Calculus OER

With Activities and Python Examples/Labs

Calculus OER

With Activities and Python Examples/Labs

Peter Keep
Moraine Valley Community College

Last revised: August 25, 2024

Contents

Chapter 1

Limits

1.1 Introduction to Limits

Text of section.

1.2 The Definition of the Limit

Almost 2,500 years ago, the Greek philosopher Zeno of Elea gifted the world with a set of philosophical paradoxes that provide the foundation for how we will begin our study of calculus. Perhaps the most famous of Zeno's paradoxes is the paradox of Achilles and the Tortoise.

1.2.1 Achilles and the Tortoise

In the paradox of Achilles and the Tortoise, the Greek hero Achilles is in a race with a tortoise. Obviously the tortoise is much slower than Achilles, and so the tortoise gets a 100m head start. The race begins, and while the tortoise moves as quickly as it can, Achilles has the obvious advantage. They both are running at a constant speed, with Achilles running faster. Achilles runs 100m, catching up to the tortoise's starting point.

In the meantime, the tortoise has moved 2 meters. Achilles has almost caught up and passed the tortoise at this point! In a *very* short time, Achilles is able to run the 2 meters to catch up to where the tortoise was. Unfortunately, in that short amount of time, the tortoise has kept on moving, and is farther along by now.

Every time Achilles catches up to where the tortoise was, the tortoise has moved farther along, and Achilles has to keep catching up.

Can Achilles, the paragon of athleticism, ever catch the tortoise?

1.2.2 A Modern Retelling

A college student is excited about having enrolled in their first calculus class. On the first day of class, they head to class. When they enter the hallway with their classroom at the end, they take a breath and excitedly head to class.

In order to get to class, though, they have to travel halfway down the hallway. Almost there.

Now, to go the rest of the way, the student will half to get to the point that is halfway between them and the door. Getting closer.

They're getting excited. Finally, their first calculus class! But to get to the class, they have to reach the point halfway between them and the door.

Their smile starts fading. They repeat the process, and go halfway from their position to the door. They're closer, but not there yet.

If they keep having to reach the new halfway point, and that halfway point is never actually *at* the door, then will they ever get there?

Halfway to the door, then halfway again, closer and closer without ever getting there.

Will the student ever get there, or are they doomed to keep getting closer and closer without ever reaching the door?

1.2.3 Defining a Limit

Objectives

- How do we formalize this idea of thinking about what happens ”as we get close” to something?

Definition 1.2.1 Limit of a Function. For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the limit as x approaches a of $f(x)$ is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x \to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$. ◊

What is x? What about $\frac{}{x}$? Or:

$$\int xx \qquad \frac{}{x}$$

1.3 Approximating Limits Numerically and Graphically

Text of section.

1.4 Evaluating Limits

Text of section.

1.5 Infinite Limits

Text of section.

1.6 End Behavior Limits

Text of section.

1.7 First Indeterminate Forms

Text of section.

1.8 The Squeeze Theorem and Other Tricks

Text of section.

1.9 Continuity and the Intermediate Value Theorem

Text of section.

Chapter 2

Derivatives

2.1 Introduction to Derivatives

2.1.1 Thinking About Slopes

2.2 Interpreting Derivatives

Text of section.

2.3 Some Early Derivative Rules

Text of section.

2.4 The Product and Quotient Rules

Text of section.

2.5 The Chain Rule

Text of section.

Chapter 3

Implicit Differentiation

3.1 Implicit Differentiation

3.1.1 Using a Derivative as an Operator

3.2 Derivatives of Inverse Functions

Text of section.

3.3 Logarithmic Differentiation

Text of section.

3.4 Related Rates

Text of section.

Chapter 4

Applications of Derivatives

4.1 Mean Value Theorem

4.1.1 Slopes

4.2 Increasing and Decreasing Functions

Text of section.

4.3 Concavity

Text of section.

4.4 Interpreting the First and Second Derivatives

Text of section.

4.5 Global Maximums and Minimums

Text of section.

4.6 Optimization

Text of section.

4.7 Linear Approximations

Text of section.

4.8 Newton's Method for Approximating Zeros

Text of section.

4.9 L'Hopital's Rule

Text of section.

