Discover Calculus II

Single-Variable Integral Calculus Topics with Motivating Activities

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Acknowledgements

There are several people that I want to thank for their help or contributions to this book.

First, thanks to Dan and Terra in the library at Moraine Valley. They've both been so supportive of the project and worked so hard to make it possible for me to find space and time to actually write the book.

In the math department at Moraine Valley, Amy, Angelina, and Lisa all were so helpful in different ways. I'm so thankful for their help with writing and editing practice problems (and solutions) and contributing their expertise to make sure that this is a teaching and learning resource first and foremost. Thanks for agreeing to help and always providing great feedback.

Thank you to the Consortium of Academic and Research Libraries in Illinois (CARLI) and the Illinois State Library for providing the opportunity through grant funding for me to write this. I've been wanting to put this type of resource together for my students for years and had never been able to find a way to carve out the time required. Your support to developing good OER practices and investment in Illinois is so appreciated.

Thanks also to the PreTeXt people: both the people behind it and the wonderful people that use it. Especially the following:

- Matt Boelkins and Mitch Keller and the rest of the Active Calculus authors,
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- Spencer Bagley,
- Oscar Levin.

I've learned so much by digging through the source code of all of your projects. Thanks so much for providing such great projects and offering up your source code for others to learn from.

And last, thank you to my students. I've loved thinking about calculus with you for so many years, and in every semester, our classes together are the highlight of my job. The way we talk about calculus together has changed over the years for the better, and I'm so thankful for your influence over how I think about calculus and how we can present calculus to people who want to learn it.

Disclosure about the Use of AI

This book has been lovingly written by a human.

Me.

Peter Keep.

I have used a lot of different tools, both for inspiration and for actually creating resources for this book. *None* of those tools has involved any form of generative AI.

I could list all of the ways that I think using generative AI in education is, at minimum, problematic. More pointedly, I believe that it is unethical. More broadly, I believe that the use of generative AI for any use-case that I have encountered to be unethical.

In my classes, I try to help students realize the joy and value of working at something and creating something and struggling with something and knowing something. Giving worth to something, even an imperfect thing. Celebrating our accomplishments, even when (especially when?) there is room to grow in those accomplishments. And so I have taken that advice in the creation of this book. I have created a book that is definitely not perfect. I have struggled to write it. There are parts of it that could be (need to be) improved.

But I was the one that created it. I struggled with it. I know it.

I hope that this book can also be a useful tool for others to use, and I have left the copyright to be about as open as possible. Others can take this, use it, can change it, add to it, subtract from it, etc.

In leaving this copyright open for others to change this book, I cannot guarantee that every version of this book is free from the mindless and joyless output from some Large Language Model. But I want to leave this note up in hopes that anyone who *does* inject some output from some generative AI product into this book will take it down. If this note, or some statement similar to it, is not present in the version of the book you are accessing, please be cautious. Find a different calculus textbook to read!

Find something written by a human. Find the words of some other mathematician who tries, maybe imperfectly, to share the ideas of calculus.

Teaching and learning is about humans communicating with each other, and only humans can do that.

Notes for Students

Hi! Thanks so much for reading this book! I hope it's a good and useful tool for you as you learn calculus. Before you dive into it, I want to try to explain some of the choices I made in putting this book together and how I hope you'll use it.

First, let's talk about all of the activities. In each section you'll find activities that lead into things like definitions or theorems. These are activities that I give my students to work on in groups. If you're using this independently, I hope you still engage with the activities: think about them, try to use them as ways of exploring the results or definitions before we state them. I want you to build intuition and I hope that these activities are helpful, even without the group exploration that would normally happen with them.

You might also notice that there aren't many proofs in this book. That's not extremely unique for an introductory calculus book, but there might be even fewer than expected. The ones that are included are ones that I think are important. Most of the included proofs show some of the kind of reasoning that we want students in calculus classes to see.

The last thing that I'll say is that I hope, most of all, that however you use this book and whatever parts of it you engage in, that it is useful for you. I hope that it helps you as you work to understand these wonderful groups of topics.

Thanks for letting me and my book be a part of your journey learning mathematics!

Notes for Instructors

Hi! I'm glad you're looking at this text, and I'm excited to share it with you. I want to outline a couple of thoughts I've had while putting this together and envisioning how an instructor could use this.

First, this book is created to be used in a relatively active classroom. In each section, for each topic, we try to motivate definitions, theorems, and other big results through the use of classroom activities. These are designed to be used in class as group-based explorations, but could be used also as guided exercises for students to complete in other contexts as well. In this sense, each section's mixture of Activities, Definitions, Theorems, and Examples serve as fairly usable lesson plans for each topic.

Chapters can be divided into two groups: one for the topics typically included in a first semester differential calculus course (Chapters 1-5) and the other for the topics typically included in a second semester integral calculus course (Chapters 6-9).

There are several different resources and formats that come along with this book. All of these are available at `discovercalculus.com`.

- The HTML option is likely the main source used, suitable for instructors to use during class sessions and students to reference outside of class. This option has the best navigation and displays interactive elements. It displays nicely on a projector for use during class sessions.

- There is a PDF option for printing, as well as PDF options for just chapters 1-5 (titled Discover Calculus 1) and chapters 6-9 (Discover Calculus 2).

- PDF printouts of just the class activities from each section are available as well. These are available in a compilation of all of the chapters, as well as subset versions for Discover Calculus 1 and Discover Calculus 2. These pdf printouts are ideal for printing to use as class handouts or a small bound workbook for each student.

Finally, there are some other resources available at `discovercalculus.com`, specifically at `discovercalculus.com/instructorresources.html`. Primarily, you can find Python notebooks that demonstrate a selection of the concepts from this book. These demonstrations are useful as class examples or as a component of the out-of-class work that a student could do. They are all self-contained, requiring no local installation of python or other software, and can be modified to fit your class as you see fit.

Thanks, again, for thinking about using Discover Calculus as a teaching and learning resource for you and your students!

Contents

Chapter 6

Applications of Integrals

6.1 Integrals as Net Change

We have some rudimentary ideas of what an integral is, but we want to challenge and expand those ideas by examining the object at the root of the definition of the Definite Integral: a Riemann Sum. We want to generalize a bit more our notion of what a Riemann sum is. So for now, let's think about how we can use a Riemann sum to think about a measurement that will *not* be an area. That's been our only real context so far, so let's try to stretch that thinking.

Estimating Movement

Activity 6.1.1 Estimating Movement.

We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 6.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

(b) When was the acceleration of the object the greatest? When was it the least?

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

So what are the big ideas in this short activity? There are a lot, and many of them are already things we know, at least to some level. So we are really focusing on adding depth to our understanding of these big ideas. Let's list them in the order that they showed up in this activity:

1. We interpret the velocity as the derivative of the position of the object. So when we interpret the value of the velocity of the object (large vs small, positive vs negative, etc.) we are interpreting these through the lens of a rate of change.

2. Acceleration is the derivative of the velocity function. While we don't have the full picture of the velocity function at any value of t, we still were interested in the rates at which velocity changes with regard to time.

3. We can estimate the total *displacement* of the object by predicting how far it traveled in eahc 30-second time interval. We might pick the starting velocity for each 30-second interval and multiply that by 30 seconds. We could alternatively pick the ending velocity of each 30-second interval. Then we can add all of these products of velocity and time together to approximate a total change in position! Doesn't this feel like a Riemann sum?

4. When we calculate displacement, the negative velocities get multiplied out to get negative changes in position for the object -- that's because a negative velocity means that the object is moving backwards. If we wanted to calculate the distance traveled, then we need to not account for negative velocities. We can just disregard the sign of the velocity on each time interval and repeat the process above. So, another Riemann sum then?

5. In order to forecast some position at time t, we just need to start with the initial position, and then calculate (or approximate) the displacement from $t = 0$ to whatever time $t \leq 300$ we care about, and then add the displacement to the intitial position.

Ok, now let's formalize those results!

Position, Velocity, and Acceleration

We know that the velocity of an object is really a rate of change of the position of that object with regard to time. Similarly, the acceleration of an object is the rate of change of the velocity of the object with regard to time. So we're really thinking about derivatives!

Definition 6.1.2 Position, Velocity, and Acceleration Functions.

For an object moving along a straight line, if $s(t)$ represents the **position** of that object at time t, then the **velocity** of the object at time t is $v(t) = s'(t)$ and the **acceleration** of the object at time t is $a(t) = v'(t) = s''(t)$.

Once we establish this relationship, we can answer questions about movement of an object using the same interpretations of derivatives that we practiced in Chapter 3 of this text.

Activity 6.1.2 A Friendly Jogger.

Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

(b) Does this time represent a maximum or minimum velocity for the jogger?

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

Displacement, Distance, and Speed

Let's revisit Activity 6.1.1. When we approximated the displacement of the object, we built a Riemann sum:

$$\sum_{k=1}^{10} v(t_k^*)\Delta t$$

We chose our t_k^* as either the time at the beginning of each 30-second interval or the time atr the end of the 30-second interval, but that was only because of the limited information that we had about different values of $v(t)$. If we had information about the $v(t)$ function at any values of t $(0 \leq t \leq 300)$, then we could pick *any* time in each 30-second time interval for our Riemann sum! We might note, though, that if we did have this kind of information about the velocity at any time in the 5-minute interval, then we would also build a more precise approximation by subdividing the time interval into smaller/shorter pieces. So maybe the Riemann sum $\sum\limits_{k=1}^{100} v(t_k^*)\Delta t$ (where we are dividing up the 5 minute interval into 100 3-second intervals) would do a better job! But why stop there? If we have the definition of the velocity function, and so we can truly obtain the velocity of the object at *any* time in the 5 minute interval, then we can use the definition of the definite integral as the limit of a Riemann sum:

$$\lim_{n\to\infty} \sum_{k=1}^{n} v(t_k^*)\Delta t = \int_{t=0}^{t=300} v(t)\ dt$$

This should work out well with our first understanding of displacement: the displacement of an object is just the difference in position from the starting time to the ending time. So we could say that if $s(t)$ is the position function, then we might expect to represent displacement from $t = a$ to $t = b$ as $s(b) - s(a)$. But isn't this just the Fundamental Theorem of Calculus, since $s'(t) = v(t)$?

Definition 6.1.3 Displacement of an Object.

If an object is moving along a straight line with velocity $v(t)$ and position $s(t)$, then the **displacement** of the object from time $t = a$ to $t = b$ is

$$\int_{t=a}^{t=b} v(t)\,dt = s(b) - s(a)$$

Let's keep revisiting the same activity. We also noticed that when we looked at the *distance* compared to the displacement, the only difference was that we were integrating the absolute value of the velocity function, since we didn't care about the sign of the velocity (the direction that the object was traveling) on each interval.

Definition 6.1.4 Distance Traveled.

If an object is moving along a straight line with velocity $v(t)$, then the **distance** traveled by the object from time $t = a$ to $t = b$ is:

$$\int_{t=a}^{t=b} |v(t)|\,dt$$

Here, we call $|v(t)|$ the **speed** of the object (instead of the velocity).

We should note that we don't have any quick and easy ways of dealing with the integral of the absolute value of a function.

$$|v(t)| = \begin{cases} -v(t) & \text{when } v(t) < 0 \\ v(t) & \text{when } v(t) \geq 0 \end{cases}$$

So, in order for us to integrate $|v(t)|$, we need to think about where the velocity passes through 0, so that we can see where it might change from positive to negative.

Activity 6.1.3 Tracking our Jogger.

Let's revisit our jogger from Activity 6.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

(b) Think back to our description of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.

Finding the Future Value of a Function

We can again think back to Activity 6.1.1 and build our last result of this section. Remember when we were looking to predict the location of our object at different times: we said it was reasonable to start at our initial position, and then add the displacement of the object from that initial time up to the time that we were interested in. So, to estimate the object's position after 150 seconds, we would calculate:

$$s(0) + \int_{t=0}^{t=150} v(t)\, dt.$$

But we said we could do this to estimate the object's position at any value for time, t.

Theorem 6.1.5 Future Position of an Object.

*For some object moving along a straight line with velocity $v(t)$ and an initial position of $s(a)$, the **future position of the object** at some time t (with $t \geq a$) is:*

$$\underbrace{s(t)}_{\substack{\text{future}\\\text{position}}} = \underbrace{s(a)}_{\substack{\text{initial}\\\text{position}}} + \underbrace{\int_{x=a}^{x=t} v(x)\, dx}_{\substack{\text{displacement from}\\a \text{ to } t}}$$

Note that we change the variable in the velocity function while we integrate: since we want our position function to be in terms of t, the ending time point that we calculate the displacement up to, we need to choose a different variable to write velocity in terms of. Mechanically, there is no difference, since we're just swapping out the variables and naming them x.

We can note that this relationship between velocity and position can exist in many other context: any pair of functions that are derivatives/antiderivatives of each other can have this relationship!

Theorem 6.1.6 Net Change and Future Value.

*Suppose the value $F(t)$ changes over time at a known rate $F'(t)$. Then the **net change** in F between $t = a$ and $t = b$ is:*

$$F(b) - F(a) = \int_{t=a}^{t=b} F'(t)\, dt.$$

*Similarly, given the initial value $F(a)$, the **future value** of F at time $t \geq a$ is:*

$$F(t) = F(a) + \int_{x=a}^{x=t} F'(x)\, dx$$

Practice Problems

1. Explain the following terms in reference to an object moving along a straight path from time $t = a$ to time $t = b$.

- **(a)** **Position** of the object at time t.
- **(b)** **Displacement** of the object.
- **(c)** **Distance** traveled by the object.
- **(d)** **Velocity** of the object at time t.

(e) **Speed** of the object at time t.

2. Consider the graph of a velocity function, $v(t)$, of some object moving along a line on the time interval $[0, 7]$.

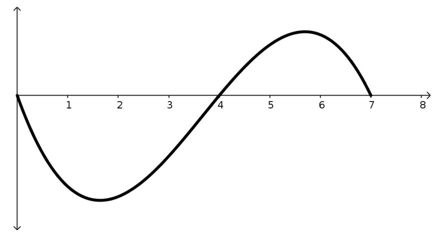


Figure 6.1.7

(a) Do you expect the displacement of the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?

(b) Write two different expressions that represent the total displacement of the object from $t = 0$ to $t = 7$.

(c) Do you expect the distance traveled by the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?

(d) Write two different expressions that represent the total distance traveled by the object from $t = 0$ to $t = 7$.

3. Let's consider an animal running along a straight path with the velocity function:

$$\begin{aligned} v(t) &= \frac{t^4}{10} - t^3 + \frac{27t^2}{10} - \frac{9t}{5} \\ &= \frac{t}{10}(t-1)(t-3)(t-6) \end{aligned}$$

on the time interval $[0, 6]$.

(a) What is the total displacement of the animal on the time interval $[0, 1]$?

(b) What is the total displacement of the animal on the time interval $[1, 3]$?

(c) What is the total displacement of the animal on the time interval $[3, 6]$?

(d) What is the total displacement of the animal on the time interval $[0, 6]$?

(e) What is the total distance traveled by the animal on the time interval $[0, 6]$?

(f) Write a short summary of the animal's movement, including notes about direction, speed, and where the animal travels.

4. Consider an object with velocity function $v(t) = t^2 - 4t + 2$ on the interval $[0, 100]$ with the initial position $s(0) = 3$.

(a) Determine the position function, $s(t)$, for $0 \le t \le 100$ using the Future Position of an Object.

(b) Determine the position function, $s(t)$, for $0 \le t \le 100$ using the

Solving Initial Value Problems strategy.

(c) Compare the results from both methods. Explain why these are equivalent.

5. Consider an object with an acceleration function $a(t) = t + \sin(2\pi t)$ for $t \geq 0$ with $v(0) = 5$.

(a) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Future Position of an Object.

(b) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Solving Initial Value Problems strategy.

(c) Can you obtain the position function, $s(t)$? Explain why or why not, based on the information given.

6. During a brake test for a heavy truck, the truck decelerates from an initial velocity of 88 ft/s with the acceleration function $a(t) = -17$ ft/s^2. Assume that the initial position of the truck is $s(0) = 0$.

(a) Find the velocity function for the truck.

(b) When does the truck stop? In this situation, the truck won't have a negative velocity (since it's just braking and not eventually going in reverse). What time interval is the velocity function relevant on?

(c) What is the total displacement of the truck on this time interval?

(d) Safety standards say that for a truck like this, it needs to be able to stop (from a speed of 88ft/s) in, at most, 200 feet.

Do we need to make changes to the braking mechanism, in order to have the acceleration function change? If so, what does the acceleration need to be (assuming it is constant and we are just replacing it with a new negative number)?

6.2 Area Between Curves

We're going to stick with our theme of thinking about a Riemann Sum, but this time we'll get back to thinking about area. First, we'll try to remind ourselves now just on what a Riemann sum is, but how we actually constructed it.

Remembering Riemann Sums

Activity 6.2.1 Remembering Riemann Sums.

Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

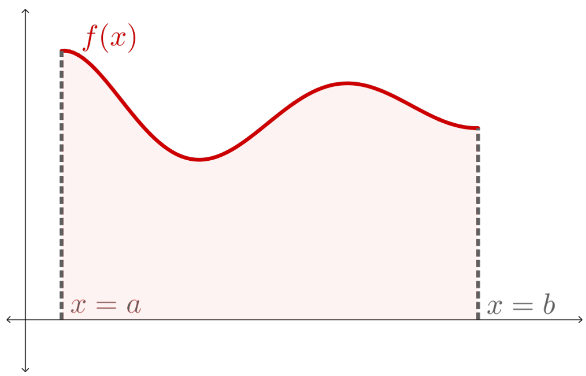


Figure 6.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

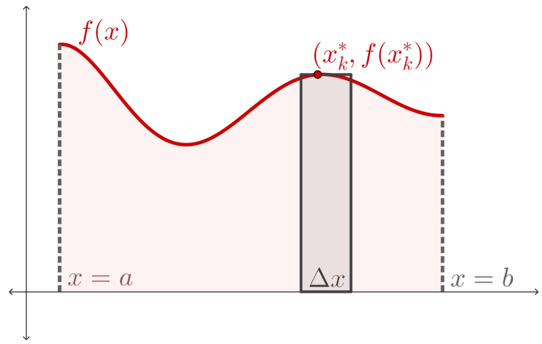


Figure 6.2.2

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

Hopefully this is helpful. If you'd like more reminders on this, you can always revisit Section 5.2 Riemann Sums and Area Approximations. For now, though, we mainly want to think about the general process we're using:

1. We slice the region from $x = a$ to $x = b$ into n pieces, and, for convenience, we choose an equal width: $\Delta x = \frac{b-a}{n}$.

2. From each of the slices, we select some x-value (called x_k^* from the kth slice). We use that to evaluate the function on each slice: $f(x_k^*)$.

3. We multiply the function value, $f(x_k^*)$, with the width of the slice, Δx, to get the measured area of each slice, $A_k = f(x_k^*)\Delta x$.

4. We can estimate the total measured area of the region by adding all of the areas of the slices together:

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

5. If we keep adding more and more slices (that keep getting thinner and thinner), then we eventually (in the limit) evaluate the area exactly:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

We're going to use this process (we'll call it the **slice-and-sum** process) for other measurements! Let's see how we can change this so slightly to measure a different area.

Building an Integral Formula for the Area Between Curves

Activity 6.2.2 Area Between Curves.

Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

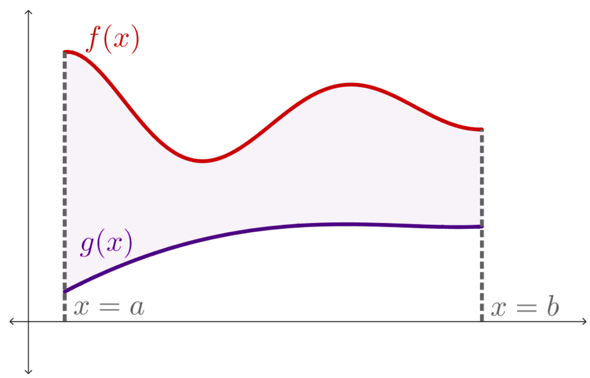


Figure 6.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

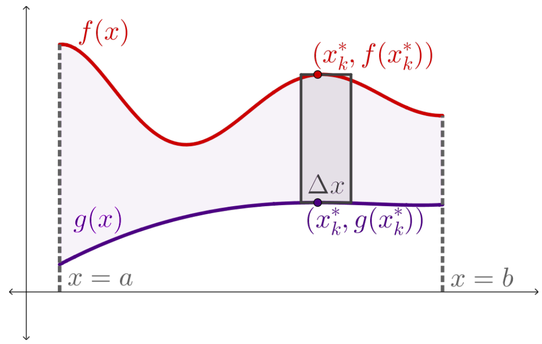


Figure 6.2.4

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Definition 6.2.5 Area Between Curves.

If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$, then the **area bounded between the curves** $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_{x=a}^{x=b} (f(x) - g(x))\ dx.$$

When we're applying this formula for the area between curves, we won't need to re-create the process from Activity 6.2.2 to *create* the integral formula. We can simply identify the functions bounding the region and the end points of the interval, and set up the integral.

We'll use the slice-and-sum process for two reasons:

1. To justify these formulas that we continue to build! While this one isn't that difficult (you could have just built the formula by thinking about the area between curves as a difference in areas under each curves), some of the formulas we play with in this chapter will not be as intuitive.

2. To help us understand what a Riemann sum actually *is*. It's a product of a function value from a subinterval multiplied by the width of that subinterval, summed up across some larger interval.

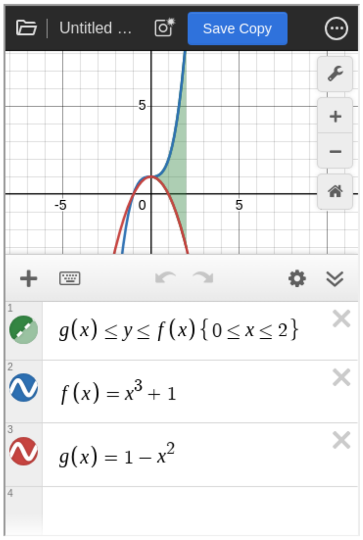
Example 6.2.6

For each of the following regions, set up an integral expression representing the area of the region. We can also practice evaluating these integrals to actually calculate the areas.

For each of these described regions, the hint will reveal a visualization of the region (using desmos). Feel free to use that to set up the integral expression!

(a) The region bounded between the graphs $y = x^3 + 1$ and $y = 1 - x^2$ from $x = 0$ to $x = 2$.

Hint.



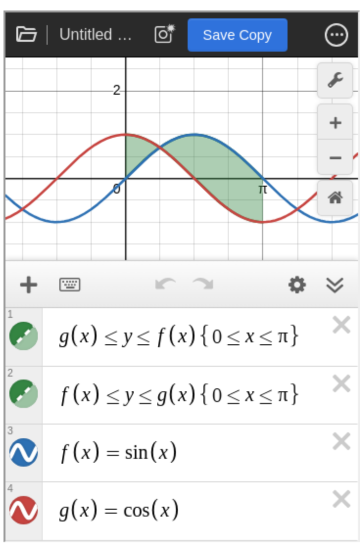
Standalone

Solution.

$$\int_{x=0}^{x=2} \left(x^3+1\right) - \left(1-x^2\right)\ dx = \int_{x=0}^{x=2} \left(x^3+x^2\right)\ dx$$
$$= \frac{20}{3}$$

(b) The region bounded between the graphs $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \pi$.

Hint.



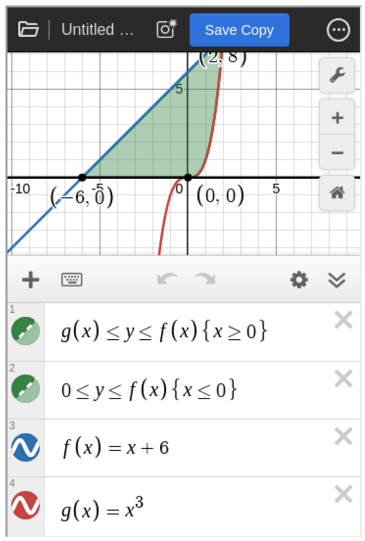
Standalone

Solution. Notice that the boundary functions intersect at $x = \frac{\pi}{4}$, and they switch order. We'll need to split this region into two different regions in order to identify the ”top” and ”bottom” boundary functions.

$$\int_{x=0}^{x=\pi/4} \cos(x) - \sin(x)\, dx + \int_{x=\pi/4}^{x=\pi} \sin(x) - \cos(x)\, dx = \left(\sqrt{2}-1\right) + \left(1+\sqrt{2}\right)$$
$$= 2\sqrt{2}$$

(c) The region bounded between the curves $y = x+6$ and $y = x^3$ and the x-axis.

Hint.



Standalone

Solution. On the interval $-6 \le x \le 0$, the region is bounded above by $y = x+6$ and below by the x-axis ($y = 0$). On the interval $0 \le x \le 2$, the region is bounded above by $y = x+6$ and below by $y = x^3$.

$$\int_{x=-6}^{x=0} (x+6) - 0\, dx + \int_{x=0}^{x=2} (x+6) - \left(x^3\right)\, dx = \int_{x=-6}^{x=0} x+6\, dx + \int_{x=0}^{x=2} 6+x-x^3\, dx$$
$$= 18 + 10$$
$$= 28$$

Changing Perspective

This last example had two interesting regions: we had to split them into two pieces because the boundary functions changed order or, in the case of the last example, changed completely to different boundary functions.

We’re going to re-do the last problem and work on trying to change our perspective a bit in order to get a single integral to evaluate the area.

Activity 6.2.3 Trying for a Single Integral.

Let’s consider the same setup as earlier: the region bounded between two curves, $y = x+6$ and $y = x^3$, as well as the x-axis (the line $y = 0$).

We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

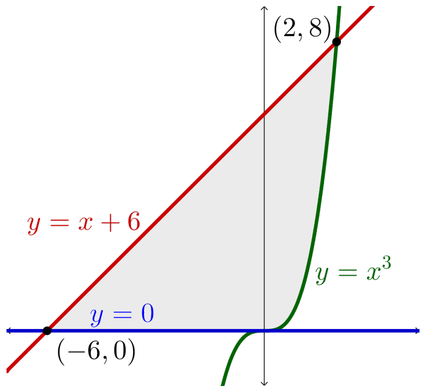


Figure 6.2.7

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths for each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

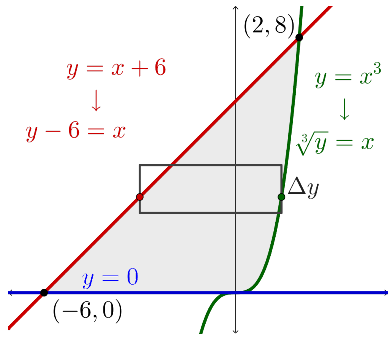


Figure 6.2.8

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

(j) Find A_k, the area of this kth rectangle.

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

We can re-write our definition of the area between curves (Definition 6.2.5) to account for this change in perspective, by thinking about these same functions in terms of y.

Definition 6.2.9 Area Between Curves (in terms of y).

If $f(y)$ and $g(y)$ are continuous functions with $f(y) \geq g(y)$ on the interval of y-values $[c, d]$, then the **area bounded between the curves** $x = f(y)$ and $x = g(y)$ from $y = c$ to $y = d$ is

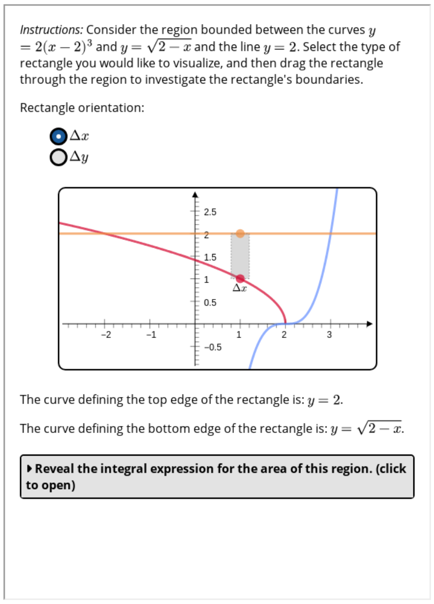
$$A = \int_{y=c}^{y=d} (f(y) - g(y))\ dy.$$

This strategy of inverting our functions to change the variable that we integrate with regard to is useful, but a tricky part of this is deciding *when* to change variables.

Something that we can look for is intersection points in the region we're working with. If, in our plan for setting up an integral, we would stack rectangles that would pass through an intersection point, then this indicates that we would need to split our region up to set up the integrals (since the boundary functions are changing). If we change the orientation of the rectangles, would they still pass through an intersection point? Are the functions that we're working with relatively easy to invert? Can we antidifferentiate these functions, or their inverted versions?

These are some of the things we'll consider as we make these decisions.

To finish things up, let's look at a nice little interactive graph that can help show the differences between finding area with regard to x (using Δx in our rectangles and dx in our integrals) and finding area with regard to y (using Δy in our rectangles and dy in our integrals).



Standalone
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Practice Problems

1. Explain how we use the "slice and sum" method to build an integral formula for the area bounded between curves. Give some details, enough to make sure you understand how the Riemann sums are constructed and how they turn into our integral formula.

2. What are some changes/considerations that we need to make when we decide to set up our integral in terms of y instead of x?

3. Set up (and practice evaluating) an integral expression representing the area of each of the regions described below.

   The hint for each problem will open a graph of the region.

   (a) The region bounded by the curves $y = x^2+1$ and $y = 4x+1$ between $x = 0$ and $x = 2$.

   (b) The region bounded by the curves $y = x$ and $y = 4 - x$ between $x = 0$ and $x = 2$

   (c) The region bounded by the curves $y = \sqrt{x} + 2$ and $y = x$ and the line $x = 0$.

   (d) The region bounded by the curves $y = \dfrac{2}{x^2+1}$ and $y = x^2$.

4. Set up and evaluate an integral representing the area of each of the regions described below. Explain whether you chose to integrate with respect to x or y, and why you made that choice.

   (a) The region bounded by the curves $y = \sin(x)$ and $y = \cos(x)$ and the line $y = 0$ between $x = 0$ and $x = \frac{\pi}{2}$.

(b) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the first quadrant.

(c) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the third quadrant.

(d) The region bounded by the curves $y = 3x$, $y = 4 - x^2$, and $y = x^2$ in the first quadrant with $0 \leq x \leq \sqrt{2}$.

(e) The region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

(f) The *other* region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

(g) The region bounded by the curves $x = 2y$ and $x = y^2 - 3$.

(h) The region(s) bounded by the curves $y = x^3$ and $y = x$.

6.3 Volumes of Solids of Revolution

Hopefully by now we're feeling pretty comfortable with the use of a Riemann sum to create an integral formula. So far, these integral formulas have matched with our intuition somewhat. We can probably justify the integral formula for displacement of an object (Definition 6.1.3 Displacement of an Object) by thinking about the fact that position is an antiderivative of velocity. We can probably convince ourselves about the integral formula for the area between curves (Definition 6.2.5 Area Between Curves) by thinking about subtracting areas, geometrically.

We're going to make a jump from a 2-dimensional measurement of size, area, to a 3-dimensional measurement of size, volume.

From Area To Volume

Here's the basic idea, in a broad overview: if we want to calculate a volume, then we are going to be working with a 3-dimensional solid. We'll use the slice-and-sum process:

1. Slice the object into uniformly thick slices along some axis.
2. For each slice, we'll approximate the volume. We can do this by thinking about the cross-sectional area. If we assume that the area is constant all the way through the slice (in the same way that we assumed earlier that the heights of our rectangles were constant), then we can simply multiply the cross-sectional area by the thickness to get the volume of each slice:
$$V_k = A(x_k^*)\Delta x.$$
3. Approximate the total volume of the solid by adding the volumes of the slices together:
$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x.$$
4. Apply a limit, where the number of slices gets infinitely big (and the thickness of each slice gets infinitely small):
$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x \\ &= \int_{x=a}^{x=b} A(x)\, dx \end{aligned}$$
Here, a and b are the x-values that define the interval we're slicing along and $A(x)$ is a formula for the cross-sectional area of the object at x.

The biggest issue here is going to be thinking about that formula for area. In order for us to do that, we're going to think about a specific type of 3-dimensional solid, built in a systematic way so that we can find the cross-sectional areas easily.

Solids of Revolution

A **solid of revolution** is a strange type of solid: we're going to define it based on a 2-dimensional region (we'll use functions in a normal xy-plane) that we then imagine revolving around a straight line axis. Maybe we define some region in the upper half of the plane, but then revolve it around the x-axis. While we imagine this revolution, we want to think about the three dimensional solid that gets "traced" by the curve spinning around the axis. Let's dive into an example to see.

Let's visualize some function $f(x)$ defined (and continuous) on the interval $[a, b]$ and with $f(x) \geq 0$ on that interval. We'll see why this is useful, but for now, we're just thinking of some function.

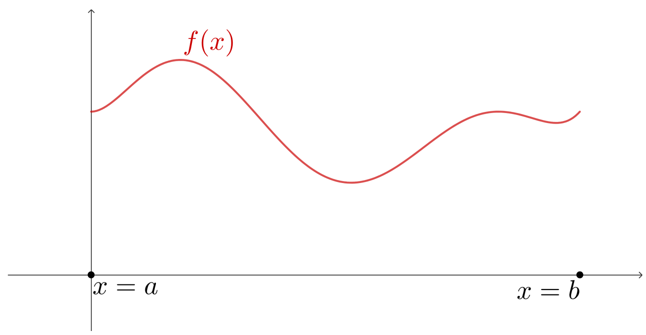


Figure 6.3.1

We're going to revolve this curve (and the region bounded between it and the x-axis) around the x-axis. This will create the following shape.

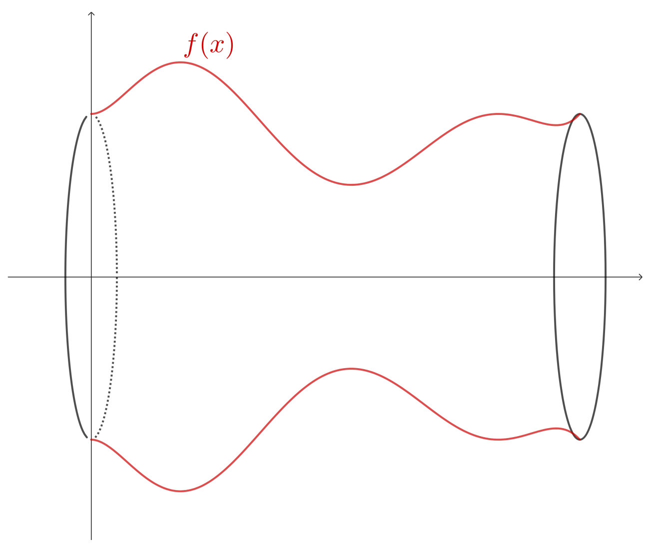


Figure 6.3.2

So our goal is to find the volume of this type of solid. The curve defining the edge of it can change, but the way that we create it will be systematic enough that we can build a formulaic integral expression for it.

As you would imagine, we'll start with a rectangle.

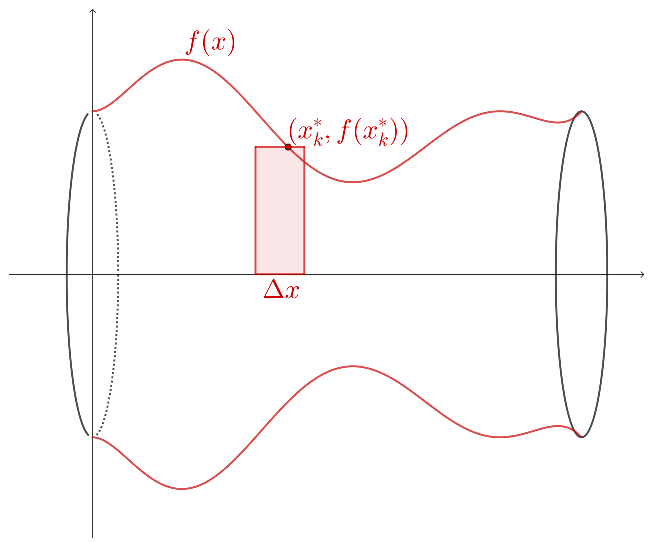


Figure 6.3.3

This rectangle will represent a single, generic slice. We really want to imagine a slice of the 3-dimensional solid, though, and so we will revolve this rectangle around the x-axis. This will create a slice of our solid of revolution. From there, we can think about the volume of this generic kth slice, and fall into the rhythm of our slice-and-sum process.

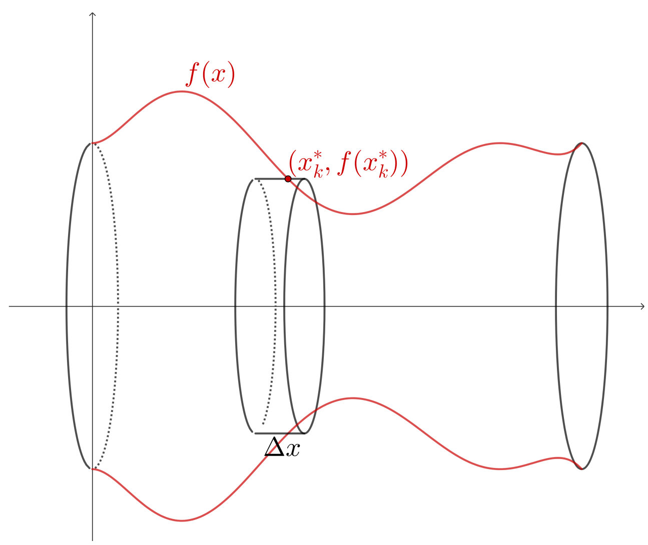


Figure 6.3.4

We want to find the volume of this specific slice. To do this, we can remove this stubby cylinder from the solid and think about it directly. We can see the thickness of the slice is represented by Δx, and so we need to think about the cross-sectional area of the ”face” of this slice.

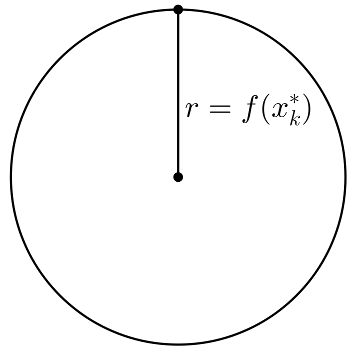


Figure 6.3.5

This is something we can easily find the area of! We know the formula for the area of a circle: $A = \pi r^2$. We'll notice that the radius of this circle is the distance from the center of our slice to the outer edge: this is the height of the rectangle in Figure 6.3.3. So we can use $r = f(x_k^*)$, giving us the cross-sectional area of the kth slice:

$$A(x_k^*) = \pi \left(f(x_k^*)\right)^2.$$

Now we can drop into our slice and sum process:

1. The volume of the kth slice is:

$$\begin{aligned} V_k &= A(x_k^*)\Delta x \\ &= \pi \left(f(x_k^*)\right)^2 \Delta x \end{aligned}$$

2. We can approximate the volume by adding the slices:

$$V \approx \sum_{k=1}^{n} \pi \left(f(x_k^*)\right)^2 \Delta x$$

Sometimes this can be hard to visualize. We're approximating the solid in Figure 6.3.2 by thinking about a bunch of these circular disks stacked next to each other.

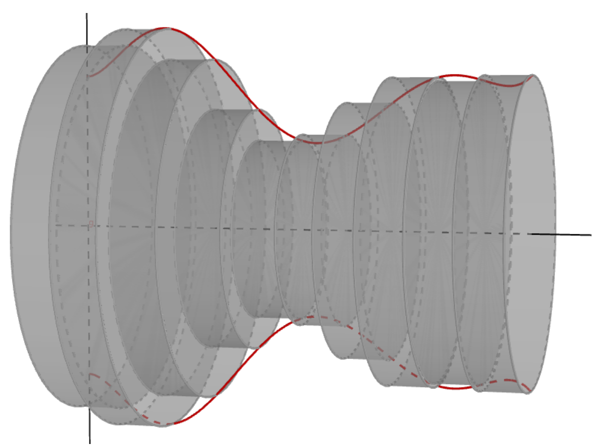


Figure 6.3.6

3. We can apply a limit to evaluate the actual volume of the solid and construct a definite integral.

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi \left(f(x_k^*)\right)^2 \Delta x$$

$$= \int_{x=a}^{x=b} \pi(f(x))^2 \, dx$$

This is great! We'll call this volume integral the **Disk Method**, since each cross section is a circular disk.

What happens if we add a second curve defining a lower boundary to the region, like we did in Section 6.2 Area Between Curves for areas?

Activity 6.3.1 Carving out a Hole in the Center.

We're going to look at the same solid as in Figure 6.3.2. But this time, when we define the 2-dimensional region that we're going to revolve around the x-axis, we're going to add a lower boundary function, $g(x)$.

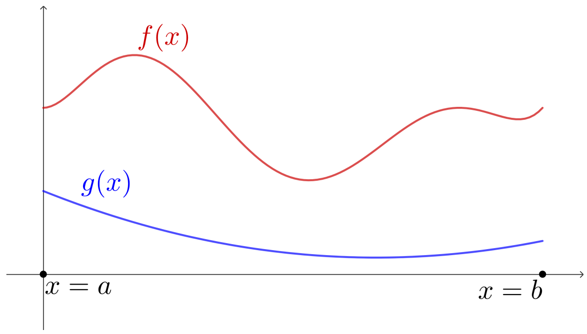


Figure 6.3.7

When we revolve this region around the x-axis, we get the following 3-dimensional solid.

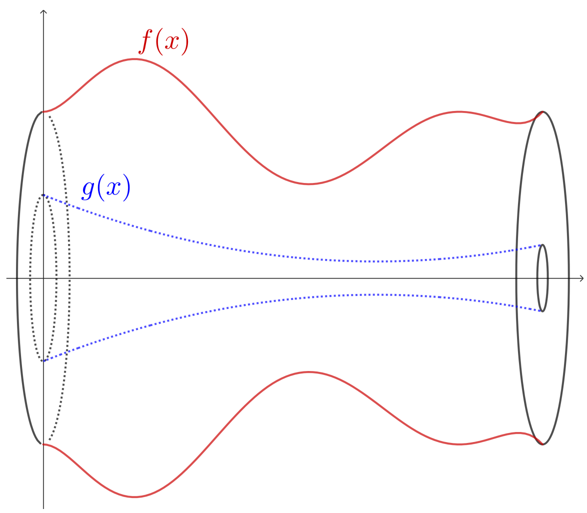


Figure 6.3.8

(a) How is a single generic slice on this solid different than the one in Figure 6.3.2?

(b) Find a formula for the area of the face of the cross-secional slice.

(c) Use the slice-and-sum process to create an integral expression representing the volume of this solid.

Definition 6.3.9 Volume by Disks/Washers.

If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, then the volume of the solid formed by revolving the region bounded between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ around the x-axis is:

$$V = \pi \int_{x=a}^{x=b} \left((f(x))^2 - (g(x))^2\right)\, dx.$$

This is called the **Washer Method**. Note that if $g(x) = 0$, then the resulting volume is:

$$V = \pi \int_{x=a}^{x=b} (f(x))^2\, dx.$$

This is called the **Disk Method**.

We'll walk through two examples where we construct these integral expressions before pretending to be too comfortable. Let's start with something similar to what we've just done.

Activity 6.3.2 Volumes by Disks/Washers.

Consider the region bounded between the curves $y = 4 + 2x - x^2$ and $y = \dfrac{4}{x+1}$. Will will create a 3-dimensional solid by revolving this region around the x-axis.

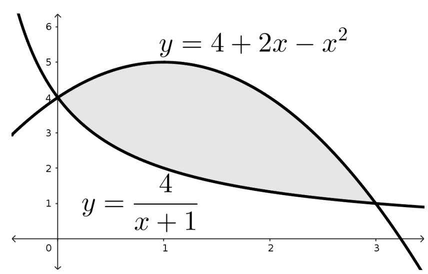


Figure 6.3.10

(a) Visualize the solid you'll create when you revolve this region around the x-axis.

