Say we have a sphere and we charge it, the charges on it repel and create a pressure. From thermal we know $dU = -P\, dV$ and the work required to assemble the sphere of radius r is $U = \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0 r}$ so then

$$dU = \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0} \left(\frac{1}{r+dr} + \frac{1}{r} \right) \quad \text{and} \quad dV = 4\pi r^2 dr$$

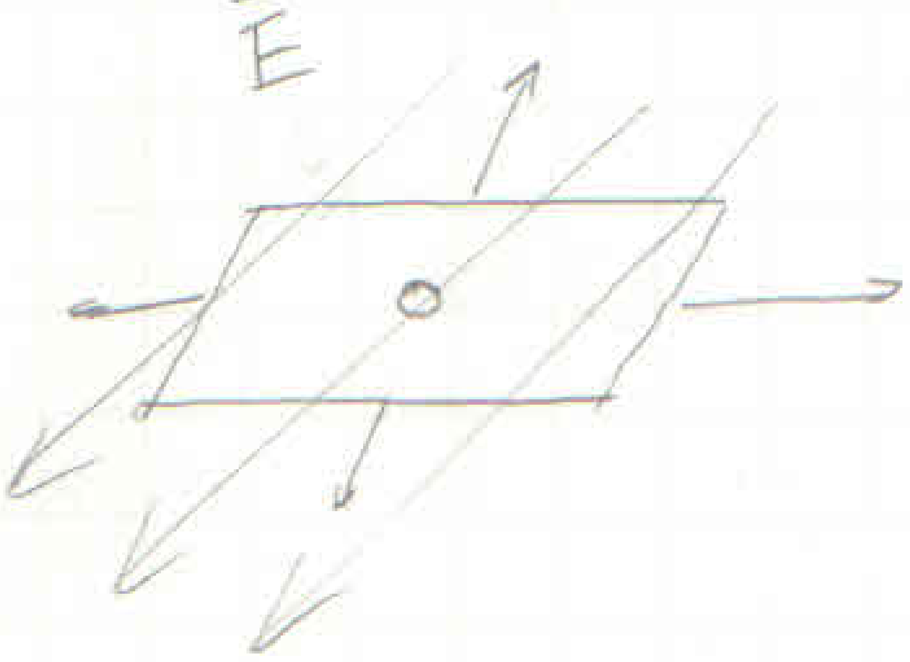
so

$$\frac{1}{2} Q^2 \frac{1}{4\pi\epsilon_0} \frac{dr}{r^2} \approx 4\pi r^2\, dr\, P$$

$$P \approx \frac{1}{2} Q^2 / \epsilon_0 (4\pi r^2)^2 = \frac{1}{2} \epsilon_0 E^2$$ which is the energy density of the electric field

thus $U = \int \frac{1}{2} E^2 \frac{1}{\epsilon_0}\, d^3x$

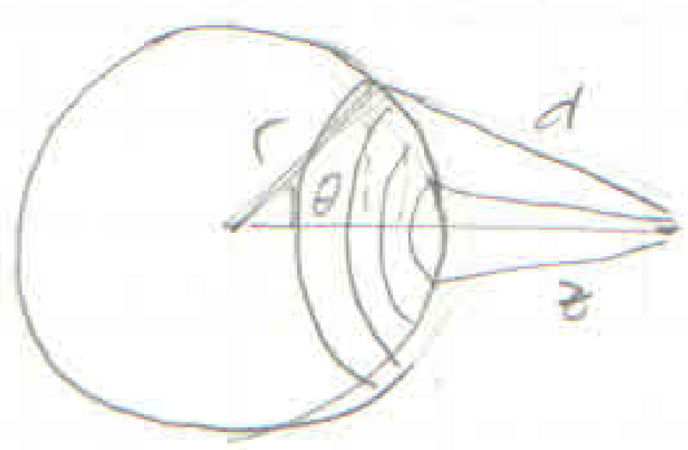
Say we have a conductor in an electric field. What is the force on a small section.



Well the conductor cannot influence itself or it will break conservation of momentum. If we remove our little chunk and freeze the charges there will be $\underline{E}$ above the plane and 0 above it? I'm not really sure what this was! Help.

Let's observe the potential $V(\vec{r}) = \int \frac{\rho(r')}{4\pi\epsilon_0 (r-r')} d^3r'$ is a scalar formula well the potential from a ring is

$V = \frac{q}{4\pi\epsilon (r^2+z^2)^{1/2}}$ then a sphere gives

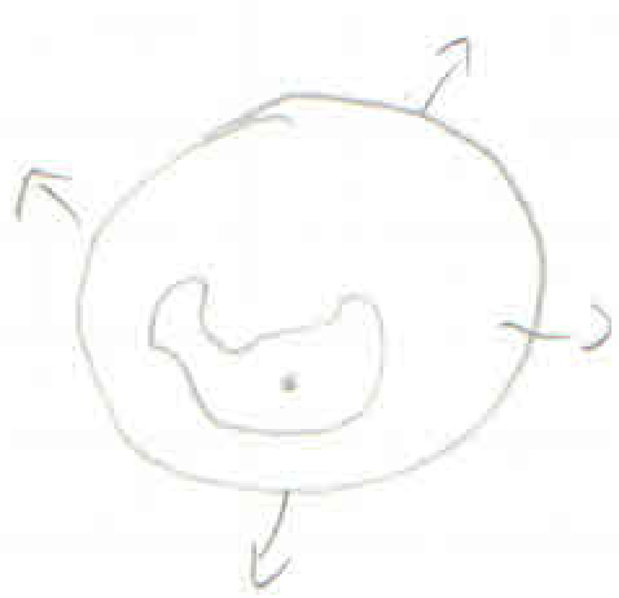


So the surface density of a sphere is σ and $dQ = \sigma 2\pi (r \sin\theta) \cdot r\, d\theta = 2\pi r^2 \sin\theta\, \sigma\, d\theta$, by law of cosines $d^2 = r^2 + z^2 - 2rz\cos\theta$ so

