Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

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Peter Keep
Moraine Valley Community College

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Acknowledgements

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First, thanks to Dan and Terra in the library at Moraine Valley. They've both been so supportive of the project and worked so hard to make it possible for me to find space and time to actually write the book.

In the math department at Moraine Valley, Amy, Angelina, and Lisa all were so helpful in different ways. I'm so thankful for their help with writing and editing practice problems (and solutions) and contributing their expertise to make sure that this is a teaching and learning resource first and foremost. Thanks for agreeing to help and always providing great feedback.

Thank you to the Consortium of Academic and Research Libraries in Illinois (CARLI) and the Illinois State Library for providing the opportunity through grant funding for me to write this. I've been wanting to put this type of resource together for my students for years and had never been able to find a way to carve out the time required. Your support to developing good OER practices and investment in Illinois is so appreciated.

Thanks also to the PreTeXt people: both the people behind it and the wonderful people that use it. Especially the following:

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- Oscar Levin.

I've learned so much by digging through the source code of all of your projects. Thanks so much for providing such great projects and offering up your source code for others to learn from.

And last, thank you to my students. I've loved thinking about calculus with you for so many years, and in every semester, our classes together are the highlight of my job. The way we talk about calculus together has changed over the years for the better, and I'm so thankful for your influence over how I think about calculus and how we can present calculus to people who want to learn it.

Disclosure about the Use of AI

This book has been lovingly written by a human.

Me.

Peter Keep.

I have used a lot of different tools, both for inspiration and for actually creating resources for this book. *None* of those tools has involved any form of generative AI.

I could list all of the ways that I think using generative AI in education is, at minimum, problematic. More pointedly, I believe that it is unethical. More broadly, I believe that the use of generative AI for any use-case that I have encountered to be unethical.

In my classes, I try to help students realize the joy and value of working at something and creating something and struggling with something and knowing something. Giving worth to something, even an imperfect thing. Celebrating our accomplishments, even when (especially when?) there is room to grow in those accomplishments. And so I have taken that advice in the creation of this book. I have created a book that is definitely not perfect. I have struggled to write it. There are parts of it that could be (need to be) improved.

But I was the one that created it. I struggled with it. I know it.

I hope that this book can also be a useful tool for others to use, and I have left the copyright to be about as open as possible. Others can take this, use it, can change it, add to it, subtract from it, etc.

In leaving this copyright open for others to change this book, I cannot guarantee that every version of this book is free from the mindless and joyless output from some Large Language Model. But I want to leave this note up in hopes that anyone who *does* inject some output from some generative AI product into this book will take it down. If this note, or some statement similar to it, is not present in the version of the book you are accessing, please be cautious. Find a different calculus textbook to read!

Find something written by a human. Find the words of some other mathematician who tries, maybe imperfectly, to share the ideas of calculus.

Teaching and learning is about humans communicating with each other, and only humans can do that.

Notes for Instructors

Notes for Students

Hi! Thanks so much for reading this book! I hope it's a good and useful tool for you as you learn calculus. Before you dive into it, I want to try to explain some of the choices I made in putting this book together and how I hope you'll use it.

First, let's talk about all of the activities. In each section you'll find activities that lead into things like definitions or theorems. These are activities that I give my students to work on in groups. If you're using this independently, I hope you still engage with the activities: think about them, try to use them as ways of exploring the results or definitions before we state them. I want you to build intuition and I hope that these activities are helpful, even without the group exploration that would normally happen with them.

You might also notice that there aren't many proofs in this book. That's not extremely unique for an introductory calculus book, but there might be even fewer than expected. The ones that are included are ones that I think are important. Most of the included proofs show some of the kind of reasoning that we want students in calculus classes to see.

The last thing that I'll say is that I hope, most of all, that however you use this book and whatever parts of it you engage in, that it is useful for you. I hope that it helps you as you work to understand these wonderful groups of topics.

Thanks for letting me and my book be a part of your journey learning mathematics!

Contents

Chapter 1

Limits

1.1 The Definition of the Limit

We're going to start this textbook by stating a definition. This is a common practice in math classes: we need to agree upon a common definition of the mathematical objects and adjectives we are thinking about. We will state a lot of definitions in this textbook.

What I hope we will do, though, is motivate these definitions. We want to arrive at a point where it makes sense to give a name to this phenomena or object that we're thinking of. Or maybe we arrive at a point where the specifics of the definition don't just come down to us out of nowhere, but feel like reasonable and obvious things to consider.

So for now, we're going to work on defining a very important and very key mathematical object that is used in calculus: the limit.

A limit is all about closeness, so let's first interact with the idea of closeness, and then work on a definition of a limit.

Defining a Limit

Activity 1.1.1 Close or Not?

We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!
Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?
Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

The balance that we need to find, as we discovered in Activity 1.1.1, is about being able to leave room for those with a very strict idea of what "close" might be. We will want to think of an idea kind of like "infinite closeness," but we're not going to frame it this way: we're going to think about a function's output being so close to some specific number that literally everyone can agree. It is so close that it is within every possible definition of closeness.

The general idea is that we want to think about the behavior of a function at inputs that are near some specific input. Is there a trend with the outputs? Are they all centered around a specific value or do they differ wildly?

Definition 1.1.1 Limit of a Function.

For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the **limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x\to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$.

We can clarify a couple of things here:

- There are two types of "close" in this definition: "arbitrarily close" and "sufficiently close." One of these is in references to x-values being close to a number and the other is in reference to function outputs being close to a specific number.

- We are concerned with the behavior of a function around, but not at, a specific x-value: $x = a$. We don't really care about what the function is doing at that input (if anything at all), and we already have words to describe that kind of behavior!

- When we talk about x-values that are near a, that might reference x-values that are a bit bigger than a or x-values that are a bit smaller than a. We can be more specific by simply changing this definition to focus on only one "side" individually.

We can go back to Activity 1.1.1 and think about how we chose x-values that were larger than $\frac{3}{4}$ and smaller than $\frac{3}{4}$. Let's define these ideas a bit more formally!

Definition 1.1.2 Left-Sided Limit.

For the function $f(x)$ defined at all x-xalues around and less than a, we say that the **left-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but less than, a. We write this as:

$$\lim_{x\to a^-} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^-$.

Definition 1.1.3 Right-Sided Limit.

For the function $f(x)$ defined at all x-xalues around and greater than a, we say that the **right-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but greater than, a. We write this as:

$$\lim_{x\to a^+} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^+$.

This should lead us to our first result in this textbook. This first result will do two things:

1. Introduce some language that we can use when we talk about limits as well as a classification that we can apply to them.

2. Introduce how we will build our results throughout the course of this text. We want to discover these results as things that are required for us

to talk about (and do) calculus together, and hopefully we can motivate each one beforehand.

In lieu of a formal activity, let's just review Definition 1.1.1 Limit of a Function pose the following questions to think about:

- Why do we put emphasis on L being a number? What could happen if it wasn't?
- Why do we put the emphasis on the number L being a *real* number? What other type(s) of number could it be?
- Why do we put emphasis on L being a *single* number? How could we have the function be close to multiple real numbers?

We can look at one of the ways that we break the definition: by having two different values that the function gets close to.

Theorem 1.1.4 Mismatched Limits.

For a function $f(x)$, if both $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then we say that $\lim_{x\to a} f(x)$ ***does not exist****.*

Approximating Limits Using Our New Definition

We have defined a new term, and now we do the typical mathematical task: define a new thing and then investigate it.

> A common joke in mathematics is that we "make up a guy to get mad at." It's really only kind of a joke, because it really is a pretty good description of what we do! Here, we defined a new object and now we'll think about it and find ways that it frustrates us or some other weird behavior about it. That's mathematics!

We will eventually get really good at thinking about limits and using them, but for now we just want to get familiar with them. Let's approximate these values that our function is near by looking at some pictures of graphs and some tables of function outputs.

Later on, we'll formalize this more. For now, we just want to use these pictures and tables to get familiar with *what* a limit even is.

Activity 1.1.2 Approximating Limits.

For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x\to 1} f(x)$ using the graph of the function $f(x)$ below.

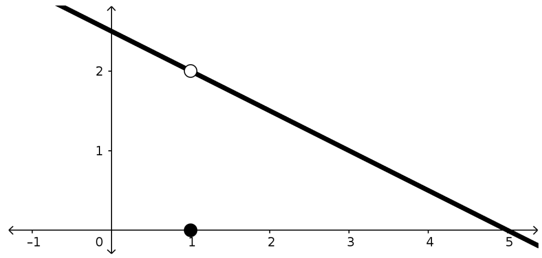


Figure 1.1.5

(b) Approximate $\lim_{x\to 2} g(x)$ using the graph of the function $g(x)$ below.

