Discover Calculus I - Activity Book

Activities for Differential Calculus Topics

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Chapter 1

Limits

1.1 The Definition of the Limit

Activity 1.1.1 Close or Not?

We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!
Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?
Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Activity 1.1.2 Approximating Limits.

For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x\to 1} f(x)$ using the graph of the function $f(x)$ below.

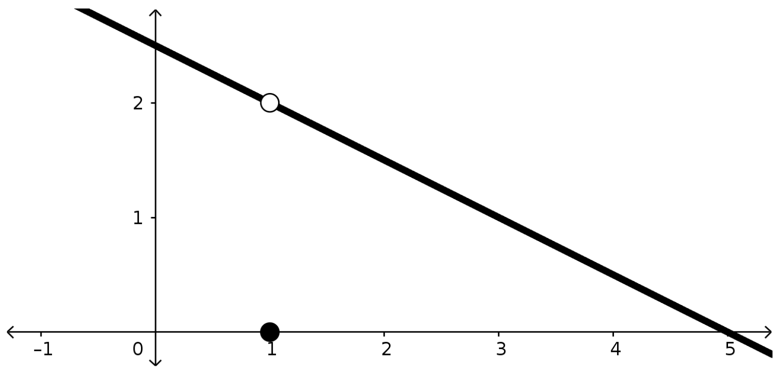


Figure 1.1.1

(b) Approximate $\lim_{x\to 2} g(x)$ using the graph of the function $g(x)$ below.

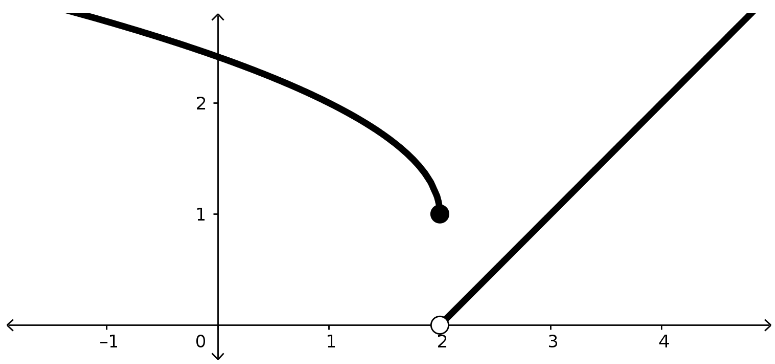


Figure 1.1.2

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x\to -1} h(x)$
- $\lim_{x\to 0} h(x)$
- $\lim_{x\to 2} h(x)$

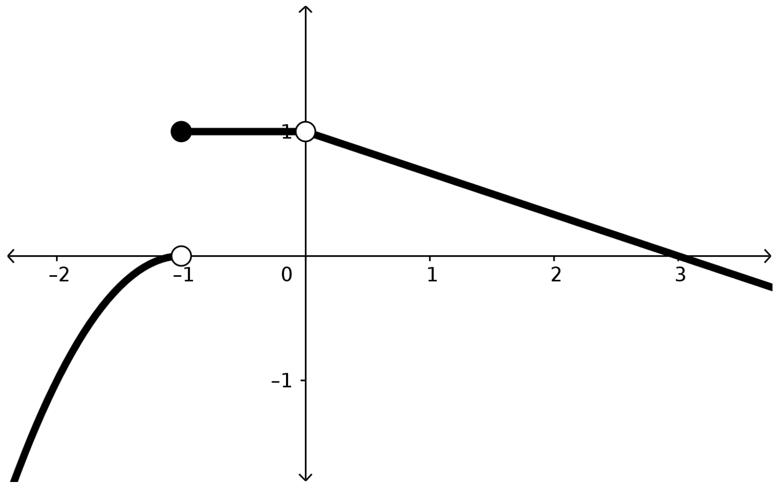


Figure 1.1.3

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.1.3 Approximating Limits Numerically.

For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.1.4

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.1.5

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.1.6

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.2 Evaluating Limits

Activity 1.2.1 From Estimating to Evaluating Limits (Part 1).

Let's consider the following graphs of functions $f(x)$ and $g(x)$.

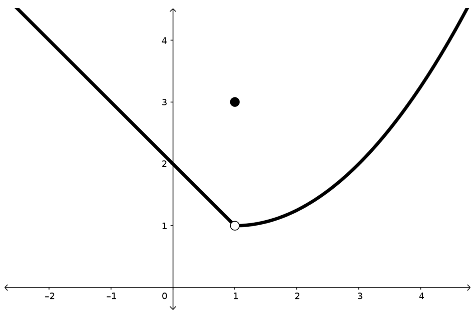


Figure 1.2.1 Graph of the function $f(x)$.

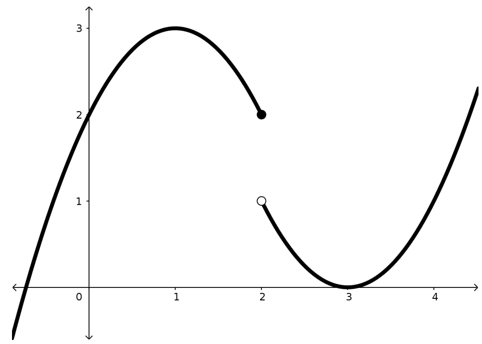


Figure 1.2.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 1^-} f(x)$
- $\lim\limits_{x \to 1^+} f(x)$
- $\lim\limits_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 2^-} g(x)$
- $\lim\limits_{x \to 2^+} g(x)$
- $\lim\limits_{x \to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.2.2 From Estimating to Evaluating Limits (Part 2).

Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

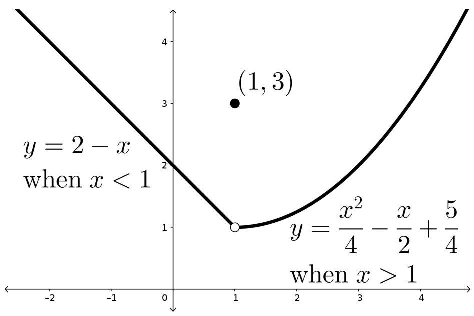


Figure 1.2.3 Graph of the function $f(x)$.

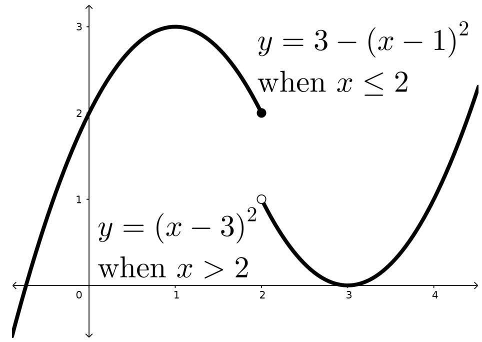


Figure 1.2.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$
- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 2^-} g(x)$
- $\lim_{x \to 2^+} g(x)$
- $\lim_{x \to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

Activity 1.2.3 Combinations of Functions.

We want to remind ourselves how we can combine functions using different operations, and how we might find outputs based on the different combinations. Our goal is to then think about how this might work with limits: how can we summarize the behavior of combinations of functions around some point?

Let's consider some functions $f(x) = x^2 + 3$ and $g(x) = x - \frac{1}{x}$. We'll say that the domain of both functions is $(0, \infty)$ for our own convenience.

(a) Let's consider the function $h(x) = f(x) + g(x)$. Describe at least two different ways of finding the value of $h(2)$.

(b) If we instead define the function $h(x) = f(x) - g(x)$, how would you describe at least two different ways of finding the value of $h(2)$?

(c) What about a scaled version of one of these functions? If we let $h(x) = 4f(x)$ and $j(x) = \frac{g(x)}{3}$, can you describe more than one way to find the value of $h(3)$ and $j(3)$?

(d) You can probably guess where we're going: we're going to define a function that is the product of f and g: $h(x) = f(x) \cdot g(x)$. Describe more than one way of evaluating $h(4)$.

(e) And finally, let's define $h(x) = \frac{f(x)}{g(x)}$. Now describe more than one way of finding $h(4)$.

(f) If $h(x) = \frac{f(x)}{g(x)}$, then are there any x-values that are in the domain of f and g (the domain is $x > 0$) that $h(x)$ cannot be defined for? Why?

Ok, we can confront this big idea: when we combine functions, we can either evaluate the combination of the functions at some x-value or evaluate each function separately and just combine the answers! Of course, there are some limitations (like when the combination isn't nicely defined because of division by 0 or something else), but this is a good framework to move forward with!

Activity 1.2.4 Limits of Polynomial Functions.

We're going to use a combination of properties from Theorem 1.2.5 and Theorem 1.2.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x \to 1} f(x)$. First, use the properties from Theorem 1.2.5 to rewrite this limit as 4 different limits added or subtracted together.

(b) Now, for each of these limits, rewrite them as products of things until you have only limits of constants and identity functions, as in Theorem 1.2.6. Evaluate your limits.

(c) Based on the definition of a limit (Definition 1.1.1), we normally say that $\lim_{x \to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x \to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x \to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x \to a} p(x)$ is the same value as $p(a)$.

1.3 First Indeterminate Forms

Activity 1.3.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x\to 1} f(x)$.

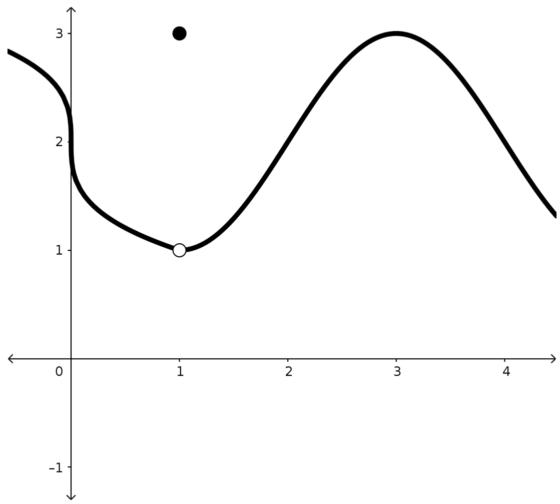


Figure 1.3.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x\to 1} g(x)$.

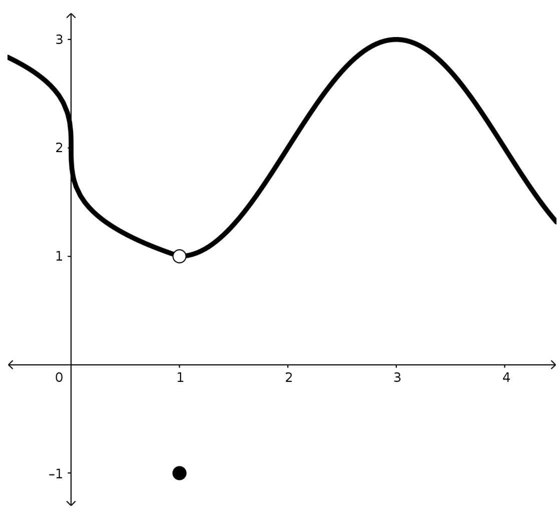


Figure 1.3.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{t\to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{t\to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Activity 1.3.2

(a) Were going to evaluate $\lim_{x\to 3}\left(\dfrac{x^2-7x+12}{x-3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \dfrac{x^2-7x+12}{x-3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \dfrac{x^2-7x+12}{x-3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3)-4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x\to -2}\left(\dfrac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim_{x \to 5} f(x) \overset{?}{\to} \frac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to rewrite your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

Activity 1.3.3

Let's consider a new limit:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

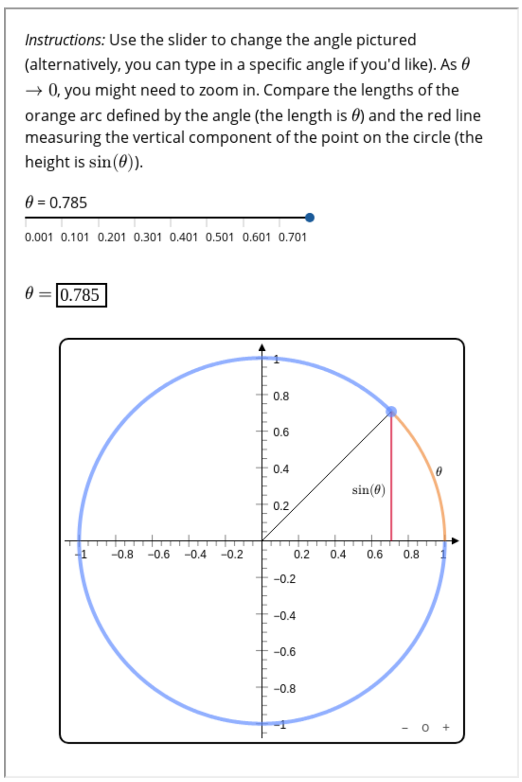
(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and rewrite, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



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(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\displaystyle\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$?

1.4 Limits Involving Infinity

Activity 1.4.1 What Happens When We Divide by 0?

