Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

Peter Keep
Moraine Valley Community College

Last revised: July 16, 2025

[1] www.discovercalculus.com

[2] creativecommons.org/licenses/by/4.0

Acknowledgements

Disclosure about the Use of AI

This book has been lovingly written by a human.

Me.

Peter Keep.

I have used a lot of different tools, both for inspiration and for actually creating resources for this book. *None* of those tools has involved any form of generative AI.

I could list all of the ways that I think using generative AI in education is, at minimum, problematic. More pointedly, I believe that it is unethical. More broadly, I believe that the use of generative AI for any use-case that I have encountered to be unethical.

In my classes, I try to help students realize the joy and value of working at something and creating something and struggling with something and knowing something. Giving worth to something, even an imperfect thing. Celebrating our accomplishments, even when (especially when?) there is room to grow in those accomplishments. And so I have taken that advice in the creation of this book. I have created a book that is definitely not perfect. I have struggled to write it. There are parts of it that could be (need to be) improved.

But I was the one that created it. I struggled with it. I know it.

I hope that this book can also be a useful tool for others to use, and I have left the copyright to be about as open as possible. Others can take this, use it, can change it, add to it, subtract from it, etc.

In leaving this copyright open for others to change this book, I cannot guarantee that every version of this book is free from the mindless and joyless output from some Large Language Model. But I want to leave this note up in hopes that anyone who *does* inject some output from some generative AI product into this book will take it down. If this note, or some statement similar to it, is not present in the version of the book you are accessing, please be cautious. Find a different calculus textbook to read!

Find something written by a human. Find the words of some other mathematician who tries, maybe imperfectly, to share the ideas of calculus.

Teaching and learning is about humans communicating with each other, and only humans can do that.

Notes for Instructors

Notes for Students

Contents

Chapter 1

Limits

1.1 Introduction to Limits

Almost 2,500 years ago, the Greek philosopher Zeno of Elea gifted the world with a set of philosophical paradoxes that provide the foundation for how we will begin our study of calculus. Perhaps the most famous of Zeno's paradoxes is the paradox of Achilles and the Tortoise.

1.1.1 Achilles and the Tortoise

In the paradox of Achilles and the Tortoise, the Greek hero Achilles is in a race with a tortoise. Obviously the tortoise is much slower than Achilles, and so the tortoise gets a 100m head start. The race begins, and while the tortoise moves as quickly as it can, Achilles has the obvious advantage. They both are running at a constant speed, with Achilles running faster. Achilles runs 100m, catching up to the tortoise's starting point.

In the meantime, the tortoise has moved 2 meters. Achilles has almost caught up and passed the tortoise at this point! In a *very* short time, Achilles is able to run the 2 meters to catch up to where the tortoise was. Unfortunately, in that short amount of time, the tortoise has kept on moving, and is farther along by now.

Every time Achilles catches up to where the tortoise was, the tortoise has moved farther along, and Achilles has to keep catching up.

Can Achilles, the paragon of athleticism, ever catch the tortoise?

1.1.2 A Modern Retelling

A college student is excited about having enrolled in their first calculus class. On the first day of class, they head to class. When they enter the hallway with their classroom at the end, they take a breath and excitedly head to class.

In order to get to class, though, they have to travel halfway down the hallway. Almost there.

Now, to go the rest of the way, the student will half to get to the point that is halfway between them and the door. Getting closer.

They're getting excited. Finally, their first calculus class! But to get to the class, they have to reach the point halfway between them and the door.

Their smile starts fading. They repeat the process, and go halfway from their position to the door. They're closer, but not there yet.

If they keep having to reach the new halfway point, and that halfway point is never actually *at* the door, then will they ever get there?

Halfway to the door, then halfway again, closer and closer without ever getting there.

Will the student ever get there, or are they doomed to keep getting closer and closer without ever reaching the door?

1.2 The Definition of the Limit

1.2.1 Defining a Limit

Activity 1.2.1 Close or Not? We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Definition 1.2.1 Limit of a Function. For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the **limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x \to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$. ♢

When we say "around $x = a$", we really just mean on either side of it. We can clarify if we want.

Definition 1.2.2 Left-Sided Limit. For the function $f(x)$ defined at all x-xalues around and less than a, we say that the **left-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but less than, a. We write this as:

$$\lim_{x \to a^-} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^-$. ♢

Definition 1.2.3 Right-Sided Limit. For the function $f(x)$ defined at all x-xalues around and greater than a, we say that the **right-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but greater than, a. We write this as:

$$\lim_{x \to a^+} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^+$. ♢

Theorem 1.2.4 Mismatched Limits. *For a function $f(x)$, if both $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then we say that $\lim_{x \to a} f(x)$ does not exist. That is, there is no single real number L that $f(x)$ is arbitrarily close to for x-values that are sufficiently close to, but not equal to, $x = a$.*

1.2.2 Approximating Limits Using Our New Definition

Activity 1.2.2 Approximating Limits. For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the graph of the function $f(x)$ below.

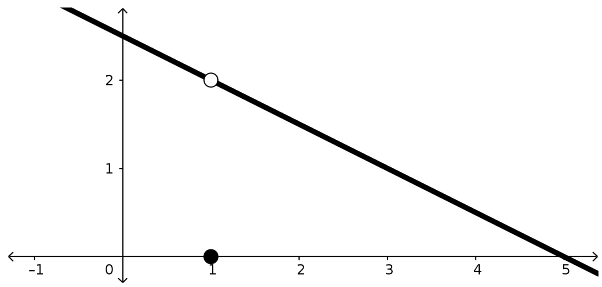


Figure 1.2.5

(b) Approximate $\lim_{x \to 2} g(x)$ using the graph of the function $g(x)$ below.

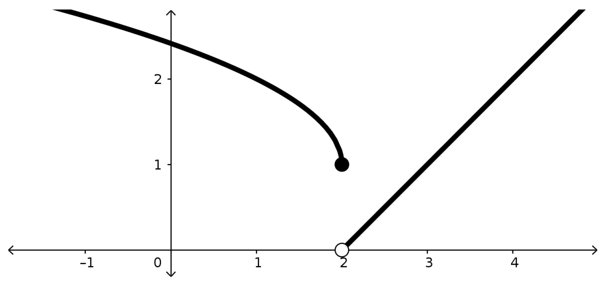


Figure 1.2.6

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x \to -1} h(x)$
- $\lim_{x \to 0} h(x)$
- $\lim_{x \to 2} h(x)$

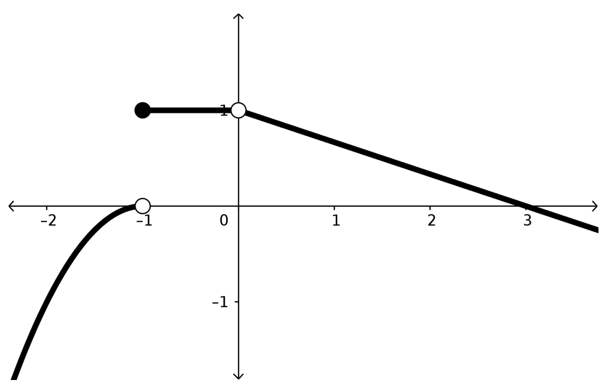


Figure 1.2.7

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.2.3 Approximating Limits Numerically. For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x\to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.2.8

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x\to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.2.9

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x\to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.2.10

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.3 Evaluating Limits

1.3.1 Adding Precision to Our Estimations

Activity 1.3.1 From Estimating to Evaluating Limits (Part 1). Let's consider the following graphs of functions $f(x)$ and $g(x)$.

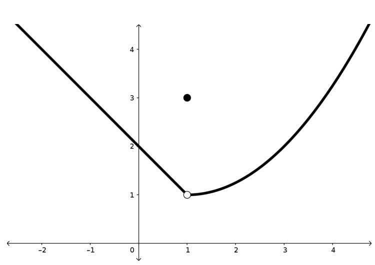


Figure 1.3.1 Graph of the function $f(x)$.

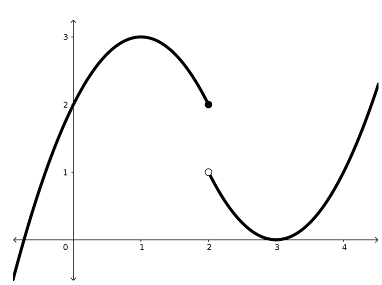


Figure 1.3.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$

- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 2^-} g(x)$
- $\lim_{x \to 2^+} g(x)$
- $\lim_{x \to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.3.2 From Estimating to Evaluating Limits (Part 2). Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

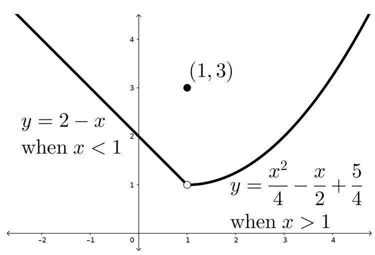


Figure 1.3.3 Graph of the function $f(x)$.

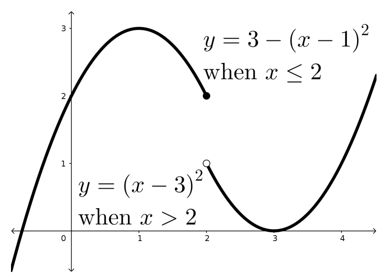


Figure 1.3.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$
- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 2^-} g(x)$
- $\lim_{x \to 2^+} g(x)$
- $\lim_{x \to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

1.3.2 Limit Properties

Theorem 1.3.5 Combinations of Limits. *If $f(x)$ and $g(x)$ are two functions defined at x-values around, but maybe not at, $x = a$ and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then we can evaluate limits of combinations of these functions.*

1. Sums: *The limit of the sum of $f(x)$ and $g(x)$ is the sum of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

2. Differences: *The limit of a difference of $f(x)$ and $g(x)$ is the difference of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$$

4. Products: *The limit of a product of $f(x)$ and $g(x)$ is the product of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

5. Quotients: *The limit of a quotient of $f(x)$ and $g(x)$ is the quotient of the limits of $f(x)$ and $g(x)$ (provided that you do not divide by 0):*

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)} \qquad (\textit{if } \lim_{x\to a} g(x) \neq 0)$$

Theorem 1.3.6 Limits of Two Basic Functions. *Let a be some real number.*

1. Limit of a Constant Function: *If k is some real number constant, then:*

$$\lim_{x\to a} k = k$$

2. Limit of the Identity Function:

$$\lim_{x \to a} x = a$$

Activity 1.3.3 Limits of Polynomial Functions. We're going to use a combination of properties from Theorem 1.3.5 and Theorem 1.3.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x \to 1} f(x)$. First, use the properties from Theorem 1.3.5 to re-write this limit as 4 different limits added or subtracted together.

Answer.

$$\lim_{x \to 1} (2x^4) - \lim_{x \to 1} (4x^3) + \lim_{x \to 1} \left(\frac{x}{2}\right) - \lim_{x \to 1} 5$$

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.3.6. Evaluate your limits.

Hint.

$$2\left(\lim_{x \to 1} x\right)^4 - 4\left(\lim_{x \to 1} x\right)^3 + \frac{1}{2}\left(\lim_{x \to 1} x\right) - \lim_{x \to 1} 5$$

(c) Based on the definition of a limit (Definition 1.2.1), we normally say that $\lim_{x \to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x \to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x \to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x \to a} p(x)$ is the same value as $p(a)$.

Theorem 1.3.7 Limits of Polynomials. *If $p(x)$ is a polynomial function and a is some real number, then:*

$$\lim_{x \to a} p(x) = p(a)$$

1.4 First Indeterminate Forms

Activity 1.4.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x \to 1} f(x)$.

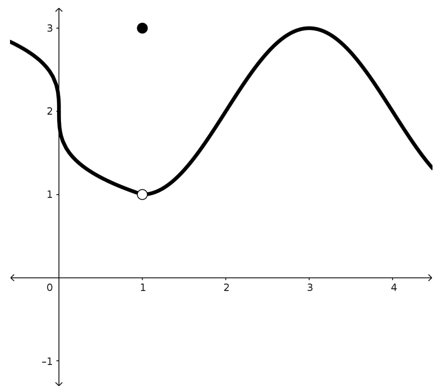


Figure 1.4.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x \to 1} g(x)$.

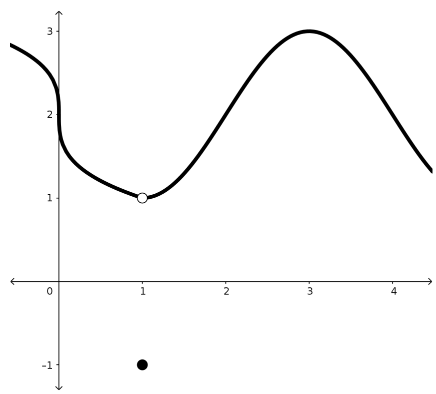


Figure 1.4.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{x \to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{x \to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Theorem 1.4.3 Limits of (Slightly) Different Functions. *If $f(x)$ and $g(x)$ are two functions defined at x-values around a (but maybe not at $x = a$ itself) with $f(x) = g(x)$ for the x-values around a but with $f(a) \neq g(a)$ then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, if the limits exist.*

1.4.1 A First Introduction to Indeterminate Forms

Definition 1.4.4 Indeterminate Form. We say that a limit has an **indeterminate form** if the general structure of the limit could take on any different value, or not exist, depending on the specific circumstances.

For instance, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that the limit $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ has an indeterminate form. We typically denote this using the informal symbol $\frac{0}{0}$, as in:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) \overset{?}{\to} \frac{0}{0}.$$

◊

Activity 1.4.2

(a) Were going to evaluate $\lim_{x\to 3}\left(\frac{x^2-7x+12}{x-3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3) - 4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x\to -2}\left(\frac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim_{x \to 5} f(x) \stackrel{?}{\to} \dfrac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

1.4.2 What if There Is No Algebra Trick?

We've seen some nice examples above where we were able to use some algebra to manipulate functions in such was as to force some shared factor in the numerator and denominator into revealing itself. From there, we were able to apply Theorem 1.4.3 and swap out our problematic function with a new one, knowing that the limit would be the same.

But what if we can't do that? What if the specific structure of the function seems *resistent* somehow to our attempts at wielding algebra?

This happens a lot, and we'll investigate some more of those types of limits in Section **??**. For now, though, let's look at a very famous limit and reason our way through the indeterminate form.

Activity 1.4.3 Let's consider a new limit:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

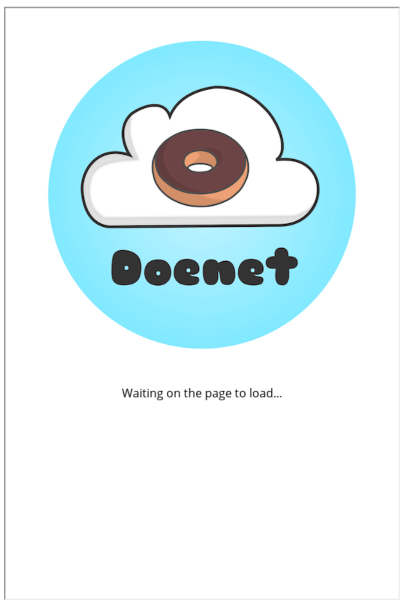
(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

Hint 1. On the unit circle, if we plot some point at an angle of θ, then the coordinates of that point can be represented with trig functions! Which ones?

Hint 2. The length of the curve defining a unit circle is 2π. This also corresponds to the angle we would use to represent moving all the way around the circle. What must the length of the portion of the circle be up to some point at an angle θ?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta}$?

1.5 Limits Involving Infinity

Two types of limits involving infinity. In both cases, we'll mostly just consider what happens when we divide by small things and what happens when we divide by big things. We can summarize this here, though:

Fractions with small denominators are big, and fractions with big denominators are small.

1.5.1 Infinite Limits

Activity 1.5.1 What Happens When We Divide by 0? First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.4), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Definition 1.5.1 Infinite Limit. We say that a function $f(x)$ has an **infinite limit** at a if $f(x)$ is arbitrarily large (positive or negative) when x is sufficiently close to, but not equal to, $x = a$.