(b) Draw a single rectangle in your region, standing perpendicular to the x-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of the solid.

(f) Can you describe how you would antidifferentiate and evaluate this integral?

Ok, so when we're creating these integrals, we are really focussing on using the rectangle we drew to show us which functions serve as the large radius compared to the small radius. In the next example, we'll see another key thing to notice from a single rectangle.

Activity 6.3.3 Another Volume.

Now lets consider another region: this time, the one bounded between the curves $y = x$ and $y = 3\sqrt{x}$. We will, again, create a 3-dimensional solid by revolving this region around the y-axis.

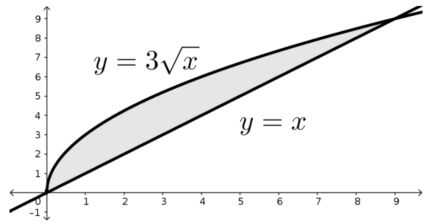


Figure 6.3.11

(a) Visualize the solid you'll create when you revolve this region around the y-axis.

(b) Draw a single rectangle in your region standing perpendicular to the y-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of this solid.

Notice that the rectangle was the clue that we were going to be using Δy when we calculated volumes. This ended up being the reason that we integrated with regard to y, since the $\Delta y \to dy$ in the integral.

A single rectangle, carefully drawn, can give us a large amount of information as we try to juggle these volumes!

Re-Orienting our Rectangles

We saw in Activity 6.3.3 that thinking about the single rectangle we draw can be helpful. We'll see that again in this next formula that we build.

Notice that, in all of the previous work we've done, we've drawn our rectangle so that it is standing *perpendicular* to the axis of revolution. This is the kind of rectangle that, when we revolve it, traces out the "washer" shape!

So what happens when we change the orientation of our rectangle? What happens when we draw a rectangle that is *parallel* to the axis of revolution? Let's consider the same region as before (the one we visualized in Figure 6.3.7) with the same rectangle as before (the one we visualized in Figure 6.3.9), but we'll revolve around the y-axis.

When we revolve this region around the y-axis, we end up with the following solid.

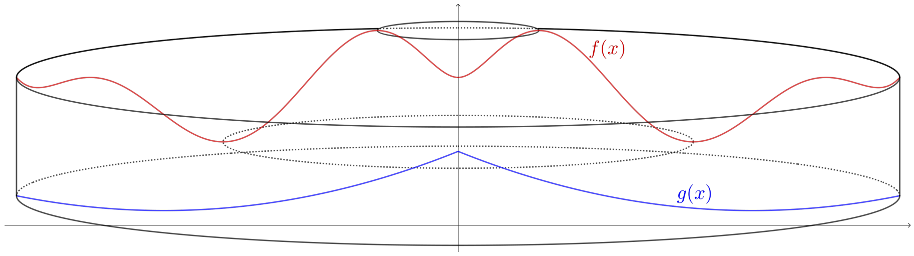


Figure 6.3.12

We want to focus on the single rectangle and the shape that it forms when we revolve it around the y-axis. From there, we can fall into our slice and sum process by thinking about how we might calculate the volume of this single sliced piece and then adding them up.

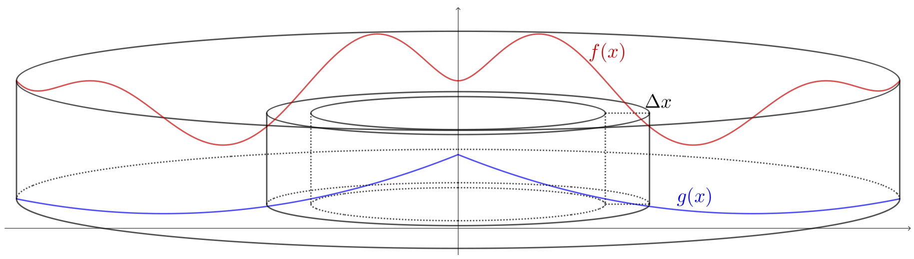


Figure 6.3.13

For this rectangle, we can notice that when we revolve it around the y-axis, we create a hollow cylinder. We'll focus more specifically on this cylinder.

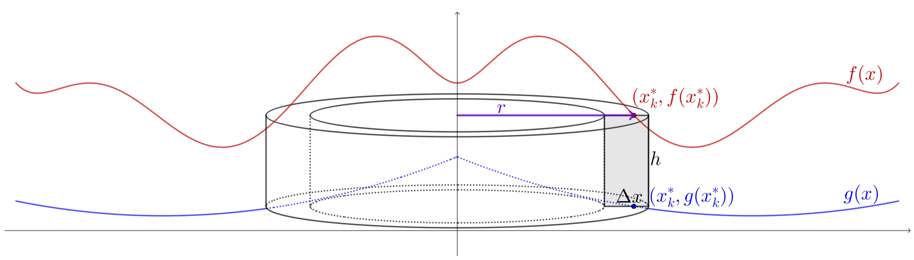


Figure 6.3.14

Let's focus more on the cylinder. We'll need to find the volume of this cylinder. We can think of this volume as really the surface area of the cylinder multiplied by the thickness. Another way to visualize it is to think about cutting the cylinder open, and unfurling it to create a rectangular solid with some thickness.

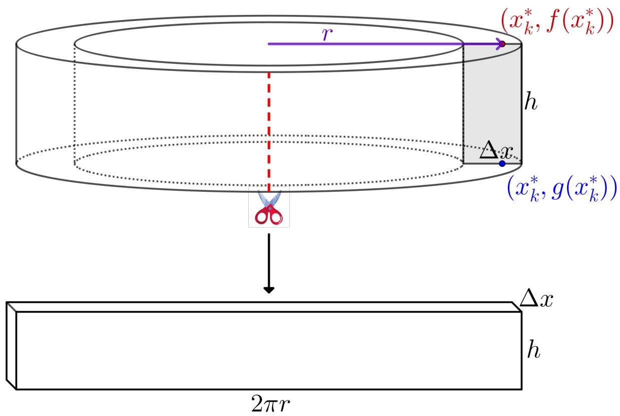


Figure 6.3.15

So we can see that to find V_k, we're going to multiply $A(x_k^*)$ and Δx again, where $A(x_k^*)$ is the area of the cross-sectional "face." In this case, we can see how we'll construct this from the unfurled cylinder.

$$
\begin{aligned}
V_k &= 2\pi r \Delta x \\
&= 2\pi(x_k^*)(f(x_k^*) - g(x_k^*))\Delta x \\
V &\approx \sum_{k=1}^{n} 2\pi(x_k^*)(f(x_k^*) - g(x_k^*))\Delta x \\
V &= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi(x_k^*)(f(x_k^*) - g(x_k^*))\Delta x \\
&= \int_{x=a}^{x=b} 2\pi x(f(x) - g(x))\, dx
\end{aligned}
$$

Definition 6.3.16 Volume by Shells.

If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$ (with $a \geq 0$), then the volume of the solid formed when the region bounded between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is revolved around the y-axis is

$$V = 2\pi \int_{x=a}^{x=b} x\left(f(x) - g(x)\right)\, dx.$$

We can apply this formula in a familiar example, and also practice changing variables.

Activity 6.3.4 Volume by Shells.

Let's consider the region bounded by the curves $y = x^3$ and $y - x + 6$ as well as the line $y = 0$. You might remember this region from Activity 6.2.3. This time, we'll create a 3-dimensional solid by revolving the region around the x-axis

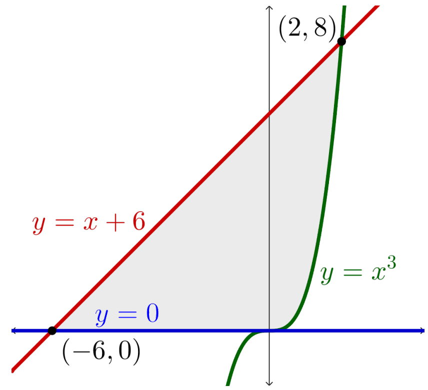


Figure 6.3.17

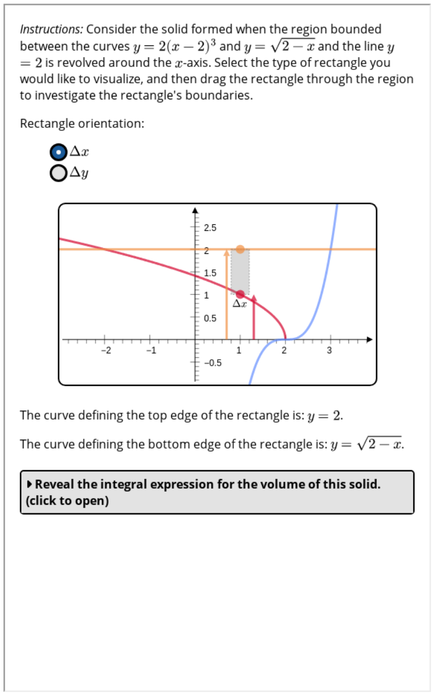
(a) Sketch one or two rectangles that are *perpendicular* to the x-axis. Then set up an integral expression to find the volume of the solid using them.

(b) Now draw a single rectangle in the region that is *parallel* to the axis of revolution. Use this rectangle to visualize the kth slice of this 3-dimensional solid. What does that single rectangle create when it is revolved around the x-axis?

(c) Set up the integral expression representing the volume of the solid.

(d) Confirm that your volumes are the same, no matter your approach to setting it up.

To finish things up, let's look at another interactive graph (similar to how we ended Section 6.2 Area Between Curves) that can help show the differences between finding volume with regard to x (using Δx in our rectangles and dx in our integrals) and finding volume with regard to y (using Δy in our rectangles and dy in our integrals), and how this choice changes our method from washers to shells depending on the axis of revolution.



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Practice Problems

1. We say that the volume of a solid can be thought of as $\displaystyle\int_{x=a}^{x=b} A(x)\ dx$ where $A(x)$ is a function describing the cross-sectional area of our solid at an x-value between $x = a$ and $x = b$. Explain how this integral formula gets built, referencing the slice-and-sum (Riemann sum) method.

2. Explain the differences and similarities between the disk and washer methods for finding volumes of solids of revolution.

3. When do we integrate with regard to x (using a dx in our integral and writing our functions with x-value inputs) and when do we integrate with regard to y (using a dy in our integral and writing our functions with y-value inputs) when we're finding volumes using disks and washers? How do we know?

4. For each of the solids described below, set up an integral using the *disk/washer method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

(a) The region bounded by the curve $y = 2x$ and the lines $y = 0$ and $x = 3$, revolved around the x-axis.

(b) The region bounded by the curve $y = e^{-2x}$ and the x-axis between $x = 0$ and $x = \ln(2)$, revolved around the x-axis.

(c) The region bounded by the curves $y = \ln(x)$ and $y = \sqrt{x}$ between $y = 0$ and $y = 1$, revolved around the y-axis.

(d) The region bounded by the curves $y = 2x + 1$ and $y = x$ between $x = 0$ and $x = 3$, revolved around the x-axis.

(e) The region bounded by the curve $y = x^3$, the x-axis, and the line $x = 2$, revolved around the y-axis.

(f) The region bounded by the curve $y = x^3$ and the y-axis between $y = 0$ and $y = 2$, revolved around the y-axis.

5. Explain where the pieces of the shell formula come from. How is this different than using disks/washers?

6. Say we're revolving a region around the x-axis to create a solid. Using the disk/washer method, we will integrate with respect to x. Using the shell method, we integrate with respect to y. Explain the difference, and why this difference occurs.

7. For each of the solids described below, set up an integral using the *shell method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

(a) The region bounded by the curve $y = 3x$ and the lines $x = 0$ and $y = 5$, revolved around the y-axis.

(b) The region bounded by the curve $y = \sqrt{x}$ and the x-axis between $x = 0$ and $x = 9$, revolved around the x-axis.

(c) The region bounded by the curves $y = 2 - x^2$ and $y = x$ and the line $x = 0$ revolved around the y-axis.

(d) The region bounded by the curves $y = sin(x^2) + 2$ and $y = x$ from $x = 0$ to $x = 1$, revolved around the y-axis.

(e) The region bounded by the curves $y = x^2 - 6x + 10$ and $y = 2 + 4x - x^2$ revolved around the y-axis.

(f) The region bounded by the curves $y = \sqrt{2x}$ and $y = 4 - x$ and the x-axis between $x = 0$ and $x = 4$, revolved around the x-axis.

8. Pick at least 2 integrals from Exercise 4 to re-write using shells instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

9. Pick at least 2 integrals from Exercise 7 to re-write using disks/washers instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

10. For each of the following solids, set up an integral expression using either the disk/washer method or the shell method. You don't need to evaluate them, but you should do some careful thinking about how you set these up, especially as you choose between methods and what variable you are integrating with.

(a) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the x-axis.

(b) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the y-axis.

(c) The region bounded by the curves $y = \frac{1}{x}$ and $y = 1 - (x - 1)^2$ in the first quadrant, revolved around the x-axis.

6.4 More Volumes: Shifting the Axis of Revolution

We have introduced some methods for creating and calculating the volume of different 3 dimensional solids of revolution.

What Changes?

Let's first consider a volume created using disks or washers.

Activity 6.4.1 What Changes (in the Washer Method) with a New Axis?

Let's revisit Activity 6.3.2 Volumes by Disks/Washers, and ask some more follow-up questions. First, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = -3$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.2?

(b) What changes, if anything, do you have to make to the area of the "face" kth washer?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

Now let's consider a volume created using shells.

Activity 6.4.2 What Changes (in the Shell Method) with a New Axis?

Let's revisit Activity 6.3.4 Volume by Shells, and ask some more follow-up questions about the shell method. Again, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = 9$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.4?

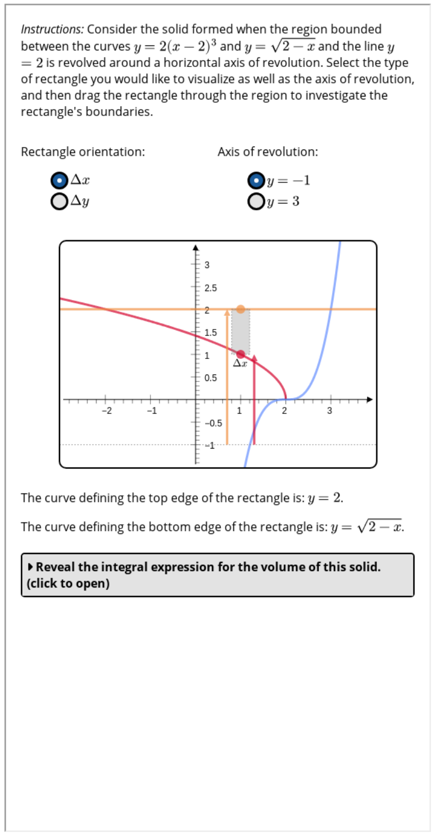
(b) What changes, if anything, do you have to make to the area of the rectangle formed by "unrolling" up kth cylinder?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

In both of these cases, we can notice that the only changes we make are to the *radii*: we just need to re-measure the distance from axis of revolution to either the ends of the rectangle (in the washer method) or the side of the rectangle (in the shell method).

Formalizing These Changes in the Washers and Shells

We can look at yet another interactive graph (similar to how we ended Section 6.2 Area Between Curves and Section 6.3 Volumes of Solids of Revolution). This time, we'll think about how our axis of revolution as well as our choice of rectangle orientation impacts how we construct the washers or shells.



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Notice that in each case, we're re-measuring the radius! Whether we're measuring the radii of a washer by thinking about how far the function outputs are away from the axis of revolution or if we're measuring the radius of a shell by thinking about how far the input variable is away from the axis of revolution, we need to rethink this and do some subtraction.

Activity 6.4.3 More Shifted Axes.

We're going to spend some time constructing *several* different volume integrals in this activity. We'll consider the same region each time, but make changes to the axis of revolution. For each, we'll want to think about what kind of method we're using (disks/washers or shells) and how the different axis of revolution gets implemented into our volume integral formulas.

Let's consider the region bounded by the curves $y = \cos(x) + 3$ and $y = \dfrac{x}{2}$ between $x = 0$ and $x = 2\pi$.

(a) Let's start with revolving this around the x-axis and thinking about the solid formed. While you set up your volume integral, think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

(b) Now let's create a different solid by revolving this region around the y-axis. Set up a volume integral, and continue to think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

(c) We'll start shifting our axis of revolution now. We'll revolve the same region around the horizontal line $y = -1$ to create a solid. Set up an integral expression to calculate the volume.

(d) Now revolve the region around the line $y = 5$ to create a solid of revolution, and write down the integral representing the volume.

(e) Let's change things up. Revolve the region around the vertical line $x = -1$ to create a new solid. Set up an integral representing the volume of that solid.

(f) We'll do one more solid. Let's revolve this region around the line $x = 7$. Set up an integral representing the volume.

Practice Problems

1. Consider the integral formula for computing volumes of a solid of revolution using the disk/washer method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius represented using the function output from the curve(s) defining the region?

2. Consider the integral formula for computing volumes of a solid of revolution using the shell method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius not represented using the function output from the curve(s) defining the region?

3. For each of the solids described below, set up an integral expression using disks/washers representing the volume of the solid.

(a) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = 2$.

(b) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = -1$.

(c) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = -2$.

(d) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = 3$.

4. For each of the solids described below, set up an integral expression using shells representing the volume of the solid.

(a) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = 2$.

(b) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = -1$.

(c) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = -2$.

(d) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = 3$.

6.5 Arc Length and Surface Area

We're going to continue to think about different applications of definite integrals: what they can measure and how we can construct these integral formulas. In this section, we're going to add two more formulas for two more measurements. Before we get far into this discussion, we want to center the important parts of our discussion.

Sure, it is worth noting that, in this section, we'll add a 1-dimensional measurement of size to our list of things an integral can measure. We have talked about a 2-dimensional measure of size (area) and a 3-dimensional measure of size (volume), but we'll add length to the list now! We'll also add a 2-dimensional extension of perimeter to the list when we talk about surface area. That's cool!

But, more importantly, we're going to see how we can construct an integral formula from a Riemann sum, and we're going to get some experience constructing a Riemann sum to measure the thing we care about. In our study of integrals, it might not actually be that important to know how to calculate the specific kinds of volumes or lengths that we're talking about. But we can get some experience with using some formulas from a pretty comfortable field (geometry) to get some experience with the slice-and-sum process. And this process is a useful one to know! We want to see that a definite integral is more than just an area under a curve, and we want to be able to look at an integral formula for some measurement or calculation and see some of the parts of that formula that could be familiar.

Anyways, let's calculate some arc lengths.

Integrals for Evaluating the Length of a Curve

When we talk about **arc length**, we might think of the length of some portion of a circle. Here, we'll use it to refer to the length of some more general curve. We'll graph a function and think about how long the curve of the graph is from some point to another point.

Activity 6.5.1 Measuring Distance.

(a) Consider the following right-triangle with the normal names of side lengths.

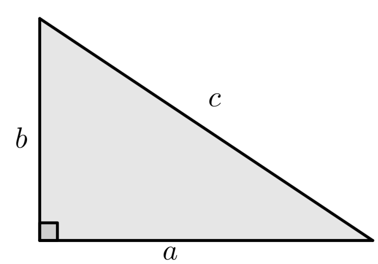


Figure 6.5.1

How do we use the Pythagorean Theorem to find the length of c?

(b) Consider the two points (x_1, y_1) and (x_2, y_2) below.

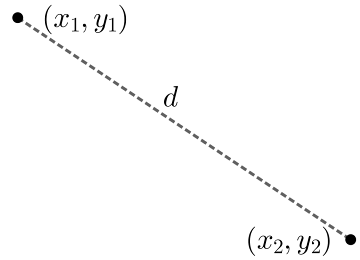


Figure 6.5.2

How do we use the distance formula to find the length of the line connecting the two point, d?

(c) How are these two things the same? How are they different?

This might be a reminder of something we already knew, but let's make sure we are certain: when we calculate distances, we're really just using the Pythagorean Theorem! We can square the vertical distance between the points and the horizontal distance between the points, and then we the length of the straight line connecting two points is:

$$\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This will be a useful formula for us to find an integral expression for the length of a curve.

If we think about the slice-and-sum technique, then we'll want to visualize the kth slice of whatever we're trying to measure. In this case, that means we'll divide the curve up into equally-wide slices and calculate the length of each subsection of the curve. We'll make a recognizable assumption: we'll assume that the curve is actually a straight line between the end points, and calculate that length.

We have made similar assumptions along the way!

1. When we calculated area, we assumed that the curve(s) defining our region were constant on each subinterval. This is what gave us rectangular slices, with flat tops and bottoms!
2. When we calculated volumes by disks and washers, we again assumed that the curve(s) were constant on each subinterval. This is what gave us constant radii on each disk/washer!
3. When we calculated volumes by shells, this assumption of constant curves made the cylindrical shells have flat tops and bottoms!

Let's visualize the kth slice.

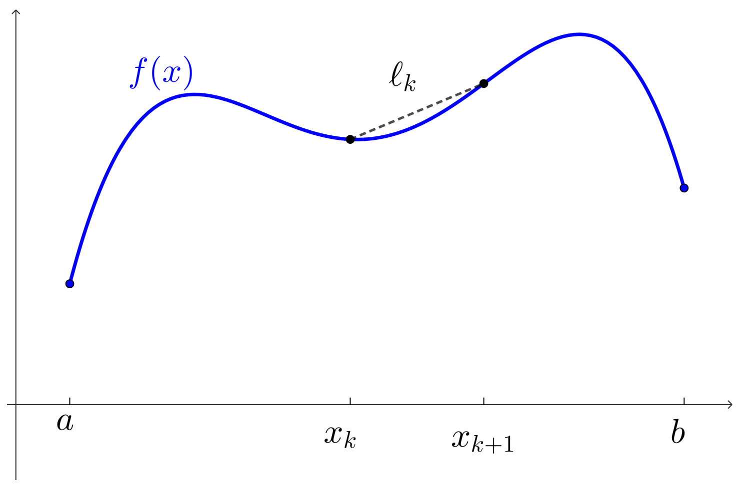


Figure 6.5.3

In order to calculate ℓ_k, the straight-line length connecting the end points of the kth subinterval, we can use the Pythagorean Theorem or distance formula (from Activity 6.5.1).

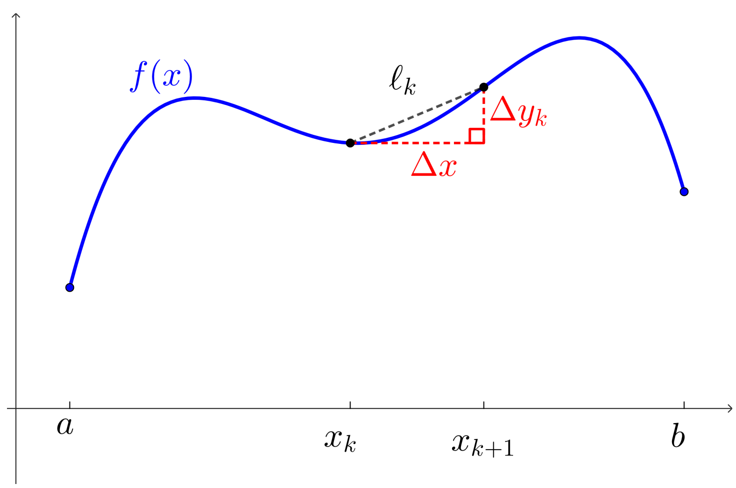


Figure 6.5.4

Let's start the slice-and-sum process.

$$\ell_k = \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$
$$\ell \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

Two notes:

1. We're using Δy_k to denote the vertical distance between the end points of the kth subinterval because we expect these to differ for each subinterval. We don't need to do this for Δx, since we've been slicing things into equally-wide subintervals this whole time.

2. This isn't a Riemann sum. This is much more important, and much more pressing.

Before we can do anything, we need to try to manipulate this sum so that it is in the form of a Riemann sum. What does this mean? What are the some of the things required for the Riemann sum structure that we don't have here? Feel free to look back at Definition 5.2.3 Riemann Sum to remind yourself what elements are needed for a Riemann sum.

Notice, first, that we need a function evaluated at any single input on the subinterval: $f(x_k^*)$. In our version, we have a function evaluated twice at very specific inputs:

$$\Delta y_k = f(x_{k+1}) - f(x_k).$$

We'll need to re-think about how we represent this part in order to get a single function output.

We also need to have this function *multiplied* by Δx. In our sum, we have Δx as a part of the function itself, under the square root. We'll want to move this Δx outside of the root. Let's start there.

We'll start by looking at the sum to approximate the length ℓ and factoring Δx outside of the root.

$$\begin{aligned}
\ell &\approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\
&= \sum_{k=1}^{n} \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_k)^2}{(\Delta x)^2}\right)} \\
&\sum_{k=1}^{n} \Delta x \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2} \\
\ell &\approx \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2}\, \Delta x
\end{aligned}$$

This looks better! We have Δx floating around at the end of our sum, ready to turn into dx once we apply the limit as $n \to \infty$.

The inside of our root, though, is still a bit messed up. We would like a single function of x_k^*, any x-value from the kth subinterval. Instead, we have a function involving the two x-values of the end points *and* we still have Δx involved in this part!

But we can notice something about $\frac{\Delta y_k}{\Delta x}$: it really is the slope of the straight line! Can we use a function to represent this? We can *absolutely* approximate this slope using a function: the derivative!

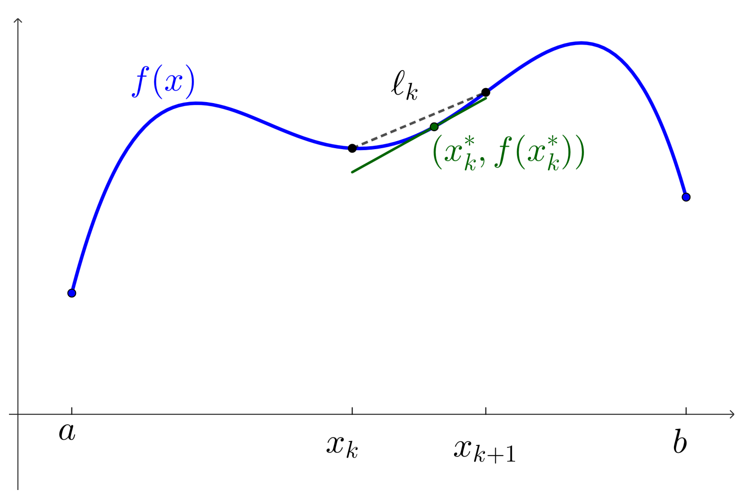


Figure 6.5.5

If we pick some point, $(x_k^*, f(x_k^*))$, on the kth subinterval, then we can approximate $\frac{\Delta y_k}{\Delta x}$ with $f'(x_k^*)$. This is a fine approximation of this slope (and the Mean Value Theorem guarantees that there is a point on the subinterval where $f'(x_k^*) = \frac{\Delta y_k}{\Delta x}$ exactly), but the real magic will happen when $n \to \infty$. The definition of the Derivative at a Point will make sure that these slopes are equal in the limit!

Let's return to our slice-and-sum process.

$$\begin{aligned}\ell &\approx \sum_{k=1}^{n} \sqrt{1+\left(\frac{\Delta y_k}{\Delta x}\right)^2}\Delta x\\ &\approx \sum_{k=1}^{n} \sqrt{1+(f'(x_k^*))^2}\Delta x\end{aligned}$$

This *is* a Riemann sum! We can apply a limit and get an integral!

$$\begin{aligned}\ell &\approx \sum_{k=1}^{n} \sqrt{1+(f'(x_k^*))^2}\Delta x\\ \ell &= \lim_{n\to\infty}\sum_{k=1}^{n} \sqrt{1+(f'(x_k^*))^2}\Delta x\\ &= \int_{x=a}^{x=b} \sqrt{1+(f'(x))^2}\,dx\end{aligned}$$

Definition 6.5.6 Length of a Curve.

If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on (a, b), then the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} \sqrt{1+(f'(x))^2}\,dx.$$

Example 6.5.7

Find an integral expression representing the length of the following curves.

(a) The curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = 2$.

Solution. Since $y' = -\dfrac{1}{x^2}$, then we can construct the following integral:

$$\begin{aligned}\ell &= \int_{x=1}^{x=2} \sqrt{1+(y')^2}\,dx\\ &= \int_{x=1}^{x=2} \sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx\\ &= \int_{x=1}^{x=2} \sqrt{1+\frac{1}{x^4}}\,dx\end{aligned}$$

Instead of worrying about actually evaluating this integral, we'll leave it like this.

If you do want to fiddle with this integral, than it might be helpful to note that we can re-write it:

$$\int_{x=1}^{x=2} \sqrt{1+\frac{1}{x^4}}\,dx = \int_{x=1}^{x=2} \sqrt{\frac{x^4+1}{x^4}}\,dx$$
$$= \int_{x=1}^{x=2} \frac{\sqrt{1+x^4}}{x^2}\,dx$$

(b) The curve $y = \sin^{-1}(x)$ from $x = -1$ to $x = 1$.

Solution. We know that $y' = \dfrac{1}{\sqrt{1-x^2}}$, so we can construct the following integral:

$$\ell = \int_{x=-1}^{x=1} \sqrt{1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2}\,dx$$
$$= \int_{x=-1}^{x=1} \sqrt{1+\frac{1}{1-x^2}}\,dx$$

We can leave this integral like this for now.

Similar to the first example, though, we can re-write this if you'd like to explore it more!

$$\int_{x=-1}^{x=1} \sqrt{1+\frac{1}{1-x^2}}\,dx = \int_{x=-1}^{x=1} \sqrt{\frac{2-x^2}{1-x^2}}\,dx$$

Integrals for Evaluating the Surface Area of a Solid

Moving from the length of some curve towards calculating the surface area of some solid of revolution won't be hard: we'll use the length formula in our procedure!

Let's build this surface area formula. Consider some function, $f(x)$, on the interval from $x = a$ to $x = b$.

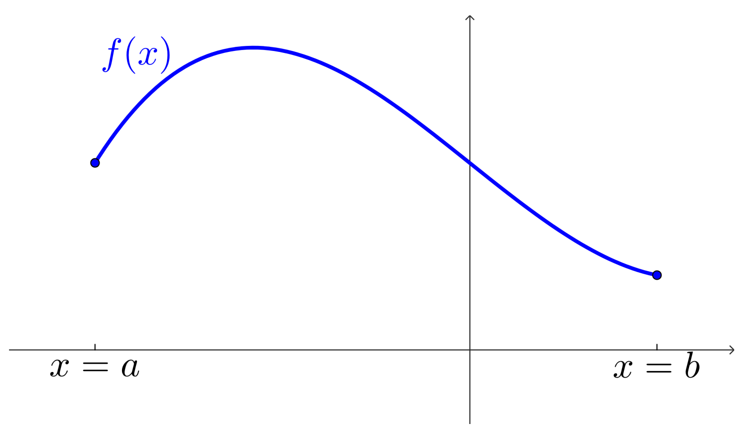


Figure 6.5.8

Instead of forming a rectangle for the kth slice, we'll do the same thing that we did for arc length: we'll connect the end points of the kth subinterval. This will create a trapezoid.

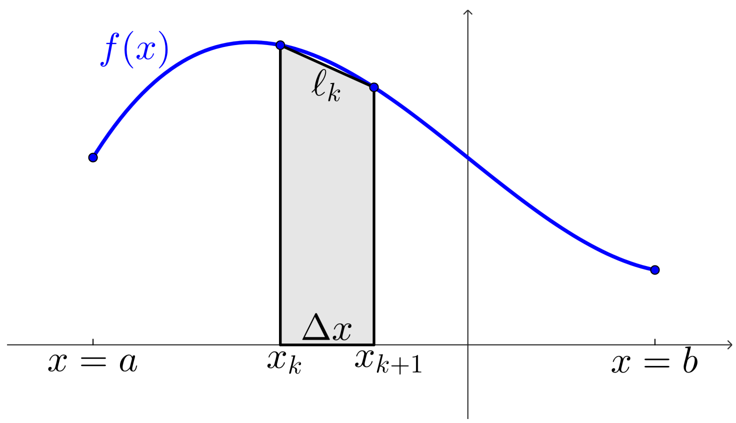


Figure 6.5.9

We'll use ℓ_k to represent the diagonal length of the line connecting the endpoints. Notice that this is going to become the arc length.

When we revolve the curve $y = f(x)$ around the x-axis, we can see not just the solid created by the curve, but the solid representing this kth slice.

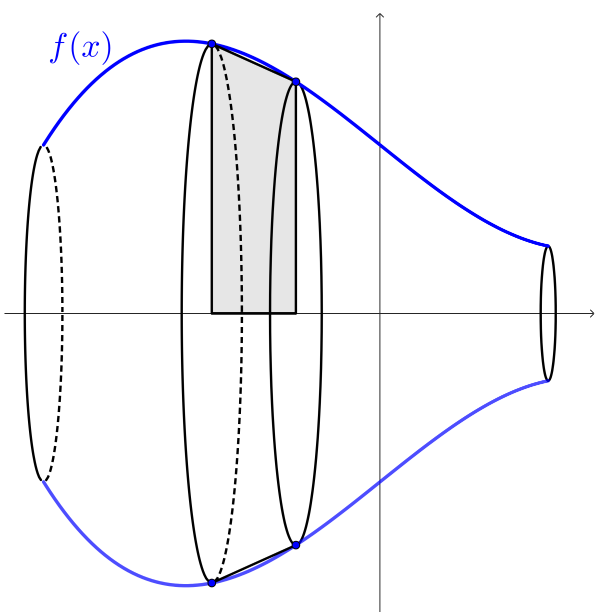


Figure 6.5.10

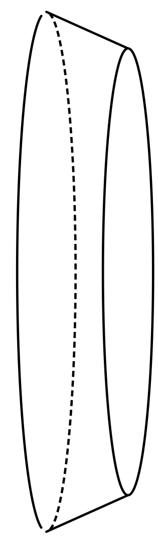


Figure 6.5.11 The kth frustum-shaped slice.

In order for us to find the surface area of this kth slice, we'll think about how "far" the diagonal line revolves. This is based on the circumference of each circular end of our slice, which means we have two radii to consider: the function outputs at both endpoints of the kth subinterval:

$$A_k = 2\pi \left(\frac{f(x_k) + f(x_{k+1})}{2} \right) \ell_k$$

This is going to become problematic, since we need only one function output evaluated at some x_k^* on the kth subinterval.

Instead, we can select some x-value on the interval and use the function output at that point to represent the radius of our kth slice.

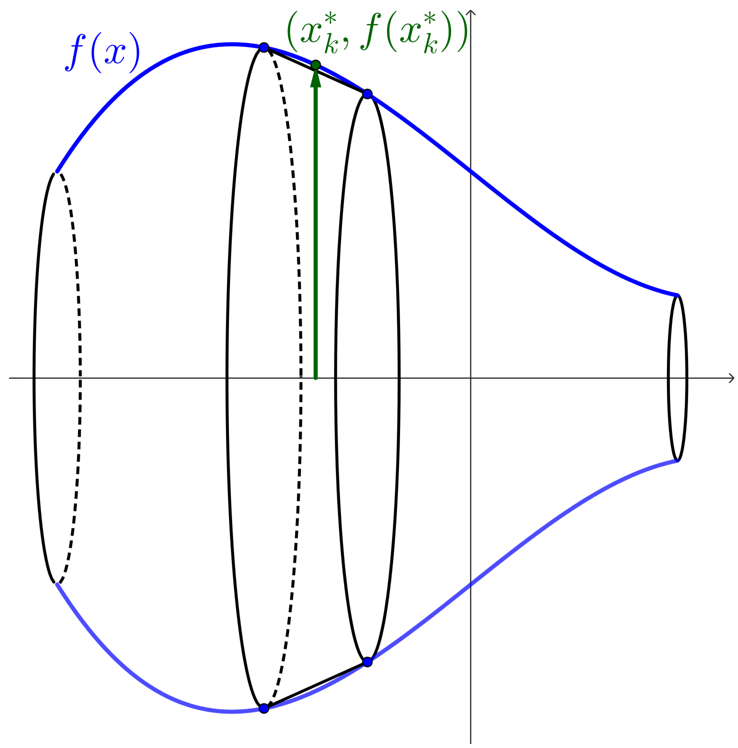


Figure 6.5.12

Instead of averaging the large and small radii from the end-points, we'll just select the one function output to represent this "average" radius. In the limit as $n \to \infty$, things will work out, since this randomly selected radius will become exactly equal the average radius in the limit since $\Delta x \to 0$.

Now we can slice and sum!

$$\begin{aligned}
A_k &= 2\pi f(x_k^*)\ell_k \\
&= 2\pi f(x_k^*)\sqrt{1+(f'(x_k^*))^2}\Delta x \\
A &\approx \sum_{k=1}^{n} 2\pi f(x_k^*)\sqrt{1+(f'(x_k^*))^2}\Delta x \\
A &= \lim_{n\to\infty}\sum_{k=1}^{n} 2\pi f(x_k^*)\sqrt{1+(f'(x_k^*))^2}\Delta x \\
&= \int_{x=a}^{x=b} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx
\end{aligned}$$

Definition 6.5.13 Surface Area.

Let $f(x)$ is continuous with $f(x) \geq 0$ on the interval $[a, b]$ and differentiable on (a, b). If the region bounded by $f(x)$ and the x-axis from $x = a$ to $x = b$ is revolved around the x-axis, then the surface area of the resulting solid is:

$$A = 2\pi \int_{x=a}^{x=b} f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Practice Problems

1. The formula for arc length of the function $f(x)$ on the interval $[a, b]$ is $L = \int_{x=a}^{x=b} \sqrt{1 + f'(x)^2}\, dx$. Explain how this definition is built, using the slice-and-sum method. Make sure to explain how we the Pythagorean Theorem is involved.

2. Why do we use $f'(x)$ in the formula for arc length?

3. For each of the following curves on intervals, evaluate the arc length.

(a) $y = -6x + 2$ on $[1, 5]$

(b) $y = \dfrac{x^{3/2}}{3}$ on $[0, 60]$

(c) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ on $[1, 3]$

4. For each of the following curves on intervals, set up an integral representing the arc length. Do not evaluate.

(a) $y = \tan^2(x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(b) $y = \ln(x^2)$ on $[1, e]$

(c) $y = \sqrt{x+1}$ on $[-1, 8]$

5. Why is the formula for arc length seemingly involved in the integral formula for surface area of a solid of revolution?

6. In the integral formula $A = \int_{x=a}^{x=b} 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx$, what does $f(x)$ represent? What about 2π?

7. For each of the following curves and intervals, find the surface area of the solid formed when the curve is revolved around the x-axis.

(a) $y = 2x + 3$ on $[0, 3]$

(b) $y = 4\sqrt{x}$ on $[4, 9]$

(c) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ on $[1, 3]$

8. For each of the following curves and intervals, set up the surface area of the solid formed when the curve is revolved around the x-axis. Do not evaluate the integral.

(a) $y = \sin\left(e^{x^2}\right) + 1$ on $[0, 2]$

(b) $y = \ln(x^2)$ on $[1, e]$

6.6 Other Applications of Integrals

We should pause and think (even briefly) about the whole point of this chapter.

Do you think you will need to know how to calculate the volume of a solid of revolution?

Is being able to think about the surface area of a solid of revolution important?

Are you going to need to calculate a bunch of arc lengths constantly in your other classes, jobs, or spare time?

No. Probably not. It's fun to think about this stuff, but the goal is not really the formulas: it's the construction.

What we really should be leaving this chapter with is a renewed view of what a definite integral *is*. After Chapter 5 Antiderivatives and Integrals, we probably thought about a definite integral as a measurement of the (signed) area "under the curve" (bounded between the curve and the x-axis). Hopefully now, though, we have a deeper view of the definite integral through this slice-and-sum process.

A definite integral is an accumulation of some function outputs multiplied by the space between the inputs.

So when we do move past these topics into other classes, jobs, or your spare time, then you *will* run into formulas involving definite integrals. Hopefully, based on this chapter, we have the tools to deconstruct those formulas to find out what they are measuring and how they are measuring it.

To end the chapter, let's look at some applications of definite integrals into formulas from other academic fields.

Physics Application: Mass

Let's consider some object made out of a material with a constant density. To calculate the mass of the object, we need to know the size (the volume of the object) and the density, and then we can multiply:

$$\text{mass} = \text{density} \cdot \text{volume}.$$

We've seen how we can use the slice-and-sum formula to calculate volumes by slicing the object into thin pieces and approximating the volume by multiplying the cross-sectional area by the thickness of the slice:

$$V_k = A(x_k^*)\Delta x.$$

We can easily construct an integral formula for the mass of an object now! We just need to find some measurement of the mass of a single slice of an object, and then start summing until we end up with a definite integral! So, for an object spanning from $x = a$ to $x = b$ with density, ρ, we have:

$$\begin{aligned}
M_k &= \rho V_k \\
&= \rho A(x_k^*)\Delta x \\
M &\approx \sum_{k=1}^{n} \rho A(x_k^*)\Delta x \\
M &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho A(x_k^*)\Delta x \\
&= \int_{x=a}^{x=b} \rho A(x)\ dx
\end{aligned}$$

This is great! We can easily extend this to other situations, as well.

Imagine if the density of our object isn't constant. Maybe we have some metal alloy, where the mixture of the metal changes as we traverse through the object. This variable density can be represented, then, as $\rho(x)$. We can replace the constant ρ with $\rho(x_k^*)$ and then eventually $\rho(x)$ to get:

$$M = \int_{x=a}^{x=b} \rho(x)A(x)\,dx.$$

Theorem 6.6.1 Mass of an Object.

For some solid spanning from $x = a$ to $x = b$ where $A(x)$ is the cross-sectional area of the object at x and $\rho(x)$ is the density of the object at x, then the total mass of the object is:

$$M = \int_{x=a}^{x=b} \rho(x)A(x)\,dx.$$

For very thin objects, like wires, we can not consider the cross-sectional area and simply use what we call the **linear density**: $\rho(x)$ measures the amount of mass per unit length (instead of mass per cubic unit of volume). In this case, our formula changes.

Theorem 6.6.2 Mass of a Thin Wire.

For some thin object (like a wire) spanning from $x = a$ to $x = b$, then if $\rho(x)$ measures the mass per unit of length ($\rho(x)$ is a linear density function), then the mass of the object is:

$$M = \int_{x=a}^{x=b} \rho(x)\,dx.$$

Example 6.6.3

For each of the following objects, set up an integral expression to calculate the mass. Evaluate the integrals!

(a) A thin wire that is 30 cm long with a density function $\rho(x) = 2 + \frac{x}{60}$, where $\rho(x)$ measures the density of the wire at x in g/cm.

Solution.

$$\begin{aligned} M &= \int_{x=0}^{x=30} \left(2 + \frac{x}{60}\right)\,dx \\ &= \left(2x + \frac{x^2}{120}\right)\Bigg|_{x=0}^{x=30} \\ &= 60 + \frac{900}{120} \\ &= 67.5 \end{aligned}$$

The mass of the wire is 67.5 g.

(b) A thin wire that is 100 cm long, with an unknown density. We assume that the density changes linearly from one end to the other, with the measured density at both ends being 5.1 g/cm and 4.7 g/cm.

Hint. Construct the density function using two points: $(0, 5.1)$ and $(100, 4.7)$. You could switch the orders and use $(0, 4.7)$ and $(100, 5.1)$ as well. What is the linear function connecting these points?

Solution. We'll use the points $(0, 5.1)$ and $(100, 4.7)$ to construct $\rho(x)$.

$$\begin{aligned}\rho(x) &= \left(\underbrace{\frac{4.7-5.1}{100-0}}_{\text{slope}}\right)(x-0)+5.1\\ &= -0.004x+5.1\end{aligned}$$

Now we can integrate:

$$\begin{aligned}M &= \int_{x=0}^{x=100} 5.1-0.004x\,dx\\ &= \left(5.1x-0.002x^2\right)\Big|_{x=0}^{x=100}\\ &= 510-20\\ &= 490\end{aligned}$$

So the mass of the wire is 490 g.