First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.
When we talk about "dividing by 0" here (and in Section 1.3), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Activity 1.4.2 What Happens When We Divide by Infinity?

Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to rebuild our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

1.5 The Squeeze Theorem

Activity 1.5.1 A Weird End Behavior Limit.

In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\underline{\qquad\qquad} \leq \sin(x) \leq \underline{\qquad\qquad}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\frac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2+1} \leq \underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } h(x)}$$

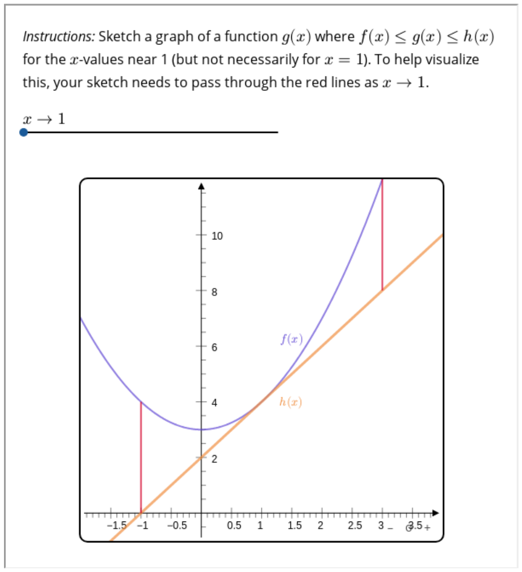
(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right)$?

Activity 1.5.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.



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(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x \to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x \to 3} g(x)$ and $\lim_{x \to 0} g(x)$?

Do we know that these limits exist? If they do, do we have information about their values?

1.6 Continuity and the Intermediate Value Theorem

Activity 1.6.1 Classification and Continuity.

Let's consider the following functions, graphed below.

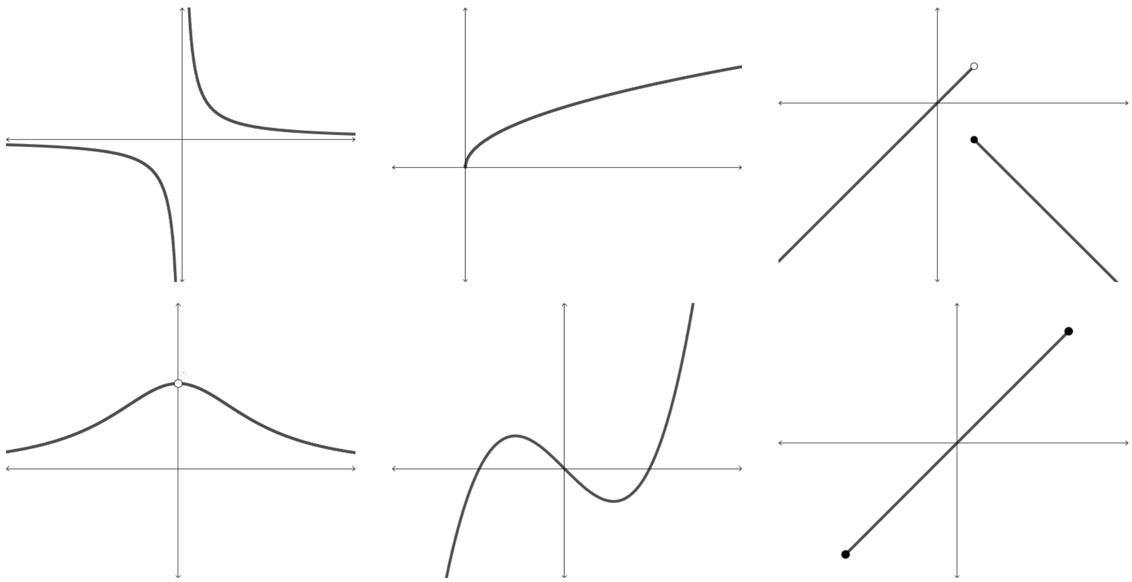


Figure 1.6.1 A variety of graphs for us to use to think about continuity.

(a) Can you point out any points on the functions above that seem like the functions might not be continuous? Note that we're not classifying each function as "continuous" or "not continuous"!

(b) We want to build towards a definition of continuity using "connectedness" as the key: a function is continuous at a point if it is connected to itself. How does that work when we think about function values and limits? Use the graphs above and the points you looked at to help!

Chapter 2

Derivatives

2.1 Introduction to Derivatives

Activity 2.1.1 Thinking about Slopes.

We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ **and...**	**Slope**
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs alongside of the curve at that point instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of in the table above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line.

Let's think about a new function, $g(x) = \sqrt{2-x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

Activity 2.1.3 Calculating a Bunch of Slopes.

Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.
Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.3 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$\begin{aligned} j'(a) &= \lim_{x \to a} \left(\frac{j(x) - j(a)}{x - a} \right) \\ &= \lim_{x \to a} \left(\frac{(x^2 - 4) - (a^2 - 4)}{x - a} \right). \end{aligned}$$

Remember, you know how this goes! You're going to do the same type of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to recalculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

2.2 Interpreting Derivatives

Activity 2.2.1 Interpreting the Derivative as a Slope.

In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \dfrac{1}{x^2}$.

(a) Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?

(b) We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.

Remember that we can write the equation of a line in two ways:

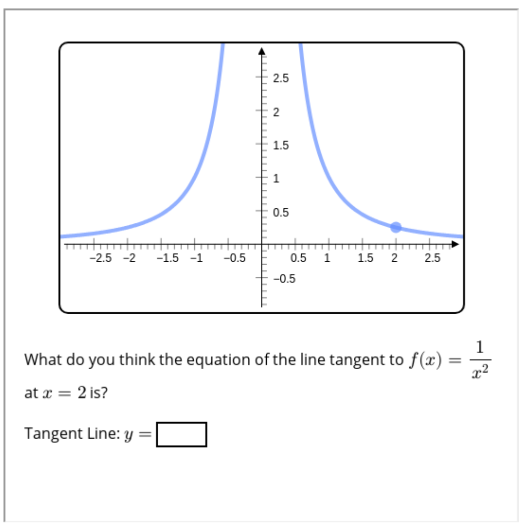
- The equation of a line with slope m that passes through the point $(a, f(a))$ is:

$$y = m(x - a) + f(a).$$

- The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:

$$y = mx + b.$$

Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \dfrac{1}{x^2}$ below to check your equation.



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(c) This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.

Use this idea of slope to predict where the y-value of our function will be at 2.01.

(d) Compare this value with $f(2.01) = \dfrac{1}{2.01^2}$. How close was it?

Activity 2.2.2 Interpreting the Derivative as a Rate of Change.

This is going to feel somewhat redundant, since we hopefully know that a slope is really just a rate of change. But we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position of the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \leq t \leq 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

Activity 2.2.3 Interpreting the Derivative as a Function.

In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.3 Some Early Derivative Rules

Activity 2.3.1 Derivatives of Power Functions.

We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.
Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\frac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

(b) Find $\frac{d}{dx}\left(x^3\right)$.

(c) Find $\frac{d}{dx}\left(x^4\right)$.

(d) Notice that in these derivative calculations, the main work is in multiplying $(x + \Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\begin{aligned}
\frac{d}{dx}\left(x^5\right) &= \lim_{\Delta x \to 0}\left(\frac{(x+\Delta x)^5 - x^5}{\Delta x}\right)\\
&= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right)\\
&= \rightsquigarrow \ldots
\end{aligned}$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x + \Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

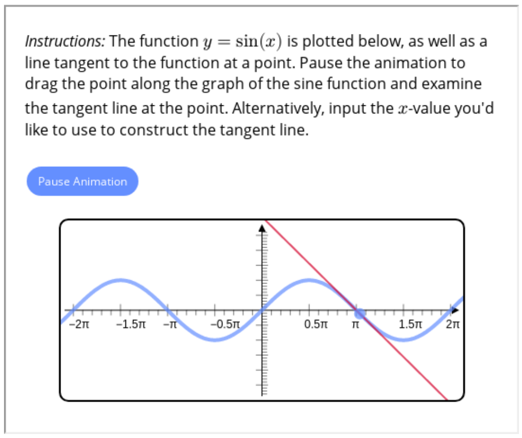
2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this is done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just whatever terms still have at least one remaining copy of Δx in them "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Activity 2.3.2 Derivatives of Trigonometric Functions.

Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing our interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



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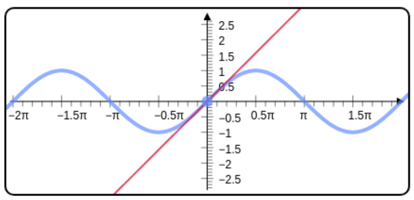
Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.

Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \sin(x)$ and y', try to think about what function y' could be.

x	$y = \sin(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		



Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!

$y' =$

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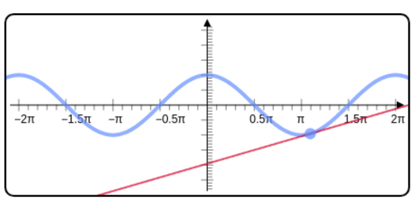
(c) Let's repeat this process using the $y = \cos(x)$ function instead.

The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.

Instructions: The function $y = \cos(x)$ is plotted below, as well as a line tangent to the function at a point. Pause the animation to drag the point along the graph of the sine function and examine the tangent line at the point. Alternatively, input the x-value you'd like to use to construct the tangent line.

Pause Animation



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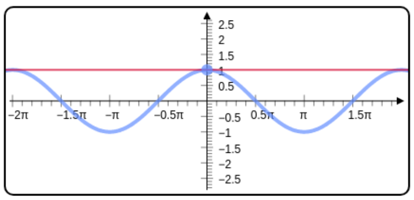
Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.

Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \cos(x)$ and y', try to think about what function y' could be.

x	$y = \cos(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		



Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!

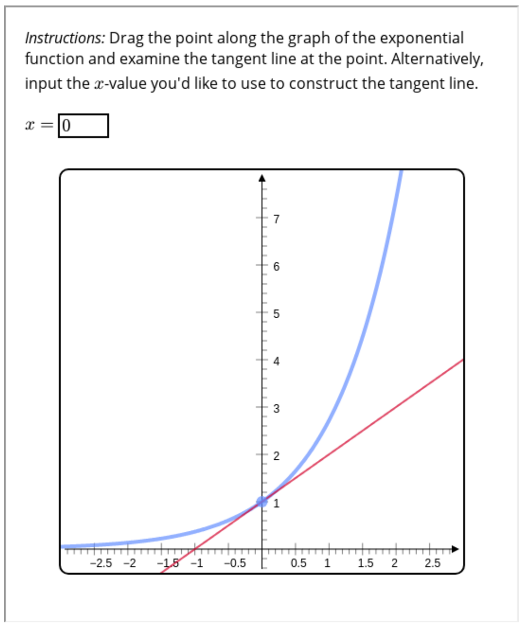
$y' =$ ☐

Standalone Embed

Activity 2.3.3 Derivative of the Exponential Function.

We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function similarly to how we thought of the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.



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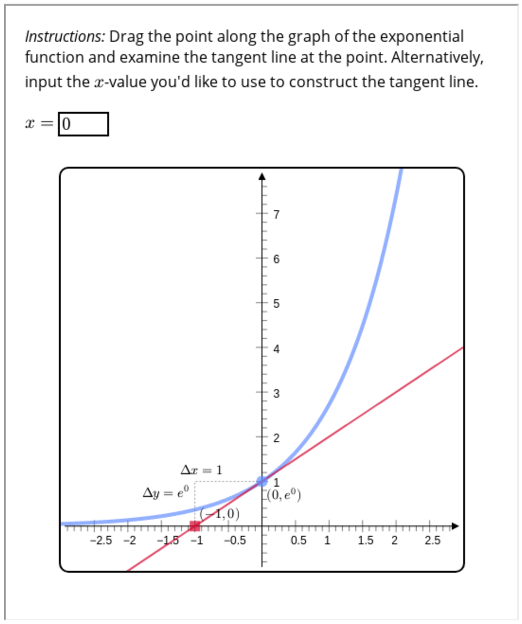
Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!



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What information does this reveal about the slopes?

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/equation for y' is then?

2.4 The Product and Quotient Rules

Activity 2.4.1 Thinking About Derivatives of Products.

Let's start with two quick facts:

$$\frac{d}{dx}\left(x^3\right) = 3x^2 \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

(a) What is $\frac{d}{dx}\left(x^3 + \sin(x)\right)$? What about $\frac{d}{dx}\left(x^3 - \sin(x)\right)$?

(b) Based on what you just explained, what is a reasonable assumption about what $\frac{d}{dx}\left(x^3 \sin(x)\right)$ might be?

(c) Let's just think about $\frac{d}{dx}\left(x^3\right)$ for a moment. What *is* x^3? Can you write this as a product? Call one of your functions $f(x)$ and the other $g(x)$. You should have that $x^3 = f(x)g(x)$ for whatever you used.