We would then say, depending on the sign of the values of $f(x)$, that:

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad\qquad \lim_{x\to a^+} f(x) = \pm\infty.$$

If the sign of both one-sided limits are the same, we can say that $\lim\limits_{x\to a} f(x) = \pm\infty$ (depending on the sign), but it is helpful to note that, by the definition of the Limit of a Function, this limit does not exist, since $f(x)$ is not arbitrarily close to a single real number. ◊

Theorem 1.5.2 Dividing by 0 in a Limit. *If* $f(x) = \dfrac{g(x)}{h(x)}$ *with* $\lim\limits_{x \to a} g(x) \neq 0$ *and* $\lim\limits_{x \to a} h(x) = 0$*, then* $f(x)$ *has an Infinite Limit at* a*. We will often denote this behavior as:*

$$\lim_{x \to a} f(x) \overset{?}{\to} \frac{\#}{0}$$

where # is meant to be some shorthand representation of a non-zero limit in the numerator (often, but not necessarily, some real number).

Evaluating Infinite Limits.

Once we know that $\lim\limits_{x \to a} f(x) \overset{?}{\to} \dfrac{\#}{0}$, we know a bunch of information right away!

- This limit doesn't exist.
- The function $f(x)$ has a vertical asymptote at $x = a$, causing these unbounded y-values near $x = a$.
- The one sided limits *must* be either ∞ or $-\infty$.
- We only need to focus on the sign of the one sided limits! And signs of products and quotients are easy to follow.

So a pretty typical process is to factor as much as we can, and check the sign of each factor (in a numerator or denominator) as $x \to a^-$ and $x \to a^+$. From there, we can find the sign of $f(x)$ in both of those cases, which will tell us the one-sided limit.

Example 1.5.3 For each function, find the relevant one-sided limits at the input-value mentioned. If you can use a two-sided limit statement to discuss the behavior of the function around this input-value, then do so.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$ and $x = -4$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$ and $x = 1$

(c) $\sec(\theta)$ and $\theta = \dfrac{\pi}{2}$

□

1.5.2 End Behavior Limits

Activity 1.5.2 What Happens When We Divide by Infinity? Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

Definition 1.5.4 Limit at Infinity. If $f(x)$ is defined for all large and positive x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently large, then we say:

$$\lim_{x \to \infty} f(x) = L.$$

Similarly, if $f(x)$ is defined for all large and negative x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently negative, then we say:

$$\lim_{x \to -\infty} f(x) = L.$$

In the case that $f(x)$ has a **limit at infinity** that exists, then we say $f(x)$ has a horizontal asymptote at $y = L$.

Lastly, if $f(x)$ is defined for all large and positive (or negative) x-values and $f(x)$ gets arbitrarily large and positive (or negative) when x gets sufficiently large (or negative), then we could say:

$$\lim_{x \to -\infty} f(x) = \pm\infty \text{ or } \lim_{x \to \infty} f(x) = \pm\infty.$$

$\Diamond$

Because the primary focus for limits at infinity is the end behavior of a function, we will often refer to these limits as **end behavior limits**.

Theorem 1.5.5 End Behavior of Reciprocal Power Functions. *If p is a positive real number, then:*

$$\lim_{x\to\infty}\left(\frac{1}{x^p}\right)=0 \text{ and } \lim_{x\to-\infty}\left(\frac{1}{x^p}\right)=0.$$

Theorem 1.5.6 Polynomial End Behavior Limits. *For some polynomial function:*

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$$

with n a positive integer (the degree) and all of the coefficients $a_0, a_1, ..., a_n$ real numbers (with $a_n \neq 0$), then

$$\lim_{x\to\infty}p(x)=\lim_{x\to\infty}a_nx^n$$

That is, the leading term (the term with the highest exponent) defines the end behavior for the whole polynomial function.

Proof. Consider the polynomial function:

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$$

where n is some integer and a_k is a real number for $k=0,1,2,...,n$. For simplicity, we will consider only the limit as $x\to\infty$, but we could easily repeat this exact proof for the case where $x\to-\infty$.

Before we consider this limit, we can factor out x^n, the variable with the highest exponent:

$$\begin{aligned}
p(x)&=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0\\
&=x^n\left(\frac{a^nx^n}{x^n}+\frac{a_{n-1}x^{n-1}}{x^n}+...+\frac{a_2x^2}{x^n}+\frac{a_1x}{x^n}+\frac{a_0}{x^n}\right)\\
&=x^n\left(a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)
\end{aligned}$$

Now consider the limit of this product:

$$\begin{aligned}
\lim_{x\to\infty}p(x)&=\lim_{x\to\infty}x^n\left(a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\\
&=\left(\lim_{x\to\infty}x^n\right)\left(\lim_{x\to\infty}a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)
\end{aligned}$$

We can see that in the second limit, we have a single constant term, a_n, followed by reciprocal power functions. Then, due to Theorem 1.5.5, we know that the second limit will by a_n, since the reciprocal power functions will all approach 0.

$$\begin{aligned}
\lim_{x\to\infty}p(x)&=\left(\lim_{x\to\infty}x^n\right)\left(\lim_{x\to\infty}a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\\
&=\left(\lim_{x\to\infty}x^n\right)(a_n+0+...+0+0+0)\\
&=\left(\lim_{x\to\infty}x^n\right)(a_n)\\
&=\lim_{x\to\infty}a_nx^n
\end{aligned}$$

And so $\lim_{x\to\infty}p(x)=\lim_{x\to\infty}a_nx^n$ as we claimed. ■

Example 1.5.7 For each function, find the limits as $x\to\infty$ and $x\to-\infty$.

(a) $\left(\dfrac{2x^2-5x+1}{x^2+8x+16}\right)$

(b) $\left(\dfrac{4x^2-x^5}{x^2-4x+3}\right)$

(c) $\dfrac{|x|}{3x}$

□

Activity 1.5.3 Matching the Limits.

(a) We're going to look at four graphs of functions, as well as a list of limit statements. Match the limit statements with the graphs that match that behavior. Note that is is possible for a limit to be relevant on more than one graph.

(b) Now consider these four function definitions. Using your knowledge of limits, as well as the matching you've already done, match the definitions of these four functions with the graphs that go with them, and then also the limits that are relevant. (These limits will already be matched with the graphs, so you don't need to do further work here).

1.6 The Squeeze Theorem

Activity 1.6.1 A Weird End Behavior Limit. In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

Hint 1. Start with Theorem 1.3.5 to think about the numerator and denominator separately.

Hint 2. Can you use Theorem 1.5.6 in the denominator?

Hint 3. Is Theorem 1.5.5 relevant?

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\underline{\qquad\qquad} \le \sin(x) \le \underline{\qquad\qquad}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\qquad\qquad\qquad}{x^2+1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2+1} \leq \underbrace{\frac{\qquad\qquad\qquad}{x^2+1}}_{\text{call this } h(x)}$$

(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim\limits_{x\to\infty}\left(\dfrac{\sin^2(x)}{x^2+1}\right)$?

Theorem 1.6.1 The Squeeze Theorem. *For some functions $f(x)$, $g(x)$, and $h(x)$ which are all defined and ordered $f(x) \leq g(x) \leq h(x)$ for x-values near $x = a$ (but not necessarily at $x = a$ itself), and for some real number L, if we know that*

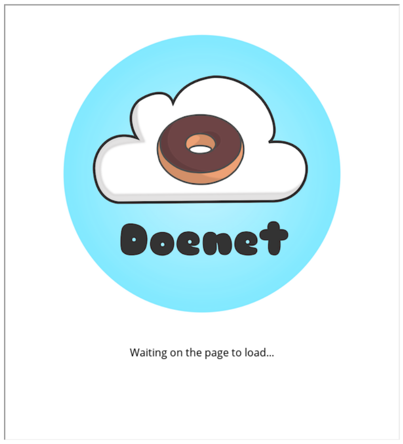
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then we also know that $\lim\limits_{x\to a} g(x) = L$.

Activity 1.6.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.



(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x\to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x\to 3} g(x)$ and $\lim_{x\to 0} g(x)$?
Do we know that these limits exist? If they do, do we have information about their values?

1.7 Continuity and the Intermediate Value Theorem

1.7.1 Continuity as Connectedness

1.7.2 Continuity as Classification

Definition 1.7.1 Continuous at a Point. The function $f(x)$ is **continuous** at an x-value in the domain of $f(x)$ if $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$, but one of the one-sided limits is equal to the function output, then we can define **directional continuity** at that point:

- We say $f(x)$ is **continuous on the left** at $x = a$ when $\lim_{x\to a^-} f(x) = f(a)$.
- We say $f(x)$ is **continuous on the right** at $x = a$ when $\lim_{x\to a^+} f(x) = f(a)$.

$\Diamond$

Definition 1.7.2 Continuous on an Interval. We say that $f(x)$ is **continuous on the interval** (a, b) if $f(x)$ is continuous at every x-value with $a < x < b$.

If $f(x)$ is continuous on the right at $x = a$ and/or continuous on the left at $x = b$, then we will say that $f(x)$ is continuous on the interval $[a, b)$, $(a, b]$, or $[a, b]$, whichever is relevant. $\Diamond$

1.7.3 Discontinuities

Where is a Function not Continuous?

Most of the functions that we consider in this text will be continuous everywhere that it makes sense: on their domain. That is, if there is a point defined at some x-value, it is likely that the function's limit matches the y-value of the point. More specifically, though:

- A function is discontinuous at any location that results in an infinite limit. These are locations where $f(x)$ is undefined and the limit is infinite (and so doesn't exist).
- A function is, in general, discontinuous wherever it is undefined. This seems silly to say! We probably could have left this unsaid.
- A function that is defined as a piecewise function could be discontinuous at locations where the pieces meet: maybe the limit doesn't exist, or maybe the function value is not defined, or maybe the limit exists and the function value is defined but they do not match.

1.7.4 Intermediate Value Theorem

Theorem 1.7.3 Intermediate Value Theorem. *If $f(x)$ is a function that is continuous on $[a, b]$ with $f(a) \neq f(b)$ and L is any real number between $f(a)$ and $f(b)$ (either $f(a) < L < f(b)$ or $f(b) < L < f(a)$), then there exists some c between a and b ($a < c < b$) such that $f(c) = L$.*

This theorem was stated as early as the 5th century BCE by Bryson of Heraclea. Back then, a really interesting problem was related to "squaring the circle." That is, given a circle with some measurable radius, can we construct a square with equal area? This is obviously true, in that we can just use a square with the side length $r\sqrt{\pi}$. What we typically mean by "construct," though, is to create this square using only a compass and straightedge (a ruler without length markings) and only a finite number of steps. This was finally proven to be impossible in 1882, approximately 2300 years later.

Bryson of Heraclea knew that the square itself existed (even if he couldn't construct it) because he was able to find a circle with area less than the square (by inscribing a circle inside of the square) and a circle with area greater than the square (where the square is inscribed in the circle). Since he posited that he could increase the size of the circle in a continuous manner (without using those words), he claimed that a square with area equal to that of the circle must exist, since the area of the circle passes through all values from the smaller area to the larger area.

Chapter 2

Derivatives

2.1 Introduction to Derivatives

We'll start this off by thinking about slopes. Before we begin, you should be able to answer the following questions:

- What *is* a slope? How could you describe it?
- How do you calculate the slope of a line between two points?
- If we have a function $f(x)$ and we pick two points on the curve of the function, what does the slope of a straight line connecting the two points tell us? What kind of behavior about $f(x)$ does this slope describe?

2.1.1 Defining the Derivative

Activity 2.1.1 Thinking about Slopes. We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ **and...**	**Slope**
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs along side of the curve there instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line. Let's think about a new function, $g(x) = \sqrt{2-x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

Definition 2.1.1 Derivative at a Point. For a function $f(x)$, we say that the **derivative** of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

provided that the limit exists.

If $f'(a)$ exists, then we say $f(x)$ is **differentiable** at a. ◊

We can investigate this definition visually. Consider the function $f(x)$ plotted below, where we will look at the point $(-1, f(-1))$. In the definition of the limit, we'll let $a = -1$, and so consider:

$$\lim_{x \to -1} \left(\frac{f(x) - f(-1)}{x - (-1)} \right).$$

Can you estimate the limit of the slope of the tangent line as $x \to -1$?

Does it look like the limit of the slope between $(-1, f(-1))$ and $(x, f(x))$ exists as $x \to -1$? What do you think it is?

2.1.2 Calculating a Bunch of Slopes at Once

Activity 2.1.3 Calculating a Bunch of Slopes. Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.

Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.4 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$\begin{aligned} j'(a) &= \lim_{x\to a}\left(\frac{j(x)-j(a)}{x-a}\right) \\ &= \lim_{x\to a}\left(\frac{(x^2-4)-(a^2-4)}{x-a}\right). \end{aligned}$$

Remember, you know how this goes! You're going to do the same sorts of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to re-calculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

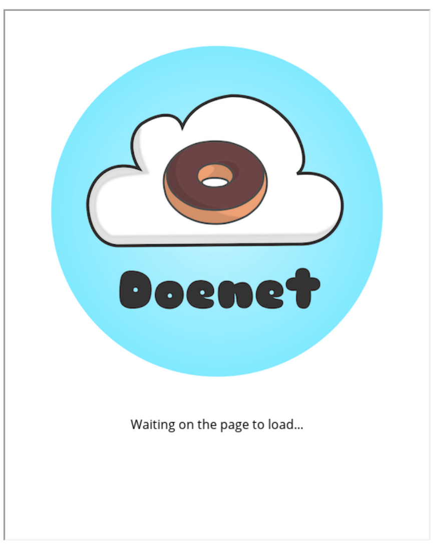
Definition 2.1.2 The Derivative Function. For a function $f(x)$, the derivative of $f(x)$, denoted $f'(x)$, is:

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

for x-values in the domain of $f(x)$ where this limit exists. ◊

This definition feels pretty different, but we can hopefully notice that this is really just calculating a slope. Notice, in the following plot, that there is a significant difference. In the visualization of the Derivative at a Point, the first point was fixed into place and the second point was the one that we moved and changed. It was the one with the variable x-value.

Notice in the following visualization that the *first* point is the one that is moveable while the *second* point is defined based on the first one (and the horizontal difference between the points, Δx). This means that we don't need to define one specific point, and can find the slope of the line tangent to $f(x)$ at some changing x-value.



2.2 Interpreting Derivatives

What is a derivative?

This can feel like a silly question, since we're calculating it and getting used to finding them. But what is it?

In this section, we just want to remind ourselves of what this object is, how we should hold it in our minds as we move through the course, and then practice being flexible with this interpretation.

2.2.1 The Derivative is a Slope

Activity 2.2.1 Interpreting the Derivative as a Slope. In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \frac{1}{x^2}$.

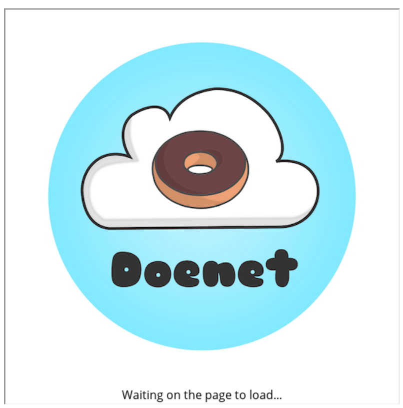
(a) Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?

(b) We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.

Remember that we can write the equation of a line in two ways:

- The equation of a line with slope m that passes through the point $(a, f(a))$ is:
$$y = m(x - a) + f(a).$$
- The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:
$$y = mx + b.$$

Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \frac{1}{x^2}$ below to check your equation.



(c) This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.

Use this idea of slope to predict where the y-value of our function will be at 2.01.

Hint. We know that slope is $\frac{\Delta y}{\Delta x}$ and we're using the fact that $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$ for small values of Δx.

Here, we have $\Delta x = 0.01$, so can you use the slope to approximate the corresponding Δy and figure out the new y-value?