Chapter 5

Antiderivatives and Integrals

5.1 Antiderivatives and Indefinite Integrals

5.2 Riemann Sums and Area Approximations

Text of section.

5.3 Definite Integral

The Definite Integral

5.4 The Fundamental Theorem of Calculus

Text of section.

5.5 More Definite Integrals

Text of section.

Chapter 6

Techniques for Antidifferentiation

6.1 u-Subtitution

6.1.1 Title

6.2 Manipulating Integrands to Reveal Substitution

Text of section.

6.3 Integration By Parts

Text of section.

6.4 Integrating Powers of Trigonometric Functions

Text of section.

6.5 Trigonometric Substitution

Text of section.

6.6 Partial Fractions

Text of section.

Chapter 7

Applications of Integrals

Text here.

7.1 Integrals as Net Change

We have some rudimentary ideas of what an integral is, but we want to challenge and expand those ideas by examining the object at the root of the definition of the definite integral: a Riemann sum.

7.1.1 Estimating Movement

Here is some text leading to our first activity.

Activity 7.1.1 Estimating Movement. We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 7.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

Hint. How do we interpret the different values of velocity? How do we interpret the sign of velocity? What about how velocity changes from one of the 30-second time points to the next?

(b) When was the acceleration of the object the greatest? When was it the least?

Hint. You can decide how to interpret the "least" acceleration: it is either where the acceleration is closes to 0, or it is the most negative value of the acceleration. These are interpreted differently, but it's a bit ambiguous what we might mean when we say "least acceleration."

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

Hint. How do we use velocity and some time interval to estimate the distance traveled? How do we estimate/assume the velocity on each 30-second time interval?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

Hint. How do we think about (or ignore) the direction of the object? Why is this important here?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

Hint. Can we limit the time intervals that we use to calculate the object's displacement? How do we use displacement and a starting point to find an ending point?

So what are the big ideas in this short activity? There are a lot, and many of them are already things we know, at least to some level. So we are really focusing on adding depth to our understanding of these big ideas. Let's list them in the order that they showed up in this activity:

1. We interpret the velocity as the derivative of the position of the object. So when we interpret the value of the velocity of the object (large vs small, positive vs negative, etc.) we are interpreting these through the lens of a rate of change.

2. Acceleration is the derivative of the velocity function. While we don't have the full picture of the velocity function at any value of t, we still were interested in the rates at which velocity changes with regard to time.

3. We can estimate the total *displacement* of the object by predicting how far it traveled in eahc 30-second time interval. We might pick the starting velocity for each 30-second interval and multiply that by 30 seconds. We could alternatively pick the ending velocity of each 30-second interval. Then we can add all of these products of velocity and time together to approximate a total change in position! Doesn't this feel like a Riemann sum?

4. When we calculate displacement, the negative velocities get multiplied out to get negative changes in position for the object -- that's because a negative velocity means that the object is moving backwards. If we wanted to calculate the distance traveled, then we need to not account for negative velocities. We can just disregard the sign of the velocity on each time interval and repeat the process above. So, another Riemann sum then?

5. In order to forecast some position at time t, we just need to start with the initial position, and then calculate (or approximate) the displacement from $t = 0$ to whatever time $t \leq 300$ we care about, and then add the displacement to the intitial position.

Ok, now let's formalize those results!

7.1.2 Position, Velocity, and Acceleration

We know that the velocity of an object is really a rate of change of the position of that object with regard to time. Similarly, the acceleration of an object is the rate of change of the velocity of the object with regard to time. So we're really thinking about derivatives!

Definition 7.1.2 For an object moving along a straight line, if $s(t)$ represents the **position** of that object at time t, then the **velocity** of theobject at time t is $v(t) = s'(t)$ and the **acceleration** of the object at time t is $a(t) = v'(t) = s''(t)$. $\Diamond$

Once we establish this relationship, we can answer questions about movement of an object using the same interpretations of derivatives that we practiced in Chapter 3 of this text.

Activity 7.1.2 A Friendly Jogger. Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

Hint. Solve $a(t) = v'(t) = 0$.

(b) Does this time represent a maximum or minimum velocity for the jogger?

