Discover Calculus - Activity Book

All of the Activities from Each Chapter

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Chapter 1

Limits

1.1 The Definition of the Limit

Activity 1.1.1 Close or Not?

We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Activity 1.1.2 Approximating Limits.

For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the graph of the function $f(x)$ below.

Figure 1.1.1

(b) Approximate $\lim_{x \to 2} g(x)$ using the graph of the function $g(x)$ below.

Figure 1.1.2

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x \to -1} h(x)$
- $\lim_{x \to 0} h(x)$
- $\lim_{x \to 2} h(x)$

Figure 1.1.3

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.1.3 Approximating Limits Numerically.

For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.1.4

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.1.5

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.1.6

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.2 Evaluating Limits

Activity 1.2.1 From Estimating to Evaluating Limits (Part 1).

Let's consider the following graphs of functions $f(x)$ and $g(x)$.

Figure 1.2.1 Graph of the function $f(x)$.

Figure 1.2.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$
- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 2^-} g(x)$
- $\lim_{x \to 2^+} g(x)$
- $\lim_{x \to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.2.2 From Estimating to Evaluating Limits (Part 2).

Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

Figure 1.2.3 Graph of the function $f(x)$.

Figure 1.2.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 1^-} f(x)$
- $\lim\limits_{x \to 1^+} f(x)$
- $\lim\limits_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 2^-} g(x)$
- $\lim\limits_{x \to 2^+} g(x)$
- $\lim\limits_{x \to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \le 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

Activity 1.2.3 Combinations of Functions.

We want to remind ourselves how we can combine functions using different operations, and how we might find outputs based on the different combinations. Our goal is to then think about how this might work with limits: how can we summarize the behavior of combinations of functions around some point?

Let's consider some functions $f(x) = x^2 + 3$ and $g(x) = x - \dfrac{1}{x}$. We'll say that the domain of both functions is $(0, \infty)$ for our own convenience.

(a) Let's consider the function $h(x) = f(x) + g(x)$. Describe at least two different ways of finding the value of $h(2)$.

(b) If we instead define the function $h(x) = f(x) - g(x)$, how would you describe at least two different ways of finding the value of $h(2)$?

(c) What about a scaled version of one of these functions? If we let $h(x) = 4f(x)$ and $j(x) = \dfrac{g(x)}{3}$, can you describe more than one way to find the value of $h(3)$ and $j(3)$?

(d) You can probably guess where we're going: we're going to define a function that is the product of f and g: $h(x) = f(x) \cdot g(x)$. Describe more than one way of evaluating $h(4)$.

(e) And finally, let's define $h(x) = \dfrac{f(x)}{g(x)}$. Now describe more than one way of finding $h(4)$.

(f) If $h(x) = \dfrac{f(x)}{g(x)}$, then are there any x-values that are in the domain of f and g (the domain is $x > 0$) that $h(x)$ cannot be defined for? Why?

Ok, we can confront this big idea: when we combine functions, we can either evaluate the combination of the functions at some x-value or evaluate each function separately and just combine the answers! Of course, there are some limitations (like when the combination isn't nicely defined because of division by 0 or something else), but this is a good framework to move forward with!

Activity 1.2.4 Limits of Polynomial Functions.

We're going to use a combination of properties from Theorem 1.2.5 and Theorem 1.2.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x \to 1} f(x)$. First, use the properties from Theorem 1.2.5 to re-write this limit as 4 different limits added or subtracted together.

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.2.6. Evaluate your limits.

(c) Based on the definition of a limit (Definition 1.1.1), we normally say that $\lim_{x \to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x \to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x \to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x \to a} p(x)$ is the same value as $p(a)$.

1.3 First Indeterminate Forms

Activity 1.3.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x \to 1} f(x)$.

Figure 1.3.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x \to 1} g(x)$.

Figure 1.3.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{t \to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{t \to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Activity 1.3.2

(a) Were going to evaluate $\lim\limits_{x\to 3}\left(\dfrac{x^2 - 7x + 12}{x - 3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.

- Now we want to find a new function that is equivalent to $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?

- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.

- This function is equivalent to $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim\limits_{x\to 1}\left(\dfrac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.

- Now we want a new function that is equivalent to $g(x) = \dfrac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2 + 3} + 2)$. We'll call this the "conjugate" of the numerator.

- In your multiplication, confirm that $(\sqrt{x^2 + 3} - 2)(\sqrt{x^2 + 3} + 2) = (x^2 + 3) - 4$.

- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.

- This function is equivalent to $g(x) = \dfrac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim\limits_{x\to -2}\left(\dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.

- Again, we want a new function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."

- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.

- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim\limits_{x \to 5} f(x) \overset{?}{\to} \dfrac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

Activity 1.3.3

Let's consider a new limit:
$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.

(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim_{\theta \to 0} \dfrac{\sin(\theta)}{\theta}$?

1.4 Limits Involving Infinity

Activity 1.4.1 What Happens When We Divide by 0?

First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.3), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Activity 1.4.2 What Happens When We Divide by Infinity?

Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

1.5 The Squeeze Theorem

Activity 1.5.1 A Weird End Behavior Limit.

In this activity, we're going to find the following limit:

$$\lim_{x \to \infty} \left(\frac{\sin^2(x)}{x^2 + 1} \right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x \to \infty} \left(\frac{1}{x^2 + 1} \right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\underline{\qquad\qquad} \leq \sin(x) \leq \underline{\qquad\qquad}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2 + 1}$. Right now we have two bits of information:

- We know $\lim\limits_{x \to \infty} \left(\dfrac{1}{x^2 + 1} \right)$.

- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1} \right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\underline{\qquad\qquad}}{x^2 + 1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2 + 1} \leq \underbrace{\frac{\underline{\qquad\qquad}}{x^2 + 1}}_{\text{call this } h(x)}$$

(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to \infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim\limits_{x \to \infty} \left(\dfrac{\sin^2(x)}{x^2 + 1} \right)$?

Activity 1.5.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.

Instructions: Sketch a graph of a function $g(x)$ where $f(x) \leq g(x) \leq h(x)$ for the x-values near 1 (but not necessarily for $x = 1$). To help visualize this, your sketch needs to pass through the red lines as $x \to 1$.

$x \to 1$

Standalone
Embed

(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x \to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x \to 3} g(x)$ and $\lim_{x \to 0} g(x)$?

Do we know that these limits exist? If they do, do we have information about their values?

1.6 Continuity and the Intermediate Value Theorem

Activity 1.6.1 Classification and Continuity.

Let's consider the following functions, graphed below.

Figure 1.6.1 A variety of graphs for us to use to think about continuity.

(a) Can you point out any points on the functions above that seem like the functions might not be continuous? Note that we're not classifying each function as "continuous" or "not continuous"!

(b) We want to build towards a definition of continuity using "connectedness" as the key: a function is continuous at a point if it is connected to itself. How does that work when we think about function values and limits? Use the graphs above and the points you looked at to help!

Chapter 2

Derivatives

2.1 Introduction to Derivatives

Activity 2.1.1 Thinking about Slopes.

We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ and...	Slope
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs along side of the curve there instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line.

Let's think about a new function, $g(x) = \sqrt{2 - x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

Activity 2.1.3 Calculating a Bunch of Slopes.

Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.

Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.3 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$j'(a) = \lim_{x \to a} \left(\frac{j(x) - j(a)}{x - a} \right)$$
$$= \lim_{x \to a} \left(\frac{(x^2 - 4) - (a^2 - 4)}{x - a} \right).$$

Remember, you know how this goes! You're going to do the same sorts of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to re-calculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

2.2 Interpreting Derivatives

> ### Activity 2.2.1 Interpreting the Derivative as a Slope.
>
> In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \dfrac{1}{x^2}$.
>
> **(a)** Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?
>
> **(b)** We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.
>
> Remember that we can write the equation of a line in two ways:
>
> - The equation of a line with slope m that passes through the point $(a, f(a))$ is:
>
> $$y = m(x - a) + f(a).$$
>
> - The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:
>
> $$y = mx + b.$$
>
> Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \dfrac{1}{x^2}$ below to check your equation.
>
> Standalone
> Embed
>
> What do you think the equation of the line tangent to $f(x) = \dfrac{1}{x^2}$ at $x = 2$ is?
>
> Tangent Line: $y = \boxed{}$
>
> **(c)** This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.
>
> Use this idea of slope to predict where the y-value of our function will be at 2.01.
>
> **(d)** Compare this value with $f(2.01) = \dfrac{1}{2.01^2}$. How close was it?

Activity 2.2.2 Interpreting the Derivative as a Rate of Change.

This is going to somewhat feel redundant, since maybe we know that a slope is really just a rate of change. But hopefully we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \leq t \leq 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

Activity 2.2.3 Interpreting the Derivative as a Function.

In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.3 Some Early Derivative Rules

Activity 2.3.1 Derivatives of Power Functions.

We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.

Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\dfrac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

(b) Find $\dfrac{d}{dx}\left(x^3\right)$.

(c) Find $\dfrac{d}{dx}\left(x^4\right)$.

(d) Notice that in these derivative calculations, the main work is in multiplying $(x + \Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\frac{d}{dx}\left(x^5\right) = \lim_{\Delta x \to 0}\left(\frac{(x + \Delta x)^5 - x^5}{\Delta x}\right)$$
$$= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right)$$
$$= \rightsquigarrow \ldots$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x + \Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this has done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just that whatever terms still have at least one remaining copy of Δx in it "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Activity 2.3.2 Derivatives of Trigonometric Functions.

Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing out interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.

Instructions: Drag the point along the graph of the sine function and examine the tangent line at the point. Alternatively, input the x-value you'd like to use to construct the tangent line.

$x = \boxed{0}$
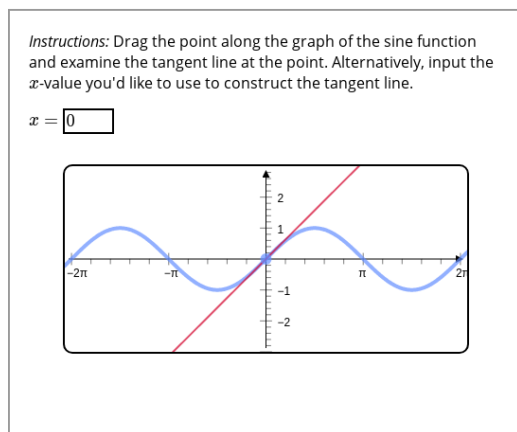
Standalone
Embed

Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.

Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \sin(x)$ and y', try to think about what function y' could be.

x	$y = \sin(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!

$y' = \boxed{}$

(c) Let's repeat this process using the $y = \cos(x)$ function instead.

The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.

Instructions: Drag the point along the graph of the cosine function and examine the tangent line at the point. Alternatively, input the x-value you'd like to use to construct the tangent line.
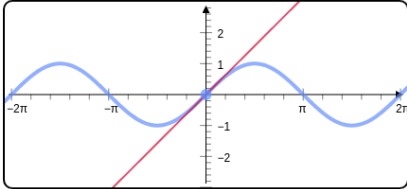
$x = \boxed{0}$

Standalone
Embed

Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.

Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \cos(x)$ and y', try to think about what function y' could be.

x	$y = \cos(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!
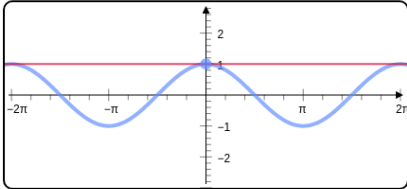
$y' = \boxed{}$

Standalone
Embed

Activity 2.3.3 Derivative of the Exponential Function.

We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function in a similar way to use thinking about the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.

Instructions: Drag the point along the graph of the sine function and examine the tangent line at the point. Alternatively, input the x-value you'd like to use to construct the tangent line.

$x = \boxed{0}$

Standalone
Embed

Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!

Instructions: Drag the point along the graph of the exponential function and examine the tangent line at the point. Alternatively, input the x-value you'd like to use to construct the tangent line.

$x =$ [0]

Standalone
Embed

What information does this reveal about the slopes?

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/equation for y' is then?

2.4 The Product and Quotient Rules

> Activity 2.4.1 Thinking About Derivatives of Products.
>
> Let's start with two quick facts:
>
> $$\frac{d}{dx}\left(x^3\right) = 3x^2 \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$
>
> **(a)** What is $\frac{d}{dx}\left(x^3 + \sin(x)\right)$? What about $\frac{d}{dx}\left(x^3 - \sin(x)\right)$?
>
> **(b)** Based on what you just explained, what is a reasonable assumption about what $\frac{d}{dx}\left(x^3\sin(x)\right)$ might be?
>
> **(c)** Let's just think about $\frac{d}{dx}\left(x^3\right)$ for a moment. What *is* x^3? Can you write this as a product? Call one of your functions $f(x)$ and the other $g(x)$. You should have that $x^3 = f(x)g(x)$ for whatever you used.
>
> **(d)** Use your example to explain why, in general, $\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$.
>
> **(e)** Let's show another way that we know this. Think about $\sin(x)$. We know two things:
>
> $$\sin(x) = (1)(\sin(x)) \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$
>
> What, though, is $\frac{d}{dx}\left(1\right) \cdot \frac{d}{dx}\left(\sin(x)\right)$?
>
> **(f)** Use all of this to reassure yourself that even though the derivative of a sum of functions is the sum of the derivatives of the functions, we will need to develop a better understanding of how the derivatives of products of functions work.

Activity 2.4.2 Building a Product Rule.

Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything. If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.
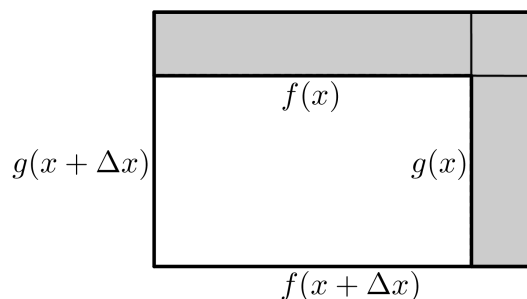
Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$$f(x+\Delta x)g(x+\Delta x) - f(x)g(x) = \underline{\hspace{4cm}}.$$

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{\rule{6cm}{0pt}}{\Delta x} \right)$$

We can split this fraction up into three fractions:

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{\Delta x \to 0} \left(\frac{\rule{4cm}{0pt}}{\Delta x} \right)$$

$$+ \lim_{\Delta x \to 0} \left(\frac{\rule{3cm}{0pt}}{\Delta x} \right)$$

$$+ \lim_{\Delta x \to 0} \left(\frac{\rule{3cm}{0pt}}{\Delta x} \right)$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

In the third limit, factor out either $\lim_{\Delta x \to 0}\left(f(x + \Delta x) - f(x)\right)$ or $\lim_{\Delta x \to 0}\left(g(x + \Delta x) - g(x)\right)$.

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x) \lim_{\Delta x \to 0} \left(\frac{\rule{3cm}{0pt}}{\Delta x} \right)$$

$$+ g(x) \lim_{\Delta x \to 0} \left(\frac{\rule{3cm}{0pt}}{\Delta x} \right)$$

$$+ \lim_{\Delta x \to 0} \left(\rule{3.5cm}{0pt} \right) \left(\lim_{\Delta x \to 0} \left(\frac{\rule{3cm}{0pt}}{\Delta x} \right) \right)$$

(f) Use Definition 2.1.2 The Derivative Function to re-write these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

Activity 2.4.3 Constructing a Quotient Rule.

We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \frac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$$u(x) = \rule{4cm}{0.4pt}$$

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$. Right? We care about finding $f'(x)$!

Use what you found for $u'(x)$ and solve for $f'(x)$.

$$f'(x) = \rule{4cm}{0.4pt}$$

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \frac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\frac{u(x)}{v(x)}$.

$$f'(x) = \rule{4cm}{0.4pt}$$

2.5 The Chain Rule

Activity 2.5.1 Composition (and Decomposition) Pictionary.

This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet-spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that will still give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?

Activity 2.5.2 Gears and Chains.