Physics Application: Work

Another example of a measurement found in physics contexts is Work. Work is, generally, the amount energy transferred to (or from, depending on perspective) an object by some force across some distance or displacement.

$$\text{Work} = \text{Force}\cdot\text{distance}$$

In general, we can use the "slice-and-sum" process when the force applied to our object is some function of the position:

$$\begin{aligned}W_k &= F(x_k^*)\Delta x\\ W &\approx \sum_{k=1}^{n} F(x_k^*)\Delta x\\ W &= \lim_{n\to\infty}\sum_{k=1}^{n} F(x_k^*)\Delta x\\ &= \int_{x=a}^{x=b} F(x)\,dx\end{aligned}$$

Theorem 6.6.4 Work Required.

If $F(x)$ is a function measuring the force applied to an object at some x-value in the interval $[a, b]$, then the work done to move the object *across the interval is:*

$$W = \int_{x=a}^{x=b} F(x)\, dx.$$

In many contexts, we can notice that the force being applied to the object is dependent on the position of the object along its path. For instance, in the 1600's, British physicist Robert Hooke claimed (and proved) that the force required to stretch or compress a spring is directly proportional to the distance that it is away from "equilibrium" (the position the spring naturally rests in).

Work: Springs

We'll first look at the springs, since the force functions are relatively simple. Robert Hooke's claim (Hooke's Law) says that the force function for a spring is always in the form $F(x) = kx$ where k is just some constant proportion connecting distance to the force required for that particular spring.

Example 6.6.5

Calculate the work required to stretch or compress the following springs.

(a) A spring stretched to 0.1 m past its equilibrium position, where a Force of $F(x) = 180x$ is applied, measured in N.

Solution.

$$\begin{aligned} W &= \int_{x=0}^{x=0.1} 180x\, dx \\ &= 90x^2 \Big|_{x=0}^{x=0.1} \\ &= 0.9 \end{aligned}$$

So the work required to stretch the spring is 0.9 J.

(b) A force of 81 N was applied to a spring to stretch it 0.8 m from its equilibrium position. Calculate the work required to stretch it 0.2 m further.

Solution.

$$\begin{aligned} F(x) &= \frac{81}{0.8}x \\ &= \frac{405x}{4} \\ W &= \int_{x=0.8}^{x=1} \frac{405x}{4}\, dx \\ &= \left(\frac{405x^2}{8}\right)\Big|_{x=0.8}^{x=1} \\ &= \frac{405}{8} - \frac{162}{5} \\ &= \frac{729}{40} \end{aligned}$$

So the work required to stretch the spring is 18.225 J.

Work: Pumping Problems

We can explore another class of examples with some more complications in the setup. We still will be thinking about force and distance, but we can re-frame them with a new example.

What if we pump liquid out of the top of a tank? How does that work? There are some fun things to think about. Let's visualize this tank below, and then talk through the construction of the formula.

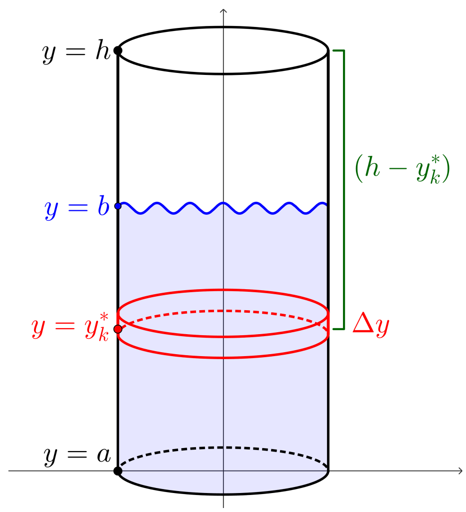


Figure 6.6.6 Diagram of a pumping problem.

We can see a couple of notable things here:

- We're going to slice up the liquid. So from $y = a$, the bottom of the tank, up to $y = b$, the height at the top of the volume of liquid, we will create k slices (each at some y_k^* y-value) with Δy representing the thickness of the slice.
- The distance portion of the formula is not represented by the width of a subinterval from the whole distance. Instead, we'll note that liquid at the top of the tank needs to be pumped a shorter distance than the liquid near the bottom of the tank. The distance part, then, is a function based on the y-value. So, if $y = h$ is the height of the tank, then $(h - y_k^*)$ is the distance that the liquid in the kth slice will move.
- To calculate the force required to pump the liquid in the kth slice, we need to know the mass of the liquid and the acceleration needed to pump the liquid out of the tank. The acceleration will be whatever the acceleration needed to overcome gravity: we'll use g, the positive gravitational constant. For the mass, we'll think about the density of the liquid (a constant, ρ) and the volume of the kth slice.
- To find the volume of the kth slice, we'll think about the cross-sectional area of the slice at y_k^* (we'll call it $A(y_k^*)$) multiplied by the thickness, Δy.

This gives us the following in the slice-and-sum process:

$$\begin{aligned} W_k &= F(y_k^*)(h - y_k^*) \\ &= (\rho A(y_k^*)\Delta y)\, g(h - y_k^*) \\ &= \rho g A(y_k^*)(h - y_k^*)\Delta y \end{aligned}$$

$$W \approx \sum_{k=1}^{n} \rho g A(y_k^*)(h - y_k^*)\Delta y$$

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho g A(y_k^*)(h - y_k^*)\Delta y \\ &= \rho g \int_{y=a}^{y=b} A(y)(h-y)\, dy \end{aligned}$$

So, when we consider these pumping problems, we really need to take note of only a few things:

- What are the limits of integration? Where *is* the liquid we're pumping (based on y-value heights)?
- What is the geometry of our tank? When we consider a single slice of the liquid, what kind of shape will we have, and how do we calculate the cross-sectional area?

Example 6.6.7

For each of the following tanks, calculate the work required to empty the liquid from the tank by pumping it through the top of the tank.

(a) Consider a cylindrical tank, similar to the one in Figure 6.6.6. The radius of the tank is 0.5 m, and the height is 2 m. The tank is half full of some liquid with density ρ.

Solution. The cross sectional area is:

$$\begin{aligned} A &= \pi r^2 \\ A(y) &= \pi(0.5)^2 \\ &= \frac{\pi}{4} \end{aligned}$$

Then the limits of integration will be from $y = 0$ to $y = 1$, since the height is 2 m and the liquid, then, reaches to 1 m.

$$\begin{aligned} W &= \rho g \int_{y=0}^{y=1} A(y)(2-y)\, dy \\ &= \frac{\rho g \pi}{4} \int_{y=0}^{y=1} 2 - y\, dy \\ &= \frac{\rho g \pi}{4} \left(2y - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1} \\ &= \frac{\rho g \pi}{4} \left(\frac{3}{2}\right) \\ &= \frac{3\rho g \pi}{8} \end{aligned}$$

(b) Now box-shaped tank, where the base of the tank is 1 meters by 3 meters and the height is 2 meter. The tank is filled all the way to the top with some liquid with density ρ.

Solution. The cross sectional area is:

$$A = \ell w$$

$$\begin{aligned} A(y) &= 1(3) \\ &= 3 \end{aligned}$$

Then the limits of integration will be from $y = 0$ to $y = 2$, since the height is 2 m and the liquid, then, reaches to the top.

$$\begin{aligned} W &= \rho g \int_{y=0}^{y=2} A(y)(2-y)\, dy \\ &= 3\rho g \int_{y=0}^{y=2} 2 - y\, dy \\ &= 3\rho g \left(2y - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=2} \\ &= 3\rho g\,(2) \\ &= 6\rho g \end{aligned}$$

(c) Consider a conical tank, where the radius of the base is 3 meters and the cone is 2 meters tall. The liquid is filled up to the 1.5 meter mark.

Solution. Finding the cross-sectional area function will be a bit harder. It is the area of a circle, but the radius changes based on the y-value. At $y = 0$, we have $r = 3$. At $y = 2$, we have $r = 0$. So the rate of change of the radius is -1.5 meters per meter of height.

$$A(y) = \pi(3 - 1.5y)^2$$

The limits of integration are from $y = 0$ to $y = 1.5$:

$$\begin{aligned} W &= \rho g \pi \int_{y=0}^{y=1.5} (3 - 1.5y)^2 (2 - y)\, dy \\ &= \rho g \pi \int_{y=0}^{y=1.5} 18 - 27y + 13.5y^2 - 2.25y^3\, dy \\ &= \rho g \pi \left(18y - \frac{27y^2}{2} + \frac{13.5y^3}{3} - \frac{2.25y^4}{4}\right)\Bigg|_{y=0}^{y=1.5} \\ &= \frac{2295\rho g \pi}{256} \end{aligned}$$

Practice Problems

1. Consider a thin wire whose density is variable and is written as $\rho(x)$ on some interval. For each of the following density functions (and their intervals), build and evaluate an integral representing the mass of the wire.

(a) $\rho(x) = \dfrac{2x+1}{10}$ on $[0, 10]$

(b) $\rho(x) = x^2 - 2$ on $[100, 200]$

2. Consider a spring that requires 100 N of force to stretch the spring 0.3 m from its equilibrium position.

(a) Using Hooke's Law, find the spring constant k such that $F(x) = kx$.

(b) How much work is required to stretch the spring 0.4 m from its equilibrium position?

(c) How much work is required to stretch the spring another 0.1 m after this?

3. Consider a spring that is stretched 0.1 m from its equilibrium position. It requires 10 N to stretch it an additional 0.2 m.

(a) Using Hooke's Law, find the spring constant k such that $F(x) = kx$.

(b) How much work is required to stretch the spring 1 m from its equilibrium position?

(c) How much work is required to compress the spring 0.2 m from is equilibrium position?

4. For each tank, assume the density of the liquid is ρ, and set up and evaluate the integral representing the work required to pump the liquid out of the top of the tank.

(a) A square-based tank that is 2 meters tall with a 0.5 m $\times$ 0.5 m base, full of liquid.

(b) A cylindrical tank where the base has a radius of 1 m and the height is 3 m, half-full of liquid.

(c) A frustum-shaped tank where the bottom radius is 4 m, the top radius is 1 m, the height is 2 m, and the tank is filled to a height of $\frac{2}{3}$ m.

Chapter 7

Techniques for Antidifferentiation

7.1 Improper Integrals

We're going to think a bit about integration with a twist: what happens when our "definite" integrals can't actually be evaluated? First, let's try to sink ourselves back into the context we've been in for a while now: what kinds of problems have we encountered so far, and how do we use our calculus intuition to get around those problems?

Activity 7.1.1 Remembering a Theme so Far.

(a) Let's say that we want to find what the y-values of some function $f(x)$ are when the x-values are "infinitely close to" some value, $x = a$. Since there is no single x-value that is "infinitely close to" a that we can evaluate $f(x)$ at, we need to do something else. How do we do this?

(b) Let's say that we want to find the rate of change of some function instantaneously at a point with $x = a$. We can't find a rate of change unless we have two points, since we need to find some differences in the outputs and inputs. How do we do this?

(c) Suppose you want to find the total area, covered by an infinite number of infinitely thin rectangles. You have a formula for finding the dimensions and areas for some finite number of rectangles, but how do we get an infinite number of them?

(d) Can you find the common calculus theme in each of these scenarios?

So moving forward, we want to remember how we typically have solved these problems. Now, let's try to identify the types of problems with integrals that we need to figure our way around.

Activity 7.1.2 Remembering the Fundamental Theorem of Calculus.

We want to think about generalizing our notion of integrals a bit. So in this activity, section, we're going to think about some of the requirements for the Fundamental Theorem of Calculus and try to loosen them up a bit to see what happens. We'll try to construct meaningful approaches to these situations that fit our overall goals of calculating area under a curve.

This practice, in general, is a really good and common mathematical process: taking some result and playing with the requirements or assumptions to see what else can happen. So it might feel like we're just fiddling with the "What if?" questions, but what we're actually doing is good mathematics!

(a) What does the Fundamental Theorem of Calculus say about evaluating the definite integral $\int_{x=a}^{x=b} f(x)\,dx$?

(b) What do we need to be true about our setup, our function, etc. for us to be able to apply this technique to evaluate $\int_{x=a}^{x=b} f(x)\,dx$?

We are going to introduce the idea of "Improper Integrals" as kind-of-but-not-quite definite integrals that we can evaluate. They are going to violate the requirements for the Fundamental Theorem of Calculus, but we'll work to salvage them in meaningful ways.

This should build a pretty good idea of a new "class" of integrals: ones that aren't *quite* definite integrals that we can evaluate with the Fundamental Theorem of Calculus, but ones that we can use limits to get at.

Improper Integrals

Definition 7.1.1 Improper Integral.

An integral is an **improper integral** if it is an extension of a definite integral whose integrand or limits of integration violate a requirement in one of two ways:

1. The interval that we integrate the function over is unbounded in width, or infinitely wide.
2. The integrand is unbounded in height, or infinitely tall, somewhere on the interval that we integrate over.

With this definition, we can think about the strategies that we got from Activity 7.1.1: we're going to identify the problems in our integral (infinite width of the interval or infinite height of the integrand function) and use a limit!

Before we formalize that, though, let's try to think about how this works by being really explicit about what this limit is actually doing.

Activity 7.1.3 Approximating Improper Integrals.

In this activity, we're going to look at two improper integrals:

1. $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\,dx$

2. $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$

(a) First, let's just clarify to ourselves what it means for an integral to be improper. Why are each of these integrals improper? Be specific!

(b) Let's focus on $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\, dx$ first. We're going to look at the slightly different integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $t > 2$, then our function is continuous and bounded on $[2, t]$, and so we can evaluate this integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$.

Find and antiderivative, $F(x)$.

(c) Now we're going to evaluate some areas for different values of t. Use your antiderivative $F(x)$ from above!

- Let's start with making $t = 99$. So we're going to evaluate:

$$\int_{x=2}^{x=99} \frac{1}{(x+1)^2}\, dx = F(99) - F(2)$$

- Now let $t = 999$. Evaluate:

$$\int_{x=2}^{x=999} \frac{1}{(x+1)^2}\, dx = F(999) - F(2)$$

- Now let $t = 9999$. Evaluate:

$$\int_{x=2}^{x=9999} \frac{1}{(x+1)^2}\, dx = F(9999) - F(2)$$

(d) Based on what you found, what do you *think* is happening when $t \to \infty$ to the definite integral

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)?$$

(e) Ok, we're going to switch our focus to the other improper integral, $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$. again, we'll look at a slightly different integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $-1 < t < 2$, then our function is continuous and bounded on $[t, 2]$, and so we can evaluate this integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$. We can just use the same antiderivative as before!

We're going to evaluate this intergal for different values of t again, but this time we'll use values that are close to -1, but slightly bigger, since we want to be in the interval $[-1, 2]$.

- Let's start with making $t = -\frac{9}{10}$. So we're going to evaluate:

$$\int_{x=-9/10}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{9}{10}\right)$$

- Now let $t = -\frac{99}{100}$. Evaluate:

$$\int_{x=-99/100}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{99}{100}\right)$$

- Now let $t = -\frac{999}{1000}$. Evaluate:

$$\int_{x=-999/1000}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{999}{1000}\right)$$

(f) Based on what you found, what do you *think* is happening when $t \to -1^+$ to the definite integral

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)?$$

We can think about putting this a bit more generally into limit notation, but we'll get to this later.

Ok, let's formalize these limits with some strategies for evaluating improper integrals!

Strategies for Evaluating Improper Integrals

Evaluating Improper Integrals (Infinite Width).

For a function $f(x)$ that is continuous on $[a, \infty)$, we can evaluate the improper integral $\displaystyle\int_{x=a}^{\infty} f(x)\, dx$:

$$\int_{x=a}^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_{x=a}^{x=t} f(x)\, dx.$$

If $f(x)$ is continuous on $(-\infty, b]$, we can evaluate the improper integral $\displaystyle\int_{-\infty}^{x=b} f(x)\,dx$:

$$\int_{-\infty}^{x=b} f(x)\,dx = \lim_{t\to-\infty}\int_{x=t}^{x=b} f(x)\,dx.$$

Finally, if $f(x)$ is continuous on $(-\infty,\infty)$ and m is some real number, then we can evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{x=m} f(x)\,dx + \int_{x=m}^{\infty} f(x)\,dx\\ &= \lim_{t\to-\infty}\int_{x=t}^{x=m} f(x)\,dx + \lim_{t\to\infty}\int_{x=m}^{x=t} f(x)\,dx\end{aligned}$$

Example 7.1.2

Evaluate the improper integral $\displaystyle\int_{x=2}^{\infty}\frac{1}{(x+1)^2}dx$ by evaluating the limit:

$$\lim_{t\to\infty}\int_{x=2}^{x=t}\frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\left(F(t)-F(2)\right).$$

Try to interpret this limit. What does it mean if this limit doesn't exist? What does it mean if the limit does exist? What does the actual number represent?

Evaluating Improper Integrals (Infinite Height).

For a function $f(x)$ that has an unbounded discontinuity (a vertical asymptote) at $x=m$ with $a<m<b$, but is otherwise continuous on $[a,b]$, then we can evaluate the improper integrals:

$$\begin{aligned}\int_{x=m}^{x=b} f(x)\,dx &= \lim_{t\to m^+}\int_{x=t}^{x=b} f(x)\,dx\\ \int_{x=a}^{x=m} f(x)\,dx &= \lim_{t\to m^-}\int_{x=a}^{x=t} f(x)\,dx\\ \int_{x=a}^{x=b} f(x)\,dx &= \int_{x=a}^{x=m} f(x)\,dx + \int_{x=m}^{x=b} f(x)\,dx\\ &= \lim_{t\to m^-}\int_{x=a}^{x=t} f(x)\,dx + \lim_{t\to m^+}\int_{x=t}^{x=b} f(x)\,dx\end{aligned}$$

Example 7.1.3

Evaluate the improper integral $\displaystyle\int_{x=-1}^{x=2}\frac{1}{(x+1)^2}dx$ by evaluating the limit:

$$\lim_{t\to-1^+}\int_{x=t}^{x=2}\frac{1}{(x+1)^2}dx = \lim_{t\to-1^+}\left(F(2)-F(t)\right).$$

Try to interpret this limit. What does it mean if this limit doesn't exist? What does it mean if the limit does exist? What does the actual number represent?

Ok, let's note that we can classify these improper integrals into two categories. We have already classified them based on the *reason* that they're improper. But now we can also classify them based on the outcome of the limit:

1. Improper integrals (of any type) whose limit exists.
2. Improper integrals (of any type) where the limit doesn't exist.

Let's define a term for this, so that our classification isn't so wordy.

Convergence and Divergence of an Improper Integral

Definition 7.1.4 Convergence of an Improper Integral.

We say that an improper integral **converges** if the limit of the appropriate definite integral exists. If the limit does not exist, then we say that the improper integral **diverges**.

All we've done here is added some language: we'll say that an improper integral diverges if the limit doesn't exist. And if the limit exists, we'll say that the improper integral "converges to _____."

Practice Problems

1. Explain what it means for an integral to be improper. What kinds of issues are we looking at?

2. Give an example of an integral that is improper due to an unbounded or infinite interval of integration (infinite width).

3. Give an example of an integral that is improper due to an unbounded integrand (infinite height).

4. What does it mean for an improper integral to "converge?" How does this connect with limits?

5. What does it mean for an improper integral to "diverge?" How does this connect with limits?

6. Why do we need to use limits to evaluate improper integrals?

7. For each of the following improper integrals:

- Explain why the integral is improper. Be specific, and point out the issues in detail.
- Set up the integral using the correct limit notation.
- Antidifferentiate and evaluate the limit.
- Explain whether the integral converges or diverges.

(a) $\displaystyle\int_{x=0}^{\infty} \frac{1}{\sqrt{x+1}}\,dx$

(b) $\displaystyle\int_{x=0}^{\infty} e^{-2x}\,dx$

(c) $\displaystyle\int_{x=-1}^{x=3} \frac{1}{x+1}\,dx$

(d) $\displaystyle\int_{-\infty}^{x=0} \sqrt{e^x}\,dx$

(e) $\displaystyle\int_{x=2}^{x=8} \frac{5}{(x-2)^3}\,dx$

(f) $\displaystyle\int_{x=1}^{x=12} \frac{dx}{\sqrt[5]{12-x}}$

8. One of the big ideas in probability is that for a curve that defines a probability density function, the area under the curve needs to be 1. What value of k makes the function $\dfrac{kx}{(x^2+3)^{5/4}}$ a valid probability distribution on the interval $[0, \infty)$?

9. Let's consider the integral $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$. This is a difficult integral to evaluate!

(a) First, compare $\sqrt{x^2+1}$ to $\sqrt{x^2}$ using an inequality: which one is bigger?

(b) Second, use this inequality to compare the function $\dfrac{\sqrt{x^2+1}}{x^2}$ to $\dfrac{1}{x}$ for $x > 0$: which one is bigger? Again, use your inequality from above to help!

(c) Now compare $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ to $\displaystyle\int_{x=1}^{\infty} \frac{1}{x}\,dx$. Which one is bigger?

(d) Explain how we can use this result to make a conclusion about whether our integral, $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ converges or diverges.

7.2 More on u-Substitution

We're going to do some more thinking about an integration topic that we've technically already introduced in Section 5.6. We're going to do a bit more with it now, and try to build some more flexibility, so definitely review that introduction section if you'd like!

Before moving on, we should work through these few examples, just to make sure we remember what we're up to.

Activity 7.2.1 Recapping u-Substitution.

We're going to consider a few integrals, and work through each of the questions for all integrals.

1. $\displaystyle\int \left(\frac{1}{\sqrt{4-3x}}\right)\,dx$
2. $\displaystyle\int x^2(5+x^3)^7\,dx$
3. $\displaystyle\int \left(\frac{\sin(x)}{\cos(x)}\right)\,dx$
4. $\displaystyle\int \left(xe^{-x^2}\right)\,dx$

(a) For each integral, explain why u-substitution is a good choice. How can you tell, just by looking at the integral, that this strategy will be a reasonable thing to try?

(b) For each integral, explain your choice of u, what that means for how we define du.

(c) For each integral, is your definition of du present in the integrand function? How do you go about making this substitution when the integrand function isn't set up perfectly?

(d) Finish the substitution and integration, and substitute back to the original variable.

Variable Substitution in Integrals

In u-substitution, we focus a lot on one specific kind of structure: composition in our integrand function and/or some function-derivative pairing present. We do this because we're undo-ing the The Chain Rule. Variable substitutions can be much more general in their goal, but this is a good one to focus on because it solves a specific problem that we might run into while integrating.

Strategy for u-Substitution.

Goal: undo the Chain Rule, or antidifferentiate functions with composition.

Process: We'll translate $\int f(x)\;dx$ to $\int g(u)\;du$ by labeling some "inside" function as u, and substituting its derivative, $du = u'\;dx$.

But we can use u-substitution more generally as a kind of grouping mechanism.

Activity 7.2.2 Turn Around Problems.

The two integrals that we're going to look at are "just" some u-substitution problems, but I like to call integrals like these **turn-around** problems. We'll see why!

(a) Consider the integral:

$$\int x\sqrt[3]{x+5}\,dx.$$

First, explain why u-substitution is reasonable here.

(b) Identify du for your chosen u-substitution. When you substitute, you should notice that there are some extra bits in this integrand function that have not been assigned to be translated over to be written in terms of u. Which parts?

(c) We need to think about how to write x in terms of u. Luckily, we already have everything we need! We have defined a link between the x variable and the u variable. We defined it as u being written as some function of x, but can we "turn around" that link to write x in terms of u?

(d) Substitute the integral to be fully written in terms of u.

(e) Before antidifferentiating, compare this integral with the original one. Specifically thinking about how we might multiply, describe the differences between the integrals with regard to composition and re-writing our integrand.

Then, go ahead and use this nicely re-written version to antidifferentiate and substitute back to x.

(f) Apply this same strategy to the following integral:

$$\int \frac{x}{x+5}\,dx.$$

This integral might be a bit trickier to find the composition in order to identify the u-substitution! Give some things a try!

(g) Compare your integral in terms of x with the substituted version, in terms of u. Why was the second one so much easier to think about or re-write?

In both of these examples, we got around not being able to multiply (using the distributive property) or divide (by splitting up our fraction into two with common denominators) by grouping some terms together with our substitution. Once we wrote $x+5$ as u in both of these, we were able to distribute $u^{1/3}$ across the two other terms, and we were able to divide $u-5$ by u through splitting the single fraction into two fractions.

The term **turn-around** problem is a good one because we're *turning around* two things:

1. The substitution itself, by solving for u instead of x.

2. The structure of the integral, by grouping $x+5$ into one term, u and expanding x into two terms, $u-5$. This allowed us to change how the

algebra would work, making it much friendlier!

Example 7.2.1

Find the following indefinite integral:

$$\int \frac{x^2+3x-1}{x-1}\,dx$$

Hint. Try letting $u = x - 1$ so that $du = dx$. Then we can say that $x = u + 1$.

Solution.

$$\begin{aligned}
u = x - 1 &\longrightarrow x = u + 1 \\
du &= dx \\
\int \frac{x^2+3x-1}{x-1}\,dx &= \int \frac{(u+1)^2+3(u+1)-1}{u}\,du \\
&= \int \frac{u^2+5u+3}{u}\,du \\
&= \int u + 5 + \frac{3}{u}\,du \\
&= \frac{u^2}{2} + 5u + 3\ln|u| + C \\
&= \frac{(x-1)^2}{2} + 5(x-1) + 3\ln|x-1| + C
\end{aligned}$$

There are some ways of re-writing this antiderivative family: we could try to group up all of the constant terms by multiplying everything out. Feel free to do this, but it is completely unnecessary.

This specific example is an interesting one, because we actually have a couple of different options with how we approach it. This is true in a lot of cases: there is very rarely only a single approach to an integral that will eventually work out. Sometimes there are approaches or more techniques that are more obvious to some people, and sometimes there are approaches that seem more easy/difficult for some people. But even still, we are often presented with many choices we could make in how we approach our integration.

Moving forward in this chapter, we'll present a whole host of strategies for how we might integrate different types of functions and how we might approach different structures that we see in the integrals we'll look at. We'll try to balance a difficult duality:

- There is rarely no single "right" way to do things! We can't summarize things with strongly worded rules like "if you function looks like this, then you have to do this to antidifferentiate."

- We would like to build some good intuition, and so having some tried-and-true strategies to fall back on will help! We can try to identify some intuitive strategies, even if they're not the only ones that will work.

All of this to simply say: we are going to present a lot of problems with a lot of solutions, and there simply isn't enough space to write out alternative approaches for each one. We will try to re-visit some integrals to think about alternative strategies when we are able to, though!

7.3 Manipulating Integrands

We've looked at how to use a variable substitution to antidifferentiate composite functions. We've already seen, though, that sometimes identifying and actually using a helpful substitution can be difficult to do. In this section, we want to introduce some different strategies for noticing and setting up useful substitutions in some specific instances.

Rewriting the Integrand

We're going to look at a few different examples or strategies that revolve around the same idea: we're going to reveal a reasonable function to antidifferentiate, whether its through finding a substitution or putting our function into some other recognizeable form.

Example 7.3.1

For each of the following integrals, re-write the integrand function using some algebraic manipulation, trigonometric identity, or some other strategy. Then, once the integrand function is in a friendlier form, antidifferentiate.

(a) $\displaystyle\int \tan^2(\theta)\, d\theta$

Hint. Can you think of a trigonometric identity that can help translate the squared tangent function into some other squared trigonometric function that we recognize as the derivative of something?

(b) $\displaystyle\int \left(\frac{x^2-9}{x+3}\right)\, dx$

Hint. Try some factoring! Can you factor and cancel?

(c) $\displaystyle\int \left(\frac{\sqrt{x}-4}{x^2}\right)\, dx$

Hint. Split this fraction into $\displaystyle\int \left(\frac{\sqrt{x}x^2}{-}\frac{4}{x^2}\right)\, dx$. Then, can you write these two terms as power functions?

(d) $\displaystyle\int \sec(x)\, dx$

Hint. This is ~~a hard one~~ an annoying one, and we'll revisit it later with a better strategy, but for now you can notice something nice happen when you multiply the numerator and denominator by $(\sec(x)+\tan(x))$:

$$\int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)\, dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\, dx.$$

This strategy is *not* intuitive, in my opinion: the nice thing to multiply seemingly comes out of nowehere!

Let's look at one more type of example, just to re-iterate what we're thinking about with these re-written functions.

Activity 7.3.1 A Negative Exponent.

Let's think about this integral:

$$\int \frac{1}{1+e^{-x}}\,dx.$$

(a) Is there any composition in this integral? Pick it out, and either explain or show that using this to guide your substitution will not be helpful.

(b) What does e^{-x} mean? What does $\frac{1}{e^{-x}}$ mean?

(c) Re-write the integral, specifically focusing on the negative exponent. You should find that the function looks worse! How can you clean that up?

(d) Why is this new integral set up so much better for the purpose of u-substitution? How could we tell this just by looking at the initial integral?

Example 7.3.2

Re-write the integrand function for $\displaystyle\int \frac{1}{x+x^{-1}}\,dx$, and then integrate using an appropriate substitution.

Hint. Try to re-write this integral by noticing that $x^{-1} = \dfrac{1}{x}$. Then try to make the resulting fraction a bit nicer to look at, since it has a fraction inside of the denominator of another fraction.

Solution.

$$\begin{aligned}
\int \frac{1}{x+x^{-1}}\,dx &= \int \frac{1}{x+\frac{1}{x}}\cdot\frac{x}{x}\,dx\\
&= \int \frac{x}{x^2+1}\,dx\\
u = x^2+1 \quad & du = 2x\,dx\\
\int \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx\\
&= \frac{1}{2}\int \frac{1}{u}\,du\\
&= \frac{1}{2}\ln|u| + C\\
&\frac{1}{2}\ln(x^2+1) + C
\end{aligned}$$

This last example is a good one to help us transition into thinking about a whole class of functions: rational functions! As a reminder, these are just polynomials divided by polynomials.

Antidifferentiating Rational Functions

Strategies for antidifferentiating rational functions are just that: strategies. There isn't one consistent rule to use to antidifferentiate these (like there is with derivatives and the Quotient Rule), but we'll find some common tactics to apply and try to build our intuition for noticing the different kinds of structure we can have in these rational functions. All of these strategies are based around cleverly re-writing our rational functions (using some algebraic manipulations) to reveal some structure. We'll try to notice the structure, so that we know what we're trying to reveal.

Activity 7.3.2 Integrating a Rational Function Three Ways.

We're going to think about the integral:

$$\int \left(\frac{x^2+3x-1}{x-1}\right)\, dx.$$

Let's find 3 different ways of integrating this. This is kind of misleading, since we're actually going to look at 2, since we've already used u-substitution to integrate this in Example 7.2.1.

(a) Let's just notice some things about this rational function.

- Are there any vertical asymptotes? How do you know where to find them?
- Are there any horizontal asymptotes? How do you know that there *aren't*?
- When you zoom really far out on the graph of this function, it looks like a different kind of function. What kind of function? Why is that?

(b) Now we're going to re-write the function itself: $\dfrac{x^2+3x-1}{x-1}$ means we're dividing (x^2+3x-1) by $(x-1)$. So let's do the division!

$$x-1\ \overline{\big)\ x^2 \quad +3x \quad -1}$$

(c) Re-write your integral using this new version of the function. Notice that we haven't done any calculus or antidifferentiating yet. Explain why this new version of this integrand function is easier to antidifferentiate. What do you get?

(d) Let's approach this integral differently. We said earlier that this function is really an "almost" linear function in disguise: when we divide the quadratic numerator by a linear denominator, we expect a linear function to be left over. In the long division, we saw this happen! We ended with a linear function and some remainder.

Let's try to uncover this linear function. If we're looking to find what linear functions multiply together to get (x^2+3x-1), then we can try factoring!

$$\frac{x^2+3x-1}{x-1} = \frac{(\qquad\qquad)(\qquad\qquad)}{x-1}$$

In order for this factoring to be useful, we want to be able to "cancel" out the $x-1$ factor in the denominator. We're really only interested in what linear factor will multiply by $(x-1)$ to get (x^2+3x-1).

$$\frac{x^2+3x-1}{x-1} = \frac{(x-1)(\qquad\qquad)}{x-1}$$

First, explain why there is no linear function factor that accomplishes this.

(e) What if were able to "almost" factor this?

If there *was* a linear factor that multiplied by $(x-1)$ to get (x^2+3x-1), then the linear portions would multiply together to get x^2. What does this mean about the first linear term of our factor?

(f) What does the constant term of our missing factor need to be? We are hoping that whatever it is can multiply by x (from $x-1$) and combine with the $-x$ (from the constant -1 multiplied by x in our missing factor) to match the $3x$ in x^2+3x-1.

What is it?

(g) Note that we have *not* factored (x^2+3x-1)! We *almost* did: we found two factors:

$$(x-1)(\qquad\qquad) = x^2+3x+\qquad.$$

How far off is the actual polynomial that we are working with, x^2+3x-1?

Write x^2+3x-1 as your two factors plus or minus some remainder.

(h) You should get the same thing that we got from using long division! Great! The rest of the integral will work the same.

Before we end, though, compare this antiderivative to the one we got in Example 7.2.1. It's different. Why? Is this a problem?

This gives us a good approach for whenever division will help us rewrite our rational function as some polynomial and a remainder.

Let's look at two more rational functions: these ones won't be good candidates to use long division, but we'll try to build some intuition for why we will need to re-write one of them to get a substitution that works.

Activity 7.3.3 Comparing Two Very Similar Integrals.

We're going to compare these two integrals:

$$\int \frac{x+2}{x^2+4x+5}\,dx \qquad\qquad \int \frac{2}{x^2+4x+5}\,dx$$

(a) Describe why $u = x^2+4x+5$ is such a useful choice for the first integral, but not for the second. How do the differences in these two integrals influence this substitution, even though the denominators are the same?

(b) Why would it be useful to have a *linear* substitution rule (instead of the *quadratic* one that we picked) for the second integral? Why would that match the structure of the numerator better?

Go ahead and integrate the first integral.

(c) We're going to write the denominator, $x^2 + 4x + 5$ in a different way, in order to get a linear function composed into something familiar.

Complete the square for this polynomial: that is, find some linear factor $(x + k)$ and a real number b such that $(x + h)^2 + b = x^2 + 4x + 3$. This should feel familiar, since we have already tried to force polynomials to factor cleverly in Activity 7.3.2.

(d) There is an intuitive substitution to pick, since we now have more obvious composition. Pick it. What kind of integral do we end up with and how do we antidifferentiate? Complete this problem!

Going forward, when you see a quadratic denominator in a rational function, what are some things you can think about and strategies you can use, based on what the rest of the function looks like? We want to summarize this a bit!

Integrating Rational Functions.

If $f(x) = \dfrac{p_n(x)}{p_m(x)}$ where $p_n(x)$ is a degree n polynomial and $p_m(x)$ is a degree m polynomial, then we can think about how we might integrate $\displaystyle\int f(x)\,dx$ based on degrees.

- If $n \geq m$ (the degree in the numerator is at least the degree in the denominator), then we can use long division to write $f(x)$ as some polynomial with degree $n - m$ and some remainder.
- If $n = m - 1$ (the degree of the numerator is one less than the degree of the denominator), then we can try a u-substitution where $u = p_m(x)$, since the derivative of $p_m(x)$ is a polynomial of degree n. If this substitution works, we can antidifferentiate to get some sort of logarithm.

  This is not guaranteed to work, but for now (without other strategies), this is something we can think about.
- If we can reduce $f(x)$ (through some transformation or substitution) to a rational function that is a constant term divided by quadratic function (or if it already is), then we can complete the square in the denominator to get to a form that can be antidifferentiated to an inverse tangent function.

In the last point, we are referencing the strategy we found in Activity 7.3.3. We have a bit more of a general version of this strategy.

Theorem 7.3.3 Generalized Inverse Tangent Forms.

$$\int \frac{1}{u^2 + a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Proof.

This is really just based on a clever substitution. Once we see this specific constant over a sum of squares, we can factor out a convenient coefficient to force the denominator to look like a sum of something squared and 1.

$$\begin{aligned}\int \frac{1}{u^2+a^2}\,du &= \frac{1}{a^2}\int \frac{1}{\frac{u^2}{a^2}+1}\,du \\ &= \frac{1}{a^2}\int \frac{1}{\left(\frac{u}{a}\right)^2+1}\,du\end{aligned}$$

Now we can let $w = \frac{u}{a}$ and $dw = \frac{1}{a}\,du$.

$$\begin{aligned}\frac{1}{a^2}\int \frac{1}{\left(\frac{u}{a}\right)^2+1}\,du &= \frac{1}{a}\int \frac{1}{\left(\frac{u}{a}\right)^2+1}\left(\frac{1}{a}\right)\,du \\ &= \frac{1}{a}\int \frac{1}{w^2+1}\,dw \\ &= \frac{1}{a}\tan^{-1}(w)+C \\ &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C\end{aligned}$$

This strategy can also be used for other inverse trigonometric derivatives. But we will use the inverse tangent form most of all, and thus we want to outline it fully.

We have much more to talk about with integration. From here, we can move on to more systematic strategies — ones that have some goals based on familiar things like operations that we might notice or specific variable substitutions that can be useful.

Practice Problems

1. Use polynomial division or some clever factoring to re-write and find the following indefinite integrals or evaluate the following definite integrals.

(a) $\displaystyle\int \left(\frac{x+4}{x-3}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x^2+4}{x-4}\right)\,dx$

(c) $\displaystyle\int \left(\frac{t^2+t+6}{t^2+1}\right)\,dt$

(d) $\displaystyle\int_{x=2}^{x=4} \left(\frac{x^3+1}{x-1}\right)\,dx$

(e) $\displaystyle\int_{x=0}^{x=1} \left(\frac{x^4+1}{x^2+1}\right)\,dx$

2. Complete the square in order to find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x^2-2x+10}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x}{x^2+4x+8}\right)\,dx$

(c) $\displaystyle\int \left(\frac{2x}{x^4+6x^2+10}\right)\,dx$

3. Find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x^{-1}+1}\right)\, dx$

(b) $\displaystyle\int \left(\frac{\sin(\theta)+\tan(\theta)}{\cos^2(\theta)}\right)\, d\theta$

(c) $\displaystyle\int \left(\frac{1-x}{1-\sqrt{x}}\right)\, dx$

(d) $\displaystyle\int \left(\frac{1}{1-\sin^2(\theta)}\right)\, d\theta$

(e) $\displaystyle\int \left(\frac{x^{2/3}-x^3}{x^{1/4}}\right)\, dx$

(f) $\displaystyle\int \left(\frac{4+x}{\sqrt{1-x^2}}\right)\, dx$

7.4 Integration By Parts

We've seen now that Introduction to u-Substitution is a useful technique for undo-ing The Chain Rule. We set up the variable substitution with the specific goal of going backwards through the Chain Rule and antidifferentiating some composition of functions.

A reasonable next step is to ask: What other derivative rules can we "undo?" What other operations between functions should we think about? This brings us to Integration by Parts, the integration technique specifically for undo-ing Product Rule.

Discovering the Integration by Parts Formula

Activity 7.4.1 Discovering the Integration by Parts Formula.

The product rule for derivatives says that:

$$\frac{d}{dx}\left(u(x)\cdot v(x)\right) = \underline{\qquad\qquad} + \underline{\qquad\qquad}.$$

We know that we intend to "undo" the product rule, so let's try to re-frame the product rule from a rule about derivatives to a rule about antiderivatives.

(a) Antidifferentiate the product rule by antidifferentiating each side of the equation.

$$\begin{aligned}\int\left(\frac{d}{dx}\left(u\cdot v\right)\right)\,dx &= \int \underline{\qquad\qquad} + \underline{\qquad\qquad}\,dx\\ \underline{\qquad\qquad} &= \int \underline{\qquad\qquad}\,dx + \int \underline{\qquad\qquad}\,dx\end{aligned}$$

(b) On the right side, we have two integrals. Since each of them has a product of functions (one function and a derivative of another), we can isolate one of them in this equation and create a formula for how to antidifferentiate a product of functions! Solve for $\int uv'\,dx$.

(c) Look back at this formula for $\int uv'\,dx$. Explain how this is really the product rule for derivatives (without just undo-ing all of the steps we have just done).

We typically use the substitutions $du = u'\,dx$ and $dv = v'\,dx$ to re-write the integrals.

Integration by Parts.

Suppose $u(x)$ and $v(x)$ are both differentiable functions. Then:

$$\int u\,dv = uv - \int v\,du.$$

When we select the parts for our integral, we are selecting a function to be labeled u and a function to be labeled as dv. We begin with one of the pieces of the product rule, a function multiplied by some other function's derivative. It is important to recognize that we do different things to these functions: for one of them, u, we need to find the derivative, du. For the other, dv, we need to find an antiderivative, b. Because of these differences, it is important to build some good intuition for how to select the parts.

Intuition for Selecting the Parts

Activity 7.4.2 Picking the Parts for Integration by Parts.

Let's consider the integral:

$$\int x \sin(x)\, dx.$$

We'll investigate how to set up the integration by parts formula with the different choices for the parts.

(a) We'll start with selecting $u = x$ and $dv = \sin(x)\, dx$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= x & v &= \underline{\qquad} \\ du &= \underline{\qquad} & dv &= \sin(x)\, dx \end{aligned}$$

(b) Now set up the integration by parts formula using your labeled pieces. Notice that the integration by parts formula gives us another integral. Don't worry about antidifferentiating this yet, let's just set the pieces up.

(c) Let's swap the pieces and try the setup with $u = \sin(x)$ and $dv = x\, dx$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= \sin(x) & v &= \underline{\qquad} \\ du &= \underline{\qquad} & dv &= x\, dx \end{aligned}$$

(d) Now set up the integration by parts formula using this setup.

(e) Compare the two results we have. Which setup do you think will be easier to move forward with? Why?

(f) Finalize your work with the setup you have chosen to find $\int x \sin(x)\, dx$.

What made things so much better when we chose $u = x$ compared to $dv = x\, dx$? We know that the new integral from our integration by parts formula will be built from the new pieces, the derivative we find from u and the antiderivative we pick from dv. So when we differentiate $u = x$, we get a constant, compared to antidifferentiating $dv = x\, dx$ and getting another power function, but with a larger exponent. We know this will be combined with a $cos(x)$ function no matter what (since the derivative and antiderivatives of $sin(x)$ will only differ in their sign). So picking the version that gets that second integral to be built from a trig function and a constant is going to be much nicer than a trig function and a power function. It was nice to pick x to be the piece that we found the derivative of!

Let's practice this comparison with another example in order to build our intuition for picking the parts in our integration by parts formula.

Activity 7.4.3 Picking the Parts for Integration by Parts.

This time we'll look at a very similar integral:

$$\int x\ln(x)\,dx.$$

Again, we'll set this up two different ways and compare them.

(a) We'll start with selecting $u = x$ and $dv = \ln(x)\,dx$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= x & v &= \_\_\_\_ \\ du &= \_\_\_\_ & dv &= \ln(x)\,dx \end{aligned}$$

(b) Ok, so here we *have to* swap the pieces and try the setup with $u = \ln(x)$ and $dv = x\,dx$, since we only know how to differentiate $\ln(x)$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= \ln(x) & v &= \_\_\_\_ \\ du &= \_\_\_\_ & dv &= x\,dx \end{aligned}$$

(c) Now set up the integration by parts formula using this setup.

(d) Why was it fine for us to antidifferentiate x in this example, but not in Activity 7.4.2?

(e) Finish this work to find $\int x\ln(x)\,dx$.

So here, we didn't actually get much choice. We couldn't pick $u = x$ in order to differentiate it (and get a constant to multiply into our second integral) since we don't know how to antidifferentiate $\ln(x)$ (yet: once we know how, it might be fun to come back to this problem and try it again with the parts flipped). But we can also notice that it ended up being fine to antidifferentiate x: the increased power from our power rule didn't really matter much when we combined it with the derivative of the logarithm, since the derivative of the log is *also a power function*! So we were able to combine those easily and actually integrate that second integral.