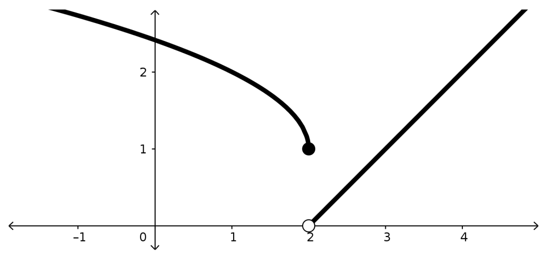


Figure 1.1.6

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x\to -1} h(x)$
- $\lim_{x\to 0} h(x)$
- $\lim_{x\to 2} h(x)$

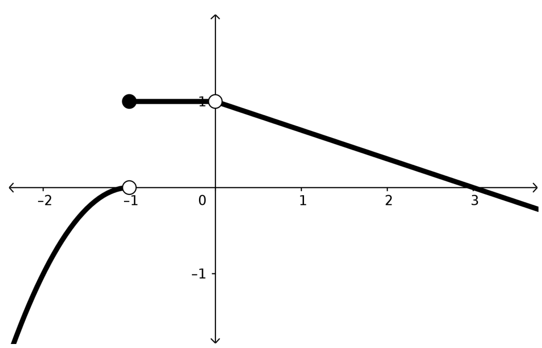


Figure 1.1.7

(d) Why do we say these are ”approximations” or ”estimations” of the limits we’re interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren’t confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

It can be hard to focus on the aspects of a graph that we really care about for the purpose of a limit. Let's build a small strategy to help us think about what we're looking at. We'll start by just considering some function, $f(x)$. Using our definition of the Limit of a Function as a guide, we'll make sure that it's defined around some x-value, $x = a$.

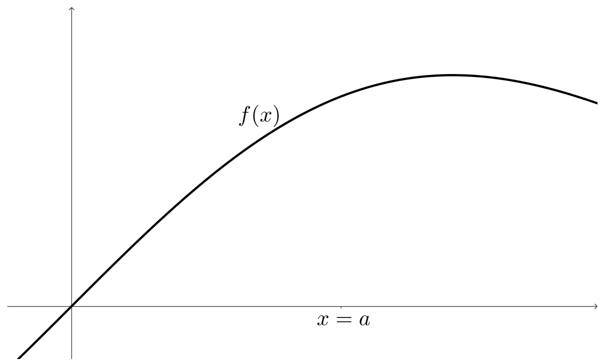


Figure 1.1.8 The function, $f(x)$.

Now we want to investigate more of our definition. We want to look at the x-values that are around, but not equal to, $x = a$.

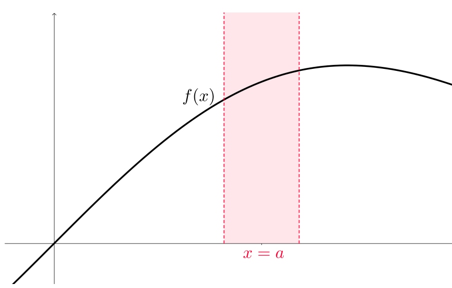


Figure 1.1.9 The x-values around $x = a$.

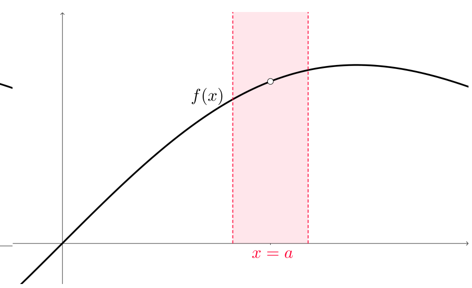


Figure 1.1.10 The x-values around, but not equal to, $x = a$.

We can see that we might as well remove any point at $x = a$ from our graph: we are only concerned with the behavior around that x-value instead of the function's behavior at it.

And now our focus can turn to the function outputs. For the x-values in this interval of inputs that we've constructed, is there some common real number that the corresponding function outputs are close to? We can visualize some interval of y-values. We'll think of this as a target: we want to build an interval of y-values that all of the function outputs from this interval of x-values land in.

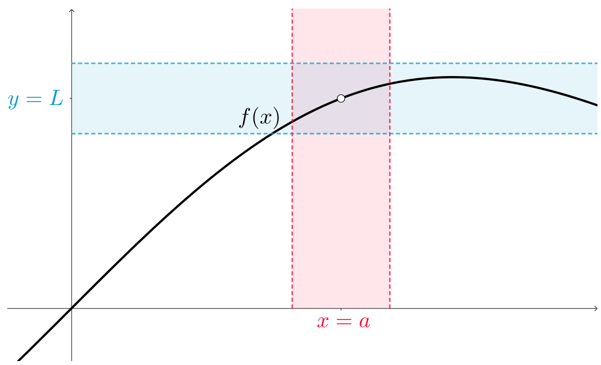


Figure 1.1.11 The corresponding function outputs $f(x)$ are all in the target interval of y-values.

This is a pretty wide range of y-values, but we can see that the graph of the function (when we limit to just the interval of x-values selected) produces function outputs that exist only in that interval. We don't *fill* the interval, but that's fine!

What we *really* care about, though, is if these function outputs are all close to the same, single, real number. What we can do is look at a more strict idea of "closeness" in the y-interval by shrinking it. In order for us to produce function outputs that are in this new, smaller, interval, we'll need to correspondingly shrink our interval of inputs to more closely surround $x = a$.

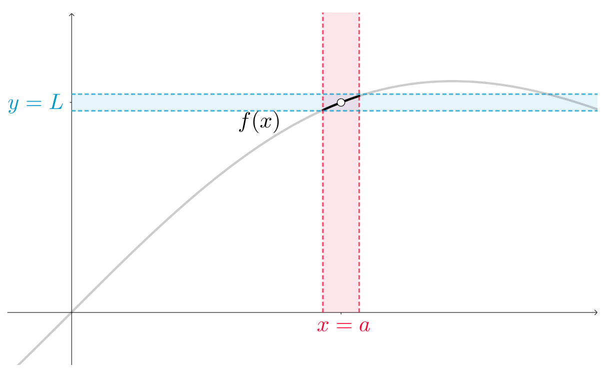


Figure 1.1.12 The x-values around, but not equal to, $x = a$.

In this visualization, we've also tried to focus on just the portion of our function that exists in this little intersection of intervals: we want to know what these functions values are close to, or more specifically if they are all close to the same thing. So we can de-emphasize the rest of our function!

All we're doing is working on a strategy to focus on the parts of this graph that matter: only the parts of the curve that are surrounding $x = a$ (but not that actual point specifically). From there, we just want to know what the function outputs are clustered around, if anything.

Let's look at this same kind of visualization for a limit that does not exist: we're going to think about the case where the one-sided limits don't match. We'll start a little further on in this visualization process: we have a function, and we can visualize an interval of x-values whose function outputs land inside a target interval of y-values.

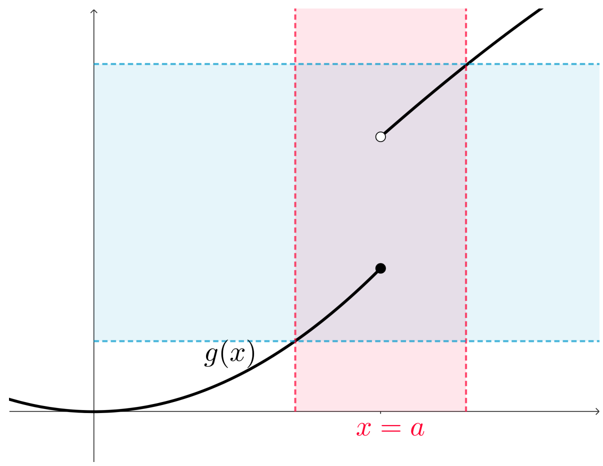


Figure 1.1.13

We can see the problem: that vertical space between the function on the left of $x = a$ and the where the function values are on the right of $x = a$ will make it so that horizontal bar cannot get much smaller. We can disregard the point at $x = a$ as well as the function outside of the interval, but once try to shrink the target interval of y-values, but we'll see the problem.

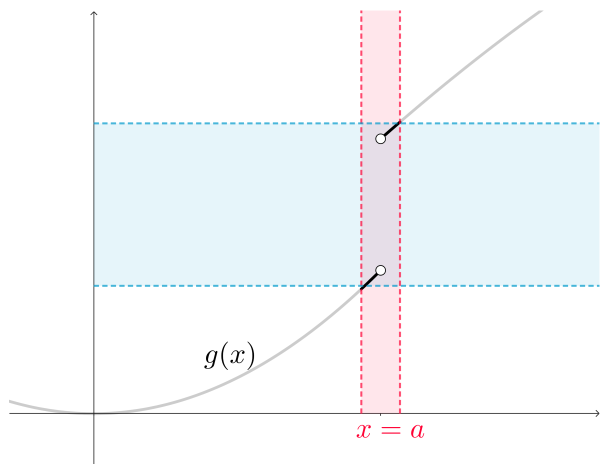


Figure 1.1.14

These function outputs are spread apart! They are not close to a single value. Instead, they're close to two! The function is close to a value on the left side, and then the function is close to a larger value on the right side.

$$\lim_{x\to a^-} g(x) \neq \lim_{x\to a^+} g(x) \text{ and so } \lim_{x\to a} g(x) \text{ does not exist.}$$

Now let's think about how we can approximate (and learn more about) limits using when we just think about the actual values of a function's inputs and corresponding outputs.