Let's think about some gears. We've got three gears, all different sizes. But the gears are linked together, and a small motor works to spin one of the gears. Since the gears are linked, when one gear spins, they all do. But since they are different sizes, they complete a different number of revolutions: the smaller ones spin more times than the larger ones, since they have a smaller circumference.
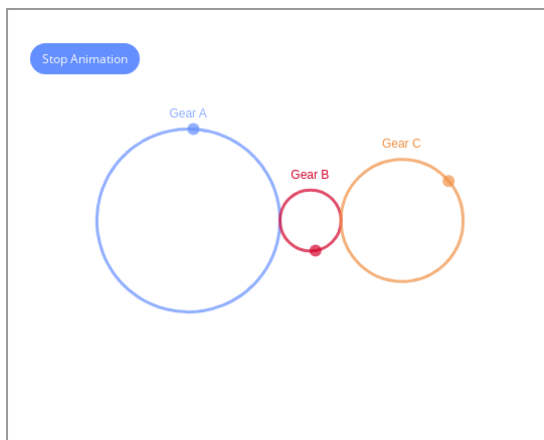
Standalone
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For our purpose, let's say that Gear A is being driven by the motor.

(a) Let's try to quantify how much "faster" Gear B is spinning compared to Gear A. How many revolutions does Gear B complete in the time it takes Gear A to complete one revolution?

(b) Now quantify the speed of Gear C compared to its neighbor, Gear B. How many revolutions does Gear C complete in the time it takes Gear B to complete one revolution?

(c) Use the above relative "speeds" to compare Gear C and Gear A: how many revolutions does Gear C complete in the time it takes Gear A to complete one revolution?

More importantly, how do we find this?

(d) Now let's translate this into some derivative notation: we've really been finding rates at which one thing changes (the speed of the gear spinning) relative to another's.

Call the speed of Gear B compared to Gear A: $\frac{dB}{dA}$. Now call the speed of Gear C compared to Gear B: $\frac{dC}{dB}$. Come up with a formula to find $\frac{dC}{dA}$.

Chapter 3

Implicit Differentiation

3.1 Implicit Differentiation

Activity 3.1.1 Thinking about the Chain Rule.

(a) Explain to someone how (and why) we use the The Chain Rule to find the following derivative:

$$\frac{d}{dx}\left(\sqrt{\sin(x)}\right).$$

(b) Let's say that $f(x) = \sin(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{f(x)}\right).$$

How is this different, or not different, than the previous derivative?

(c) Let's say that we have some other function, $g(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{g(x)}\right).$$

How is this different, or not different, than the previous derivatives?

(d) What is the difference between the following derivatives:

$$\frac{d}{dx}\left(\sqrt{x}\right) \qquad \frac{d}{dx}\left(\sqrt{y}\right) \qquad \frac{d}{dy}\left(\sqrt{y}\right)$$

Activity 3.1.2 Slopes on a Circle.

Visualize the unit circle. Feel free to draw one, or find the picture above. We're going to think about slopes on this circle.

(a) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly horizontal. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(b) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly vertical. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(c) Find the point(s) where $x = \frac{1}{2}$. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(d) For the unit circle defined by the equation $x^2 + y^2 = 1$, apply the derivative to both sides of this equation to get the following:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}\,(1)$$
$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\,(1)$$

Carefully consider each of these derivatives (each of the terms). Which of these will you need to apply the Chain Rule for?

(e) Differentiate. Solve for $\frac{dy}{dx}$ or y', whichever notation you decide to use.

(f) Go back to the first few questions, and try to answer them again:

(a) Find the locations of any horizontal tangent lines (where $\frac{dy}{dx} = 0$).

(b) Find the locations of any vertical tangent lines (where $\frac{dy}{dx}$ doesn't exist, or where you would divide by 0).

(c) Find the values of $\frac{dy}{dx}$ for the points on the circle where $x = \frac{1}{2}$.

Activity 3.1.3 .

Let's consider a new curve:

$$\sin(x) + \sin(y) = x^2 y^2.$$

$$\sin(x) + \sin(y) = x^2 y^2$$

Figure 3.1.1

(a) We are going to find $\frac{dy}{dx}$ or y'. Let's dive into differentiation:

$$\frac{d}{dx}\left(\sin(x) + \sin(y)\right) = \frac{d}{dx}\left(x^2 y^2\right)$$
$$\frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right) = \frac{d}{dx}\left(x^2 y^2\right)$$

Think carefully about these derivatives. For each of the three, how will you approach it? What kinds of nuances or rules or strategies will you need to think about? Why?

(b) Implement your ideas or strategies from above to differentiate each term.

(c) Now we need to solve for $\frac{dy}{dx}$ or y', whichever you are using. While this equation can look complicated, we can notice something about the "location" of $\frac{dy}{dx}$ or y' in our equation.

Why do we always know that $\frac{dy}{dx}$ or y' will be *multiplied* on a term whenever it shows up?

(d) Now that we are confident that we will *always* know that we are multiplying this derivative, we can employ a consistent strategy:

   (a) Rearrange our equation so that every term with a $\frac{dy}{dx}$ or y' is on one side, and everything without is on the other.

   (b) Now we are guaranteed that $\frac{dy}{dx}$ or y' is a common factor: factor it out.

   (c) Now we can solve for $\frac{dy}{dx}$ or y' by dividing!

Solve for $\frac{dy}{dx}$ or y' in your equation!

(e) Build the equation of a line that lays tangent to the curve at the origin. Does the value of $\frac{dy}{dx}$ at $(0,0)$ look reasonable to you?

3.2 Derivatives of Inverse Functions

> ### Activity 3.2.1 Building the Derivative of the Logarithm.
>
> We're going to accomplish two things here:
>
> 1. By the end of this activity, we'll have a nice way of thinking about $\frac{d}{dx}\left(\ln(x)\right)$, and we will now be able to differentiate functions involving logarithms!
>
> 2. Throughout this activity, we're going to develop a way of approaching derivatives of inverse functions more generally. Then we can apply this framework to other functions!
>
> Let's think about this logarithmic function!
>
> **(a)** We have stated (a couple of times now) how we define the log function:
>
> $$y = e^x \longleftrightarrow x = \ln(y).$$
>
> This arrow goes both directions: the log function is the inverse of the exponential, but the exponential is the inverse of the log function!
>
> Can you re-write the relationship $y = \ln(x)$ using its inverse (the exponential)?
>
> **(b)** For your inverted $y = \ln(x)$ from above (it should be $x = \underline{\qquad\qquad}$), apply a derivative operator to both sides, and use implicit differentiation to find $\frac{dy}{dx}$ or y'.
>
> **(c)** A weird thing that we can notice is that when we use implicit differentiation, it is common to end up with our derivative written in terms of both x and y variables. This makes sense for our earlier examples: we needed specific coordinates of the point on the circle, for instance, to find the slope there.
>
> But if $y = \ln(x)$, we want $\frac{dy}{dx}$ or y' to be a function of x:
>
> $$f(x) = \ln(x) \longrightarrow f'(x) = \underline{\qquad\qquad}.$$
>
> Your derivative is written with only y values.
>
> Since $y = \ln(x)$, replace any instance of y with the log function. What do you have left?
>
> **(d)** Remember that $y = \ln(x)$. Substitute this into your equation for $\frac{dy}{dx}$. Can you write this in a pretty simplistic way?
>
> **(e)** Before we go much further, we should be a bit careful. What is the domain of this derivative? What are the values of x where $\frac{d}{dx}\left(\ln(x)\right)$ makes sense to think about?

Activity 3.2.2 Finding the Derivative of the Inverse Sine Function.

We're going to do the same trick, except that there will be a couple of small differences due to thinking specifically about trigonometric functions.

Let's think about the function $y = \sin^{-1}(x)$. We know that this is equivalent to $x = \sin(y)$ (for y-values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

(a) Move the point around the portion of the unit circle in the graph below. Convince yourself that:

- $\sin(y) = x$
- $\sin(y) \geq 0$ when $0 \leq y \leq \frac{\pi}{2}$
- $\sin(y) < 0$ when $-\frac{\pi}{2} \leq y < 0$

Standalone
Embed

What is $\cos(y)$ in this figure? Does the sign change depending on the value of y?

(b) Use implicit differentiation and the equation $x = \sin(y)$ to find $\frac{dy}{dx}$ or y'

(c) If you still have your derivative written in terms of y, make sure to write $\cos(y)$ in terms of x!

(d) Let's think about the domain of this derivative: what x-values make sense to think about?

Think about this both in terms of what x-values reasonably fit into your formula of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$ as well as the domain of the inverse sine function in general.

(e) Notice that in the denominator of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$, you have a square root. Based on that information (and the visual above), what do you expect to be true about the sign of the derivative of the inverse sine function?

Confirm this by playing with the graph of $y = \sin^{-1}(x)$ below.

Instructions: Move the point on the graph of $y = \sin^{-1}(x)$ and think about what must be true about the derivative $\frac{dy}{dx}$.

Standalone
Embed

(f) Investigate the behavior of $\frac{dy}{dx}$ at the end-points of the function: at $x = -1$ and $x = 1$. What do the slopes look like they're doing, graphically?

How does this work when you look at the function you built above? What happens when you try to find $\left.\frac{dy}{dx}\right|_{x=-1}$ or $\left.\frac{dy}{dx}\right|_{x=1}$?

Activity 3.2.3 Building the Derivatives for Inverse Tangent and Secant.

(a) Consider the triangle representing the case when $y = \tan^{-1}(x)$.

Figure 3.2.1

For $x = \tan(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ based on the triangle above, but we will refer back to the version with the $\sec(y)$ in it later.

(b) Consider the triangle representing the case when $y = \sec^{-1}(x)$.

Figure 3.2.2

For $x = \sec(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ and $\tan(y)$ based on the triangle above, but we will refer back to the version with the functions of y in it later.

(c) Here's a graph of just the unit circle for angles $[0, \pi]$. We are choosing to focus on this region, since these are the angles that the inverse tangent and inverse secant functions will return to us. We want to investigate the signs of $\tan(y)$ and $\sec(y)$.

Instructions: For angles y in $[0, \pi]$, find the signs of $\tan(y)$ and $\sec(y)$. When are each of them positive or negative?

1

$\sin(y)$

y

$\cos(y) > 0$

▸ **Hint (click to open)**

Standalone
Embed

(d) Go back to our derivative expressions for both the inverse tangent and inverse secant functions. What do we know about the signs of these derivatives?

(e) Confirm your idea about the sign of the derivatives by investigating the graphs of each function.

Instructions: Move the points on the graphs of $y = \tan^{-1}(x)$ and $y = \sec^{-1}(x)$ and think about what must be true about the signs of the derivatives.

Graph of $y = \tan^{-1}(x)$ Graph of $y = \sec^{-1}(x)$

Standalone
Embed

(f) How do we need to write these derivatives, when we write them in terms of x to account for the sign of the derivative?

Activity 3.2.4 Connecting These Inverse Functions.

We're going to look at a graph of $y = \cos^{-1}(x)$, but we're specifically going to try to compare it to the graph of $y = \sin^{-1}(x)$. We'll use some graphical transformations to make these functions match up, and then later we'll think about derivatives.

(a) Ok, consider the graph of $y = \cos^{-1}(x)$ and a transformed version of the inverse sine function. Apply some graphical transformations to make these match!

Instructions: Fill in values in the inverse sine function below to change the plot. Try to find values that will make it line up with the plot of $y = \cos^{-1}(x)$.

$y = \boxed{1}\ \sin^{-1}(x) + \boxed{0}$

Hm...What values can you use to make these curves match? What kinds of transformations should you apply to the $y = \sin^{-1}(x)$ function in order to make it match $y = \cos^{-1}(x)$?

Standalone
Embed

(b) It might be fun to think about another reason that this connection between $\sin^{-1}(x)$ and $\cos^{-1}(x)$ exists.

Consider this triangle:

Figure 3.2.3

We're going to think about these inverse trigonometric functions as angles: let $\alpha = \cos^{-1}(x)$ and $\beta = \sin^{-1}(x)$. We can re-write these as:

$$\cos(\alpha) = x$$
$$\sin(\beta) = x.$$

Can you fill in your triangle using this information?

Why does $\alpha + \beta = \frac{\pi}{2}$? Convince yourself that this is what we did with the graphical transformations above, as well.

(c) Use this equation above, re-writing $\cos^{-1}(x)$ as some expression involving the inverse sine function, and then find

$$\frac{d}{dx}\left(\cos^{-1}(x)\right).$$

We could repeat this task to try to connect the graph of $y = \tan^{-1}(x)$ with $y = \cot^{-1}(x)$ as well as the graph of $y = \sec^{-1}(x)$ with $y = \csc^{-1}(x)$, but we can hopefully see what will happen. In each case, we have the same kind of connection that we saw in the triangle, since these are cofunctions of each other!

We can summarize by believing that:

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{d}{dx}\left(\sin^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{d}{dx}\left(\tan^{-1}(x)\right) = -\frac{1}{x^2+1}$$
$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{d}{dx}\left(\sec^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

3.3 Logarithmic Differentiation

Activity 3.3.1 Returning to the Power Rule.

Back in Section 2.3 we built an explanation for why $\frac{d}{dx}(x^n) = nx^{n-1}$ that relied on thinking about exponents as repeated multiplication: it relied on n being some positive integer. We said, at the time, that the Power Rule generalizes and works for *any* integer, but did so without explanation. Let's consider $y = x^n$ where n is just some real number without any other restrictions.

(a) Apply a logarithm to both sides of this equation:

$$\ln(y) = \ln(x^n)$$

Now use one of the Properties of Logarithms to re-write this equation.

(b) Use implicit differentiation to find $\frac{dy}{dx}$ or y'.

(c) Explain to yourself why this is equivalent to the Power Rule that we built so long ago:

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(d) Let's get weird. What if we have a not-quite-power function? Where the thing in the exponent isn't simply a number, but another variable?

Let's use the same technique to think about $y = x^x$ and its derivative. First, though, confirm that this is not a power function (and so we cannot use the Power Rule to find the derivative) and is also not an exponential function (and so the derivative isn't itself or itself scaled by a log).

(e) Now apply a log to both sides:
$$\ln(y) = \ln(x^x).$$

Re-write this using the same log property as before, and then use implicit differentiation to find $\frac{dy}{dx}$ or y'.

(f) Explain to yourself why logs are friends, especially when trying to differentiate functions in the form of $y = (f(x))^{g(x)}$.

Activity 3.3.2 Logarithmic Differentiation with Products and Quotients.

Let's say we've got some function that has products and quotients in it. Like, a lot. Consider the function:

$$y = \frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}.$$

(a) Work out a general strategy for how you would find y' directly. Where would you have to use Quotient Rule? What are the pieces? Where would you have to use Product Rule? What are the pieces? Where would you have to use Chain Rule? What are the pieces?

To be clear: do not actually differentiate this. Just look at it in horror and try to outline a plan that some other fool would use.

Click on the "Solution" below to see what the fool did.

(b) Let's instead use logarithmic differentiation. First, apply the log to both sides to get:

$$\ln(y) = \ln\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right).$$

Since this function is just a bunch of products of things with exponents all put into some big quotient, we can use our log properties to re-write this!

(c) We should have:

$$\ln(y) = 3\ln(x-4) + \frac{1}{2}\ln(3x+1) - 7\ln(x+1) - 3\ln(x+5).$$

Confirm this.

(d) Now differentiate both sides! You'll have to use some Chain Rule (but not a lot)! Refer back to Fact 3.3.1 for help here.

(e) Solve for $\frac{dy}{dx}$ or y'.

(f) While this is not a *nice* looking expression for the derivative, spend some time confirming that this was a nicer *process* then differentiating directly. This is because logs are friends.

Chapter 4

Applications of Derivatives

4.1 Mean Value Theorem

4.2 Increasing and Decreasing Functions

> ### Activity 4.2.1 How Should We Think About Direction?
>
> Our goal in this activity is to motivate some new terminology and results that will help us talk about the "direction" of a function and some interesting points on a function (related to the direction of a function). For us to do this, we'll look at some different examples of functions and try to think about some unifying ideas.
>
>
>
> **Figure 4.2.1**
>
> These examples do not cover all of the possibilities of how a function can act, but will hopefully provide us enough fertile ground to think about some different situations.
>
> **(a)** In each graph, find and identify:
>
> - The intervals where the function is increasing.
> - The intervals where the function is decreasing.

- The points (or locations) around and between these intervals, the points where the function changes direction or the direction terminates.

(b) Make a conjecture about the behavior of a function at any point where the function changes direction.