(d) Use your example to explain why, in general, $\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$.

(e) Let's show another way that we know this. Think about $\sin(x)$. We know two things:

$$\sin(x) = (1)(\sin(x)) \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

What, though, is $\frac{d}{dx}(1) \cdot \frac{d}{dx}\left(\sin(x)\right)$?

(f) Use all of this to reassure yourself that even though the derivative of a sum of functions is the sum of the derivatives of the functions, we will need to develop a better understanding of how the derivatives of products of functions work.

Activity 2.4.2 Building a Product Rule.

Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything. If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

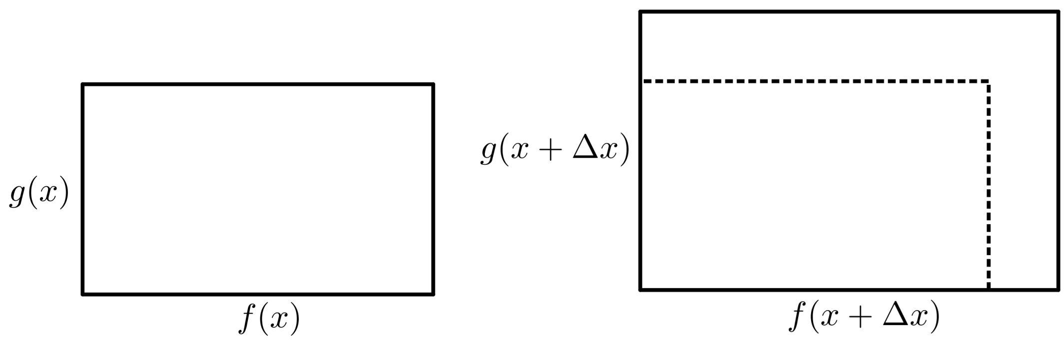


Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.

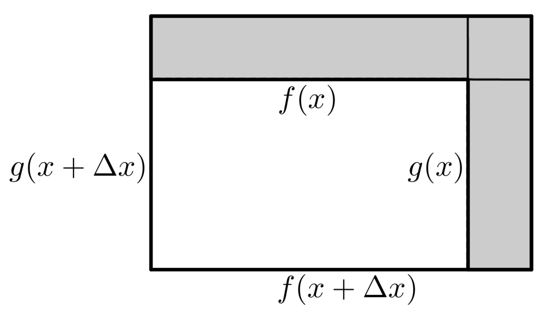


Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$$f(x+\Delta x)g(x+\Delta x) - f(x)g(x) = \underline{\qquad\qquad\qquad\qquad}.$$

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right)$$

We can split this fraction up into three fractions:

$$\begin{aligned} \frac{d}{dx}\left(f(x)g(x)\right) &= \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \\ &+ \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \\ &+ \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \end{aligned}$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

In the third limit, factor out either $\lim\limits_{\Delta x \to 0} \left(f(x + \Delta x) - f(x)\right)$ or $\lim\limits_{\Delta x \to 0} \left(g(x + \Delta x) - g(x)\right)$.

$$\begin{aligned} \frac{d}{dx}\left(f(x)g(x)\right) &= f(x) \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \\ &+ g(x) \lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \\ &+ \lim_{\Delta x \to 0} \left(\boxed{} \right) \left(\lim_{\Delta x \to 0} \left(\frac{\boxed{}}{\Delta x} \right) \right) \end{aligned}$$

(f) Use Definition 2.1.2 The Derivative Function to rewrite these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

Activity 2.4.3 Constructing a Quotient Rule.

We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \dfrac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$$u(x) = \underline{\hspace{8cm}}$$

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$, right? We care about finding $f'(x)$!

Use what you found for $u'(x)$ and solve for $f'(x)$.

$$f'(x) = \underline{\hspace{8cm}}$$

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \dfrac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\dfrac{u(x)}{v(x)}$.

$$f'(x) = \underline{\hspace{8cm}}$$

2.5 The Chain Rule

Activity 2.5.1 Composition (and Decomposition) Pictionary.

This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

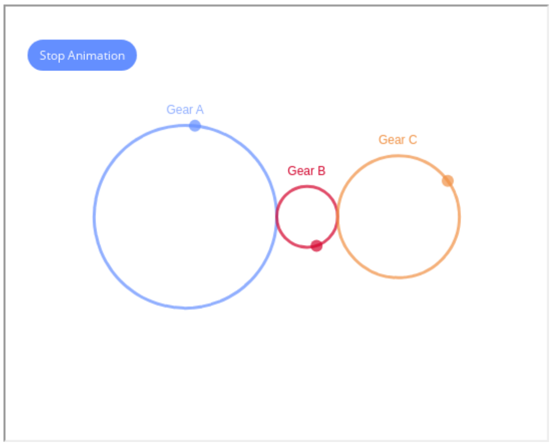
(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that still will give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?

Activity 2.5.2 Gears and Chains.

Let's think about some gears. We've got three gears, all different sizes. But the gears are linked together, and a small motor works to spin one of the gears. Since the gears are linked, when one gear spins, they all spin. But since they are different sizes, they complete a different number of revolutions: the smaller ones spin more times than the larger ones, since they have a smaller circumference.



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For our purpose, let's say that Gear A is being driven by the motor.

(a) Let's try to quantify how much "faster" Gear B is spinning compared to Gear A. How many revolutions does Gear B complete in the time it takes Gear A to complete one revolution?

(b) Now quantify the speed of Gear C compared to its neighbor, Gear B. How many revolutions does Gear C complete in the time it takes Gear B to complete one revolution?

(c) Use the above relative "speeds" to compare Gear C and Gear A: how many revolutions does Gear C complete in the time it takes Gear A to complete one revolution?

More importantly, how do we find this?

(d) Now, let's translate this into some derivative notation: we've really been finding rates at which one thing changes (the speed of the gear spinning) relative to another's.

Call the speed of Gear B compared to Gear A: $\frac{dB}{dA}$. Now, call the speed of Gear C compared to Gear B: $\frac{dC}{dB}$. Come up with a formula to find $\frac{dC}{dA}$.

Chapter 3

Implicit Differentiation

3.1 Implicit Differentiation

Activity 3.1.1 Thinking about the Chain Rule.

(a) Explain to someone how (and why) we use the The Chain Rule to find the following derivative:

$$\frac{d}{dx}\left(\sqrt{\sin(x)}\right).$$

(b) Let's say that $f(x) = \sin(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{f(x)}\right).$$

How is this different, or not different, than the previous derivative?

(c) Let's say that we have some other function, $g(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{g(x)}\right).$$

How is this different, or not different, than the previous derivatives?

(d) What is the difference between the following derivatives:

$$\frac{d}{dx}\left(\sqrt{x}\right) \qquad \frac{d}{dx}\left(\sqrt{y}\right) \qquad \frac{d}{dy}\left(\sqrt{y}\right)$$

Activity 3.1.2 Slopes on a Circle.

Visualize the unit circle. Feel free to draw one, or find the picture above. We're going to think about slopes on this circle.

(a) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly horizontal. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(b) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly vertical. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(c) Find the point(s) where $x = \frac{1}{2}$. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(d) For the unit circle defined by the equation $x^2 + y^2 = 1$, apply the derivative to both sides of this equation to get the following:

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(1) \end{aligned}$$

Carefully consider each of these derivatives (each of the terms). Which of these will you need to apply the Chain Rule for?

(e) Differentiate. Solve for $\frac{dy}{dx}$ or y', whichever notation you decide to use.

(f) Go back to the first few questions, and try to answer them again:

(a) Find the locations of any horizontal tangent lines (where $\frac{dy}{dx} = 0$).

(b) Find the locations of any vertical tangent lines (where $\frac{dy}{dx}$ doesn't exist, or where you would divide by 0).

(c) Find the values of $\frac{dy}{dx}$ for the points on the circle where $x = \frac{1}{2}$.

Activity 3.1.3 A New Curve.

Let's consider a new curve:

$$\sin(x) + \sin(y) = x^2y^2.$$

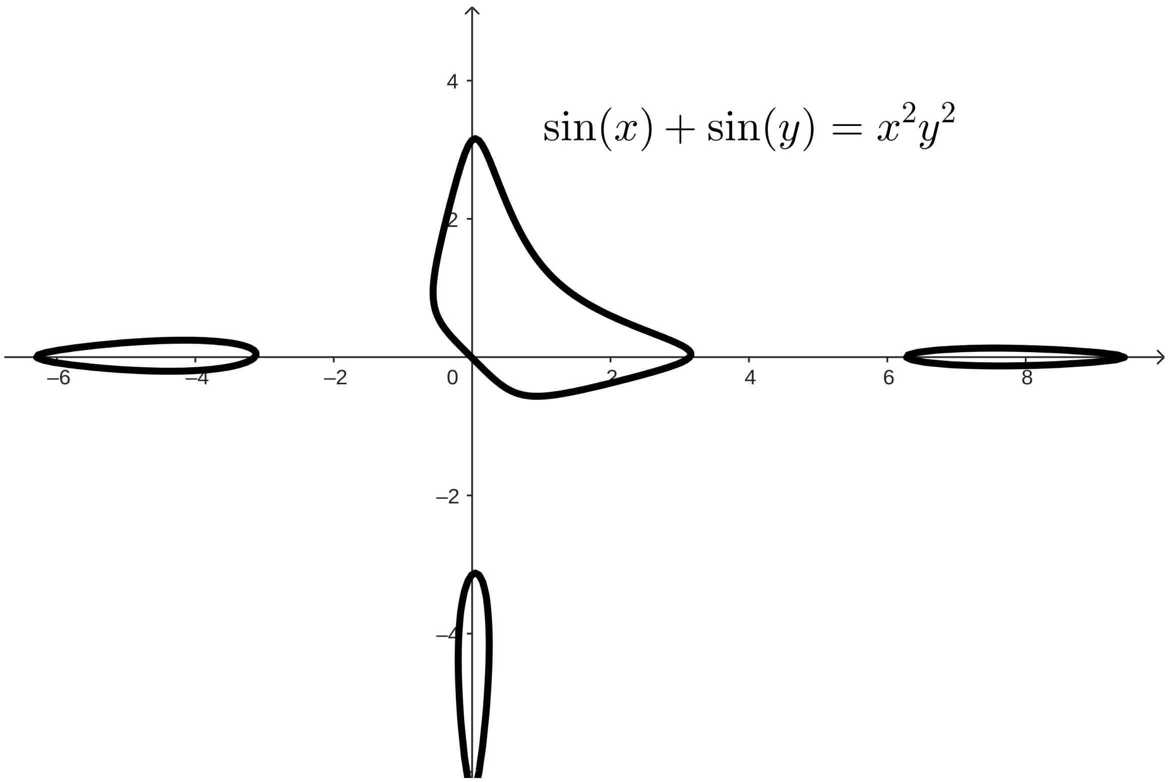


Figure 3.1.1

(a) We are going to find $\frac{dy}{dx}$ or y'. Let's dive into differentiation:

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)+\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right)\\ \frac{d}{dx}\left(\sin(x)\right)+\frac{d}{dx}\left(\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right)\end{aligned}$$

Think carefully about these derivatives. For each of the three, how will you approach it? What kinds of nuances or rules or strategies will you need to think about? Why?

(b) Implement your ideas or strategies from above to differentiate each term.

(c) Now we need to solve for $\frac{dy}{dx}$ or y', whichever you are using. While this equation can look complicated, we can notice something about the "location" of $\frac{dy}{dx}$ or y' in our equation.

Why do we always know that $\frac{dy}{dx}$ or y' will be *multiplied* on a term whenever it shows up?

(d) Now that we are confident that we will *always* know that we are multiplying this derivative, we can employ a consistent strategy:

(a) Rearrange our equation so that every term with a $\frac{dy}{dx}$ or y' is on one side, and everything without is on the other.

(b) Now we are guaranteed that $\frac{dy}{dx}$ or y' is a common factor: factor it out.

(c) Now we can solve for $\frac{dy}{dx}$ or y' by dividing!

Solve for $\frac{dy}{dx}$ or y' in your equation!

(e) Build the equation of a line that lays tangent to the curve at the origin. Does the value of $\frac{dy}{dx}$ at $(0, 0)$ look reasonable to you?

3.2 Derivatives of Inverse Functions

Activity 3.2.1 Building the Derivative of the Logarithm.

We're going to accomplish two things here:

1. By the end of this activity, we'll have a nice way of thinking about $\frac{d}{dx}(\ln(x))$, and we will now be able to differentiate functions involving logarithms!
2. Throughout this activity, we're going to develop a way of approaching derivatives of inverse functions more generally. Then we can apply this framework to other functions!

Let's think about this logarithmic function!

(a) We have stated (a couple of times now) how we define the log function:

$$y = e^x \longleftrightarrow x = \ln(y).$$

This arrow goes both directions: the log function is the inverse of the exponential, but the exponential is the inverse of the log function!