Activity 1.1.3 Approximating Limits Numerically.

For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim\limits_{x\to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.1.15

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.1.16

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.1.17

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Overall, there's a common theme here: in either representation (graphically or numerically), we're making a best guess at the behavior of the function values around a point. We have limited information in these estimations, and so we're doing the best we can: in graphs, we're trying our best to make sense of the lack of precision in in the scales of our visual, and in the numerical tables we are only given a limited number of points to think about. In both cases, we are hoping to see more information to add more confidence to these estimations.

We want to make the jump from estimating these limits to evaluating them, and for that to happen, we'll need to add more information and more precision about the behavior of our function.

Practice Problems

1. Explain in your own words the meaning of:

$$\lim_{x \to a^-} f(x) = L.$$

2. Explain in your own words the meaning of:

$$\lim_{x \to a^+} f(x) = L.$$

3. Explain in your own words the meaning of:

$$\lim_{x \to a} f(x) = L.$$

4. Say we know that $\lim_{x \to 3^-} f(x) = 2$ and $\lim_{x \to 3^+} f(x) = 2$. What do we know (specifically or in general, if anything) about each of the following?

(a) $f(3)$

(b) $f(2.999)$

(c) $f(3.001)$

(d) $\lim_{x\to 3} f(x)$

5. Which of the following is possible? Explain why or why not, and any other conclusions that we can draw.

(a) For some function $f(x)$, $\lim_{x\to 5} f(x) = 6$ and $f(5) = -3$

(b) For some function $g(x)$, $\lim_{x\to 4^-} f(x) = -\dfrac{3}{2}$ and $\lim_{x\to 4^+} g(X) = \dfrac{4}{7}$.

(c) For some function $\ell(t)$, $\lim_{t\to\alpha} \ell(t) = 2$ and $\lim_{t\to\alpha^+} \ell(t) = 1$.

(d) For some function $r(\theta)$, $\lim_{\theta\to 0} r(\theta)$ does not exist, $\lim_{\theta\to 0^-} r(\theta) = \pi$, and $\lim_{\theta\to 0^+} r(\theta) = -\dfrac{\pi}{2}$.

(e) For some function $j(w)$, $j(4) = \pi$ while $\lim_{w\to 4} j(w)$ does not exist.

6. Fill in the following tables in order to satisfy the requirements listed. Afterwards, include a sentence or two justifying your choices.

(a) *Requirements:* $\lim_{x\to 1} f(x) = 3$

x	____	0.93	____	1	____	____	1.04
$f(x)$	____	____	____	____	____	____	____

(b) *Requirements:* $\lim_{x\to -5^-} f(x) = 2$, $f(-5) = 6$, and $\lim_{x\to -5} f(x)$ doesn't exist.

x	-5.2	____	____	-5	____	4.98	____
$f(x)$	____	____	____	____	____	____	____

(c) *Requirements:* $f(7)$ does not exist and $\lim_{x\to 7} f(x) = 3$

x	____	____	6.985	7	____	____	7.002
$f(x)$	____	____	____	____	____	____	____

(d) *Requirements:* $\lim_{x\to 0^-} f(x) = \pi$ and $\lim_{x\to 0^+} f(x) = e$.

x	____	-0.14	____	0	____	____	0.5
$f(x)$	____	____	____	____	____	____	____

7. From the following tables, estimate/report each of the requested values. Explain your choices.

(a) *Requested:* $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$, and $f(1)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.4	2.48	2.4998	9	2.5004	2.52	2.8

(b) *Requested:* $\lim_{x\to 8^-} f(x)$, $\lim_{x\to 8^+} f(x)$, $\lim_{x\to 8} f(x)$, and $f(8)$

x	7.9	7.99	7.999	8	7.001	7.01	7.1
$f(x)$	-1.5	-1.9	-1.999	-2	7.0001	7.2	7.5

(c) *Requested:* $\lim_{x\to\pi^-} f(x)$, $\lim_{x\to\pi^+} f(x)$, $\lim_{x\to\pi} f(x)$, and $f(\pi)$

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$f(x)$	-3	-3	-3	does not exist	-3	-3	-3

8. For each of the listed requirements, sketch a graph of a function that satisfies each. Afterwards, include a sentence or two justifying you sketch.

(a) *Requirements:* $f(6) = 0$, $\lim_{x\to 6} f(x) = -2$, $\lim_{x\to -2^-} f(x) = 1$, and $\lim_{x\to -2} f(x)$ does not exist.

(b) *Requirements:* $\lim_{\omega\to 0} \rho(\omega) = 8$, $\lim_{\omega\to 2} \rho(\omega) = -2$, and $\rho(2)$ does not exist.

(c) *Requirements:* $\lim_{t\to -3^-} q(t) = 0$, $\lim_{t\to -3^+} q(t) = 4$, $\lim_{t\to -1} q(t) = 9$, and $q(-1) = 9$.

9. From the graph of $f(x)$ below, estimate of each of the requested values. Explain each of your choices.

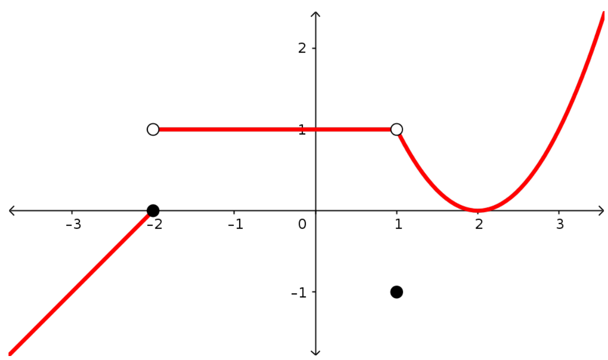


Figure 1.1.18 The function $f(x)$.

(a) $\lim_{x\to -2^-} f(x)$

(b) $\lim_{x\to -2^+} f(x)$

(c) $\lim_{x\to -2} f(x)$

(d) $\lim_{x\to 0^-} f(x)$

(e) $\lim_{x\to 0^+} f(x)$

(f) $\lim_{x\to 0} f(x)$

(g) $\lim_{x\to 1^-} f(x)$

(h) $\lim_{x\to 1^+} f(x)$

(i) $\lim_{x\to 1} f(x)$

(j) $\lim\limits_{x\to 2^-} f(x)$

(k) $\lim\limits_{x\to 2^+} f(x)$

(l) $\lim\limits_{x\to 2} f(x)$

1.2 Evaluating Limits

Adding Precision to Our Estimations

Activity 1.2.1 From Estimating to Evaluating Limits (Part 1).

Let's consider the following graphs of functions $f(x)$ and $g(x)$.

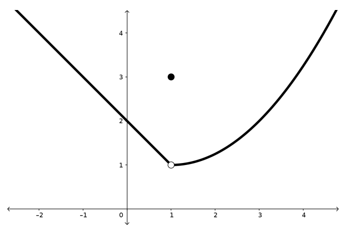


Figure 1.2.1 Graph of the function $f(x)$.

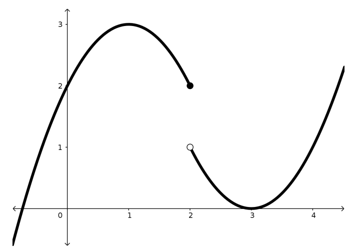


Figure 1.2.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x\to 1^-} f(x)$
- $\lim\limits_{x\to 1^+} f(x)$
- $\lim\limits_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x\to 2^-} g(x)$
- $\lim\limits_{x\to 2^+} g(x)$
- $\lim\limits_{x\to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

We're going to repeat this process, but with a slight change to the representation of each function. Hopefully this will be illuminating in our attempt to add more precision to our estimations.

Activity 1.2.2 From Estimating to Evaluating Limits (Part 2).

Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

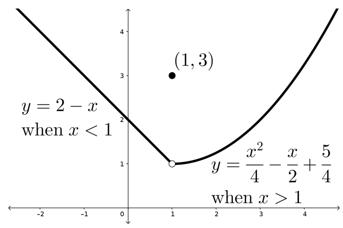


Figure 1.2.3 Graph of the function $f(x)$.

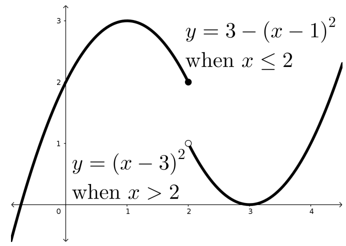


Figure 1.2.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2-x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3-(x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

These two examples are hopefully helpful for us to see that when we are given the actual rule for a function f that connects x to the corresponding output $f(x)$, we are able to move past estimation. We suddenly have whatever level of precision we'd like, since we can immediately see what is happening with every x input to produce the corresponding $f(x)$ output.