Hint. You can use the First Derivative Test or the Second Derivative Test here!

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

7.1.3 Displacement, Distance, and Speed

Let's revisit Activity 7.1.1. When we approximated the displacement of the object, we built a Riemann sum:

$$\sum_{k=1}^{10} v(t_k^*)\Delta t$$

We chose our t_k^* as either the time at the beginning of each 30-second interval or the time atr the end of the 30-second interval, but that was only because of the limited information that we had about different values of $v(t)$. If we had information about the $v(t)$ function at any values of t ($0 \leq t \leq 300$), then we could pick *any* time in each 30-second time interval for our Riemann sum! We might note, though, that if we did have this kind of information about the velocity at any time in the 5-minute interval, then we would also build a more precise approximation by subdividing the time interval into smaller/shorter pieces. So maybe the Riemann sum $\sum_{k=1}^{100} v(t_k^*)\Delta t$ (where we are dividing up the 5 minute interval into 100 3-second intervals) would do a better job! But why stop there? If we have the definition of the velocity function, and so we can truly obtain the velocity of the object at *any* time in the 5 minute interval, then we can use the definition of the definite integral as the limit of a Riemann sum:

$$\lim_{n\to\infty} \sum_{k=1}^{n} v(t_k^*)\Delta t = \int_{t=0}^{t=300} v(t)\ dt$$

This should work out well with our first understanding of displacement: the displacement of an object is just the difference in position from the starting time to the ending time. So we could say that if $s(t)$ is the position function, then we might expect to represent displacement from $t = a$ to $t = b$ as $s(b) - s(a)$. But isn't this just the Fundamental Theorem of Calculus, since $s'(t) = v(t)$?

Definition 7.1.3 If an object is moving along a straight line with velocity $v(t)$ and position $s(t)$, then the **displacement** of the object from time $t = a$ to $t = b$ is

$$\int_{t=a}^{t=b} v(t)\ dt = s(b) - s(a)$$

◊

Let's keep revisiting the same activity. We also noticed that when we looked at the *distance* compared to the displacement, the only difference was that we were integrating the absolute value of the velocity function, since we didn't care about the sign of the velocity (the direction that the object was traveling) on each interval.

Definition 7.1.4 If an object is moving along a straight line with velocity $v(t)$, then the **distance** traveled by the object from time $t = a$ to $t = b$ is:

$$\int_{t=a}^{t=b} |v(t)|\ dt$$

Here, we call $|v(t)|$ the **speed** of the object (instead of the velocity). ◊

We should note that we don't have any quick and easy ways of dealing with the integral of the absolute value of a function.

$$v(t)| = \begin{cases} -v(t) & \text{when } v(t) < 0 \\ v(t) & \text{when } v(t) \geq 0 \end{cases}$$

So, in order for us to integrate $|v(t)|$, we need to think about where the velocity passes through 0, so that we can see where it might change from positive to negative.

Activity 7.1.3 Tracking our Jogger. Let's revisit our jogger from Activity 7.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

Hint. Set up and evaluate a definite integral here, using the velocity function.

(b) Think back to our descroption of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.

Hint. Remember that we're really calculating:

$$\left|\int_{t=0}^{t=c} v(t)\ dt\right| + \left|\int_{t=c}^{t=3} v(t)\ dt\right|$$

7.1.4 Finding the Future Value of a Function

We can again think back to Activity 7.1.1 and build our last result of this section. Remember when we were looking to predict the location of our object at different times: we said it was reasonable to start at our initial position, and then add the displacement of the object from that initial time up to the time that we were interested in. So, to estimate the object's position after 150 seconds, we would calculate:

$$s(0) + \int_{t=0}^{t=150} v(t)\ dt.$$

But we said we could do this to estimate the object's position at any value for time, t.