It is common for students to want to place functions into sort of hierarchy or classification guidelines for choosing the parts. Some students have found that the acronym LIPET (logs, inverse trig, power functions, exponentials, and trig functions) can be a useful tool for selecting the parts. When you have two different types of functions, it might help to select u to be whichever function shows up first in that list.

Example 7.4.1

Integrate the following:

(a) $\displaystyle\int x^2 e^x\,dx$

Hint. It doesn't matter whether we differentiate or antidifferentiate e^x, since we'll get the same thing. Let's pick $u = x^2$ so that we can differentiate it.

Solution.

$$\begin{array}{ll} u = x^2 & v = e^x \\ du = 2x\ dx & dv = e^x\ dx \end{array}$$

$$\int x^2 e^x\ dx = x^2 e^x - \int 2xe^x\ dx$$

We need to do more integration by parts!

$$\begin{array}{ll} u = 2x & v = e^x \\ du = 2\ dx & dv = e^x\ dx \end{array}$$

$$\begin{aligned} \int x^2 e^x\ dx &= x^2 e^x - \left(2xe^x - \int 2e^x\ dx\right) \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

(b) $\int 2x \tan^{-1}(x)\ dx$

Hint. We don't know how to antidifferentiate $\tan^{-1}(x)$, but we do know how to differentiate it!

Solution.

$$\begin{array}{ll} u = \tan^{-1}(x) & v = x^2 \\ du = \frac{1}{x^2+1}\ dx & dv = 2x\ dx \end{array}$$

$$\begin{aligned} \int 2x \tan^{-1}(x)\ dx &= x^2 \tan^{-1}(x) - \int \frac{x^2}{x^2+1}\ dx \\ &= x^2 \tan^{-1}(x) - \int \frac{(x^2+1)-1}{x^2+1}\ dx \quad \text{Alternatively, use long division.} \\ &= x^2 \tan^{-1}(x) - \int 1 - \frac{1}{x^2+1}\ dx \\ &= x^2 \tan^{-1}(x) - x + \tan^{-1}(x) + C \end{aligned}$$

Some Flexible Choices for Parts

We're going to look at a couple of examples where we can showcase some of the flexibility we have with our choices of parts. First, we'll revisit Example 7.4.1.In this example, when we got to that second integral, we noticed that for the fraction $\frac{x^2}{x^2+1}$, we could either do some long division (since the degrees in the numerator and denominator are the same) or do some clever re-writing of the numerator. Either way, we know that this fraction is *almost* 1...It's really $1\pm$ some bit (in this case, the extra bit was a fraction $\frac{1}{x^2+1}$).

What if we chose our parts differently? Not the u and dv parts, though, since we still haven't figured out how to antidifferentiate $\tan^{-1}(x)$. But we get one more choice!

Once we choose u, we don't really get a separate choice for du: it's simply the derivative of u with regard to x multiplied by the differential dx. But consider our choice of dv, and the subsequent process of finding v. Yes, there's only one possible answer, but in a much more real sense, there isn't just one possible answer. There are an infinite number of them! We know, due to the Mean Value Theorem and then later due to Theorem 4.1.7, that there are an infinite number of antiderivatives, all differing by at most a constant term. So let's pick a more appropriate antiderivative!

Example 7.4.2

Integrate $\displaystyle\int 2x\tan^{-1}(x)\,dx$, this time making a more intentional choice for v.

Hint. Note that if we pick $v = x^2 + 1$, then the second integral will be just delightful.

Solution.

$$\begin{array}{ll} u = \tan^{-1}(x) & v = x^2+1 \\ du = \frac{1}{x^2+1}\,dx & dv = 2x\,dx \end{array}$$

$$\begin{aligned} \int 2x\tan^{-1}(x)\,dx &= (x^2+1)\tan^{-1}(x) - \int \frac{x^2+1}{x^2+1}\,dx \\ &= x^2\tan^{-1}(x) + \tan^{-1}(x) - \int dx \\ &= x^2\tan^{-1}(x) + \tan^{-1}(x) - x + C \end{aligned}$$

So we get the same thing, but didn't have to think through the long division or the forced factoring. But the trade off here is that we almost *have to* see this coming to notice it. This flexibility doesn't always come into play for us. But we can look at a different kind of flexibility.

We've looked at integrals with both $\ln(x)$ and $\tan^{-1}(x)$. For these, and for other inverse functions specifically, we pick them to be the u part in our integration by parts problems because we don't know how do antidifferentiate them.

So let's look at $\displaystyle\int \ln(x)\,dx$, and we'll solve this integral by, specifically, differentiating $\ln(x)$ instead of antidifferentiating it.

Example 7.4.3 Antidifferentiating the Log Function.

Integrate $\displaystyle\int \ln(x)\,dx$.

Hint. Pick $u = \ln(x)$, since we can differentiate it. What does that leave for dv?

Solution 1.

$$\begin{array}{ll} u = \ln(x) & v = x \\ du = \frac{1}{x}\,dx & dv = \ dx \end{array}$$

$$\begin{aligned} \int \ln(x)\,dx &= x\ln(x) - \int \frac{x}{x}\,dx \\ &= x\ln(x) - x + C \end{aligned}$$

Solution 2. An alternate approach is to use a substitution first. We're going to be using a lot of different variable names here, so let's use a t-substitution. Let $t = \ln(x)$ so that $dt = \frac{1}{x}\,dx$. In order to induce this derivative of the log, let's multiply by $\frac{x}{x}$ inside the integral:

$$\int \ln(x)\,dx = \int \frac{x}{x}\ln(x)\,dx$$

$$= \int \underbrace{x \ln(x)}_{t} \underbrace{\left(\frac{1}{x}\right) dx}_{dt}$$

$$t = \ln(x) \longrightarrow x = e^t$$

$$\int x \ln(x) \left(\frac{1}{x}\right) dx = \int te^t \, dt$$

This integral can be done using the standard integration by parts!

$$u = tv = e^t$$

$$du = dtdv = e^t \, dt$$

$$\int te^t \, dt = uv - \int v \, du$$

$$= te^t - \int e^t \, dt$$

$$= te^t - e^t + C$$

Now we can substitute back to x:

$$te^t - e^t + C = x \ln(x) - x + C.$$

We can use this same strategy to find antiderivatives of $\tan^{-1}(x)$, $\sin^{-1}(x)$, and eventually $\sec^{-1}(x)$.

For $\displaystyle\int \sec^{-1}(x) \, dx$, we'll need to use this same tactic of setting $u = \sec^{-1}(x)$ and $dv = \, dx$, but then later on we'll need to use a technique called Trigonometric Substitution to finish the problem.

Now that we know the antiderivative family for $\ln(x)$, we can revisit the problem in Activity 7.4.3, $\displaystyle\int x \ln(x) \, dx$, and try to work through the integration by parts when $u = x$ and $dv = \ln(x) \, dx$.

Example 7.4.4

Integrate $\displaystyle\int x \ln(x) \, dx$.

Solution.

$$\begin{aligned} u &= x & v &= x\ln(x) - x \\ du &= \, dx & dv &= \ln(x) \, dx \end{aligned}$$

$$\int x \ln(x) \, dx = x(x \ln(x) - x) - \int x \ln(x) - x \, dx$$

$$= x^2 \ln(x) - x^2 - \int x \ln(x) \, dx + \int x \, dx$$

$$= x^2 \ln(x) - x^2 + \frac{x^2}{2} - \int x \ln(x) \, dx$$

Note that this last integral is really recognizable: it's the one we started with! Let's "solve" this equation for that integral by adding it to both sides of our equation.

$$\begin{aligned} 2\int x\ln(x)\,dx &= x^2\ln(x) - \frac{x^2}{2} \\ \int x\ln(x)\,dx &= \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C \end{aligned}$$

Solving for the Integral

In this last example, we ended up seeing the original integral repeated when we did integration by parts. This is a useful technique, especially when we deal with functions that have a kind of "repeating" structure to their derivatives or antiderivatives. We'll look at a couple of classic integrals where we see this kind of technique employed. Let's have you explore this idea.

Activity 7.4.4 Squared Trig Functions.

Let's look at two integrals. We'll talk about both at the same time, since they're similar.

$$\int \sin^2(x)\,dx \qquad\qquad \int \cos^2(x)\,dx$$

(a) What does it mean to "square" a trig function? Write these integrals in a different way, where the meaning of the "squared" exponent is more clear. What do you notice about the structure of these integrals, the operation in the integrand function? What does this mean about our choice of integration technique?

(b) If you were to use integration by parts on these integrals, does your choice of u and dv even matter here? Why not?

(c) Apply the integration by parts formula to each. What do you notice?

(d) Instead of applying another round of integration by parts to the resulting integral, use the Pythagorean identities to re-write these integrals:

$$\begin{aligned} \sin^2(x) &= 1 - \cos^2(x) \\ \cos^2(x) &= 1 - \sin^2(x) \end{aligned}$$

(e) You should notice that in your equation for the integration of $\sin^2(x)\,dx$, you have another copy of $\int \sin^2(x)\,dx$. Similarly, in your equation for the integration of $\cos^2(x)\,dx$, you have another copy of $\int \cos^2(x)\,dx$.

Replace these integrals with a variable, like I (for "integral"). Can you "solve" for this variable (integral)?

This "solving for the integral" approach works well, but works best when we can see it coming. Notice that it happened here due to the repeating structure of the derivatives of the sine and cosine functions, as well as the

Pythagorean identities. We can see some more examples of this in play with similar functions!

Example 7.4.5

For each of the following integrals, use integration by parts to solve.

(a) $\displaystyle\int \sin(x)\cos(x)\,dx$

Hint. This one is pretty straight forward, since it doesn't really matter what we select as our parts. Notice, though, that this isn't the only way we can approach this! We can use u-substitution, or even re-write this using a trigonometric identity.

Solution.

$$\begin{array}{ll} u = \sin(x) & v = \sin(x) \\ du = \cos(x)\,dx & dv = \cos(x)\,dx \end{array}$$

$$\begin{aligned} \int \sin(x)\cos(x)\,dx &= \sin^2(x) - \int \sin(x)\cos(x)\,dx \\ 2\int \sin(x)\cos(x)\,dx &= \sin^2(x) \\ \int \sin(x)\cos(x)\,dx &= \frac{\sin^2(x)}{2} + C \end{aligned}$$

(b) $\displaystyle\int e^x\cos(x)\,dx$

Solution.

$$\begin{array}{ll} u = e^x & v = \sin(x) \\ du = e^x\,dx & dv = \cos(x)\,dx \end{array}$$

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx$$

$$\begin{array}{ll} u = e^x & v = -\cos(x) \\ du = e^x\,dx & dv = \sin(x)\,dx \end{array}$$

$$\begin{aligned} \int e^x\cos(x)\,dx &= e^x\sin(x) - \int e^x\sin(x)\,dx \\ \int e^x\cos(x)\,dx &= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx \\ 2\int e^x\cos(x)\,dx &= e^x\sin(x) + e^x\cos(x) \\ \int e^x\cos(x)\,dx &= \frac{e^x\sin(x) + e^x\cos(x)}{2} + C \end{aligned}$$

Notice that we can come up with a bunch of different examples that are similar to Example 7.4.5. If we put trigonometric functions inside our integral, we'll have some options with how we approach them! We can use u-substitution, since the derivatives of trigonometric functions are other trigonometric functions. In Example 7.4.5, for instance, we could write $u = \sin(x)$ and $du = \cos(x)\,dx$, or even chose $u = \cos(x)$ and $du = -\sin(x)\,dx$.

The real issues will come when our integrand is not just a product of two trigonometric functions, but when they are products of trigonometric functions raised to exponents. We'll have some combinations of these products (which maybe makes us think about integration by parts) and composition (which points towards u-substitution). In the next section, we'll develop some strategies to deal with these kinds of integrals.

Practice Problems

1. Explain how we build the Integration by Parts formula, as well as what the purpose of this integration strategy is.

2. How do you choose options for u and dv? What are some good strategies to think about?

3. Let's say that you make a choice for u and dv and begin working through the Integration by Parts strategy. How can you tell if you've made a poor choice for your parts? Can you *always* tell?

4. Integrate the following.

(a) $\displaystyle\int 3x\sin(x)\,dx$

(b) $\displaystyle\int 5xe^x\,dx$

(c) $\displaystyle\int x^2e^{-x}\,dx$

(d) $\displaystyle\int x^2\ln(x)\,dx$

(e) $\displaystyle\int x^2\cos(x)\,dx$

(f) $\displaystyle\int x^3e^{-x}\,dx$

(g) $\displaystyle\int x\sin(x)\cos(x)\,dx$

(h) $\displaystyle\int e^x\sin(x)$

(i) $\displaystyle\int \sin^{-1}(x)\,dx$

(j) $\displaystyle\int \tan^{-1}(x)\,dx$

5. Evaluate the following definite integrals.

(a) $\displaystyle\int_{x=1}^{x=e} x\ln(x)\,dx$

(b) $\displaystyle\int_{x=0}^{x=\pi/4} x\cos(2x)\,dx$

(c) $\displaystyle\int_{x=0}^{x=\ln(5)} xe^x\,dx$

6. In this problem, we'll consider the integral $\displaystyle\int \sin^2(x)\,dx$. We'll integrate this in two different ways!

(a) We know that:

$$\int \sin^2(x)\, dx = \int \sin(x)\sin(x)\, dx.$$

Use the Integration by Parts strategy, and especially note that you can solve for the integral (Solving for the Integral).

(b) We can use a trigonometric identity to re-write the integral:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}.$$

So we have:

$$\int \sin^2(x)\, dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\, dx.$$

Use u-substitution.

(c) Were your answers the same or different? Should they be the same? Why or why not? Are they connected somehow?

7. For these next problems, we'll use $x = u^2$ and $dx = 2u\, du$ to substitute into the integral as written. Then use Integration by Parts.

(a) $\int \sin(\sqrt{x})\, dx$

(b) $\int e^{\sqrt{x}}\, dx$

7.5 Integrating Powers of Trigonometric Functions

Let's remind ourselves of two example problems that we've done in the past.

In Example 5.6.3, we performed a u-substitution, but needed to work to re-write our whole integrand in terms of u. Specifically, we found that in the numerator, there was an x^3, but $du = 2x\ dx$. We were substituting out a linear function of x in the numerator, but the actual function was cubic. This wasn't a problem: we re-wrote $x^3 = x^2 \cdot x$, and noticed that the extra x^2 was able to be substituted, since we could re-write out subsitution rule: we noted that $u = x^2 + 1$ is equivalent to $x^2 = u - 1$. This meant that even though we had an extra factor of x^2 "in" the part that we were using for substituting in the differential du, we were still able to translate the whole function to be written in terms of u.

Then, more recently, in Example 7.4.5, we noted that we could use a mix of methods to integrate this:

$$\int \sin(x)\cos(x)\ dx.$$

One on hand, we can look at the structure of the integrand and notice that we have a product of two functions! Integration by parts was a fine strategy to employ, and that's what we did in the example. On the other hand, we noticed that since we have this function-derivative pairing, a u-substitution was also appropriate.

In this section, we'll explore more combinations of trigonometric functions and build a strategy for antidifferentiating them that includes some ideas from both of these previous examples.

Building a Strategy for Powers of Sines and Cosines

Activity 7.5.1 Compare and Contrast.

Let's do a quick comparison of two integrals, keeping the above examples in mind. Consider these two integrals:

$$\int \sin^4(x)\cos(x)\ dx \qquad\qquad \int \sin^4(x)\cos^3(x)\ dx$$

(a) Consider the first integral, $\int \sin^4(x)\cos(x)\ dx$. Think about and set up a good technique for antidifferentiating. Without actually solving the integral, explain why this technique will work.

(b) Now consider the second integral, $\int \sin^4(x)\cos^3(x)\ dx$. Does the same integration strategy work here? What happens when you apply the same thing?

(c) We know that $\sin(x)$ and $\cos(x)$ are related to each other through derivatives (each is the derivative of the other, up to a negative). Is there some other connection that we have between these functions? We might especially notice that we have a $\cos^2(x)$ left over in our integral. Can we write this in terms of $\sin(x)$, so that we can write it in terms of u?

(d) Why would this strategy not have worked if we were looking at the integrals $\int \sin^4(x)\cos^2(x)\,dx$ or $\displaystyle\int \sin^4(x)\cos^4(x)\,dx$? What, specifically, did we need in order to use this combination of substitution and trigonometric identity to solve the integral?

Integrating Powers of Sine and Cosine.

For integrals in the form $\displaystyle\int \sin^p(x)\cos^q(x)\,dx$ where p and q are real number exponents:

- If q, the exponent on $\cos(x)$ is odd, we should use $u = \sin(x)$ and $du = \cos(x)\,dx$. Then we can apply the Pythagorean Identity $\cos^2(x) = 1 - \sin^2(x)$.
- If p, the exponent on $\sin(x)$ is odd, we should use $u = \cos(x)$ and $du = -\sin(x)\,dx$. Then we can apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$.
- If both p and q are even, we can either use Integration by Parts or use the following power-reducing trigonometric identities:

$$\begin{aligned}\sin^2(x) &= \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{\cos(2x)}{2}\\ \cos^2(x) &= \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{\cos(2x)}{2}\end{aligned}$$

A strange note, here, is that we typically pick our u-substitution based on looking to see a suitable candidate for u: we look for functions that are composed "inside" of other functions or we look for a function whose derivative is in the integral (the "function-derivative pair" that we talk about in Section 5.6). Here, though, we're selecting our substitution based on du: we're looking to see which function we can set aside one copy of for the differential, and then have an even power left over so that we can apply the Pythagorean Identity to translate the rest.

Example 7.5.1

For each of the following, identify an appropriate substitution, make a note of which trigonometric identity you'll use, and then integrate.

(a) $\displaystyle\int \sin^5(x)\cos^6(x)\,dx$

Hint. Notice that the exponent on $\sin(x)$ is odd: if you let $u = \cos(x)$, you'll end up with $\sin^4(x)$ left over in your integral, and you can write it as $(\underbrace{1-\cos^2(x)}_{\sin^2(x)})^2$.

(b) $\displaystyle\int \sin^{4/3}(x)\cos^5(x)\,dx$

Hint. This one seems scary at first, because of the fraction exponent. Notice, though, that you have *no hope* of converting fractional exponents of sine functions into cosine functions easily.

So pick $u = \sin(x)$ and try to convert any remaining cosines using $\cos^2(x) = 1 - \sin^2(x)$.

(c) $\displaystyle\int \sin^3(x)\cos^9(x)\,dx$

Hint. You get a choice here! Both exponents are odd, so picking either function as u will leave you with an even exponent on the other function to use the Pythagorean Identity on. Is there a choice of u that will be easier than the other choice?

Building a Strategy for Powers of Secants and Tangents

Activity 7.5.2 Compare and Contrast (Again).

We're going to do another Compare and Contrast, but this time we're only going to consider one integral:

$$\int \sec^4(x)\tan^3(x)\,dx.$$

We're going to employ another strategy, similar to the one for Integrating Powers of Sine and Cosine.

(a) Before you start thinking about this integral, let's build the relevant version of the Pythagorean Identity that we'll use. Our standard version of this is:

$$\sin^2(x)+\cos^2(x)=1.$$

Since we want a version that connects $\tan(x)$, which is also written as $\frac{\sin(x)}{\cos(x)}$, with $\sec(x)$, or $\frac{1}{\cos(x)}$, let's divide everything in the Pythagorean Identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{}+\frac{\cos^2(x)}{}=\frac{1}{}$$

$$\underline{}+\underline{}=\underline{}$$

(b) Now start with the integral. We're going to use two different processes here, two different u-substitutions. First, set $u=\tan(x)$. Complete the substitution and solve the integral.

(c) Now try the integral again, this time using $u=\sec(x)$ as your substitution.

(d) For each of these integrals, why were the exponents set up *just right* for u-substitution each time? How does the structure of the derivatives of each function play into this?

(e) Which substitution would be best for the integral $\int \sec^4(x)\tan^4(x)\,dx$. Why?

(f) Which substitution would be best for the integral $\int \sec^3(x)\tan^3(x)\,dx$. Why?

Integrating Powers of Secant and Tangent.

For integrals in the form $\int \sec^p(x)\tan^q(x)\,dx$ where p and q are real number exponents:

- If q, the exponent on $\tan(x)$, is odd, we can use $u=\sec(x)$ and $du=\sec(x)\tan(x)\,dx$. Then we can apply the Pythagorean Identity $\tan^2(x)=\sec^2(x)-1$.
- If p, the exponent on $\sec(x)$, is even, we can use $u=\tan(x)$ and $du=\sec^2(x)\,dx$. Then we can apply the Pythagorean Identity $\sec^2(x)=\tan^2(x)+1$.

- If p is odd and q is even, we can use Integration by Parts.

Example 7.5.2

For each of the following, identify an appropriate substitution, make a note of which trigonometric identity you'll use, and then integrate.

(a) $\displaystyle\int \tan^3(x)\sec^8(x)\ dx$

Hint. You have some choices here! If you use $u = \sec(x)$, then you'll end up with a remaining $tan^2(x)$ to convert using a Pythagorean identity. Alternatively, if you use $u = \tan(x)$, you'll end up needing to convert the remaining $\sec^6(x) = (\sec^2(x))^3$.

(b) $\displaystyle\int \tan^{-5/2}(x)\sec^6(x)\ dx$

Hint. Another fraction exponent that is pushing us towards using $u = \tan(x)$. There will still be a $\sec^4(x)$ to convert after substituting in du.

(c) $\displaystyle\int \tan^5(x)\sec^3(x)\ dx$

Hint. The odd exponent on tangent is fine, since we can use $u = \sec(x)$ to leave us with a $\tan^4(x)$ to convert.

Practice Problems

1. For an integral $\displaystyle\int \sin^a(x)\cos^b(x)\ dx$, how do you know whether to use $u = \sin(x)$ or $u = \cos(x)$ as the substitution?

2. For an integral $\displaystyle\int \tan^a(x)\sec^b(x)\ dx$, how do you know whether to use $u = \tan(x)$ or $u = \sec(x)$ as the substitution?

3. Integrate the following.

(a) $\displaystyle\int \sin^3(x)\cos^2(x)\ dx$

(b) $\displaystyle\int \sin^2(x)\cos^3(x)\ dx$

(c) $\displaystyle\int \sin^3(x)\cos^3(x)\ dx$

(d) $\displaystyle\int \tan^4(x)\sec^4(x)\ dx$

(e) $\displaystyle\int \tan^3(x)\sec^3(x)\ dx$

(f) $\displaystyle\int \tan^3(x)\sec^4(x)\ dx$

4. Integrate the following.

(a) $\displaystyle\int \sin^{3/4}(x)\cos^5(x)\, dx$

(b) $\displaystyle\int \tan^5(x)\sec^{-1/2}(x)\, dx$

(c) $\displaystyle\int \sin^{3/4}(x)\cos^5(x)\, dx$

(d) $\displaystyle\int \tan^5(x)\sec^{-1/2}(x)\, dx$

5. Consider the integral $\displaystyle\int \sin^2(x)\, dx$.

(a) Use the trigonometric identity:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

(c) Which of these techniques do you think was easier to implement and use? Why is that?

6. Consider the integral $\displaystyle\int \cos^4(x)\, dx$.

(a) Use the trigonometric identity:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

(c) Which of these techniques did you prefer? Why?

7. Integrate the following integrals.

(a) $\displaystyle\int \tan^2(x)\, dx$

(b) $\displaystyle\int \sec^3(x)\, dx$

(c) $\displaystyle\int \tan^5(x)\, dx$

(d) $\displaystyle\int \sin^5(x)\, dx$

7.6 Trigonometric Substitution

We're going to look at an integral that requires a variable substitution, but our goal for the substitution will be a bit different. We're going to focus on the structure of our integrand function, but we won't be focusing on composition. Instead, we're going to focus on some trigonometric identities that we've used already:

$$1 - \sin^2(\theta) = \cos^2(\theta)$$
$$\sec^2(\theta) - 1 = \tan^2(\theta)$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Activity 7.6.1 Difference of Squares.

Consider the integral:

$$\int \sqrt{1 - x^2}\, dx.$$

(a) First, convince yourself that a normal u-substitution will not be an effective strategy for integration in this case. Why not?

(b) Second, convince yourself that $\sqrt{1 - x^2} \neq \sqrt{1} - \sqrt{x^2}$. Why can we not distribute roots across sums and differences like this? When *can* we "distribute" roots across multiple things?

(c) Our goal, then, is to utilize a substitution (using trigonometric functions) to somehow transform this difference of squared terms under the square root into a single product of squared things under the square root.

Which trigonometric identities from our list of them above utilize differences of thing squared, and equate them to a single term?

Can you use the order of the subtraction to help guide which substitution we should use?

(d) When we do a variable substitution in an integral, we are not only finding a way of transforming x to be in terms of some other variable (in this case, θ). We also need to transform the differential, dx. Based on your substitution of $x = T(\theta)$, what is dx?

(e) Perform your substitution! Use your substitution $x = T(\theta)$ and $dx = T'(\theta)\, d\theta$. Note that we have picked this substitution with a very specific goal: we are hoping to notice a Pythagorean identity.

After you have performed your substitution, apply the relevant Pythagorean identity to the **radicand**: the bit of our function underneath the radical or root. What integral are we left with (in terms of θ)?

This new integral is something we can antidifferentiate now! We have already done this one in Activity 7.4.4 Squared Trig Functions. So we can end up with:

$$\begin{aligned}\int \sqrt{1 - x^2}\, dx &= \int \cos^2(\theta)\, d\theta \\ &= \frac{\theta + \sin(\theta)\cos(\theta)}{2} + C\end{aligned}$$

It is up to us, now, to translate this antiderivative family to be written in terms of x. We can utilize our substitution to do this, but let's first think about how this variable substitution works a bit more.

Another Type of Variable Substitution

We're going to employ another variable substitution, in the same way that we use u-substitution. The main difference is the goal: we're going to select our substitution not based on uncovering the composition in our function (like in u-substitution). Instead, we'll focus on selecting a trigonometric function in order to utilize the relevant Pythagorean identity to re-write our sum or difference of squares.

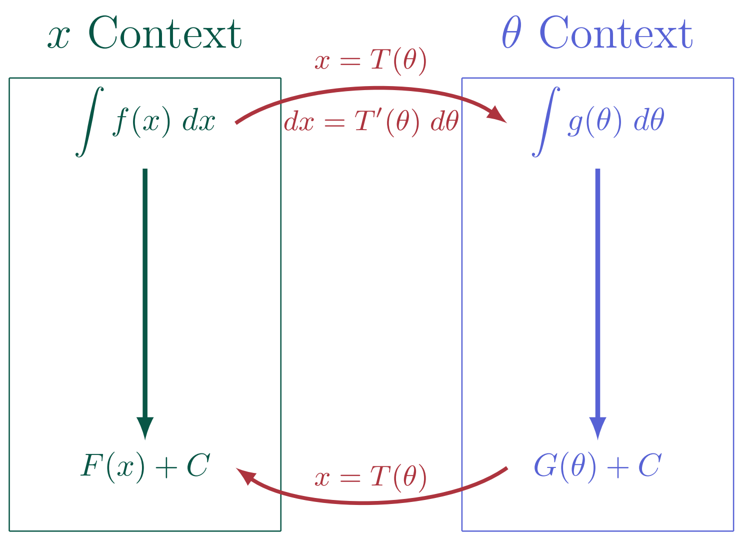


Figure 7.6.1 General idea of how this variable substitution works.

Ok, but how do we choose which trigonometric function to use in our substitution? Since we're focussing on sums or differences of squared terms, we can think of the different arrangements of terms, connect them with different Pythagorean identities, and set up some strategies for picking a trigonometric substitution.

$$\begin{aligned}
1 - x^2 &\longleftrightarrow 1 - \sin^2(\theta) = \cos^2(\theta)\\
x^2 - 1 &\longleftrightarrow \sec^2(\theta) - 1 = \tan^2(\theta)\\
x^2 + 1 &\longleftrightarrow \tan^2(\theta) + 1 = \sec^2(\theta)
\end{aligned}$$

We can note that the sum is commutative, so we can treat $1 + x^2$ the same that we treat $x^2 + 1$.

We'll also notice that the constant term can differ: we can scale our Pythagorean identities by some constant easily to make sure that they match.

$$\begin{aligned}
a^2 - x^2 &\longleftrightarrow a^2 - (a\sin(\theta))^2 = (a\cos(\theta))^2\\
x^2 - a^2 &\longleftrightarrow (a\sec(\theta))^2 - a^2 = (a\tan(\theta))^2\\
x^2 + a^2 &\longleftrightarrow (a\tan(\theta))^2 + a^2 = (a\sec(\theta))^2
\end{aligned}$$

This can be confusing, and we want to keep thinking about how we might recognize these structures to pick a substitution. Yes, we can recognize these

Pythagorean identities. We can rely on the order of subtraction or noticing addition. But we can also think about this geometrically. The Pythagorean identities come from the Pythagorean Theorem, relating the squared lengths of the sides of a right triangle together. Let's visualize our substitutions geometrically.

We'll consider three triangles, each with side lengths of x and a. The third side length will vary between $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 + a^2}$ (or the equivalent $\sqrt{a^2 + x^2}$) based on which length is representing the hypotenuse.

Activity 7.6.2 Trig Substitution Geoemtry.

We're going to consider three triangles, and we're going to fill in side lengths. In each of these, we'll assume that the lengths x and a are real numbers and are positive.

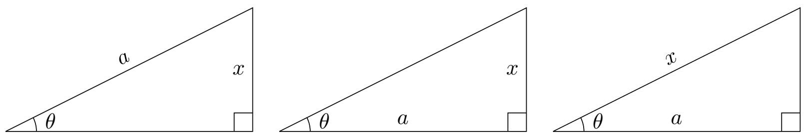


Figure 7.6.2 Three triangles to guide our trigonometric substitutions.

(a) Use the Pythagorean theorem to label the missing side length in each of the three triangles.

(b) For each triangle, explain how you can tell which side length represents the hypotenuse when you see the lengths x, a, and then the missing lengths you found above: $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 + a^2}$.

(c) For each triangle, find a trigonometric function of θ that connects lengths x and a to each other.

Solve each for x to reveal the relevant substitution.

(d) For each substitution, find the corresponding substitution for the differential, dx.

This gives us a nice strategy for substitution!

Trigonometric Substitution.

We have three (typical) ways of using trigonometric substitution to transform a sum or difference of squared terms into a product of squares.

- For an integral containing $(a^2 - x^2)$, we can use the following triangle to build our substitution:

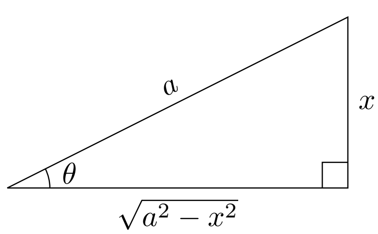


Figure 7.6.3

This results in using the following substitution and identity.

$$\begin{aligned} x &= a\sin(\theta) \\ dx &= a\cos(\theta)\ d\theta \\ a^2 - (a\sin(\theta))^2 &= (a\cos(\theta))^2 \end{aligned}$$

- For an integral containing $(x^2 - a^2)$, we can use the following triangle to build our substitution:

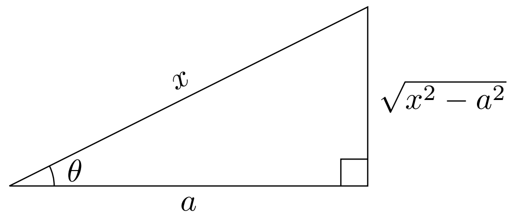


Figure 7.6.4

This results in using the following substitution and identity.

$$\begin{aligned} x &= a\sec(\theta) \\ dx &= a\sec(\theta)\tan(\theta)\ d\theta \\ (a\sec(\theta))^2 - a^2 &= (a\tan(\theta))^2 \end{aligned}$$

- For an integral containing $(x^2 + a^2)$, we can use the following triangle to build our substitution:

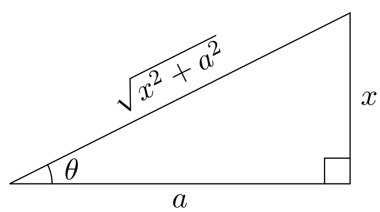


Figure 7.6.5

This results in using the following substitution and identity.

$$\begin{aligned} x &= a\tan(\theta) \\ dx &= a\sec^2(\theta)\ d\theta \\ (a\tan(\theta))^2 + a^2 &= (a\sec(\theta))^2 \end{aligned}$$

Two things to note before we move on:

1. There are really 6 main trigonometric substitutions. If you go back to Activity 7.6.2 and place the angle θ in the opposite corner of the triangle, the substitutions you build will all be using the “co-functions”: cosine, cosecant, and cotangent. Each of these has a very similar structure with regard to derivatives (for the differential substitution) and Pythagorean identities. Each is equivalent to the respective sine, secant, and tangent substitutions. We often choose to use sine, secant, and tangent just due to familiarity.

2. We can use the triangle as a kind of key for our substitution! After

antidifferentiating, we have some antiderivative family written in terms of an angle θ: we can use the triangle to substitution trigonometric functions of θ to be written in terms of x.

Example 7.6.6

We can finish the substitution we started in Activity 7.6.1. We used the substitution $x = \sin(\theta)$, but we can now construct the relevant triangle. Since we were hoping so use a substitution to re-write the difference of squares, $\sqrt{1-x^2}$, we had the following triangle:

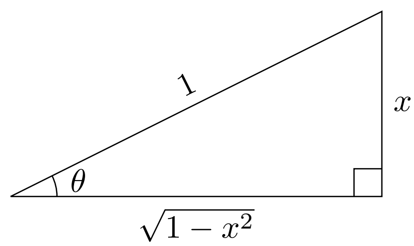


Figure 7.6.7 Substitution used in Activity 7.6.1.

We can see that $\sin(\theta) = \frac{x}{1}$ or $x = \sin(\theta)$, which was our substitution. But we were also left with the following antiderivative:

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \cos^2(\theta)\,d\theta \\ &= \frac{\theta + \sin(\theta)\cos(\theta)}{2} + C\end{aligned}$$

Now we can substitute that antiderivative! We can see from our triangle that $\cos(\theta) = \sqrt{1-x^2}$, $\sin(\theta) = x$ (this was also our original substitution anyways), and we also can invert our substitution to get $\theta = \sin^{-1}(x)$.

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \cos^2(\theta)\,d\theta \\ &= \frac{\theta + \sin(\theta)\cos(\theta)}{2} + C \\ &= \frac{1}{2}\left(\sin^{-1}(x) + x\sqrt{1-x^2}\right) + C\end{aligned}$$

Activity 7.6.3 Practicing Trigonometric Substitution.

Let's look at three integrals, and practice the kind of thinking we'll need to use to apply trigonometric substitution to them.

1. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$
2. $\displaystyle\int \frac{2}{(4-x^2)^{3/2}}\,dx$
3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

For each integral, do the following:

(a) Identify the term (or terms) that signify that trigonometric substitution might be a reasonable strategy.

(b) Use that portion of the integral to compare three side lengths of a triangle. Which one is the largest (and so must represent the length of the hypotenuse)?

(c) Construct the triangle, label an angle θ, and use a trigonometric function to connect the two single-term side lengths. (Feel free to change the angle you label in order to use the sine, secant, or tangent functions instead of their co-functions).

(d) Define your substitution (for both x and the differential dx), and identify the Pythagorean identity that will be relevant for the integral.

(e) Substitute and antidifferentiate!

(f) Use your triangle to substitute your antiderivative back in terms of x.

Trigonometric substitution is pretty involved technique! Setting up the substitution is definitely not trivial. Because our substitution involves trigonometric functions, we end up with integrals of trigonometric functions that we then have to work to antidifferentiate. And substituting back to x relies on us having set up a robust substitution strategy from the beginning.

It can sometimes seem like this strategy is barely relevant: the goal of it is so focussed on the specific structure of the Pythagorean identities, and these might not feel very present.

A friend of mine, though, says that once we start recognizing sums and differences of squares as being connected to Pythagoras, it's hard to *not* see them.

For instance, we can go back to Theorem 7.3.3 and see the sum of squares in the denominator. Instead of doing any tricky factoring to get the u-substitution to work, we could instead try a trigonometric substitution and get the same thing!

Another friend of mine says that trigonometric substitution only exists so that we can evaluate arc length integrals (Integrals for Evaluating the Length of a Curve).

Whatever the case, this new substitution strategy should, at the very least, generalize the concept of a variable substitution in an integral to show that we can define these for a variety of purposes, all based on the kinds of structures that we're seeing in the integrand function itself.

Practice Problems

1. Explain how trigonometric substitution helps to convert sums or differences of squares to products of squares. Why is this helpful? When is it helpful?

2. Draw a right triangle with $\sqrt{x^2-4}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{x^2-4}$?

3. Draw a right triangle with $\sqrt{4-x^2}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{4-x^2}$?

4. Draw a right triangle with $\sqrt{x^2+4}$ as one of the hypotenuse. What are the lengths of the other two sides? What would be an appropriate substitution for an integral containing $\sqrt{x^2+4}$?

5. Integrate the following using an appropriate trigonometric substitution.

(a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

(b) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

(c) $\displaystyle\int \frac{1}{(9x^2+1)^{3/2}}\,dx$

(d) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

(e) $\displaystyle\int \sqrt{49-x^2}\,dx$

(f) $\displaystyle\int \frac{1}{x(x^2-1)^{3/2}}\,dx$ (for $x>1$)

(g) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(h) $\displaystyle\int \frac{x^2}{(x^2+81)^2}\,dx$

6. Complete the square and then integrate.

(a) $\displaystyle\int \frac{1}{x^2-8x+62}\,dx$

(b) $\displaystyle\int \frac{x^2-8x+16}{(-x^2+8x+9)^{3/2}}\,dx$

7.7 Partial Fractions

In this last integration technique, we'll consider more rational functions. We've already thought about rational functions a bit (Integrating Rational Functions), but here we'll add some more detail to a special type of rational function. Let's not spoil anything. Instead, we'll just jump into a quick comparison.

Activity 7.7.1 Comparing Rational Integrands.

We're going to compare three integrals:

$$\int \frac{2}{x^2+4x+5}\,dx$$

$$\int \frac{2}{x^2+4x+3}\,dx$$

$$\int \left(\frac{1}{x+1}-\frac{1}{x+3}\right)\,dx$$

(a) Start with the first integral:

$$\int \frac{2}{x^2+4x+5}\,dx.$$

How would you approach integrating this?

(b) Try the same tactic on the second integral:

$$\int \frac{2}{x^2+4x+3}\,dx.$$

You don't need to complete this integral, but think about how you might proceed.

(c) Think about the third integral:

$$\int \left(\frac{1}{x+1}-\frac{1}{x+3}\right)\,dx.$$

How would you integrate this?

(d) The third integral is unique from the other two in that is has two terms. Let's combine them together to see how we could write this integral to compare it more closely to the other two.

Subtract $\frac{1}{x+1}-\frac{1}{x+3}$ using common denominators and compare your re-written integral to the other two.

(e) Which of these integrals and/or representations of an integral is easiest to work with? Which one is most annoying to work with? Why?

We're going to try to take advantage of the re-written structure in Activity 7.7.1: when we can decompose a "large" rational function into a sum of "smaller" rational functions, we'll be more likely to be able to antidifferentiate the "smaller" pieces.

The real trick, here, is going to be recognizing *when* we can do this and building a process for *how* we do this.

When?

In order to recognize *when* we will employ this strategy, we should think about what we're doing: We are attempting to write one fraction as a sum or difference of others. We're "undo-ing" fraction addition, here. We can remember that when we add (or subtract) fractions, we need to find a common denominator and re-write our fraction terms as equivalent versions with this same denominator. This is typically done by just "scaling" the numerator and denominator of a fraction by a factor (often then other denominator). For instance:

$$\begin{aligned}\frac{3}{5} - \frac{1}{3} &= \frac{3}{5}\left(\frac{3}{3}\right) - \frac{1}{3}\left(\frac{5}{5}\right)\\ &= \frac{9}{15} - \frac{5}{15}\\ &= \frac{4}{15}\end{aligned}$$

This same thing happens when we think about rational functions:

$$\begin{aligned}\frac{1}{x} + \frac{2}{x-1} &= \frac{1}{x}\left(\frac{x-1}{x-1}\right) + \frac{2}{x-1}\left(\frac{x}{x}\right)\\ &= \frac{x-1}{x(x-1)} + \frac{2x}{x(x-1)}\\ &= \frac{3x-1}{x(x-1)} \quad \text{or} \quad \frac{3x-1}{x^2-x}\end{aligned}$$

Now, we can notice something about these (common) denominators: since we scaled each fraction before adding/subtracting them, the denominator is a product of factors.

So right away we know that this method will only work well when we can factor the denominator of a rational integrand.

We also know (from Integrating Rational Functions) that once the degree of the numerator is at least the same (or larger) than the degree of the denominator, we can re-write things using division.

So, we will use this strategy of re-writing rational function integrands as sums or differences of "smaller" rational functions when:

1. The denominator can be factored.
2. The degree of the denominator is larger than the degree of the numerator.

How?

The actual process for finding the smaller "partial" fractions is not complicated. Once we have the denominator of our rational function factored, we can see what the possible denominators that had to "combine" to make a "common" denominator were.

Our general process will be simple: set up these possible rational functions and put unknown placeholder numerators in place, making sure that the degree is smaller than the degree of the denominator. Then, we add these possible rational functions up and compare the numerator (with the unknown placeholders) to the numerator that we want (from our actual function that we're integrating).

There are more tricks along the way, but this process is simple to think about. What we'll find, though, is that the process is full of algebra, which can be tedious.

Let's look at two small examples to see how this could work.

Activity 7.7.2 First Examples of Partial Fractions.

(a) Consider the integral:

$$\int \frac{6x-16}{x^2-4x+3}\,dx.$$

First, confirm that this would be *very* annoying to try to use u-substitution on, even though we have a linear numerator and quadratic denominator.

(b) Notice that the denominator can be factored:

$$\int \frac{6x-16}{(x-3)(x-1)}\,dx.$$

If this integrand function were a sum of two "smaller" rational functions, what would their denominators be? What do we know about their numerators?

(c) Use some variables (it's typical to use capital letters like A, B, C, etc.) to represent the numerators, and then add the partial fractions together. What do you get? How does this rational function compare to $\dfrac{6x-16}{(x-3)(x-1)}$?

(d) Set up an equation connecting the numerators, and solve for your unknown variables. What are the two fractions that added together to get $\dfrac{6x-16}{(x-3)(x-1)}$?

(e) Antidifferentiate to solve the integral $\displaystyle\int \frac{6x-16}{(x-3)(x-1)}\,dx$.

(f) Let's do the same thing with a new integral:

$$\int \frac{3x^2-2x+3}{x(x^2+1)}\,dx.$$

What are the partial fraction forms? What kinds of numerators should we expect to see? Use variables to represent these.

(g) Add the partial fractions together and set up an equation for the numerators to solve. What are the two partial fractions after you solve for the unknown coefficients?

(h) Antidifferentiate and solve the integral $\displaystyle\int \frac{3x^2-2x+3}{x(x^2+1)}\,dx$.

So we can see the basics of how this will work:

1. Figure out the denominators of the fractions we could add to get the function we're integrating.

2. Construct the partial fractions using placeholders for the numerators (making sure to keep the degree of the numerators smaller than the degree of the denominators).

3. Add these placeholder fractions up, and see what the coefficients would

have to be in order to make them add to the function we're integrating.

4. Antidifferentiate the smaller rational functions.

More Specific Strategies

We're going to investigate these partial fractions a bit more, and focus on two cases: linear denominators, and quadratic denominators. This will limit the scope of our work enough that this doesn't get too wild, and also works well with the kinds of things we know how to antidifferentiate: we can antidifferentiate any rational function with a linear denominator and we know how to antidifferentiate many rational functions with quadratic denominators.