In order for us to formalize this evaluation of limits, we're going to think about some properties of this limit object.

Limit Properties

Activity 1.2.3 Combinations of Functions.

We want to remind ourselves how we can combine functions using different operations, and how we might find outputs based on the different combinations. Our goal is to then think about how this might work with limits: how can we summarize the behavior of combinations of functions around some point?

Let's consider some functions $f(x) = x^2+3$ and $g(x) = x-\dfrac{1}{x}$. We'll say that the domain of both functions is $(0,\infty)$ for our own convenience.

(a) Let's consider the function $h(x) = f(x) + g(x)$. Describe at least two different ways of finding the value of $h(2)$.

(b) If we instead define the function $h(x) = f(x) - g(x)$, how would you describe at least two different ways of finding the value of $h(2)$?

(c) What about a scaled version of one of these functions? If we let $h(x) = 4f(x)$ and $j(x) = \dfrac{g(x)}{3}$, can you describe more than one way to find the value of $h(3)$ and $j(3)$?

(d) You can probably guess where we're going: we're going to define a function that is the product of f and g: $h(x) = f(x) \cdot g(x)$. Describe more than one way of evaluating $h(4)$.

(e) And finally, let's define $h(x) = \dfrac{f(x)}{g(x)}$. Now describe more than one way of finding $h(4)$.

(f) If $h(x) = \dfrac{f(x)}{g(x)}$, then are there any x-values that are in the domain of f and g (the domain is $x > 0$) that $h(x)$ cannot be defined for? Why?

Ok, we can confront this big idea: when we combine functions, we can either evaluate the combination of the functions at some x-value or evaluate each function separately and just combine the answers! Of course, there are some limitations (like when the combination isn't nicely defined because of division by 0 or something else), but this is a good framework to move forward with!

Maybe this activity was obvious for you, but it might not have been! This isn't something that we always think about with functions, even if (deep down) we know it to be true.

A nice extension that we can make is that moving past functions evaluated at a specific x-value towards descriptions of the behavior of functions *around* that specific x-value.

We'll apply this same kind of thinking (combining things by looking at each piece individually first, and then combining the answers together) to limits of combinations of functions.

Theorem 1.2.5 Combinations of Limits.

If $f(x)$ and $g(x)$ are two functions defined at x-values around, but maybe not at, $x = a$ and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then we can evaluate limits of combinations of these functions.

1. Sums: *The limit of the sum of $f(x)$ and $g(x)$ is the sum of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. Differences: *The limit of a difference of $f(x)$ and $g(x)$ is the difference of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$$

4. Products: *The limit of a product of $f(x)$ and $g(x)$ is the product of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$$

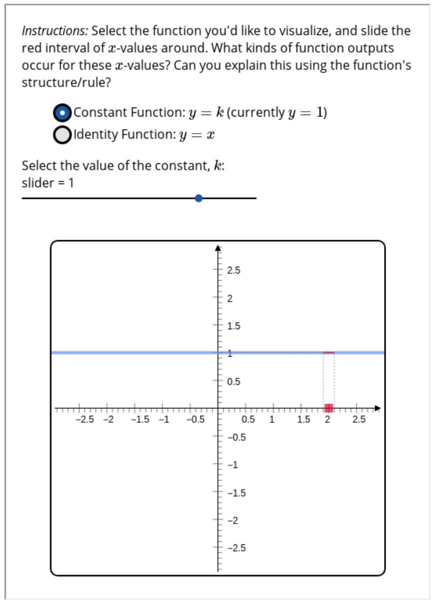
5. Quotients: *The limit of a quotient of $f(x)$ and $g(x)$ is the quotient of the limits of $f(x)$ and $g(x)$ (provided that you do not divide by 0):*

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x \to a} f(x)\right)}{\left(\lim_{x \to a} g(x)\right)} \qquad (\textit{if } \lim_{x \to a} g(x) \neq 0)$$

We can summarize these properties: when we are thinking about our basic operations on functions, we can evaluate limits by just looking at the limits of each component function individually and then piecing those individual limit values back together.

This kind of structural “building-block” behavior is a really important one in mathematics. Whenever we define some new mathematical object, properties like this are typically good ideas for us to check in order to learn more about the object we’ve defined.

Ok, let’s move on. We’re going to turn our attention to something more concrete. We’re going to think of two function types: constant functions and the identity function.



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Theorem 1.2.6 Limits of Two Basic Functions.

Let a be some real number.

1. Limit of a Constant Function: *If k is some real number constant, then:*
$$\lim_{x \to a} k = k$$
2. Limit of the Identity Function:
$$\lim_{x \to a} x = a$$

These two functions might seem pretty simplistic (most functions that we think of are more complicated than these), but we can use these to build more functions!

Activity 1.2.4 Limits of Polynomial Functions.

We're going to use a combination of properties from Theorem 1.2.5 and Theorem 1.2.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x \to 1} f(x)$. First, use the properties from Theorem 1.2.5 to re-write this limit as 4 different limits added or subtracted together.

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.2.6. Evaluate your limits.

(c) Based on the definition of a limit (Definition 1.1.1), we normally say that $\lim_{x \to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x \to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x \to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x \to a} p(x)$ is the same value as $p(a)$.

This leads us to an important result about a whole class of functions: polynomials! We can (finally) evaluate the limit of a polynomial without having to think too carefully about the distinction between the behavior of the function *around* $x = a$ and the behavior of the function *at* $x = a$.

Theorem 1.2.7 Limits of Polynomials.

If $p(x)$ is a polynomial function and a is some real number, then:

$$\lim_{x \to a} p(x) = p(a)$$

This result really just says that polynomials are friendly functions for limits: sure, a limit is really about the behavior of the function outputs around but not at $x = a$, but for polynomial functions, specifically, we can wave our hands and say "Ah, who cares, it's all the same anyways!"

Some questions that we might ask:

1. Are there other functions that have the same nice result about them that Theorem 1.2.7 says for polynomials?
2. Are there some typical functions that we'll work with where this result *doesn't* work (and we actually have to be aware of the behavior around a point instead of at it)?
3. What are we even going to use these limits for, anyways? Why do we care about these?

The answers to these questions will come slowly but surely, and we'll hopefully be able to start using these limits as a tool to think about more interesting and important topics soon: we just need to make sure we're familiar with them first.

Practice Problems

1. Given $\lim_{x \to 3} f(x) = 5$ and $\lim_{x \to 3} g(x) = -2$, evaluate the following limits. If the limit doesn't exist, explain why. Write out a few steps and explanations to justify your work.

(a) $\lim_{x \to 3} \left(6f(x) - \frac{g(x)}{3}\right)$

(b) $\lim_{x \to 3} (f(x))^2 g(x)$

(c) $\lim_{x \to 3^-} \left(\frac{4g(x)}{f(x)} + 3f(x)\right)$

2. Evaluate each limit. Justify your answers.

(a) $\lim_{x\to 0}(4x^3 - 6x^2 + 7x - 10)$

(b) $\lim_{x\to -2}(9 - 3x + x^2 + 3x^3 + x^4)$

(c) $\lim_{t\to a}\left(9t^2 + 3at - 1\right)$ where a is some real number

(d) $\lim_{s\to 1}\left(\frac{5s^2 - 6s + 1}{s^2 - 4}\right)$

(e) $\lim_{t\to 2}\left(\frac{4t - 5}{6 + t^2}\right)$

(f) $\lim_{z\to 2}\frac{z^2 - 4}{z + 2}$

3. Evaluate each limit. If the limit does not exist, explain why not.

(a) Let $f(x) = \begin{cases} 3x - 2 & \text{if } x < -1 \\ x^2 + x - 4 & \text{if } x \geq -1 \end{cases}.$

Evaluate $\lim_{x\to -1^-} f(x)$, $\lim_{x\to -1^+} f(x)$, and $\lim_{x\to -1} f(x)$.

(b) Let $g(x) = \begin{cases} 6 + x & \text{if } x < -3 \\ 6 & \text{if } x = -3 \\ \frac{2x+15}{3} & \text{if } x > -3 \end{cases}.$

Evaluate $\lim_{x\to -3^-} g(x)$, $\lim_{x\to -3^+} g(x)$, and $\lim_{x\to -3} g(x)$.

(c) Let $s(t) = \begin{cases} t^2 + 1 & \text{if } t < 1 \\ 1 - t^2 & \text{if } t \geq 1 \end{cases}.$

Evaluate $\lim_{t\to 1^-} s(t)$, $\lim_{t\to 1^+} s(t)$, and $\lim_{t\to 1} s(t)$

(d) Let $r(\theta) = \begin{cases} \theta - 1 + \theta^3 & \text{if } \theta < 2 \\ 3\theta + 2 & \text{if } \theta \geq 2 \end{cases}.$

Evaluate $\lim_{\theta\to 2^-} r(\theta)$, $\lim_{\theta\to 2^+} r(\theta)$, and $\lim_{\theta\to 2} r(\theta)$.