Can you rewrite the relationship $y = \ln(x)$ using its inverse (the exponential)?

(b) For your inverted $y = \ln(x)$ from above (it should be $x =$ ________), apply a derivative operator to both sides, and use implicit differentiation to find $\frac{dy}{dx}$ or y'.

(c) A weird thing that we can notice is that when we use implicit differentiation, it is common to end up with our derivative written in terms of both x and y variables. This makes sense for our earlier examples: we needed specific coordinates of the point on the circle, for instance, to find the slope there.

But if $y = \ln(x)$, we want $\frac{dy}{dx}$ or y' to be a function of x:

$$f(x) = \ln(x) \longrightarrow f'(x) = \_\_\_\_\_\_\_\_ .$$

Your derivative is written with only y values.

Since $y = \ln(x)$, replace any instance of y with the log function. What do you have left?

(d) Remember that $y = \ln(x)$. Substitute this into your equation for $\frac{dy}{dx}$. Can you write this in a pretty simplistic way?

(e) Before we go much further, we should be a bit careful. What is the domain of this derivative?

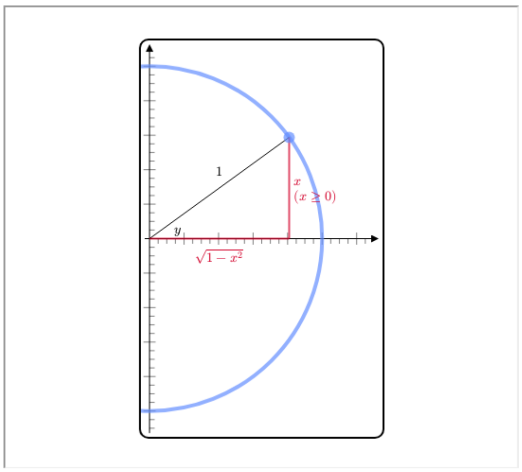
What are the values of x where $\frac{d}{dx}(\ln(x))$ makes sense to think about?

Activity 3.2.2 Finding the Derivative of the Inverse Sine Function.

We're going to do the same trick, except that there will be a couple of small differences due to thinking specifically about trigonometric functions.
Let's think about the function $y = \sin^{-1}(x)$. We know that this is equivalent to $x = \sin(y)$ (for y-values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

(a) Move the point around the portion of the unit circle in the graph below. Convince yourself that:

- $\sin(y) = x$
- $\sin(y) \geq 0$ when $0 \leq y \leq \frac{\pi}{2}$
- $\sin(y) < 0$ when $-\frac{\pi}{2} \leq y < 0$



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What is $\cos(y)$ in this figure? Does the sign change depending on the value of y?

(b) Use implicit differentiation and the equation $x = \sin(y)$ to find $\frac{dy}{dx}$ or y'

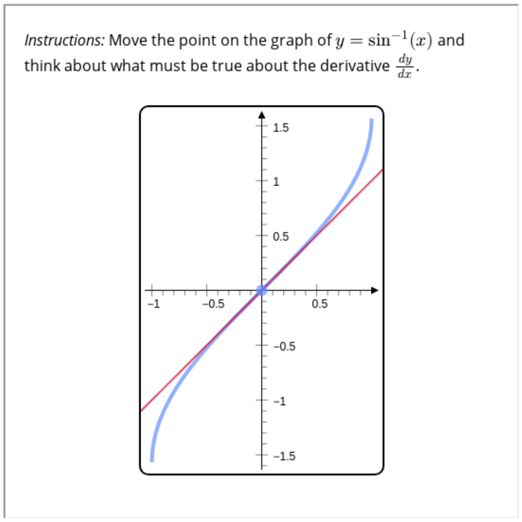
(c) If you still have your derivative written in terms of y, make sure to write $\cos(y)$ in terms of x!

(d) Let's think about the domain of this derivative: what x-values make sense to think about?

Think about this both in terms of what x-values reasonably fit into your formula of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$ as well as the domain of the inverse sine function in general.

(e) Notice that in the denominator of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$, you have a square root. Based on that information (and the visual above), what do you expect to be true about the sign of the derivative of the inverse sine function?

Confirm this by playing with the graph of $y = \sin^{-1}(x)$ below.



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(f) Investigate the behavior of $\frac{dy}{dx}$ at the end-points of the function: at $x = -1$ and $x = 1$. What do the slopes look like they're doing, graphically?

How does this work when you look at the function you built above? What happens when you try to find $\left.\dfrac{dy}{dx}\right|_{x=-1}$ or $\left.\dfrac{dy}{dx}\right|_{x=1}$?

Activity 3.2.3 Building the Derivatives for Inverse Tangent and Secant.

(a) Consider the triangle representing the case when $y = \tan^{-1}(x)$.

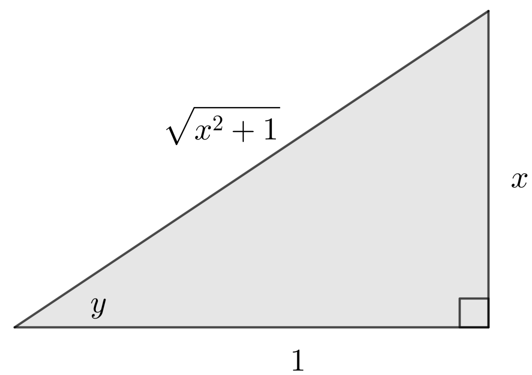


Figure 3.2.1

For $x = \tan(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ based on the triangle above, but we will refer back to the version with the $\sec(y)$ in it later.

(b) Consider the triangle representing the case when $y = \sec^{-1}(x)$.

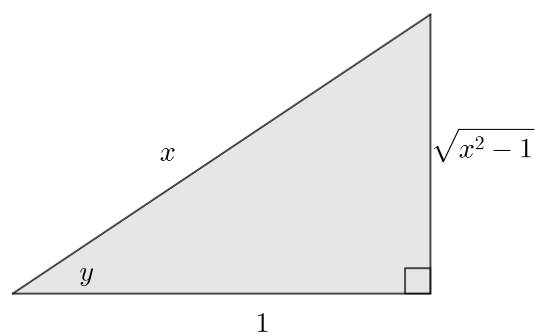
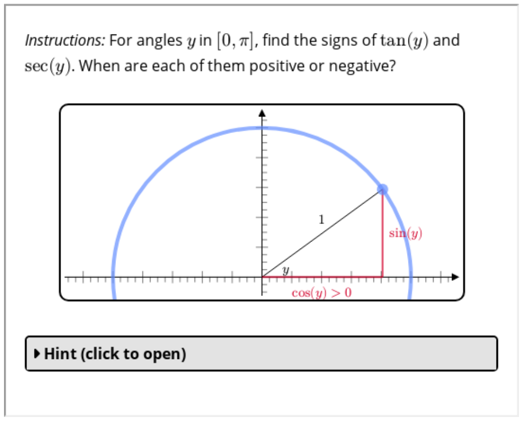


Figure 3.2.2

For $x = \sec(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ and $\tan(y)$ based on the triangle above, but we will refer back to the version with the functions of y in it later.

(c) Here's a graph of just the unit circle for angles $[0, \pi]$. We are choosing to focus on this region, since these are the angles that the inverse tangent and inverse secant functions will return to us. We want to investigate the signs of $\tan(y)$ and $\sec(y)$.



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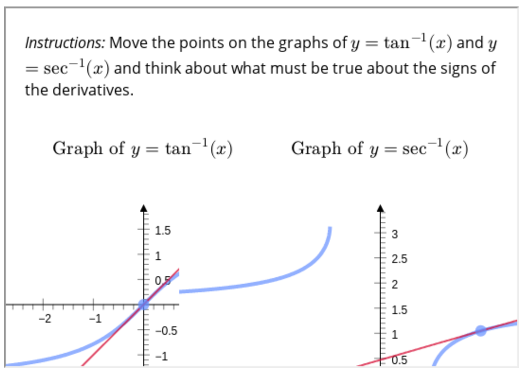
(d) Go back to our derivative expressions for both the inverse tangent and inverse secant functions. What do we know about the signs of these derivatives?

(e) Confirm your idea about the sign of the derivatives by investigating the graphs of each function.



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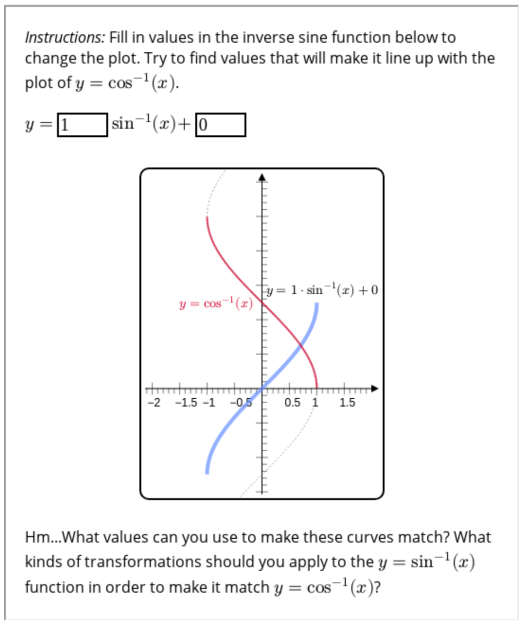
(f) How do we need to write these derivatives, when we write them in terms of x to account for the sign of the derivative?

Activity 3.2.4 Connecting These Inverse Functions.

We're going to look at a graph of $y = \cos^{-1}(x)$, but we're specifically going to try to compare it to the graph of $y = \sin^{-1}(x)$. We'll use some graphical transformations to make these functions match up, and then later we'll think about derivatives.

(a) Ok, consider the graph of $y = \cos^{-1}(x)$ and a transformed version of the inverse sine function. Apply some graphical transformations to make these match!



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(b) It might be fun to think about another reason that this connection between $\sin^{-1}(x)$ and $\cos^{-1}(x)$ exists.

Consider this triangle:

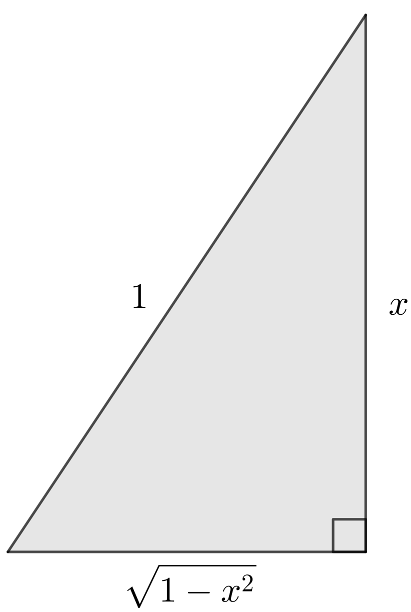


Figure 3.2.3

We're going to think about these inverse trigonometric functions as angles: let $\alpha = \cos^{-1}(x)$ and $\beta = \sin^{-1}(x)$. We can rewrite these as:

$$\cos(\alpha) = x$$
$$\sin(\beta) = x.$$

Can you fill in your triangle using this information?

Why does $\alpha + \beta = \frac{\pi}{2}$? Convince yourself that this is what we did with the graphical transformations above, as well.

(c) Use this equation above, rewriting $\cos^{-1}(x)$ as some expression involving the inverse sine function, and then find

$$\frac{d}{dx}\left(\cos^{-1}(x)\right).$$

We could repeat this task to try to connect the graph of $y = \tan^{-1}(x)$ with $y = \cot^{-1}(x)$ as well as the graph of $y = \sec^{-1}(x)$ with $y = \csc^{-1}(x)$, but we can hopefully see what will happen. In each case, we have the same kind of connection that we saw in the triangle, since these are cofunctions of each other!

We can summarize by believing that:

$$\begin{aligned}
\frac{d}{dx}\left(\cos^{-1}(x)\right) &= -\frac{d}{dx}\left(\sin^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\cot^{-1}(x)\right) &= -\frac{d}{dx}\left(\tan^{-1}(x)\right) = -\frac{1}{x^2+1} \\
\frac{d}{dx}\left(\csc^{-1}(x)\right) &= -\frac{d}{dx}\left(\sec^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

3.3 Logarithmic Differentiation

Activity 3.3.1 Returning to the Power Rule.

Back in Section 2.3 we built an explanation for why $\frac{d}{dx}(x^n) = nx^{n-1}$ that relied on thinking about exponents as repeated multiplication: it relied on n being some positive integer. We said, at the time, that the Power Rule generalizes and works for *any* integer, but did so without explanation. Let's consider $y = x^n$ where n is just some real number without any other restrictions.

(a) Apply a logarithm to both sides of this equation:

$$\ln(y) = \ln(x^n)$$

Now use one of the Properties of Logarithms to rewrite this equation.

(b) Use implicit differentiation to find $\frac{dy}{dx}$ or y'.