(c) Look at the highest and lowest points on each function. You can even include the points that are highest and lowest just compared to the points around it. Make a conjecture about the behavior of the function at these points.

Activity 4.2.2 Comparing Critical Points.

Let's think about four different functions:

- $f(x) = 4 + 3x - x^2$
- $g(x) = \sqrt[3]{x+1} + 1 + x$
- $h(x) = (x-4)^{2/3}$
- $j(x) = 1 - x^3 - x^5$

Our goal is to find the critical points on the interval $(-\infty, \infty)$ and then to try to figure out if these critical points are local maximums or local minimums or just points that the function increases or decreases through.

(a) To start, we're going to be finding critical points. Without looking at a picture of the graph of the function, find the derivative.

Are there any x-values (in the domain of the function) where the derivative doesn't exist? We are normally looking for things like division by 0 here, but we could be finding more than that. Check out When Does a Derivative Not Exist? to remind yourself if needed.

Are there any x-values (in the domain of the function) where the derivative is 0?

(b) Now that we have the critical points for the function, let's think about where the derivative might be positive and negative. These will correspond to the direction of a function, based on Theorem 4.1.5 Sign of the Derivative and Direction of a Function.

Write out the intervals of x-values around and between your list of critical points. For each interval, what is the sign of the derivative? What do these signs mean about the direction of your function?

(c) Without looking at the graph of your function:

- What changes about how your function increases up to or decreases down to a critical point based on whether the derivative was 0 or the derivative didn't exist?

- Does your function change direction at a critical point? What will that look like, whether it does or does not change direction?

(d) Give a basic sketch of your graph. It might be helpful to find the y-values for any critical points you've got. Then you can sketch your function increasing/decreasing in the intervals between these points.

In your sketch, include enough detail to tell whether the derivative is 0 or doesn't exist at each critical point.

(e) Compare your sketch to the actual graph of the function (you can find all of the graphs in the hint).

Activity 4.2.3 First Derivative Test Graphically.

Let's focus on looking at a picture of a derivative, $f'(x)$, and trying to collect information about the function $f(x)$. This is what we've done already, except that we've done it by thinking about the representation of $f'(x)$ as a function rule written out with algebraic symbols. Here we'll focus on connecting all of that to the picture of the graphs.

For all of the following questions, refer to the plot below. You can add information with the hints whenever you need to. Don't reveal the picture of $f(x)$ until you're really ready to check what you know.

Instructions: Move the point on the graph of $f'(x)$ and connect it to the behavior of $f(x)$. Reveal the hints to think more about interpreting what you see on the graph of $f'(x)$. Finally, click the button to show the graph of $f(x)$ to check your understanding.

Graph of $f'(x)$

Standalone
Embed

▸ **Hint: How do we interpret the height of the point on $f'(x)$?**
(click to open)

▸ **Hint: What do we learn about $f(x)$ from the graph of $f'(x)$?**
(click to open)

Check your understanding: Click the button to reveal the graph of $f(x)$.

Show Graph of f(x)

(a) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is increasing.

(b) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is decreasing.

(c) Find the x-values of the critical points of $f(x)$. Once you've estimated these, classify them as local maximums, local minimums, or neither. Explain your reasoning.

(d) What do you think the graph of $f(x)$ looks like? Do your best to sketch it or explain it before revealing it!

(e) Why could we estimate the x-values of the critical numbers of $f(x)$, but not find the actual coordinates? How come we can't find the y-value based on looking at the graph of $f'(x)$?

4.3 Concavity

Activity 4.3.1 Compare These Curves.

(a) Consider the two curves pictured below. Compare and contrast them. What characteristics of these functions are the same? What characteristics of these functions are different?

Instructions: Compare the functions $f(x)$ and $g(x)$ on the interval $[0, 1]$.

Hint 1: Click the button below to add tangent lines to the graphs of $f(x)$ and $g(x)$.

Show Tangent Lines

Graph of $f(x)$ Graph of $g(x)$

Hint 2: Click the button below to see the graphs of the derivative functions, $f'(x)$ and $g'(x)$.

Show Graphs of Derivatives

(b) Explain the similarities you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(c) Explain the differences you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(d) Make a conjecture about the *rate of change* of both f' and g'. We'll call these things **second derivative** functions, $f''(x)$ and $g''(x)$.

Activity 4.3.2 Finding a Function's Concavity.

We're going to consider a pretty complicated function to work with, and employ a strategy similar to what we did with the first derivative:

- Our goal is to find the sign of $f''(x)$ on different intervals and where that sign *changes*.

- To do' this, we'll find the places that $f''(x) = 0$ or where $f''(x)$ doesn't exist. These are the critical points of $f'(x)$.

- From there, we can build a little sign chart, where we split up the x-values based on the domain of f and the critical numbers of f'. Then we can attach each interval of x-values with a sign of the second derivative, f'', on that interval.

- We'll interpret these signs. For any intervals where $f''(x) > 0$, we know that $f'(x)$ must be increasing and so $f(x)$ is concave up. Similarly, for any intervals where $f''(x) < 0$, we know that $f'(x)$ must be decreasing and so $f(x)$ is concave down.

Let's consider the function
$$f(x) = \ln(x^2 + 1) - \frac{x^2}{2}.$$

(a) Find the first derivative, $f'(x)$, and use some algebra to confirm that it is really:
$$f'(x) = -\frac{x(x+1)(x-1)}{x^2 + 1}.$$

While we have this first derivative, we *could* find the critical points of $f(x)$ and then classify those critical points using the First Derivative Test.

For our goal of finding the intervals where $f(x)$ is concave up and concave down, and then finding the inflection points of f, let's move on.

(b) Find the second derivative, $f''(x)$, and confirm that this is really:
$$f''(x) = -\frac{x^4 + 4x^2 - 1}{(x^2 + 1)^2}.$$

(c) Our goal is to find the x-values where $f''(x) = 0$ or where $f''(x)$ doesn't exist.

Note that in order to find where $f''(x) = 0$, we are really looking at the x-values that make the numerator 0:
$$x^4 + 4x^2 - 1 = 0.$$

Then, to find where $f''(x)$ doesn't exist, we are finding the x-values that make the denominator 0:
$$(x^2 + 1)^2 = 0.$$

Solve these equations.

(d) You have two critical points of $f'(x)$ (let's just call them x_1 and x_2). These are possible inflection points of $f(x)$, but we need to check to see if the concavity changes at these points.

Fill in the following sign chart and interpret.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''					
f					

Let's confirm what we've just calculated graphically. Below, we have three graphs:

1. $f(x) = \ln(x^2 + 1) - \dfrac{x^2}{2}$

2. $f'(x) = -\dfrac{x(x+1)(x-1)}{x^2+1}$

3. $f''(x) = -\dfrac{x^4 + 4x^2 - 1}{(x^2+1)^2}$

Move the point on any graph and make sure the statements about signs, directions, and concavity match what you found! You should notice that signs of the first and second derivative change at the local maximums/minimums and the inflection points that we found.

Instructions: Compare what you've found to what is happening in the plots. Turn on or off information using the checkboxes.

☐ Show the local maximums and local minimums of $f(x)$.

☐ Show the inflection points of $f(x)$.

- The outputs of $f(x)$ are increasing and concave down.
- This is because the the slopes of $f(x)$ are positive and decreasing.
- This is because the concavity of $f(x)$ is negative.

- The outputs of $f'(x)$ are positive and decreasing.
- This is because the the slopes of $f'(x)$ are negative.

- The outputs of $f''(x)$ are negative.

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4.4 Global Maximums and Minimums

Activity 4.4.1 When Would We Not Have Maximums or Minimnums?

In this section, we're going to define these global maximums and then, most importantly, try to predict when these global maximums or global minimums might actually exist for a function.

To start, maybe we should come up with some examples of functions that do not have them!

(a) Come up with some situations where a function does not have some combination of global maximum/minimums. Sketch some graphs!

(b) Come up with some examples of graphs of functions that are bounded (do not ever have y-values that tend towards infinity in a limit) that do not have some combination of global maximum/minimums.

(c) For the examples of graphs that you have built or collected, features of the functions that allow for the examples you picked? If you could impose some requirements that would "fix" the examples you found (so that they had both a global maximum and a global minimum), what requirements could you use?

4.5 Optimization

Activity 4.5.1 Constructing a Can.

A typical can of pop is 355 ml, and has around 15 ml of headspace (air). This results in a can that can hold approximately 23 cubic inches of volume.

Let's say we want to construct this can in the most efficient way, where we use the least amount of material. How could we do that? What are the dimensions of the can?

(a) First, let's think of our can and try to translate this to some geometric shape with variable names. Collect as much information as we can about this setup! What is the shape? What are the names of the dimensions?

(b) What is the actual measurement that we are trying to optimize? Are we finding a maximum or a minimum?

(c) What other information about the can do we know? How do we translate this into a constraint, or a connection between our variables?

(d) What kinds of values can our variables take on? Is there a smallest value for either? A largest? Are there other limitations to these?

(e) Now we need to set up a calculus statement. This part mostly relies on us finding a way to build a single-variable function defining the surface area. Build that function, and then write down the calculus statement.

(f) Do some calculus to find the global maximum or minimum, and solve the optimization problem.

(g) What is the relationship between r and h, here? How do they compare? What about the height and diameter of our can?

Is this relationship noticeable in a typical can of pop?

4.6 Linear Approximations

Activity 4.6.1 Approximating an Exponential Function.

Let's consider the function $f(x) = e^x$. We're going to build the linear approximation, $L(x)$, but we also want to focus on understanding what the "center" is, and how we think about accuracy of our estimations.

(a) We first need to find a "good" center for our linear approximation. We have two real requirements here:

(a) We need the center to be some x-value that will be "close" to the inputs we're most interested in. We know that $L(x) \approx f(x)$ for x-values "near" the center, so we should keep this in mind. We don't have a specific input that we're interested in (we are not specifically focused on estimating $f(7.35)$ for instance), so we don't need to worry about this for now.

(b) We are going to need to evaluate the function and its derivative at the center: we use $f'(a)$ to find the slope and $f(a)$ to find a y-value for a point on the line. We'd like to choose a center that will make evaluating these functions reasonable, if we can!

We are going to choose a center of 0: why?

(b) Build a linear approximation of $f(x) = e^x$ centered at $x = 0$.

(c) Use your linear approximation to estimate the value of $\sqrt[10]{e} = e^{\frac{1}{10}}$.

(d) Let's visualize this approximation a bit:

Instructions: The red line is the linear approximation of the exponential function, centered at $x = 0$. Move the point around to see how well the linear approximation actually estimates the y-value outputs from the exponential function.

$L(x) \approx f(x)$ for x-values close to 0.
$x + 1 \approx e^x$

$L(1) \approx f(1)$
$2 \approx e^1$

Standalone
Embed

Are you confident in your approximation of $\sqrt[10]{e}$? Would you be more or less confident in an approximation of $\frac{1}{e}$? Why?

(e) Is your estimate of $\sqrt[10]{e}$ too big or too small? How can you tell, without even calculating the actual value of $\sqrt[10]{e}$?

How can you tell that *every* estimate that you get out of *any* linear approximation of e^x (no matter what the center is) is going to be too small?

Activity 4.6.2 Approximating some Values.

Pick one of the following values to approximate:

- $\sin(-0.023)$

- $\ln(2)$

- $\sqrt{8}$

- $\sqrt[3]{10}$

Throughout the rest of this activity, use your value to build a linear approximation of some relevant function and estimate the value you chose.

(a) To build a linear approximation of some function at some center, we need two things:

(a) A function.

(b) A center.

What function will you be using for $f(x)$? Why that one?

(b) What center are you choosing? Why that one?

(c) Build your linear approximation at your center! You should end up with an actual linear function. It might be helpful to plot this linear function and your actual function to confirm that you have actually built a tangent line.

(d) Use your linear approximation function to estimate your value! Report the estimate, and comment on the accuracy of your estimate. Without calculating the actual value, can you tell if this is close or not? Do you have an overestimate or underestimate?

Activity 4.6.3 Walking in the Footsteps of Ancient Mathematicians.

Let's travel all the way back to the first (or maybe second) century AD and recreate Heron's method to approximate the value of $\sqrt{2}$. We'll develop this using modern calculus, and simple linear approximation.

We're going to re-frame the problem, and instead we're going to try to use a linear approximation of $f(x) = x^2 - 2$ to approximate the x-value where $f(x) = 0$. We know enough about quadratic functions to know that there are two values: $x = -\sqrt{2}$ and $x = \sqrt{2}$.

(a) We're going to build a linear approximation of $f(x) = x^2 - 2$, and we need a reasonable center. Honestly, any integer will work, since we can evaluate f and f' really easily, but we want to find one that is close to $\sqrt{2}$. Let's center our approximation at $x = 2$.

Find $f'(x)$, and then construct the linear approximation:

$$L(x) = f'(2)(x - 2) + f(2).$$

(b) Now we know that $L(x) \approx f(x)$ for x-values near our center, $x = 2$. What if we estimate the x-value where $f(x) = 0$ by solving $L(x) = 0$ instead? Since $L(x) \approx f(x)$, the x-value where $L(x) = 0$ should make $f(x)$ pretty close to 0 at least.

Solve $L(x) = 0$.

(c) Ok, this might be kind of close to the value of $\sqrt{2}$, right? Let's visualize this.

Hm...so this isn't that good of an approximation yet. We can check this by looking at the actual value of our function at $x = \frac{3}{2}$ and seeing if it's close to 0.

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 2$$
$$= \frac{9}{4} - 2$$
$$\frac{1}{4}$$

This...isn't that close to 0.

So let's try this again. This time, though, let's center our *new* linear approximation at $x = \frac{3}{2}$.

(d) Now set this *new* linear approximation equal to 0 and solve $L(x) = 0$ to estimate the solution to $f(x) = 0$.

(e) We can keep repeating this process, and that's exactly what the mathematicians we talked about discovered.

Say we've build a linear approximation at some x-value (we'll call it x_{old}).

$$L(x) = f'(x_{\text{old}})(x - x_{\text{old}}) + f(x_{\text{old}}).$$

Set this equal to 0 and solve.

(f) Let's visualize these calculations.

Instructions: This is a visualization of our iterative approximation of $\sqrt{2}$ by estimating solutions to $x^2 - 2 = 0$. Click the buttons below to move forward one iteration/step or backward one interation/step. For each step, explore how we use a linear approximation (centered at our previous estimation) to calculate a new estimation of the x-intercept.

Go forward a step.   Go backward a step.

This is step number 0, where our estimate is:

$$\sqrt{2} \approx 2$$

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Something kind of strange happens in the last two steps. Why does the value of our estimation not change? What happens to our estimate?

4.7 L'Hôpital's Rule

Activity 4.7.1 Building L'Hôpital's Rule.

We're going to take a closer look at the indeterminate form, $\frac{0}{0}$, and use our new ideas of linear approximation to think about how these types of things work.

We're going to be working with the following limit:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are differentiable at $x = a$ (since we're going to want to build linear approximations of them).

(a) Write out the linear approximations for both $f(x)$ and $g(x)$, both centered at $x = a$. We'll call them $L_f(x)$ and $L_g(x)$.

(b) Describe how well or how poorly these linear approximations estimate the values from our functions $f(x)$ and $g(x)$? What happens to these approximations as we get close to the center $x = a$? What happens in the limit as $x \to a$?

(c) Let's re-write our limit. We can replace $f(x)$ with our formula for its linear approximation, $L_f(x)$ and replace $g(x)$ with its linear approximation, $L_g(x)$:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\rule{0pt}{1.5em}\hspace{6em} \right)$$

(d) Up until now, we have not thought about indeterminate forms at all. Let's start now.

If this limit is a $\frac{0}{0}$ indeterminate form, then that means that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

Since our functions are, by definition, differentiable at $x = a$, then they also have to be continuous at $x = a$. What does this mean about the values of $f(a)$ and $g(a)$?

(e) Use this new information about the values of $f(a)$ and $g(a)$ to revisit the limit. We re-wrote $\lim_{x \to a} \frac{f(x)}{g(x)}$ by replacing each function with its linear approximation. What happens with the algebra when we know this information about $f(a)$ and $g(a)$?