Definition 7.1.5 Future Position of an Object. For some object moving along a straight line with velocity $v(t)$ and an intitial position of $s(a)$, the **future position of the object** at some time t (with $t \geq a$) is:

$$\underbrace{s(t)}_{\substack{\text{future}\\\text{position}}} = \underbrace{s(0)}_{\substack{\text{initial}\\\text{position}}} + \underbrace{\int_{x=a}^{x=t} v(x)\ dx}_{\substack{\text{displacement from}\\ a \text{ to } t}}$$

Note that we change the variable in the velocity function while we integrate: since we want our position function to be in terms of t, the ending time point that we calculate the displacemetn up to, we need to choose a different variable to write velocity in terms of. Mechanically, there is no difference, since we're just swapping out the variables and naming them x. ◊

We can note that this relationship between velocity and position can exist in many other context: any pair of functions that are derivatives/antiderivatives of each other can have this relationship!

Definition 7.1.6 Net Change and Future Value. Suppose the value $F(t)$ changes over time at a known rate $F'(t)$. Then the **net change** in F between $t = a$ and $t = b$ is:

$$F(b) - F(a) = \int_{t=a}^{t=b} F'(t)\ dt.$$

Similarly, given the initial value $F(a)$, the **future value** of F at time $t \geq a$ is:

$$F(t) = F(a) + \int_{x=a}^{x=t} F'(x)\ dx$$

◊

7.2 Area Between Curves

Let's remember Riemann Sums.

7.2.1 Remembering Riemann Sums

Text here.

Activity 7.2.1 Remembering Riemann Sums. Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

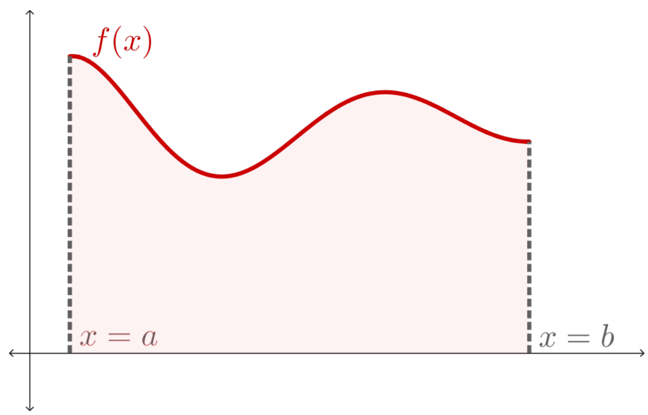

Figure 7.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

Hint. These points are just general ones, and you don't have to come up with actual numbers for the x-values or the corresponding y-values. Instead, just draw them in on the curve, somewhere in each of the subintervals.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

Hint. You won't have any numbers to calculate here, really: instead, see if you can calculate the widths by thinking about the total width of your interval. Then calculate the heights by thinking about the points you created.

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

Hint. Try to think about the accuracy of your area approximation by looking at it visually. Are there any places where your approximation looks far away from the actual area we're thinking about?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

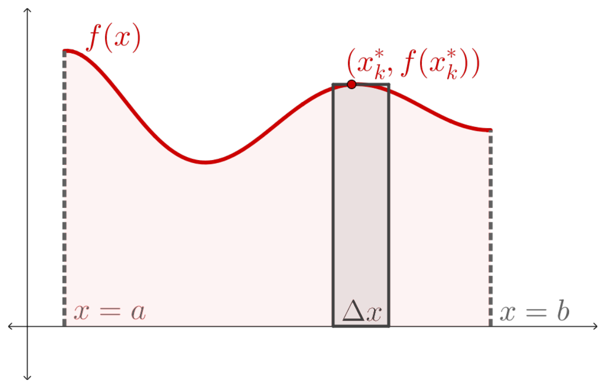

Figure 7.2.2

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

7.2.2 Building an Integral Formula for the Area Between Curves

Activity 7.2.2 Area Between Curves. Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