Partial Fraction Type: Simple Linear Factors

For rational functions where the denominator has some linear factor like $(x-k)$, we can set up a partial fraction with just a constant coefficient in the numerator:

$$\frac{A}{x-k}.$$

We have outlined a pretty reasonable approach for these in Activity 7.7.2 when we worked on the integral

$$\int \frac{6x-16}{(x-3)(x-1)}\, dx.$$

We can make note of something useful, though. At one point during the process, we had set up the two partial fractions, added them together, and then we said that we wanted to find the values of A and B that made the general partial fraction forms we had set up match with the actual function we were hoping to integrate.

$$\begin{aligned}\frac{A}{x-3}+\frac{B}{x-1} &= \frac{A(x-1)+B(x-3)}{(x-3)(x-1)}\\ &= \frac{6x-16}{(x-3)(x-1)}\end{aligned}$$

What this meant for us was that we can set the numerators equal to each other and solve for A and B:

$$A(x-1)+B(x-3) = 6x-16.$$

Let's pause here.

We are hoping to find the values of A and B that will make this equation work out for all values of x. But we can also make use of the fact that this equation will be true for some *specific* values of x. For instance, we can evaluate this equation at $x = 2$:

$$\begin{aligned}A(2-1)+B(2-3) &= 6(2)-16\\ A-B &= -4\end{aligned}$$

This just generates another equation connecting A and B that we could use like we did in in Activity 7.7.2.

But is there a nicer, more convenient x-value to use? Can we force this to generate a nicer equation involving our unknown coefficients? Can we find some x-values that will make the factors $(x-1)$ and $(x-3)$ go to 0, for instance?

$$A\underbrace{(x-1)}_{x=1}+B\underbrace{(x-3)}_{x=3} = 6x-16$$

Let's try it!

$$\begin{aligned}
x = 1: \qquad & A(1-1) + B(1-3) = 6(1) - 16 \\
& -2B = -10 \\
& B = 5 \\
x = 3: \qquad & A(3-1) + B(3-3) = 6(3) - 16 \\
& 2A = 2 \\
& A = 1
\end{aligned}$$

These are the same values we had in Activity 7.7.2! This strategy works well for *any* simple linear factors, and definitely helps to reduce the amount of algebra required.

Simple Linear Factors.

If a rational function has a simple linear factor in the denominator in the form $(x-k)$, then the corresponding partial fraction is

$$\frac{A}{x-k}$$

where A is a real number constant with $A \neq 0$.

When we antidifferentiate these, we will end up with a logarithmic function.

Partial Fraction Type: Irreducible Quadratic Factors

So what about our other example in Activity 7.7.2? We had a quadratic factor (that couldn't be factored nicely). We'll call these **irreducible quadratic** factors. The real difference was just the structure of the numerator, where we accounted for the option of the numerator being some linear function, $Ax+B$.

There isn't much more to say here, since the algebra can be frustrating to deal with. It can be helpful to save some of these coefficients for last: that way, we can find some of the "easier" ones (from the Simple Linear Factors cases) first, and hopefully the remaining coefficients won't be too difficult to find.

Irreducible Quadratic Factors.

If a rational function has an irreducible quadratic factor in the denominator in the form (ax^2+bx+c), then the corresponding partial fraction is

$$\frac{Ax+B}{ax^2+bx+c}$$

where A and B are real number constants, and A and B cannot both be 0. That is, if one of A or B is 0, then the other cannot be.

When we antidifferentiate these, we can expect a logarithmic function, an inverse tangent function, or some combination of the two. We can get other antiderivatives, but those will be beyond the scope of this introductory text.

Partial Fraction Type: Repeated Linear Factors

Let's look at one last case. It's a bit of a weird one, so we'll explore it in its own example.

Activity 7.7.3 Fiddling with Repeated Factors.

Let's sit with the following integral:

$$\int \frac{3x+5}{x^2+2x+1}\,dx.$$

Before we start, we can think about how annoying it would be to try to start with a u-substitution where $u = x^2 + 2x + 1$.

(a) Factor the denominator! What's different about the factors in this denominator compared to the ones in Activity 7.7.2?

(b) Why can't we use these two factors to create two partial fractions with Simple Linear Factors?

(c) Ok, instead, we're going to do some algebra that is reminiscent of what we have done before in Section 7.3.

Can you write the numerator, $3x + 5$, as some coefficient on the factor $(x + 1)$ with some constant "remainder?"

$$3x + 5 = \boxed{}(x + 1) + \boxed{}$$

(d) Why is this re-forming of the numerator useful? What does that do, when we write it over the factored denominator? Why did we choose $(x + 1)$ as the factor to use for our re-writing?

Feel free to show why this is helpful!

(e) Integrate $\int \frac{3x+5}{x^2+2x+1}\,dx$ using your clever re-writing. Explain why this is a friendlier form.

This is something we can do, algebraically, for every fraction with a "repeated" factor like this. But, more importantly, we can incorporate this idea into how we think about partial fractions.

Repeated Linear Factors.

If a rational function has a repeated linear factor in the denominator in the form $(x - k)^n$, where mn is some integer greater than 1, then the corresponding partial fractions are

$$\frac{A_1}{x-k} + \frac{A_2}{(x-k)^2} + \ldots + \frac{A_n}{(x-k)^n}$$

where $A_1, A_2, \ldots, A_n$ are real numbers and $A_n \neq 0$.

When we antidifferentiate these, we can expect to use the rule and maybe find a logarithm function.

At this point, we can spend our time going through one or two examples where we put this all together.

Example 7.7.1

For each of the following integrals, set up the relevant partial fraction forms, solve for the unknown coefficients, and then antidifferentiate.

(a) $\displaystyle\int \frac{5x^2+27x+51}{(x-2)(x+3)^3}\,dx$

Hint 1. Your partial fraction forms will look like this:

$$\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}+\frac{D}{(x+3)^3}.$$

Hint 2. You can find the values for A and D pretty easily by thinking about convenient x-values, like we talked about in Partial Fraction Type: Simple Linear Factors.

Solution. We'll re-write our integral using the partial fraction forms:

$$\int \frac{5x^2+27x+51}{(x-2)(x+3)^3}\,dx = \int \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}+\frac{D}{(x+3)^3}\,dx.$$

When we combine these fractions to compare the numerators, we end up with the following equation:

$$A(x+3)^3+B(x-2)(x+3)^2+C(x-2)(x+3)+D(x-2) = 5x^2+27x+51$$

We can evaluate this at $x=2$ and $x=-3$ to reveal the values of A and D:

$$\begin{aligned} x=2: \qquad A(5)^3 &= 5(4)+27(2)+51 \\ A &= \frac{125}{125} = 1 \\ x=-3: \qquad D(-5) &= 5(9)+27(-3)+51 \\ D &= \frac{15}{-5} = -3 \end{aligned}$$

Now that we know $A=1$ and $D=-3$, we can put those into our equation connecting the numerators, and solve for B and C.

$$(x+3)^3+B(x-2)(x+3)^2+C(x-2)(x+3)-3(x-2) = 5x^2+27x+51$$

If we just consider the cubic terms, then on the left side of the equation we have x^3+Bx^3, and there are no cubic terms on the right side. This means that $1+B=0$ and so $B=-1$.

Similarly, we can consider just the constant terms of the (updated) equation:

$$(x+3)^3-(x-2)(x+3)^2+C(x-2)(x+3)-3(x-2) = 5x^2+27x+51$$

We can see that on the left side, we'll have $(3)^3-(-2)(3)^2+C(-2)(3)-3(-2)$ and on the right side, the constant term is 51.

$$\begin{aligned} 27+18-6C+6 &= 51 \\ 51-6C &= 51 \end{aligned}$$

$$C = 0$$

Finally, we have our new, re-written, integral. We can antidifferentiate.

$$\begin{aligned}\int \frac{5x^2+27x+51}{(x-2)(x+3)^3}\,dx &= \int \frac{1}{x-2} - \frac{1}{x+3} - \frac{3}{(x+3)^3}\,dx \\ &= \ln|x-2| - \ln|x+3| + \frac{3}{2(x+3)^2} + C\end{aligned}$$

(b) $\displaystyle\int \frac{6x^2+25x+27}{(x+1)(x^2+4x+5)}\,dx$

Hint 1. Your partial fraction forms will look like this:

$$\frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}.$$

Hint 2. You'll be able to easily find A by thinking about convenient x-values, but not B or C.

Solution. Let's, again, re-write our integral using the partial fraction forms we set up:

$$\int \frac{6x^2+25x+27}{(x+1)(x^2+4x+5)}\,dx = \int fracAx+1 + \frac{Bx+C}{x^2+4x+5}\,dx$$

Our equation for the combined numerator is:

$$A(x^2+4x+5) + (Bx+C)(x+1) = 6x^2+25x+27.$$

We can find A be evaluating at $x=-1$.

$$\begin{aligned} x=-1: \qquad A(1-4+5) &= 6(1)-25+27 \\ A &= \frac{8}{2} = 4\end{aligned}$$

Now, knowing that $A=2$, we can re-write our equation to solve for B and C.

$$4(x^2+4x+5) + (Bx+C)(x+1) = 6x^2+25x+27$$

We can collect the quadratic terms, and see the following equation:

$$4x^2 + Bx^2 = 6x^2$$

So $B=2$.

Similarly, we can collect the constant terms:

$$4(5) + C = 27.$$

It is easy to see that $C=7$. So we have our newly re-written integral:

$$\int \frac{6x^2+25x+27}{(x+1)(x^2+4x+5)}\,dx = \int \frac{4}{x+1} + \frac{2x+7}{x^2+4x+5}\,dx$$

The first term is pretty straight-forward to integrate: we'll get a log. The second one, though, will take some work. Let's consider it by itself:

$$\int frac2x+7x^2+4x+5\;dx.$$

We can start with a u-substitution of $u = x^2+4x+5$, giving us $dx = 2x+4\;dx$. Let's re-write the numerator as $2x+4+3$ in order to make this work:

$$\begin{aligned}\int frac2x+7x^2+4x+5\;dx &= \int frac(2x+4)+3x^2+4x+5\;dx\\ &= \int \frac{2x+4}{x^2+4x+5}\;dx + \int \frac{3}{x^2+4x+5}\;dx\end{aligned}$$

Now, the first of these will work with our stated substitution. The second one, though, will require a different strategy. Let's complete the square to get the inverse tangent form (Theorem 7.3.3). For all three integrals, then, we get:

$$\begin{aligned}\int \frac{6x^2+25x+27}{(x+1)(x^2+4x+5)}\;dx &= \int \frac{4}{x+1}+\frac{2x+7}{x^2+4x+5}\;dx\\ &= \int \frac{4}{x+1}+\frac{2x+4}{x^2+4x+5}+\frac{3}{x^2+4x+5}\;dx\\ &= \int \frac{4}{x+1}+\frac{2x+4}{x^2+4x+5}+\frac{3}{(x+2)^2+1}\;dx\\ &= 4\ln|x+1|+\ln|x^2+4x+5|+3\tan^{-1}(x+2)+C\end{aligned}$$

There are more things that we can think about, but it really ends up being just extensions of what we've done. For instance, we could think about repeated quadratic factors or irreducible polynomials that have larger degrees, but the general principles are the same: we set up a placeholder numerator that has a degree less than the denominator and try to solve for the unknown coefficients.

There are really only two limitations for us:

1. As we increase the number of coefficients, it becomes very tedious to solve for them. It isn't *difficult*, really: just a lot of algebra.

2. As we increase the degree of the kinds of denominators we see, we run out of approaches for antidifferentiation. We could spend *much* more time talking about integrating more rational functions or dive into the much of irrational coefficients (or even non-real ones), but this serves as a good stopping point for our purposes.

Practice Problems

1. Why do we use partial fraction decomposition on some integrals of rational functions? Give an example and explain why it is helpful in your example.

2. For each rational function described, write out the corresponding partial fraction forms.

(a) $\dfrac{p(x)}{(x-4)(x+2)(x-1)}$ where $p(x)$ is some polynomial with degree less than 3.

(b) $\dfrac{p(x)}{(x+1)^2(3x-5)^3}$ where $p(x)$ is some polynomial with degree less than 5.

(c) $\dfrac{p(x)}{(x^2+1)(x^2+2x+5)}$ where $p(x)$ is some polynomial with degree less than 4.

(d) $\dfrac{p(x)}{x^4-1}$ where $p(x)$ is some polynomial with degree less than 4.

3. Consider the following integral, with the partial fraction forms written out:

$$\int \frac{x^3+6x^2-x}{(x-2)(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x-2}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\right)\,dx.$$

(a) Write an equation connecting the numerators.

(b) Find (and use) a specific x-value to input into the equation to solve for A.

(c) Find (and use) a specific x-value to input into the equation to solve for B.

(d) Why can you not use this strategy to solve for coefficients C or D?

(e) Find the cubic terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for C.

(f) Find the constant terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for D.

(g) Integrate!

4. Explain why partial fractions is not an appropriate technique for the following integral:

$$\int \frac{x^2+x}{x^2-x+1}\,dx.$$

How should we approach this integral, instead?

5. Integrate the following.

(a) $\displaystyle\int \frac{2}{(x-1)(x+3)}\,dx$

(b) $\displaystyle\int \frac{4x+1}{(x-4)(x+5)}\,dx$

(c) $\displaystyle\int \frac{2x^2-15x+32}{x(x^2-8x^2+64)}$

(d) $\displaystyle\int \frac{1}{(x+2)(x-2)}\,dx$

(e) $\displaystyle\int \frac{20x}{(x-1)(x^2+4x+5)}\,dx$

(f) $\displaystyle\int \frac{x^2}{(x-2)^3}\,dx$

6. In the problems we are looking at in this section, we're limiting ourselves to, at most, irreducible quadratic factors in the denominator. In problems with simple linear factors, repeated linear factors, or irreducible quadratic factors, what types of antiderivative functions do you expect to see? Explain.

7. For each of the following integrals, we will do some preliminary work before using partial fractions to integrate. Really, we'll perform a specific u-substitution that will give us some resulting integral to use partial fractions on.

(a) $\displaystyle\int \frac{4e^{2x}}{(e^{2x}+3)(e^{2x}-5)}\,dx$ where we use $u = e^{2x}$.

(b) $\displaystyle\int \sqrt{e^x+1}\,dx$ where we use $u = \sqrt{e^x+1}$.

(c) $\displaystyle\int \frac{\sqrt{x}+3}{\sqrt{x}(x-1)}\,dx$ where we use $u = \sqrt{x}$.

Chapter 8

Infinite Series

8.1 Introduction to Infinite Sequences

Sequences as Functions

Before we move on to our actual goal of analyzing infinite series, we will construct infinite sequences. The big thing to remember here is that, when we build and analyze these sequences, we are are really building and analyzing functions. We want to keep this idea of sequences as functions in the forefront, since it will help us as we think about accumulating these function values into infinite series.

Activity 8.1.1 Building our First Sequences.

We might already have some familiarity with sequences. Here, we'll focus less on some of the detailed mechanics and just think about these sequences as functions.

(a) Describe a sequence of numbers where you use a consistent rule/function to build each term (each number) based only on the *previous term* in the sequence. You will need to decide on some first term to start your sequence.

(b) Describe a different sequence of numbers using the same rule to generate new terms/numbers from the previous one. What do you need to do to make these two sequences different from each other?

(c) Describe a new sequence of numbers where you use a consistent rule/function to build each term based on its position in the sequence (i.e. the first term will be some rule/function based on the input 1, the second will be based on 2, you'll use 3 to get the third term, etc.). We will call the position of each term in the sequence the *index*.

(d) Describe another, new, sequence of numbers where you use a consistent rule/function to build each term based on its index. This time, make the terms get smaller in size as the index increases.

Definition 8.1.1 Explicit Formula.

An infinite sequence defined using an **explicit formula** is one where the kth term of the sequence is defined as a function output of k, the term's index.
Using notation, we might say that $a_k = f(k)$ where:

- a is the ``name'' of the sequence (similar to how f and g are common names of functions).
- k is the index of the term, typically a non-negative integer.
- $f(k)$ is the function that we use to generate the terms.

Definition 8.1.2 Recursion Relation.

A sequence is defined using a **recursion relation** is one where the kth term of the sequence is defined as a function output of the previous term, the $(k-1)$st term. The sequence also needs some initial term to base the subsequent terms from.
Using notation, we might say that $a_k = f(a_{k-1})$.

These definitions are relatively limited. You might, for instance, know of a *very* famous sequence that is defined recursively by having each term being the sum of the *two* previous terms. Our study of sequences will be brief and all pointing towards infinite series, so there are a lot of nuances about sequences that we will skip.

Activity 8.1.2 Returning to our First Sequences.

Let's return back to the four sequences we created in Activity 8.1.1.

(a) For each of the sequences, how are we going to define them? Explicit formulas? Recursion relations? How do you know?

(b) Now, for each sequence, define the sequence formally using either an explicit formula or recursion relation, whichever matches with how you described the sequence in Activity 8.1.1.

Example 8.1.3 Practice Writing some Terms.

For each of the following sequences, write out the first handful of terms. There isn't a set amount, but you should write out enough to get a feel for the sequence structure and how the different ways of defining the sequences work. In each, you can start the index k at 1 and count upwards $(k = 1, 2, 3, ...)$.

(a) $a_1 = \frac{1}{3}$ and $a_k = 2(a_{k-1})^2$

(b) $b_k = \frac{\sin(k)}{k^2}$

(c) $c_k = \sqrt[k]{k+1}$

(d) $d_k = \frac{k+e^k}{e^k}$

Activity 8.1.3 Describing These Sequences.

Let's look at the sequences from Example 8.1.3. Go through the following tasks for each sequence.

(a) What do you think each sequence is "counting towards" (if anything)?

(b) Can you show that the sequence is counting towards what you think it is with a limit (or show that it's not counting towards anything)?

Activity 8.1.4 Write the Sequence Rules.

We'll look at some sequences by writing out the first handful of terms. From there, our goal is to write out more terms and eventually define each sequence fully.
For each sequence, write an explicit formula and a recursion relation to define the sequence. You can choose whether to start your index at $k = 0$ or $k = 1$.

(a) $\{a_k\} = \left\{4, \frac{2}{3}, \frac{1}{9}, \frac{1}{54}, ...\right\}$

(b) $\{b_k\} = \left\{\frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, ..., \right\}$

(c) $\{c_k\} = \left\{\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, ...\right\}$

(d) What kinds of connections do you notice between the explicit formulas and the recursion relations for these sequences?

Before moving on, we should think about a couple of notes:

- When we add something recursively (where we add the same thing repetitively to get from the kth term to the $(k+1)$st term), this is the same thing as multiplication in an explicit formula!

$$\underbrace{3 + 3 + ... + 3}_{k \text{ times}} = 3k$$

- Similarly, when we multiply something recursively, we can think about this as an exponential in the explicit formula!

$$\underbrace{3 \cdot 3 \cdot 3 \cdots 3}_{k \text{ times}} = 3^k$$

- In general, it can be pretty difficult to find either of these formulas for a given sequence of numbers. In fact, in any sequence where only the first few terms are given, we can find an infinite number of formulas that provide those first few terms and then deviate on to any other numbers. We cannot easily extrapolate onto only one "pattern" or formula. Because of this, we'll try to limit our work as much as we can to situations where we have the formula defining the sequences.

Graphing Sequences

We have tried introducing and talking about sequences as special types of functions, mapping natural number inputs to real number outputs. If we are committed to thinking about sequences as functions, with maybe some special context, then we should really investigate how one of our primary representations of functions (graphs) manifests itself in this new context.

There really is not too much to think about here! We can focus on the domain of these functions. If we define a sequence $\{a_k\}$ explicitly, then we have some function $a_k = f(k)$, and we can plot this sequence function in the same way that we normally would any other function $g(x)$. We will use the horizontal axis for the inputs and the vertical axis to represent the outputs, and try to visualize the graph as the set of all of the pairs of inputs with their (single) corresponding output.

The only new feature, then, is that these functions have only non-negative integer inputs. So when we plot the points, we do not get some nice curve acting as a visual representation of the function: we get discrete points floating on the 2-dimensional plane, each with some consistent horizontal spacing between them.

Consider, for example the following function and sequence:

$$f(x) = \frac{x}{x^2+1} \text{ for } x \geq 0$$

and

$$a_k = \frac{k}{k^2+1} \text{ for } k = 0, 1, 2, ...$$

We can graph the curve $y = f(x)$ in the normal way, as a smooth curve starting at the point at $x = 0$.

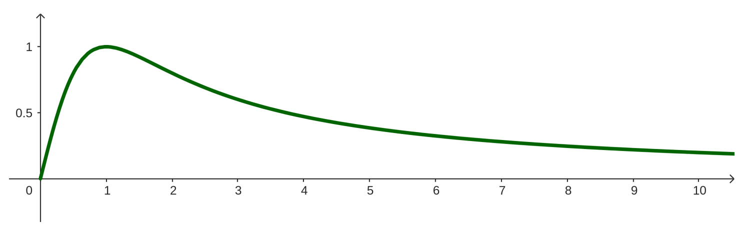


Figure 8.1.4 The function $f(x) = \dfrac{x}{x^2+1}$ plotted on the interval $[0, \infty)$.

When we plot the sequence $\{a_k\}$, though, we will visualize just the points on $f(x)$ at non-negative integer inputs.

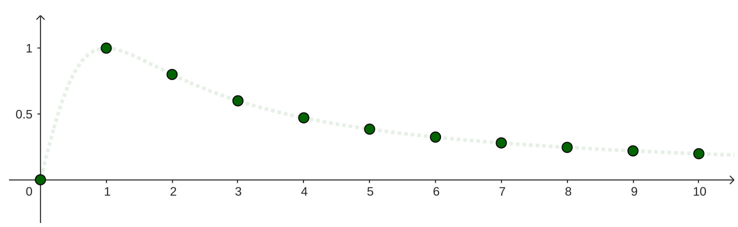


Figure 8.1.5 The sequence $\{a_k\}_{k=0}^{\infty}$ for $a_k = \dfrac{k}{k^2+1}$

We will typically not plot these with the smooth curve of the corresponding continuous function plotted, but in this first example it is useful to highlight how we think about this sequence as a function.

Let's continue to think about these sequences as just functions in a special kind of context. How does this discrete context change how we talk about functions and what kinds of terminology we use?

Sequence Terminology

If a sequence is a function (and we're saying in this introductory section that it is), then we can think of all of the different terminology and adjectives that we use to describe functions. How many of them are relevant to sequences?

- Continuous?
- Differentiable?
- Integrable?
- Increasing?
- Decreasing?

For now, we'll talk about sequences in two ways: their direction and the size of their terms.

Definition 8.1.6 Direction of a Sequence.

We say that a sequence $\{a_k\}_{k=1}^{\infty}$ is **increasing** if, for all $k = 1, 2, 3...$, $a_{k+1} > a_k$. If $a_{k+1} \geq a_k$ for all $k = 1, 2, 3, ...$ then we say that $\{a_k\}$ is **non-decreasing**.
We say that a sequence $\{a_k\}_{k=1}^{\infty}$ is **decreasing** if, for all $k = 1, 2, 3...$, $a_{k+1} < a_k$. If $a_{k+1} \leq a_k$ for all $k = 1, 2, 3, ...$ then we say that $\{a_k\}$ is **non-increasing**.
We say that a_k is constant if $a_{k+1} = a_k$, but this is a very boring sequence and we will likely not think terribly hard about these kinds of sequences.
Sometimes we might say that a sequence is **eventually non-increasing** if there is some $K > 1$, and the sequence is non-increasing for $k = K, K+1, K+2, ...$, and similarly for **eventually non-decreasing**.

A good example of a sequence that is eventually decreasing is the one we plotted in Figure 8.1.5. We can see that the sequence increases from $k = 0$ to $k = 1$ (since $a_0 < a_1$), but then decreases after that.

We could think about the corresponding continuous function (the one plotted in Figure 8.1.5) and find the point at which our function starts decreasing: we just need to refer back to Theorem 4.2.6 First Derivative Test to find the interval(s) for which $f(x) = \dfrac{x}{x^2+1}$ decreases.

Definition 8.1.7 Monotonic Sequences.

For the sequence $\{a_k\}_{k=1}^{\infty}$, we say that it is **monotonic** if the sequence is either non-increasing or non-decreasing.

We can, again, include a little modifier to talk about a sequence being **eventually monotonic**.

Definition 8.1.8 Bounded Sequences.

We say that a sequence $\{a_k\}_{k=1}^{\infty}$ is **bounded below** if there is some real number M such that $a_k \geq M$ for all $k = 1, 2, 3, ...$
Similarly we say that a sequence $\{a_k\}_{k=1}^{\infty}$ is **bounded above** if there is some real number N such that $a_k \leq N$ for all $k = 1, 2, 3, ...$
If a sequence has both an upper bound and a lower bound, then we

often just say that the sequence is **bounded**.

Lastly, we'll focus on the end-behavior of a sequence. We'll think about convergence of a sequence in the same way that we did for Improper Integrals: does the limit exist?

Definition 8.1.9 Sequence Convergence.

For the sequence $\{a_k\}$, if L is some real number and $\lim_{k\to\infty} a_k = L$, the we say that the sequence $\{a_k\}$ **converges** to L. If this limit does not exist, we say that the sequence $\{a_k\}$ **diverges**.

Theorem 8.1.10 Monotone Convergence Theorem.

If $\{a_k\}$ is a sequence that is both monotonic and bounded, then it must converge.

This theorem seems to be a bit obvious to many students: why would we care about this, when we can just find a limit of the explicit formula for a sequence? We'll see throughout the rest of this chapter that this theorem is one of the most important and most useful results in our study of infinite sequences and infinite series. For now, though, let's use it to find the limits of some recursively defined sequences.

Some Cool Recursive Examples

Let's re-visit one of the recursively defined sequences that we've seen already and then think about a couple of other interesting ones. Before we do that, though, we should recognize why we need to treat recursively defined sequences a bit differently than ones defined explicitly.

In an explicit formula, the terms themselves are a function of k, the index. This means that we can simply apply a limit as $k \to \infty$ to understand whether or not the sequence converges and what it might converge to. These limits could be tricky, but we have the tools to evaluate them! In a recursion relation, though, each term is not a function of the index, which means we can't easily apply a limit as $k \to \infty$ to the term definition.

We'll be able to apply a limit, but it will feel a bit different: we're going to go into the limit work under the assumption that the limit exists. Let's see how it goes.

Example 8.1.11

Let's re-visit the first sequence from Example 8.1.3: $\{a_k\}_{k=1}^{\infty}$ where $a_1 = \frac{1}{3}$ and $a_k = 2(a_{k-1})^2$.

(a) Let's start by assuming that the sequence converges. That means that there exists some real number L such that

$$\lim_{k\to\infty} a_k = L.$$

What would this L be, if it exists? A key thing to note is that if $\lim_{k\to\infty} a_k$ exists (and we have a symbol, L, for it) then we can say that

$$\lim_{k\to\infty} a_{k-1} = L.$$

Whether or not this is obvious to you is not a mark of your understanding, but we need to make sure that this ends up being obvious to you. If it's not, that's ok! But it is an indicator that you should take a couple of minutes to think about this. Once you are convinced that these two limits are the same thing, move on to the next part.

(b) Let's now apply a limit to the sequence definition:

$$\begin{aligned} a_k &= 2(a_{k-1})^2 \\ \lim_{k\to\infty} a_k &= \lim_{k\to\infty} 2(a_{k-1})^2 \\ \underbrace{\lim_{k\to\infty} a_k}_{L} &= 2\left(\underbrace{\lim_{k\to\infty} a_{k-1}}_{L}\right)^2 \\ L &= 2L^2 \\ 0 &= 2L^2 - L \\ 0 &= L(2L-1) \end{aligned}$$

And so we have two solutions to this equation: $L = 0$ and $L = \frac{1}{2}$. This is strange: how can a sequence have more than one value that it converges to?

It's because we have yet to take into account the initial term, a_1! Depending on this value, the sequence might converge or not, and if it does converge, then there are two options for what the sequence can converge to, based on the value of a_1.

(c) You can do the next part on your own, but I want you to pick different numbers for a_1 and write out some terms of the resulting sequence. You should find that some of them look like they're converging to 0, one of them will converge to $\frac{1}{2}$ (it's a fun hunt to find which one), and some will diverge.

Solution. You should find that if $a_1 = \frac{1}{2}$, then the sequence is constant and converges to $\frac{1}{2}$. If $0 \le |a_1| < \frac{1}{2}$ then the sequence seems like it'll converge to 0. And if $|a_1| > \frac{1}{2}$, then it looks like the sequence diverges.

(d) Now it is up to us to show that this sequence, with $a_1 = \frac{1}{3}$, does converge. Sure, we have some evidence and a good conjecture that it converges to $\frac{1}{2}$, but that is just our good guess based on what we have seen in the first handful of numbers.

We will attempt to convince ourselves that this sequence is both monotonic and bounded. We'll begin with boundedness.

It should be clear that $a_k > 0$, since as long as $a_{k-1} \neq 0$, then $(a_{k-1})^2 > 0$. Since we start with $a_1 \neq 0$, we are guaranteed to get non-zero values from the formula for a new term! Great news, we have a lower bound.

Let's show that $\frac{1}{2}$ is an upper bound: $a_k < \frac{1}{2}$ when

$$2(a_{k-1})^2 < \frac{1}{2}$$

$$(a_{k-1})^2 < \frac{1}{4}$$
$$a_{k-1} < \frac{1}{2}$$

Since $a_1 < \frac{1}{2}$, we know that each successive term will also be less than $\frac{1}{2}$. So we have an upper bound!

So the sequence $\{a_k\}$ is bounded. Now we just need to convince ourselves that this sequence is monotonic. We know that our terms are bounded above by $\frac{1}{2}$, and I hope that this means we can convince ourselves that since our terms are smaller than this, which would produce a constant sequence, then all of our terms are probably decreasing.

Let's show this by showing that $a_{k+1} - a_k < 0$:

$$\begin{aligned} a_{k+1} - a_k &= 2(a_k)^2 - a_k \\ &= a_k(2a_k - 1) \end{aligned}$$

We can solve for when this is negative! It shouldn't be hard to show that $a_k(2a_k - 1) < 0$ when $0 < a_k < \frac{1}{2}$. And we've already shown this is true in our case!

So $\{a_k\}$ is bounded and monotonic and must therefore converge because of the Monotone Convergence Theorem. Because $a_1 < \frac{1}{2}$, we know that this sequence doesn't converge to $\frac{1}{2}$, and so must converge to the only other option: 0.

There are some other fun ways of doing this same thing for other recursive examples. The argument above is relatively bulky to use, and so we understandably will not think about recursively defined sequences very much: we'll leave that topic for another course where we have more time to really explore them. If you are interested in trying this same argument with other sequences though, we'll end this section with two more fun examples.

Example 8.1.12

(a) Consider the sequence defined by $b_k = \sqrt{2 + b_{k-1}}$ with $b_1 = \sqrt{2}$. Does this sequence converge? To what?

Hint. Write out some terms to get a feel for things! Then, assuming that the sequence converges to some real number, L, think about what happens when you apply a limit as $k \to \infty$: we should get the equation $L = \sqrt{2 + L}$.

(b) Consider the sequence defined by $c_k = \frac{1}{2(c_{k-1})+1}$ with $c_1 = 1$. Does this sequence converge? To what?

Hint. Write out some terms to get a feel for things! Then, assuming that the sequence converges to some real number, L, think about what happens when you apply a limit as $k \to \infty$: we should get the equation $L = \frac{1}{2L+1}$.

8.2 Introduction to Infinite Series

Let's try to introduce the idea of an infinite series using a framework that we know and are (maybe) comfortable with: integrals!

With an integral, we have a nice way of evaluating integrals of nicely behaved functions with finite limits of integration (Fundamental Theorem of Calculus (Part 2)).

Then, when we talked about improper integrals, we built a nice way to think about evaluating integrals with unbounded limits of integration (Evaluating Improper Integrals (Infinite Width)). How will we use this to think about infinite series, a sum of the infinitely many terms from an infinite sequence?

Partial Sums

If we approach infinite series in a manner similar to improper integrals, then we will need to do a couple of things.

1. Truncate the infinite series at some finite ending point. This is what we did with the integral, when we replaced the infinity with some real number variable t. We might use n for the series "ending index."
2. Find a formula for this truncated/finite version. For the integrals, we could use the Fundamental Theorem of Calculus (Part 2) for this! For series, we'll need to do something else.
3. Apply a limit as t (or n in the case of infinite series) goes off to infinity!

Activity 8.2.1 How Do We Think About Infinite Series?

Let's consider the following sequence:

$$\left\{\frac{9}{10^k}\right\}_{k=1}^{\infty}$$

(a) White out the first 5 terms of the sequence.

(b) What does this sequence converge to? Show this with a limit!

(c) Now we'll construct a new sequence, this time by adding things up. We're going to be working with the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{k=n}\left(\frac{9}{10^k}\right).$$

Write out the first five terms of this sequence: S_1, S_2, S_3, S_4, S_5.

(d) Can you come up with an explicit formula for S_n?

(e) Does $\{S_n\}$ converge or diverge? Use a limit to find what it converges to!

(f) What do you think this means for the infinite series $\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right)$? Does the infinite series converge or diverge?

This is hopefully a nice little introduction to how we'll think about infinite series: we'll consider, instead, the sequence of sums where we add up more and more terms. This is also a nice first example, because we really just showed that

$$0.999... = 1$$

since

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right) &= \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+...\\ &= 0.9+0.09+0.009+...\\ 1.\end{aligned}$$

But more importantly, we now have a good strategy for thinking about infinite series as sequences of *partial sums*.

Definition 8.2.1 Partial Sum.

For an infinite series $\sum_{k=1}^{\infty} a_k$, we call $S_n = \sum_{k=1}^{n} a_k$ the nth **Partial Sum** of the infinite series.

Definition 8.2.2 Series Convergence.

We say that the infinite series $\sum_{k=1}^{\infty} a_k$ **converges** to the real number L if the sequence $\{S_n\}_{n=1}^{\infty}$ converges to L (where $\lim_{n\to\infty} S_n = L$), where $S_n = \sum_{k=1}^{n} a_k$ is the nth partial sum of the infinite series.
If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges (the limit $\lim_{n\to\infty} S_n$ does not exist), then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ **diverges**.

Visualizing the Sequence of Partial Sums

Since we'll think about an infinite series $\sum_{k=0}^{\infty} a_k$ as the sequence of its partial sums, $\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty}$, then we can think about visualizing an infinite series as really the same thing as visualizing a sequence in general (Graphing Sequences).

Let's consider, as a first (visual) example, the infinite series:

$$\sum_{k=0}^{\infty}\frac{3}{k+1}.$$

We can think about the two important sequences that we'll consider:

$$\underbrace{\left\{\frac{3}{n+1}\right\}_{n=0}^{\infty}}_{\{a_n\},\text{ the sequence of terms}} \quad\text{and}\quad \underbrace{\left\{\sum_{k=0}^{n}\frac{3}{k+1}\right\}_{n=0}^{\infty}}_{\{S_n\},\text{ the sequence of partial sums}}$$

We can plot the sequence of terms, $\left\{\dfrac{3}{n+1}\right\}_{n=0}^{\infty}$, and visualize the limit $\lim_{n\to\infty} a_n = 0$. This sequence of terms converges to 0.

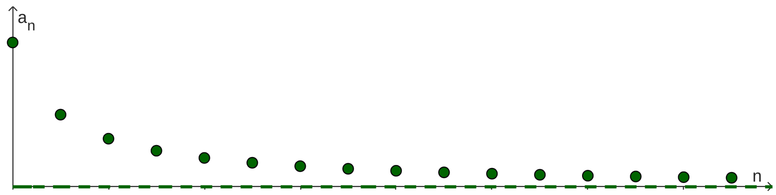


Figure 8.2.3 $\{a_n\}$, the sequence of terms in the series.

Then, we can compare this with the plot of the partial sums, $\{S_n\}$ where:

$$S_n = \sum_{k=0}^{n} \frac{3}{k+1}.$$

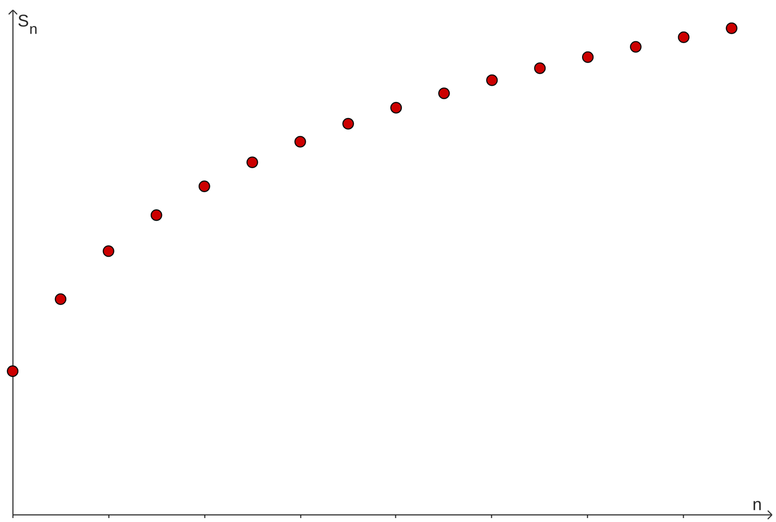


Figure 8.2.4 $\{S_n\}$, the sequence of partial sums for the series.

This image is fine, but not very good at showing how the sequence of terms and the sequence of partial sums are related to each other. We should note that each point in Figure 8.2.4 is the accumulation of the heights of the preceding points in Figure 8.2.3. We can visualize this to make it easier by overlaying some information onto the plot of partial sums in Figure 8.2.4.

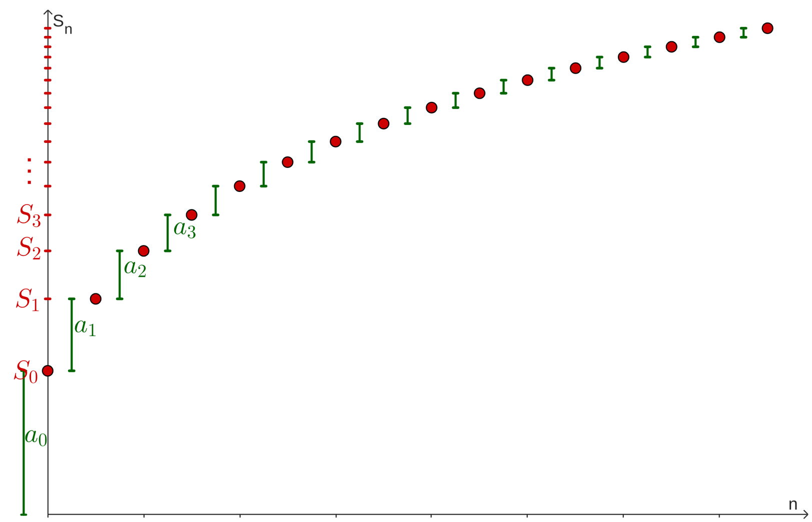


Figure 8.2.5 $\{S_n\}$ and $\{a_n\}$ visualized together.

Hopefully this does a good job of illustrating the connections between the two:

$$
\begin{aligned}
S_0 &= a_0 \\
S_1 &= a_0 + a_1 \\
&= S_0 + a_1 \\
S_2 &= a_0 + a_1 + a_2 \\
&= S_1 + a_2 \\
S_3 &= a_0 + a_1 + a_2 + a_3 \\
&= S_2 + a_3 \\
&\vdots \\
S_n &= a_0 + a_1 + \ldots + a_n \\
&= S_{n-1} + a_n
\end{aligned}
$$

Note 8.2.6 Finding Explicit Formulas.

We had noted earlier (in Section 8.1) that it was hard to find explicit formulas (or recursion relations) for sequences where we had the first few terms.

This remains true when we think about finding formulas for the sequences of partial sums. Notice that it is easy to find the location of the horizontal asymptote in Figure 8.2.3 (by evaluating $\lim_{n\to\infty} a_n$), but that we did not attempt to find one to for the partial sums in Figure 8.2.4 or Figure 8.2.5.

If you'd like to try this, then we need to find a formula for S_n. Try to find the first several partial sums by adding up terms in the series. Then try to find a formula to predict the next partial sum. This will definitely not be easy!

Ok, actually, this will be an impossible task. There is no closed-form formula for this. We cannot simply find $\lim_{n\to\infty} S_n$ in the way that we've found the limit of the sequence of terms.

Special Series

Let's look at three examples where we can think about partial sums and play with our new idea of series convergence.

Example 8.2.7

For each of the following series, write out a few of the terms of the series. Then write out the corresponding partial sums. Use these to find a formula for S_n, the nth partial sum. Then make a claim about whether or not the series converges and what it converges to.

(a) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right)$

Solution.

$$\begin{aligned}
S_0 &= 1\\
S_1 &= 1+\frac{1}{2}=\frac{3}{2}\\
S_2 &= 1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}\\
S_3 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}\\
S_n &= 1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}=\frac{2^{n+1}-1}{2^n}\\
&= 2-\frac{1}{2^n}\\
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)\\
&= 2
\end{aligned}$$

The series converges to 2.

(b) $\displaystyle\sum_{k=2}^{\infty}\left(\frac{k}{\ln(k)}-\frac{k+1}{\ln(k+1)}\right)$

Solution.

$$\begin{aligned}
S_2 &= \frac{2}{\ln(2)}-\frac{3}{\ln(3)}\\
S_3 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}-\frac{4}{\ln(4)}\\
&= \frac{2}{\ln(2)}-\frac{4}{\ln(4)}\\
S_4 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}\underbrace{-\frac{4}{\ln(4)}+\frac{4}{\ln(4)}}_{0}-\frac{5}{\ln(5)}\\
&= \frac{2}{\ln(2)}-\frac{5}{\ln(5)}\\
S_5 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}\underbrace{-\frac{4}{\ln(4)}+\frac{4}{\ln(4)}}_{0}\underbrace{-\frac{5}{\ln(5)}+\frac{6}{\ln(6)}}_{0}-\frac{6}{\ln(6)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\ln(2)} - \frac{6}{\ln(6)} \\
S_n &= \frac{2}{\ln(2)} \underbrace{- \frac{3}{\ln(3)} + \frac{3}{\ln(3)}}_{0} \underbrace{- \frac{4}{\ln(4)} + \frac{4}{\ln(4)}}_{0} \underbrace{- \frac{5}{\ln(5)} + \dots + \frac{n}{\ln(n)}}_{0} - \frac{n+1}{\ln(n+1)} \\
&= \frac{2}{\ln(2)} - \frac{n+1}{\ln(n+1)} \\
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{2}{\ln(2)} - \frac{n+1}{\ln(n+1)} \\
&= -\infty
\end{aligned}
$$

So the series $\sum_{k=2}^{\infty} \left(\frac{k}{\ln(k)} - \frac{k+1}{\ln(k+1)}\right)$ diverges.

(c) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{2}{k^2-1}\right)$

Hint. This one is tricky! It's hard to notice anything unless we write out the series term formula a bit differently. Use Partial Fractions to re-write $\frac{2}{k^2-1}$ as $\frac{1}{k-1} - \frac{1}{k+1}$.

Solution.

$$\sum_{k=2}^{\infty} \left(\frac{2}{k^2-1}\right) = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

$$
\begin{aligned}
S_2 &= 1 - \frac{1}{3} \\
S_3 &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \\
S_4 &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \\
&= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \\
S_5 &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \\
&= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \\
S_n &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots + \underbrace{\frac{1}{n-2} - \frac{1}{n}}_{a_{n-1}} + \underbrace{\frac{1}{n-1} - \frac{1}{n+1}}_{a_n}
\end{aligned}
$$

$$
\begin{aligned}
&1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+2} \\
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+2}\right) \\
&= \frac{3}{2}
\end{aligned}
$$

These examples are a bit misleading: we often won't be able to do this kind of thing! For most infinite series, we will struggle to find an explicit formula for the nth partial sum. In these examples, though, we took advantage of some specific structure.

In this first example (as well as the example in Activity 8.2.1), we noticed that because of the exponential function defining the terms, we were able to find some nice patterns in the partial sums. We'll explore this a bit more later in Section 8.6.