(d) Compare this value with $f(2.01) = \frac{1}{2.01^2}$. How close was it?

2.2.2 The Derivative is a Rate of Change

Activity 2.2.2 Interpreting the Derivative as a Rate of Change. This is going to somewhat feel redundant, since maybe we know that a slope is really just a rate of change. But hopefully we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \leq t \leq 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

Hint. What is the rate at which the position (in feet) of the vehicle changes per unit time (in seconds)? What do we call this, and what are the units?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

2.2.3 The Derivative is a Limit

Look back at the definition of Derivative at a Point. The end of it is interesting: "provided that the limit exists." We need to keep in mind that this is a limit, and so a derivative exists or fails to exist whenever that limit exists or fails to exist.

What are some ways that a limit fails to exist?

- A limit doesn't exist if the left-side limit and the right-side limit do not match: Theorem 1.2.4 Mismatched Limits.
- A limit doesn't exist if it is an Infinite Limit.

What do each of these situations look like when we're considering the limit of slopes?

When Does a Derivative Not Exist?

1. A derivative doesn't exist at points where the slopes on either side of the point don't match.
2. A derivative doesn't exist at points with vertical tangent lines.
3. A derivative doesn't exist at points where the function is not continuous.

2.2.4 The Derivative is a Function

Activity 2.2.3 Interpreting the Derivative as a Function. In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.2.5 Notation for Derivatives

So far we've been using the "prime" notation to represent derivatives: the derivative of $f(x)$ is $f'(x)$. We will continue to use this notation, but we'll introduce a bunch of other ways of writing notation to represent the derivative.

Each new notation will emphasize some aspect of the derivative that will serve to be useful, even though they all represent essentially the same thing.

Function	**Derivative**	**Derivative at $x = a$**	**Emphasis**
$f(x)$	$f'(x)$	$f'(a)$	The derivative is a function. The function takes in x-value inputs and returns the slope of f at that x-value.
y	y'	$y'\Big\vert_{x=a}$	We can find slopes on any curve, not just functions. This is sometimes also used as a way to simplify the notation, especially when we want to manipulate equations involving y'.
y	$\frac{dy}{dx}$	$\frac{dy}{dx}\Big\vert_{x=a}$	The derivative is a slope. It is $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, and we use dx and dy (called **differentials**) to represent Δx and Δy as the limits as $\Delta x \to 0$. This notation is also useful to tell us what the rate of change is: what is changing (in this case y) and what is it changing based on (in this case x).
$f(x)$	$\frac{d}{dx}\left(f(x)\right)$	$\frac{d}{dx}\left(f(x)\right)\Big\vert_{x=a}$	The derivative is an action that we do to some function. We can call it an **operator**, although we won't formally define that term in this text. We'll look at this idea more in Section 3.1. We can specify what we are expecting the input variable to be, based on the differential dx in the denominator.

2.3 Some Early Derivative Rules

We are going to break this topic into two parts:

1. We will try to find some common patterns or connections between derivatives and specific functions. For instance, when we use Definition 2.1.2 The Derivative Function to build a derivative, are there patterns in the work of evaluating that limit that will allow us to get through the limit work quickly? Can we group some functions together based on how we might deal with the limit?

2. We will try to think about derivatives a bit more generally and show how we can build some basic properties to help us think about differentiating variations of the functions that we recognize.

2.3.1 Derivatives of Common Functions

Activity 2.3.1 Derivatives of Power Functions. We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.

Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\frac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

Hint. Remember that the graph of $x^2 - 4$ has the same shape as the graph of x^2, but has been shifted down by 4 units. Why does this vertical shift not change the value of the derivative at any x-value?

(b) Find $\frac{d}{dx}\left(x^3\right)$.

Hint. Remember that $(x + \Delta x)^3 = (x + \Delta x)(x + \Delta x)(x + \Delta x)$

(c) Find $\frac{d}{dx}\left(x^4\right)$.

Hint. Remember that $(x+\Delta x)^4 = (x+\Delta x)(x+\Delta x)(x+\Delta x)(x+\Delta x)$

(d) Notice that in these derivative calculations, the main work is in multiplying $(x+\Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\begin{aligned}\frac{d}{dx}\left(x^5\right) &= \lim_{\Delta x \to 0}\left(\frac{(x+\Delta x)^5 - x^5}{\Delta x}\right)\\ &= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right)\\ &= \rightsquigarrow \ldots\end{aligned}$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x+\Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this has done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just that whatever terms still have at least one remaining copy of Δx in it "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Triangle binomial theorem for coefficients.

Theorem 2.3.1 Power Rule for Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

We have shown that this is true for $n = 2, 3, 4, ...$, but this is also true for *any* value of n (including $n = 1$, non-integers, and non-positives). We will prove this more formally later (in Section 3.3), and until then we will be free to use this result.

Example 2.3.2 Let's confirm this Power Rule for two examples that we are familiar with.

(a) Find the derivative $\frac{d}{dx}(\sqrt{x})$ using the limit definition of the derivative function. Note that $\sqrt{x} = x^{1/2}$ and $\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$.

(b) Find the derivative $\frac{d}{dx}\left(\frac{1}{x}\right)$ using the limit definition of the derivative function. Note that $\frac{1}{x} = x^{-1}$ and $-\frac{1}{x^2} = -x^{-2}$.

□

In this activity, we also found one other result.

Theorem 2.3.3 Derivative of a Constant Function. *If $y = k$ where k is some real number constant, then $y' = 0$. Another way of saying this is:*

$$\frac{d}{dx}(k) = 0.$$

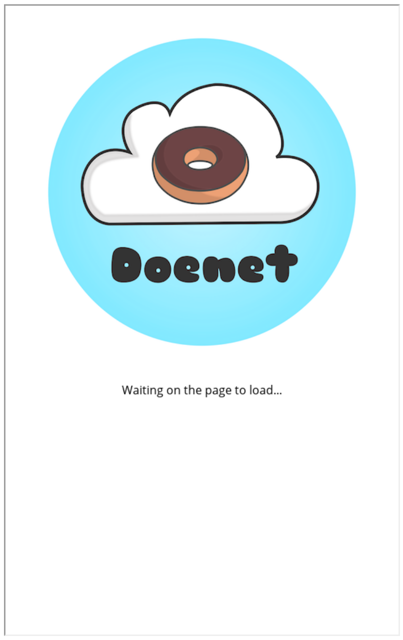
Activity 2.3.2 Derivatives of Trigonometric Functions. Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing out interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.

Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

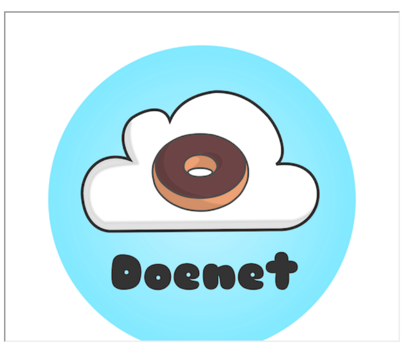
Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



(c) Let's repeat this process using the $y = \cos(x)$ function instead.

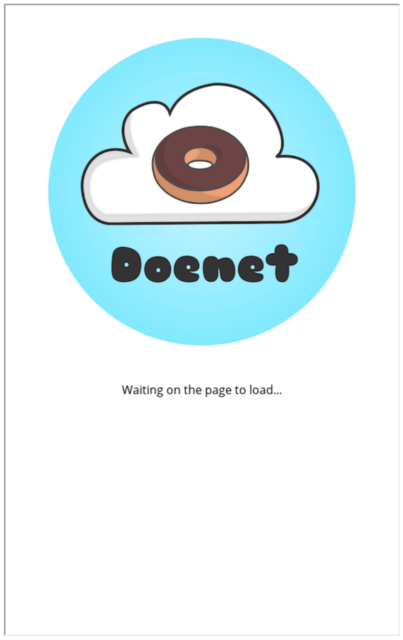
The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



Theorem 2.3.4 Derivatives of the Sine and Cosine Functions.

$$\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$$

$$\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$$

Proof. In order to show why $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{dx}(\cos(\theta)) = -\sin(\theta)$, we will work with the limit definitions of both. Consider both:

$$\frac{d}{d\theta}(\sin(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\sin(\theta + \Delta\theta) - \sin(\theta)}{\Delta\theta}\right)$$
$$\frac{d}{d\theta}(\cos(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta}\right)$$

Our goal is to re-write the numerators in both of these limits as something more usable. So far, we have been evaluating these types of limits (First Indeterminate Forms) using some algebraic manipulations. Instead of using algebra, we will use geometry.

Consider the unit circle below. We have plotted the angle θ and are reminded that the point on the circle that corresponds with the terminal side of the angle θ has coordinates $(\cos(\theta), \sin(\theta))$. We can label the sides of the triangle pictured below.

Now we consider a second point on the circle, this one formed by the terminal side of the angle $(\theta + \Delta\theta)$. This point has coordinates $(\cos(\theta + \Delta\theta), \sin(\theta + \Delta\theta))$. Note, below, that we want to find expressions for $\sin(\theta + \Delta\theta) - \sin(\theta)$ and $\cos(\theta + \Delta\theta) - \cos(\theta)$. We can find these geometrically.

Note, then, that the two triangles look to be similar triangles. In fact, we will find that in the limit as $\Delta\theta \to 0$, the length h matches the arc length $\Delta\theta$ perfectly, and thus lays at a right angle to the terminal side of the angle $\theta + \Delta\theta$.

Since as $\Delta\theta \to 0$ we have $h \to \Delta\theta$, we can find the other side lengths as well: $(\sin(\theta+\Delta\theta)-\sin(\theta)) \to \Delta\theta\cos\theta$ and $(\cos(\theta+\Delta\theta)-\cos(\theta)) \to \Delta\theta\sin\theta$. So then $(\cos(\theta+\Delta\theta)-\cos(\theta)) \to -\Delta\theta\sin\theta$.

Consider, then, the limits involved in the derivative calculations that we built earlier.

$$
\begin{aligned}
\frac{d}{d\theta}(\sin(\theta)) &= \lim_{\Delta\theta\to 0}\left(\frac{\sin(\theta+\Delta\theta)-\sin(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta\to 0}\left(\frac{\Delta\theta\cos(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta\to 0}(\cos(\theta))\\
&= \cos(\theta)\\
\frac{d}{d\theta}(\cos(\theta)) &= \lim_{\Delta\theta\to 0}\left(\frac{\cos(\theta+\Delta\theta)-\cos(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta\to 0}\left(\frac{-(\cos(\theta)-\cos(\theta+\Delta\theta))}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta\to 0}\left(\frac{-\Delta\sin(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta\to 0}(-\sin(\theta))\\
&= -\sin(\theta)
\end{aligned}
$$

So we have shown that $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$ as we claimed. ■

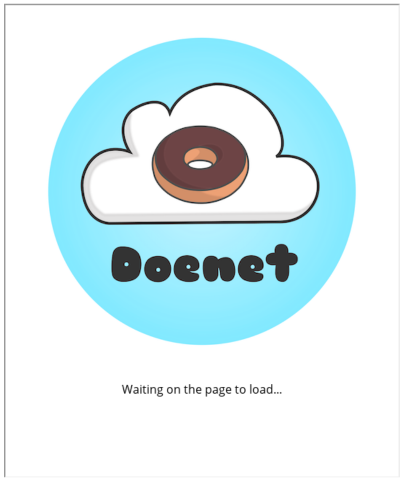
Activity 2.3.3 Derivative of the Exponential Function. We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function in a similar way to use thinking about the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.

Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. Are there any x-values where the slope is negative? Are there any where the slope is equal to 0? What happens to the slopes as x increases?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!



What information does this reveal about the slopes?

Hint. Especially it might be helpful to think about the slopes and their relationship to the y-value of the point we are building the tangent line at.

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/equation for y' is then?

Theorem 2.3.5 Derivative of the Exponential Function.

$$\frac{d}{dx}\left(e^x\right) = e^x$$

2.3.2 Some Properties of Derivatives in General

Theorem 2.3.6 Combinations of Derivatives. *If $f(x)$ and $g(x)$ are differentiable functions, then the following statements about their derivatives are true.*

1. Sums: *The derivative of the sum of $f(x)$ and $g(x)$ is the sum of the derivatives of $f(x)$ and $g(x)$:*

$$\begin{aligned}\frac{d}{dx}\left(f(x)+g(x)\right) &= \left(\frac{d}{dx}f(x)\right)+\left(\frac{d}{dx}g(x)\right)\\ &= f'(x)+g'(x)\end{aligned}$$

2. Differences: *The derivative of the difference of $f(x)$ and $g(x)$ is the difference of the derivatives of $f(x)$ and $g(x)$:*

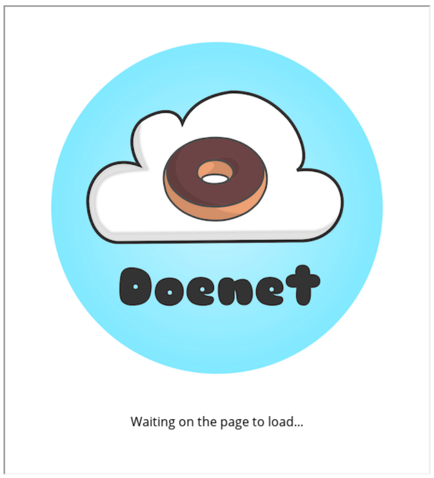
$$\frac{d}{dx}\left(f(x)-g(x)\right) = \left(\frac{d}{dx}f(x)\right)-\left(\frac{d}{dx}g(x)\right)$$

$$= f'(x) - g'(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\begin{aligned}\frac{d}{dx}\left(kf(x)\right) &= k\left(\frac{d}{dx}f(x)\right) \\ &= kf'(x)\end{aligned}$$

We can think about each of these properties through the lense of how combining these functions impacts the slopes. For instance, if we wanted to visualize the property about coefficients (that $\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}\left(f(x)\right)$), we can visualize this coefficient as a scaling factor.



Example 2.3.7 Putting These Together. Find the following derivatives:

(a) $\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right) &= \frac{d}{dx}\left(4x^5\right) - \frac{d}{dx}\left(\frac{5x}{2}\right) + \frac{d}{dx}\left(6\cos(x)\right) - \frac{d}{dx}(1) \\ &= 4\frac{d}{dx}\left(x^5\right) - \frac{5}{2}\frac{d}{dx}(x) + 6\frac{d}{dx}\left(\cos(x)\right) - \frac{d}{dx}(1) \\ &= 4(5x^4) - \frac{5}{2}(1) + 6(-\sin(x)) - 0 \\ &= 20x^4 - \frac{5}{2} - 6\sin(x)\end{aligned}$$

□

2.4 The Product and Quotient Rules

We saw in Theorem 2.3.6 Combinations of Derivatives that when we want to find the derivative of a sum or difference of functions, we can just find the derivatives of each function separately, and then combine the derivatives back together (by adding or subtracting). This, hopefully, is pretty intuitive: of course a slope of a sum of things is just the slopes of each thing added together!

In this section, we want to think about derivatives of product and quotients of functions. What happens when we differentiate a function that is really just two functions multiplied together?

Activity 2.4.1 Thinking About Derivatives of Products. Let's start with two quick facts:

$$\frac{d}{dx}\left(x^3\right) = 3x^2 \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

(a) What is $\frac{d}{dx}\left(x^3 + \sin(x)\right)$? What about $\frac{d}{dx}\left(x^3 - \sin(x)\right)$?

(b) Based on what you just explained, what is a reasonable assumption about what $\frac{d}{dx}\left(x^3 \sin(x)\right)$ might be?

Hint. Does it seem reasonable that we could just multiply the derivatives together, and say that $\frac{d}{dx}\left(x^3 \sin(x)\right)$ was the same thing as

$$\frac{d}{dx}\left(x^3\right) \cdot \frac{d}{dx}\left(\sin(x)\right)?$$

(c) Let's just think about $\frac{d}{dx}\left(x^3\right)$ for a moment. What *is* x^3? Can you write this as a product? Call one of your functions $f(x)$ and the other $g(x)$. You should have that $x^3 = f(x)g(x)$ for whatever you used.