(c) Explain to yourself why this is equivalent to the Power Rule that we built so long ago:

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(d) Let's get weird. What if we have a not-quite-power function? Where the thing in the exponent isn't simply a number, but another variable?

Let's use the same technique to think about $y = x^x$ and its derivative. First, though, confirm that this is not a power function (and so we cannot use the Power Rule to find the derivative) and is also not an exponential function (and so the derivative isn't itself or itself scaled by a log).

(e) Now apply a log to both sides:

$$\ln(y) = \ln(x^x).$$

Rewrite this using the same log property as before, and then use implicit differentiation to find $\frac{dy}{dx}$ or y'.

(f) Explain to yourself why logs are friends, especially when trying to differentiate functions in the form of $y = (f(x))^{g(x)}$.

Activity 3.3.2 Logarithmic Differentiation with Products and Quotients.

Let's say we've got some function that has products and quotients in it. Like, a lot. Consider the function:

$$y = \frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}.$$

(a) Work out a general strategy for how you would find y' directly. Where would you have to use Quotient Rule? What are the pieces? Where would you have to use Product Rule? What are the pieces? Where would you have to use the Chain Rule? What are the pieces?

To be clear: do not actually differentiate this. Just look at it in horror and try to outline a plan that some other fool would use.

(b) Let's instead use logarithmic differentiation. First, apply the log to both sides to get:

$$\ln(y) = \ln\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right).$$

Since this function is just a bunch of products of things with exponents all put into some big quotient, we can use our log properties to rewrite this!

(c) We should have:

$$\ln(y) = 2\ln(x-4) + \frac{1}{2}\ln(3x+1) - 7\ln(x+1) - 3\ln(x+5).$$

Confirm this.

(d) Now differentiate both sides! You'll have to use some Chain Rule (but not a lot)! Refer back to Fact 3.3.1 for help here.

(e) Solve for $\frac{dy}{dx}$ or y'.

(f) While this is not a *nice* looking expression for the derivative, spend some time confirming that this was a nicer *process* than differentiating directly. This is because logs are friends.

Chapter 4

Applications of Derivatives

4.1 Mean Value Theorem

4.2 Increasing and Decreasing Functions

Activity 4.2.1 How Should We Think About Direction?

Our goal in this activity is to motivate some new terminology and results that will help us talk about the "direction" of a function and some interesting points on a function (related to the direction of a function). For us to do this, we'll look at some different examples of functions and try to think about some unifying ideas.

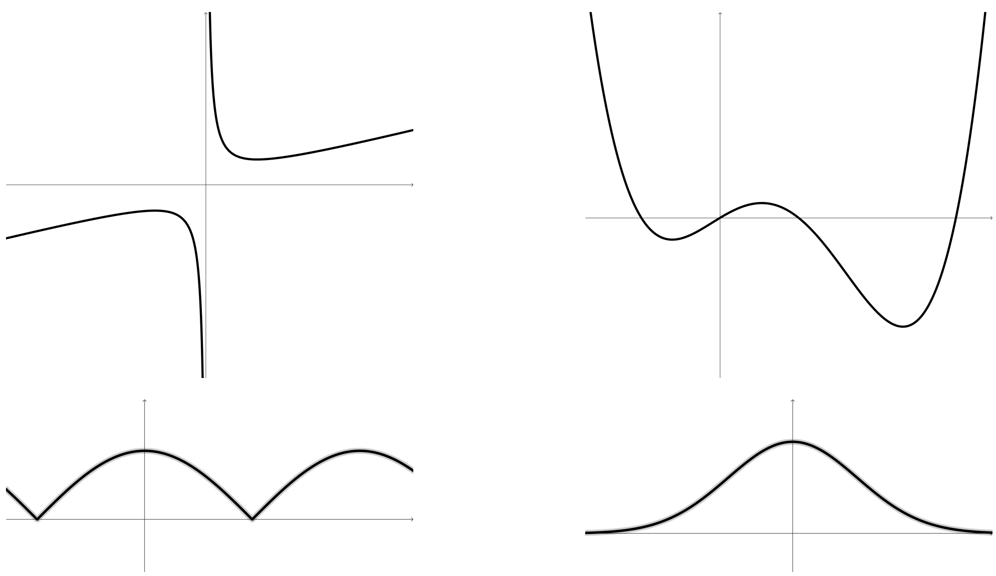


Figure 4.2.1

These examples do not cover all of the possibilities of how a function can act, but will hopefully provide us enough fertile ground to think about some different situations.

(a) In each graph, find and identify:

- The intervals where the function is increasing.
- The intervals where the function is decreasing.

- The points (or locations) around and between these intervals, the points where the function changes direction or the direction terminates.

(b) Make a conjecture about the behavior of a function at any point where the function changes direction.

(c) Look at the highest and lowest points on each function. You can even include the points that are highest and lowest just compared to the points around it. Make a conjecture about the behavior of the function at these points.

Activity 4.2.2 Comparing Critical Points.

Let's think about four different functions:

- $f(x) = 4 + 3x - x^2$
- $g(x) = \sqrt[3]{x+1} + 1 + x$
- $h(x) = (x-4)^{2/3}$
- $j(x) = 1 - x^3 - x^5$

Our goal is to find the critical points on the interval $(-\infty, \infty)$ and then to try to figure out if these critical points are local maximums or local minimums or just points that the function increases or decreases through.

(a) To start, we're going to be finding critical points. Without looking at a picture of the graph of the function, find the derivative.

Are there any x-values (in the domain of the function) where the derivative doesn't exist? We are normally looking for things like division by 0 here, but we could be finding more than that. Check out When Does a Derivative Not Exist? to remind yourself if needed.

Are there any x-values (in the domain of the function) where the derivative is 0?

(b) Now that we have the critical points for the function, let's think about where the derivative might be positive and negative. These will correspond to the direction of a function, based on Theorem 4.1.5 Sign of the Derivative and Direction of a Function.

Write out the intervals of x-values around and between your list of critical points. For each interval, what is the sign of the derivative? What do these signs mean about the direction of your function?

(c) Without looking at the graph of your function:

- What changes about how your function increases up to or decreases down to a critical point based on whether the derivative was 0 or the derivative didn't exist?
- Does your function change direction at a critical point? What will that look like, whether it does or does not change direction?

(d) Give a basic sketch of your graph. It might be helpful to find the y-values for any critical points you've got. Then you can sketch your function increasing/decreasing in the intervals between these points.

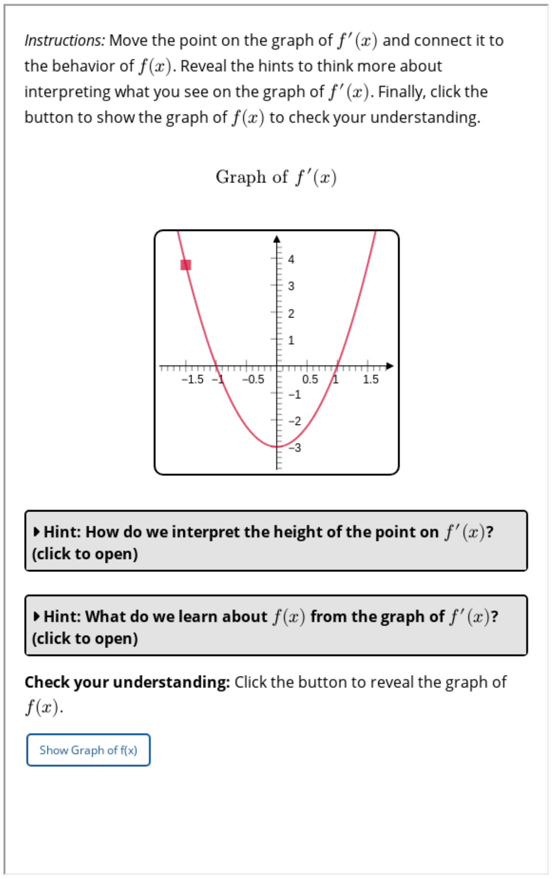
In your sketch, include enough detail to tell whether the derivative is 0 or doesn't exist at each critical point.

(e) Compare your sketch to the actual graph of the function (you can find all of the graphs in the hint).

Activity 4.2.3 First Derivative Test Graphically.

Let's focus on looking at a picture of a derivative, $f'(x)$, and trying to collect information about the function $f(x)$. This is what we've done already, except that we've done it by thinking about the representation of $f'(x)$ as a function rule written out with algebraic symbols. Here we'll focus on connecting all of that to the picture of the graphs.

For all of the following questions, refer to the plot below. You can add information with the hints whenever you need to. Don't reveal the picture of $f(x)$ until you're really ready to check what you know.



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(a) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is increasing.

(b) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is decreasing.

(c) Find the x-values of the critical points of $f(x)$. Once you've estimated these, classify them as local maximums, local minimums, or neither. Explain your reasoning.

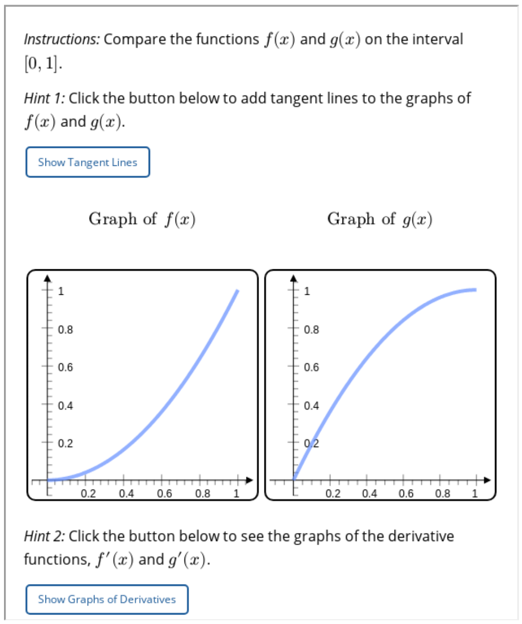
(d) What do you think the graph of $f(x)$ looks like? Do your best to sketch it or explain it before revealing it!

(e) Why could we estimate the x-values of the critical numbers of $f(x)$, but not find the actual coordinates? How come we can't find the y-value based on looking at the graph of $f'(x)$?

4.3 Concavity

Activity 4.3.1 Compare These Curves.

(a) Consider the two curves pictured below. Compare and contrast them. What characteristics of these functions are the same? What characteristics of these functions are different?



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(b) Explain the similarities you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(c) Explain the differences you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(d) Make a conjecture about the *rate of change* of both f' and g'. We'll call these things **second derivative** functions, $f''(x)$ and $g''(x)$.

Activity 4.3.2 Finding a Function's Concavity.

We're going to consider a pretty complicated function to work with, and employ a strategy similar to what we did with the first derivative:

- Our goal is to find the sign of $f''(x)$ on different intervals and where that sign *changes*.
- To do' this, we'll find the places that $f''(x) = 0$ or where $f''(x)$ doesn't exist. These are the critical points of $f'(x)$.
- From there, we can build a little sign chart, where we split up the x-values based on the domain of f and the critical numbers of f'. Then we can attach each interval of x-values with a sign of the second derivative, f'', on that interval.
- We'll interpret these signs. For any intervals where $f''(x) > 0$, we know that $f'(x)$ must be increasing and so $f(x)$ is concave up. Similarly, for any intervals where $f''(x) < 0$, we know that $f'(x)$ must be decreasing and so $f(x)$ is concave down.

Let's consider the function

$$f(x) = \ln(x^2 + 1) - \frac{x^2}{2}.$$

(a) Find the first derivative, $f'(x)$, and use some algebra to confirm that it is really:

$$f'(x) = -\frac{x(x+1)(x-1)}{x^2+1}.$$

While we have this first derivative, we *could* find the critical points of $f(x)$ and then classify those critical points using the First Derivative Test.

For our goal of finding the intervals where $f(x)$ is concave up and concave down, and then finding the inflection points of f, let's move on.

(b) Find the second derivative, $f''(x)$, and confirm that this is really:

$$f''(x) = -\frac{x^4 + 4x^2 - 1}{(x^2+1)^2}.$$

(c) Our goal is to find the x-values where $f''(x) = 0$ or where $f''(x)$ doesn't exist.

Note that in order to find where $f''(x) = 0$, we are really looking at the x-values that make the numerator 0:

$$x^4 + 4x^2 - 1 = 0.$$

Then, to find where $f''(x)$ doesn't exist, we are finding the x-values that make the denominator 0:

$$(x^2 + 1)^2 = 0.$$

Solve these equations.

(d) You have two critical points of $f'(x)$ (let's just call them x_1 and x_2). These are possible inflection points of $f(x)$, but we need to check to see if the concavity changes at these points.