So we have a really nice result here! In the $\frac{0}{0}$ indeterminate form, we can replace the ratio of the y-values from our functions with the ratio of slopes (coming from the first derivatives) of our functions.

In general, we'll put a step in between, where we find $f'(x)$ and $g'(x)$ first before trying to evaluate these derivatives at $x = a$.

Chapter 5

Antiderivatives and Integrals

5.1 Antiderivatives and Indefinite Integrals

Activity 5.1.1 Find a Function Where....

For each of the following derivatives, find a function $f(x)$ whose first (or second) derivative matches the listed derivative.

(a) $f'(x) = 4x^3 + 6$

(b) $f'(x) = 8x^7 - x^4 + x$

(c) $f'(x) = \sqrt{x} - \dfrac{1}{x^2}$

(d) $f''(x) = x^3 + \cos(x)$

(e) $f''(x) = e^x - \dfrac{1}{\sqrt[3]{x}}$

(f) Go back through each of the above derivatives, and find a different option for $f(x)$ that still works. Make sure that it is something completely unique, and not just an equivalent function that is written differently.

Why are able to find multiple answers in these questions, but not when we are given a function and need to find a derivative?

Activity 5.1.2 A File Sorting Speed Test.

A computer program is trying to sort a group of computer files based on their size. The program isn't very efficient, and the time that it takes to sort the files increases when it tries to sort more files.

The time that it takes, measured in seconds, based on the total, cumulative size of the files g, measured in gigabytes, is modeled by a function $T(g)$. We don't know the function, but we do know that the time increases at an instantaneous rate of $0.0001g$ seconds when the total size, g increases slightly.

(a) We can build a function for $T'(g)$. What is it?

(b) Find all of the possibilities for the function modeling the time, T, that it takes the computer program to sort files with a total size of g.

(c) What does your constant C represent, here? You can interpret it graphically, interpret it by thinking about derivatives, but you should also interpret it in terms of the time that it takes this program to sort these files by size.

(d) Let's say that we feed some number of files totaling up to 1.4GB in size into this program. It takes 0.24 seconds to sort the files by size.

Find the function, $T(g)$, that models the how quickly this program sorts these files.

5.2 Riemann Sums and Area Approximations

Activity 5.2.1 Approximating Areas.

We're going to consider two different functions, and some areas based on them. Let's think about two functions: $f(x) = 2x + 1$ and $g(x) = x^2 + 1$. For both of these functions, we'll focus on the interval $[0, 2]$. Instead of thinking about the function only, we'll be considering the two-dimensional region bounded between the graph of our function and the x-axis between $x = 0$ and $x = 2$.

(a) Find the area of the region bounded between the graph of $f(x) = 2x+1$ and the x-axis between $x = 0$ and $x = 2$.

Figure 5.2.1

How did you evaluate this area? What kind(s) of shape(s) did you think about?

(b) Estimate the area of the region bounded between the graph of $g(x) = x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$.

Figure 5.2.2

   Archimedes of Syracuse discovered how to calculate this area exactly, without estimation, around 300 BC, writing his results in the now-famous "Quadrature of the Parabola." This is, notably, before the formalization of Calculus (during the 1600's). It might be unfair to say that Archimedes proved this "without using calculus," though, since his technique, the "Method of Exhaustion," is really a version of what we do in calculus, but without a formal framework of limits.

How did you estimate this area? What kind(s) of shape(s) did you think about?

(c) Come up with an upper and lower bound for this area. In other words, give an underestimate and overestimate for the actual area we would like to know.

How did you come up with these estimates? How "good" do you think your estimates are? Can you come up with "better" (or closer) ones?

Activity 5.2.2 Approximating the Area using Rectangles.

We're going to stick with the function $g(x) = x^2 + 1$ on the interval $[0, 2]$, and keep thinking about the area bounded by the curve and the x-axis on this interval. We're going to approximate the area in a couple of different tries, each one more accurate than the one before. By the end of this activity, we'll have a pretty good process built!

(a) Let's start with approximating this region with a single rectangle. We're going to define the rectangle by picking some x-value in the interval $[0, 2]$. Then, we'll use the point at that x-value to define the height of our rectangle.

Essentially, we are picking a single point on the our function on the interval and our approximation is pretending that the single point we picked is representative of the whole function on the interval.

Instructions: Input an x-value to construct a rectangle for approximating the area.

Input a value in the interval $[0, 2]$: $x = \boxed{}$

Your rectangle has a width of 2 and a height of NaN, and your approximation of the area is NaN.

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(b) Can you try re-picking an x-value, and trying to find one that gets you an area approximation that is pretty good?

(c) We're going to use more rectangles. Let's jump up to 3 rectangles. If we split up the interval between $x = 0$ and $x = 2$ into 3 rectangles, we can make them all the same width, and pick an x-value that we can use to get a representative point for each of the 3 rectangles.

We'll need to pick 3 x-values this time.

Instructions: Input three x-values to construct rectangles for approximating the area.

Input a value in the interval $\left[0, \frac{2}{3}\right]$: $x_1 = \boxed{}$

Input a value in the interval $\left[\frac{2}{3}, \frac{4}{3}\right]$: $x_2 = \boxed{}$

Input a value in the interval $\left[\frac{4}{3}, 2\right]$: $x_3 = \boxed{}$

Your rectangles all have widths of $\frac{2}{3}$ and heights of NaN, NaN, and NaN.

The areas are NaN, NaN, and NaN, so your approximation of the total area is NaN.

Standalone
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(d) Can you try re-picking your x-values, and trying to find one that gets you an area approximation that is pretty good?

(e) Let's scale this up a bit. Pick a good number for your number of rectangles. We'll call this value n.

(If you're working in a classroom, maybe it would be good to pick the number of groups or the number of students, or some other number between 10 and 20 or something like that.)

For your value n, we're going to divide up the interval between $x = 0$ and $x = 2$ into n pieces. These will be the intervals that we pick from to get our rectangles. What are the subintervals? What are the widths of each subinterval (and then the widths of the rectangles)? Call this with Δx.

(f) For each subinterval, pick an x-value in the subinterval to represent it.

(g) Evaluate the function $f(x) = x^2 + 1$ at each of the x-values you picked. These are the heights of your rectangles!

(h) Find the areas of each rectangle by multiplying the height of each rectangle by Δx, the width of each rectangle.

(i) Add these areas up to get a total approximation of the actual area!

What do you think: is it worth fiddling with what x-value to pick from each subinterval to try to get a better approximation? If n is large, do you think it matters how we pick the x-values from each subinterval?

5.3 The Definite Integral

Activity 5.3.1 Weird Areas.

Let's think about a simple linear function, $f(x) = 4 - 3x$. We'll both approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 0$ to $x = 3$:

$$\int_{x=0}^{x=3} (4 - 3x) \, dx$$

Figure 5.3.1

(a) Explain why we do not need to think about Riemann sums in order for us to calculate the shaded in area. How would you calculate this without using calculus?

Calculate the area!

(b) Let's approximate this area using a Riemann sum. Calculate L_3, the Left Riemann sum with $n = 3$ rectangles.

(c) Let's approximate this area a second time, but with a different selection strategy for our x-values. Calculate R_3, the Right Riemann sum with $n = 3$ rectangles.

(d) Compare your answers for L_3 and R_3. They should be *very* different. Why? What is happening that makes R_3 specifically such a weird value?

(e) Do you need to go back and re-calculate the area geometrically (from the first part of this activity)? Explain why your answer for $\int_{x=0}^{x=3} (4 - 3x) \, dx$ *should* be negative, based on the Riemann sums we calculated.

(f) Find a value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x) \, dx = 0.$$

(g) Find a *different* value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x) \, dx = 0.$$

Is there a second way of making this area 0?

Activity 5.3.2 Weird Areas - Part 2.

We're going to think about the same region, kind of.

Figure 5.3.2

Let's think about the same linear function, $f(x) = 4 - 3x$, but this time we'll approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 3$ to $x = 0$:

$$\int_{x=3}^{x=0} (4 - 3x) \; dx$$

(a) Use geometry to calculate the area. Compare this to the result from Activity 5.3.1.

(b) Let's approximate this using a Riemann sum. Calculate M_3, the Midpoint Riemann sum with $n = 3$ rectangles.

(c) Do you need to go back and re-calculate the area geometrically (from the first part of this activity)? Explain why your answer for $\int_{x=3}^{x=0} (4 - 3x) \; dx$ *should* be positive, based on the Riemann sums we calculated.

5.4 The Fundamental Theorem of Calculus

Activity 5.4.1 Interpreting the Graph of a Derivarive.

Let's look at a picture of a graph of the first derivative, $f'(x)$, and try to get some information about $f(x)$ from it. Use the following graph of $f'(x)$, the first derivative, to answer the questions about $f(x)$.

Instructions: Move the point on the graph of $f'(x)$ and connect it to the behavior of $f(x)$. Click the button to show the graph of $f(x)$ to check your understanding.

Graph of $f'(x)$

Check your understanding: Click the button to reveal the graph of $f(x)$.

Show Graph of f(x)

Standalone
Embed

Since we don't have a huge amount of detail, you'll likely have to estimate the x-values for intervals and points in the following questions, but that's ok! Estimate away! Just make sure you know what you're looking for in the graph of $f'(x)$ to answer these questions.

(a) List the intervals on which $f(x)$ is increasing. What about decreasing?

(b) Find the x-values of any local maximums and/or local minimums of $f(x)$.

(c) List the intervals on which $f(x)$ is concave up. What about concave down?

(d) Find the x-values of any inflection points of $f(x)$.

Activity 5.4.2 Interpreting Area.

First, we're going to define a bit of a weird function. Sometimes it's called the **Area function**:

$$A(x) = \int_{t=0}^{t=x} g(t)\, dt.$$

This is a strange function, because we're defining the function as an integral of another function. Specifically, note that the *input* for our area function $A(x)$ is the ending limit of integration: we're calculating the signed area "under" the curve of $g(t)$ from $t = 0$ up to some variable ending point $t = x$.

We can visualize this function by looking at the areas we create as we change x. For now, get used to just seeing the area "under" g when we move the point around. The areas themselves are the outputs of the function $A(x)$.

Instructions: Move the point on the graph of $g(t)$ and connect it to the behavior of $A(x)$ where

$$A(x) = \int_{t=0}^{t=x} g(t)\, dt$$

Click the button to show the graph of $A(x)$ to check your understanding.

Graph of $g(t)$

$(x, g(x))$

Check your understanding: Click the button to reveal the graph of $A(x)$.

Show Graph of A(x)

Standalone
Embed

Now we can think about this area function, and try to connect it to the graph of $g(t)$.

(a) List the intervals on which $A(x)$ is increasing. What about decreasing?

(b) Find the x-values of any local maximums and/or local minimums of $A(x)$.

(c) List the intervals on which $A(x)$ is concave up. What about concave down?

(d) Find the x-values of any inflection points of $A(x)$.

(e) Compare your answers here to your answers about the behavior of $f(x)$ based on the (same) graph of $f'(x)$ in Activity 5.4.1.

What does this mean about the connection between areas and derivatives, or areas and anti-derivatives?

Activity 5.4.3 Evaluating Areas and Antiderivatives.

In this short activity, we'll just collect information about antiderivatives and this new area function,

$$A(x) = \int_{t=a}^{t=x} f(t)\, dt$$

for a function $f(t)$ that is continuous on the interval $a \le t \le x$.
For our purposes in this activity, let's say that $f(x) = x + \cos(x)$.

(a) From the Fundamental Theorem of Calculus (Part 1), we know that $A(x)$ is an antiderivative of $f(x)$, since $A'(x) = f(x)$.

Write out the function $A(x)$, and then name/write out one *other* antiderivative of $f(x)$, some $F(x)$.

(b) We know that all of the antiderivatives of a function are connected to each other.

Describe the connection between $A(x)$ and your $F(x)$.

(c) What is the value of $A(a)$? What is the value of $F(a)$? How are they different from each other?

(d) What is the value of $A(b)$? What is the value of $F(b)$? How are they different from each other?

(e) What about the differences: $A(b) - A(a)$ compared to $F(b) - F(a)$?

5.5 More Results about Definite Integrals

Activity 5.5.1 Symmetry in Functions and Integrals.

First, let's take a moment to remind ourselves (or see for the first time) what two types of "symmetry" we'll be considering. We call them "even" and "odd" symmetry, but sometimes we think of them as a "reflective" symmetry and a "rotational" symmetry in the graphs of our functions.

Instructions: Use the selection options and drag the points to remind yourself how even and odd symmetry looks graphically in both the way the functions are represented and also with how the areas/integrals are impacted.

Select which type of symmetry you'd like to visualize:

● Even Symmetry
○ Odd Symmetry

Show integrals

Standalone
Embed

(a) Convince yourself that you know what we mean when we say that a function is **even symmetric** on an interval if $f(-x) = f(x)$ on the interval.

Similarly, convince yourself that you know what we mean when we say that a function is **odd symmetric** on an interval if $f(-x) = -f(x)$ on the interval.

(b) Now let's think about areas. Before we visualize too much, let's start with a small question: How does the height of a function impact the area defined by a definite integral? It should be helpful to think about Riemann sums and areas of rectangles here.

The important question then, is how does a function being even or odd symmetric tell us information about areas defined by definite integrals of that function?

Activity 5.5.2 Connecting Symmetric Integrals.

We're going to do some sketching here, and I want you to be clear about something: your sketches can be absolutely terrible. It's ok! They just need to embody the kind of symmetry we're talking about. You will probably sketch something and notice that your areas aren't to scale (or maybe even the wrong sign!), and that's fine.

It might be helpful to practice sketching graphs accurately, but don't worry if that part is a struggle.

(a) Sketch a function $f(x)$ with the following properties:

- $f(x)$ is *even symmetric* on the interval $[-6, 6]$
- $\displaystyle\int_{x=0}^{x=6} f(x)\, dx = 4$
- $\displaystyle\int_{x=-6}^{x=-2} f(x)\, dx = -1$

(b) Find the values of the following integrals:

- $\displaystyle\int_{x=0}^{x=2} f(x)\, dx$
- $\displaystyle\int_{x=-6}^{x=6} f(x)\, dx$

(c) Sketch a function $g(x)$ with the following properties:

- $g(x)$ is *odd symmetric* on the interval $[-9, 9]$
- $\displaystyle\int_{x=0}^{x=4} g(x)\, dx = 5$
- $\displaystyle\int_{x=-9}^{x=0} g(x)\, dx = 2$

(d) Find the values of the following integrals:

- $\displaystyle\int_{x=-9}^{x=-4} g(x)\, dx$
- $\displaystyle\int_{x=-4}^{x=9} g(x)\, dx$

Activity 5.5.3 Visualizing the Average Height of a Function.

We are going to build a formula to find the "average height" or "average value" of a function $f(x)$ on the interval $[a, b]$. We're going to look at a function and try to find the average height. Along the way, we'll think a bit about areas!

(a) Consider the following function. Find the average height of the function on the interval pictured!

Instructions: Move the purple horizontal line up or down until you think it is at the vertical location of the average value of $f(x)$ on the interval $[a, b]$. Check your answer using the button on the bottom of the graph.

Check your answer!

Move the horizontal line up or down until you think it is close to the average height of the function.

▶ Hint! (click to open)

Standalone
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(b) How does the area "under" the curve $f(x)$ on the interval compare to the area of the rectangle formed by the average height line?

(c) How do you define the two areas?

(d) Set up an equation connecting the two areas, and solve for the average height of $f(x)$.

5.6 Introduction to u-Substitution

Activity 5.6.1 A Hard Integral.

We're going to be thinking about two integrals here, but before we do, we should remind ourselves about how we can "re-phrase" an integration question.

If we are asked to find $\int f(x)\,dx$, then we are really being asked to find some function $F(x)$ whose derivative is $f(x)$. Of course, we're actually being asked to find *all* of the possible functions that fit this requirement, but we know that the constant of integration covers all of the differences.

This means that we can (and should?) check our answers pretty consistently: just find a derivative and check that it matches the integrand!

(a) Find $\int 3x^2 e^{x^3}\,dx$.

(b) Explain the role of $3x^2$ in the integral. Explain why the positioning of x^3 matters.

(c) Let's try another one.
$$\int 7x^6 \cos(x^7)\,dx$$

(d) How can you tell, in general, that the derivative you're looking at is one that was produced through the Chain Rule and not the Product Rule?