(d) Use your example to explain why, in general, $\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$.

(e) Let's show another way that we know this. Think about $\sin(x)$. We know two things:

$$\sin(x) = (1)(\sin(x)) \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

What, though, is $\frac{d}{dx}(1) \cdot \frac{d}{dx}\left(\sin(x)\right)$?

(f) Use all of this to reassure yourself that even though the derivative of a sum of functions is the sum of the derivatives of the functions, we will need to develop a better understanding of how the derivatives of products of functions work.

A thing that I like to think about is this: if $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then every function's derivative would be 0.

In the example with the $\sin(x)$ function, we noticed that we could write the function as $(1)(\sin(x))$. This is true of every function!

If $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then we could say that for any function $f(x)$ with a derivative $f'(x)$:

$$\begin{aligned}\frac{d}{dx}\left(f(x)\right) &= \frac{d}{dx}\left(1 \cdot f(x)\right)\\ &= \frac{d}{dx}(1)\,\frac{d}{dx}\left(f(x)\right)\\ &= 0 \cdot f'(x)\\ &= 0.\end{aligned}$$

This, obviously, can't be true! We know of *tons* of functions that have non-zero slopes...*most* of them do!

So, we hopefully have some motivation for building a rule to that helps us think about derivatives of products of functions.

2.4.1 The Product Rule

Activity 2.4.2 Building a Product Rule. Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything.

If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

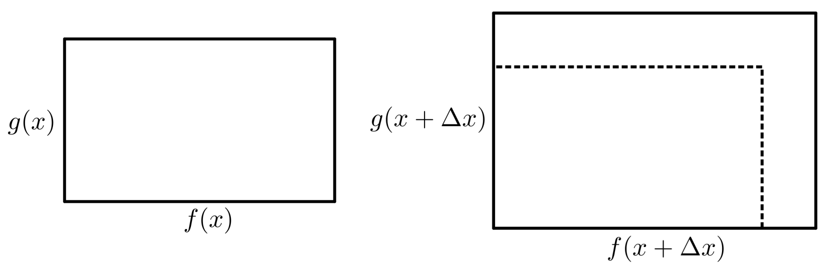


Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.

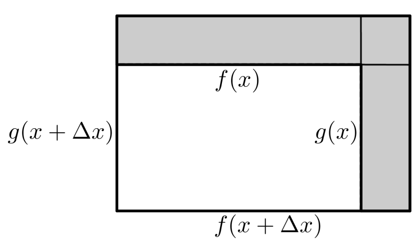


Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$f(x+\Delta x)g(x+\Delta x)-f(x)g(x) =$ ______.

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\begin{aligned}
\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x \to 0} & \left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right) \\
&= \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)
\end{aligned}$$

We can split this fraction up into three fractions:

$$\begin{aligned}
\frac{d}{dx}\left(f(x)g(x)\right) = & \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\
& + \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\
& + \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)
\end{aligned}$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

In the third limit, factor out either $\lim_{\Delta x \to 0}\left(f(x+\Delta x) - f(x)\right)$ or $\lim_{\Delta x \to 0}\left(g(x+\Delta x) - g(x)\right)$.

$$\begin{aligned}
\frac{d}{dx}\left(f(x)g(x)\right) = & f(x) \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\
& + g(x) \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\
& + \lim_{\Delta x \to 0}\left(\boxed{}\right)\left(\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\right)
\end{aligned}$$

(f) Use Definition 2.1.2 The Derivative Function to re-write these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

We can investigate this visual a bit more dynamically: see the differences in area as $\Delta x \to 0$. What parts are left, when $\Delta x \to 0$? What areas aren't visible?

Theorem 2.4.3 Product Rule. *If $u(x)$ and $v(x)$ are functions that are differentiable at x and $f(x) = u(x) \cdot v(x)$, then:*

$$\frac{d}{dx}(f(x)) = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

For convenience, this is often written as:

$$\frac{d}{dx}(u \cdot v) = u'v + uv' \qquad \text{or} \qquad \frac{d}{dx}(u \cdot v) = v\left(\frac{du}{dx}\right) + u\left(\frac{dv}{dx}\right).$$

Example 2.4.4 Use the Product Rule to find the following derivatives.

(a) $\frac{d}{dx}\left(x^3 \sin(x)\right)$

Hint. Use $u = x^3$ and $v = \sin(x)$. Now find u' and v', and use:

$$\frac{d}{dx}(uv) = u'v + uv'.$$

(b) $\frac{d}{dx}\left((x^3 + 4x)e^x\right)$

(c) $\frac{d}{dx}(\sqrt{x}\cos(x))$

□

2.4.2 What about Dividing?

So we can differentiate a product of functions, and the obvious next question should be about division: if we needed to build a reasonable way of differentiating a product, shouldn't we also need to build a new way of thinking about derivatives of quotients?

A nice thing that we can do is think about division as really just multiplication. For instance, if we want to differentiate $\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right)$, then we can just think about this quotient as really a product:

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}(\sin(x))\right).$$

Now we can just apply product rule!

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^2}(\sin(x))\right) &= \frac{d}{dx}\left(x^{-2}\sin(x)\right) \\
u = \sin(x) &\quad v = x^{-2} \\
u' = \cos(x) &\quad v' = -2x^{-3} \\
\frac{d}{dx}\left(\sin(x)x^{-2}\right) &= x^{-2}\cos(x) + (-2x^{-3}\sin(x)) \\
&= \frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}
\end{aligned}$$

This works great! We can *always* think about quotients as just products of reciprocals! But the problem is that we can't always differentiate these reciprocals (for now). We were able to differentiate $\frac{1}{x^2}$ by re-writing this as just a power function (with a negative exponent).

What about this flipped example:

$$\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right)?$$

In order for us to do the same thing, we need to re-write this as

$$\frac{d}{dx}\left(x^2\left(\sin(x)\right)^{-1}\right)$$

but we don't know how to differentiate $\left(\sin(x)\right)^{-1}$ (yet).

So let's try to build a general way of differentiating quotients.

Activity 2.4.3 Constructing a Quotient Rule. We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \dfrac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$u(x) =$

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$. Right? We care about finding $f'(x)$! Use what you found for $u'(x)$ and solve for $f'(x)$.

$f'(x) =$

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \dfrac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\dfrac{u(x)}{v(x)}$.

$f'(x) =$

This formula is fine, but a little clunky. We'll try to re-write it in some nice ways, but it is a bit more complex than the Product Rule.

Theorem 2.4.5 Quotient Rule. *If $u(x)$ and $v(x)$ are differentiable at x and $f(x) = \dfrac{u(x)}{v(x)}$ then:*

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}.$$

For convenience, this is often written as:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

Example 2.4.6 Use the Quotient Rule to find the following derivatives.

(a) $\dfrac{d}{dx}\left(\dfrac{\sin(x)}{x^2}\right)$

Once you have this derivative, confirm that it is the same as $\dfrac{\cos(x)}{x^2} - \dfrac{2\sin(x)}{x^3}$, the way that we found it using the Product Rule.

(b) $\dfrac{d}{dx}\left(\dfrac{x^2}{\sin(x)}\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{x+4}{x^2+1}\right)$

□

2.4.3 Derivatives of (the Rest of the) Trigonometric Functions

You might remember that of the six main trigonometric functions, we can write four of them in terms of the other two: here are the different trigonometric functions written in terms of sine and cosine functions:

$$\tan(x) = \left(\frac{\sin(x)}{\cos(x)}\right)$$
$$\sec(x) = \left(\frac{1}{\cos(x)}\right)$$
$$\cot(x) = \left(\frac{\cos(x)}{\sin(x)}\right)$$
$$\csc(x) = \left(\frac{1}{\sin(x)}\right)$$

Example 2.4.7 Find the derivatives of the remaining trigonometric functions.

(a) $\dfrac{d}{dx}(\tan(x))$

Hint. Write $\dfrac{d}{dx}(\tan(x)) = \dfrac{d}{dx}\left(\dfrac{\sin(x)}{\cos(x)}\right)$ and use the Quotient Rule.

(b) $\dfrac{d}{dx}(\sec(x))$

Hint. Write $\dfrac{d}{dx}(\sec(x)) = \dfrac{d}{dx}\left(\dfrac{1}{\cos(x)}\right)$ and use the Quotient Rule.

(c) $\dfrac{d}{dx}(\cot(x))$

Hint. Write $\dfrac{d}{dx}(\cot(x)) = \dfrac{d}{dx}\left(\dfrac{\cos(x)}{\sin(x)}\right)$ and use the Quotient Rule.

(d) $\dfrac{d}{dx}(\csc(x))$

Hint. Write $\dfrac{d}{dx}(\csc(x)) = \dfrac{d}{dx}\left(\dfrac{1}{\cos(x)}\right)$ and use the Quotient Rule.

□

2.5 The Chain Rule

We've been building up some intuition and rules to help us think about differentiating different functions and combinations of functions. We can find derivatives of scaled functions, sums of functions, differences of functions, products of functions, and also quotients of functions.

In this section, we'll look at our last operation between functions: composition.

2.5.1 Composition and Decomposition

An important part of finding derivatives of products and quotients is identifying the component functions that are being multiplied/divided (often labeled $u(x)$ or just u and $v(x)$ or just v). From there, we find the derivatives of each of the component functions, and then use the formula from the Product Rule or Quotient Rule to put the pieces together.

Thinking about derivatives of composed functions will be the same: we'll need to identify what functions are being composed inside of other functions, and use those pieces in some formulaic way to represent the derivative. On that note, let's remind ourselves and practice working with composition (and decomposition) of functions.

Activity 2.5.1 Composition (and Decomposition) Pictionary. This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet-spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that will still give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?

A big thing to notice here is that when we pick the pieces of functions that we think were composed inside of each other, there's not a single obvious answer. This is pretty different compared to, say, using the Quotient Rule. In these quotients, we have a natural division (literally) between the pieces. Here, it's much more subjective for us when we decide to label an "inside" function and an "outside" function.

We will build up our intuition to find a good balance for how we pick these.

2.5.2 The Chain Rule, Intuitively

Before we build the Chain Rule for differentiating composed functions, we should talk about some notation. Earlier (in Subsection 2.2.5) we talked about the derivative notation, $\frac{dy}{dx}$. One of the things we mentioned is that while we know that the derivative is an instantaneous rate of change, this notation is helpful to tell us *what* is changing with regard to *what*.

In $\frac{dy}{dx}$, we are calculating how much the y-variable changes when x increases. If we talked about $\frac{df}{dt}$, then we are discussing how much f changes for an increase in t, whatever these variables represent.

Activity 2.5.2 Gears and Chains. Let's think about some gears. We've got three gears, all different sizes. But the gears are linked together, and a small motor works to spin one of the gears. Since the gears are linked, when one gear spins, they all do. But since they are different sizes, they complete a different number of revolutions: the smaller ones spin more times than the larger ones, since they have a smaller circumference.

For our purpose, let's say that Gear A is being driven by the motor.

(a) Let's try to quantify how much "faster" Gear B is spinning compared to Gear A. How many revolutions does Gear B complete in the time it takes Gear A to complete one revolution?

(b) Now quantify the speed of Gear C compared to its neighbor, Gear B. How many revolutions does Gear C complete in the time it takes Gear B to complete one revolution?

(c) Use the above relative "speeds" to compare Gear C and Gear A: how many revolutions does Gear C complete in the time it takes Gear A to complete one revolution?

More importantly, how do we find this?

(d) Now let's translate this into some derivative notation: we've really been finding rates at which one thing changes (the speed of the gear spinning) relative to another's.

Call the speed of Gear B compared to Gear A: $\frac{dB}{dA}$. Now call the speed of Gear C compared to Gear B: $\frac{dC}{dB}$. Come up with a formula to find $\frac{dC}{dA}$.

So what we need to do now is to somehow translate this intuitive idea of multiplying rates of change to build a strategy for thinking about derivatives of composed functions.

We can think of these linked gears as functions: Gear C changes based on what is happening with Gear B, which changes based on Gear A. We can translate Gear A to be an input variable, like x. Then Gear B is a function based on that: we can call it $g(x)$. Then Gear C is a function that takes in the position of Gear B (the function $g(x)$), and so we can think of it as $f(g(x))$.

To build the derivative rule for composite functions, we need to find how the "outside" function changes as the "inside" function changes ($\frac{dC}{dB}$ in this case) and multiply that by how the "inside" function changes as the input variable changes ($\frac{dB}{dA}$ here).

Theorem 2.5.1 The Chain Rule. *For the composite function $y = f(g(x))$, if we define $u = g(x)$ and $y = f(u)$, then, as long as both f and g are differentiable at u and x respectively:*

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{du}\left(f(u)\right) \cdot \frac{d}{dx}\left(g(x)\right).$$

Alternatively, this can be written as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \textit{or} \qquad \frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

2.5.3 Doing is Different than Knowing

It is lovely to know that the Chain Rule is really just linking the two rates of change together to connect a function with an input variable through a middle processing function. That's great!

But doing the Chain Rule is different than just knowing it, so let's walk through a first example. Let's find the following derivative:

$$\frac{d}{dx}\left(\sin(x^2)\right)$$

We'll call the "inside" function $u = x^2$, so we can really write the whole function (normally we're calling this y) as $y = \sin(u)$.

$$\begin{aligned}\frac{d}{dx}\left(\underbrace{\sin(\overbrace{x^2}^{u})}_{y}\right) &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}\left(\sin(u)\right) \cdot \frac{d}{dx}\left(x^2\right)\end{aligned}$$

What we can notice, here, is that $\sin(u)$ is just a function of some variable u, and we want to find $\frac{dy}{du}$, the rate at which $y = \sin(u)$ changes with regard to its input variable. This might feel a bit strange, since u isn't just an input variable: it *means* something, since we have that $u = x^2$. This is fine! The extra $\frac{du}{dx}$ that we multiply will take care of linking this derivative to the input variable x.

$$\frac{d}{dx}\left(\underbrace{\sin(\overbrace{x^2}^{u})}_{y}\right) = \frac{d}{du}\left(\sin(u)\right) \cdot \frac{d}{dx}\left(x^2\right)$$

$$
\begin{aligned}
&= \cos(u)\cdot 2x \\
&= \cos(x^2)\cdot 2x \\
&= 2x\cos(x^2)
\end{aligned}
$$

After we finished differentiating $\frac{d}{du}(\sin(u))$, you'll notice that we used the fact that $u = x^2$ to write our combination of derivatives (the derivative of the "outside" function and the derivative of the "inside" function) in terms of the same input variable again.

The last line, rewriting $\cos(x^2)\cdot 2x$ as $2x\cos(x^2)$, is just for aesthetics.

Now you're ready to try some more examples! In each, focus on identifying a natural selection for the "inside" function, u.

Example 2.5.2 Use the Chain Rule to differentiate the following:

(a) $\dfrac{d}{dx}\left(\sqrt{x^2+4}\right)$

Hint. Notice that x^2+4 is composed under the square root. Use $u = x^2+4$.

(b) $\dfrac{d}{dx}\left(e^{\tan(x)}\right)$

Hint. Try letting $u = \tan(x)$, since it's composed inside the exponent of the exponential function.

(c) $\dfrac{d}{dx}\left(\sin^5(x)\right)$

Hint. You could think about this as $\dfrac{d}{dx}\left(\sin(x)\sin(x)\sin(x)\sin(x)\sin(x)\right)$ and try to use a very annoying product rule, but it might be easier to think about this as $\dfrac{d}{dx}\left((\sin(x))^5\right)$.

□

2.5.4 Generalizing the Derivative of the Exponential

Earlier, we looked at the specific derivative for $f(x) = e^x$ (Theorem 2.3.5), but we haven't talked about derivatives of other exponential functions. What about things like $y = 2^x$ or $y = \left(\frac{1}{2}\right)^x$? We can use a nice fact about exponentials and *logarithms*. We'll think more about log functions later (starting in Section 3.2), but we can think a bit about them now.