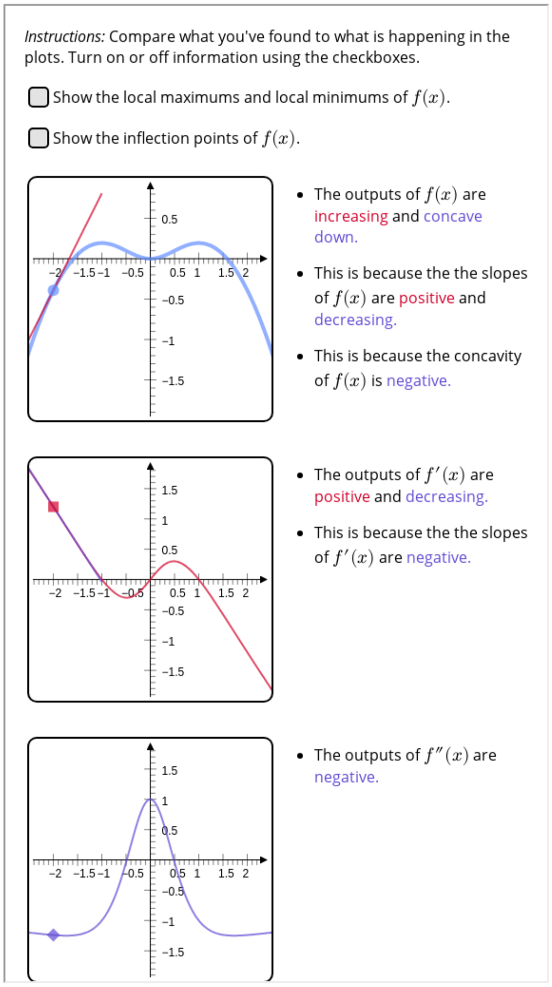
Fill in the following sign chart and interpret.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''					
f					

Let's confirm what we've just calculated graphically. Below, we have three graphs:

1. $f(x) = \ln(x^2+1) - \dfrac{x^2}{2}$
2. $f'(x) = -\dfrac{x(x+1)(x-1)}{x^2+1}$
3. $f''(x) = -\dfrac{x^4+4x^2-1}{(x^2+1)^2}$

Move the point on any graph and make sure the statements about signs, directions, and concavity match what you found! You should notice that signs of the first and second derivative change at the local maximums/minimums and the inflection points that we found.



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4.4 Global Maximums and Minimums

Activity 4.4.1 When Would We Not Have Maximums or Minimums?

In this section, we're going to define these global maximums and then, most importantly, try to predict when these global maximums or global minimums might actually exist for a function.
To start, maybe we should come up with some examples of functions that do not have them!

(a) Come up with some situations where a function does not have some combination of global maximum/minimums. Sketch some graphs!

(b) Come up with some examples of graphs of functions that are bounded (do not ever have y-values that tend towards infinity in a limit) that do not have some combination of global maximum/ minimums.

(c) For the examples of graphs that you have built or collected, what are the features of the functions that allow for the examples you picked? If you could impose some requirements that would "fix" the examples you found (so that they had both a global maximum and a global minimum), what requirements could you use?

4.5 Optimization

Activity 4.5.1 Constructing a Can.

A typical can of pop is 355 ml, and has around 15 ml of headspace (air). This results in a can that can hold approximately 23 cubic inches of volume.
Let's say we want to construct this can in the most efficient way, where we use the least amount of material. How could we do that? What are the dimensions of the can?

(a) First, let's think of our can and try to translate this to some geometric shape with variable names. Collect as much information as we can about this setup! What is the shape? What are the names of the dimensions?

(b) What is the actual measurement that we are trying to optimize? Are we finding a maximum or a minimum?

(c) What other information about the can do we know? How do we translate this into a constraint, or a connection between our variables?

(d) What kinds of values can our variables take on? Is there a smallest value for either? A largest? Are there other limitations to these?

(e) Now we need to set up a calculus statement. This part mostly relies on us finding a way to build a single-variable function defining the surface area. Build that function, and then write down the calculus statement.

(f) Do some calculus to find the global maximum or minimum, and solve the optimization problem.

(g) What is the relationship between r and h, here? How do they compare? What about the height and diameter of our can?

Is this relationship noticeable in a typical can of pop?

4.6 Linear Approximations

Activity 4.6.1 Approximating an Exponential Function.

Let's consider the function $f(x) = e^x$. We're going to build the linear approximation, $L(x)$, but we also want to focus on understanding what the "center" is, and how we think about accuracy of our estimations.

(a) We first need to find a "good" center for our linear approximation. We have two real requirements here:

(a) We need the center to be some x-value that will be "close" to the inputs we're most interested in. We know that $L(x) \approx f(x)$ for x-values "near" the center, so we should keep this in mind. We don't have a specific input that we're interested in (we are not specifically focused on estimating $f(7.35)$ for instance), so we don't need to worry about this for now.

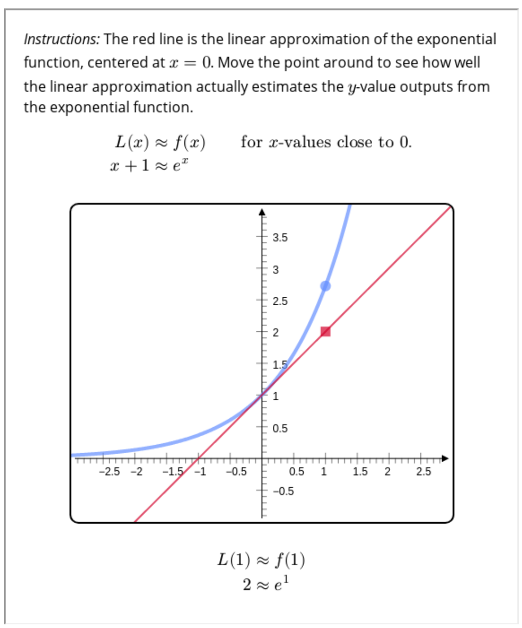
(b) We are going to need to evaluate the function and its derivative at the center: we use $f'(a)$ to find the slope and $f(a)$ to find a y-value for a point on the line. We'd like to choose a center that will make evaluating these functions reasonable, if we can!

We are going to choose a center of 0: why?

(b) Build a linear approximation of $f(x) = e^x$ centered at $x = 0$.

(c) Use your linear approximation to estimate the value of $\sqrt[10]{e} = e^{\frac{1}{10}}$.

(d) Let's visualize this approximation a bit:



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Are you confident in your approximation of $\sqrt[10]{e}$? Would you be more or less confident in an approximation of $\frac{1}{e}$? Why?

(e) Is your estimate of $\sqrt[10]{e}$ too big or too small? How can you tell, without even calculating the actual value of $\sqrt[10]{e}$?

How can you tell that *every* estimate that you get out of *any* linear approximation of e^x (no matter what the center is) is going to be too small?

Activity 4.6.2 Approximating some Values.

Pick one of the following values to approximate:

- $\sin(-0.023)$
- $\ln(2)$
- $\sqrt{8}$
- $\sqrt[3]{10}$

Throughout the rest of this activity, use your value to build a linear approximation of some relevant function and estimate the value you chose.

(a) To build a linear approximation of some function at some center, we need two things:

   (a) A function.

   (b) A center.

   What function will you be using for $f(x)$? Why that one?

(b) What center are you choosing? Why that one?

(c) Build your linear approximation at your center! You should end up with an actual linear function. It might be helpful to plot this linear function and your actual function to confirm that you have actually built a tangent line.

(d) Use your linear approximation function to estimate your value! Report the estimate, and comment on the accuracy of your estimate. Without calculating the actual value, can you tell if this is close or not? Do you have an overestimate or underestimate?

Activity 4.6.3 Walking in the Footsteps of Ancient Mathematicians.

Let's travel all the way back to the first (or maybe second) century AD and re-create Heron's method to approximate the value of $\sqrt{2}$. We'll develop this using modern calculus, and simple linear approximation.

We're going to reframe the problem, and instead we're going to try to use a linear approximation of $f(x) = x^2 - 2$ to approximate the x-value where $f(x) = 0$. We know enough about quadratic functions to know that there are two values: $x = -\sqrt{2}$ and $x = \sqrt{2}$.

(a) We're going to build a linear approximation of $f(x) = x^2 - 2$, and we need a reasonable center. Honestly, any integer will work, since we can evaluate f and f' really easily, but we want to find one that is close to $\sqrt{2}$. Let's center our approximation at $x = 2$.

Find $f'(x)$, and then construct the linear approximation:

$$L(x) = f'(2)(x - 2) + f(2).$$

(b) Now we know that $L(x) \approx f(x)$ for x-values near our center, $x = 2$. What if we estimate the x-value where $f(x) = 0$ by solving $L(x) = 0$ instead? Since $L(x) \approx f(x)$, the x-value where $L(x) = 0$ should make $f(x)$ pretty close to 0 at least.

Solve $L(x) = 0$.

(c) Ok, this might be kind of close to the value of $\sqrt{2}$, right? Let's visualize this.

Hm...so this isn't that good of an approximation yet. We can check this by looking at the actual value of our function at $x = \frac{3}{2}$ and seeing if it's close to 0.

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \left(\frac{3}{2}\right)^2 - 2 \\ &= \frac{9}{4} - 2 \\ \frac{1}{4} \end{aligned}$$

This...isn't that close to 0.

So let's try this again. This time, though, let's center our *new* linear approximation at $x = \frac{3}{2}$.

(d) Now set this *new* linear approximation equal to 0 and solve $L(x) = 0$ to estimate the solution to $f(x) = 0$.

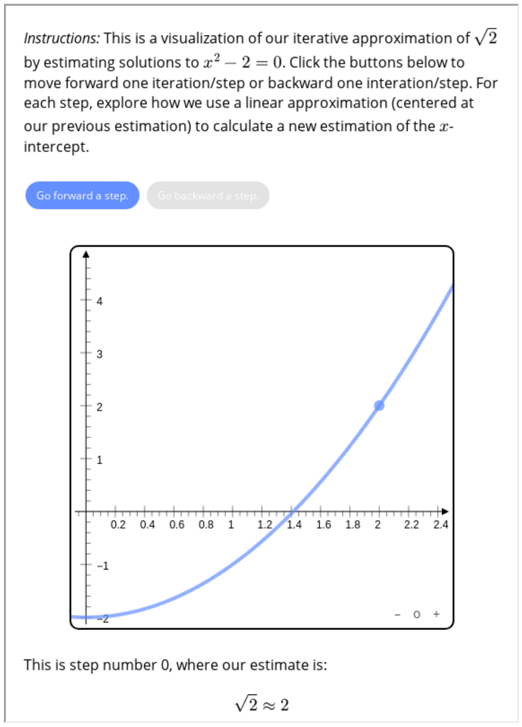
(e) We can keep repeating this process, and that's exactly what the mathematicians we talked about discovered.

Say we've built a linear approximation at some x-value (we'll call it x_{old}).

$$L(x) = f'(x_{\text{old}})(x - x_{\text{old}}) + f(x_{\text{old}}).$$

Set this equal to 0 and solve.

(f) Let's visualize these calculations.



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Something kind of strange happens in the last two steps. Why does the value of our estimation not change? What happens to our estimate?

4.7 L'Hôpital's Rule

Activity 4.7.1 Building L'Hôpital's Rule.

We're going to take a closer look at the indeterminate form, $\frac{0}{0}$, and use our new ideas of linear approximation to think about how these types of things work.

We're going to be working with the following limit:

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are differentiable at $x = a$ (since we're going to want to build linear approximations of them).

(a) Write out the linear approximations for both $f(x)$ and $g(x)$, both centered at $x = a$. We'll call them $L_f(x)$ and $L_g(x)$.

(b) Describe how well or how poorly these linear approximations estimate the values from our functions $f(x)$ and $g(x)$? What happens to these approximations as we get close to the center $x = a$? What happens in the limit as $x \to a$?

(c) Let's rewrite our limit. We can replace $f(x)$ with our formula for its linear approximation, $L_f(x)$ and replace $g(x)$ with its linear approximation, $L_g(x)$:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left(\frac{\qquad\qquad}{\qquad\qquad} \right)$$

(d) Up until now, we have not thought about indeterminate forms at all. Let's start now.

If this limit is a $\frac{0}{0}$ indeterminate form, then that means that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

Since our functions are, by definition, differentiable at $x = a$, then they also have to be continuous at $x = a$. What does this mean about the values of $f(a)$ and $g(a)$?

(e) Use this new information about the values of $f(a)$ and $g(a)$ to revisit the limit. We rewrote $\lim_{x\to a} \frac{f(x)}{g(x)}$ by replacing each function with its linear approximation. What happens with the algebra when we know this information about $f(a)$ and $g(a)$?

So we have a really nice result here! In the $\frac{0}{0}$ indeterminate form, we can replace the ratio of the y-values from our functions with the ratio of slopes (coming from the first derivatives) of our functions.

In general, we'll put a step in between, where we find $f'(x)$ and $g'(x)$ first before trying to evaluate these derivatives at $x = a$.