$$V(z) = \int_0^{\pi} \frac{2\pi r^2 \sin\theta\, \sigma\, d\theta}{4\pi\epsilon_0 \sqrt{r^2+z^2-2rz\cos\theta}} \qquad \begin{aligned} u &= r^2+z^2-2rz\cos\theta \\ du &= -2rz\sin\theta \end{aligned}$$

$$= \int_{(r-z)^2}^{(r+z)^2} \frac{Q}{4\pi\epsilon_0} \frac{1}{2rz} \frac{du}{\sqrt{u}} = \frac{Q}{4\pi\epsilon_0} \frac{1}{2rz} \left(|r+z| - |r-z| \right)$$

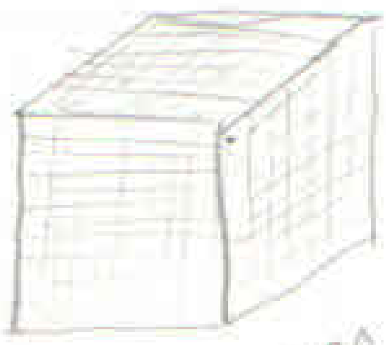
Return briefly to the charge in a cavity of a conducting sphere.



How do we know the surface charge is uniform? Well if one takes the outer sphere to infinity it has no effect on the cavity => the field cancels from charge on the cavity => must be uniform for surface effect to be 0

Say we wanted to compute the potential of a 1000x1000x1000 grid of charge at every point in the grid

1000



1000

1000

The amount of computations for each point is $\frac{n^3 \cdot n^3}{2} = \frac{1}{2} n^6$, so far us $10^{6 \cdot 3} = 10^{18}$ so then that would take an average computer ~3yrs

But what about the convolution? We can compute this very easily with Fourier transforms since we can use FFTs and convolute back. This is really strong. An FFT is of order $n^3 \log(n^3)$.

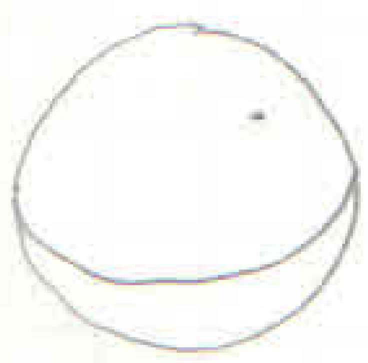
Let's go back to potential $\nabla V = -\vec{E}$ and $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, $\nabla \times \vec{E} = 0$
$\Rightarrow \nabla \cdot \nabla V = -\rho/\varepsilon_0 \Rightarrow \nabla^2 V = -\rho/\varepsilon$ is the laplace equation.

Say for a moment that $\nabla^2 V = 0$, then the solution to this is linear in 1.d so if charge density is 0 we get a line.

For those who have not done complex Variables note

$$\nabla^2 f(z) = \frac{\partial^2}{\partial xx} f(x+iy) + \frac{\partial^2}{\partial y \partial y} f(x+iy) = f''(z) - f''(z) = 0$$

What else can we say about the average potential for a point on a conducting sphere



$$V_{avg} = \frac{1}{4\pi r^2} \iint V(r)\, dA = \frac{1}{4\pi} \iint V(r) \frac{\cancel{r^2}}{\cancel{r^2}} \sin\theta \, d\theta \, d\varphi$$

$$\frac{d V_{avg}}{dr} = \frac{1}{4\pi} \iint \nabla V(r) \cdot \hat{r} \sin\theta \, d\theta \, d\varphi = \frac{1}{4\pi r^2} \iint_S \vec{\nabla} V \cdot d\vec{a}$$

$$= \frac{-1}{4\pi r^2} \int_S \vec{E} \cdot d\vec{a} = \frac{-1}{4\pi r^2 \varepsilon_0} \int_S \rho \, da = \frac{q}{4\pi\varepsilon_0 r^2} = 0.$$

$\Rightarrow \rho$ is constant $\Rightarrow$ surface density is uniform! Say for some general conductor we want to solve for the potential on it's surface

say we have two solutions that fit here, how do we know they are the same.

$\nabla^2 V_1(r) = 0$, $\nabla^2 V_2(r) = 0$ and so $\nabla(V_1(r) - V_2(r)) = 0$
so since the potential inside acts like an average of the boundary at equilibrium, we know there cannot be maximums inside the boundary. However since $V_1(r) - V_2(r)$ is 0 on the boundary, and by linearity of the laplacian has $V_1(r) - V_2(r)$ as a solution. If there is any nonzero value of $V_1(r) - V_2(r)$ then it could not be an average of 0. Thus $V_1(r) - V_2(r) = 0$.