A big fact to recall is that a logarithm is a way of finding an exponent with a specific property. If we want to find the exponent that we would need to put on the number e to give us 9 as an answer, we could use $\ln(9)$.

$$e^{\ln(9)} = 9$$

This is just because logs are defined in this circular way: they are, by definition, the exponent you would need to output whatever number is inside the log.

This means that if we want to think about the number 2, but written in a different way, we can think of $e^{\ln(2)}$.

Ok, but why would we *ever* use this? This seems like a ridiculous way to write a number as basic as 2!

Consider the following:

$$2^x = \Big(\underbrace{e^{\ln(2)}}_{=2}\Big)^x$$

But we also might notice that we can re-write this using an exponent rule! We know that in general: $\left(a^b\right)^c = a^{b\cdot c}$. Let's re-write this exponential function:

$$\begin{aligned} 2^x &= \left(e^{\ln(2)}\right)^x \\ &= e^{\ln(2)\cdot x} \end{aligned}$$

Remember, $\ln(2)$ is just a number: it's specifically the number you have to put in the exponent on e to get 2. So this is just a coefficient on x. We can differentiate and use the Chain Rule!

$$\begin{aligned} \frac{d}{dx}\left(2^x\right) &= \frac{d}{dx}\left(e^{\ln(2)\cdot x}\right) \\ &= e^{\ln(2)\cdot x} \cdot \ln(2) \end{aligned}$$

Now we can remember that $e^{\ln(2)\cdot x}$ is really $\left(e^{\ln(2)}\right)^x$ which is just 2^x.

So we get $\frac{d}{dx}\left(2^x\right) = 2^x \ln(2)$. We can notice that we can re-create this with *any* (reasonable) value for the base of this exponential function.

Theorem 2.5.3 Derivative of the Generalized Exponential Function. *If $b > 0$ and $b \neq 1$, then:*

$$\frac{d}{dx}\left(b^x\right) = b^x \ln(b).$$

Chapter 3

Implicit Differentiation

3.1 Implicit Differentiation

Let's quickly recap what we've done with this new calculus object, the derivative:

1. We defined the derivative at a point (Definition 2.1.1) to find the slope of a line touching a graph of a function $f(x)$ at a single point. We call this the "slope of the tangent line" at a point.

2. Once we calculated this slope, we quickly found a way to think about the derivative as a *function* (Definition 2.1.2) that connects x-values with the slope of the line tangent to $f(x)$ at that x-value.

3. We thought about how we could interpret the derivative as more than just a slope (Section 2.2). We can think about this as an instantaneous rate of change, and use it to add detail to how we think about mathematical models of different things.

4. We spent some time building up shortcuts, noticing patterns, and generalizing ways of finding these derivative functions for specific functions (Section 2.3) as well as combinations of those functions (Section 2.4 and Section 2.5).

Our goal, now, is to generalize this a bit. What happens when we push past the restriction of only dealing with *functions*? Can we think of some reasonable *non-functions* that might produce curves? Might we think about tangent lines and slopes in these contexts?

3.1.1 Explicit vs. Implicit Definitions

Definition 3.1.1 Explicitly and Implicitly Defined Curves. A function or curve that is defined **explicitly** is one where the relationship between the variables is stated in with an equation like $y = f(x)$. Here, x is the input variable and we can find each corresponding value of the y-variable by applying some operations to x. As an example, we might consider the following function:

$$y = 3x + 1.$$

A function or curve that is defined **implicitly** us one where the relationship between the variables is stated with an equation connecting the variables, but not necessarily one which is "solved" for a single variable. Here, the relationship between variables is not stated with the typical "input" and "output" variables. As an example, we might consider the same function as above, but defined as:

$$y - 3x - 1 = 0.$$

Often, an implicitly defined curve is one where we *cannot* solve for a single variable by isolating it. ◊

A classic and important implicitly defined curve is the unit circle:

$$x^2 + y^2 = 1.$$

We can try to isolate for y and write this as an explicitly defined curve, and end up with:

$$y = \sqrt{1 - x^2}.$$

Unfortunately, this only displays the upper half of the circle, since the square root will only output positive values. In this case, we can get around this by defining the circle with two functions.

As we move forward, let's work with this circle using the implicitly defined version ($x^2 + y^2 = 1$). How might we find a slope of a line tangent to this circle at some point?

3.1.2 Using a Derivative as an Operator

Let's recall back to Subsection 2.2.5 Notation for Derivatives. We talked about how we can use the notation $\frac{d}{dx}(f(x))$ as a kind of action: the notation says "find the derivative of $f(x)$ with respect to x." When we say "with respect to x," we mean that we are treating x as an input variable, and trying to find out how f changes based on changes to that input. The notation says, "find the rate at which $f(x)$ changes as x increases."

Because this notation is a call to action, we can use it when we're dealing with an equation. We can call back to our early algebra days, where we learn that whatever we do to one side of an equation needs to be done to the other side as well, in order to maintain the equality.

We can apply this to our derivative actions: we can differentiate both sides of an equation!

Activity 3.1.1 Thinking about the Chain Rule.

(a) Explain to someone how (and why) we use the The Chain Rule to find the following derivative:

$$\frac{d}{dx}\left(\sqrt{\sin(x)}\right).$$

(b) Let's say that $f(x) = \sin(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{f(x)}\right).$$

How is this different, or not different, than the previous derivative?

(c) Let's say that we have some other function, $g(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{g(x)}\right).$$

How is this different, or not different, than the previous derivatives?

(d) What is the difference between the following derivatives:

$$\frac{d}{dx}\left(\sqrt{x}\right) \qquad \frac{d}{dx}\left(\sqrt{y}\right) \qquad \frac{d}{dy}\left(\sqrt{y}\right)$$

Hint. When do we need to use the Chain Rule? When do we need to use some linking derivative to connect the function we're looking at with the variable we care about?

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{x}\right) &= \frac{d}{dx}\left(x^{1/2}\right)\\ &= \frac{1}{2}(x^{-1/2})\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{y}\right) &= \frac{d}{dy}\left(y^{1/2}\right)\cdot\frac{dy}{dx}\\ &= \frac{1}{2}(y^{-1/2})\cdot\frac{dy}{dx}\\ &= \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}\\ \text{or } &\frac{y'}{2\sqrt{y}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dy}\left(\sqrt{y}\right) &= \frac{d}{dx}\left(x^{1/2}\right)\\ &= \frac{1}{2}(y^{-1/2})\\ &= \frac{1}{2\sqrt{y}}\end{aligned}$$

Because we'll be applying our derivative notation to pieces of some equation, we'll need to be very aware of where we need to apply the Chain Rule.

Now we can look at some examples of implicitly defined curves and think about questions about the derivative. Let's start with our circle.

Activity 3.1.2 Slopes on a Circle. Visualize the unit circle. Feel free to draw one, or find the picture above. We're going to think about slopes on this circle.

(a) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly horizontal. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(b) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly vertical. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(c) Find the point(s) where $x = \frac{1}{2}$. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

Hint. There are two points to consider here: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

(d) For the unit circle defined by the equation $x^2 + y^2 = 1$, apply the derivative to both sides of this equation to get the following:

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(1) \end{aligned}$$

Carefully consider each of these derivatives (each of the terms). Which of these will you need to apply the Chain Rule for?

(e) Differentiate. Solve for $\frac{dy}{dx}$ or y', whichever notation you decide to use.

Hint 1. Make sure to use the Chain Rule when necessary!

Hint 2. $\dfrac{d}{dx}\left(y^2\right) = \dfrac{d}{dy}\left(y^2\right)\dfrac{dy}{dx} = 2y \cdot \dfrac{dy}{dx}$ or $2yy'$

(f) Go back to the first few questions, and try to answer them again:

(a) Find the locations of any horizontal tangent lines (where $\frac{dy}{dx} = 0$).

(b) Find the locations of any vertical tangent lines (where $\frac{dy}{dx}$ doesn't exist, or where you would divide by 0).

(c) Find the values of $\frac{dy}{dx}$ for the points on the circle where $x = \frac{1}{2}$.

Example 3.1.2 Let's repeat some of this process, but using a new curve. Consider the curve defined by the equation:

$$y^2 = x^3 - x + 1.$$

This curve is a special curve with some interesting mathematical properties, and is actually a part of a family of curves called **elliptic curves**. For now, let's just consider it as a fun curve to look at, and use implicit differentiation to think about it.

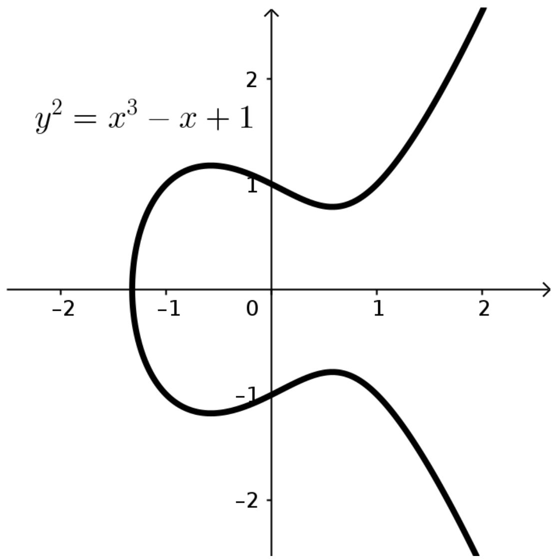


Figure 3.1.3

(a) Mark the locations on the curve where it looks like the curve will have horizontal tangent lines. How many did you find?

(b) Mark the locations on the curve where it looks like the curve will have vertical tangent lines. How many did you find?

(c) Find the point(s) where $x = 0$. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(d) For the elliptic curve defined by the equation $y^2 = x^3 - x + 1$, apply the derivative to both sides of this equation:

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^3 - x + 1\right)$$
$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

Carefully consider each of these derivatives (each of the terms). Which of these will you need to apply the Chain Rule for?

(e) Differentiate. Solve for $\frac{dy}{dx}$ or y', whichever notation you decide to use.

Hint 1. Make sure to use the Chain Rule when necessary!

Hint 2. $\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$ or $2yy'$

(f) Go back to the first few questions, and try to answer them again:

(a) Find the locations of any horizontal tangent lines (where $\frac{dy}{dx} = 0$).

(b) Find the locations of any vertical tangent lines (where $\frac{dy}{dx}$ doesn't exist, or where you would divide by 0).

(c) Find the values of $\frac{dy}{dx}$ for the points on the curve where $x = 0$.

□

This example was pretty similar to the first activity: in both of these, we use the Chain Rule to differentiate $\frac{d}{dx}\left(y^2\right)$. Let's look at another example.

Activity 3.1.3 . Let's consider a new curve:

$$\sin(x) + \sin(y) = x^2y^2.$$

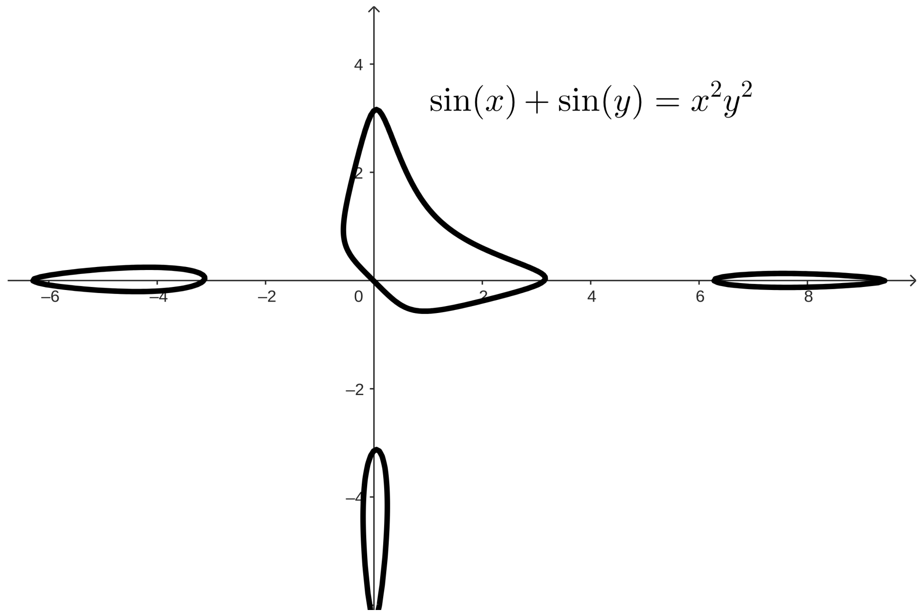


Figure 3.1.4

(a) We are going to find $\frac{dy}{dx}$ or y'. Let's dive into differentiation:

$$\begin{aligned} \frac{d}{dx}\left(\sin(x) + \sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \end{aligned}$$

Think carefully about these derivatives. For each of the three, how will you approach it? What kinds of nuances or rules or strategies will you need to think about? Why?

Hint. Are any of these derivatives involving a variable other than x, the input variable (based on our $\frac{dy}{dx}$ notation, since we are thinking about how y changes *with regard to* x).

Are any of these derivatives involving any other combination of functions? Are there products and/or quotients that we need to think about?

(b) Implement your ideas or strategies from above to differentiate each term.

Hint. We need to apply the Chain Rule to $\frac{d}{dx}\left(\sin(y)\right)$ and then we need to apply the Product Rule $\frac{d}{dx}\left(x^2y^2\right)$. Notice that when we find the derivative of y^2 for the Product Rule, we need to use the Chain Rule!

Solution.

$$\begin{aligned} \frac{d}{dx}\left(\sin(x) + \sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \cos(x) + \cos(y) \cdot \frac{dy}{dx} &= 2xy^2 + 2x^2y \cdot \frac{dy}{dx} \\ \text{or } \cos(x) + y'\cos(y) &= 2xy^2 + 2x^2yy' \end{aligned}$$

(c) Now we need to solve for $\frac{dy}{dx}$ or y', whichever you are using. While this equation can look complicated, we can notice something about the "location" of $\frac{dy}{dx}$ or y' in our equation.

Why do we always know that $\frac{dy}{dx}$ or y' will be *multiplied* on a term whenever it shows up?

(d) Now that we are confident that we will *always* know that we are multiplying this derivative, we can employ a consistent strategy:

(a) Rearrange our equation so that every term with a $\frac{dy}{dx}$ or y' is on one side, and everything without is on the other.

(b) Now we are guaranteed that $\frac{dy}{dx}$ or y' is a common factor: factor it out.

(c) Now we can solve for $\frac{dy}{dx}$ or y' by dividing!

Solve for $\frac{dy}{dx}$ or y' in your equation!

(e) Build the equation of a line that lays tangent to the curve at the origin. Does the value of $\frac{dy}{dx}$ at $(0,0)$ look reasonable to you?

3.1.3 Some Summary and Strategy

Let's wrap this up with some general strategy and summary of what we've seen.

The first thing we can notice is that we have talked through how to employ two of the three big derivative rules: we used Chain Rule throughout these examples, and in Activity 3.1.3 we needed to use the Product Rule in order to differentiate $\frac{d}{dx}\left(x^2y^2\right)$. We have a glaring omission from our examples so far, though. Where is the Quotient Rule?

In these implicitly defined curves, we can manipulate the equations to never have to think about division! Consider the curve:

$$\frac{\sin(x)}{x} + \frac{\sin(y)}{x} = xy^2.$$

Graph this curve in a graphing utility like desmos. Does it look any different than the curve in Activity 3.1.3?

The only difference, really, is that the curve with the division is not defined at $x = 0$. As long as we keep those domain issues in mind, we can multiply everything by x to get our familiar equation:

$$\sin(x) + \sin(y) = x^2y^2.$$

And there we go, we never have to think about the Quotient Rule in these kinds of contexts!

So we really only need a strategy for using the Chain Rule and the Product Rule.

Strategy for Implicit Differentiation.

- We use the *Chain Rule* whenever we differentiate something like $\frac{d}{dy}\left(f(y)\right)$. We differentiate whatever the function is, and multiply by the derivative of y: $f'(y)y'$.