Then in these other two examples, we noticed that once we could write each term as really a difference of two fractions that have a really similar structure, we got these "repeat" values from term to term where the opposite signs made things add up to 0. These are called "telescoping series," and they're mostly fun examples to think about partial sum formulas. We'll see some pop up later though, and Partial Fraction Decomposition is a nice trick to keep in mind for these kinds of things.

8.3 The Divergence Test and the Harmonic Series

The Relationship Between a Sequence and Series

We have looked at both infinite sequences and infinite series so far, and, to make things complicated, we're really thinking about an infinite series (of terms from an infinite sequence) as an infinite sequence (of partial sums of the series). We've looked at how to visualize these (in both Graphing Sequences and Visualizing the Sequence of Partial Sums).

Let's first start with defining a new series. This is a relatively important one by itself (it *does* have its own name), but it's mostly an important series because it leads us into some new and interesting ways of thinking about series in general.

Definition 8.3.1 Harmonic Series.

We call the series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

the **Harmonic Series**.

You might not recognize this, but we've worked with a version of this before. The example series that we plotted in Figure 8.2.5 was

$$\sum_{k=0}^{\infty} \frac{3}{k+1}.$$

We can notice that if we re-index this by starting at $k = 1$ instead of $k = 0$, we were really just looking at a scaled version of the harmonic series.

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{3}{k+1} &= \sum_{k=1}^{\infty} \frac{3}{k} \\ &= 3\left(\sum_{k=1}^{\infty} \frac{1}{k}\right)\end{aligned}$$

Activity 8.3.1 Investigating the Harmonic Series.

(a) Write out the first several terms of the harmonic series, terms from $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$. Write however many you need to get a feel for how the terms work.

(b) Can you find out how many terms you would have to go "into" the series before the term was less than 0.00000001?

(c) Can you do this same kind of thing, no matter how small? For instance, how many terms would you have to go into the series before the term was less than some real number ε where $\varepsilon > 0$?

(d) Remind/explain/convince yourself that what we've really done is show that $\lim_{k\to\infty} \frac{1}{k} = 0$. This isn't a new or terribly interesting fact, but make sure that you understand why the argument above shows this.

(e) Let's do something very similar, but with $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}_{n=1}^{\infty}$, the sequence of partial sums, instead. Write out the first few partial sums. There's no specific number that you *need* to write, but make sure to write enough partial sums to get a feel for how the partial sums work.

(f) Can you find out how many terms you need to add up until the partial sum is larger than 1?

(g) Can you find out how many terms you need to add up until the partial sum is larger than 5?

(h) Can you find out how many terms you need to add up until the partial sum is larger than 10?

(i) Do you think that for any positive number S, we can always find some partial sum $\sum_{k=1}^{n} \frac{1}{k} > S$? What do you think this would mean about

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}?$$

To actually show that for any $S > 0$ we could always find an $n > 1$ where

$$\sum_{k=1}^{n} \frac{1}{k} > S$$

is an extremely difficult task! We will show that the Harmonic Series diverges in a different way, but for now I want us to notice these contradictory results: we have a series whose terms get small, but whose partial sums do not seem to converge.

We have $\frac{1}{k} \to 0$ but it seems like $\lim_{n\to\infty} S_n$ does not exist. Is this behavior special to the Harmonic Series? Is this something we should make note of? Is there some other connection between the terms of a series and the behavior of the partial sums of the series that we need to note?

Let's continue to think about this strange series, but actually prove that the series itself diverges.

Theorem 8.3.2 The Harmonic Series Diverges.

The ***Harmonic Series****,*

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

diverges.

Proof.

Let's assume, for the sake of eventual contradiction, that the harmonic series converges. Our goal in this proof is to show that this assumption (convergence) logically leads to an internal contradiction. This would mean that the assumption (convergence) cannot be true.

So, let's assume that the harmonic series converges.

Based on our definition of series convergence (Definition 8.2.2), there exists some real number S such that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = S.$$

We're going to think about this number, S, and show that there cannot be such a number.

First, let's write out what S is:

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$$

We're now going to systematically change the numbers being added together in order to create some number that is smaller than S: we're going to take all of the odd terms and make them as small as the next term after it:

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots \\ S &> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \ldots \end{aligned}$$

Note, though, that we can group together these duplicate terms and add them. Let's do that!

$$\begin{aligned} S &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \ldots \\ S &> 1 + \frac{1}{2} + \frac{1}{3} + \ldots \end{aligned}$$

But we should recognize this new series that is smaller than S...it's the harmonic series! Which, by our initial assumption, is also S!

Ok, so what we have shown is that if the harmonic series converges, then it converges to some number S that has the contradictory property of being smaller than itself.

There is no such number.

This is a contradiction, then, and so the harmonic series must diverge.

This is a strange result, and one that has been brought up again and again by mathematicians throughout history. We'll see that this series is notable because of its use later on in this chapter, but for now we can simply note that it is strange to see a series of terms that get so small (and so quickly) and yet the sum of those terms diverges.

The connection between the terms of a series and the behavior of the infinite series itself is maybe more mysterious than we initially thought. Since we will likely not have "access" to the formula for the partial sums (Note 8.2.6), we will want to explore these kinds of connections as much as we can. They will be the things to help us analyze an infinite series.

The Divergence Test

Theorem 8.3.3 Divergence Test.

For an infinite series $\sum_{k=0}^{\infty} a_k$*, if the infinite series converges then*

$$\lim_{k\to\infty} a_k = 0.$$

This is equivalent to saying that if

$$\lim_{k\to\infty} a_k \neq 0$$

then the infinite series $\sum_{k=0}^{\infty} a_k$ *diverges.*

Proof.

We will prove the claim that if an infinite series converges, then its sequence of terms must converge to 0.

This result will fall out of a simple exploration of what partial sums are. We noted in Section 8.2 that we can write any partial sum as the sum of the previous partial sum and the next term:

$$\begin{aligned} S_0 &= a_0 \\ S_1 &= a_0 + a_1 \\ &= S_0 + a_1 \\ S_2 &= a_0 + a_1 + a_2 \\ &= S_1 + a_2 \\ S_3 &= a_0 + a_1 + a_2 + a_3 \\ &= S_2 + a_3 \\ &\vdots \\ S_n &= a_0 + a_1 + \ldots + a_n \\ &= S_{n-1} + a_n \end{aligned}$$

Let's now say that the series we are dealing with converges. This means that $\lim_{n\to\infty} S_n = S$ for some real number S.

What, then, would the limit of S_{n-1} be as $n \to \infty$?

It has to also be S! If the partial sums converge, then these two partial sums must converge to each other n increases:

$$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n.$$

So, since $S_n = S_{n-1} + a_n$, we can investigate the limit of a_n:

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= 0 \end{aligned}$$

So of course the nth term has to converge to 0 in the limit!

Example 8.3.4

Apply the Divergence Test to the following series and interpret the results.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k^{15} - 4k^{10} + 10k^4}{e^{2k}}$

Hint. We can do a couple of things here! There is a nice result about limits of polynomials that we can use in the numerator (Polynomial End Behavior Limits). We could also get this same result using some other techniques, like what we use to prove that theorem. Then we can use L'Hôpital's Rule to evaluate the limit, since we have a $\frac{\infty}{\infty}$ indeterminate form.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+1}$

Hint. These terms are strange! The $(-1)^{k+1}$ part really just impacts the sign of the terms, since it is either 1 or -1 depending on if k is even or odd.

We can consider only one sign (maybe the positive), and then try to make a conclusion about the alternating terms. Do they go to 0?

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[k]{k}}$

Hint. This is similar: focus on only the positive terms for now. But that denominator is also strange! If you want to focus only on the denominator, you can use the following friendly rearrangement:

$$\begin{aligned}\lim_{k\to\infty} \sqrt[k]{k} &= \lim_{k\to\infty} k^{\frac{1}{k}}\\ &= \lim_{k\to\infty} e^{\ln\left(k^{1/k}\right)}\\ &= \lim_{k\to\infty} e^{\frac{1}{k}\ln(k)}\\ &= e^{\lim_{k\to\infty} \frac{\ln(k)}{k}}\end{aligned}$$

Now you can use L'Hôpital's Rule to evaluate this part!

8.4 The Integral Test

Infinite Series As a Kind of Integral

Let's start here with a connection between objects. We've already thought about the connection between an infinite series and the sequence of terms of the series (Theorem 8.3.3 Divergence Test). Now we'll think about the connections between two objects that are similar to each other in that they both represent an accumulation of function values.

Earlier (in Sequences as Functions) we tried to describe sequences as just a special kind of function: the domain is the set of non-negative integer (or positive integers, depending on whether we start our index at $n = 0$ or $n = 1$) and we map these inputs to real number outputs. And now we want to think about what it might mean to accumulate the values of these kinds of functions.

Function value accumulation is what we've been looking at lately! That's what integration is! We are trying to accumulate all of the function values and weigh them based on their "width." In the context of continuous functions, that means we start approximating this accumulation by looking at some finite number of function values that we pick, and we give them some Δx width between them. That's our Riemann sum:

$$\sum_{k=1}^{n} f(x_k^*)\ \Delta x$$

And from there, we work on making that space between function values get smaller (as the number of function values we use gets higher). So when n is the number of function values, we can let $n \to \infty$ and correspondingly we get $\Delta x \to dx$, the differential in our integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\ \Delta x = \int_{x=a}^{x=b} f(x)\ dx.$$

And this is how we've talked about infinite series so far, even adopting the same notions of convergence and thinking about how we extend a familiar idea (in this case adding numbers, compared to integrating a function) out to infinity: we just keep walking our (finite) ending point out to infinity using a limit!

So this brings us to this comparison of the same types of objects across these two different contexts.

Table 8.4.1 Comparisons of Calculus Objects in Continuous and Discrete Contexts

Object	Continuous Context	Discrete Context
Function	$f(x)$	a_k
Graph	Figure 8.1.4	Figure 8.1.5
Finite Accumulation	Definite Integral $A(t) = \int_{x=0}^{x=t} f(x)\ dx$	Partial Sum $S_n = \sum_{k=0}^{n} a_k$
Infinite Accumulation	Improper Integral $\int_{x=0}^{\infty} f(x)\ dx$	Infinite Series $\sum_{k=0}^{\infty} a_k$

So in this section, we'll investigate this link between infinite series and improper integrals as the same kind of object occurring in different contexts. Intuitively, then, they'll be related to each other, under the right conditions.

The Integral Test

The Integral Test is really about connecting the behavior of an integral and a series, and the way that we'll do it is by trying to visualize what an infinite series is (a sum of function values, where the function inputs are spaced apart by 1) and linking that to a Riemann sum. From there, we'll use the Monotone Convergence Theorem on the sequence of partial sums to show that the series converges.

Activity 8.4.1 Integrals and Infinite Series.

We're going to work with a graph of a continuous function, and we're going to start with a couple of conditions:

1. Our function will be continuous wherever it's defined.
2. Our function will be decreasing on its domain.
3. All of the function outputs will be positive.

Let's not worry about picking a specific function for this, but we will visualize a graph of one that meets these three requirements.

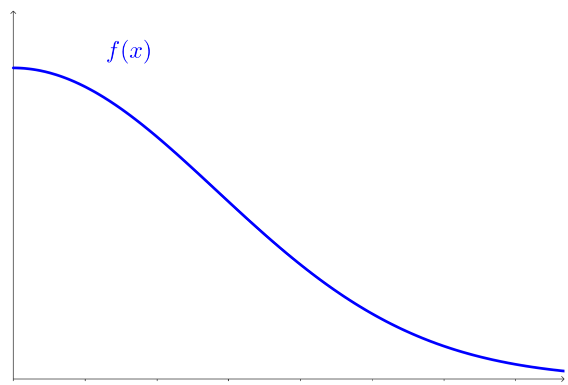


Figure 8.4.2

We can then visualize the sequence of terms, $a_k = f(k)$ for $k = 0, 1, 2, \ldots,$.

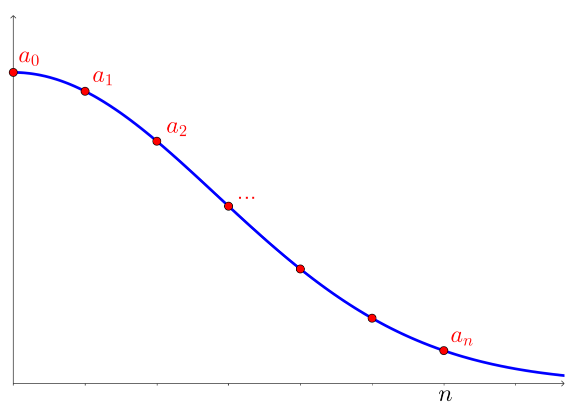


Figure 8.4.3

(a) How does the partial sum, $\sum_{k=0}^{n} a_k$ compare to the Riemann sum for $f(x)$ from $x = 0$ to $x = n$ with n rectangles?

(b) We're going to visualize the accumulation of $f(x)$ from $x = 0$ to $x = n$ by thinking about the integral:

$$\int_{x=0}^{x=n} f(x)\, dx.$$

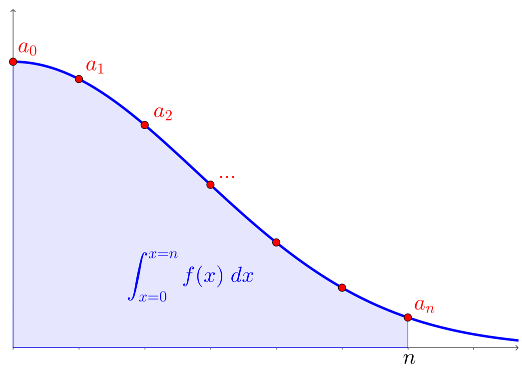


Figure 8.4.4

How does this area compare to the Riemann sum you thought of above? Compare them with an inequality and make sure you can explain why this has to be true.

(c) Remove the first term of the series, a_0, and instead think of the sum $\sum_{k=1}^{n} a_k$. Can you still think of this as a Riemann sum to approximate the area from the integral $\int_{x=0}^{x=n} f(x)\, dx$?

How does this new Riemann sum compare to the area formed by the integral? Compare them with an inequality and make sure you can explain why this has to be true.

(d) We have thought about two sums, and we can connect them:

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k.$$

Use the sums to bound the integral:

$$\underline{\qquad\qquad} < \int_{x=0}^{x=n} f(x)\, dx < \underline{\qquad\qquad}$$

(e) Similarly, use the integral to bound the sum:

$$\underline{\qquad\qquad} < \sum_{k=0}^{n} a_k < \underline{\qquad\qquad}$$

These bounds are going to be super useful! Discovering them is the main task for finding the connections between improper integrals and infinite series. These inequalities might seem kind of strange at first, but we're going to apply a limit to everything as $n \to \infty$, and then think about our definitions of convergence (Definition 7.1.4 and Definition 8.2.2).

Theorem 8.4.5 Integral Test.

If $\displaystyle\sum_{k=0}^{\infty} a_k$ is an infinite series with $a_k > 0$ for all $k \geq 0$ and $f(x)$ is a continuous and decreasing function with $f(k) = a_k$ for all $k \geq 0$, then we can compare the behaviors of $\displaystyle\sum_{k=0}^{\infty} a_k$ and $\displaystyle\int_{x=0}^{\infty} f(x)\,dx$: the integral and the series are guaranteed to either both diverge or both converge.

Proof.

The proof of this will come in two parts. First, we'll prove that $\displaystyle\sum_{k=0}^{\infty} a_k$ converges when $\displaystyle\int_{x=0}^{\infty} f(x)\,dx$ converges.

Then, we'll prove that $\displaystyle\sum_{k=0}^{\infty} a_k$ diverges when $\displaystyle\int_{x=0}^{\infty} f(x)\,dx$ diverges.

1. Let's start with the assumption that $\displaystyle\int_{x=0}^{\infty} f(x)\,dx$ converges. We know, based on Definition 7.1.4, that this means that $\displaystyle\lim_{n\to\infty}\int_{x=0}^{x=n} f(x)\,dx$ exists. We also know, since $f(x) > 0$, that

   $$\lim_{n\to\infty}\int_{x=0}^{x=n} f(x)\,dx > \int_{x=0}^{x=n} f(x)\,dx.$$

   This means:

   $$\sum_{k=0}^{n} a_k < a_0 + \int_{x=0}^{x=n} f(x)\,dx$$
   $$\sum_{k=0}^{n} a_k < a_0 + \lim_{n\to\infty}\int_{x=0}^{x=n} f(x)\,dx$$
   $$\sum_{k=0}^{n} a_k < a_0 + \int_{x=0}^{\infty} f(x)\,dx$$

   This means that the partial sum, $S_n = \displaystyle\sum_{k=0}^{n} a_k$ has an upper bound.

   We also know that, since $a_k > 0$ for all $k = 0, 1, 2, \ldots$, then $S_{n+1} > S_n$. This means that the sequence of partial sums, $\{S_n\}_{n=0}^{\infty}$ is both monotonic and bounded, and therefore must converge (by the Monotone Convergence Theorem).

   Thus, $\displaystyle\sum_{k=0}^{\infty} a_k$ converges.

2. Now, we can start with the assumption that the integral $\displaystyle\int_{x=0}^{\infty} f(x)\,dx$ diverges. Since we know that $f(x)$ is positive, then we know that

   $$\lim_{n\to\infty}\int_{x=0}^{x=n} f(x)\,dx = \infty.$$

   We can re-consider the inequalities from Activity 8.4.1:

   $$\int_{x=0}^{x=n} f(x)\,dx < \sum_{k=0}^{n} a_k$$

$$\lim_{n\to\infty}\int_{x=0}^{x=n} f(x)\,dx < \lim_{n\to\infty}\sum_{k=0}^{n} a_k$$
$$\infty < \sum_{k=0}^{\infty} a_k$$

So then $\sum_{k=0}^{\infty} a_k$ must also diverge.

This is everything we need to prove! Note that we could replicate this proof by swapping the role of the series and the integral to get the same conclusion.

So now we have a more formal link between these two objects. We have some intuition from Table 8.4.1 that these are pretty similar objects: one of them is an accumulation of function values in a continuous context, while the other is an accumulation of function values in a discrete context.

We can think about this even more formally! We reminded ourselves at the beginning of this section that, when we think about the limit of a Riemann sum (in the *continuous context*) that as $n \to \infty$, we get $\Delta x \to dx$, where dx is essentially the distance between inputs. This produces our integral.

But now, in the *discrete context*, we have something different. Without being too formal, we can think about the corresponding limit of Δk as we slice this up further. Because our functions are in the discrete context, our inputs have a minimum distance between each other: they're all 1 unit apart! So here, in the limit where we expect $\Delta x \to dx$, we get $\Delta k \to 1$. And similarly, in a typical definite integral, we are adding up an infinite number of function outputs between some starting input and stopping input. Now in the discrete context, we don't get that! We get our normal partial sum:

$$\sum_{k=0}^{n} a_k \underbrace{\Delta k}_{1} = \sum_{k=0}^{n} a_k.$$

So the Integral Test is pretty obvious, really: these corresponding objects retain the same type of behavior when we translate them back and forth between the continuous context and the discrete context.

Great! Let's apply this, now.

Example 8.4.6

For each of the following infinite series, decide whether it is possible (and reasonable) to use the Integral Test. If it is, apply the test and interpret the conclusions.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+1}$

Hint. This would connect with the integral $\displaystyle\int_{x=0}^{\infty} \frac{1}{x^2+1}\ dx$. Does the function, $f(x) = \dfrac{1}{x^2+1}$, meet the requirements of the Integral Test? Does it look like something you could antidifferentiate?

Solution. This is a fine opportunity to apply the Integral Test. The Integral Test says that we can link the behavior of the integral and the series:

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1} \sim \int_{x=0}^{\infty} \frac{1}{x^2+1}\, dx.$$

Let's think about the integral!

$$\begin{aligned}
\int_{x=0}^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{t\to\infty} \int_{x=0}^{x=t} \frac{1}{x^2+1}\, dx \\
&= \lim_{t\to\infty} \left(\tan^{-1}(x)\right)\Big|_{x=0}^{x=t} \\
&= \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(0) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

This integral converges.

Conclusion: Since the integral $\int_{x=0}^{\infty} \frac{1}{x^2+1}\, dx$ converges, we know that the infinite series $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ also converges.

(b) $\sum_{k=0}^{\infty} \frac{1}{e^{k^2}}$

Hint. We can re-write $\frac{1}{e^{k^2}}$ to e^{-k^2}, and so we're thinking about the integral $\int_{x=0}^{\infty} e^{-x^2}\, dx$. Does this function meet the conditions of the Integral Test? Can we antidifferentiate?

Solution. Unfortunately, this integral is going to be very difficult for us! The function $f(x) = e^{-x^2}$ has an antiderivative on the interval $[0, \infty)$ (it's a continuous function, and so it is integrable according to the Fundamental Theorem of Calculus (Part 1)).

This function, though, doesn't have what we call an **elementary antiderivative**: any antideriative of e^{-x^2} can't be written as a combination of our basic function types.

This means that we're unable to integrate this using our typical techniques, and (for now), we don't know if the integral converges or not.

Conclusion: We won't apply the Integral Test, and so we don't know whether the series $\sum_{k=0}^{\infty} \frac{1}{e^{k^2}}$ converges or not.

(c) $\sum_{k=1}^{\infty} \frac{k}{e^{k^2+1}}$

Hint. The Integral Test would connect this series to $\int_{x=1}^{\infty} \frac{x}{e^{x^2+1}}\, dx$. Does the function $f(x) = \frac{x}{e^{x^2+1}}$ meet the requirements of the Integral Test? Could we antidifferentiate?

Solution. Let's apply the Integral Test. we'll connect the behavior of the integral and the series:

$$\sum_{k=1}^{\infty} \frac{k}{e^{k^2+1}} \sim \int_{x=1}^{\infty} \frac{x}{e^{x^2+1}}\, dx.$$

We'll consider the integral, and use a u-substitution where $u = -(x^2+1)$ and $du = -2x\, dx$.

$$\begin{aligned}
\int_{x=1}^{\infty} \frac{x}{e^{x^2+1}}\, dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{x}{e^{x^2+1}}\, dx \\
&= \lim_{t\to\infty} -\frac{1}{2} \int_{x=1}^{x=t} \frac{-2x}{e^{x^2+1}}\, dx \\
&= -\frac{1}{2} \lim_{t\to\infty} \int_{u=-2}^{u=-(t^2+1)} e^u\, du \\
&= -\frac{1}{2} \lim_{t\to\infty} \left(e^u\right)\Big|_{u=-2}^{u=-(t^2+1)} \\
&= -\frac{1}{2} \lim_{t\to\infty} e^{-t^2+1} - e^{-2} \\
&= 0 + \frac{1}{2e^2}
\end{aligned}$$

This integral converges.

Conclusion: The integral $\displaystyle\int_{x=1}^{\infty} \frac{x}{e^{x^2+1}}\, dx$ converges, and so we know that the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2+1}}$ also converges.

(d) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$

Hint. We're considering the function $f(x) = \dfrac{1}{x\ln(x)}$ and the integral $\displaystyle\int_{x=2}^{\infty} \frac{1}{x\ln(x)}\, dx$. Does this work for the Integral Test?

Solution. If we apply the Integral Test, we're connecting the following series and integral:

$$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)} \sim \int_{x=2}^{\infty} \frac{1}{x\ln(x)}\, dx.$$

We'll consider the integral and use the substitution $u = \ln(x)$ so that $du = \dfrac{1}{x}\, dx$.

$$\begin{aligned}
\int_{x=2}^{\infty} \frac{1}{x\ln(x)}\, dx &= \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{x\ln(x)}\, dx \\
&= \lim_{t\to\infty} \int_{u=\ln(2)}^{u=\ln(t)} \frac{1}{u}\, du
\end{aligned}$$

$$= \lim_{t\to\infty} (\ln|u|)\Big|_{u=\ln(2)}^{u=\ln(t)}$$
$$= \lim_{t\to\infty} \ln|\ln(t)| - \ln(\ln(2))$$
$$= \infty$$

This integral diverges.

Conclusion: We found that the integral $\displaystyle\int_{x=2}^{\infty} \frac{1}{x\ln(x)}\,dx$ diverges, which means that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ also diverges.

As we develop more strategies and tests for series convergence, we should pause and summarize our test.

Integral Test Strategy.

We want to use this for functions that are relatively easy to antidifferentiate. Looking for u-substitution is a good idea, and sometimes we can straightforwardly apply integration by parts.

We'll find, though, that this test will mostly be used to introduce a family of infinite series and build up our intuition about partial sums (since we're really using the integrals to find a bound on our monotonic sequence of partial sums).

Why Do We Need These Conditions?

Before we wrap things up, let's just add some commentary on the conditions or requirements of the Integral Test. This is really a part of a broader discussion on conditions/requirements in mathematical results in general, but we'll limit ourselves to just this specific test. There are three conditions that we can consider: positive, decreasing, and continuous.

1. *Positive:* We need $f(x)$ and a_k to be positive in order for us to get monotonic sequences. Since we know that $a_k > 0$, we guarantee ourselves that $S_{n+1} > S_n$, since we're really adding another positive term. The same thing is true for the integrals, where we guarantee that
$$\int_{x=0}^{x=n+1} f(x)\,dx > \int_{x=0}^{x=n} f(x)\,dx.$$

2. *Decreasing:* This one serves two purposes. First, we use the direction of the function to get some ideas on how the Left and Right Riemann sum compare to the areas: with a decreasing function, the Left sum will always overestimate the integral while the Right sum will underestimate it.

   We could have just required our function to be **monotonic**, though, since that would guarantee that one of those Riemann sums overestimated the integral while the other underestimated it: we really don't care about the order. So why would we need it, specifically, to be decreasing? Easy: if $\{a_k\}$ is positive and *increasing*, then $\lim\limits_{k\to\infty} a_k \neq 0$, and we can just apply the Divergence Test.

3. *Continuous:* This one is pretty simple. Continuity guarantees that an antiderivative of our function exists on the interval we're looking at. We might not be able to actually find it easily, but at least we know there is one! Without continuity, we can't antidifferentiate easily and it doesn't make sense to think about the integral.

We talked about why it's nice to have a monotonic sequence of terms, but what happens when they aren't? Briefly, we can say that there are plenty of examples where the series and the integral may not behave similarly. For an easy to see example, let's consider the following series:

$$\sum_{k=0}^{\infty} \sin^2(\pi \cdot k).$$

If we try to think about the corresponding integral, we're considering:

$$\int_{x=0}^{\infty} \sin^2(\pi \cdot x)\, dx.$$

Can you see what the problem is? The issue becomes evident when we plot the sequence of terms and the continuous function on the same axes.

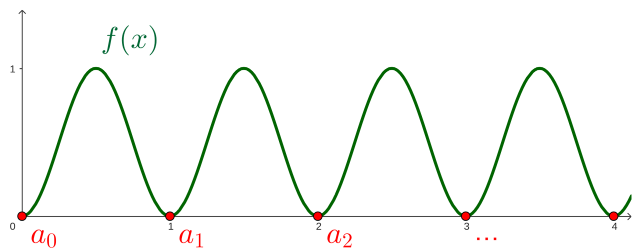


Figure 8.4.7

We can see that if we define the sequence of terms using $a_k = \sin^2(\pi \cdot k)$, then $a_k = 0$ for $k = 0, 1, 2, ...$ This means that

$$\sum_{k=0}^{\infty} \sin^2(\pi \cdot k) = 0 + 0 + 0 + ... = 0.$$

Meanwhile, we can see that the integral $\displaystyle\int_{x=0}^{\infty} \sin^2(\pi \cdot x)\, dx$ will diverge to ∞, since every oscillation adds the same positive area over and over. The limit will not exist!

In the Integral Test, and in general, we want to (and need to) be careful about the conditions we apply: we want them to be general enough that we can actually use the test, but specific enough to protect against strange counter-examples.

Overall, we're really just trying to connect an integral in one context to an integral in another context. We have talked, briefly, about how we can think of an infinite series as a kind of improper integral for some discrete function (since we are summing up the function values multiplied by the minimal distance between inputs).

Earlier (in Section 5.6 Introduction to u-Substitution and Section 7.6 Trigonometric Substitution), we tried to visualize the variable substitutions (with Figure 5.6.1, Figure 5.6.2, and Figure 7.6.1) as a translation of functions between one context and another. We can visualize the Integral Test in a similar way:

we're translating our integral from a Discrete context into a Continuous context, figuring out whether the integral converges in this context, and then taking the conclusions and applying them to the integral in the Discrete context.

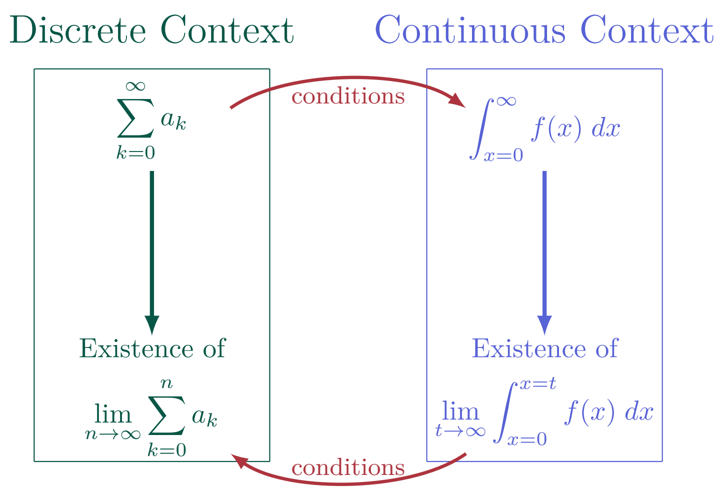


Figure 8.4.8 Integral Test, visualized as a transformation between contexts.

In both the discrete context and the continuous context, we care about whether the limit of the partial sum (or the limit of the definite integral) exists. Under the conditions of the Integral Test, though, we can translate our infinite series to the continuous context and represent it as a traditional improper integral. If we can get conclusions about whether the improper integral converges or not (based on the behavior of the limit of the definite integral), then we can bring that conclusion back to the discrete context and apply it to the limit of the partial sum.

This process works both ways! We could use information about an infinite series to learn about whether the corresponding improper integral converges or not! But for now, we are likely much more confident working with the integral in the continuous context, and so it makes most sense to think about the Integral Test using the directions pictured in Figure 8.4.10.

8.5 Alternating Series and Conditional Convergence

Before we move too far forward, let's circle back to a point made in Why Do We Need These Conditions?. In the Integral Test, we required the terms of our series (and the continuous function we connected it with) to be positive. This was really just a mechanism that allowed us to say, in our proof, that the sequence of partial sums was monotonic. When we accumulate more of a positive thing, the total gets bigger. This is half of what we needed for us to employ the Monotone Convergence Theorem. And because this is such a useful tool, we'll see more of this "positive term series"condition showing up in the tools we use to see if a series converges.

But that makes this a perfect time to stop and ask a hallowed mathematical question: *What happens if that property isn't there?* What happens when our series does not only have positive terms?

We definitely have fewer tools to use, since we don't get anything that relies on applying the Monotone Convergence Theorem to partial sums. So instead, we'll take a brief detour into something we call **Alternating Series** (a series whose terms alternate in sign).

Activity 8.5.1 Which is More Likely to Converge?

We're going to try to think about what might be different when we analyze an alternating series compared to a series with only positive (or non-negative) terms.

Let's say that $\{a_k\}_{k=0}^{\infty}$ is some sequence of positive real numbers. Now let's consider the two series:

$$\sum_{k=0}^{\infty} a_k \qquad \text{vs} \qquad \sum_{k=0}^{\infty} (-1)^k a_k$$

(a) Let's first consider the sequences of terms: $\{a_k\}$ compared with $\{(-1)^k a_k\}$. Is either of these more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(b) Now let's think of the partial sums:

$$\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty} \qquad \text{vs} \qquad \left\{\sum_{k=0}^{n} (-1)^k a_k\right\}_{n=0}^{\infty}$$

Is either of these sequences more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(c) Now make a conjecture about which infinite series is more likely to converge:

$$\sum_{k=0}^{\infty} a_k \qquad \text{vs} \qquad \sum_{k=0}^{\infty} (-1)^k a_k$$

Remember that $a_k > 0$ for $k = 0, 1, 2, \ldots$, so the only differences are the changes in sign.

Defining Alternating Series, and the Main Result

Ok so hopefully we have convinced ourselves that a series that has terms that alternate in sign might have an "easier"time converging than a series with only positive terms. Let's start with a definition (so that we can continue to refer to these types of series easily), and then move towards the main result.

Definition 8.5.1 Alternating Series.

An infinite series $\displaystyle\sum_{k=0}^{\infty} a_k$ is called an **Alternating Series** when $a_k = (-1)^k|a_k|$ or $a_k = (-1)^{k+1}|a_k|$ for all $k = 0, 1, 2, \ldots$. That is, the sign of the terms alternates:

$$\sum_{k=0}^{\infty} a_k = |a_0| - |a_1| + |a_2| - +\ldots$$

So this is the type of series we're thinking of: the terms perfectly alternate in sign.

> At the beginning of this section, we talked about removing conditions and seeing what happens. We are not fully generalizing to terms that are just different in sign, since we still have a specific format that the terms need to hold to: switching between positive, negative, positive negative, etc. Series that have a less consistent pattern between positive and negative signs are harder to think about, and so we will have to be satisfied with loosening the restrictions on signs of the terms without fully removing any conditions.

Now, we will bring into focus one of the main results about alternating series before investigating these types of series (and how they converge) further.

To lead into this new result, let's remind ourselves of a few things:

1. Remember the Divergence Test. We, specifically, want to remember that, in general, we don't know anything about a series if $\lim_{k\to\infty} a_k = 0$.

2. We have typically been looking at infinite series where we impose a further restriction on the terms: we looked at infinite series with only *positive* terms. We can get a lot more information about these series!

3. Now we are looking at infinite series with a different kind of structure on their terms: the signs alternate. So it won't be a surprise when we get to add information about them!

The big idea that we'll use the extra information about these alternating series (based on how we defined them) to get more information from the limit of the terms. Normally we don't get any information when $\lim_{k\to\infty} a_k = 0$: the series of those terms could converge or diverge, and we can't tell! But with this new structure of the terms (the alternating signs), we'll actually be able to tell something from the limit of the terms being 0.

Let's look at it! We're going to think about visualizing the partial sums.

In Figure 8.2.5, we looked at the partial sums of an infinite series and saw how the terms made up the differences between those partial sums. We're going to think about this same picture, but think about it through the lense of:

1. an alternating series (with terms that alternate in sign) where...

2. the terms of the alternating series approach 0 in the limit: $\lim_{k\to\infty} a_k = 0$. Note that this means that the *size* of the terms must go to 0 in the limit, $|a_k| \to 0$.

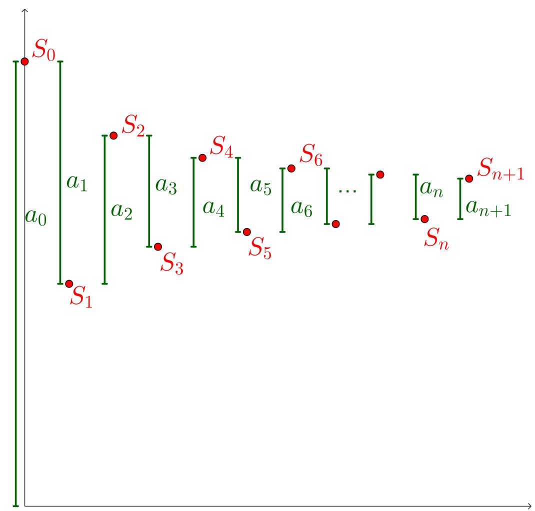


Figure 8.5.2 Partial sums of an alternating series.

Let's note a couple of things:

- All of the "even-indexed"terms (a_0, a_2, a_4, etc.) are positive, while all of the "odd-indexed"terms (a_1, a_3, a_5, etc.) are negative.
- This means that all of the "even-indexed"partial sums are big, while all of the "odd-indexed partial sums"are small. Our sequence of partial sums bounces up to an even-index and bounces down to an odd-index.

  As long as the size of the terms (the size of the differences between partial sums) is decreasing like we have pictured, then each "next"even-indexed partial sum is a bit smaller than the "previous"even-indexed partial sum. The same thing is true for the odd-indexed partial sums.
- The terms themselves represent the distance between these successive partial sums: the difference between S_n and S_{n+1} is the term a_{n+1}.

So, as long as the sizes of the terms are decreasing (or, as long as the distance between partial sums is decreasing consistently), then what happens when the terms (the distance between partial sums) goes to 0?

The even-indexed partial sums and the odd-indexed partial sums approach each other!

Theorem 8.5.3 Alternating Series Test.

If $\sum_{k=0}^{\infty} a_k$ *is an alternating series and the size of the terms* $|a_k|$ *is decreasing, then if* $\lim_{k\to\infty} a_k = 0$ *then* $\sum_{k=0}^{\infty} a_k$ *converges.*

Proof.

This proof will follow the discussion before the statement of the theorem. Mostly, we will just fill in some details and provide some further justification for why what we were noticing must be true.

Let's start with the conditions of the test:

- We are considering an Alternating Series, $\sum_{k=0}^{\infty} a_k$. For our purposes, we'll assume that we have something like

$$\sum_{k=0}^{\infty} a_k = |a_0| - |a_1| + |a_2| - |a_3| + -...$$

  where the even-indexed terms are the positive ones. This could be flipped and it wouldn't make a difference.

- The size of the terms are decreasing. That is, $|a_k| < |a_{k+1}$ for all $k = 0, 1, 2, ...$
- The limit $\lim_{k\to\infty} a_k = 0$. Note that this also means that $\lim_{k\to\infty} |a_k| = 0$.

We're going to show that, under these conditions, the alternating series we're considering must converge. The way that we'll do this is, no surprise, by invoking Theorem 8.1.10 Monotone Convergence Theorem. We're going to do it by considering the partial sums in halves: the even-indexed ones and the odd-indexed ones.

First, consider the sequence $\{S_{2n}\}_{n=0}^{\infty} = \{S_0, S_2, S_4, ...\}$. The difference between successive terms in this sequence (successive even-indexed partial sums) is:

$$S_{2n+2} - S_{2n} = -|a_{2n+1}| + |a_{2n+2}|.$$

Since the terms of the alternating series are decreasing in size, we know that $|a_{2n+2}| < |a_{2n+1}|$, which means that $S_{2n+2} - S_{2n} < 0$, and so $S_{2n} > S_{2n+2}$.

All of this which is to say, $\{S_{2n}\}_{n=0}^{\infty}$ is a decreasing sequence.

We can apply the same reasoning to the sequence $\{S_{2n+1}\}_{n=0}^{\infty} = \{S_1, S_3, S_5, ...\}$. We know the differences between successive odd-indexed partial sums is:

$$S_{2n+3} - S_{2n+1} = -|a_{2n+3}| + |a_{2n+2}|.$$

This time, though, $|a_{2n+2}| > |a_{2n+3}|$ and so the difference is positive: $S_{2n+3} - S_{2n+1} > 0$ which means that $S_{2n+3} > S_{2n+1}$. The sequence $\{S_{2n+1}\}_{n=0}^{\infty}$ is an increasing sequence.

We're getting close! We have monotonic sequences. Now we just need to show bounds, and then we'll show that each of these sequences converges. Then, we'll show that they converge to the same thing.

Getting an upper bound on the odd-indexed partial sums and a lower bound on the even-indexed partial sums is pretty easy. Let's consider subsequent partial sums, S_{2n} and S_{2n+1}.

$$\begin{aligned} S_{2n+1} - S_{2n} &= -|a_{2n+1}| \\ S_{2n+1} - S_{2n} &< 0 \\ S_{2n+1} &< S_{2n} \end{aligned}$$

Ok so this is easy: we can just pick any odd-indexed partial sum to be the lower bound on the even-indexed partial sums, and vice versa.

So S_0 is an upper bound on $\{S_{2n+1}\}_{n=0}^{\infty}$, since every other S_{2n} is less than S_0, and all of S_{2n+1} partial sums are less than S_{2n}:

$$\begin{aligned} &S_0 \geq S_{2n} && \text{for } n = 0, 1, 2, ... \\ &S_0 \geq S_{2n} > S_{2n+1} && \text{for } n = 0, 1, 2, ... \\ &S_0 > S_{2n+1} && \text{for } n = 0, 1, 2, ... \end{aligned}$$

Similarly, we can say the same thing about S_1 being a lower bound for the even-indexed partial sums:

$$\begin{aligned} &S_1 \leq S_{2n+1} && \text{for } n = 0, 1, 2, ... \\ &S_1 \leq S_{2n+1} < S_{2n} && \text{for } n = 0, 1, 2, ... \\ &S_1 < S_{2n} && \text{for } n = 0, 1, 2, ... \end{aligned}$$

So we have shown that the sequences $\{S_{2n}\}_{n=0}^{\infty}$ and $\{S_{2n+1}\}_{n=0}^{\infty}$ are both monotonic and bounded and so both of these sequences must converge.

Now we can show that they converge to the same thing.

Since $\{S_{2n}\}_{n=0}^{\infty}$ converges, let's say that there is some number S_E where

$$\lim_{n\to\infty} S_{2n} = S_E.$$

Similarly, there is a number S_O where

$$\lim_{n\to\infty} S_{2n+1} = S_O.$$

Now we can use the fact that the limit of the terms is 0:

$$\begin{aligned}\lim_{k\to\infty} a_k &= 0\\ \lim_{n\to\infty} a_{2n+1} &= 0\\ \lim_{n\to\infty} (S_{2n+1} - S_{2n}) &= 0\\ S_O - S_E &= 0\end{aligned}$$

So the numbers that these two sequences of partial sums converge to are actually equal to each other.

So, finally, we know that under the conditions we started with, the alternating series must converge.

We can actually get another result really easily from this one! It's about how we might approximate the value that an infinite series converges to.

We know that if a series converges, it's because the limit of the partial sums exists. So we "just"need to find what the real number, S, is when

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k = S.$$

We said earlier, though, that this is pretty hard to do! We can always approximate this limit, though, by looking at a *big* partial sum: if the limit exists, then probably adding up the first 10,000 terms will be a pretty good approximation (since we don't expect the partial sums to change, much, as n gets bigger). But how many terms is enough to give us a good enough approximation?

That's a hard question to answer, until we think about alternating series.

Activity 8.5.2 Approximating an Alternating Series.

Let's look, again, at the picture of the partial sums of an alternating series in Figure 8.5.2. We're going to assume that the series converges, which means that:

- $\lim_{n\to\infty} S_n$ exists.
- $\lim_{n\to\infty} a_n = 0$.

Let's add to our figure.

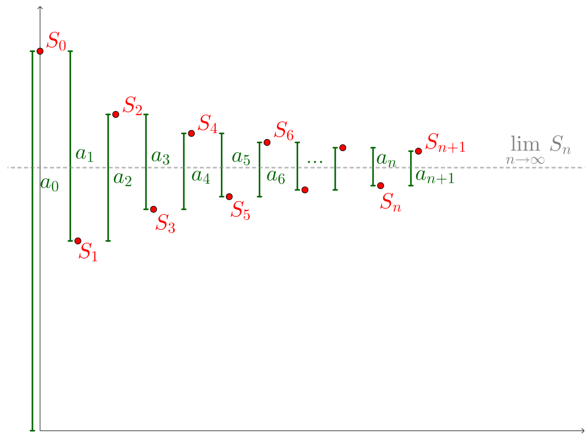


Figure 8.5.4

(a) Why are the even-indexed partial sums sitting above the odd-indexed partial sums?

(b) Why are the even-indexed partial sums sitting above the horizontal line, $\lim_{n\to\infty} S_n$?

(c) Why are the odd-indexed partial sums sitting below the horizontal line, $\lim_{n\to\infty} S_n$?

(d) If we were trying to approximate the value of $\lim_{n\to\infty} S_n$, how can we use the partial sums to build an interval that approximates the value?