Activity 5.6.2 Picking the Pieces of a Substitution.

We're going to look at three integrals. Instead of working through them one-at-a-time, we'll look at all three simultaneously, where we can practice identifying, substituting, and antidifferentiating all at the same time.

(a) Let's consider these three integrals:

- $\displaystyle \int \frac{3x^2 + 1}{(x^3 + x - 2)^2}\, dx$

- $\displaystyle \int \cos(x)\sqrt{\sin(x)}\, dx$

- $\displaystyle \int \frac{(\ln(x))^3}{x}\, dx$

For each of these integrals, identify the substitution: define u as some function of x.

(b) For each substitution, define $du = u'\, dx$.

(c) For each integral, use your substitution (for both u and the differential du) to re-write the integral.

(d) Antidifferentiate each integral, and then use your substitution to write each integral back in terms of x.

Activity 5.6.3 Compare Two Integrals.

Let's compare two integrals, and use this to build a more general strategy for performing u-substitution.

(a) Consider the following integral:

$$\int -4x^3 \sec^2(1 - x^4)\, dx$$

Select and justify a choice for u.

(b) Perform the u-substitution and antidifferentiate, and then substitute back to write your anti-derivative in terms of x.

(c) Compare that integral to this one:

$$\int x^3 \sec^2(1 - x^4)\, dx$$

What is different about this new integral? What has remained the same? How does that impact your choice for u, or *does* it impact your choice for u?

(d) Has that changed what du should be?

(e) Ok so we've noticed an issue here. There are *plenty* of good ways of solving this problem, where du doesn't "show up" perfectly in our integral. In this case, we have that we're missing a necessary coefficient. We have the x^3 part, but we are missing the -4.

Try to re-write our integral with a -4 coefficient in there. We'll do that by multiplying the integrand function by 1, disguised as $\frac{-4}{-4}$ or $\left(-\frac{1}{4}\right)(-4)$.

(f) Now we can use the same u-substitution as before, and integrate in a similar way! Notice, though, that we will retain the coefficient of $-\frac{1}{4}$.

(This should be reasonable: our integral is $-\frac{1}{4}$ of the original one, since our coefficient was 1 to the original's -4.)

Go ahead and integrate!

Chapter 6

Applications of Integrals

6.1 Integrals as Net Change

Activity 6.1.1 Estimating Movement.

We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 6.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

(b) When was the acceleration of the object the greatest? When was it the least?

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

Activity 6.1.2 A Friendly Jogger.

Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

(b) Does this time represent a maximum or minimum velocity for the jogger?

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

Activity 6.1.3 Tracking our Jogger.

Let's revisit our jogger from Activity 6.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

(b) Think back to our description of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.
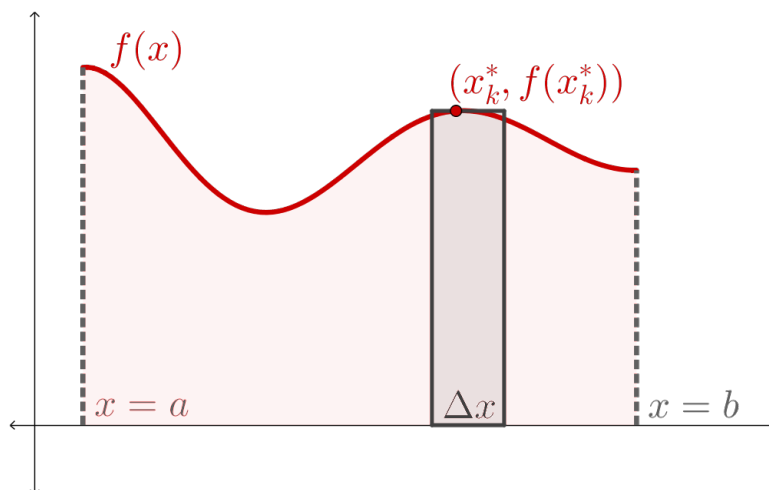
6.2 Area Between Curves

Activity 6.2.1 Remembering Riemann Sums.

Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

Figure 6.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

Figure 6.2.2

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

Activity 6.2.2 Area Between Curves.

Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

Figure 6.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

Figure 6.2.4

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Activity 6.2.3 Trying for a Single Integral.

Let's consider the same setup as earlier: the region bounded between two curves, $y = x + 6$ and $y = x^3$, as well as the x-axis (the line $y = 0$). We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

Figure 6.2.5

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths for each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

$(2,8)$

$y = x + 6$
$\downarrow$
$y - 6 = x$

$y = x^3$
$\downarrow$
$\sqrt[3]{y} = x$

Δy

$y = 0$
$(-6, 0)$

Figure 6.2.6

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

(j) Find A_k, the area of this kth rectangle.

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

6.3 Volumes of Solids of Revolution

Activity 6.3.1 Carving out a Hole in the Center.

We're going to look at the same solid as in Figure 6.3.2. But this time, when we define the 2-dimensional region that we're going to revolve around the x-axis, we're going to add a lower boundary function, $g(x)$.

Figure 6.3.1

When we revolve this region around the x-axis, we get the following 3-dimensional solid.
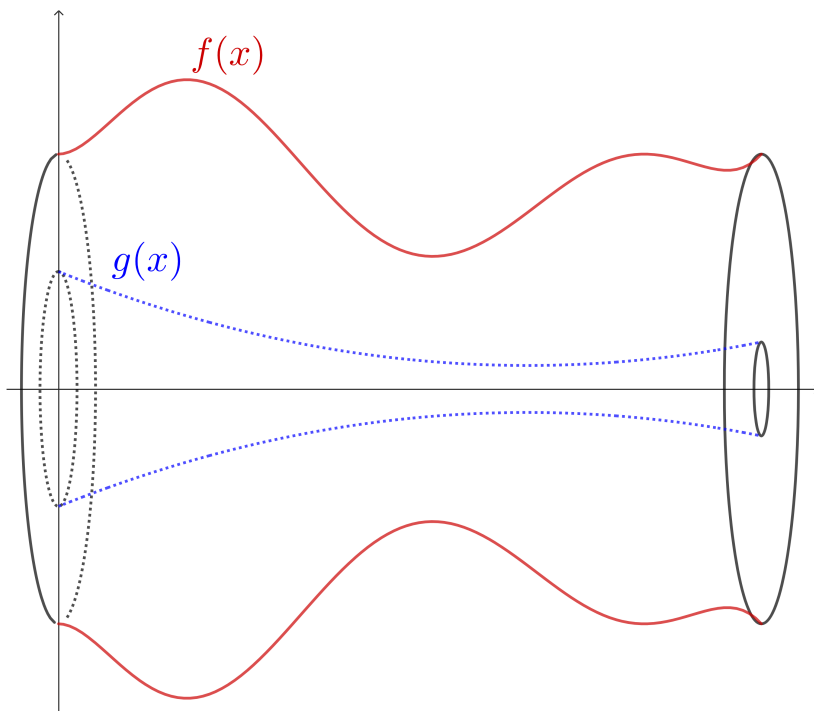

Figure 6.3.2

(a) How is a single generic slice on this solid different than the one in Figure 6.3.2?

(b) Find a formula for the area of the face of the cross-secional slice.

(c) Use the slice-and-sum process to create an integral expression representing the volume of this solid.

Activity 6.3.2 Volumes by Disks/Washers.

Consider the region bounded between the curves $y = 4 + 2x - x^2$ and $y = \dfrac{4}{x+1}$. Will will create a 3-dimensional solid by revolving this region around the x-axis.

Figure 6.3.3

(a) Visualize the solid you'll create when you revolve this region around the x-axis.

(b) Draw a single rectangle in your region, standing perpendicular to the x-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of the solid.

(f) Can you describe how you would antidifferentiate and evaluate this integral?

Activity 6.3.3 Another Volume.

Now lets consider another region: this time, the one bounded between the curves $y = x$ and $y = 3\sqrt{x}$. We will, again, create a 3-dimensional solid by revolving this region around the y-axis.

Figure 6.3.4

(a) Visualize the solid you'll create when you revolve this region around the y-axis.

(b) Draw a single rectangle in your region standing perpendicular to the y-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of this solid.

Activity 6.3.4 Volume by Shells.

Let's consider the region bounded by the curves $y = x^3$ and $y - x + 6$ as well as the line $y = 0$. You might remember this region from Activity 6.2.3. This time, we'll create a 3-dimensional solid by revolving the region around the x-axis

Figure 6.3.5

(a) Sketch one or two rectangles that are *perpendicular* to the x-axis. Then set up an integral expression to find the volume of the solid using them.

(b) Now draw a single rectangle in the region that is *parallel* to the axis of revolution. Use this rectangle to visualize the kth slice of this 3-dimensional solid. What does that single rectangle create when it is revolved around the x-axis?

(c) Set up the integral expression representing the volume of the solid.

(d) Confirm that your volumes are the same, no matter your approach to setting it up.

6.4 More Volumes: Shifting the Axis of Revolution

Activity 6.4.1 What Changes (in the Washer Method) with a New Axis?

Let's revisit Activity 6.3.2 Volumes by Disks/Washers, and ask some more follow-up questions. First, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = -3$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.2?

(b) What changes, if anything, do you have to make to the area of the "face" kth washer?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

Activity 6.4.2 What Changes (in the Shell Method) with a New Axis?

Let's revisit Activity 6.3.4 Volume by Shells, and ask some more follow-up questions about the shell method. Again, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = 9$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.4?

(b) What changes, if anything, do you have to make to the area of the rectangle formed by "unrolling" up kth cylinder?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

Activity 6.4.3 More Shifted Axes.

We're going to spend some time constructing *several* different volume integrals in this activity. We'll consider the same region each time, but make changes to the axis of revolution. For each, we'll want to think about what kind of method we're using (disks/washers or shells) and how the different axis of revolution gets implemented into our volume integral formulas.

Let's consider the region bounded by the curves $y = \cos(x) + 3$ and $y = \dfrac{x}{2}$ between $x = 0$ and $x = 2\pi$.

(a) Let's start with revolving this around the x-axis and thinking about the solid formed. While you set up your volume integral, think carefully about which method you'll be using (disks/ washers or shells) as well as which variable you are integrating with regard to (x or y).

(b) Now let's create a different solid by revolving this region around the y-axis. Set up a volume integral, and continue to think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

(c) We'll start shifting our axis of revolution now. We'll revolve the same region around the horizontal line $y = -1$ to create a solid. Set up an integral expression to calculate the volume.

(d) Now revolve the region around the line $y = 5$ to create a solid of revolution, and write down the integral representing the volume.

(e) Let's change things up. Revolve the region around the vertical line $x = -1$ to create a new solid. Set up an integral representing the volume of that solid.

(f) We'll do one more solid. Let's revolve this region around the line $x = 7$. Set up an integral representing the volume.

6.5 Arc Length and Surface Area

Activity 6.5.1 Measuring Distance.

(a) Consider the following right-triangle with the normal names of side lengths.

Figure 6.5.1

How do we use the Pythagorean Theorem to find the length of c?

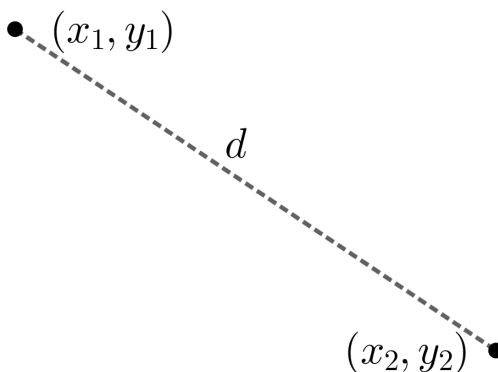(b) Consider the two points (x_1, y_1) and (x_2, y_2) below.

Figure 6.5.2

How do we use the distance formula to find the length of the line connecting the two point, d?

(c) How are these two things the same? How are they different?

6.6 Other Applications of Integrals

Chapter 7

Techniques for Antidifferentiation

7.1 Improper Integrals

Activity 7.1.1 Remembering a Theme so Far.

(a) Let's say that we want to find what the y-values of some function $f(x)$ are when the x-values are "infinitely close to" some value, $x = a$. Since there is no single x-value that is "infinitely close to" a that we can evaluate $f(x)$ at, we need to do something else. How do we do this?

(b) Let's say that we want to find the rate of change of some function instantaneously at a point with $x = a$. We can't find a rate of change unless we have two points, since we need to find some differences in the outputs and inputs. How do we do this?

(c) Suppose you want to find the total area, covered by an infinite number of infinitely thin rectangles. You have a formula for finding the dimensions and areas for some finite number of rectangles, but how do we get an infinite number of them?

(d) Can you find the common calculus theme in each of these scenarios?

Activity 7.1.2 Remembering the Fundamental Theorem of Calculus.

We want to think about generalizing our notion of integrals a bit. So in this activity, section, we're going to think about some of the requirements for the Fundamental Theorem of Calculus and try to loosen them up a bit to see what happens. We'll try to construct meaningful approaches to these situations that fit our overall goals of calculating area under a curve.

This practice, in general, is a really good and common mathematical process: taking some result and playing with the requirements or assumptions to see what else can happen. So it might feel like we're just fiddling with the "What if?" questions, but what we're actually doing is good mathematics!

(a) What does the Fundamental Theorem of Calculus say about evaluating the definite integral $\int_{x=a}^{x=b} f(x)\,dx$?

(b) What do we need to be true about our setup, our function, etc. for us to be able to apply this technique to evaluate $\int_{x=a}^{x=b} f(x)\,dx$?

We are going to introduce the idea of "Improper Integrals" as kind-of-but-not-quite definite integrals that we can evaluate. They are going to violate the requirements for the Fundamental Theorem of Calculus, but we'll work to salvage them in meaningful ways.

Activity 7.1.3 Approximating Improper Integrals.

In this activity, we're going to look at two improper integrals:

1. $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\, dx$

2. $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$

(a) First, let's just clarify to ourselves what it means for an integral to be improper. Why are each of these integrals improper? Be specific!

(b) Let's focus on $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\, dx$ first. We're going to look at the slightly different integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $t > 2$, then our function is continuous and bounded on $[2, t]$, and so we can evaluate this integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$.

Find and antiderivative, $F(x)$.

(c) Now we're going to evaluate some areas for different values of t. Use your antiderivative $F(x)$ from above!

- Let's start with making $t = 99$. So we're going to evaluate:

$$\int_{x=2}^{x=99} \frac{1}{(x+1)^2}\, dx = F(99) - F(2)$$

- Now let $t = 999$. Evaluate:

$$\int_{x=2}^{x=999} \frac{1}{(x+1)^2}\, dx = F(999) - F(2)$$

- Now let $t = 9999$. Evaluate:

$$\int_{x=2}^{x=9999} \frac{1}{(x+1)^2}\, dx = F(9999) - F(2)$$

(d) Based on what you found, what do you *think* is happening when $t \to \infty$ to the definite integral

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)?$$

(e) Ok, we're going to switch our focus to the other improper integral, $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$. again, we'll look at a slightly different integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $-1 < t < 2$, then our function is continuous and bounded on $[t, 2]$, and so we can evaluate this integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)$$

where $F(x)$ is an antiderivative of $f(x) = \frac{1}{(x+1)^2}$. We can just use the same antiderivative as before!

We're going to evaluate this intergal for different values of t again, but this time we'll use values that are close to -1, but slightly bigger, since we want to be in the interval $[-1, 2]$.

- Let's start with making $t = -\frac{9}{10}$. So we're going to evaluate:

$$\int_{x=-9/10}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{9}{10}\right)$$

- Now let $t = -\frac{99}{100}$. Evaluate:

$$\int_{x=-99/100}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{99}{100}\right)$$

- Now let $t = -\frac{999}{1000}$. Evaluate:

$$\int_{x=-999/1000}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{999}{1000}\right)$$

(f) Based on what you found, what do you *think* is happening when $t \to -1^+$ to the definite integral

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)?$$

We can think about putting this a bit more generally into limit notation, but we'll get to this later.

7.2 More on u-Substitution

Activity 7.2.1 Recapping u-Substitution.

We're going to consider a few integrals, and work through each of the questions for all integrals.