1.3 First Indeterminate Forms

We're going to really focus on one of the main aspects of a limit in this next activity. The activity should serve two purposes:

1. We'll review a really important property or aspect of what a limit is!
2. We'll look at this thing that we already know from a slightly different perspective (or maybe just a specific perspective), and we'll discover a really important and helpful result from it!

Activity 1.3.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x\to 1} f(x)$.

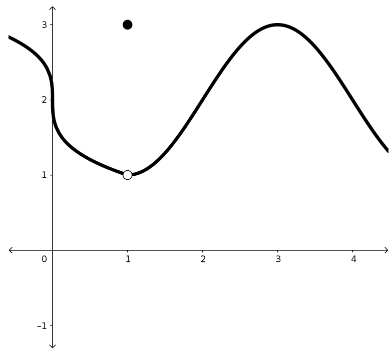


Figure 1.3.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x\to 1} g(x)$.

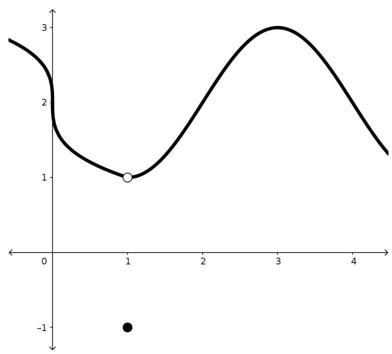


Figure 1.3.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{x\to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{x\to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \le 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

This is an important thing to notice: we can change our function by changing the value of the function at $x = a$ without changing the value of the limit as $x \to a$.

Theorem 1.3.3 Limits of (Slightly) Different Functions.

If $f(x)$ and $g(x)$ are two functions defined at x-values around a (but maybe not at $x = a$ itself) with $f(x) = g(x)$ for the x-values around a but with $f(a) \neq g(a)$ then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$, if the limits exist.

Why will this be helpful? At the end of Section 1.2 we found that some functions (polynomials) are great: the limit of these functions is the same as the function value at the point (Theorem 1.2.7). This is a special case, and many functions won't be so nice to work with. But maybe we could use Theorem 1.3.3 to swap out an annoying-to-work-with function for a nice-to-work-with function!

A First Introduction to Indeterminate Forms

So before we begin applying this result, we will focus on a situation where we need it. We're going to do something strange: define a situation before we experience it.

Definition 1.3.4 Indeterminate Form.

We say that a limit has an **indeterminate form** if the general structure of the limit could take on any different value, or not exist, depending on the specific circumstances.
For instance, if $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, then we say that the limit $\lim\limits_{x \to a} \left(\dfrac{f(x)}{g(x)} \right)$ has an indeterminate form. We typically denote this using the informal symbol $\frac{0}{0}$, as in:

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) \stackrel{?}{\to} \frac{0}{0}.$$

Ok, so why do we need this definition? What does the word "indeterminate" even mean, here?

We're going to see in this next activity that this kind of $\dfrac{0}{0}$ form of a limit can actually lead to very different behavior: we call it indeterminate because we cannot determine, based solely on the form $\dfrac{0}{0}$, what the limit is or even if it will exist.

Activity 1.3.2

(a) Were going to evaluate $\lim\limits_{x \to 3} \left(\dfrac{x^2 - 7x + 12}{x - 3} \right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?

- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3) - 4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x\to -2}\left(\dfrac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim_{x\to 5} f(x) \overset{?}{\to} \dfrac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

This is great: we're applying Theorem 1.3.3 because in each algebraic manipulation, we change the domain of the function by removing some factor from the denominator!

These three algebra tricks are all we'll look at for now. In reality, there are plenty of little tricky manipulations we can use to slightly change functions, but if we focused on trying to build one for every situation we could run into, we'd spend the rest of this text just outlining different algebra tricks for different situations.

Algebra Tricks for Indeterminate Forms.

For limits with the $\frac{0}{0}$ indeterminate form, we can apply the following algebraic tricks:

1. *Factor and cancel:* This works well when we have polynomials divided by polynomials.
2. *Conjugates:* This works well when we have some difference of square roots in the numerator or denominator.
3. *Combine fractions with common denominators:* This works well when we have some subtraction with fractions inside of a numerator or denominator of another fraction.

What if There Is No Algebra Trick?

We've seen some nice examples above where we were able to use some algebra to manipulate functions in such was as to force some shared factor in the numerator and denominator into revealing itself. From there, we were able to apply Theorem 1.3.3 and swap out our problematic function with a new one, knowing that the limit would be the same.

But what if we can't do that? What if the specific structure of the function seems *resistent* somehow to our attempts at wielding algebra?

This happens a lot, and we'll investigate some more of those types of limits in Section **??**. For now, though, let's look at a very famous limit and reason our way through the indeterminate form.

Activity 1.3.3

Let's consider a new limit:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

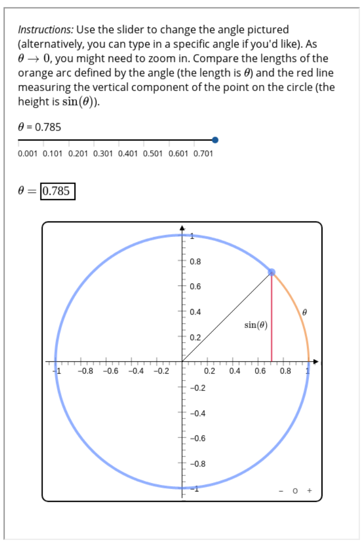
(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



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(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\displaystyle\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$?

Practice Problems

1. Explain, in your own words, why Theorem 1.3.3 is true.

2. Consider the following limit:

$$\lim_{x \to 3} \left(\frac{2x^2 - 5x - 3}{x^2 + 5x - 24} \right).$$

(a) Confirm that this limit has an indeterminate form.

(b) Evaluate the limit.

(c) When you were evaluating the limit, you likely "cancelled" a factor of $(x-3)$ from the numerator and denominator. Why might you have known, before factoring anything, that $(x-3)$ would be a factor shared in the numerator and denominator?

For instance, how did you know it wasn't going to be $(x-4)$ or $(x+1)$ or something else?

3. Consider the following limit:

$$\lim_{x\to 4}\left(\frac{\frac{2}{x+1}-\frac{x-2}{5}}{x-4}\right).$$

(a) Confirm that this limit has an indeterminate form.

(b) Evaluate the limit.

(c) When you were evaluating the limit, you likely "cancelled" a factor of $(x-4)$ from the numerator and denominator. Why might you have known, before factoring anything, that $(x-4)$ would be a factor shared in the numerator and denominator?

For instance, how did you know it wasn't going to be $(x-3)$ or $(x+1)$ or something else?

4. Use the algebra tricks from Algebra Tricks for Indeterminate Forms to evaluate each limit.

(a) $\displaystyle\lim_{x\to 2}\left(\frac{x^2-4}{x^2+3x-10}\right)$

(b) $\displaystyle\lim_{x\to -1}\left(\frac{2x^2+x-1}{x+1}\right)$

(c) $\displaystyle\lim_{x\to 4}\left(\frac{4-x}{x-4}\right)$

(d) $\displaystyle\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right)$

(e) $\displaystyle\lim_{x\to 2}\left(\frac{\sqrt{x^2-1}-\sqrt{x+1}}{x-2}\right)$

(f) $\displaystyle\lim_{h\to 0}\left(\frac{\sqrt{a+h}-\sqrt{a}}{h}\right)$ where a is some non-negative real number

(g) $\displaystyle\lim_{t\to 3}\left(\frac{\frac{1}{t}-\frac{1}{3}}{t-3}\right)$

(h) $\displaystyle\lim_{t\to 6}\left(\frac{\frac{t+1}{t-1}-\frac{7}{11-t}}{t-6}\right)$

(i) $\displaystyle\lim_{h\to 0}\left(\frac{\frac{1}{a+h}-\frac{1}{a}}{h}\right)$ where a is some real number

1.4 Limits Involving Infinity

Two types of limits involving infinity. In both cases, we'll mostly just consider what happens when we divide by small things and what happens when we divide by big things. We can summarize this here, though:

Fractions with small denominators are big, and fractions with big denominators are small.

Infinite Limits

When we talked about indeterminate forms in Section 1.3, we noticed that the function value wasn't defined, since we divided by 0. Specifically, we were looking at the $\frac{0}{0}$ form in the limit. What happens in other cases when we divide by 0 with a *nonzero* numerator?

Activity 1.4.1 What Happens When We Divide by 0?

First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.3), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

So we want to formalize this kind of behavior. We know that the limits that we're looking at don't exist (since there isn't a single, real number that the function outputs are all close to), but there is definitely some sort of consistent behavior that we'd like to signify.