  This generalizes more: any time the variable in our derivative notation does not match the variable in the function/term, we need to use the Chain Rule:

$$\frac{d}{dy}\left(e^x\right) \qquad \frac{d}{dt}\left(\sin(x)\right) \qquad \frac{d}{dx}\left(y^4\right)$$

- We use the *Product Rule* whenever we differentiate a term with some combination of x and y variables. More generally, we can use the Product Rule any time we have a combination of at least two variables. We have to treat these as different kinds of functions getting multiplied!

$$\frac{d}{dy}\left(xe^y\right) \qquad \frac{d}{dt}\left(\cos(t)\sin(x)\right) \qquad \frac{d}{dx}\left(y^4\sqrt{x+1}\right)$$

From here on out, we will use the ideas of implicit differentiation to accomplish two things:

1. We have a bit more flexibility with how we think of derivatives! We do not need to be restricted to only thinking about functions in order to think about rates of change or slopes at a point. We can think about any curve, any relationship between variables, and think about the relationship between them based on how one changes with regard to the other.

2. Implicit differentiation will be a very useful tool. Even when we have functions that can be written explicitly, they might be hard to deal with -- overly complex or maybe involving functions that we aren't used to. It is absolutely possible, and a really useful strategy, to re-write the relationship between variables implicitly! We can maybe create a version of these equations that we can differentiate!

We're going to use this second idea first: in the next section we'll be thinking about inverse functions. We do not have any idea of how to think about derivatives of logarithmic functions, like $y = \ln(x)$.

We can re-write this:

$$y = \ln(x) \longleftrightarrow x = e^y.$$

This second representation is something we can differentiate!

Similarly, if we wanted to think about the derivative of $y = \tan^{-1}(x)$, we might instead think about re-writing this:

$$y = \tan^{-1}(x) \longleftrightarrow x = \tan(y).$$

There are some weird issues to think about with the domains and ranges of these functions, but this is how we'll approach these derivatives next.

3.2 Derivatives of Inverse Functions

We should start here by saying: we're going to be thinking about inverse functions, and so maybe it will be helpful to recap some facts about inverse functions.

- If $y = f(x)$ is some function, then we can use the inverse function to represent this relationship between variables: $x = f^{-1}(y)$.Some examples:

- $y = e^x \longleftrightarrow x = \ln(y)$. That is, the logarithm function "solves" for the exponent (sometimes this is easier to just say that the logarithm *is* the exponent).
- $y = \sin(\theta) \longleftrightarrow \theta = \sin^{-1}(y)$. That is, this inverse sine function (or sometimes $\arcsin(y)$) finds the angle at which sine of that angle is y. With these trigonometric functions, we need to make some restrictions: because there are an infinite number of angles that will produce the same output of the sine function (reflecting the angle across the y-axis will do it, as will adding any multiple of 2π), we restrict the angles that the inverse sine function can output to being in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- Based on this re-representation above, we can always compose a function and its inverse to get the identity function, $y = x$. In general, if $y = f(x)$ has an inverse function f^{-1}, then $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x$. Similarly, we can compose in the opposite order: $\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x$. This can be a bit trickier to think about for the inverse trigonometric functions, since this only works on intervals of x where that inverse is defined. So we end up with strange things like:

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = \sin^{-1}(-1) = -\frac{\pi}{2}.$$

This is because the inverse sine function finds only angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the angles $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$ are *coterminal* (and so have the same output from the sine function).

With these small facts in mind, we can think about derivatives of inverse functions.

3.2.1 Wielding Implicit Differentiation

We're going to do a very cool thing: in order to find derivatives of inverse functions, we can invert the relationship between x and y, and then use Implicit Differentiation to find $\frac{dy}{dx}$.

Activity 3.2.1 Building the Derivative of the Logarithm. We're going to accomplish two things here:

1. By the end of this activity, we'll have a nice way of thinking about $\frac{d}{dx}(\ln(x))$, and we will now be able to differentiate functions involving logarithms!
2. Throughout this activity, we're going to develop a way of approaching derivatives of inverse functions more generally. Then we can apply this framework to other functions!

Let's think about this logarithmic function!

(a) We have stated (a couple of times now) how we define the log function:

$$y = e^x \longleftrightarrow x = \ln(y).$$

This arrow goes both directions: the log function is the inverse of the exponential, but the exponential is the inverse of the log function!

Can you re-write the relationship $y = \ln(x)$ using its inverse (the exponential)?

(b) For your inverted $y = \ln(x)$ from above (it should be $x =$ ____), apply a derivative operator to both sides, and use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Where do we have to use Chain Rule?

(c) A weird thing that we can notice is that when we use implicit differentiation, it is common to end up with our derivative written in terms of both x and y variables. This makes sense for our earlier examples: we needed specific coordinates of the point on the circle, for instance, to find the slope there.

But if $y = \ln(x)$, we want $\frac{dy}{dx}$ or y' to be a function of x:

$$f(x) = \ln(x) \longrightarrow f'(x) = \underline{\qquad\qquad}.$$

Your derivative is written with only y values.

Since $y = \ln(x)$, replace any instance of y with the log function. What do you have left?

(d) Remember that $y = \ln(x)$. Substitute this into your equation for $\frac{dy}{dx}$. Can you write this in a pretty simplistic way?

Hint. Remember that $e^{\ln(x)} = x$, since these functions are inverses of each other!

(e) Before we go much further, we should be a bit careful. What is the domain of this derivative?

What are the values of x where $\frac{d}{dx}(\ln(x))$ makes sense to think about?

Theorem 3.2.1 Derivative of the Logarithmic Function.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

for $x > 0$.

Further, since $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ *(for* $b \geq 0$ *and* $b \neq 1$*), we can say that:*

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$$

for $x > 0$.

3.2.2 Derivatives of the Inverse Trigonometric Functions

Let's try a similar thing with inverse trigonometric functions. We'll start with the inverse sine function, $y = \sin^{-1}(x)$.

Activity 3.2.2 Finding the Derivative of the Inverse Sine Function. We're going to do the same trick, except that there will be a couple of small differences due to thinking specifically about trigonometric functions.

Let's think about the function $y = \sin^{-1}(x)$. We know that this is equivalent to $x = \sin(y)$ (for y-values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

(a) Move the point around the portion of the unit circle in the graph below. Convince yourself that:

- $\sin(y) = x$

- $\sin(y) \geq 0$ when $0 \leq y \leq \frac{\pi}{2}$
- $\sin(y) < 0$ when $-\frac{\pi}{2} \leq y < 0$

What is $\cos(y)$ in this figure? Does the sign change depending on the value of y?

(b) Use implicit differentiation and the equation $x = \sin(y)$ to find $\frac{dy}{dx}$ or y'

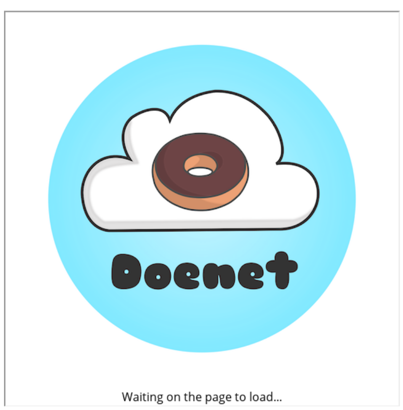
(c) If you still have your derivative written in terms of y, make sure to write $\cos(y)$ in terms of x!

(d) Let's think about the domain of this derivative: what x-values make sense to think about?

Think about this both in terms of what x-values reasonably fit into your formula of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$ as well as the domain of the inverse sine function in general.

(e) Notice that in the denominator of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$, you have a square root. Based on that information (and the visual above), what do you expect to be true about the sign of the derivative of the inverse sine function?

Confirm this by playing with the graph of $y = \sin^{-1}(x)$ below.



(f) Investigate the behavior of $\frac{dy}{dx}$ at the end-points of the function: at $x = -1$ and $x = 1$. What do the slopes look like they're doing, graphically?

How does this work when you look at the function you built above? What happens when you try to find $\left.\frac{dy}{dx}\right|_{x=-1}$ or $\left.\frac{dy}{dx}\right|_{x=1}$?

Let's repeat the process to find the derivatives of $y = \tan^{-1}(x)$ and $y = \sec^{-1}(x)$.

Activity 3.2.3 Building the Derivatives for Inverse Tangent and Secant.

(a) Consider the triangle representing the case when $y = \tan^{-1}(x)$.

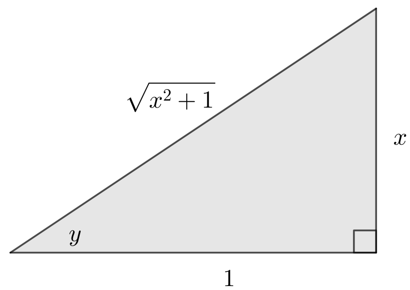


Figure 3.2.2

For $x = \tan(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ based on the triangle above, but we will refer back to the version with the $\sec(y)$ in it later.

(b) Consider the triangle representing the case when $y = \sec^{-1}(x)$.

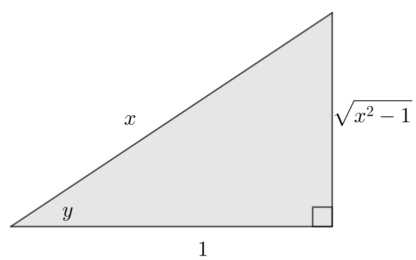


Figure 3.2.3

For $x = \sec(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ and $\tan(y)$ based on the triangle above, but we will refer back to the version with the functions of y in it later.

(c) Here's a graph of just the unit circle for angles $[0, \pi]$. We are choosing to focus on this region, since these are the angles that the inverse tangent and inverse secant functions will return to us. We want to investigate the signs of $\tan(y)$ and $\sec(y)$.

(d) Go back to our derivative expressions for both the inverse tangent and inverse secant functions. What do we know about the signs of these derivatives?

Hint. Notice that in $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2(y)}$, we know that the derivative must be positive. Even when $\sec(y) < 0$, we are squaring it.

In $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{\sec(y)\tan(y)}$, we know that the derivative must also always be positive. Whenever $\sec(y) < 0$, we have $\tan(y) < 0$, and so the product must be positive.

(e) Confirm your idea about the sign of the derivatives by investigating the graphs of each function.

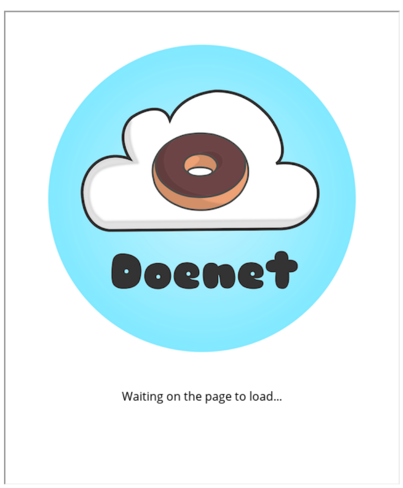
(f) How do we need to write these derivatives, when we write them in terms of x to account for the sign of the derivative?

Hint. Use an absolute value in the formula for $\frac{d}{dx}\left(\sec^{-1}(x)\right)$!

It is important to think carefully about how things might change when we start thinking about other trigonometric functions. For instance, what happens when we think about $y = \cos^{-1}(x)$ instead? We *could* repeat the process from Activity 3.2.2 with $y = \cos^{-1}(x)$ instead (and we'll do that for $y = \tan^{-1}(x)$), but for now let's think about the graph of $y = \cos^{-1}(x)$.

Activity 3.2.4 Connecting These Inverse Functions. We're going to look at a graph of $y = \cos^{-1}(x)$, but we're specifically going to try to compare it to the graph of $y = \sin^{-1}(x)$. We'll use some graphical transformations to make these functions match up, and then later we'll think about derivatives.

(a) Ok, consider the graph of $y = \cos^{-1}(x)$ and a transformed version of the inverse sine function. Apply some graphical transformations to make these match!



(b) It might be fun to think about another reason that this connection between $\sin^{-1}(x)$ and $\cos^{-1}(x)$ exists.

Consider this triangle:

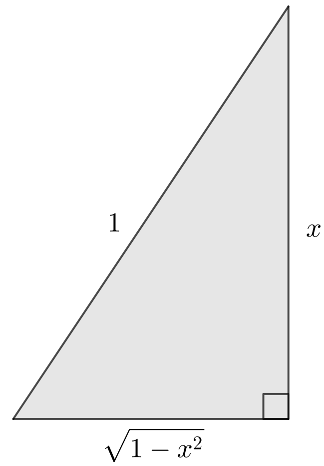


Figure 3.2.4

We're going to think about these inverse trigonometric functions as angles: let $\alpha = \cos^{-1}(x)$ and $\beta = \sin^{-1}(x)$. We can re-write these as:

$$\cos(\alpha) = x$$
$$\sin(\beta) = x.$$

Can you fill in your triangle using this information?

Why does $\alpha + \beta = \frac{\pi}{2}$? Convince yourself that this is what we did with the graphical transformations above, as well.

(c) Use this equation above, re-writing $\cos^{-1}(x)$ as some expression involving the inverse sine function, and then find

$$\frac{d}{dx}\left(\cos^{-1}(x)\right).$$

Hint.

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{d}{dx}\left(-\sin^{-1}(x) + \frac{\pi}{2}\right)$$

We could repeat this task to try to connect the graph of $y = \tan^{-1}(x)$ with $y = \cot^{-1}(x)$ as well as the graph of $y = \sec^{-1}(x)$ with $y = \csc^{-1}(x)$, but we can hopefully see what will happen. In each case, we have the same kind of connection that we saw in the triangle, since these are cofunctions of each other!

We can summarize by believing that:

$$\begin{aligned}
\frac{d}{dx}\left(\cos^{-1}(x)\right) &= -\frac{d}{dx}\left(\sin^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\cot^{-1}(x)\right) &= -\frac{d}{dx}\left(\tan^{-1}(x)\right) = -\frac{1}{x^2+1} \\
\frac{d}{dx}\left(\csc^{-1}(x)\right) &= -\frac{d}{dx}\left(\sec^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

Theorem 3.2.5 Derivatives of the Inverse Trigonometric Functions.

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \textit{Domain: } -1 < x < 1$$

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \textit{Domain: } -1 < x < 1$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1} \qquad \textit{Domain: all Real numbers}$$

$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{x^2+1} \qquad \textit{Domain: all Real numbers}$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \textit{Domain: } x < -1 \textit{ and } x > 1$$

$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}} \qquad \textit{Domain: } x < -1 \textit{ and } x > 1$$

3.3 Logarithmic Differentiation

We're going to start with a quick fact about logs and their derivatives. The derivative rule for $\frac{d}{dx}(\ln(x))$ is still relatively new for us, so it is ok to still be getting comfortable with it, but we should notice this nice fact.

Fact 3.3.1 Derivative of the Log of a Function.

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)} \quad (\textit{when } f(x) > 0)$$

Note that there's nothing really special going on here: this is just an application of the The Chain Rule to the Derivative of the Logarithmic Function.

Throughout this section, the goal is to convince any open-minded readers of one thing:

Logs are friends.

Let us be informal and technically not quite correct but hopefully clear in this. Logs really are friendly mathematical objects. They were *created* to be friendly objects! In a time when doing arithmetic with large numbers was difficult due to a lack of computing technology, logs were introduced to make arithmetic easier.

The general idea is that, if there is some kind of hierarchy of operations, then logs transform operations between things into different operations that are lower on the hierarchy of operations. So logs turn things like products (repeated addition) and quotients (repeated subtraction) into addition and subtraction. Logs turn exponents (repeated multiplication) into coefficients.

Using math notation, we can write the following log properties.

Fact 3.3.2 Properties of Logarithms. *We will make use of the following properties of logarithms.*

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^y) = y\ln(x)$

Because of the domain of the log function, we need $x, y > 0$ for these properties to make sense. We will use them relatively loosely, with functions that take on negative and positive values, and not worry too much about the domain issues when we don't need to.