Chapter 5

Antiderivatives and Integrals

5.1 Antiderivatives and Indefinite Integrals

Activity 5.1.1 Find a Function Where....

For each of the following derivatives, find a function $f(x)$ whose first (or second) derivative matches the listed derivative.

(a) $f'(x) = 4x^3 + 6$

(b) $f'(x) = 8x^7 - x^4 + x$

(c) $f'(x) = \sqrt{x} - \dfrac{1}{x^2}$

(d) $f''(x) = x^3 + \cos(x)$

(e) $f''(x) = e^x - \dfrac{1}{\sqrt[3]{x}}$

(f) Go back through each of the above derivatives, and find a different option for $f(x)$ that still works. Make sure that it is something completely unique, and not just an equivalent function that is written differently.

Why are you able to find multiple answers in these questions, but not when we are given a function and need to find a derivative?

Activity 5.1.2 A File Sorting Speed Test.

A computer program is trying to sort a group of computer files based on their size. The program isn't very efficient, and the time that it takes to sort the files increases when it tries to sort more files.

The time that it takes, measured in seconds, based on the total, cumulative size of the files g, measured in gigabytes, is modeled by a function $T(g)$. We don't know the function, but we do know that the time increases at an instantaneous rate of $0.0001g$ seconds when the total size, g increases slightly. That is, the rate of change is a function of the file size, g, itself.

(a) We can build a function for $T'(g)$. What is it?

(b) Find all of the possibilities for the function modeling the time, T, that it takes the computer program to sort files with a total size of g.

(c) What does your constant C represent, here? You can interpret it graphically, interpret it by thinking about derivatives, but you should also interpret it in terms of the time that it takes this program to sort these files by size.

(d) Let's say that we feed some number of files totaling up to 1.4GB in size into this program. It takes 0.24 seconds to sort the files by size.

Find the function, $T(g)$, that models how quickly this program sorts these files.

5.2 Riemann Sums and Area Approximations

Activity 5.2.1 Approximating Areas.

We're going to consider two different functions, and some areas based on them. Let's think about two functions: $f(x) = 2x + 1$ and $g(x) = x^2 + 1$. For both of these functions, we'll focus on the interval $[0, 2]$. Instead of thinking about the function only, we'll be considering the two-dimensional region bounded between the graph of our function and the x-axis between $x = 0$ and $x = 2$.

(a) Find the area of the region bounded between the graph of $f(x) = 2x+1$ and the x-axis between $x = 0$ and $x = 2$.

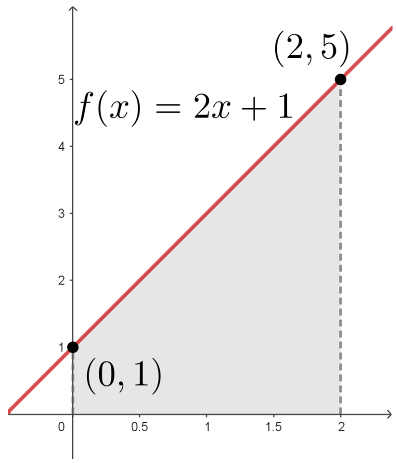


Figure 5.2.1

How did you evaluate this area? What kind(s) of shape(s) did you think about?

(b) Estimate the area of the region bounded between the graph of $g(x) = x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$.

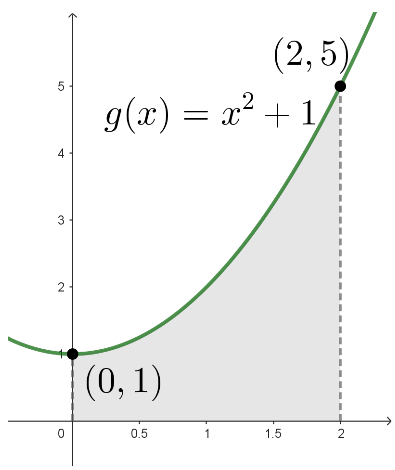


Figure 5.2.2

Archimedes of Syracuse discovered how to calculate this area exactly, without estimation, around 300 BC, writing his results in the now-famous "Quadrature of the Parabola." This is, notably, before the formalization of Calculus (during the 1600's). It might be unfair to say that Archimedes proved this "without using calculus," though, since his technique, the "Method of Exhaustion," is really a version of what we do in calculus, but without a formal framework of limits.

How did you estimate this area? What kind(s) of shape(s) did you think about?

(c) Come up with an upper and lower bound for this area. In other words, give an underestimate and overestimate for the actual area we would like to know.

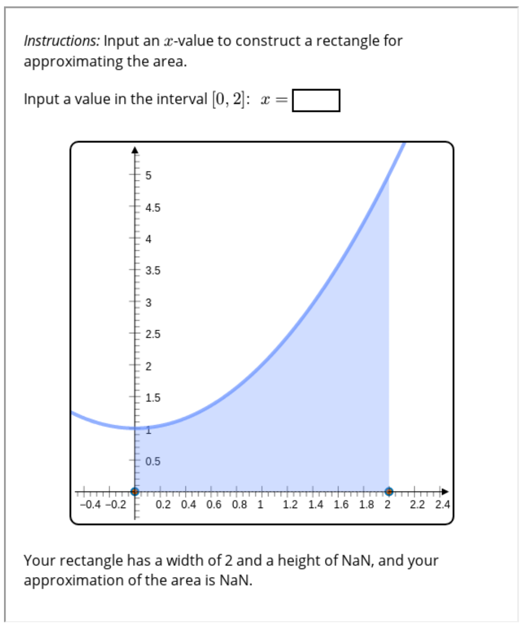
How did you come up with these estimates? How “good” do you think your estimates are? Can you come up with “better” (or closer) ones?

Activity 5.2.2 Approximating the Area using Rectangles.

We're going to stick with the function $g(x) = x^2 + 1$ on the interval $[0, 2]$, and keep thinking about the area bounded by the curve and the x-axis on this interval. We're going to approximate the area in a couple of different tries, each one more accurate than the one before. By the end of this activity, we'll have a pretty good process built!

(a) Let's start with approximating this region with a single rectangle. We're going to define the rectangle by picking some x-value in the interval $[0, 2]$. Then, we'll use the point at that x-value to define the height of our rectangle.

Essentially, we are picking a single point on our function on the interval, and our approximation is pretending that the single point we picked is representative of the whole function on the interval.



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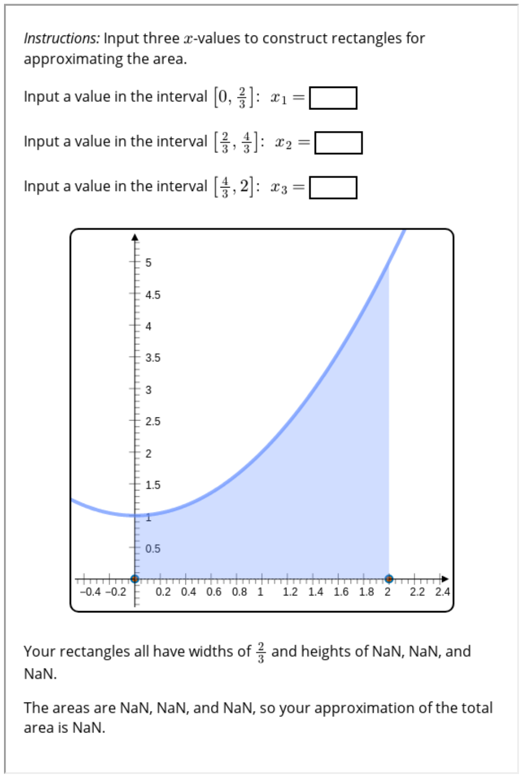
(b) Can you try repicking an x-value, and try to find one that gets you an area approximation that is pretty good?

(c) We're going to use more rectangles. Let's jump up to 3 rectangles. If we split up the interval between $x = 0$ and $x = 2$ into 3 rectangles, we can make them all the same width, and pick an x-value that we can use to get a representative point for each of the 3 rectangles.

We'll need to pick 3 x-values this time.



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(d) Can you try repicking your x-values, and trying to find one that gets you an area approximation that is pretty good?

(e) Let's scale this up a bit. Pick a good number for your number of rectangles. We'll call this value n.

(If you're working in a classroom, maybe it would be good to pick the number of groups or the number of students, or some other number between 10 and 20 or something like that.)

For your value n, we're going to divide up the interval between $x = 0$ and $x = 2$ into n pieces. These will be the intervals that we pick from to get our rectangles. What are the subintervals? What are the widths of each subinterval (and then the widths of the rectangles)? Call this with Δx.

(f) For each subinterval, pick an x-value in the subinterval to represent it.

(g) Evaluate the function $f(x) = x^2 + 1$ at each of the x-values you picked. These are the heights of your rectangles!

(h) Find the areas of each rectangle by multiplying the height of each rectangle by Δx, the width of each rectangle.

(i) Add these areas up to get a total approximation of the actual area!

What do you think: is it worth fiddling with what x-value to pick from each subinterval to try to get a better approximation? If n is large, do you think it matters how we pick the x-values from each subinterval?

5.3 The Definite Integral

Activity 5.3.1 Weird Areas.

Let's think about a simple linear function, $f(x) = 4 - 3x$. We'll both approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 0$ to $x = 3$:

$$\int_{x=0}^{x=3} (4 - 3x)\, dx$$

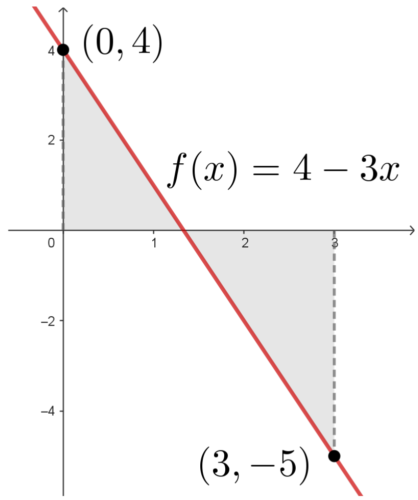


Figure 5.3.1

(a) Explain why we do not need to think about Riemann sums in order for us to calculate the shaded area. How would you calculate this without using calculus?

Calculate the area!

(b) Let's approximate this area using a Riemann sum. Calculate L_3, the Left Riemann sum with $n = 3$ rectangles.

(c) Let's approximate this area a second time, but with a different selection strategy for our x-values. Calculate R_3, the Right Riemann sum with $n = 3$ rectangles.

(d) Compare your answers for L_3 and R_3. They should be *very* different. Why? What is happening that makes R_3 specifically such a weird value?

(e) Do you need to go back and recalculate the area geometrically (from the first part of this activity)? Explain why your answer for $\int_{x=0}^{x=3} (4 - 3x)\, dx$ *should* be negative, based on the Riemann sums we calculated.

(f) Find a value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x)\, dx = 0.$$

(g) Find a *different* value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x)\, dx = 0.$$

Is there a second way of making this area 0?

Activity 5.3.2 Weird Areas - Part 2.

We're going to think about the same region, kind of.

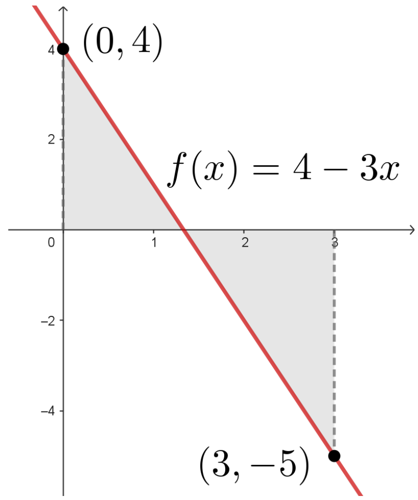


Figure 5.3.2

Let's think about the same linear function, $f(x) = 4 - 3x$, but this time we'll approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 3$ to $x = 0$:

$$\int_{x=3}^{x=0} (4 - 3x)\, dx$$

(a) Use geometry to calculate the area. Compare this to the result from Activity 5.3.1.

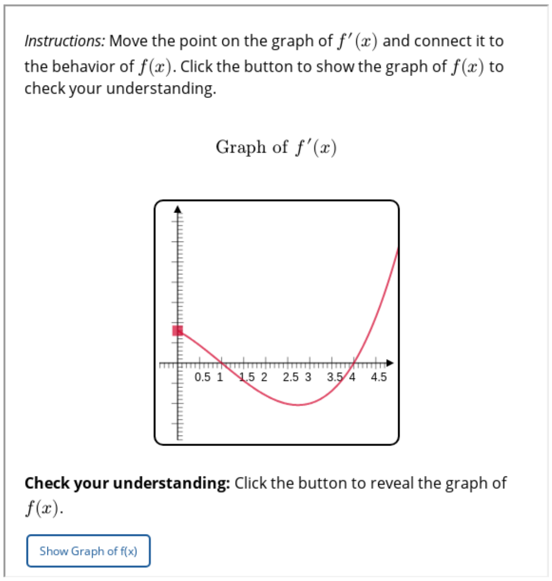
(b) Let's approximate this using a Riemann sum. Calculate M_3, the Midpoint Riemann sum with $n = 3$ rectangles.