Definition 1.4.1 Infinite Limit.

We say that a function $f(x)$ has an **infinite limit** at a if $f(x)$ is arbitrarily large (positive or negative) when x is sufficiently close to, but not equal to, $x = a$.
We would then say, depending on the sign of the values of $f(x)$, that:

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad\qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

If the sign of both one-sided limits are the same, we can say that $\lim_{x \to a} f(x) = \pm\infty$ (depending on the sign), but it is helpful to note that, by the definition of the Limit of a Function, this limit does not exist, since $f(x)$ is not arbitrarily close to a single real number.

We know, from our adage "*Fractions with small denominators are big, and fractions with big denominators are small,*" that this type of behavior happens when the denominator is tiny compared to the numerator. We can summarize this:

Theorem 1.4.2 Dividing by 0 in a Limit.

If $f(x) = \dfrac{g(x)}{h(x)}$ with $\lim_{x \to a} g(x) \neq 0$ and $\lim_{x \to a} h(x) = 0$, then $f(x)$ has an Infinite Limit at a. We will often denote this behavior as:

$$\lim_{x \to a} f(x) \overset{?}{\to} \frac{\#}{0}$$

where $\#$ is meant to be some shorthand representation of a non-zero limit in the numerator (often, but not necessarily, some real number).

These kinds of limits are great, in that they're consistent to identify. We just look for this tiny denominator compared to the numerator, and go from there. We also know a lot about these types of limits, and can summarize this below.

Evaluating Infinite Limits.

Once we know that $\lim_{x \to a} f(x) \overset{?}{\to} \dfrac{\#}{0}$, we know a bunch of information right away!

- This limit doesn't exist.
- The function $f(x)$ has a vertical asymptote at $x = a$, causing these unbounded y-values near $x = a$.
- The one sided limits *must* be either ∞ or $-\infty$.

- We only need to focus on the sign of the one sided limits! And signs of products and quotients are easy to follow.

So a pretty typical process is to factor as much as we can, and check the sign of each factor (in a numerator or denominator) as $x \to a^-$ and $x \to a^+$. From there, we can find the sign of $f(x)$ in both of those cases, which will tell us the one-sided limit.

Example 1.4.3

For each function, find the relevant one-sided limits at the input-value mentioned. If you can use a two-sided limit statement to discuss the behavior of the function around this input-value, then do so.

(a) $\left(\dfrac{2x^2-5x+1}{x^2+8x+16}\right)$ and $x=-4$

Solution. Let's first factor the denominator: we want to see the factor $(x+4)$, where we divide by 0.

$$\left(\frac{2x^2-5x+1}{x^2+8x+16}\right) = \left(\frac{2x^2-5x+1}{(x+4)^2}\right)$$

Now, when $x \to 4$, we have $\left(\dfrac{2x^2-5x+1}{(x+4)^2}\right) \stackrel{?}{\to} \frac{13}{0}$. This is our $\frac{\#}{0}$ form that tells us we have an infinite limit. We're going to look at the one-sided limits, but let's notice something:

- For x-values close to 4, we expect that the numerator will be close to 13. No matter what "side" of 13 it is, it's still going to be positive.
- For x-values close to 4, we expect the denominator to be close to 0. Depending on what side of 0 it is, it could be positive or negative.

So in our one-sided limits, we know that these both will be infinite limits (the function values will either approach $-\infty$ or ∞). The only difference in these is the sign. So let's check the signs!

First, we'll consider $x \to -4^-$. That's $x < -4$. We know the numerator is positive, since it is close to 13, and we also know that the factor $x+4<0$.

$$\begin{aligned}
\lim_{x\to-4^-}\left(\frac{2x^2-5x+1}{(x+4)^2}\right) &= \lim_{x\to-4^-}\left(\frac{\overbrace{2x^2-5x+1}^{\approx 13}}{(\underbrace{x+4}_{\approx 0})^2}\right) \\
&= \frac{\oplus}{(\ominus)^2} \\
&= \oplus
\end{aligned}$$

So we know that $\lim\limits_{x\to-4^-}\left(\dfrac{2x^2-5x+1}{(x+4)^2}\right)$ is an infinite limit, and we know that it is positive.

$$\lim_{x\to-4^-}\left(\frac{2x^2-5x+1}{(x+4)^2}\right) = \infty$$

Now we can check the signs of these factors when $x \to -4^+$ (and so $x > -4$). The numerator is still close to 13, and so still negative. Now, $(x+4)$ is positive, and is still being squared.

$$\begin{aligned}
\lim_{x\to -4^+}\left(\frac{2x^2-5x+1}{(x+4)^2}\right) &= \lim_{x\to -4^+}\left(\frac{\overbrace{2x^2-5x+1}^{\approx 13}}{(\underbrace{x+4}_{\approx 0})^2}\right)\\
&= \frac{\oplus}{(\oplus)^2}\\
&= \oplus
\end{aligned}$$

Since this is an infinite limit and is also positive, we know that:

$$\lim_{x\to -4^+}\left(\frac{2x^2-5x+1}{(x+4)^2}\right) = \infty.$$

Since the function approaches ∞ on both sides, we can say that

$$\lim_{x\to -4}\left(\frac{2x^2-5x+1}{(x+4)^2}\right) = \infty,$$

but we know that this limit does not exist (since the function values are not close to a single real number).

(b) $\left(\dfrac{4x^2-x^5}{x^2-4x+3}\right)$ and $x=1$

Solution. We can start this in a similar way: factor the denominator to see the places where we divide by 0:

$$\left(\frac{4x^2-x^5}{x^2-4x+3}\right) = \left(\frac{4x^2-x^5}{(x-3)(x-1)}\right).$$

Now, when we think about the limit as $x \to 1^-$, we're thinking about $x < 1$. We can check the signs, again! The numerator has $4x^2 - x^5 \to 3$, and so for x-values slightly smaller than 1, this numerator is close to 3, and so positive. Similarly, $(x-3) \to -2$, and so for x-values close to but less than 1, this is negative. Then, we can see that $(x-1) \to 0$. This is the part that gives us the $\frac{\#}{0}$ form.

For $x < 1$, we have $(x-1) < 0$, and so this is negative.

$$\lim_{x\to 1^-}\left(\frac{\overbrace{4x^2-x^5}^{\oplus}}{(\underbrace{x-3}_{\ominus})(\underbrace{x-1}_{\ominus})}\right) \to \oplus$$

And so this limit is a positive infinite limit:

$$\lim_{x\to 1^-}\left(\frac{4x^2-x^5}{(x-3)(x-1)}\right) = \infty.$$

Now we can similarly check the signs when $x \to 1^+$, which is when $x > 1$. Note that, since x is still close to 1, the numerator and the factor $(x-3)$ will retain their sign. But, for the factor $(x-1)$, we get $(x-1) > 0$.

$$\lim_{x\to 1^-} \left(\frac{\overbrace{4x^2 - x^5}^{\oplus}}{\underbrace{(x-3)}_{\ominus}\underbrace{(x-1)}_{\oplus}} \right) \to \ominus$$

And so this limit is a negative infinite limit:

$$\lim_{x\to 1^+} \left(\frac{4x^2 - x^5}{(x-3)(x-1)} \right) = -\infty.$$

(c) $\sec(\theta)$ and $\theta = \dfrac{\pi}{2}$

Solution. We can think of $\sec(\theta)$ as a reciprocal: $\dfrac{1}{\cos(\theta)}$. Now, we can see that $\cos\left(\frac{\pi}{2}\right) = 0$, hence this is an infinite limit.

Let's visualize $\cos\left(\frac{\pi}{2}\right)$, so that we can tell the sign of this denominator when θ is on either side of $\frac{\pi}{2}$.

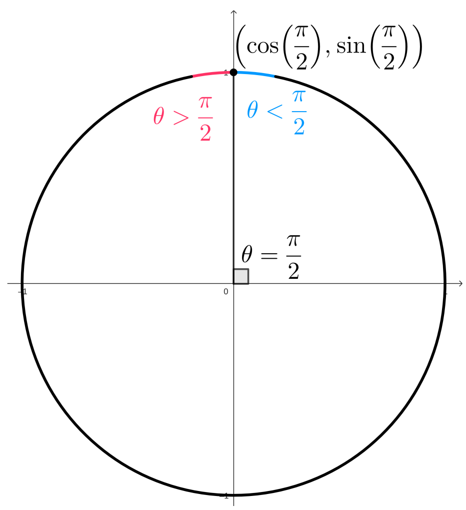


Figure 1.4.4

We can see that for $\theta < \frac{\pi}{2}$, we are looking at a point in the first quadrant with a positive horizontal component. So, in this case, $\cos(\theta) > 0$.

For the case when $\theta > \frac{\pi}{2}$, though, we are looking in the second quadrant with a negative horizontal component. So we see that $\cos(\theta) < 0$.