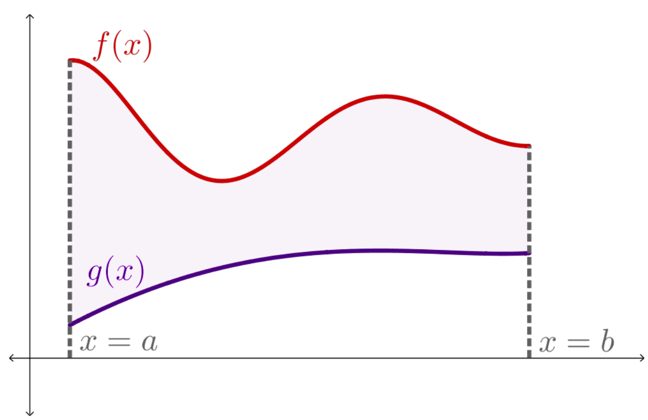

Figure 7.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

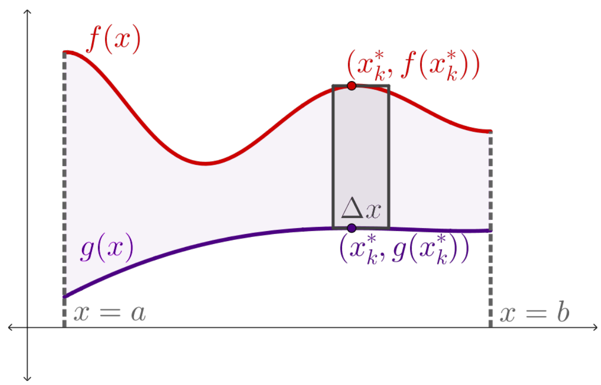

Figure 7.2.4

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Example here.

7.2.3 Changing Perspective

Text here.

Activity 7.2.3 Trying for a Single Integral. Let's consider the same setup as earlier: the region bounded between two curves, $y = x + 6$ and $y = x^3$, as well as the x-axis (the line $y = 0$). We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

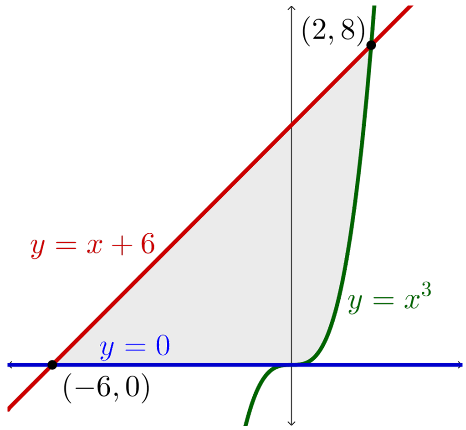

Figure 7.2.5

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

Hint. $\Delta y = \dfrac{8-0}{4}$

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

Hint. You're really just putting your y-values into the equation $y = x + 6$ and solving for x. Or you can solve for $f^{-1}(y)$ in general, by solving for x while leaving y as a variable.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths for each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

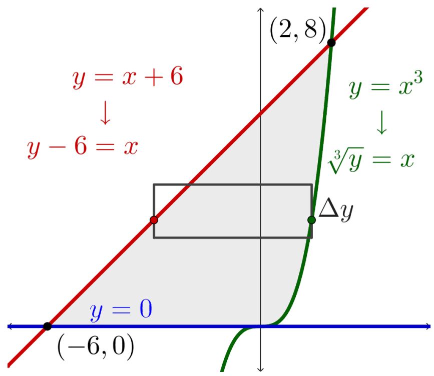

Figure 7.2.6

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

(j) Find A_k, the area of this kth rectangle.

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

7.3 Volumes of Solids of Revolution

Text of section.

7.4 More Volumes of Solids of Revolution

Text of section.

7.5 Arc Length

Text of section.

7.6 Surface Area

Text of section.

7.7 Other Applications of Integrals

Text of section.

7.8 Improper Integrals

Text of section.

Chapter 8

Infinite Series

8.1 Introduction to Infinite Sequences

8.1.1 Title

8.2 Introduction to Infinite Series

Text of section.

8.3 The Divergence Test and the Harmonic Series

Text of section.

8.4 Alternating Series and Conditional Convergence

Text of section.

8.5 The Integral Test

Text of section.

8.6 Common Series Types

Text of section.

8.7 Comparison Tests

Text of section.

8.8 The Ratio and Root Tests

Text of section.

Chapter 9

Power Series

9.1 Polynomial Approximations of Functions

9.2 Taylor Series

Text of section.

9.3 Properties of Power Series

Text of section.

9.4 How to Build Taylor Series

Text of section.

9.5 How to Use Taylor Series

Text of section.

Appendix A

More on Limits

Text here.

Colophon

This book was authored in PreTeXt.