Theorem 8.5.5 Approximations of Alternating Series.

If $\displaystyle\sum_{k=0}^{\infty} a_k$ *is a converging alternating series, then the value* S *that the series converges to is bound between consecutive partial sums.*
Another way of saying this is that the partial sum S_n *approximates the actual value of* $\displaystyle\sum_{k=0}^{\infty} a_k$ *with a maximum error of* $|a_{n+1}|$.

Convergence, More Carefully

Let's circle back to an important point from Activity 8.5.1: an alternating series is more likely to converge than its positive-term counterpart.

Let's look at a classic example of this: the alternating harmonic series.

Activity 8.5.3 The Alternating Harmonic Series Converges.

Let's consider the alternating harmonic series, as written below:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + -...$$

(a) First, confirm that this series does converge, using the Alternating

Series Test.

(b) Find $S_6 = \sum_{k=1}^{k=6} \frac{(-1)^{k+1}}{k}$.

(c) Use Theorem 8.5.5 Approximations of Alternating Series to create a bound on this estimation. Report an interval that you know that the actual value that the alternating harmonic series converges to is in.

(d) Use technology to find S_{1000}. Compare this value to $\ln(2)$. Would it surprise you for someone to claim that the alternating harmonic series converges to $\ln(2)$?

So we have some evidence that says that the alternating harmonic series converges to $\ln(2)$. Let's look at another "version" of this "same" series. You'll notice that we're using scare-quotes on "version" and "same," and that's what we're going to investigate.

Activity 8.5.4 The Alternating Harmonic Series Converges (Again).

Now let's consider a new, rearrranged, version of the alternating harmonic series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$$

We can write this in summation notation as:

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \right).$$

(a) First, confirm that all of the terms from the alternating harmonic series will eventually show up in this series. Convince yourself that we are truly adding (and subtracting) all of the same values with just a different order.

(b) Does this new series converge? Check, using the Alternating Series Test.

(c) Add up the first few terms of this series to find the value of a partial sum. You can choose how many terms you add. Does it look like it will also converge to $\ln(2)$?

(d) Use technology to add up many terms of this series. Can you convince yourself what this series converges to?

This is strange! We have two series that seem to be the same thing (one is just a re-arranged version of the other) that both converge, but they converge to different things! One seemingly converges to $\ln(2)$ while the other converges to half of that: $\frac{\ln(2)}{2}$. Very strange!

This doesn't seem to follow the *normal* rules of addition: we lose the normal associative property of addition, where the order or the way that we group terms to add typically doesn't matter. Here it does!

It turns out (and we won't prove this) that this type of convergence happens only for alternating series, and further it only happens for alternating series whose positive-term counterpart diverges.

Definition 8.5.6 Conditional (and Absolute) Convergence.

If $\sum_{k=0}^{\infty} a_k$ is a converging alternating series, then we say that the alternating series $\sum_{k=0}^{\infty} a_k$ **converges conditionally** if $\sum_{k=0}^{\infty} |a_k|$ diverges.

If, instead, the positive-term series $\sum_{k=0}^{\infty} |a_k|$ converges as well, then we say that the alternating series $\sum_{k=0}^{\infty} a_k$ **converges absolutely**.

We won't go into this more, but a very cool result is that if an alternating series converges conditionally, then we can find a rearrangement of the terms that will converge to *literally any real number that you'd like.* We can rearrange the terms in such a way that the series converges to 0 or to π or to anything else you'd like.

But wait, it gets even stranger: you can also rearrange the terms in such a way that the resulting series diverges to ∞ or to $-\infty$. These series really test the limits of what we mean when we say that a series converges, and so I hope that it seems reasonable to classify them separately of the "normally behaved" series.

So now, for a series whose terms alternate in size, we can:

- Classify whether the series converges or not (Theorem 8.5.3 Alternating Series Test).
- Further classify any converging alternating series to see whether the value it converges to is invariant to re-arrangement. (Definition 8.5.6 Conditional (and Absolute) Convergence).
- Approximate the value that the series converges to using the nth partial sum, and also include an error bound on that approximation using the $|a_{n+1}|$ (Theorem 8.5.5).

8.6 Common Series Types

In this section, we'll stop and recap some of the common series types that we should recognize moving forward. We'll look at the structure of these series (mainly the functions defining the *terms* of the series) as well as the convergence criteria for them.

Look back to Activity 8.2.1. We noticed that we were able to find an explicit formula for the nth partial sum, which allowed us to evaluate $\lim_{n\to\infty} S_n$. We noticed this again in Example 8.2.7.

But there are some differences between *why* we were able to find formulas for the nth partial sum in each of these examples. Let's first focus on the infinite series with terms defined by exponential functions.

Geometric Series

We're going to name these series and define them explicitly. The name, **geometric**, comes from the idea of a **geometric mean**: each term in the series is the geometric mean of the term before it and after it.

> The geometric mean is a way of finding a kind of average. Instead of adding up the values and dividing by the number of values, we multiply the values and then take an nth root, where n is the number of things averaged.
>
> For instance, a geometric mean of the numbers 1, 4, and 5 is:
>
> $$\sqrt[3]{1 \cdot 4 \cdot 5} = \sqrt[3]{20}$$
>
> This is approximately 2.714. We can compare that to the arithmetic mean:
>
> $$\frac{1+4+5}{3} = \frac{10}{3}.$$
>
> These are just different kinds of measures of "center" of a list of values, although you might be most familiar with the arithmetic mean.

Definition 8.6.1 Geometric Series.

For real numbers a and r with $a, r \neq 0$, we say that the series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots$$

is a **geometric series**. We call r the **constant ratio** and a the **initial term**.

We noticed in Section 8.2 Introduction to Infinite Series that these kinds of series, with their exponential structure in the terms, make it relatively easy to find patterns or explicit formulas for the partial sums. Since we can find a formula for S_n based on n, we can find $\lim_{n\to\infty} S_n$. This was something we noted then, and said that it was a pretty rare property.

Let's generalize this a bit, and come up with a general formula for these partial sums. Once we have that, we will be able to find out what any geometric series converges to (if it does converge).

Activity 8.6.1 Building a Convergence Formula for Geometric Series.

We're going to think of constructing two different ways of thinking about how much area of of a circle has been shaded. We can pretend we have a circle with area that is 1, where the radius is $r = \sqrt{\frac{1}{\pi}}$, giving

$$A = \pi \left(\sqrt{\frac{1}{\pi}}\right)^2 = 1.$$

Then we can describe the areas we're looking at as almost a percentages of the total area.

(a) We are going to split our circle into two parts, with r amount of the area left unshaded and so $1 - r$ area shaded. We'll shade in some angular sector.

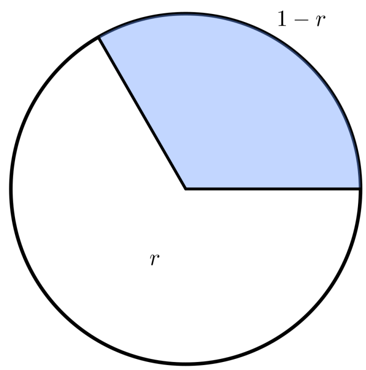


Figure 8.6.2

This part is easy: how much of the area is shaded?

(b) This next step will set the stage for how we think about this problem now: we're going to divide the remaining white area up into the same proportional pieces: we'll shade in a ratio of $1 - r$ of the remaining white space and leave a ratio of r of the white space unshaded.

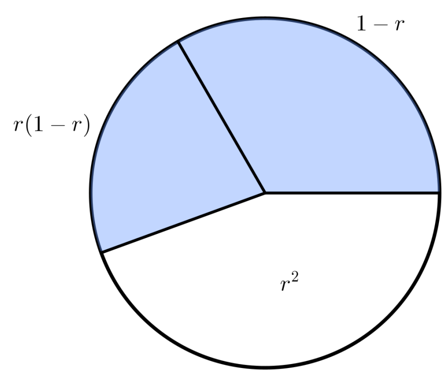


Figure 8.6.3

Can you describe two ways of calculating the total amount of shaded area?

(c) We'll repeat the process: shade in more, where the ratio of shaded area to unshaded area is $1 - r$ to r.

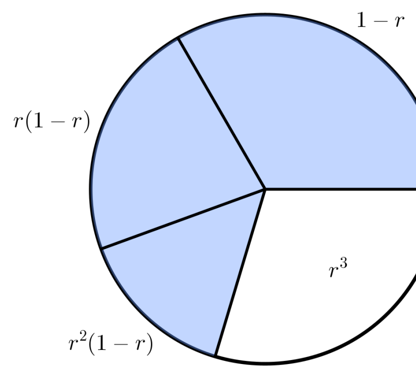


Figure 8.6.4

Can you describe two ways of calculating the total amount of shaded area?

(d) Now we're going to repeat this process until we've done it a total of $n+1$ times.

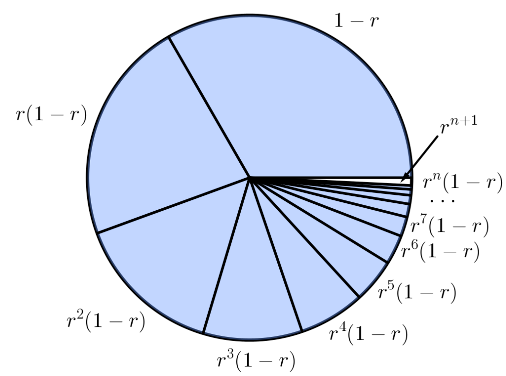


Figure 8.6.5

Can you describe two ways of calculating the total amount of shaded area?

(e) In the limit as $n \to \infty$, how much of the area is shaded in?

Notice that $(1-r)$ is likely a common factor in one of your ways of calculating this area. Convince yourself, then, that:

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{n} r^k &= \lim_{n\to\infty}(1+r+r^2+...+r^n)\\ &= \frac{1}{1-r}\end{aligned}$$

In order for us to move towards a formal statement of the geometric series convergence criteria, we can note that in the above activity, $0 \le r < 1$. We can extend this to negative values, with $|r| < 1$. We also can note that we could scale the area, maybe call it a, to get a similar formula.

Theorem 8.6.6 Geometric Series Convergence Criteria.

A geometric series $\sum_{k=0}^{\infty} ar^k$ *converges to* $\frac{a}{1-r}$ *when* $|r| < 1$ *and diverges if* $|r| \geq 1$.

p-Series

Another type of structure that we can take advantage of is power functions. This way, we can leverage the Integral Test (since antidifferentiating using the Power Rule for Antiderivatives will be easy) to classify whether or not these series converge.

Let's start by naming these. We'll focus on power functions with negative exponents, or reciprocal power functions.

Definition 8.6.7 p-Series.

For a real number p, we say that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

is a p-**series**. We mostly will be concerned about the case where $p > 0$, making the terms of the series be reciprocal power functions.

Now, we just need to think about integration, and the convergence classification comes quickly from there.

Theorem 8.6.8 p-Series Convergence Criteria.

A p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.

Proof.

Let's divide this into four cases: when $p \leq 0$, when $0 < p < 1$, when $p = 1$, and when $p > 1$.

Case 1: $p \leq 0$

Note that for $\frac{1}{k^p}$ with $p < 0$, we can write this as $k^{|p|}$. Now we can consider the limit of the terms, in order to use the Divergence Test.

$$\lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{|p|}$$

Since this limit is non-zero (since it is either ∞ or 1, depending on whether $p = 0$ or not), the series diverges by the Divergence Test.

Case 2: $0 < p < 1$

When $0 < p < 1$, we can apply the Integral Test to the series. It is worth showing that the conditions of the test are met, but this is left up to the reader.

So now we will consider the integral $\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$ as a way of seeing whether the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges or diverges.

$$\begin{aligned}\int_{x=0}^{\infty} \frac{1}{x^p}\, dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x^p}\, dx \\ &= \lim_{t\to\infty} \left(\frac{x^{1-p}}{(1-p)}\right)\Bigg|_{x=1}^{x=t} \\ &= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

We can note that since $0 < p < 1$, that $1 - p > 0$. This means that when $t \to \infty$, $t^{1-p} \to \infty$ as well.

$$\int_{x=0}^{\infty} \frac{1}{x^p}\, dx \;= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = \infty$$

This integral diverges, and so then does the series.

Case 3: $p = 1$

This is the Harmonic Series! This series diverges (Theorem 8.3.2).

Case 4: $p > 1$

We can repeat the proof from *Case 2*, but we will end with a different conclusion based on the sign of the exponent! Let us, again, apply the Integral Test.

Consider the integral $\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$ as a way of seeing whether the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges or diverges.

$$\begin{aligned}\int_{x=0}^{\infty} \frac{1}{x^p}\, dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x^p}\, dx\\ &= \lim_{t\to\infty} \left(\frac{x^{1-p}}{(1-p)}\right)\Bigg|_{x=1}^{x=t}\\ &= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

Now, though, we have $p > 1$ which means that $1 - p < 0$. This means that $t^{1-p} = \frac{1}{t^{|p-1|}}$. So now we will consider the limit, and note that as $t \to \infty$, we get $\frac{1}{t^{|p-1|}} \to 0$.

$$\int_{x=0}^{\infty} \frac{1}{x^p}\, dx \;= \lim_{t\to\infty} \frac{1}{(1-p)t^{|p-1|}} - \frac{1}{1-p} = -\frac{1}{1-p}$$

This integral converges, and so then does the series. We remember, though, that the series converges to something different than the integral, and so we do not know what the series converges to.

Recapping Our Mathematical Objects

It's a good idea to pause and try to make sure we understand what these infinite series are. We have talked a lot about a whole bunch of objects in this chapter so far: infinite sequences, partial sums, sequences of partial sums, infinite series, integrals, limits, etc. We want to make sure that we can keep track of the ways in which we use these and talk about them. The following activity is brief, but can help make sure we understand some of the interactions we've talked about so far.

Activity 8.6.2 (Im)Possible Combinations.

When we have thought about infinite series, we have thought about three different mathematical objects: the sequence of terms of the series, the sequence of partial sums of the series, and the infinite series itself. As a reminder, if we had an infinite series

$$\sum_{k=1}^{\infty} a_k$$

we can say that:

- $\{a_k\}_{k=1}^{\infty}$ is the sequence of terms of the series
- $S_n = \sum_{k=1}^{n} a_k$ is a partial sum and $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series

For each of these three objects—the terms, the partial sums, and the series—we have some notion of what it means for that object to converge or diverge.

Consider the following table of all of the different combinations of convergence and divergence of the three objects. For each combination, decide whether this combination is possible or impossible. If it is possible, give an example of an infinite series whose terms, partial sums, and the series itself converge/diverge appropriately. If it is impossible, give an explanation of why.

Table 8.6.9 (Im)Possible Combinations

$\{a_k\}_{k=1}^{\infty}$	$\{S_n\}_{n=1}^{\infty}$	$\sum_{k=1}^{\infty} a_k$	**(Im)Possible?**	**Example or Explanation**
Converges	Converges	Converges		
Converges	Converges	Diverges		
Converges	Diverges	Converges		
Converges	Diverges	Diverges		
Diverges	Converges	Converges		
Diverges	Converges	Diverges		
Diverges	Diverges	Converges		
Diverges	Diverges	Diverges		

Moving forward, we'll want to commit these families of series to memory, as well as their convergence criteria.

Here's some justification: for an infinite series like $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$, we previously (in Example 8.4.8) compared this series to the integral $\int_{x=0}^{\infty} \frac{1}{x^2+1}\, dx$. This worked well, since we could pretty easily antidifferentiate and conclude that the integral converged.

Now, though, we have another connection to make: doesn't this series *almost* look like a p-Series? It's very close to a reciprocal power function, where the only thing that's "off" is the "+1" in the denominator.

We can hopefully think about changing this example slightly: what about the series $\sum_{k=0}^{\infty} \frac{1}{k^3+1}$? This one could still be compared to the integral $\int_{x=0}^{\infty} \frac{1}{x^3+1}\, dx$, but this integral will be harder for us to integrate. But if we think about this as *almost* a p-Series, then we might still be able to have some intuition about it's behavior: it *looks* kind of like a converging p-series, so shouldn't it also converge?

Our next section will develop this kind of comparison, where instead of comparing an "integral" in the discrete context to one in the continuous context (like we do in the Integral Test), we can compare an "integral" in the discrete context to a similar one in a similar context.

8.7 Comparison Tests

So far, our strategies for thinking about infinite series have been focused around drawing a connection between the infinite series we care about and some other mathematical object:

- The Divergence Test draws a connection (even thought it's a limited one) between the terms of the series and the series itself.
- The Alternating Series Test draws a (stronger) connection between the terms of, specifically, an Alternating Series and the series itself.
- The Integral Test draws a connection between the series and a corresponding integral.

Now we'll work on building the most important series convergence test mechanism: we'll draw a link between the series we care about and some other series that we already know about.

This is helpful for three reasons:

1. We already have a couple of types of series that we know about (Section 8.6), and we can keep adding to that list.
2. We can take advantage of similar structure or common term formulas when we see them to essentially say, "This series kind of looks like one that I recognize. I wonder if they act the same?"
3. We don't always have to integrate things using the Integral Test! Integrating can be hard!

Comparing Partial Sums

We're going to start by trying to do the same thing we did when we build the Integral Test: show that the partial sums are monotonic and bounded and then make use of the Monotone Convergence Theorem.

Activity 8.7.1 Comparisons to Bound Partial Sums.

This activity is mostly going to be thinking about proof mechanisms, and so it might be helpful to review Activity 8.4.1 Integrals and Infinite Series. If you want to see more, then the proof of Integral Test will provide some further details on why the inequalities we built were useful.

(a) In the Integral Test, how did we guarantee that our sequence of partial sums was monotonic?

(b) How did we know that, as long as the corresponding integral converged, then our sequence of partial sums was bounded?

(c) How did we know that, as long as our integral diverged, then our sequence of partial sums had to diverge as well?

(d) What happens if we swap out the integral we're connecting our series to with a different series?

For these inequalities to be useful, what do we need to be true?

This is it! We have everything we need to construct another link: this time between the infinite series we care about and another infinite series that we think acts the same.

To illustrate what we've done, let's think about two sequences of terms: $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{i}nfty$ where $a_k \leq b_k$ for $k \geq 0$. We can think of the graphs, pictured below.

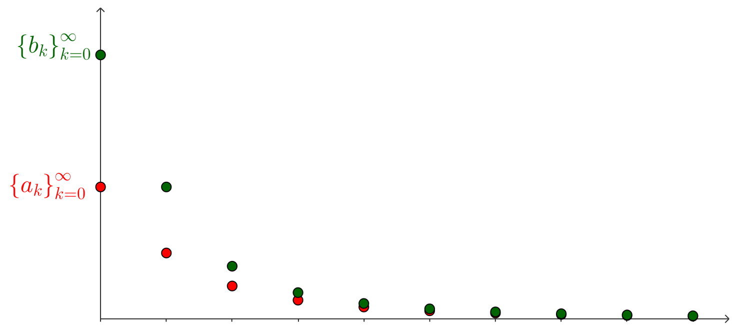


Figure 8.7.1 The "smaller" sequence of terms $\{a_k\}_{k=0}^{\infty}$ graphed alongside the "bigger" sequence of terms $\{b_k\}_{k=0}^{\infty}$.

Now we can think about the graphs of these partial sums. Let's first think about what might happen if the series $\sum\limits_{k=0}^{\infty} b_k$ converges. We'll plot the partial sums, and the sequence of partial sums has to converge to something. But then we can think about the sequence of partial sums from the a_k terms: the smaller terms, of course, will build smaller partial sums.

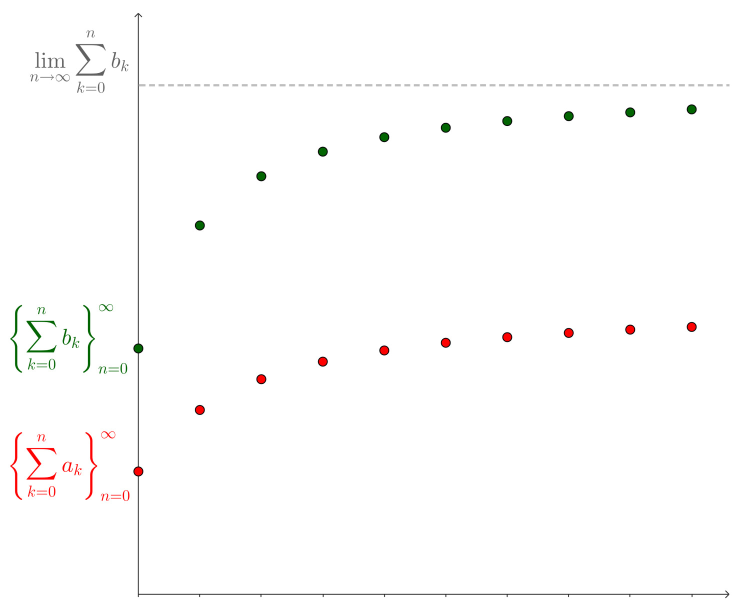


Figure 8.7.2 Comparison of partial sums when $\sum\limits_{k=0}^{\infty} b_k$ converges.

So we can pretty easily use $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} b_k$ as an upper bound on the sequence

$\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty}$! And now we can say that the sequence of partial sums for $\sum_{k=0}^{\infty} a_k$ is monotonic and bounded, and so it must converge.

Then we can ask about the diverging case. This time, we'll say that the smaller series, $\sum_{k=0}^{\infty} a_k$ diverges.

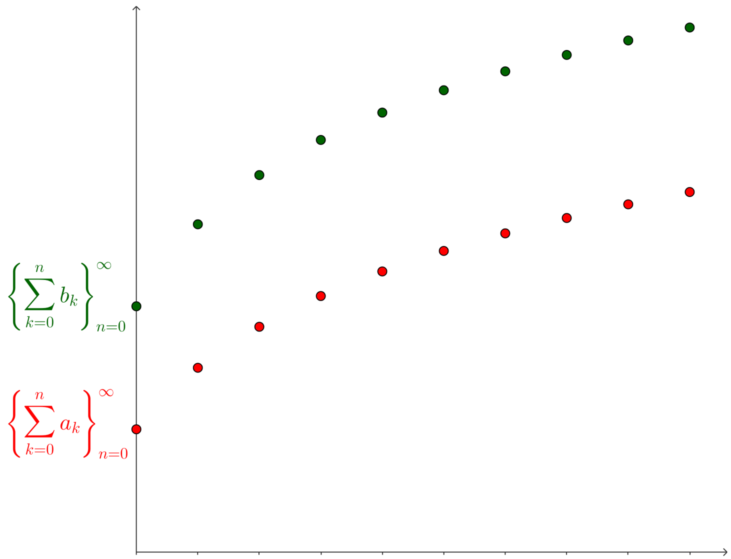


Figure 8.7.3 Comparison of partial sums when $\sum_{k=0}^{\infty} a_k$ diverges.

We can think that this "smaller" sequence of partial sums is "pushing" the "larger" sequence of partial sums up to infinity with it.

These two cases make up our first comparison mechanic between two infinite series.

Theorem 8.7.4 Direct Comparison Test.

If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are infinite series with positive terms ($a_k > 0$ and $b_k > 0$ for $k \geq 0$) with the ordering $a_k \leq b_k$ for $k \geq 0$, then:

- *If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ also diverges.*
- *If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ also converges.*

At the end of Section 8.6 Common Series Types, we referenced the series:

$$\sum_{k=0}^{\infty} \frac{1}{k^3+1}.$$

Let's use our new test!

Example 8.7.5

Does the series $\sum_{k=0}^{\infty} \frac{1}{k^3+1}$ converge or diverge? How do you know?

Hint 1. If we're going to use our new Direct Comparison Test, we need to identify two things:

1. Some intuition on whether we think our series converges or not.
2. An appropriate series to compare to. Likely, this will be either a Geometric Series or a p-Series, since we have clear convergence criteria for each of those.

We can do this in any order: sometimes we might use the structure of the series we're looking at to give us a good candidate to compare to, and that might tell us the behavior we think we're looking for. Other times we might have good intuition about convergence/divergence of the series which will tell us whether we need to find a series that is smaller or larger to compare to.
What do you think? Do we have a suitable comparison?

Hint 2. Compare to the p-Series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. Here, $p = 3$. Based on this, do we need to show that $\frac{1}{k^3+1}$ is greater than or less than $\frac{1}{k^3}$?
Show this!
Does the change in the starting index matter?

Solution. Let's compare our series to the converging p-series with $p = 3$:

$$\sum_{k=0}^{\infty} \frac{1}{k^3+1} \sim \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

We want to show that $\frac{1}{k^3+1} \leq \frac{1}{k^3}$ for $k \geq 1$.
Let's start with comparing the denominators, and move from there.

$$k^3 + 1 > k^3 \text{ for any value of } k$$

Now we can think about reciprocals:

$$\frac{1}{k^3+1} < \frac{1}{k^3} \text{ for all } k \geq 1.$$

This means that, since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, then $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ must also converge.
We can just add the term where $k = 0$ to get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k^3+1} &= \overbrace{\frac{1}{0^3+1}}^{k=0} + \sum_{k=1}^{\infty} \frac{1}{k^3+1} \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k^3+1} \end{aligned}$$

So then we know that our series, $\sum_{k=0}^{\infty} \frac{1}{k^3+1}$ must converge.

A quick note: in this example, we thought about building the link

$$\sum_{k=1}^{\infty} \frac{1}{k^3+1} \sim \sum_{k=1}^{\infty} \frac{1}{k^3}$$

and then arguing that changing the index to start at $k = 0$ by adding in a single term wouldn't change the behavior of the infinite series.

In general, we don't need to add in that last argument. We can assume from here on out that changing our starting index won't impact the behavior of the series (as long as we're avoiding things like division by 0 and such). So showing that $\frac{1}{k^3+1} < \frac{1}{k^3}$ for $k \geq 1$ is enough for us!

(Un)Helpful Comparisons

Activity 8.7.2 (Un)Helpful Comparisons.

We're going to consider a handful of infinite series, here:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(3^k)}$
2. $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k+1}}{k-2}$

(a) Pick a series that is reasonable to use as a comparison for each of the series listed. Remember, we want:

- A series that is recognizable (probably a Geometric Series or a *p*-Series), so that we know the behavior of it: we need to know whether the series we're comparing to converges or diverges!
- A series that is similar enough to the series we're working with that we can construct an inequality comparing the term structure. It can be hard to compare functions that are seemingly unrelated to each other!
- A series that has terms that are either larger or smaller than our series, depending on whether we are trying to show that our series converges or diverges.

(b) Build the comparison from the series we start with to the one you picked. What kinds of conclusions can you make?

(c) We're going to change the series we're considering to two slightly different series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{3^k}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k-1}}{k+2}$

How do these small changes impact the inequalities you built?

(d) How do these changes in the inequalities change the conclusions we can draw from the Direct Comparison Test?

(e) What do you *think* is happening with these series: do you think that these small changes are enough to change the behavior of the series (i.e. whether it converges or diverges)?

There are some ways around this. When we find a comparison that we think is reasonable, but has the wrong inequality for us to make a conclusion, we can do one of two things:

1. Try a different comparison.
2. Try a different test.

We'll address the second strategy momentarily.

Note 8.7.6

In fact, we might argue that we could start with a different test! It might be more useful to skip using the Direct Comparison Test in favor of just using the Limit Comparison Test, which we'll see soon, or the Ratio Test or Root Test, which we'll talk about in the next section.
I think that the only real reason to think about the Direct Comparison Test is that

- Sometimes it can be pretty quick to use. Sometimes.
- It is much easier to see *why* it works, since we can make a nice argument about monotonic and bounded sequences of partial sums. From there, the Limit Comparison Test (coming up next) isn't a big jump conceptually. But it might be harder to start with.

So let's think about one of the series from Activity 8.7.2:

$$\sum_{k=1}^{\infty} \frac{2k}{3^k}$$

We, intuitively, tried to compare this a related geometric series:

$$\sum_{k=1}^{\infty} \frac{2k}{3^k} \sim \underbrace{\sum_{k=1}^{\infty} \frac{2}{3^k}}_{\text{converges}}$$

Unfortunately, we weren't able to make this connection, since $\dfrac{2k}{3^k} \geq \dfrac{2}{3^k}$ for $k = 1, 2, 3 \ldots$

We could, instead, still think that this is a converging series but compare it to the converging p-series, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$. We'll actually compare it to $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2}$, but this will converge as well (since the coefficient will just scale the value that the series converges to).

Note, first, that $3^k \geq k^3$ for $k \geq 3$. At $k = 3$ we have $3^3 = 3^3$, but after this intersection point, the exponential will be larger than the power function.

$$\begin{aligned}
3^k &\geq k^3 \text{ for } k = 3, 4, 5, \ldots \\
\frac{1}{3^k} &\leq \frac{1}{k^3} \text{ for } k = 3, 4, 5 \ldots \\
\frac{2k}{3^k} &\leq \frac{2k}{k^3} \text{ for } k = 3, 4, 5 \ldots
\end{aligned}$$

$$\frac{2k}{3^k} \leq \frac{2}{k^2} \text{ for } k = 3, 4, 5...$$

And there we go! We have $\frac{2k}{3^k} \leq \frac{2}{k^2}$ for $k = 3, 4, 5...$ and $\sum_{k=3}^{\infty} \frac{2}{k^2}$ converges, so then $\sum_{k=3}^{\infty} \frac{2k}{3^k}$ must also converge.

Of course, we can add two more real numbers (when $k = 1$ and $k = 2$), and the resulting series will still converge. So, $\sum_{k=1}^{\infty} \frac{2k}{3^k}$ converges, just like we originally thought.

Let's not get ahead of ourselves: this quick change from one comparison to another is rarely easy! Here are some questions we can ask:

- Where did $\sum_{k=1}^{\infty} \frac{1}{k^2}$ come from? How could we have anticipated that as being useful?
- Didn't we think, originally, that this was *almost* a geometric series? Why are we switching to comparing to a p-series?
- Is there some systematic way for us to think about what to compare to? Something other than appealing to growth rates?

These questions will remain largely unanswered in this text, other than the following (unhelpful) solution: getting good at thinking about inequalities and comparisons means that inequalities and comparisons become easier.

This is not intended to be a self-indulged brag: *I am not that good at thinking about inequalities and comparisons either!* It's just meant to have you confront the fact that this test, while understandable, is not always useful in practice. There are better paths forward!

Let's follow those instead.

Limit Comparison

Let's revisit some of our intuition from earlier. When we talked about the series in Example 8.7.5, we made the claim that it probably acted like the p-series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. Why did we choose this series?

Similarly, how did we pick our comparisons in Activity 8.7.2?

We had thought about what parts of the functions defining the terms of the series would take over as $k \to \infty$. We were thinking about which parts of these terms ends up dominating, in behavior, over the other parts.

We're thinking about limits, really!

We're going to use this intuition that we have about limits and relative growth rates to come up with another way of thinking about whether two things act similarly. There are a few ways that we could approach this, but it will be useful to think about this comparison of relative growth rates as ratios.

What if we think about the ratio of the functions defining the terms of the series we're comparing, and see how this ratio acts in the limit as $k \to \infty$?

Activity 8.7.3 Ratios for Comparison.

Let's start with some functions: we'll consider $f(x)$ and $g(x)$ as two functions that are continuous when $x \geq 0$ with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

All of this is so that we can think about $\dfrac{f(x)}{g(x)}$ and know that we have an indeterminate form. We could put the requirement of differentiability on these functions (so that we could think about L'Hôpital's Rule), but we don't need to do that.

We're going to now consider the limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

(a) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $g(x) \to 0$ with a faster growth rate than $f(x)$ does? In this case, we might say that:

$$f(x) >> g(x).$$

(b) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $f(x) \to 0$ with a faster growth rate than $g(x)$ does? In this case, we might say that:

$$g(x) >> f(x).$$

(c) If the functions $f(x)$ and $g(x)$ eventually act equivalently, then what does the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

(d) If the function $f(x)$ eventually acts like some scaled version of $g(x)$, then what does the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

So we have three outcomes of our comparisons using ratios of functions in the limit:

1. The numerator function could, eventually, be so much larger than the denominator function that the limit of the ratio is infinite.
2. The denominator could, eventually, be so much larger than the numerator function that the limit of the ratio is zero.
3. The numerator and denominator could, eventually, act so similarly to each other (or like scaled versions of each other) that the limit is some real number that isn't 0.

This is the motivation for the next comparison test! We'll just think about the functions defining the terms of a series instead, and we'll make conclusions about the series themselves instead of the functions defining the terms.

Theorem 8.7.7 Limit Comparison Test.

If $\displaystyle\sum_{k=0}^{\infty} a_k$ *and* $\displaystyle\sum_{k=0}^{\infty} b_k$ *are infinite series with positive terms (*$a_k > 0$ *and*

$b_k > 0$ for $k \geq 0$), then we can consider $\lim\limits_{k\to\infty} \frac{a_k}{b_k}$.

- *If* $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = 0$, *then:*
  - *If* $\sum\limits_{k=0}^{\infty} a_k$ *diverges, then* $\sum\limits_{k=0}^{\infty} b_k$ *diverges as well.*
  - *If* $\sum\limits_{k=0}^{\infty} b_k$ *converges, then* $\sum\limits_{k=0}^{\infty} a_k$ *converges as well.*
- *If* $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = \infty$, *then:*
  - *If* $\sum\limits_{k=0}^{\infty} a_k$ *converges, then* $\sum\limits_{k=0}^{\infty} b_k$ *converges as well.*
  - *If* $\sum\limits_{k=0}^{\infty} b_k$ *diverges, then* $\sum\limits_{k=0}^{\infty} a_k$ *diverges as well.*
- *If* $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = L$ *where L is some non-zero real number, then* $\sum\limits_{k=0}^{\infty} a_k$ *and* $\sum\limits_{k=0}^{\infty} b_k$ *will either both converge or both diverge.*

Example 8.7.8

For each of the following infinite series, try to select an appropriate comparison series, and then apply a comparison test to make conclusions about whether the series converges or diverges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{1/k}}{\sqrt{k^2+k}}$

Hint 1. What happens, in the limit as $k \to \infty$ to $e^{1/k}$? How does $\sqrt{k^2+k}$ act in the limit: does the k term influence much, compared to k^2?

Hint 2. One of the comparisons we can try is to link the behavior of these two series:

$$\sum_{k=1}^{\infty} \frac{e^{1/k}}{\sqrt{k^2+k}} \sim \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2}} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Solution. In order to link these series, we'll apply a limit comparison test:

$$\sum_{k=1}^{\infty} \frac{e^{1/k}}{\sqrt{k^2+k}} \sim \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2}} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

So let's investigate the limit of the ratio of the terms.

$$\lim_{k\to\infty} \frac{\left(\frac{e^{1/k}}{\sqrt{k^2+k}}\right)}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} \left(\frac{e^{1/k}}{\sqrt{k^2+k}}\right)\left(\frac{k}{1}\right)$$

$$\begin{aligned}
&= \lim_{k\to\infty} \frac{ke^{1/k}}{\sqrt{k^2+k}} \\
&= \left(\underbrace{\lim_{k\to\infty} e^{1/k}}_{\to e^0=1} \right) \left(\lim_{k\to\infty} \frac{k}{\sqrt{k^2+k}} \right) \\
&= \lim_{k\to\infty} \frac{k}{\sqrt{k^2+k}}
\end{aligned}$$

This limit is one that we can think about using Theorem 1.4.7: we can apply the limit in the denominator under the root, and notice that the whole thing is really dependent on the behavior of k^2.

We'll apply a technique that is used in the proof of the theorem.

$$\begin{aligned}
\lim_{k\to\infty} \frac{k}{\sqrt{k^2+k}} &= \lim_{k\to\infty} \frac{k}{\sqrt{k^2\left(1+\frac{k}{k^2}\right)}} \\
&= \lim_{k\to\infty} \frac{k}{\sqrt{k^2}\sqrt{1+\frac{1}{k}}} \\
&= \lim_{k\to\infty} \frac{k}{k\sqrt{1+\frac{1}{k}}} \\
&= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k}}}
\end{aligned}$$

Now, since $\dfrac{1}{k} \to 0$ as $k \to \infty$ we end up with:

$$\lim_{k\to\infty} \frac{\left(\frac{e^{1/k}}{\sqrt{k^2+k}}\right)}{\left(\frac{1}{k}\right)} = 1.$$

Conclusions: Since we're comparing our series to the diverging p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, and the limit comparison test says that these two infinite series must have the same behavior (since the limit of the ratio of the term functions was 1), then we can conclude that the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{e^{1/k}}{\sqrt{k^2+k}}$ also diverges.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$

Hint. This is a hard one to come up with a reasonable comparison. Try writing out some terms and getting a feel for what kinds of things are happening structurally:

- Is there something you can describe, recursively, about how we get from one term in the series to the next?
- Are there consistent operations that we're applying to terms?

- Does this remind you of anything?

Solution. We can compare this to a converging geometric series, $\sum_{k=0}^{\infty} \frac{1}{2^k}$. So, we want to draw the following link:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sim \sum_{k=0}^{\infty} \frac{1}{2^k}.$$

The limit comparison test follows as such:

$$\lim_{k\to\infty} \frac{\left(\frac{1}{k!}\right)}{\left(\frac{1}{2^k}\right)} = \lim_{k\to\infty} \frac{2^k}{k!}$$

We cannot use L'Hopital's Rule here, since $k!$ is not a continuous function for real numbers (since it only takes in non-negative integer inputs) and so it is not differentiable.

We can, instead, appeal to growth rates: $k!$ approaches infinity much faster than 2^k, and so this fraction has a much larger denominator.

$$\lim_{k\to\infty} \frac{2^k}{k!} = 0.$$

Conclusions: By the Limit Comparison Test, this means that $\frac{1}{k!} << \frac{1}{2^k}$, and so, since $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is a converging geometric series, then the infinite series $\sum_{k=0}^{\infty} \frac{1}{k!}$ must also converge.

There isn't anything special about the series we're comparing to: We could have used any mix of the following comparisons to find the same conclusions:

- $\sum_{k=1}^{\infty} \frac{e^{1/k}}{\sqrt{k^2+k}} \sim \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$
- $\sum_{k=0}^{\infty} \frac{1}{k!} \sim \sum_{k=1}^{\infty} \frac{1}{k^2}$
- $\sum_{k=0}^{\infty} \frac{1}{k!} \sim \sum_{k=1}^{\infty} \frac{1}{10^k}$

There's no magic series to compare to: one of the nice things about the limit comparison test is that we can compare a series to another one not based on similar structure, but based on intuited behavior: if you think a series converges, compare it to a converging p-series if you'd like!

There certainly are some instances where it is reasonable to pick a specific series to compare to based on the structure.

Example 8.7.9

Consider the following series:

$$\sum_{k=1}^{\infty} \frac{2k^2 - k + 3\sqrt{k}}{3k^{7/3} + 4k^{5/3} - k^{2/5}}.$$

Perform a test and state a conclusion about whether or not this series converges.

Hint 1. There are a lot of power functions here! Which ones do you think are most important in deciding how quickly the terms approach 0?

Hint 2. This isn't a p-Series, but it might act like one. Compare it to a relevant p-Series!

Solution. We can note that the numerator is really driven by the quadratic term, k^2, while the denominator's behavior is determined by $k^{7/3}$, the power function with the highest exponent. We can make the following comparison:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2k^2 - k + 3\sqrt{k}}{3k^{7/3} + 4k^{5/3} - k^{2/5}} &\sim \sum_{k=1}^{\infty} \frac{k^2}{k^{7/3}} \\ &\sim \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}\end{aligned}$$

Note that this is a diverging p-series, and so, in performing this comparison, we think that our series also converges. Let's show this using a limit comparison.

$$\begin{aligned}\lim_{k\to\infty} \frac{\left(\frac{1}{k^{1/3}}\right)}{\left(\frac{2k^2-k+3\sqrt{k}}{3k^{7/3}+4k^{5/3}-k^{2/5}}\right)} &= \lim_{k\to\infty} \left(\frac{1}{k^{1/3}}\right)\left(\frac{3k^{7/3} + 4k^{5/3} - k^{2/5}}{2k^2 - k + 3\sqrt{k}}\right) \\ &= \lim_{k\to\infty} \frac{3k^{7/3} + 4k^{5/3} - k^{2/5}}{2k^{7/3} - k^{4/3} + 3k^{5/6}} \\ &= \frac{3}{2}\end{aligned}$$

e*Conclusion* : Since this limit is a non-zero real number, we can conclude that our two series have the same behavior. So, since $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ diverged, then we know that the series $\sum_{k=1}^{\infty} \frac{2k^2 - k + 3\sqrt{k}}{3k^{7/3} + 4k^{5/3} - k^{2/5}}$ must also diverge.

This example is not unique! Note that out selection of the p-series to use as comparison is based on what we know will happen in the limit that we eventually use in the test itself!

Theorem 8.7.10 Rational Comparison Theorem.

If a_k is a rational function of k, $a_k = \frac{p(k)}{q(k)}$ where both $p(k)$ and $q(k)$ are polynomial functions, then:

- *If* $\deg(q(k) - p(k)) > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges.*
- *If* $\deg(q(k) - p(k)) \leq 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Note 8.7.11

We can extend this result pretty easily by loosening up the "rational function" requirement. If we have combinations of power functions (even with non-integer exponents), this works as well!

So we have a great tool for analyzing series that look similar to p-series: as long as the most aggressive pieces of our terms are defined by power functions, then we can connect the series to the relevant p-series and use the same convergence criteria through arguments about the degrees of these power functions.

What about infinite series that are seemingly not connected to a p-series? We saw other series in this section that acted more like a geometric series:

- $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(3^k)}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{3^k}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$

We can set up some comparisons, like we did in this section, for each of these series individually. But is there a result similar to Theorem 8.7.10 for series that act like a geometric series?

It turns out that the answer is a resounding, "Yes!" We just need to think about what aspects of a geometric series we're looking for.

8.8 The Ratio and Root Tests

We have just learned about one of the big and important tools that are used for testing convergence. The Limit Comparison Test is super useful for rational functions and things that act like p-Series, since these power functions and related function types behave so nicely, and are so friendly to work with, in the end behavior limit. The algebra typically works well, and we can analyze the limits pretty easily.

In this section, we're going to try to draw a similar connection between our other family of common series, the Geometric Series, and series that act similarly. So for us to begin, we want to think about what it might mean for a series to have terms that act similarly to the terms of a geometric series.

Activity 8.8.1 Reminder about Geometric Series.

We are going to build some convergence tests that try to link some infinite series to the family of geometric series and show that even though a series is *not* geometric, it might act enough like one to be considered "eventually geometric-ish."

But first, what does it mean for a series to be a geometric series?

(a) Describe a defining characteristic of a geometric series. What makes it geometric?

(b) Can you describe this characteristic in another way? For instance, if you described a geometric series using a characteristic about the Explicit Formula, can you describe the same characteristic in the context of the Recursion Relation instead? Or vice versa?

(c) Write out a generalized and simplified form of the term a_k of a geometric series explicitly and recursively. In each case, solve for r, the ratio between terms.

Eventually Geometric-ish

We're going to use these two guiding features of a geometric series to determine if a series is *geometric-ish*. That is, if a series is not actually a geometric series, do the terms act like terms from a geometric series in the limit? Is there some *eventually* (almost) constant ratio between consecutive terms? Is there one in the limit as $k \to \infty$? If the terms aren't actually exponential functions of k, do they kind of act like that in the limit as $k \to \infty$? If so, shouldn't they act like geometric series and converge with the same criteria?

Theorem 8.8.1 Root Test.

Let $\displaystyle\sum_{k=0}^{\infty} a_k$ *be an infinite series with* $a_k > 0$ *for* $k \geq 0$ *and consider* $\displaystyle\lim_{k\to\infty} \sqrt[k]{a_k}$.