All of this to say:

$$\lim_{\theta\to \frac{\pi}{2}^-} \sec(\theta) = \infty$$

$$\lim_{\theta\to \frac{\pi}{2}^+} \sec(\theta) = -\infty$$

These are limits where $f(x) \to \pm\infty$ when $x \to a$ where a is a real number. What about the other way around? Is there a case where $x \to \pm\infty$, and what would we be looking at in the behavior of $f(x)$?

End Behavior Limits

Activity 1.4.2 What Happens When We Divide by Infinity?

Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.
When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

Definition 1.4.5 Limit at Infinity.

If $f(x)$ is defined for all large and positive x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently large, then we say:

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ is defined for all large and negative x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently negative, then we say:

$$\lim_{x\to-\infty} f(x) = L.$$

In the case that $f(x)$ has a **limit at infinity** that exists, then we say $f(x)$ has a horizontal asymptote at $y = L$.
Lastly, if $f(x)$ is defined for all large and positive (or negative) x-values and $f(x)$ gets arbitrarily large and positive (or negative) when x gets sufficiently large (or negative), then we could say:

$$\lim_{x\to-\infty} f(x) = \pm\infty \text{ or } \lim_{x\to\infty} f(x) = \pm\infty.$$

Because the primary focus for limits at infinity is the end behavior of a function, we will often refer to these limits as **end behavior limits**.

We're going to think about end-behavior of many types of functions, but we want to start with some small examples and build from there. So we're going to start by thinking about power functions. We're going to think about power functions in two ways:

1. Power functions in the form of x^p (with $p > 0$). It shouldn't be too much work to convince yourself that these functions always have limits where, as $x \to \pm\infty$, $x^p \to \pm\infty$ as well.
2. Reciprocal power functions, in the form $\dfrac{1}{x^p}$ (still with $p > 0$). We already should have an idea of what's going to happen in these, based on Activity 1.4.2.

Theorem 1.4.6 End Behavior of Reciprocal Power Functions.

If p is a positive real number, then:

$$\lim_{x\to\infty}\left(\frac{1}{x^p}\right) = 0 \textit{ and } \lim_{x\to-\infty}\left(\frac{1}{x^p}\right) = 0.$$

Our last result is one that you might already know, but we'll provide some more justification for this. We can use our knowledge of end-behavior limits for both type of power functions to think about the end-behavior limits of polynomials in general!

Theorem 1.4.7 Polynomial End Behavior Limits.

For some polynomial function:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

with n a positive integer (the degree) and all of the coefficients

$a_0, a_1, ..., a_n$ real numbers (with $a_n \neq 0$), then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

That is, the leading term (the term with the highest exponent) defines the end behavior for the whole polynomial function.

Proof.

Consider the polynomial function:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

where n is some integer and a_k is a real number for $k = 0, 1, 2, ..., n$. For simplicity, we will consider only the limit as $x \to \infty$, but we could easily repeat this exact proof for the case where $x \to -\infty$.

Before we consider this limit, we can factor out x^n, the variable with the highest exponent:

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 \\ &= x^n \left(\frac{a^n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + ... + \frac{a_2x^2}{x^n} + \frac{a_1x}{x^n} + \frac{a_0}{x^n}\right) \\ &= x^n \left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

Now consider the limit of this product:

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \lim_{x\to\infty} x^n \left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

We can see that in the second limit, we have a single constant term, a_n, followed by reciprocal power functions. Then, due to Theorem 1.4.6, we know that the second limit will by a_n, since the reciprocal power functions will all approach 0.

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n + 0 + ... + 0 + 0 + 0) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n) \\ &= \lim_{x\to\infty} a_n x^n \end{aligned}$$

And so $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ as we claimed.

Instead of spending our time thinking about these polynomial end-behavior limits too specifically (since we might already be familiar with this result), and just focus on using these polynomials in the middle of larger problems.

Example 1.4.8

For each function, find the limits as $x \to \infty$ and $x \to -\infty$.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$

Hint 1. You can think about the limit in the numerator and the limit in the denominator. Based on Theorem 1.4.7, which terms will be the ones to dictate the behavior of the numerator and denominator?

Hint 2. What happens when you think about just those dominant terms in the numerator and denominator and reduce the fraction? What is left?

Solution. We'll start with the same kind of factoring that is used in the proof of Theorem 1.4.7.

$$\begin{aligned}\left(\frac{2x^2-5x+1}{x^2+8x+16}\right) &= \left(\frac{2x^2}{x^2}\right)\left(\frac{1-\frac{5}{2x}+\frac{1}{2x^2}}{1+\frac{8}{x}+\frac{16}{x^2}}\right)\\ &= 2\left(\frac{1-\frac{5}{2x}+\frac{1}{2x^2}}{1+\frac{8}{x}+\frac{16}{x^2}}\right)\end{aligned}$$

Now we can apply the limits as $x \to \infty$ and $x \to -\infty$, since the reciprocal power functions will all $\to 0$.

$$\lim_{x\to\infty} 2\left(\frac{1-\frac{5}{2x}+\frac{1}{2x^2}}{1+\frac{8}{x}+\frac{16}{x^2}}\right) = 2(1) = 2$$
$$\lim_{x\to-\infty} 2\left(\frac{1-\frac{5}{2x}+\frac{1}{2x^2}}{1+\frac{8}{x}+\frac{16}{x^2}}\right) = 2(1) = 2$$

Alternatively, we could have done the following:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{2x^2-5x+1}{x^2+8x+16}\right) &= \frac{\lim_{x\to\infty}(2x^2-5x+1)}{\lim_{x\to\infty}(x^2+8x+16)}\\ &= \frac{\lim_{x\to\infty} 2x^2}{\lim_{x\to\infty} x^2}\\ &= \lim_{x\to\infty}\frac{2x^2}{x^2}\\ &= \lim_{x\to\infty} 2 = 2\end{aligned}$$

The same process could be used to show that:

$$lim_{x\to-\infty}\left(\frac{2x^2-5x+1}{x^2+8x+16}\right) = 2.$$

(b) $\left(\dfrac{4x^2-x^5}{x^2-4x+3}\right)$

Hint. Be careful about which term (in the numerator) will persist!

(c) $\dfrac{|x|}{3x}$

Hint. It might be helpful to remind yourself of the definition of the absolute value function:

$$|x| = \begin{cases}-x & \text{when } x < 0\\ x & \text{when } x \geq 0\end{cases}.$$

This means you can replace $|x|$ with either x or $-x$ in the limits as $x \to \infty$ and $x \to -\infty$.

Practice Problems

1. Use the graph of the function $f(x)$ below to determine each limit.

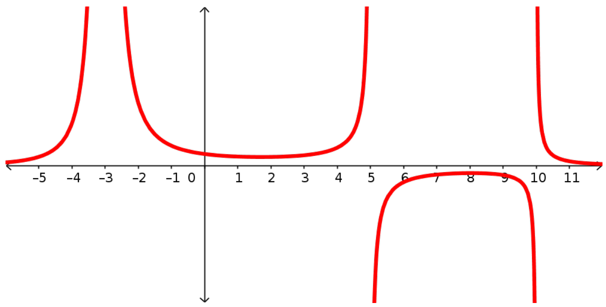


Figure 1.4.9 The function $f(x)$.

(a) $\lim\limits_{x\to -3^-} f(x)$

(b) $\lim\limits_{x\to -3^+} f(x)$

(c) $\lim\limits_{x\to -3} f(x)$

(d) $\lim\limits_{x\to 5^-} f(x)$

(e) $\lim\limits_{x\to 5^+} f(x)$

(f) $\lim\limits_{x\to 5} f(x)$

(g) $\lim\limits_{x\to 10^-} f(x)$

(h) $\lim\limits_{x\to 10^+} f(x)$

(i) $\lim\limits_{x\to 10} f(x)$

2. Use the graph of the function $g(x)$ below to determine each limit.

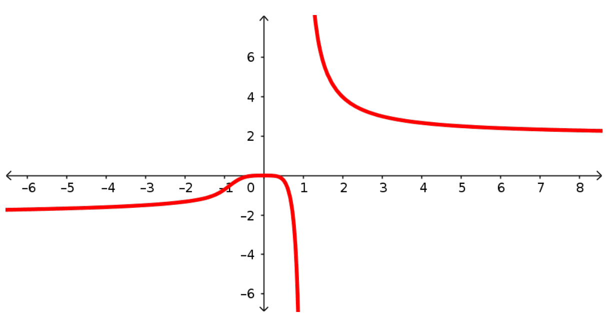


Figure 1.4.10 The function $g(x)$.