1. $\displaystyle \int \left(\frac{1}{\sqrt{4 - 3x}} \right) \, dx$

2. $\displaystyle \int x^2 (5 + x^3)^7 \, dx$

3. $\displaystyle \int \left(\frac{\sin(x)}{\cos(x)} \right) \, dx$

4. $\displaystyle \int \left(x e^{-x^2} \right) \, dx$

(a) For each integral, explain why u-substitution is a good choice. How can you tell, just by looking at the integral, that this strategy will be a reasonable thing to try?

(b) For each integral, explain your choice of u, what that means for how we define du.

(c) For each integral, is your definition of du present in the integrand function? How do you go about making this substitution when the integrand function isn't set up perfectly?

(d) Finish the substitution and integration, and substitute back to the original variable.

Activity 7.2.2 Turn Around Problems.

The two integrals that we're going to look at are "just" some u-substitution problems, but I like to call integrals like these **turn-around** problems. We'll see why!

(a) Consider the integral:

$$\int x \sqrt[3]{x + 5} \, dx.$$

First, explain why u-substitution is reasonable here.

(b) Identify du for your chosen u-substitution. When you substitute, you should notice that there are some extra bits in this integrand function that have not been assigned to be translated over to be written in terms of u. Which parts?

(c) We need to think about how to write x in terms of u. Luckily, we already have everything we need! We have defined a link between the x variable and the u variable. We defined it as u being written as some function of x, but can we "turn around" that link to write x in terms of u?

(d) Substitute the integral to be fully written in terms of u.

(e) Before antidifferentiating, compare this integral with the original one. Specifically thinking about how we might multiply, describe the differences between the integrals with regard to composition and re-writing our integrand.

Then, go ahead and use this nicely re-written version to antidifferentiate and substitute back to x.

(f) Apply this same strategy to the following integral:

$$\int \frac{x}{x + 5} \, dx.$$

This integral might be a bit trickier to find the composition in order to identify the u-substitution! Give some things a try!

(g) Compare your integral in terms of x with the substituted version, in terms of u. Why was the second one so much easier to think about or re-write?

7.3 Manipulating Integrands

Activity 7.3.1 A Negative Exponent.

Let's think about this integral:

$$\int \frac{1}{1 + e^{-x}} \, dx.$$

(a) Is there any composition in this integral? Pick it out, and either explain or show that using this to guide your substitution will not be helpful.

(b) What does e^{-x} mean? What does $\frac{1}{e^{-x}}$ mean?

(c) Re-write the integral, specifically focusing on the negative exponent. You should find that the function looks worse! How can you clean that up?

(d) Why is this new integral set up so much better for the purpose of u-substitution? How could we tell this just by looking at the initial integral?

Activity 7.3.2 Integrating a Rational Function Three Ways.

We're going to think about the integral:

$$\int \left(\frac{x^2 + 3x - 1}{x - 1} \right) \, dx.$$

Let's find 3 different ways of integrating this. This is kind of misleading, since we're actually going to look at 2, since we've already used u-substitution to integrate this in Example 7.2.1.

(a) Let's just notice some things about this rational function.

- Are there any vertical asymptotes? How do you know where to find them?
- Are there any horizontal asymptotes? How do you know that there *aren't*?
- When you zoom really far out on the graph of this function, it looks like a different kind of function. What kind of function? Why is that?

(b) Now we're going to re-write the function itself: $\dfrac{x^2 + 3x - 1}{x - 1}$ means we're dividing $(x^2 + 3x - 1)$ by $(x - 1)$. So let's do the division!

$$x - 1 \,\overline{\big)\, x^2 \quad +3x \quad -1}$$

(c) Re-write your integral using this new version of the function. Notice that we haven't done any calculus or antidifferentiating yet. Explain why this new version of this integrand function is easier to antidifferentiate. What do you get?

(d) Let's approach this integral differently. We said earlier that this function is really an "almost" linear function in disguise: when we divide the quadratic numerator by a linear denominator, we expect a linear function to be left over. In the long division, we saw this happen! We ended with a linear function and some remainder.

Let's try to uncover this linear function. If we're looking to find what linear functions multiply together to get $(x^2 + 3x - 1)$, then we can try factoring!

$$\frac{x^2 + 3x - 1}{x - 1} = \frac{(\underline{\hspace{1.8cm}})(\underline{\hspace{1.8cm}})}{x - 1}$$

In order for this factoring to be useful, we want to be able to "cancel" out the $x - 1$ factor in the denominator. We're really only interested in what linear factor will multiply by $(x - 1)$ to get $(x^2 + 3x - 1)$.

$$\frac{x^2 + 3x - 1}{x - 1} = \frac{(x - 1)(\underline{\hspace{1.8cm}})}{x - 1}$$

First, explain why there is no linear function factor that accomplishes this.

(e) What if were able to "almost" factor this?

If there *was* a linear factor that multiplied by $(x - 1)$ to get $(x^2 + 3x - 1)$, then the linear portions would multiply together to get x^2. What does this mean about the first linear term of our factor?

(f) What does the constant term of our missing factor need to be? We are hoping that whatever it is can multiply by x (from $x - 1$) and combine with the $-x$ (from the constant -1 multiplied by x in our missing factor) to match the $3x$ in $x^2 + 3x - 1$.

What is it?

(g) Note that we have *not* factored $(x^2 + 3x - 1)$! We *almost* did: we found two factors:

$$(x - 1)() = x^2 + 3x + .$$

How far off is the actual polynomial that we are working with, $x^2 + 3x - 1$?

Write $x^2 + 3x - 1$ as your two factors plus or minus some remainder.

(h) You should get the same thing that we got from using long division! Great! The rest of the integral will work the same.

Before we end, though, compare this antiderivative to the one we got in Example 7.2.1. It's different. Why? Is this a problem?

Activity 7.3.3 Comparing Two Very Similar Integrals.

We're going to compare these two integrals:

$$\int \frac{x+2}{x^2+4x+5}\,dx \qquad\qquad \int \frac{2}{x^2+4x+5}\,dx$$

(a) Describe why $u = x^2 + 4x + 5$ is such a useful choice for the first integral, but not for the second. How do the differences in these two integrals influence this substitution, even though the denominators are the same?

(b) Why would it be useful to have a *linear* substitution rule (instead of the *quadratic* one that we picked) for the second integral? Why would that match the structure of the numerator better?

Go ahead and integrate the first integral.

(c) We're going to write the denominator, $x^2 + 4x + 5$ in a different way, in order to get a linear function composed into something familiar.

Complete the square for this polynomial: that is, find some linear factor $(x + k)$ and a real number b such that $(x+h)^2 + b = x^2 + 4x + 3$. This should feel familiar, since we have already tried to force polynomials to factor cleverly in Activity 7.3.2.

(d) There is an intuitive substitution to pick, since we now have more obvious composition. Pick it. What kind of integral do we end up with and how do we antidifferentiate? Complete this problem!

7.4 Integration By Parts

Activity 7.4.1 Discovering the Integration by Parts Formula.

The product rule for derivatives says that:

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = \underline{\qquad} + \underline{\qquad}.$$

We know that we intend to "undo" the product rule, so let's try to re-frame the product rule from a rule about derivatives to a rule about antiderivatives.

(a) Antidifferentiate the product rule by antidifferentiating each side of the equation.

$$\int \left(\frac{d}{dx}(u \cdot v)\right) \, dx = \int \underline{\qquad} + \underline{\qquad} \, dx$$

$$\underline{\qquad} = \int \underline{\qquad} \, dx + \int \underline{\qquad} \, dx$$

(b) On the right side, we have two integrals. Since each of them has a product of functions (one function and a derivative of another), we can isolate one of them in this equation and create a formula for how to antidifferentiate a product of functions! Solve for $\int uv' \, dx$.

(c) Look back at this formula for $\int uv' \, dx$. Explain how this is really the product rule for derivatives (without just undo-ing all of the steps we have just done).

What made it so useful to pick $u = x$ instead of $dv = x \, dx$ in this case? Since we know that we are going to get another integral, one that specifically has the new derivative and new antiderivative that we find from the parts we picked, we noticed that differentiating the function x was much nicer than antidifferentiating it: we get a constant that we multiply by the trig function in this new integral, instead of a power function with an even bigger exponent. We can also notice that when it comes to the trig function, it doesn't really matter if we differentiate it or antidifferentiate it. In both cases, we get a $cos(x)$ in our new integral, with the only difference being whether it is positive or negative.

Activity 7.4.2 Picking the Parts for Integration by Parts.

Let's consider the integral:

$$\int x \sin(x) \, dx.$$

We'll investigate how to set up the integration by parts formula with the different choices for the parts.

(a) We'll start with selecting $u = x$ and $dv = \sin(x) \, dx$. Fill in the following with the rest of the pieces:

$$u = x \qquad\qquad v = \underline{\hspace{2cm}}$$
$$du = \underline{\hspace{2cm}} \qquad dv = \sin(x) \, dx$$

(b) Now set up the integration by parts formula using your labeled pieces. Notice that the integration by parts formula gives us another integral. Don't worry about antidifferentiating this yet, let's just set the pieces up.

(c) Let's swap the pieces and try the setup with $u = \sin(x)$ and $dv = x \, dx$. Fill in the following with the rest of the pieces:

$$u = \sin(x) \qquad\qquad v = \underline{\hspace{2cm}}$$
$$du = \underline{\hspace{2cm}} \qquad dv = x \, dx$$

(d) Now set up the integration by parts formula using this setup.

(e) Compare the two results we have. Which setup do you think will be easier to move forward with? Why?

(f) Finalize your work with the setup you have chosen to find $\int x \sin(x) \, dx$.

What made things so much better when we chose $u = x$ compared to $dv = x \, dx$? We know that the new integral from our integration by parts formula will be built from the new pieces, the derivative we find from u and the antiderivative we pick from dv. So when we differentiate $u = x$, we get a constant, compared to antidifferentiating $dv = x \, dx$ and getting another power function, but with a larger exponent. We know this will be combined with a $cos(x)$ function no matter what (since the derivative and antiderivatives of $sin(x)$ will only differ in their sign). So picking the version that gets that second integral to be built from a trig function and a constant is going to be much nicer than a trig function and a power function. It was nice to pick x to be the piece that we found the derivative of!

Activity 7.4.3 Picking the Parts for Integration by Parts.

This time we'll look at a very similar integral:

$$\int x \ln(x) \; dx.$$

Again, we'll set this up two different ways and compsare them.

(a) We'll start with selecting $u = x$ and $dv = \ln(x) \; dx$. Fill in the following with the rest of the pieces:

$$u = x \qquad\qquad v = \underline{\hphantom{XXXXXXX}}$$
$$du = \underline{\hphantom{XXXXXXX}} \qquad dv = \ln(x) \; dx$$

(b) Ok, so here we *have to* swap the pieces and try the setup with $u = \ln(x)$ and $dv = x \; dx$, since we only know how to differentiate $\ln(x)$. Fill in the following with the rest of the pieces:

$$u = \ln(x) \qquad\qquad v = \underline{\hphantom{XXXXXXX}}$$
$$du = \underline{\hphantom{XXXXXXX}} \qquad dv = x \; dx$$

(c) Now set up the integration by parts formula using this setup.

(d) Why was it fine for us to antidifferentiate x in this example, but not in Activity 7.4.2?

(e) Finish this work to find $\int x \ln(x) \; dx$.

So here, we didn't actually get much choice. We couldn't pick $u = x$ in order to differentiate it (and get a constant to multiply into our second integral) since we don't know how to antidifferentiate $\ln(x)$ (yet: once we know how, it might be fun to come back to this problem and try it again with the parts flipped). But we can also notice that it ended up being fine to antidifferentiate x: the increased power from our power rule didn't really matter much when we combined it with the derivative of the logarithm, since the derivative of the log is *also a power function*! So we were able to combine those easily and actually integrate that second integral.

Activity 7.4.4 Squared Trig Functions.

Let's look at two integrals. We'll talk about both at the same time, since they're similar.

$$\int \sin^2(x)\,dx \qquad\qquad\qquad \int \cos^2(x)\,dx$$

(a) What does it mean to "square" a trig function? Write these integrals in a different way, where the meaning of the "squared" exponent is more clear. What do you notice about the structure of these integrals, the operation in the integrand function? What does this mean about our choice of integration technique?

(b) If you were to use integration by parts on these integrals, does your choice of u and dv even matter here? Why not?

(c) Apply the integration by parts formula to each. What do you notice?

(d) Instead of applying another round of integration by parts to the resulting integral, use the Pythagorean identities to re-write these integrals:

$$\sin^2(x) = 1 - \cos^2(x)$$
$$\cos^2(x) = 1 - \sin^2(x)$$

(e) You should notice that in your equation for the integration of $\sin^2(x)\,dx$, you have another copy of $\int \sin^2(x)\,dx$. Similarly, in your equation for the integration of $\cos^2(x)\,dx$, you have another copy of $\int \cos^2(x)\,dx$.

Replace these integrals with a variable, like I (for "integral"). Can you "solve" for this variable (integral)?

7.5 Integrating Powers of Trigonometric Functions

Activity 7.5.1 Compare and Contrast.

Let's do a quick comparison of two integrals, keeping the above examples in mind. Consider these two integrals:

$$\int \sin^4(x) \cos(x) \, dx \qquad\qquad \int \sin^4(x) \cos^3(x) \, dx$$

(a) Consider the first integral, $\int \sin^4(x) \cos(x) \, dx$. Think about and set up a good technique for antidifferentiating. Without actually solving the integral, explain why this technique will work.

(b) Now consider the second integral, $\int \sin^4(x) \cos^3(x) \, dx$. Does the same integration strategy work here? What happens when you apply the same thing?

(c) We know that $\sin(x)$ and $\cos(x)$ are related to each other through derivatives (each is the derivative of the other, up to a negative). Is there some other connection that we have between these functions? We might especially notice that we have a $\cos^2(x)$ left over in our integral. Can we write this in terms of $\sin(x)$, so that we can write it in terms of u?

(d) Why would this strategy not have worked if we were looking at the integrals $\int \sin^4(x) \cos^2(x) \, dx$ or $\int \sin^4(x) \cos^4(x) \, dx$? What, specifically, did we need in order to use this combination of substitution and trigonometric identity to solve the integral?

Activity 7.5.2 Compare and Contrast (Again).

We're going to do another Compare and Contrast, but this time we're only going to consider one integral:

$$\int \sec^4(x) \tan^3(x) \, dx.$$

We're going to employ another strategy, similar to the one for Integrating Powers of Sine and Cosine.

(a) Before you start thinking about this integral, let's build the relevant version of the Pythagorean Identity that we'll use. Our standard version of this is:

$$\sin^2(x) + \cos^2(x) = 1.$$

Since we want a version that connects $\tan(x)$, which is also written as $\frac{\sin(x)}{\cos(x)}$, with $\sec(x)$, or $\frac{1}{\cos(x)}$, let's divide everything in the Pythagorean Identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{\rule{2cm}{0.4pt}} + \frac{\cos^2(x)}{\rule{2cm}{0.4pt}} = \frac{1}{\rule{2cm}{0.4pt}}$$

$$\rule{2cm}{0.4pt} + \rule{2cm}{0.4pt} = \rule{2cm}{0.4pt}$$

(b) Now start with the integral. We're going to use two different processes here, two different u-substitutions. First, set $u = \tan(x)$. Complete the substitution and solve the integral.

(c) Now try the integral again, this time using $u = \sec(x)$ as your substitution.

(d) For each of these integrals, why were the exponents set up *just right* for u-substitution each time? How does the structure of the derivatives of each function play into this?

(e) Which substitution would be best for the integral $\int \sec^4(x) \tan^4(x) \, dx$. Why?

(f) Which substitution would be best for the integral $\int \sec^3(x) \tan^3(x) \, dx$. Why?

7.6 Trigonometric Substitution

Activity 7.6.1 Difference of Squares.

Consider the integral:

$$\int \sqrt{1 - x^2} \, dx.$$

(a) First, convince yourself that a normal u-substitution will not be an effective strategy for integration in this case. Why not?

(b) Second, convince yourself that $\sqrt{1 - x^2} \neq \sqrt{1} - \sqrt{x^2}$. Why can we not distribute roots across sums and differences like this? When *can* we "distribute" roots across multiple things?