3.3.1 Logs Are Friends!

Ok, so how will we use these new-found friends? We're going to think about some functions (and combinations of functions) that are new to us and that aren't so clear for us to use things like the Product, Quotient, or Chain Rule. We'll try to use logs to re-write our functions into some easier to approach implicitly defined relationships in order for us to differentiate.

But first, let's build an explanation for the Power Rule for Derivatives.

Activity 3.3.1 Returning to the Power Rule. Back in Section 2.3 we built an explanation for why $\frac{d}{dx}(x^n) = nx^{n-1}$ that relied on thinking about exponents as repeated multiplication: it relied on n being some positive integer. We said, at the time, that the Power Rule generalizes and works for *any* integer, but did so without explanation.

Let's consider $y = x^n$ where n is just some real number without any other restrictions.

(a) Apply a logarithm to both sides of this equation:

$$\ln(y) = \ln(x^n)$$

Now use one of the Properties of Logarithms to re-write this equation.

(b) Use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Remember that when you solve for $\frac{dy}{dx}$ or y', you might have some y-variables in your derivative. Replace them with $y = x^n$.

(c) Explain to yourself why this is equivalent to the Power Rule that we built so long ago:

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(d) Let's get weird. What if we have a not-quite-power function? Where the thing in the exponent isn't simply a number, but another variable?

Let's use the same technique to think about $y = x^x$ and its derivative. First, though, confirm that this is not a power function (and so we cannot use the Power Rule to find the derivative) and is also not an exponential function (and so the derivative isn't itself or itself scaled by a log).

(e) Now apply a log to both sides:

$$\ln(y) = \ln(x^x).$$

Re-write this using the same log property as before, and then use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Don't forget that in order to find $\frac{dy}{dx} x\ln(x)$, we need to use the Product Rule.

(f) Explain to yourself why logs are friends, especially when trying to differentiate functions in the form of $y = (f(x))^{g(x)}$.

This idea that we've just implemented (applying a logarithm to make some function more friendly and then using implicit differentiation to differentiate) is often called **logarithmic differentiation**. It works so well because *logs are friends*.

3.3.2 Wow, So Friendly!

This is incredible! We can now differentiate a whole new class of functions! Functions raised to functions, what could we think of next!

How about products and quotients of functions? I know, I know, we have The Product and Quotient Rules...what about *big* products and quotients? Annoying ones. Complicated ones.

Activity 3.3.2 Logarithmic Differentiation with Products and Quotients. Let's say we've got some function that has products and quotients in it. Like, a lot. Consider the function:

$$y = \frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}.$$

(a) Work out a general strategy for how you would find y' directly. Where would you have to use Quotient Rule? What are the pieces? Where would you have to use Product Rule? What are the pieces? Where would you have to use Chain Rule? What are the pieces?

To be clear: do not actually differentiate this. Just look at it in horror and try to outline a plan that some other fool would use.

Click on the "Solution" below to see what the fool did.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right) \\
&= \frac{(x+1)^7(x+5)^3\,\frac{d}{dx}\left(\overbrace{(x-4)^2\sqrt{3x+1}}^{\text{Product Rule}}\right) - (x-4)^2\sqrt{3x+1}\,\frac{d}{dx}\left(\overbrace{(x+1)^7(x+5)^3}^{\text{Product Rule}}\right)}{((x+1)^7(x+5)^3)^2} \\
&= \frac{(x+1)^7(x+5)^3\left(2(x-4)\sqrt{3x+1} + \frac{3(x-4)^2}{2\sqrt{3x+1}}\right) - (x-4)^2\sqrt{3x+1}\left(7(x+1)^6(x+5)^3 + 3(x+1)^7(x\right.}{(x+1)^{14}(x+5)^6}
\end{aligned}$$

What now? Can we "simplify" this somehow? Maybe, but I am *not* doing any more of this!

(b) Let's instead use logarithmic differentiation. First, apply the log to both sides to get:

$$\ln(y) = \ln\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right).$$

Since this function is just a bunch of products of things with exponents all put into some big quotient, we can use our log properties to re-write this!

(c) We should have:

$$\ln(y) = 3\ln(x-4) + \frac{1}{2}\ln(3x+1) - 7\ln(x+1) - 3\ln(x+5).$$

Confirm this.

(d) Now differentiate both sides! You'll have to use some Chain Rule (but not a lot)! Refer back to Fact 3.3.1 for help here.

(e) Solve for $\frac{dy}{dx}$ or y'.

(f) While this is not a *nice* looking expression for the derivative, spend some time confirming that this was a nicer *process* then differentiating directly. This is because logs are friends.

So how do we wrap this up? I hope we can see that logs are a useful and powerful tool: we can use logarithmic differentiation to transform our functions to become "easier to work with" versions of themselves: we put everything on a log-scale and allow our properties of logarithms to make the operations become a bit more accessible.

This is a commonly used trick in many applications of calculus. Specifically, this is used often enough in statistics that there is a whole paradigm of estimation (called Maximum Likelihood Estimation) that uses a log-transformed version of a likelihood function and then applies some basic calculus ideas (that we'll cover in Section **??**) to perform some powerful estimations.

While I hope that we end up leaving this section knowing that *logs are friends*, we can probably add a second (and related result) that we're using over and over.

Using the Chain Rule is probably easier than any other option.

We apply logs in order to re-write these functions in a friendly way *because* we would rather use the Chain Rule than any combination of other derivative strategies.

Chapter 4

Applications of Derivatives

4.1 Mean Value Theorem

Let's begin here with some weird questions. The questions aren't weird because of what they're asking. Instead, they're weird because of the logic of how we interpret them compared to how we *want* to interpret them.

1. Why is the derivative of a constant function 0?
2. Why do $y = x^2 + 7$ and $y = x^2 - 3$ have the same derivative?
3. If a function is only increasing on the interval $(0, 1)$, what do we know about the derivative at any of these x-values in $(0, 1)$?

These questions are ones we can think through and answer.

Here are some answers for these first three questions:

1. A constant function has all of the same y-values for any x-value in the domain: of course the slope everywhere is 0! There isn't any change in the y-values!
2. We can differentiate these functions term by term: we know that the x^2 term has a derivative of $2x$, and then for each function's constant, the derivative has to be 0. So it doesn't matter what each constant was, the derivatives wouldn't rely on that value!
3. If a function is increasing on an interval, then it means that for any pair of x-values, $x_1 < x_2$, we get y-values in the same order: $f(x_1) < f(x_2)$. If we find the limit of slopes as $x_1 \to x_2$, we'll always get a positive slope, so the derivative has to be positive.

What's tricky is that these don't always say what we *want* to say. For instance, I might sometimes want to say the following:

1. If you know a function's derivative is 0, then you know the function is constant. Another way of saying this is that the *only* functions whose derivative is 0 are constant functions.
2. Similarly, we might want to say that if you know two functions that have the same derivative, then the only difference between the functions is a constant. There aren't any other ways for functions to be different with the same derivative.

3. We might want to say that if you know the derivative is positive on an interval, that means that the function has to be increasing.

Do you see the (slight) difference?

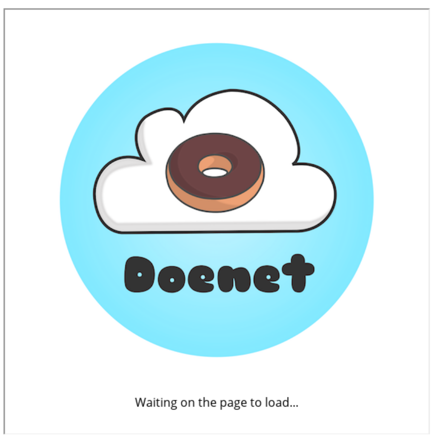
What we'll secretly see is that all of these statements are actually correct but require a result for us to say them. The Mean Value Theorem will be the result that we use to build important and helpful results throughout the rest of this course.

4.1.1 Slopes

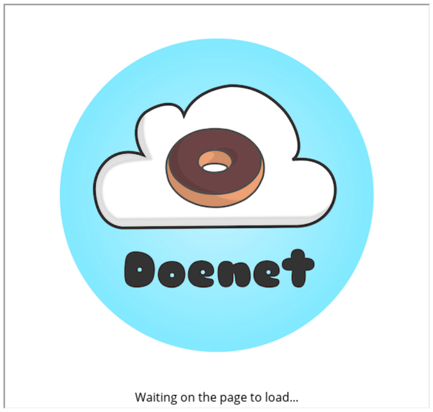
We have two different kinds of slopes that we think of right now: a slope between two points, and a slope at a single point.

We can translate this into a rate of change! We think of two rates of change: an average rate of change on some interval and the instantaneous rate of change at some point.

We will try to connect these two different slopes/rates of change for "well-behaved" functions. Let's take a look at an example. In the graph below, we have a curve where we are considering some closed interval, as well as a point within that interval. Both slopes are visualized and calculated: the slope between the ending points of the interval is the average slope, while the slope of the line tangent to the curve at a point is the instantaneous slope. Move the point/interval around and get a feel for how these two different slopes relate (or don't relate!) to each other.



If you move the interval/point around enough, you'll see that sometimes these two slopes are really similar and sometimes they're very different. But what if the point in the middle of the interval wasn't so "set" at being stuck in the exact middle of the interval? What if we stuck the interval in place, and then had the freedom to move the point anywhere in the interval?



Do you think we will *always* be able to do this? What kinds of functions will have these points where the slope at the point matches the average slope on the interval?

4.1.2 The Mean Value Theorem

Theorem 4.1.1 Mean Value Theorem. *For a function $f(x)$ that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is some value $x = c$ with $a < c < b$ where:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This theorem is really just guaranteeing the existence of at least one of the points we found above: a point where the slope of the line tangent to the curve is the same as the slope between the endpoints of the interval. We can (and, very briefly, we will) use this theorem to find the point that is guaranteed to exist, but we will, more generally, use this theorem as a tool for connection.

We will try to use it as a way of connecting derivatives to the behavior of a function. The Mean Value Theorem gives us this equation where on one side we have a derivative, and on the other side we have function outputs. This is really similar to the definition of the Derivative at a Point, except that in this case we have no limit: we just get to use the behavior of the function on an interval around the point.

Secretly, the Mean Value Theorem is the driving force behind most of the results we will look at from here on out, at least when the requirements include continuity *and* differentiability on an interval. You can almost guarantee that if a theorem or result has these two requirements, then the Mean Value Theorem is likely at work.

Let's look at one example, at least, before we move on.

Example 4.1.2 Let's say that a person is planning on biking to their college campus, 8 miles away. At 2:39pm they send a text to their friend with a selfie of them on their bike about to start their trip, captioned "Beautiful day for a ride!" At 3:27pm, they post a picture on their social media of them in front of the bike rack on campus.

(a) What was the average velocity of the student between 2:39pm and 3:27pm?

Solution. The total trip took 48 minutes (or 0.8 hours), and the student traveled a total distance of 8 miles.

The student's average velocity is $\dfrac{8}{0.8} = 10$ miles per hour.

(b) The Mean Value Theorem connects average slopes with slopes of tangent lines. What does that mean, in general, in this context?

Solution. In this case, the average rate of change of the function on the interval is the average velocity of the biker. So then the instantaneous rate of change must correspond with an instantaneous velocity, of their velocity at some specific point in time.

(c) Make a claim about the instantaneous velocity of the student on their bike at some point in time.

Solution. We know that at some point between 2:39pm and 3:27pm, the cyclist had to be traveling at exactly 10 miles per hour.

□

Example 4.1.3

(a) For a function $f(x) = \sqrt{x} + 1$ on the interval $[1, 4]$, find the point that the Mean Value Theorem guarantees the existence of, and explain what it is.

Solution. Let's calculate the average slope on the interval:

$$\begin{aligned} \frac{f(4) - f(1)}{4 - 1} &= \frac{(\sqrt{4} + 1) - (\sqrt{1} + 1)}{3} \\ &= \frac{1}{3} \end{aligned}$$

So we know that there is some x-value between 1 and 4 where $f'(x) = \dfrac{1}{3}$.

The derivative is $f'(x) = \dfrac{1}{2\sqrt{x}}$, so we can solve for x:

$$\begin{aligned} f'(x) &= \frac{1}{3} \\ \frac{1}{2\sqrt{x}} &= \frac{1}{3} \\ 2\sqrt{x} &= 3 \\ \sqrt{x} &= \frac{3}{2} \\ x &= \frac{9}{4} \end{aligned}$$

So the point guaranteed to exist by the Mean Value Theorem is located at $\left(\frac{9}{4}, \frac{5}{2}\right)$, where the slope of the tangent line is $f'\left(\frac{9}{4}\right) = \frac{1}{3}$.

□

These examples are fine, but the real power of the Mean Value Theorem comes in how we can use it to general more interesting results. Let's check those out!

4.1.3 More Results due to the Mean Value Theorem

First, let's remind ourselves of what it means for a function to be increasing or decreasing on an interval.

Definition 4.1.4 Increasing and Decreasing on an Interval. A function $f(x)$ is **increasing** on the interval (a, b) if, for every pair of x-values x_1 and x_2 (with $x_1 < x_2$), $f(x_1) < f(x_2)$.

If $f(x_1) > f(x_2)$, then we say that f is **decreasing** on the interval (a, b). ♢

Note that we classify a function as increasing or decreasing based on comparing two function outputs. This is a perfect time to use the Mean Value Theorem, since it can connect these pairs of function outputs with a derivative!

Theorem 4.1.5 Sign of the Derivative and Direction of a Function. *If f is a function that is differentiable on the interval (a, b) and $f'(x) > 0$ for all x in the interval (a, b), then f must be increasing on (a, b).*

Similarly, if $f'(x) < 0$ for all x in the interval (a, b), then f must be decreasing on (a, b).

Proof. Before we begin, let's just agree to think about only the case where $f'(x) > 0$ on the interval (a, b). The other case (where $f'(x) < 0$) ends up being the exact same argument, with some changes in signs and directions of inequalities. So we'll say $f'(x) > 0$ for all x-values in the interval (a, b).

Ok, let's begin!

Let's pick two arbitrary x-values from the interval (a, b). Call them x_1 and x_2, and we'll make sure that we name them in a way where $x_1 < x_2$. Now, f must be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). We also know that $f'(x) > 0$ for every x in the interval $(x_1 < x < x_2)$.

The Mean Value Theorem says that there is some $x = c$ in (x_1, x_2) with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Equivalently, this means that

$$f'(c)(x_2 - x_1) = f(x_2) - f(x_1).$$

Notice that $f'(c) > 0$ (since the derivative is positive everywhere in the interval) and $(x_2 - x_1)gt0$ (by the wat we named these x-values). This means that $f'(c)(x_2 - x_1) > 0$, and so $(f(x_2) - f(x_1)) > 0$ as well.

Ok so notice what just happened: we arbitrarily chose x-values x_1 and x_2 and noticed that for any of these pairs where $x_1 < x_2$, we found that $f(x_1) < f(x_2)$. This is exactly what it means for f to be increasing on the interval (a, b) (based on Definition 4.1.4). ■

We'll use this result pretty extensively in the next couple of topics. It is helpful for us to use this to connect the signs of a derivative with our intuition about what that must mean for the direction of a function.

Theorem 4.1.6 If a Function's Derivative is 0, it's Constant. *If $f(x)$ is a function defined on (a, b) with $f'(x) = 0$ for all x-values in the interval (a, b), then $f(x)$ is a constant function.*

Proof. We can almost exactly replicate the proof from Theorem 4.1.5 here!

Let's pick two arbitrary x-values from the interval (a, b). Call them x_1 and x_2, and we'll again make sure that we name them in a way where $x_1 < x_2$. Now, f must be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). This time, we know that $f'(x) = 0$ for every x in the interval $(x_1 < x < x_2)$.

The Mean Value Theorem says that there is some $x = c$ in (x_1, x_2) with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Equivalently, this means that

$$f'(c)(x_2 - x_1) = f(x_2) - f(x_1).$$

Notice that $f'(c) = 0$ (since the derivative is zero everywhere in the interval). This means that $f'(c)(x_2 - x_1) = 0$, and so $(f(x_2) - f(x_1)) = 0$ as well.