(a) $\lim\limits_{x\to -\infty} g(x)$

(b) $\lim\limits_{x\to \infty} g(x)$

3. Use the graph of the function $h(x)$ below to determine each limit.

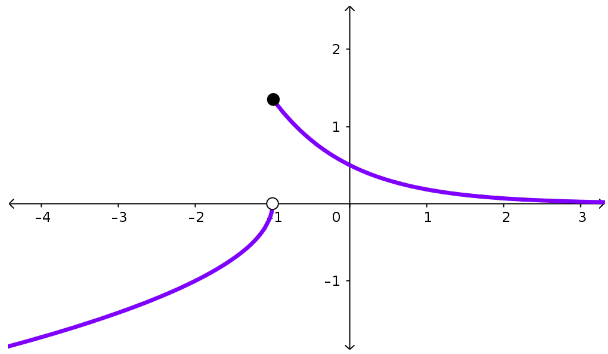


Figure 1.4.11 The function $h(x)$.

(a) $\lim_{x\to-\infty} h(x)$

(b) $\lim_{x\to\infty} h(x)$

4. For each limit, fill in the table in a way that determines the limit, and report your answer.

(a) $\lim_{x\to 4^-} \left(\dfrac{3x}{x-4}\right)$

x	_____	_____	_____	4
$\frac{3x}{x-4}$	_____	_____	_____	does not exist

(b) $\lim_{x\to 4^+} \left(\dfrac{3x}{x-4}\right)$

x	4	_____	_____	_____
$\frac{3x}{x-4}$	does not exist	_____	_____	_____

(c) $\lim_{x\to 1} \left(\dfrac{x-3}{(2x-4)(x-1)^2}\right)$

x	_____	_____	_____	1	_____	_____	_____
$\frac{x-3}{(2x-4)(x-1)^2}$	_____	_____	_____	does not exist	_____	_____	_____

1.5 The Squeeze Theorem

We won't get enough time to spend thinking about all of the possible techniques that we could use to evaluate limits, but in this section we'll investigate one more.

Here, we'll introduce a new limit involving a type of function that we've not used in limits so far: trigonometric functions.

Weird Functions, Weird Behavior

Ok, trigonometric functions aren't actually weird. But we want to look at a function that is slightly more complicated than the ones we've looked at so far.

Activity 1.5.1 A Weird End Behavior Limit.

In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\underline{\qquad\qquad} \leq \sin(x) \leq \underline{\qquad\qquad}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2+1} \leq \underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } h(x)}$$

(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim\limits_{x\to\infty}\left(\dfrac{\sin^2(x)}{x^2+1}\right)$?

Squeeze Theorem

This strategy is a really nice one to use when we know the behavior of some well-behaved "bounding" functions. We can try to off-load the task of summarizing the behavior of a strangely behaved function to these bounding functions, and follow them! As long as they approach each other, than the strangely behaved function has to have the same behavior.

Let's formalize this result carefully.

Theorem 1.5.1 The Squeeze Theorem.

For some functions $f(x)$, $g(x)$, and $h(x)$ which are all defined and ordered $f(x) \leq g(x) \leq h(x)$ for x-values near $x = a$ (but not necessarily at $x = a$ itself), and for some real number L, if we know that

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then we also know that $\lim_{x\to a} g(x) = L$.

Sometimes this theorem is called the Sandwich Theorem, since the upper bounding function and the lower bounding function act as the slices of bread, while the strangely behaved middle function acts like the toppings on the sandwich.

We choose not to use that naming convention in this text in order to preserve the flexibility of the definition of a sandwich. An open-faced sandwich is still a sandwich (in the opinion of the textbook author), but this result clearly doesn't hold when we only have one bounding function.

This theorem can be difficult to use, primarily because building the bounds for a function is difficult. In Activity 1.5.1, we were able to build the boundary functions by simply thinking about the bounds on the $\sin(x)$. This worked well, but we were only able to do this because of our familiarity with this function. With other functions, these bounds are harder to just *come up with*. This is especially true in that we need the bounds to accomplish multiple things at once:

- We need them to be ordered correctly with regard to the function we care about: one above it and one below it.
- We need the limits of these functions to be things we can actually evaluate! This is the whole point: we use these (hopefully easier) limits to evaluate the (probably hard) limit that we're interested in.
- We need the limits of these functions to be the same as $x \to a$, otherwise we're not certain about where our function actually is.

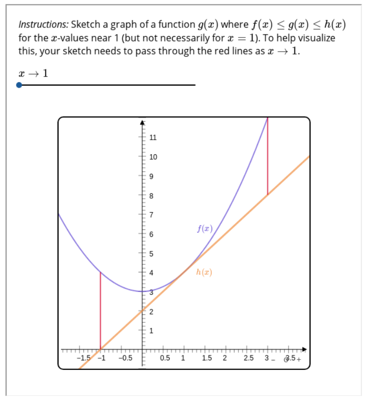
In practice, we'll try to build any bounding functions with some assistance, or start with bounding functions already stated.

Let's see another way of thinking about this result using our graphical intuition.

Activity 1.5.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.



Standalone Embed

(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x\to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x\to 3} g(x)$ and $\lim_{x\to 0} g(x)$?

Do we know that these limits exist? If they do, do we have information about their values?

Practice Problems

1. For each of the following statements, discuss the possible existence of the limit in question. Explain in detail how we might know whether the limit exists or not, or why it is impossible to tell.

(a) We want to know about $\lim_{x\to 4} f(x)$, and we know that $a(x) \le f(x) \le b(x)$ for all x-values, with $a(4) = 3$ and $b(4) = 3$.

(b) We want to know about $\lim_{x\to 0} f(x)$, and we know that $\lim_{x\to 0} g(x) = 5$ for the function $g(x) \ge f(x)$ for x-values around 0.

(c) We want to know about $\lim_{x\to -1} f(x)$, and we know that $a(x) \le f(x) \le b(x)$ for x-values around -1, with $\lim_{x\to -1} a(x) = 9$ and $\lim_{x\to -1} b(x) = 9$.

2. We want to use the Squeeze Theorem to evaluate $\lim_{x\to 0} \left(x^2 \sin(\pi/x)\right)$.

(a) Explain why we know that $-1 \le \sin(\pi/x) \le 1$ for any non-zero x-values. Why does this inequality not hold for $x = 0$?

(b) Use the inequality from (a) to build an inequality of functions $f(x) \le x^2 \sin(\pi/x) \le g(x)$ for non-zero x-values.

(c) With these functions $f(x)$ and $g(x)$ from (b), find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.

(d) Explain how the Squeeze Theorem tells us what the value of $\lim_{x\to 0} x^2 \sin(\pi/x)$ is.

1.6 Continuity and the Intermediate Value Theorem

What does it mean for a function to be continuous?

This might be a familiar concept to you, and we all probably have some idea of what the word "continuous" means (or should mean), both in a colloquial context and in a math-specific context. Instead of focusing on some of the traditional ideas like "can you draw the graph without picking up your pencil", we're going to formalize the idea of continuity a bit.

And the first thing that we'll have to do is shift our intuition from what continuity is describing. We won't (at least at first) use continuity as a classification of functions, as a whole. Instead, we're going to think of continuity as a description of some "local" behavior.

Continuity as Connectedness

We're going to try to think about what we want to describe when we define continuity. The description of "drawing a graph without picking up a pencil" is a global property, and also a pretty vague and ambiguous one.

Activity 1.6.1 Classification and Continuity.

Let's consider the following functions, graphed below.

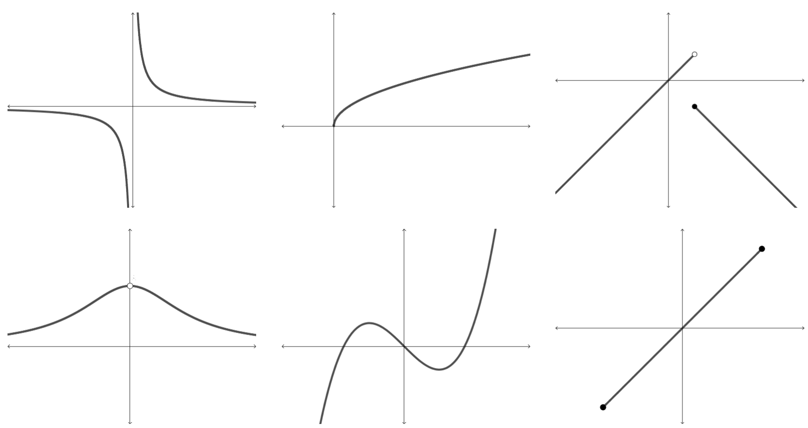


Figure 1.6.1 A variety of graphs for us to use to think about continuity.

(a) Can you point out any points on the functions above that seem like the functions might not be continuous? Note that we're not classifying each function as "continuous" or "not continuous"!

(b) We want to build towards a definition of continuity using "connectedness" as the key: a function is continuous at a point if it is connected to itself. How does that work when we think about function values and limits? Use the graphs above and the points you looked at to help!

There are definitely some unknowns here. One of the questions that students ask a lot during an investigation like this is what we do about end-points. Is a function continuous at a closed ending-point in its domain? Is it connected to itself? To answer questions like this, we should try to write out what we mean by connectedness.