(c) Our goal, then, is to utilize a substitution (using trigonometric functions) to somehow transform this difference of squared terms under the square root into a single product of squared things under the square root.

Which trigonometric identities from our list of them above utilize differences of thing squared, and equate them to a single term?

Can you use the order of the subtraction to help guide which substitution we should use?

(d) When we do a variable substitution in an integral, we are not only finding a way of transforming x to be in terms of some other variable (in this case, θ). We also need to transform the differential, dx. Based on your substitution of $x = T(\theta)$, what is dx?

(e) Perform your substitution! Use your substitution $x = T(\theta)$ and $dx = T'(\theta) \, d\theta$. Note that we have picked this substitution with a very specific goal: we are hoping to notice a Pythagorean identity.

After you have performed your substitution, apply the relevant Pythagorean identity to the **radicand**: the bit of our function underneath the radical or root. What integral are we left with (in terms of θ)?

This new integral is something we can antidifferentiate now! We have already done this one in Activity 7.4.4 Squared Trig Functions. So we can end up with:

$$\int \sqrt{1 - x^2} \, dx = \int \cos^2(\theta) \, d\theta$$
$$= \frac{\theta + \sin(\theta)\cos(\theta)}{2} + C$$

It is up to us, now, to translate this antiderivative family to be written in terms of x. We can utilize our substitution to do this, but let's first think about how this variable substitution works a bit more.

Activity 7.6.2 Trig Substitution Geoemtry.

We're going to consider three triangles, and we're going to fill in side lengths. In each of these, we'll assume that the lengths x and a are real numbers and are positive.

Figure 7.6.1 Three triangles to guide our trigonometric substitutions.

(a) Use the Pythagorean theorem to label the missing side length in each of the three triangles.

(b) For each triangle, explain how you can tell which side length represents the hypotenuse when you see the lengths x, a, and then the missing lengths you found above: $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 + a^2}$.

(c) For each triangle, find a trigonometric function of θ that connects lengths x and a to each other.

Solve each for x to reveal the relevant substitution.

(d) For each substitution, find the corresponding substitution for the differential, dx.

Activity 7.6.3 Practicing Trigonometric Substitution.

Let's look at three integrals, and practice the kind of thinking we'll need to use to apply trigonometric substitution to them.

1. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

2. $\displaystyle\int \frac{2}{(4-x^2)^{3/2}}\,dx$

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

For each integral, do the following:

(a) Identify the term (or terms) that signify that trigonometric substitution might be a reasonable strategy.

(b) Use that portion of the integral to compare three side lengths of a triangle. Which one is the largest (and so must represent the length of the hypotenuse)?

(c) Construct the triangle, label an angle θ, and use a trigonometric function to connect the two single-term side lengths. (Feel free to change the angle you label in order to use the sine, secant, or tangent functions instead of their co-functions).

(d) Define your substitution (for both x and the differential dx), and identify the Pythagorean identity that will be relevant for the integral.

(e) Substitute and antidifferentiate!

(f) Use your triangle to substitute your antiderivative back in terms of x.

7.7 Partial Fractions

Activity 7.7.1 Comparing Rational Integrands.

We're going to compare three integrals:

$$\int \frac{2}{x^2 + 4x + 5} \, dx$$

$$\int \frac{2}{x^2 + 4x + 3} \, dx$$

$$\int \left(\frac{1}{x+1} - \frac{1}{x+3} \right) \, dx$$

(a) Start with the first integral:
$$\int \frac{2}{x^2 + 4x + 5} \, dx.$$

How would you approach integrating this?

(b) Try the same tactic on the second integral:
$$\int \frac{2}{x^2 + 4x + 3} \, dx.$$

You don't need to complete this integral, but think about how you might proceed.

(c) Think about the third integral:
$$\int \left(\frac{1}{x+1} - \frac{1}{x+3} \right) \, dx.$$

How would you integrate this?

(d) The third integral is unique from the other two in that is has two terms. Let's combine them together to see how we could write this integral to compare it more closely to the other two.

Subtract $\dfrac{1}{x+1} - \dfrac{1}{x+3}$ using common denominators and compare your re-written integral to the other two.

(e) Which of these integrals and/or representations of an integral is easiest to work with? Which one is most annoying to work with? Why?

Activity 7.7.2 First Examples of Partial Fractions.

(a) Consider the integral:
$$\int \frac{6x - 16}{x^2 - 4x + 3}\, dx.$$
First, confirm that this would be *very* annoying to try to use u-substitution on, even though we have a linear numerator and quadratic denominator.

(b) Notice that the denominator can be factored:
$$\int \frac{6x - 16}{(x - 3)(x - 1)}\, dx.$$
If this integrand function were a sum of two "smaller" rational functions, what would their denominators be? What do we know about their numerators?

(c) Use some variables (it's typical to use capital letters like A, B, C, etc.) to represent the numerators, and then add the partial fractions together. What do you get? How does this rational function compare to $\dfrac{6x - 16}{(x - 3)(x - 1)}$?

(d) Set up an equation connecting the numerators, and solve for your unknown variables. What are the two fractions that added together to get $\dfrac{6x - 16}{(x - 3)(x - 1)}$?

(e) Antidifferentiate to solve the integral $\displaystyle\int \frac{6x - 16}{(x - 3)(x - 1)}\, dx$.

(f) Let's do the same thing with a new integral:
$$\int \frac{3x^2 - 2x + 3}{x(x^2 + 1)}\, dx.$$
What are the partial fraction forms? What kinds of numerators should we expect to see? Use variables to represent these.

(g) Add the partial fractions together and set up an equation for the numerators to solve. What are the two partial fractions after you solve for the unknown coefficients?

(h) Antidifferentiate and solve the integral $\displaystyle\int \frac{3x^2 - 2x + 3}{x(x^2 + 1)}\, dx$.

Chapter 8

Infinite Series

8.1 Introduction to Infinite Sequences

Activity 8.1.1 Building our First Sequences.

We might already have some familiarity with sequences. Here, we'll focus less on some of the detailed mechanics and just think about these sequences as functions.

(a) Describe a sequence of numbers where you use a consistent rule/function to build each term (each number) based only on the *previous term* in the sequence. You will need to decide on some first term to start your sequence.

(b) Describe a different sequence of numbers using the same rule to generate new terms/numbers from the previous one. What do you need to do to make these two sequences different from each other?

(c) Describe a new sequence of numbers where you use a consistent rule/function to build each term based on its position in the sequence (i.e. the first term will be some rule/function based on the input 1, the second will be based on 2, you'll use 3 to get the third term, etc.). We will call the position of each term in the sequence the *index*.

(d) Describe another, new, sequence of numbers where you use a consistent rule/function to build each term based on its index. This time, make the terms get smaller in size as the index increases.

Activity 8.1.2 Returning to our First Sequences.

Let's return back to the four sequences we created in Activity 8.1.1.

(a) For each of the sequences, how are we going to define them? Explicit formulas? Recursion relations? How do you know?

(b) Now, for each sequence, define the sequence formally using either an explicit formula or recursion relation, whichever matches with how you described the sequence in Activity 8.1.1.

Activity 8.1.3 Describing These Sequences.

Let's look at the sequences from Example 8.1.3. Go through the following tasks for each sequence.

(a) What do you think each sequence is "counting towards" (if anything)?

(b) Can you show that the sequence is counting towards what you think it is with a limit (or show that it's not counting towards anything)?

Activity 8.1.4 Write the Sequence Rules.

We'll look at some sequences by writing out the first handful of terms. From there, our goal is to write out more terms and eventually define each sequence fully.

For each sequence, write an explicit formula and a recursion relation to define the sequence. You can choose whether to start your index at $k = 0$ or $k = 1$.

(a) $\{a_k\} = \left\{4, \frac{2}{3}, \frac{1}{9}, \frac{1}{54}, \ldots\right\}$

(b) $\{b_k\} = \left\{\frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, \ldots, \right\}$

(c) $\{c_k\} = \left\{\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, \ldots\right\}$

(d) What kinds of connections do you notice between the explicit formulas and the recursion relations for these sequences?

8.2 Introduction to Infinite Series

Activity 8.2.1 How Do We Think About Infinite Series?

Let's consider the following sequence:

$$\left\{ \frac{9}{10^k} \right\}_{k=1}^{\infty}$$

(a) White out the first 5 terms of the sequence.

(b) What does this sequence converge to? Show this with a limit!

(c) Now we'll construct a new sequence, this time by adding things up. We're going to be working with the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{k=n} \left(\frac{9}{10^k} \right).$$

Write out the first five terms of this sequence: S_1, S_2, S_3, S_4, S_5.

(d) Can you come up with an explicit formula for S_n?

(e) Does $\{S_n\}$ converge or diverge? Use a limit to find what it converges to!

(f) What do you think this means for the infinite series $\sum_{k=1}^{\infty} \left(\frac{9}{10^k} \right)$? Does the infinite series converge or diverge?

This is hopefully a nice little introduction to how we'll think about infinite series: we'll consider, instead, the sequence of sums where we add up more and more terms. This is also a nice first example, because we really just showed that

$$0.999... = 1$$

since

$$\sum_{k=1}^{\infty} \left(\frac{9}{10^k} \right) = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$$
$$= 0.9 + 0.09 + 0.009 + ...$$
$$1.$$

But more importantly, we now have a good strategy for thinking about infinite series as sequences of *partial sums.*

8.3 The Divergence Test and the Harmonic Series

Activity 8.3.1 Investigating the Harmonic Series.

(a) Write out the first several terms of the harmonic series, terms from $\left\{\dfrac{1}{k}\right\}_{k=1}^{\infty}$. Write however many you need to get a feel for how the terms work.

(b) Can you find out how many terms you would have to go "into" the series before the term was less than 0.00000001?

(c) Can you do this same kind of thing, no matter how small? For instance, how many terms would you have to go into the series before the term was less than some real number ε where $\varepsilon > 0$?

(d) Remind/explain/convince yourself that what we've really done is show that $\lim\limits_{k \to \infty} \dfrac{1}{k} = 0$. This isn't a new or terribly interesting fact, but make sure that you understand why the argument above shows this.

(e) Let's do something very similar, but with $\left\{\sum\limits_{k=1}^{n} \dfrac{1}{k}\right\}_{n=1}^{\infty}$, the sequence of partial sums, instead. Write out the first few partial sums. There's no specific number that you *need* to write, but make sure to write enough partial sums to get a feel for how the partial sums work.

(f) Can you find out how many terms you need to add up until the partial sum is larger than 1?

(g) Can you find out how many terms you need to add up until the partial sum is larger than 5?

(h) Can you find out how many terms you need to add up until the partial sum is larger than 10?

(i) Do you think that for any positive number S, we can always find some partial sum $\sum\limits_{k=1}^{n} \dfrac{1}{k} > S$?

What do you think this would mean about

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k}?$$

To actually show that for any $S > 0$ we could always find an $n > 1$ where

$$\sum_{k=1}^{n} \frac{1}{k} > S$$

is an extremely difficult task! We will show that the Harmonic Series diverges in a different way, but for now I want us to notice these contradictory results: we have a series whose terms get small, but whose partial sums do not seem to converge.

We have $\dfrac{1}{k} \to 0$ but it seems like $\lim\limits_{n \to \infty} S_n$ does not exist. Is this behavior special to the Harmonic Series? Is this something we should make note of? Is there some other connection between the terms of a series and the behavior of the partial sums of the series that we need to note?

8.4 The Integral Test

We're going to work with a graph of a continuous function, and we're going to start with a couple of conditions:

1. Our function will be continuous wherever it's defined.

2. Our function will be decreasing on its domain.

3. All of the function outputs will be positive.

Let's not worry about picking a specific function for this, but we will visualize a graph of one that meets these three requirements.
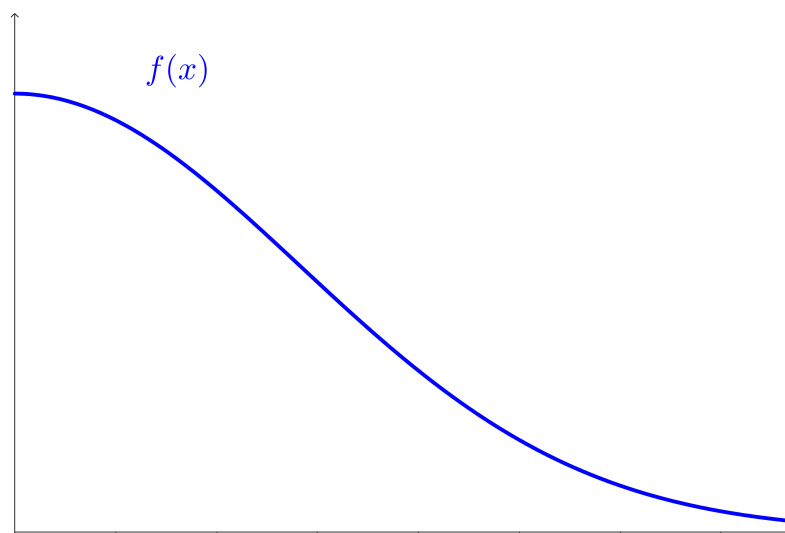
$f(x)$

Figure 8.4.1

We can then visualize the sequence of terms, $a_k = f(k)$ for $k = 0, 1, 2, ...,$.

Figure 8.4.2

(a) How does the partial sum, $\displaystyle\sum_{k=0}^{n} a_k$ compare to the Riemann sum for $f(x)$ from $x = 0$ to $x = n$ with n rectangles?

(b) We're going to visualize the accumulation of $f(x)$ from $x = 0$ to $x = n$ by thinking about the integral:
$$\int_{x=0}^{x=n} f(x) \ dx.$$

Figure 8.4.3

How does this area compare to the Riemann sum you thought of above? Compare them with an inequality and make sure you can explain why this has to be true.

(c) Remove the first term of the series, a_0, and instead think of the sum $\sum\limits_{k=1}^{n} a_k$. Can you still think of this as a Riemann sum to approximate the area from the integral $\int_{x=0}^{x=n} f(x)\, dx$?

How does this new Riemann sum compare to the area formed by the integral? Compare them with an inequality and make sure you can explain why this has to be true.

(d) We have thought about two sums, and we can connect them:

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k.$$

Use the sums to bound the integral:

$$\boxed{} < \int_{x=0}^{x=n} f(x)\, dx < \boxed{}$$

(e) Similarly, use the integral to bound the sum:

$$\boxed{} < \sum_{k=0}^{n} a_k < \boxed{}$$

These bounds are going to be super useful! Discovering them is the main task for finding the connections between improper integrals and infinite series. These inequalities might seem kind of strange at first, but we're going to apply a limit to everything as $n \to \infty$, and then think about our definitions of convergence (Definition 7.1.4 and Definition 8.2.2).

8.5 Alternating Series and Conditional Convergence

Activity 8.5.1 Which is More Likely to Converge?

We're going to try to think about what might be different when we analyze an alternating series compared to a series with only positive (or non-negative) terms.

Let's say that $\{a_k\}_{k=0}^{\infty}$ is some sequence of positive real numbers. Now let's consider the two series:

$$\sum_{k=0}^{\infty} a_k \qquad\qquad \text{vs} \qquad\qquad \sum_{k=0}^{\infty}(-1)^k a_k$$

(a) Let's first consider the sequences of terms: $\{a_k\}$ compared with $\{(-1)^k a_k\}$. Is either of these more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(b) Now let's think of the partial sums:

$$\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty} \qquad\qquad \text{vs} \qquad\qquad \left\{\sum_{k=0}^{n}(-1)^k a_k\right\}_{n=0}^{\infty}$$

Is either of these sequences more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(c) Now make a conjecture about which infinite series is more likely to converge:

$$\sum_{k=0}^{\infty} a_k \qquad\qquad \text{vs} \qquad\qquad \sum_{k=0}^{\infty}(-1)^k a_k$$

Remember that $a_k > 0$ for $k = 0, 1, 2, ...$, so the only differences are the changes in sign.

Activity 8.5.2 Approximating an Alternating Series.

Let's look, again, at the picture of the partial sums of an alternating series in Figure 8.5.2. We're going to assume that the series converges, which means that:

- $\lim_{n \to \infty} S_n$ exists.

- $\lim_{n \to \infty} a_n = 0$.