(c) Do you need to go back and recalculate the area geometrically (from the first part of this activity)? Explain why your answer for $\int_{x=3}^{x=0} (4 - 3x)\, dx$ *should* be positive, based on the Riemann sums we calculated.

5.4 The Fundamental Theorem of Calculus

Activity 5.4.1 Interpreting the Graph of a Derivative.

Let's look at a picture of a graph of the first derivative, $f'(x)$, and try to get some information about $f(x)$ from it. Use the following graph of $f'(x)$, the first derivative, to answer the questions about $f(x)$.



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Since we don't have a huge amount of detail, you'll likely have to estimate the x-values for intervals and points in the following questions, but that's ok! Estimate away! Just make sure you know what you're looking for in the graph of $f'(x)$ to answer these questions.

(a) List the intervals on which $f(x)$ is increasing. What about decreasing?

(b) Find the x-values of any local maximums and/or local minimums of $f(x)$.

(c) List the intervals on which $f(x)$ is concave up. What about concave down?

(d) Find the x-values of any inflection points of $f(x)$.

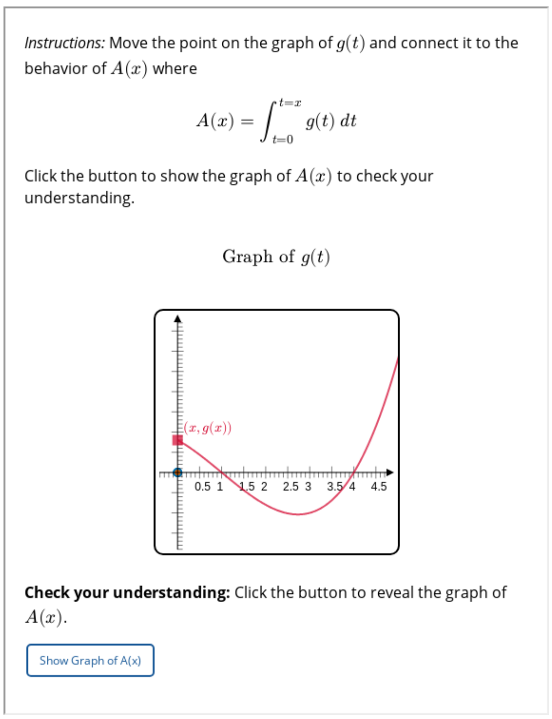
Activity 5.4.2 Interpreting Area.

First, we're going to define a bit of a weird function. Sometimes it's called the **Area function**:

$$A(x) = \int_{t=0}^{t=x} g(t)\, dt.$$

This is a strange function, because we're defining the function as an integral of another function. Specifically, note that the *input* for our area function $A(x)$ is the ending limit of integration: we're calculating the signed area "under" the curve of $g(t)$ from $t = 0$ up to some variable ending point $t = x$.

We can visualize this function by looking at the areas we create as we change x. For now, get used to just seeing the area "under" g when we move the point around. The areas themselves are the outputs of the function $A(x)$.



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Now we can think about this area function, and try to connect it to the graph of $g(t)$.

(a) List the intervals on which $A(x)$ is increasing. What about decreasing?

(b) Find the x-values of any local maximums and/or local minimums of $A(x)$.

(c) List the intervals on which $A(x)$ is concave up. What about concave down?

(d) Find the x-values of any inflection points of $A(x)$.

(e) Compare your answers here to your answers about the behavior of $f(x)$ based on the (same) graph of $f'(x)$ in Activity 5.4.1.

What does this mean about the connection between areas and derivatives, or areas and anti-derivatives?

Activity 5.4.3 Evaluating Areas and Antiderivatives.

In this short activity, we'll just collect information about antiderivatives and this new area function,

$$A(x) = \int_{t=a}^{t=x} f(t)\, dt$$

for a function $f(t)$ that is continuous on the interval $a \leq t \leq x$.
For our purposes in this activity, let's say that $f(x) = x + \cos(x)$.

(a) From the Fundamental Theorem of Calculus (Part 1), we know that $A(x)$ is an antiderivative of $f(x)$, since $A'(x) = f(x)$.

Write out the function $A(x)$, and then name/write out one *other* antiderivative of $f(x)$, some $F(x)$.

(b) We know that all of the antiderivatives of a function are connected to each other.

Describe the connection between $A(x)$ and your $F(x)$.

(c) What is the value of $A(a)$? What is the value of $F(a)$? How are they different from each other?

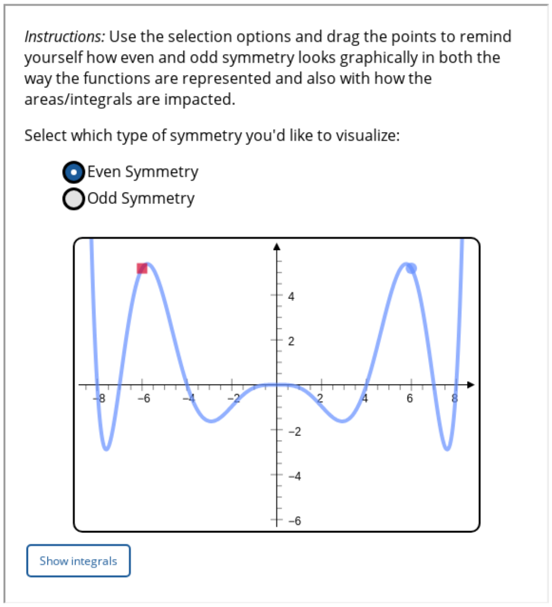
(d) What is the value of $A(b)$? What is the value of $F(b)$? How are they different from each other?

(e) What about the differences: $A(b) - A(a)$ compared to $F(b) - F(a)$?

5.5 More Results about Definite Integrals

Activity 5.5.1 Symmetry in Functions and Integrals.

First, let's take a moment to remind ourselves (or see for the first time) what two types of "symmetry" we'll be considering. We call them "even" and "odd" symmetry, but sometimes we think of them as a "reflective" symmetry and a "rotational" symmetry in the graphs of our functions.



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(a) Convince yourself that you know what we mean when we say that a function is **even symmetric** on an interval if $f(-x) = f(x)$ on the interval.

Similarly, convince yourself that you know what we mean when we say that a function is **odd symmetric** on an interval if $f(-x) = -f(x)$ on the interval.

(b) Now let's think about areas. Before we visualize too much, let's start with a small question: How does the height of a function impact the area defined by a definite integral? It should be helpful to think about Riemann sums and areas of rectangles here.

The important question then, is how does a function being even or odd symmetric tell us information about areas defined by definite integrals of that function?

Activity 5.5.2 Connecting Symmetric Integrals.

We're going to do some sketching here, and I want you to be clear about something: your sketches can be absolutely terrible. It's ok! They just need to embody the kind of symmetry we're talking about. You will probably sketch something and notice that your areas aren't to scale (or maybe even the wrong sign!), and that's fine.
It might be helpful to practice sketching graphs accurately, but don't worry if that part is a struggle.

(a) Sketch a function $f(x)$ with the following properties:

- $f(x)$ is *even symmetric* on the interval $[-6, 6]$
- $\displaystyle\int_{x=0}^{x=6} f(x)\, dx = 4$
- $\displaystyle\int_{x=-6}^{x=-2} f(x)\, dx = -1$

(b) Find the values of the following integrals:

- $\displaystyle\int_{x=0}^{x=2} f(x)\, dx$
- $\displaystyle\int_{x=-6}^{x=6} f(x)\, dx$

(c) Sketch a function $g(x)$ with the following properties:

- $g(x)$ is *odd symmetric* on the interval $[-9, 9]$
- $\displaystyle\int_{x=0}^{x=4} g(x)\, dx = 5$
- $\displaystyle\int_{x=-9}^{x=0} g(x)\, dx = 2$

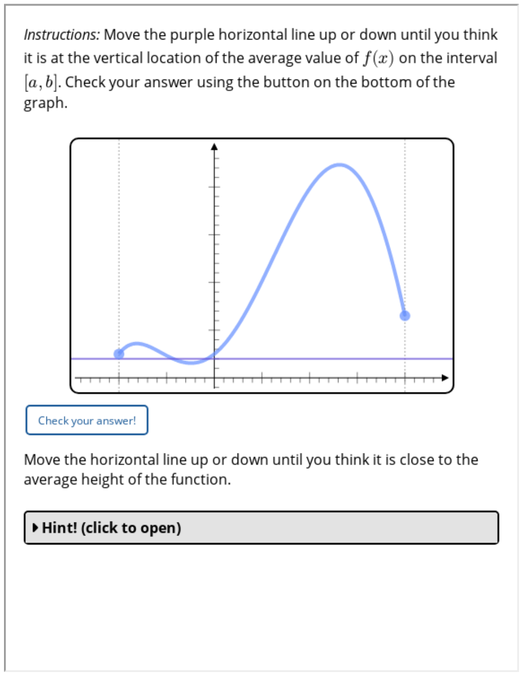
(d) Find the values of the following integrals:

- $\displaystyle\int_{x=-9}^{x=-4} g(x)\, dx$
- $\displaystyle\int_{x=-4}^{x=9} g(x)\, dx$

Activity 5.5.3 Visualizing the Average Height of a Function.

We are going to build a formula to find the“average height” or “average value” of a function $f(x)$ on the interval $[a, b]$. We’re going to look at a function and try to find the average height. Along the way, we’ll think a bit about areas!

(a) Consider the following function. Find the average height of the function on the interval pictured!



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(b) How does the area “under” the curve $f(x)$ on the interval compare to the area of the rectangle formed by the average height line?

(c) How do you define the two areas?

(d) Set up an equation connecting the two areas, and solve for the average height of $f(x)$.

5.6 Introduction to u-Substitution

Activity 5.6.1 A Hard Integral.

We're going to be thinking about two integrals here, but before we do, we should remind ourselves about how we can "rephrase" an integration question.

If we are asked to find $\int f(x)\,dx$, then we are really being asked to find some function $F(x)$ whose derivative is $f(x)$. Of course, we're actually being asked to find *all* of the possible functions that fit this requirement, but we know that the constant of integration covers all of the differences.

This means that we can (and maybe should) check our answers pretty consistently: just find a derivative and check that it matches the integrand!

(a) Find $\int 3x^2 e^{x^3}\,dx$.

(b) Explain the role of $3x^2$ in the integral. Explain why the positioning of x^3 matters.

(c) Let's try another one.

$$\int 7x^6 \cos(x^7)\,dx$$

(d) How can you tell, in general, that the derivative you're looking at is one that was produced through the Chain Rule and not the Product Rule?

Activity 5.6.2 Picking the Pieces of a Substitution.

We're going to look at three integrals. Instead of working through them one-at-a-time, we'll look at all three simultaneously, where we can practice identifying, substituting, and antidifferentiating all at the same time.

(a) Let's consider these three integrals:

- $\displaystyle\int \frac{3x^2+1}{(x^3+x-2)^2}\,dx$
- $\displaystyle\int \cos(x)\sqrt{\sin(x)}\,dx$
- $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$

For each of these integrals, identify the substitution: define u as some function of x.

(b) For each substitution, define $du = u'\,dx$.

(c) For each integral, use your substitution (for both u and the differential du) to rewrite the integral.

(d) Antidifferentiate each integral, and then use your substitution to write each integral back in terms of x.

Activity 5.6.3 Compare Two Integrals.

Let's compare two integrals, and use this to build a more general strategy for performing u-substitution.

(a) Consider the following integral:

$$\int -4x^3 \sec^2(1-x^4)\, dx$$

Select and justify a choice for u.

(b) Perform the u-substitution and antidifferentiate, and then substitute back to write your antiderivative in terms of x.

(c) Compare that integral to this one:

$$\int x^3 \sec^2(1-x^4)\, dx$$

What is different about this new integral? What has remained the same? How does that impact your choice for u, or *does* it impact your choice for u?

(d) Has that changed what du should be?

(e) Ok, so we've noticed an issue here. There are *plenty* of good ways of solving this problem, where du doesn't "show up" perfectly in our integral. In this case, we have that we're missing a necessary coefficient. We have the x^3 part, but we are missing the -4.

Try to rewrite our integral with a -4 coefficient in there. We'll do that by multiplying the integrand function by 1, disguised as $\frac{-4}{-4}$ or $\left(-\frac{1}{4}\right)(-4)$.

(f) Now we can use the same u-substitution as before, and integrate in a similar way! Notice, though, that we will retain the coefficient of $-\frac{1}{4}$.

(This should be reasonable: our integral is $-\frac{1}{4}$ of the original one, since our coefficient was 1 to the original's -4.)

Go ahead and integrate!