- *If there is some real number* r *with* $r = \displaystyle\lim_{k\to\infty} \sqrt[k]{a_k}$ *and* $0 \leq r < 1$, *then the series* $\displaystyle\sum_{k=0}^{\infty} a_k$ *converges.*

- *If there is some real number* r *with* $r = \lim_{k\to\infty} \sqrt[k]{a_k}$ *and* $r >$ *or if* $\lim_{k\to\infty} \sqrt[k]{a_k}$ *does not exist, then the series* $\sum_{k=0}^{\infty} a_k$ *diverges.*
- *If* $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$ *then the Root Test fails and is inconclusive.*

Theorem 8.8.2 Ratio Test.

Let $\sum_{k=0}^{\infty} a_k$ *be an infinite series with* $a_k > 0$ *for* $k \geq 0$ *and consider* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

- *If there is some real number* r *with* $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *and* $0 \leq r < 1$, *then the series* $\sum_{k=0}^{\infty} a_k$ *converges.*
- *If there is some real number* r *with* $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *and* $r >$ *or if* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *does not exist, then the series* $\sum_{k=0}^{\infty} a_k$ *diverges.*
- *If* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$ *then the Ratio Test fails and is inconclusive.*

So these tests are good and fine, but when do we use them? How do we know that they can be helpful? The key is to notice behavior in the terms of a series that look "kind of" geometric: we're looking for k up in the exponent on things and we're looking for repeated multiplication.

Activity 8.8.2 When Are These Tests Useful?

We're going to look at a couple of small examples where we can re-write some expressions into friendlier forms, and try to connect these re-writing strategies to the Ratio and Root Tests.

(a) Re-write the following expression into a friendlier form. Explain why this new form is friendlier.

$$\sqrt[k]{\frac{2^{k+1}}{7^{3k}}}$$

(b) Re-write the following expression into a friendlier form. Explain why this new form is friendlier.

$$\sqrt[k]{\frac{(k+1)^k}{4^{2k+1}}}$$

(c) Re-write the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{(5^{k+2})(6^{k-3})}{(5^{k-1})(6^{k+1})}$$

(d) Re-write the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{103!}{99!}$$

(e) Re-write the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{(2k+4)!}{(2k+2)!}$$

(f) Why do you think the Ratio Test especially will be useful for series whose terms include factorials and exponentials?

Why do you think the Root Test will be useful for series whose terms include exponentials and functions raised to functions (of k)?

Example 8.8.3

For each infinite series, apply one of the Ratio or Root tests and interpret the conclusions of the test.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2}{(k+1)^k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln(k+1)}{e^{2k}}$

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(k+2)}{(k!)(3^k)}$

Hint. This is an alternating series! We can show that this series converges using the Alternating Series Test, and so we really need to test for absolute convergence.

That works perfectly, though, since the Ratio and Root tests only test series with positive terms. So test the series:

$$\sum_{k=0}^{\infty} \frac{(k+2)}{(k!)(3^k)}$$

Chapter 9

Power Series

9.1 Polynomial Approximations of Functions

Before we start, it might be helpful to remind ourselves of the way we have used linear functions to approximate other functions in Section 4.6 Linear Approximations.

What Do We Want From a Polynomial Approximation?

It's always good to lay out our expectations clearly. We want to make sure that, when we create this new *thing*, that we have some clear idea of what we're trying to accomplish in its creation. So let's start with our Linear Approximation of a Function. What were the important properties of this linear function we created?

1. The function's output, $f(x)$, matched the output from the linear approximation, $L(x)$, at the center $x = a$.

$$\begin{aligned} L(x) &= f'(a)(x-a) + f(a) \\ L(a) &= f'(a)(a-a) + f(a) \\ & f(a) \end{aligned}$$

2. The function's first derivative, $f'(x)$, matched the slope of the linear approximation, $L'(x)$, at $x = a$.

$$\begin{aligned} L'(x) &= \frac{d}{dx}\left(f'(a)(x-a) + f(a)\right) \\ &= f'(a) \\ L'(a) &= f'(a) \end{aligned}$$

This seems like a good structure! We have $f(a) = L(a)$, giving us that nice intersection between our approximation and the function we're approximating at the center. Then we used the slope of the function to estimate how our approximating function should move away from that center. The only real problem is that our function probably doesn't have a *constant* slope, while a constant slope is the defining characteristic of the line we're using.

So how do we extend this, then, into a better approximation? Well, an easy next step is to make the slope of our approximating function change as we move away from the center, $x = a$. That way, maybe the slopes change in a way that's similar to the slopes of the actual function, and our approximating curve (not a line anymore) can follow the function for a bit longer.

What we're saying is that we want a function where the second derivative is $f''(a)$, just like our first derivative matched $f'(a)$. Instead of using letters like L for linear and Q for quadratic, let's just call these approximations by their function type (polynomials!) and degree. So $p_1(x) = f'(a)(x-a) + f(a)$, the first-degree polynomial approximation.

Now, to find $p_2(x)$, we're going to add a term to the first-degree polynomial:

$$p_2(x) = \boxed{}(x-a)^2 + f'(a)(x-a) + f(a)$$

Let's differentiate this function twice, and force it to match $f''(a)$ at $x = a$.

$$\begin{aligned} p_2(x) &= \boxed{}(x-a)^2 + f'(a)(x-a) + f(a) \\ p_2'(x) &= 2(\boxed{})(x-a) + f'(a) \\ p_2''(x) &= 2(\boxed{}) \end{aligned}$$

What do we need to fill in the blank to make this match $f''(a)$?

$$\begin{aligned} p_2''(x) &= 2\left(\frac{f''(a)}{2}\right) \\ &= f''(a) \end{aligned}$$

So we get:

$$p_2(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a).$$

What if we wanted a higher degree? Like, $p_3(x)$? Let's repeat the same process and see what happens!

$$\begin{aligned} p_3(x) &= \boxed{}(x-a)^3 + \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a) \\ p_3'(x) &= 3(\boxed{})(x-a)^2 + f''(a)(x-a) + f'(a) \\ p_3''(x) &= 3(2)(\boxed{})(x-a) + f''(a) \\ p_3'''(x) &= 3(2)(\boxed{}) \end{aligned}$$

If we, again, want this third derivative to match $f'''(a)$ (so that the rate at which the slope changes as we move away from the center changes in the same way that it does on f...whew, that is going to be hard to interpret!), then we need to fill the blank in with $\dfrac{f'''(a)}{3(2)}$.

How Do We Build a Polynomial Approximation?

Let's jump into a generalization of what we've just done. Answer these few questions with yourself, just to make sure you can see where we're going:

1. Why, in the coefficients for each term, do we have the derivative that matches the degree of the term? (First derivative for the first degree term, second derivative for the second degree term, and third derivative for the third degree term.)

2. Why, in the coefficients for each term, do we divide? What are we dividing by, and why do we need these numbers specifically?

3. If we add a 4th, 5th, and 6th term, what will we divide those 4th, 5th, and 6th derivatives ($f^{(4)}(a)$, $f^{(5)}(a)$, and $f^{(6)}(a)$) by?

Definition 9.1.1 Polynomial Approximation.

If $f(x)$ is a function that is n-times differentiable at $x = a$ (that is, the function/derivative values $f(a)$, $f'(a)$, $f''(a)$, ..., $f^{(n)}(a)$ all exist), then the nth degree **polynomial approximation of** $f(x)$ **centered at** $x = a$ is:

$$\begin{aligned} p_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

Activity 9.1.1 Build a Polynomial.

We're going to use the formula in Definition 9.1.1 to construct two different polynomials that approximate two different approximations. Then, we'll use them to approximate things!

(a) We're going to start with approximating the function $f(x) = \sin(x)$ centered at $x = 0$. Let's choose to look at a 5th degree polynomial.

This means we'll need to find the first five derivatives of $f(x) = \sin(x)$. Then, we'll evaluate our function and the five derivatives at the center. After that, we can divide by the relevant factorial in order to create the coefficients of our polynomial.

Fill out the following chart to produce these coefficients.

Table 9.1.2 Coefficients for Polynomial Approximation

k	$f^{(k)}(x)$		$f^{(k)}(a)$		$\frac{f^{(k)}(a)}{k!}$
$k=0$	$f(x) = \sin(x)$		$f(0) =$		
$k=1$	$f'(x) =$		$f'(0) =$		
$k=2$	$f''(x) =$		$f''(0) =$		
$k=3$	$f'''(x) =$		$f'''(0) =$		
$k=4$	$f^{(4)}(x) =$		$f^{(4)}(0) =$		
$k=5$	$f^{(5)}(x) =$		$f^{(5)}(0) =$		

(b) Now we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x-a)^k$ for $k = 0, 1, \ldots, 5$. These (added together) will form your polynomial, $p_5(x)$.

(c) Approximate $f(1) = \sin(1)$ using your polynomial.

(d) Let's repeat this for another function. Let's build a 5th degree polynomial approximation of $g(x) = e^x$ centered at $x = 0$. We can construct the coefficients in the same way.

Table 9.1.3 Coefficients for Polynomial Approximation

k	$g^{(k)}(x)$	$g^{(k)}(a)$	$\dfrac{g^{(k)}(a)}{k!}$
$k=0$	$g(x)=e^x$	$g(0)=$	
$k=1$	$g'(x)=$	$g'(0)=$	
$k=2$	$g''(x)=$	$g''(0)=$	
$k=3$	$g'''(x)=$	$g'''(0)=$	
$k=4$	$g^{(4)}(x)=$	$g^{(4)}(0)=$	
$k=5$	$g^{(5)}(x)=$	$g^{(5)}(0)=$	

(e) And now, again, we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x-a)^k$ for $k=0,1,\ldots,5$. These (added together) will form your polynomial, $p_5(x)$.

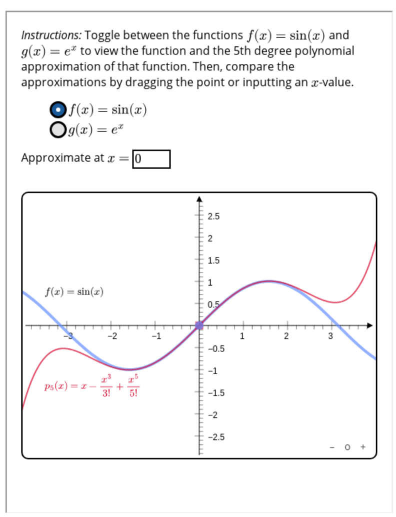
(f) Approximate $g(-3)=\dfrac{1}{e^3}$ using your polynomial.

Now that we know how to build and use these, we should probably think about accuracy. Are the estimations coming from these polynomials even accurate? How do we talk about that?

We won't formally define this too much for now: instead, we'll just look at things visually and see if we can figure out what might go into how we talk about accuracy of our estimations.

Activity 9.1.2 How Good Are Our Approximations?

We're going to think more carefully about our approximations of $\sin(1)$ and e^{-3} from Activity 9.1.1. In order for us to do this, let's visualize the function and the 5th degree polynomial for it.



Standalone
Embed

(a) How good of a job did the polynomial approximation do when approximating $\sin(1)$? How can you tell, visually?

(b) How good of a job did the polynomial approximation do when approximating e^{-3}? How can you tell, visually?

(c) How does the relationship between the "center" and the x-value that we're approximating at impact the accuracy of our approximation?

(d) How do you think you could make these approximations better (without changing the center)?

So we have a couple of main ideas about the accuracy of our approximations. We don't need to formalize them, but we can use them as a guiding rule for how we talk about these polynomial approximations.

Accuracy in Polynomial Approximations.

- Approximations using x-values closer to the center are likely to be more accurate than approximations using x-values farther away from the center.
- Polynomials with larger degrees give more accurate approximations than polynomials with smaller degrees at the same x-values.

Are These Partial Sums?

We have been using some familiar language here...we're talking about these "approximations" improving as we increase some parameter, n. We have some intuition that when we increase n, these approximations "approach" an object (in this case, a function) in some sense.

We're adding more and more terms to this sum as we increase n. Is a polynomial approximation just a partial sum?

Are we just going to look at a limit as $n \to \infty$ and see what happens?

Activity 9.1.3 Partial Sums of What?

Let's revisit our 5th degree polynomial approximations from Activity 9.1.1.

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

These approximations work well for x-values that are close to 0, but we will not be more formal than that.

(a) Make a conjecture about what the 7th degree polynomial approximations are for each of these functions.

What about the 15th degree?

(b) Make a conjecture about what the general formula would be for these terms. If you were to write these out using summation notation, what would they look like?

(c) Why does the polynomial approximation for the sine function only have odd-exponent terms?

(d) Make a conjecture about what a polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$ would be.

We'll explore these polynomials more, but we're specifically interested in the infinite-series version of these things. We'll give these polynomials a name, for easy reference: **Taylor polynomials**.

> These polynomials are named after the 17th century British mathematician, Brook Taylor. Taylor is mostly famous for his contributions in this area of calculus, but he also sat on the committee that mediated a major dispute between Isaac Newton and Gottfried Leibniz centered around their individual claims to get credit for the "creation" of calculus.
>
> Taylor was not able to give an unbiased judgment, due to his public support of Isaac Newton.

The main result to come out of this has to do with the relationship between these polynomials and the function they are approximating.

Theorem 9.1.4 Taylor's Theorem.

If $f(x)$ is a function that is n-times differentiable at $x = a$, then there is some remainder function, $R_n(x)$, such that:

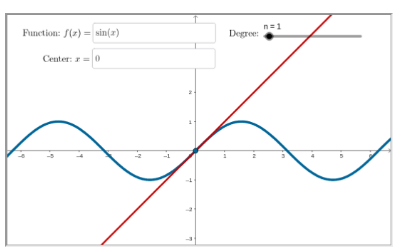
$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right) + R_n(x)(x-a)^n$$

where $\lim_{x \to a} R_n(x) = 0$.
That is, the remainder is small for x-values close to the center, $x = a$.

This result is great, in that it gives us the confidence to approximate these functions. We can add on to this the idea that we expect $R_n(x)$ to get smaller as $n \to \infty$. This should lead us to some idea of convergence, which we can think about here.

Activity 9.1.4 How Do These Polynomials Converge?

We're going to end here by thinking about these polynomials as some partial sum from an infinite series. If there is an infinite series, we should be prepared to think about convergence!
We're going to think about convergence in the same way that we have already: as an end behavior limit of the partial sums. So let's spend our time investigating this end behavior by visualizing polynomial approximations as the degree increases.



Standalone

(a) What happens to the polynomial approximation of $\sin(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(b) Does this behavior change if we centered our approximation elsewhere?

(c) What happens to the polynomial approximation of e^x centered at $x = 0$ as the degree $n \to \infty$?

(d) Does this behavior change if we centered our approximation elsewhere?

(e) What happens to the polynomial approximation of $\cos(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(f) Does this behavior change if we centered our approximation elsewhere?

(g) What happens to the polynomial approximation of $\ln(x)$ centered at $x = 2$ as the degree $n \to \infty$?

(h) Describe the difference in what you're seeing with the log function compared to the other functions we've thought about. Describe how the polynomial approximations converge: do they converge to the log function? How? More importantly, *where*?

(i) Does this behavior change if we centered our approximation elsewhere?

Ok, so this is pretty interesting! For some functions, like $\sin(x)$, e^x, and $\cos(x)$, it seems like the polynomials that we built will end up matching (converging to) the functions pretty much everywhere: as $n \to \infty$ we can get any function value from the polynomial, approximated to whatever accuracy we'd like!

But that's different for the log function. We were only able to get our polynomial to match the function's behavior on a specific interval of x-values. No matter how high the degree of the polynomial was, we weren't able to get it close to approximating something like $\ln(10)$, for instance. Unless we changed the center, of course!

This is going to bring up some great questions about how these partial sums converge to functions. We'll talk all about that in the next section!

9.2 Power Series Convergence

At the end of Section 9.1 Polynomial Approximations of Functions, we saw that these polynomial approximations can be thought of as partial sums of some larger infinite series. These infinite series are begging us to think about different notions of convergence, and at the end of the section, we saw that the polynomials, as the degree increases off to infinity, converge to match the function they are approximating, but this might be dependent on some interval of x-values. A domain, in a sense.

In this section, we'll investigate what it means for a power series, these infinite series of power functions, to converge. Let's define a power series, and then we can think about convergence from there.

Definition 9.2.1 Power Series.

A **power series** centered at $x = a$ is an infinite series in the form:

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

where $\{c_k\}_{k=0}^{\infty}$ is a sequence of real numbered coefficients.

We have a good idea of how we can build these sequences of coefficients in order for the power series we construct to converge to specific functions that we are interested in.

Before we state this formally, let's write down what we mean when we talk about convergence of power series.

One last thing: we have a kind of closure, so far, in our series. If we add up an infinite amount of numbers, then the infinite series might converge. If it does, it converges to a number (since sums of numbers are numbers). In a power series, though, we are adding up an infinite number of functions of x. If this series converges, it will converge to a function of x (since sums of functions are functions). So every power series is really a function.

Interval of Convergence

Activity 9.2.1 Polynomial Division.

We're going to do some fiddling with polynomials, and hopefully use this as a bridge to connect how we think of polynomials and power series with how we think about our traditional infinite series and the notions of convergence that we've already built.

(a) We're going to factor some polynomials, but we might end up using some division. First, we'll confirm some factors that we already know.

$$x^2 - 1 = (x-1)(x+1)$$

We'll confirm this by using division.

$$x - 1 \overline{\smash{\big)}\, x^2 \qquad -1}$$

(b) Now let's factor $x^3 - 1$. If the factors for this polynomial isn't as familiar, it might be helpful to know that $(x-1)$ is also a factor of x^3-1.

$$x-1 \, \overline{\big) \, x^3 \qquad\qquad -1}$$

(c) Let's try another one. Complete the following division.

$$x-1 \, \overline{\big) \, x^4 \qquad\qquad -1}$$

(d) Can you generalize this? Find the formula for $\dfrac{x^n-1}{x-1}$ for some positive integer n.

$$x-1 \, \overline{\big) \, x^n \qquad\qquad -1}$$

(e) Now that we have good evidence that

$$\sum_{k=0}^{\infty} x^k = \frac{x^n-1}{x-1},$$

We can apply a limit as $n \to \infty$.

$$\begin{aligned}\sum_{k=0}^{\infty} x^k &= \lim_{n\to\infty} \sum_{k=0}^{n} x^k \\ &= \lim_{n\to\infty} \frac{x^n-1}{x-1}\end{aligned}$$

For what values of x will this limit exist?

(f) Revisit Theorem 8.6.6 Geometric Series Convergence Criteria. Is there any difference for what we've just done compared to this result that we already know?

In this activity, we see that we can re-think about our Geometric Series family of series as a power series! Then, instead of saying that we have some requirements on the "ratio" for the geometric series to converge, we can say that the power series $g(x) = \displaystyle\sum_{k=0}^{\infty} x^k$ converges for x-values in the interval $(-1, 1)$.

If we do this same thing with our other common series, the p-series, then we'd not have a power series, but something slightly different:

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}.$$

This series definitely converges for real x-values in the interval $(1, \infty)$.

This function is called the **Riemann zeta function**, and is hugely important to many different fields of mathematics. We often care about this function when it has complex-number inputs (instead of just real-number inputs). It is also the focal point of one of the most famous unsolved mathematical questions, the Riemann hypothesis.

We noticed in Activity 9.1.4 How Do These Polynomials Converge? that the polynomials built to approximate the natural log function does seem to converge to the function as $n \to \infty$, but only for specific x-values. Hopefully we have some nice ideas as to why that happened: there is a vertical asymptote, and so maybe the "distance" from the center that this polynomial could approximate $\ln(x)$ at is limited!

In general, we can notice that this isn't new: we have families of infinite series that have specific values of variables for which they converge.

A big thing to notice is that these power series, by definition, include exponentials in them (since x^k is a power function of x but is also an exponential function of k). This means that they're great candidates to use the Ratio Test. Since we have a variable x (and we don't know if this variable is taking on positive or negative values), we'll need to test these series for absolute convergence.

Activity 9.2.2 Some Power Series and their Convergence.

Let's consider a couple of power series and apply some convergence tests to them in order for us to find out how it might converge.

(a) Consider the power series:

$$\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2}.$$

In order for us to apply the Ratio Test, we'll actually need to consider the positive-term version:

$$\sum_{k=1}^{\infty} \frac{|x-1|^k}{k^2}.$$

Apply the Ratio Test. What do you get in the limit of the ratio between terms?

(b) What kind of result from the Ratio Test guarantees convergence for the series? What are the x-values that guarantee convergence?

(c) The Ratio Test is inconclusive when the limit is equal to 1. What x-values does this happen at? Consider the power series evaluated at each of these x-values. Do these series converge or diverge?

(d) Consider the power series:

$$\sum_{k=1}^{\infty} \frac{(x-1)^k}{\sqrt{k}}.$$

Find the interval of x-values for which this series converges and test the end points of the interval in the same way as earlier. If it differs, explain why.

Definition 9.2.2 Interval of Convergence.

For a power series $\displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at $x = a$, the interval of

x-values for which the power series converges is called the **Interval of Convergence**. The distance from the center to endpoints of the interval is called the **Radius of Convergence**

So this is how we'll think about convergence! When we think about power series and their convergence, we're specifically thinking about convergence for specific inputs. These series are really families of infinite series, and we can try to explain their convergence criteria.

And for a power series, there is always some $x-$value for which if converges. For the power series

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$

centered at $x = a$, then as long as $\{c_k\}$ is a sequence of real numbers, then the series converges at $x = a$. We get $f(a) = c_0$, the constant term. This should match with how we thought about these series originally! They came from polynomial approximations of our functions, where of course the polynomials needed to match the function value at the center. We were thinking about tangent lines, tangent quadratics, tangent cubics, etc. They *need* to be tangent, and so they "converge" at that single center x-value at least.

Example 9.2.3

For the power series $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1}$, find the interval of convergence. Along the way, it will likely be helpful to identify the center and the radius of convergence.

Solution. Let's apply the Ratio Test. We'll technically be applying this to $\displaystyle\sum_{k=0}^{\infty} \frac{3^k|x|^k}{2k+1}$.

$$\begin{aligned}\lim_{k\to\infty} \frac{3^{k+1}|x|^{k+1}}{2(k+1)+1} \cdot \frac{2k+1}{3^k|x|^k} &= \lim_{k\to\infty} \frac{3|x|(2k+1)}{2k+3} \\ &= 3|x|\end{aligned}$$

For us to conclude that this series converges, we need the limit from the ratio test to be less than 1.

$$3|x| < 1 \to |x| < \frac{1}{3}$$

This series is centered at 0 with a radius of convergence of $\frac{1}{3}$. So we know that this series converges for $-\dfrac{1}{3} < x < \dfrac{1}{3}$.

Since the Ratio Test is inconclusive when $x = -\dfrac{1}{3}$ and $x = \dfrac{1}{3}$, we'll test those individually.

When $x = -\frac{1}{3}$

$$\sum_{k=0}^{\infty} \frac{3^k \left(-\frac{1}{3}\right)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

We can apply the Alternating Series Test!

$$\lim_{k\to\infty} \frac{1}{2k+1} = 0$$

This series converges!

When $x = \frac{1}{3}$

$$\sum_{k=0}^{\infty} \frac{3^k \left(\frac{1}{3}\right)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{1}{2k+1}$$

We can apply the Rational Comparison Theorem! Since the difference in degrees is 1, we know that this series could be compared to the Harmonic series, and so it diverges.

So the interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

Operations on Power Series

Theorem 9.2.4 Operations on Power Series.

For two power series $\sum_{k=0}^{\infty} c_k x^k$ *and* $\sum_{k=0}^{\infty} d_k x^k$ *that converge to* $f(x)$ *and* $g(x)$ *(respectively) on the interval of convergence* I*, we can consider the following operations to combine power series.*

- Sum: $\sum_{k=0}^{\infty} (c_k + d_k) x^k$ *converges to* $f(x) + g(x)$ *on* I*.*
- Difference: $\sum_{k=0}^{\infty} (c_k - d_k) x^k$ *converges to* $f(x) - g(x)$ *on* I*.*
- Product: *If* bx^n *is a power function, then* $\sum_{k=0}^{\infty} b(c_k) x^{k+n}$ *converges to* $(bx^m) f(x)$ *on* I*.*
- Composition: *If* bx^n *is a power function, then* $\sum_{k=0}^{\infty} c_k (b^k)(x^{nk})$ *converges to* $f(bx^n)$ *when* bx^n *is in* I*.*

We can do something similar with some of our calculus operations: differentiation and integration.

Theorem 9.2.5 Differentiating and Integrating Power Series.

If $\sum_{k=0}^{\infty} c_k (x-a)^k$ *converges to* $f(x)$ *on an interval of convergence with* *a radius $R > 0$, then:*

- $\displaystyle\sum_{k=0}^{\infty} kc_k(x-a)^{k-1}$ *converges to* $f'(x)$.
- $\displaystyle\sum_{k=0}^{\infty} \frac{c_k(x-a)^{k+1}}{k+1}$ *converges to* $F(x)$*, an antiderivative of* $f(x)$.

Both of these converge on an interval of convergence centered at $x = a$ with radius R.

Note 9.2.6

We're being weird about naming the interval of convergence for these. The issue is that when we differentiate, we might lose closed endpoints of an interval. Similarly, when we antidifferentiate, we could add endpoints to an open interval.

We can see this in some of the examples that follow, but the intervals of convergence are going to be identical except for possibly at the endpoints.

Let's finish this here by revisiting the power series from Example 9.2.3.

Example 9.2.7

Let's re-consider the power series $\displaystyle\sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1}$. We know, from Example 9.2.3, that this power series converges to $f(x)$ for x-values in the interval $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

(a) Find a power series that converges to $x^7 f(x)$. What is the interval of convergence?

Solution.

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1} \\
x^7 f(x) &= x^7 \sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{3^k(x^k)(x^7)}{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{3^k x^{k+7}}{2k+1}
\end{aligned}$$

The interval of convergence doesn't change: $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

(b) Find a power series that converges to $f(x^2)$. What is the interval of convergence?

Solution.

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1} \\
f(x^2) &= \sum_{k=0}^{\infty} \frac{3^k((x^2)^k)}{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{3^k(x^{2k})(x^7)}{2k+1}
\end{aligned}
$$

The interval of convergence is $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

(c) Find a power series that converges to $f'(x)$. What is the interval of convergence?

Solution.

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1} \\
f'(x) &= \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{3^k(x^k)}{2k+1}\right) \\
&= \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{3^k(x^k)}{2k+1}\right) \\
&= \sum_{k=0}^{\infty} \frac{3^k \frac{d}{dx}\left(x^k\right)}{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{k3^k(x^{k-1})}{2k+1}
\end{aligned}
$$

Since k is in the coefficient, starting the index at $k = 0$ means that the “first” term is 0: we can re-index if we’d like, but it is not necessary.

$$
\sum_{k=1}^{\infty} \frac{k3^k(x^{k-1})}{2k+1} \qquad \text{or} \qquad \sum_{k=0}^{\infty} \frac{(k+1)3^{k+1}(x^k)}{2k+3}
$$

In any of these cases, the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

These operations aren’t that useful without a purpose. Similarly, these power series aren’t that interesting without knowing what functions they converge to.

We’ll now start putting some things we’ve learned together.

- We have ideas of power series representations for $\sin(x)$, $\cos(x)$, and e^x.
- We know the geometric series, which we can use to get a power series representation for $\frac{1}{1-x}$.
- We have a general way of building terms of these power series, using the polynomial approximations and the formula for building those terms.

- We have ways of combining, differentiating, and anti-differentiating power series.

We'll move forward and demonstrate some really great ways of constructing power series representations of some new functions. And then we'll show some very fun uses of these power series representations to illustrate how friendly they are to work with.

9.3 How to Build Taylor Series

Let's first clear the air: we've been talking about Taylor Polynomials that are partial sums of some larger infinite series. Then we talked about Power Series, a generic structure of an infinite series of power functions of x. We want to combine these together, and name the specific power series that these Polynomial Approximations are partial sums of.

Definition 9.3.1 Taylor Series.

If $f(x)$ is an infinitely differentiable function at $x = a$, then the **Taylor series** expansion of $f(x)$ centered at $x = a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

If the Taylor series expansion of $f(x)$ is centered at $x = 0$, some people name these specifically, calling them **Maclaurin series**. Colin Maclaurin made good use of Taylor series, and so is credited with having this special case named after him.

Now that we have named these Taylor series, we want to work on constructing these Taylor series for a given function. In this section, we'll investigate a couple of ways that we can do this.

Constructing Directly

We saw in Section 9.1 Polynomial Approximations of Functions that we can use the idea of a polynomial partial sum to build out a Taylor series representation for a function.

We'll need to build a bunch of terms for some Taylor polynomial, and then we can hopefully extrapolate afterwards. We want to come up with a general formula to define these terms, so that we can write the explicit formula in the infinite series, but we know that this, in general, is a difficult task! It's *hard* to find these explicit formulas for the terms in these series.

It's not impossible, though! You can remind yourself of how we've done this! We found (and speculated about) the Taylor series for $f(x) = e^x$, $f(x) = \sin(x)$, and $f(x) = \cos(x)$ (all centered at $x = 0$) in Activity 9.1.1 Build a Polynomial and Activity 9.1.3 Partial Sums of What?.

Activity 9.3.1 Constructing a Taylor Series Directly.

Let's work on this! To be honest, the goal here is to realize how difficult this can be.

Let's consider the function $f(x) = \tan(x)$. We're going to build a Taylor polynomial approximating $f(x)$ centered at $x = 0$.

(a) We're going to build a 4th degree polynomial approximating $f(x) = \tan(x)$. In order to do this, we'll need to find the coefficients for the 5 terms.

Table 9.3.2 Coefficients for Polynomial Approximation

k	$f^{(k)}(x)$	$f^{(k)}(a)$	$\frac{f^{(k)}(a)}{k!}$
$k=0$	$f(x) = \tan(x)$	$f(0) =$	
$k=1$	$f'(x) =$	$f'(0) =$	
$k=2$	$f''(x) =$	$f''(0) =$	
$k=3$	$f'''(x) =$	$f'''(0) =$	
$k=4$	$f^{(4)}(x) =$	$f^{(4)}(0) =$	
$k=5$	$f^{(5)}(x) =$	$f^{(5)}(0) =$	

(b) Now we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x-a)^k$ for $k = 0, 1, ..., 5$. These (added together) will form your polynomial, $p_5(x)$.

(c) Can you find a general explicit formula for the terms? If you're having a hard time with this, what could you do to try to make this easier? Why is this difficult?

We're going to stop there. This series is famously difficult to work with. We won't generate the whole Taylor series this way, and this strategy really doesn't work well for functions like this (where the derivatives are pretty annoying to find).

Connections To Other Taylor Series

A (hopefully) more useful strategy is to collect the Taylor series that we know, and then try to build a series for a function based on how that function relates. We saw this in Theorem 9.2.4 Operations on Power Series and Theorem 9.2.5 Differentiating and Integrating Power Series.

Activity 9.3.2 Connecting to Another Series.

Let's start with some "known" series.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{on } (-1,1)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{on } (-\infty,\infty)$$

We're going to build a Taylor series for a logarithmic function:

$$y = \ln(1+x).$$

(a) In order to start, we're going to find a Taylor series for $g(x) = \frac{1}{1+x}$. Which of the "known" series above is this most similar to? How is it different?

(b) Find a Taylor series for $g(x) = \frac{1}{1+x}$ by using Theorem 9.2.4. Can you connect this series to the known series by multiplying or composing something?

(c) Notice that the function we're trying to get a Taylor series for, $y = \ln(1+x)$, is an antiderivative of $g(x) = \frac{1}{1+x}$. Antidifferentiate the series you found above, using Theorem 9.2.5 Differentiating and Integrating Power Series. We know that at $x = 0$, $y = 0$, so use this information to find specific constant that we need when we antidifferentiate.

(d) What is the interval of convergence for this new series? How does that relate to the interval of convergence of the "known" series you started with?

We're going to try this again with a different function as the goal!

Activity 9.3.3 Connecting to Yet Another Series.

Here, again, are some *updated* "known" series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{on } (-1,1)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{on } (-\infty,\infty)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} \qquad \text{on } (-1,1]$$

We're going to build a Taylor series for a logarithmic function:

$$y = \tan^{-1}(x).$$

(a) In order to start, we're going to find a Taylor series for $g(x) = \frac{1}{1+x^2}$. Which of the "known" series above is this most similar to? How is it different?

(b) Find a Taylor series for $g(x) = \frac{1}{1+x^2}$ by using Theorem 9.2.4. Can you connect this series to the known series by multiplying or composing something?

(c) Notice that the function we're trying to get a Taylor series for, $y = \tan^{-1}(x)$, is an antiderivative of $g(x) = \frac{1}{1+x^2}$. Antidifferentiate the series you found above, using Theorem 9.2.5 Differentiating and Integrating Power Series. We know that at $x = 0$, $y = 0$, so use this information to find specific constant that we need when we antidifferentiate.

(d) What is the interval of convergence for this new series? How does that relate to the interval of convergence of the "known" series you started with?

Now we can update our list of "known" series!

Known Taylor Series.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{on } (-1,1)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{on } (-\infty,\infty)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} \qquad \text{on } (-1,1]$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad \text{on } [-1,1]$$

Most importantly, we can use these two activities as a framework for a more general way of constructing Taylor series representations of functions. If we can link the function we're interested in to one of the functions with a "known" Taylor series, then we can use the link between functions as a way of linking the respective Taylor series.

That is, if we know the Taylor series for $f(x)$ and we want to find the Taylor series for a different function $g(x)$ where $g(x)$ is some sort of transformed version of $f(x)$, then we can list the transformations or changes to $f(x)$ in order to produce $g(x)$, and apply those transformations or changes to the Taylor series for $f(x)$.

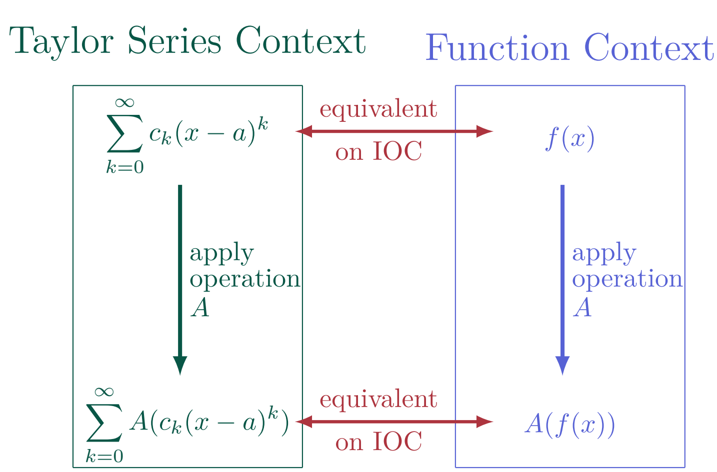


Figure 9.3.3 Constructing a Taylor series from a related series.

So when the operation A is something from either Theorem 9.2.4 Operations on Power Series or Theorem 9.2.5 Differentiating and Integrating Power Series, then we can apply these operations to the Taylor series of $f(x)$ in order to construct the Taylor series of $A(f(x))$. Interestingly enough, we can note that the intervals of convergence can be mostly constructed from these same transformations: there might be some changes to the inclusion of the endpoints based on differentiation and integration.

Now that we have a convenient way of constructing Taylor series for conveniently linked functions, we should think about how we might use these Taylor series representations of functions.

9.4 How to Use Taylor Series

In this final topic for this chapter, we'll pause for a moment to figure out what we could actually *use* these Taylor series for. This next statement is the personal opinion of the author: I do not typically care to focus on the applications of mathematical topics. In this text, you'll maybe notice that there is less of a focus on applications of derivatives and integrals into other areas and more focus on what these calculus objects actually are.

We'll try to focus on the flexible uses and applications of Taylor series as a way of allowing us the freedom to think about calculus topics in hard-to-work-with contexts, or to extend our uses of calculus ideas into new contexts.

Approximations

Taylor series have, as a part of their structure, a built-in approximation strategy. If we want to approximate a series, we can use a partial sum! Approximations of functions using Taylor series, then, are easily approximated using the polynomials that build the partial sums.

We can use this as a way of approximating the functions themselves or approximating function outputs (by evaluating our Taylor series at specific x-values).

Activity 9.4.1 Approximating π and Other Values.

Let's pick a couple of values that are just based on functions with Known Taylor Series and approximate them.

(a) If we note that $\tan^{-1}(1) = \frac{\pi}{4}$, then we can say that $\pi = 4\tan^{-1}(1)$.

Find an infinite series that converges to $\tan^{-1}(1)$, and use that to construct a series that converges to π.

(b) If we note that $\frac{1}{e} = e^{-1}$, then find an infinite series that converges to $\frac{1}{e}$.

(c) Find an infinite series that converges to $\sin(1)$.

(d) Note that each of these three series are alternating series! We can approximate them using a partial sum, and then we can get an error bound for that partial sum by looking at the size of the next term in the infinite series (Theorem 8.5.5 Approximations of Alternating Series).

Approximate the value of each infinite series using a partial sum with the same number of terms. You can pick the number of terms you use. Then, compare the margin of error. Which approximations are most/least accurate? Why do you think that is?

Integrals

We have built a whole host of antidifferentiation strategies in Chapter 7, but there are plenty of functions that are resistant to those specific techniques. In fact, most functions do not have what we call **elementary antiderivatives**: antiderivative functions that can be written as some combination of the basic, named function types.

A typical example is $\sin(x^2)$. Let's look at an integral involving this function!

Activity 9.4.2 Integrating using Taylor Series.

Let's consider the definite integral:

$$\int_{x=0}^{x=1} \sin(x^2)\, dx.$$

We know this function is continuous everywhere, and so it is continuous on the interval $[0,1]$. So the Fundamental Theorem of Calculus applies. All we need to do is find an antiderivative, evaluate it at the end-points of the interval, and subtract.
Great! Easy!
Except that we can't write out the family of antiderivatives for this function.
So let's convert over to the Taylor series context and solve our problem there.

(a) Create a Taylor series for the function $f(x) = \sin(x^2)$.

(b) Find the interval of convergence for this Taylor series. Specifically, we want to know if this series converges to $\sin(x^2)$ on the interval $[0,1]$, since that's the interval we're integrating over.

(c) If $f(x) = \sin(x^2)$, find a Taylor series representation of $F(x)$, an antiderivative of $f(x) = \sin(x^2)$.

(d) Evaluate this antidervative at the endpoints and subtract:

$$F(1) - F(0).$$

(e) Use Theorem 8.5.5 Approximations of Alternating Series to approximate the value of $\displaystyle\int_{x=0}^{x=1} \sin(x^2)\, dx$ with a maximum error of 10^{-5}.

We got lucky here, in that the alternating series was reasonable to approximate with some specified error bound (based on Theorem 8.5.5), but this isn't technically unique to alternating series! There are some other ways of building error bounds on our partial sum approximations, but we won't spend our time building them.

Euler's Formula

Three of the Taylor series that we've looked at have been pretty similar:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

It's probably worth stopping for a moment to remember *why* the exponential function's Taylor series has "all" of the terms, while the ones for the sine and cosine functions have only the odd and even terms (respectively), and also why the terms for the sine and cosine function alternate in sign.

If you want to refresh yourself on why these series have these kinds of structures, take a look back at Section 9.1, where we built the partial sums of these series and conjectured about the infinite series for each.

Before we begin, let's remember a fact about i, the imaginary unit where $i^2 = -1$.

Fact 9.4.1 Powers of i. *For $i = \sqrt{-1}$ where $i^2 = -1$, we can think about the following powers of i:*

$$\begin{array}{llll} i^0 = 1 & i^4 = 1 & i^8 = 1 & i^{4k} = 1 \\ i^1 = i & i^5 = i & i^9 = i & i^{4k+1} = i \\ i^2 = -1 & i^6 = -1 & i^{10} = -1 & i^{4k+2} = -1 \\ i^3 = -i & i^7 = -i & i^{11} = -i & i^{4k+3} = -i \end{array}$$

(where k is some integer)

This structure (flipping back and forth between two things, and changing the sign as we do it) might feel familiar! It *almost* feels like the same thing that happens with derivatives of the sine and cosine functions, right? Hmm....

Activity 9.4.3 Constructing Euler's Formula.

(a) Compose (ix) inside of the e^x function and it's Taylor series to come up with a Taylor series representation for e^{ix}.

(b) Write out several terms of this new series. You should write out at least 6 of them, but more is fun, too!

Now, use Fact 9.4.1 Powers of i to re-write these terms.

(c) Which terms have are real coefficients? Which terms have imaginary coefficients? If you group these together, do you recognize these terms?

(d) Write e^{ix} in terms of sines and cosines.

Theorem 9.4.2 Euler's Formula.

For i, the imaginary unit,

$$e^{ix} = \cos(x) + i\sin(x).$$

This is a really powerful theorem for a couple of reasons, but we'll just touch on a few of them. The first, and maybe least important, is a really cool identity, called **Euler's Identity**. It's just Euler's Formula evaluated at $x = \pi$:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

$$e^{i\pi} = -1$$
$$e^{i\pi} + 1 = 0$$

This last statement is the one most used. It's often cited as one of the most "beautiful" or interesting equations in mathematics, since it combines 5 of the most important numbers:

- e, the natural exponential base, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
- i, the imaginary unit where $i = \sqrt{-1}$ and $i^2 = -1$.
- π, the constant ratio between the circumference and diameter of every circle.
- 1, the multiplicative identity, where $1(x) = x$.
- 0, the additive identity, where $0 + x = x$.

These 5 numbers are all combined into one connected equation, even though they are seemingly not connected at all in their "origins". It's no wonder so many people love this equation!

> As a fun scavenger hunt, see if you can find the faculty in the mathematics department at Moraine Valley Community College that has a tattoo of this identity!

Complex Analysis

Euler's Formula is a really nice way of linking some of our well-known functions to their complex-function extensions. We have complex exponentials, and we can use this same formula to create a complex-valued logarithmic function.

From here, mathematicians were able to think about how the concepts of calculus could be applied to functions that took in complex-valued inputs and outputs complex-valued outputs. This area of mathematics proved to be rich in applications and also hugely interesting.

Trigonometric Identities

Trigonometric identities are used in a ton of different areas of mathematics (including in this textbook) to try to re-write some expressions into friendlier forms. One of the big struggles for mathematics students, though, is trying to remember them when needed. The author of this textbook thinks, as a matter of personal opinion, that memorizing facts and identities like this is overrated in general, but being able to construct these when necessary is more helpful.

A problem with many trigonometric identities is that the construction or explanation of them often relies on tricky geometry involving lots of triangles.

Euler's Formula provides a really nice way of thinking of these identities, since we can construct them from the definition of the complex exponential.

Example 9.4.3

Let's construct the sum of angle identites for sine and cosine. We, eventually, want to have some formula for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$. In order to do this, let's evaluate e^{ix} at $x = \alpha+\beta$ and see what happens.

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

So we can see that the "real" part of this is the sum of angles for the cosine function, and the "imaginary" part is the sum of angles for the sine function.
But now we can think about $e^{i(\alpha+\beta)}$ by considering that a sum of exponents could really be a product of exponentials:

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$$

Let's look at this second version.

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))\,(\cos(\beta) + i\sin(\beta)) \\ &= \cos(\alpha)\cos(\beta) + i\sin(\alpha)\cos(\beta) + i\cos(\alpha)\sin(\beta) - \sin(\alpha)\sin(\beta) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i\,(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \end{aligned}$$

Now we can remember that this should be equivalent to $\cos(\alpha+\beta) + i\sin(\alpha+\beta)$, from earlier:

$$\cos(\alpha+\beta)+i\sin(\alpha+\beta) = (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))+i\,(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta))$$

So we can equate the real parts and the imaginary parts separately to get:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \end{aligned}$$

This strategy is super useful for creating these identities, since we can often take advantage of exponent properties to write two different equivalent expressions to connect.

These are only a few applications of Taylor series, but hopefully they are enough to show a sufficiently wide range of applications or uses. Taylor series, whether they're used to get polynomial approximations of objects or to translate some function into a different context, are some of the most used applications of calculus, and are a wonderful way to think about connections between some of the calculus concepts that we've talked about.

Colophon

This book was authored in PreTeXt.