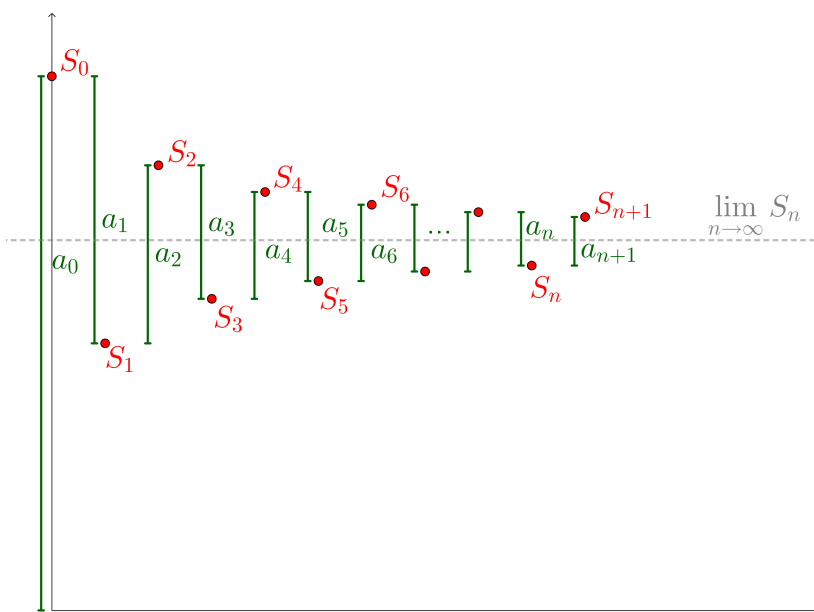Let's add to our figure.

Figure 8.5.1

(a) Why are the even-indexed partial sums sitting above the odd-indexed partial sums?

(b) Why are the even-indexed partial sums sitting above the horizontal line, $\lim_{n \to \infty} S_n$?

(c) Why are the odd-indexed partial sums sitting below the horizontal line, $\lim_{n \to \infty} S_n$?

(d) If we were trying to approximate the value of $\lim_{n \to \infty} S_n$, how can we use the partial sums to build an interval that approximates the value?

Activity 8.5.3 The Alternating Harmonic Series Converges.

  (a)

Activity 8.5.4 The Alternating Harmonic Series Converges (Again).

   (a)

8.6 Common Series Types

Activity 8.6.1 (Im)Possible Combinations.

When we have thought about infinite series, we have thought about three different mathematical objects: the sequence of terms of the series, the sequence of partial sums of the series, and the infinite series itself. As a reminder, if we had an infinite series

$$\sum_{k=1}^{\infty} a_k$$

we can say that:

- $\{a_k\}_{k=1}^{\infty}$ is the sequence of terms of the series

- $S_n = \sum_{k=1}^{n} a_k$ is a partial sum and $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series

For each of these three objects—the terms, the partial sums, and the series—we have some notion of what it means for that object to converge or diverge.

Consider the following table of all of the different combinations of convergence and divergence of the three objects. For each combination, decide whether this combination is possible or impossible. If it is possible, give an example of an infinite series whose terms, partial sums, and the series itself converge/diverge appropriately. If it is impossible, give an explanation of why.

Table 8.6.1 (Im)Possible Combinations

$\{a_k\}_{k=1}^{\infty}$	$\{S_n\}_{n=1}^{\infty}$	$\sum_{k=1}^{\infty} a_k$	(Im)Possible?	Example or Explanation
Converges	Converges	Converges		
Converges	Converges	Diverges		
Converges	Diverges	Converges		
Converges	Diverges	Diverges		
Diverges	Converges	Converges		
Diverges	Converges	Diverges		
Diverges	Diverges	Converges		
Diverges	Diverges	Diverges		

8.7 Comparison Tests

Activity 8.7.1 Comparisons to Bound Partial Sums.

This activity is mostly going to be thinking about proof mechanisms, and so it might be helpful to review Activity 8.4.1 Integrals and Infinite Series. If you want to see more, then the proof of Integral Test will provide some further details on why the inequalities we built were useful.

(a) In the Integral Test, how did we guarantee that our sequence of partial sums was monotonic?

(b) How did we know that, as long as the corresponding integral converged, then our sequence of partial sums was bounded?

(c) How did we know that, as long as our integral diverged, then our sequence of partial sums had to diverge as well?

(d) What happens if we swap out the integral we're connecting our series to with a different series? For these inequalities to be useful, what do we need to be true?

Activity 8.7.2 (Un)Helpful Comparisons.

We're going to consider a handful of infinite series, here:

1. $\sum\limits_{k=1}^{\infty} \dfrac{2}{k(3^k)}$

2. $\sum\limits_{k=3}^{\infty} \dfrac{\sqrt{k+1}}{k-2}$

(a) Pick a series that is reasonable to use as a comparison for each of the series listed. Remember, we want:

- A series that is recognizable (probably a Geometric Series or a p-Series), so that we know the behavior of it: we need to know whether the series we're comparing to converges or diverges!

- A series that is similar enough to the series we're working with that we can construct an inequality comparing the term structure. It can be hard to compare functions that are seemingly unrelated to each other!

- A series that has terms that are either larger or smaller than our series, depending on whether we are trying to show that our series converges or diverges.

(b) Build the comparison from the series we start with to the one you picked. What kinds of conclusions can you make?

(c) We're going to change the series we're considering to two slightly different series:

(a) $\sum\limits_{k=1}^{\infty} \dfrac{2k}{3^k}$

(b) $\sum\limits_{k=3}^{\infty} \dfrac{\sqrt{k-1}}{k+2}$

How do these small changes impact the inequalities you built?

(d) How do these changes in the inequalities change the conclusions we can draw from the Direct Comparison Test?

(e) What do you *think* is happening with these series: do you think that these small changes are enough to change the behavior of the series (i.e. whether it converges or diverges)?

Activity 8.7.3 Ratios for Comparison.

Let's start with some functions: we'll consider $f(x)$ and $g(x)$ as two functions that are continuous when $x \geq 0$ with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

All of this is so that we can think about $\dfrac{f(x)}{g(x)}$ and know that we have an indeterminate form. We could put the requirement of differentiability on these functions (so that we could think about L'Hôpital's Rule), but we don't need to do that.

We're going to now consider the limit:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}.$$

(a) What would the limit $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ look like if $g(x) \to 0$ with a faster growth rate than $f(x)$ does? In this case, we might say that:

$$f(x) >> g(x).$$

(b) What would the limit $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ look like if $f(x) \to 0$ with a faster growth rate than $g(x)$ does? In this case, we might say that:

$$g(x) >> f(x).$$

(c) If the functions $f(x)$ and $g(x)$ eventually act equivalently, then what does the limit $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ look like?

(d) If the function $f(x)$ eventually acts like some scaled version of $g(x)$, then what does the limit $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ look like?

8.8 The Ratio and Root Tests

Activity 8.8.1 Reminder about Geometric Series.

We are going to build some convergence tests that try to link some infinite series to the family of geometric series and show that even though a series is *not* geometric, it might act enough like one to be considered "eventually geometric-ish."
But first, what does it mean for a series to be a geometric series?

(a) Describe a defining characteristic of a geometric series. What makes it geometric?

(b) Can you describe this characteristic in another way? For instance, if you described a geometric series using a characteristic about the Explicit Formula, can you describe the same characteristic in the context of the Recursion Relation instead? Or vice versa?

(c) Write out a generalized and simplified form of the term a_k of a geometric series explicitly and recursively. In each case, solve for r, the ratio between terms.

Chapter 9

Power Series

9.1 Polynomial Approximations of Functions

> **Activity 9.1.1 Build a Polynomial.**
>
> We're going to use the formula in Definition 9.1.1 to construct two different polynomials that approximate two different approximations. Then, we'll use them to approximate things!
>
> **(a)** We're going to start with approximating the function $f(x) = \sin(x)$ centered at $x = 0$. Let's choose to look at a 5th degree polynomial.
>
> This means we'll need to find the first five derivatives of $f(x) = \sin(x)$. Then, we'll evaluate our function and the five derivatives at the center. After that, we can divide by the relevant factorial in order to create the coefficients of our polynomial.
>
> Fill out the following chart to produce these coefficients.
>
> **Table 9.1.1 Coefficients for Polynomial Approximation**
>
> | k | $f^{(k)}(x)$ | $f^{(k)}(a)$ | $\dfrac{f^{(k)}(a)}{k!}$ |
> |---|---|---|---|
> | $k = 0$ | $f(x) = \sin(x)$ | $f(0) =$ | |
> | $k = 1$ | $f'(x) =$ | $f'(0) =$ | |
> | $k = 2$ | $f''(x) =$ | $f''(0) =$ | |
> | $k = 3$ | $f'''(x) =$ | $f'''(0) =$ | |
> | $k = 4$ | $f^{(4)}(x) =$ | $f^{(4)}(0) =$ | |
> | $k = 5$ | $f^{(5)}(x) =$ | $f^{(5)}(0) =$ | |
>
> **(b)** Now we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x - a)^k$ for $k = 0, 1, ..., 5$. These (added together) will form your polynomial, $p_5(x)$.
>
> **(c)** Approximate $f(1) = \sin(1)$ using your polynomial.
>
> **(d)** Let's repeat this for another function. Let's build a 5th degree polynomial approximation of $g(x) = e^x$ centered at $x = 0$. We can construct the coefficients in the same way.

Table 9.1.2 Coefficients for Polynomial Approximation

k	$g^{(k)}(x)$	$g^{(k)}(a)$	$\dfrac{g^{(k)}(a)}{k!}$
$k = 0$	$g(x) = e^x$	$g(0) =$	
$k = 1$	$g'(x) =$	$g'(0) =$	
$k = 2$	$g''(x) =$	$g''(0) =$	
$k = 3$	$g'''(x) =$	$g'''(0) =$	
$k = 4$	$g^{(4)}(x) =$	$g^{(4)}(0) =$	
$k = 5$	$g^{(5)}(x) =$	$g^{(5)}(0) =$	

(e) And now, again, we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x - a)^k$ for $k = 0, 1, ..., 5$. These (added together) will form your polynomial, $p_5(x)$.

(f) Approximate $g(-3) = \dfrac{1}{e^3}$ using your polynomial.

Activity 9.1.2 How Good Are Our Approximations?

We're going to think more carefully about our approximations of $\sin(1)$ and e^{-3} from Activity 9.1.1. In order for us to do this, let's visualize the function and the 5th degree polynomial for it.

Instructions: Toggle between the functions $f(x) = \sin(x)$ and $g(x) = e^x$ to view the function and the 5th degree polynomial approximation of that function. Then, compare the approximations by dragging the point or inputting an x-value.

⦿ $f(x) = \sin(x)$
◯ $g(x) = e^x$

Approximate at $x =$ [0]
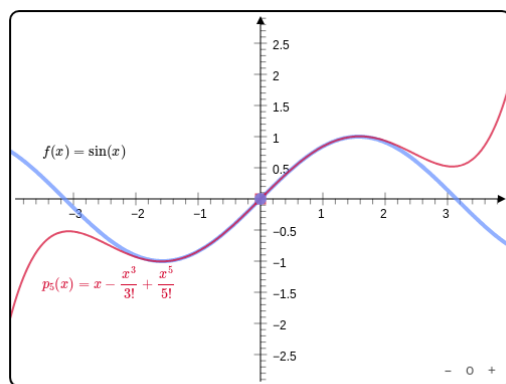
$f(x) = \sin(x)$

$p_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

Standalone
Embed

(a) How good of a job did the polynomial approximation do when approximating $\sin(1)$? How can you tell, visually?

(b) How good of a job did the polynomial approximation do when approximating e^{-3}? How can you tell, visually?

(c) How does the relationship between the "center" and the x-value that we're approximating at impact the accuracy of our approximation?

(d) How do you think you could make these approximations better (without changing the center)?

Activity 9.1.3 Partial Sums of What?

Let's revisit our 5th degree polynomial approximations from Activity 9.1.1.

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

These approximations work well for x-values that are close to 0, but we will not be more formal than that.

(a) Make a conjecture about what the 7th degree polynomial approximations are for each of these functions.

What about the 15th degree?

(b) Make a conjecture about what the general formula would be for these terms. If you were to write these out using summation notation, what would they look like?

(c) Why does the polynomial approximation for the sine function only have odd-exponent terms?

(d) Make a conjecture about what a polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$ would be.

Activity 9.1.4 How Do These Polynomials Converge?

We're going to end here by thinking about these polynomials as some partial sum from an infinite series. If there is an infinite series, we should be prepared to think about convergence!

We're going to think about convergence in the same way that we have already: as an end behavior limit of the partial sums. So let's spend our time investigating this end behavior by visualizing polynomial approximations as the degree increases.
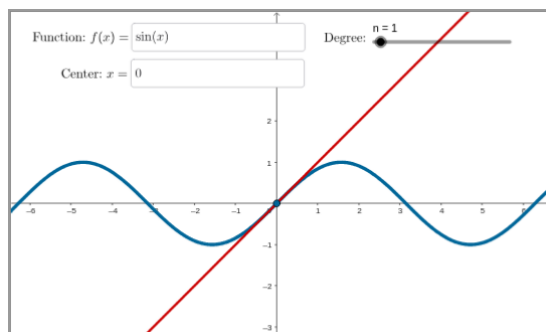
Standalone

(a) What happens to the polynomial approximation of $\sin(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(b) Does this behavior change if we centered our approximation elsewhere?

(c) What happens to the polynomial approximation of e^x centered at $x = 0$ as the degree $n \to \infty$?

(d) Does this behavior change if we centered our approximation elsewhere?

(e) What happens to the polynomial approximation of $\cos(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(f) Does this behavior change if we centered our approximation elsewhere?

(g) What happens to the polynomial approximation of $\ln(x)$ centered at $x = 2$ as the degree $n \to \infty$?

(h) Describe the difference in what you're seeing with the log function compared to the other functions we've thought about. Describe how the polynomial approximations converge: do they converge to the log function? How? More importantly, *where*?

(i) Does this behavior change if we centered our approximation elsewhere?

9.2 Power Series Convergence

Activity 9.2.1 Polynomial Division.

We're going to do some fiddling with polynomials, and hopefully use this as a bridge to connect how we think of polynomials and power series with how we think about our traditional infinite series and the notions of convergence that we've already built.

(a) We're going to factor some polynomials, but we might end up using some division. First, we'll confirm some factors that we already know.

$$x^2 - 1 = (x - 1)(x + 1)$$

We'll confirm this by using division.

$$x - 1 \ \overline{\big) \ x^2 \qquad -1}$$

(b) Now let's factor $x^3 - 1$. If the factors for this polynomial isn't as familiar, it might be helpful to know that $(x - 1)$ is also a factor of $x^3 - 1$.

$$x - 1 \ \overline{\big) \ x^3 \qquad -1}$$

(c) Let's try another one. Complete the following division.

$$x - 1 \ \overline{\big) \ x^4 \qquad -1}$$

(d) Can you generalize this? Find the formula for $\dfrac{x^n - 1}{x - 1}$ for some positive integer n.

$$x - 1 \ \overline{\big) \ x^n \qquad -1}$$

(e) Now that we have good evidence that

$$\sum_{k=0}^{\infty} x^k = \frac{x^n - 1}{x - 1},$$

We can apply a limit as $n \to \infty$.

$$\sum_{k=0}^{\infty} x^k = \lim_{n \to \infty} \sum_{k=0}^{n} x^k$$
$$= \lim_{n \to \infty} \frac{x^n - 1}{x - 1}$$

For what values of x will this limit exist?

(f) Revisit Theorem 8.6.2 Geometric Series Convergence Criteria. Is there any difference for what we've just done compared to this result that we already know?

Activity 9.2.2 Some Power Series and their Convergence.

Let's consider a couple of power series and apply some convergence tests to them in order for us to find out how it might converge.

(a) Consider the power series:

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k^2}.$$

In order for us to apply the Ratio Test, we'll actually need to consider the positive-term version:

$$\sum_{k=0}^{\infty} \frac{|x-1|^k}{k^2}.$$

Apply the Ratio Test. What do you get in the limit of the ratio between terms?

(b)

9.3 How to Build Taylor Series

Text of section.

9.4 How to Use Taylor Series

Text of section.