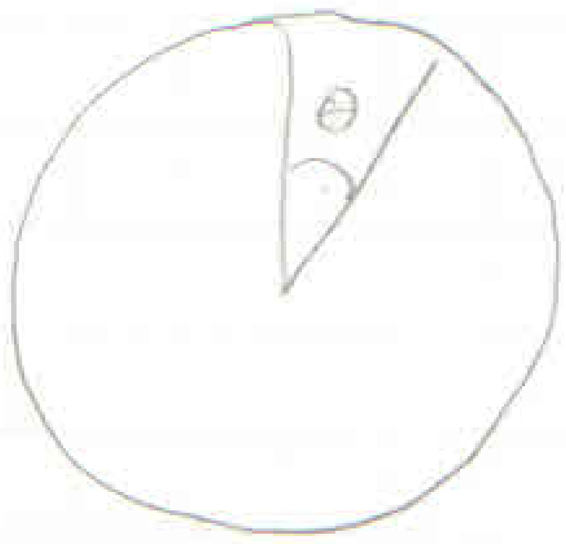


let this box be our area of intecest

Then we have $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

By variable seperation $\frac{\partial^2 V}{\partial x^2} = k_x^2$, $\frac{\partial^2 V}{\partial z^2} = k_z^2$
s.t. $k_x^2 + k_y^2 + k_z^2 = 0$ $\quad \frac{\partial^2 V}{\partial y^2} = k_y^2$

We must select which are positive, negative, or zero from the boundary conditions. What about the sphere though.



say $V(r,\theta,\varphi) = R(r)\,\Theta(\theta)$ then $\nabla^2 V = 0$

$$\Rightarrow 1/r^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \ell(\ell+1),\quad \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -\ell(\ell+1)$$