This means that every y-value from the function is the same as every other one, since we picked these points arbitrarily. So f must be a constant function! ■

Theorem 4.1.7 Equal Derivatives Correspond with Related Functions. *For two functions f and g, both differentiable on (a, b), if $f'(x) = g'(x)$ for all x-values on (a, b), then we know that $f(x) = g(x) + C$ for some real number constant C. That is, the only differences in f and g are due to a difference in the constant term.*

Proof. This one is pretty easy to prove: we're going to use a fun little trick where we connect this theorem to Theorem 4.1.6.

Let's think about these two functions f and g with $f'(x) = g'(x)$, and let's think about a function $h(x) = f(x) - g(x)$. Now we can notice that $h'(x) = f'(x) - g'(x)$ which has to be 0.

Oh wow, $h(x)$ is a function with $h'(x) = 0$ on the interval (a, b). It must be a constant function (based on Theorem 4.1.6)! Let's call it $h(x) = C$, where C is some real number constant.

This means that $f(x) - g(x) = C$, and we can see that the only difference between these two functions is a constant. ■

We'll use this result a bit later on, but it's a great example of how we can use the Mean Value Theorem to connect information about the derivative to information about how a function might work.

Let me interject my own opinion here: I think the Mean Value Theorem is extremely important. But I also don't think that it is that important for you to *use*.

I think you should know the general idea of connecting the slopes of the line tangent to the curve and the average slope on an interval. I think you should remember the picture we looked at. I think you should be content to see some results later on in the course and smile fondly about how the Mean Value Theorem is under the surface, churning away and getting us cool results to think about. I think we should be happy that the Mean Value Theorem is here and we can recognize it as, maybe, the most important result in this textbook.

But we don't need to pretend that we need to actually *use* it directly...we aren't going to need to compute a lot or anything like that. We'll just *know* it.

4.2 Increasing and Decreasing Functions

Activity 4.2.1 How Should We Think About Direction? Our goal in this activity is to motivate some new terminology and results that will help us talk about the "direction" of a function and some interesting points on a function (related to the direction of a function). For us to do this, we'll look at some different examples of functions and try to think about some unifying ideas.

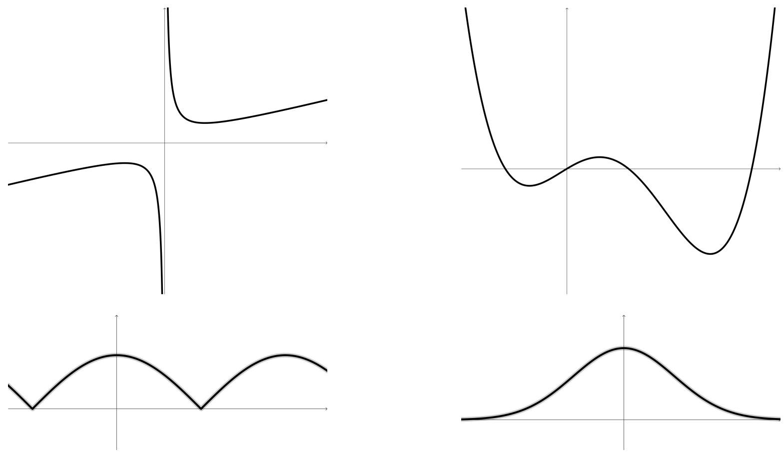


Figure 4.2.1

These examples do not cover all of the possibilities of how a function can act, but will hopefully provide us enough fertile ground to think about some different situations.

(a) In each graph, find and identify:

- The intervals where the function is increasing.
- The intervals where the function is decreasing.
- The points (or locations) around and between these intervals, the points where the function changes direction or the direction terminates.

(b) Make a conjecture about the behavior of a function at any point where the function changes direction.

Hint. What do you think has to be true about the derivative at these points?

(c) Look at the highest and lowest points on each function. You can even include the points that are highest and lowest just compared to the points around it. Make a conjecture about the behavior of the function at these points.

Hint. What do you think has to be true about the derivative at these points?

We want to turn this little bit of thinking and exploring into some useful definitions for us. To craft these definitions, we need to start with thinking about what we care about and why we might care about it.

4.2.1 Critical Points, Local Maximums, and Local Minimums

Let's start by saying what we're really looking for is the highest and lowest points on a function. These points are interesting, have useful applications, and are difficult to find in general without calculus. We hopefully noticed, though, that these points always end up showing up at the same kinds of locations! They're points where the direction of a function changes (or terminates).

We also noticed that there are some common characteristics of those points. They're points where the derivative was either 0 or didn't exist. So we'll start by defining these points, and then we'll define the idea of ”highest” and ”lowest” points. Then we'll put together a result that we hopefully noticed here: that the highest and lowest points show up at these points where the derivative is 0 or doesn't exist.

Definition 4.2.2 Critical Point of a Function. We say that a point $(c, f(c))$ on the graph of $y = f(x)$ is a **critical point** of the function f if $f'(c) = 0$ or $f'(c)$ doesn't exist.

If $(c, f(c))$ is a critical point of f, then we sometimes will call $x = c$ a **critical number** and $y = f(c)$ a **critical value**. $\Diamond$

So these are the points we will look for to find the ”highest” and ”lowest” points.

Now we need to define this idea so that we can build the result that tells us how to find these highest and lowest points.

Definition 4.2.3 Local Maximum/Minimum. A point $(c, f(c))$ is a **local maximum** of $f(x)$ if there is some open interval of real numbers (a, b) around $x = c$ (that is, $a < c < b$) and $f(c) \geq f(x)$ for all x-values in the intersection of (a, b) and the domain of f.

Similarly, a point $(c, f(c))$ is a **local minimum** of $f(x)$ if there is some open interval of real numbers (a, b) around $x = c$ (that is, $a < c < b$) and $f(c) \leq f(x)$ for all x-values in the intersection of (a, b) and the domain of f.

These are really just slightly technical ways of saying that $f(c)$ is either the highest or lowest y-value produced by the function f for the x-values near $x = c$. $\Diamond$

If you look around online, or in other textbooks, you'll find different definitions of these same kinds of pints. Some of those definitions have silly stipulations, like saying that an ending point of a function cannot be a local maximum/local minimum.

This seems to come from some sense that the derivative must be defined on each ”side” of a local max/min. In this book, we'll not worry about this restriction, and instead just look at the highest and lowest points relative to the other points near it.

Now we want to build the connection between these points. In Activity 4.2.1, we pointed out that the highest and lowest points on a function all had the common theme of showing up at places where the derivative was 0 or didn't exist.

Theorem 4.2.4 Every Local Maximum/Minimum Occurs at a Critical Point. *Every local maximum or local minimum of f occurs at a critical point of f.*

Another way of saying this is that if $(c, f(c))$ is a local maximum or local minimum of f, then it must also be a critical point of f.

WAIT! STOP! Before you move on, let's make sure we understand what this theorem is saying. Or maybe what this theorem is *not* saying.

Notice that we are not saying that every critical point is a local maximum or local minimum! This is a classic ”every square is a rectangle but not every rectangle is a square” situation.

Every local maximum/minimum occurs at a critical point, but not every critical point is a local maximum/minimum.

4.2.2 Direction of a Function (and Where it Changes)

Let's build up a way of classifying critical points as local maximums, local minimums, or neither.

Activity 4.2.2 Comparing Critical Points. Let's think about four different functions:

- $f(x) = 4 + 3x - x^2$
- $g(x) = \sqrt[3]{x+1} + 1 + x$
- $h(x) = (x-4)^{2/3}$
- $j(x) = 1 - x^3 - x^5$

Our goal is to find the critical points on the interval $(-\infty, \infty)$ and then to try to figure out if these critical points are local maximums or local minimums or just points that the function increases or decreases through.

(a) To start, we're going to be finding critical points. Without looking at a picture of the graph of the function, find the derivative.

Are there any x-values (in the domain of the function) where the derivative doesn't exist? We are normally looking for things like division by 0 here, but we could be finding more than that. Check out When Does a Derivative Not Exist? to remind yourself if needed.

Are there any x-values (in the domain of the function) where the derivative is 0?

(b) Now that we have the critical points for the function, let's think about where the derivative might be positive and negative. These will correspond to the direction of a function, based on Theorem 4.1.5 Sign of the Derivative and Direction of a Function.

Write out the intervals of x-values around and between your list of critical points. For each interval, what is the sign of the derivative? What do these signs mean about the direction of your function?

(c) Without looking at the graph of your function:

- What changes about how your function increases up to or decreases down to a critical point based on whether the derivative was 0 or the derivative didn't exist?
- Does your function change direction at a critical point? What will that look like, whether it does or does not change direction?

(d) Give a basic sketch of your graph. It might be helpful to find the y-values for any critical points you've got. Then you can sketch your function increasing/decreasing in the intervals between these points.

In your sketch, include enough detail to tell whether the derivative is 0 or doesn't exist at each critical point.

(e) Compare your sketch to the actual graph of the function (you can find all of the graphs in the hint).

Hint.

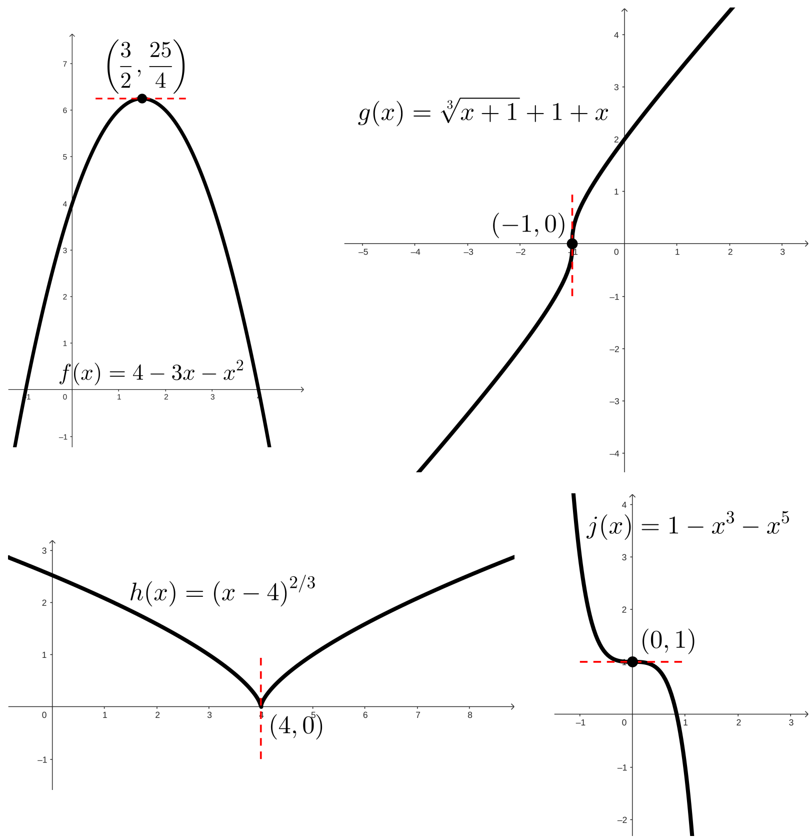


Figure 4.2.5

This is great, we have a nice strategy for thinking about critical points!

Something we can notice: in finding these critical points (as well as thinking about the domain of the function), we found *all* of the locations where the derivative is both not positive and not negative. This is a weird way of saying that all of the intervals in between the critical points we found and any breaks in the domain of the function (like if there were vertical asymptotes or holes or something) are places where the derivative is positive or negative.

Even more exciting: if the derivative function we found is continuous, then the Intermediate Value Theorem says that it will *only* change signs at these critical points (or places like vertical asymptotes or holes). So this means that we can always construct a little chart or something, think about the x-values around and at critical points or other breaks in the domain, and then look at what the derivative does as we move through those intervals and x-values.

This will serve as a nice way of thinking about what's going on with our functions!

Theorem 4.2.6 First Derivative Test. *If $(c, f(c))$ is a critical point of f and we can evaluate the derivative f' on either side of this point, then we can use the signs of the first derivative to classify the critical point:*

- *If the sign of f' changes from positive to negative as x passes through $x = c$, then $(c, f(c))$ is a local maximum.*
- *If the sign of f' changes from negative to positive as x passes through $x = c$, then $(c, f(c))$ is a local minimum.*
- *If the sign of f' does not change as x passes through $x = c$, then the function f increases or decreases (depending on whether $f' > 0$ or $f' < 0$) through $x = c$.*

We will often lay these results out in a chart or table, like the following:

x		c	
f'	$\oplus$		$\ominus$
f	$\nearrow$		$\searrow$
		local max	

x		c	
f'	$\ominus$		$\oplus$
f	$\searrow$		$\nearrow$
		local min	

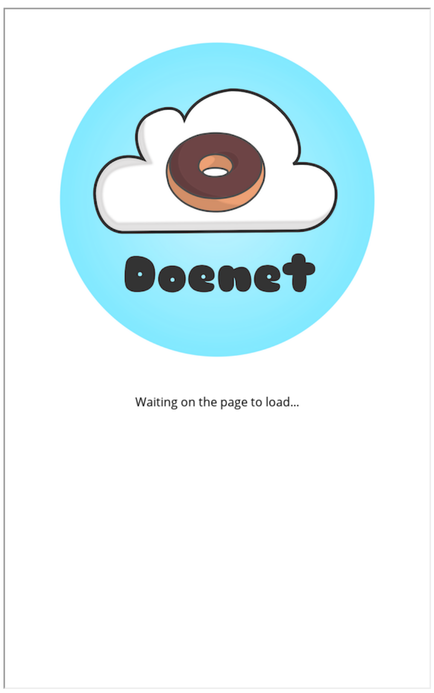
x		c	
f'	$\oplus$		$\oplus$
f	$\nearrow$		$\nearrow$
		increasing through	

x		c	
f'	$\ominus$		$\ominus$
f	$\searrow$		$\searrow$
		decreasing through	

4.2.3 Using the Graph of the First Derivative

Activity 4.2.3 First Derivative Test Graphically. Let's focus on looking at a picture of a derivative, $f'(x)$, and trying to collect information about the function $f(x)$. This is what we've done already, except that we've done it by thinking about the representation of $f'(x)$ as a function rule written out with algebraic symbols. Here we'll focus on connecting all of that to the picture of the graphs.

For all of the following questions, refer to the plot below. You can add information with the hints whenever you need to. Don't reveal the picture of $f(x)$ until you're really ready to check what you know.



(a) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is increasing.

(b) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is decreasing.

(c) Find the x-values of the critical points of $f(x)$. Once you've estimated these, classify them as local maximums, local minimums, or neither. Explain your reasoning.

(d) What do you think the graph of $f(x)$ looks like? Do your best to sketch it or explain it before revealing it!

(e) Why could we estimate the x-values of the critical numbers of $f(x)$, but not find the actual coordinates? How come we can't find the y-value based on looking at the graph of $f'(x)$?

Reading the graphs of functions is, in general, an important skill. But it's an especially important idea to be able to read and understand the graph of a function like a derivative and then interpret what we are seeing into some other context.

So for us to really excel here, we'll want to focus on the fact that a first derivative tells about the *slope* or *direction* of a function. Whatever y-values we find on the graph of a $f'(x)$ needs to be interpreted as a slope or rate of change of $f(x)$. Then we can string these slopes or rates of changes together to try to think about the behavior of $f(x)$ by knowing how the y-values are changing as we move along the curve of $y = f(x)$!

4.2.4 Strange Domains

We'll look at two more examples, both of them using functions whose domain is *not* $(-\infty, \infty)$.

Example 4.2.7

(a) Consider the function

$$f(x) = \frac{x^2}{(x-3)^2}.$$

Find the domain of f, the critical points of f, and then the intervals where f is increasing/decreasing. Then, classify any critical points local maximums/minimums if necessary.

Hint 1. $f'(x) = -\dfrac{6x}{(x-3)^3}$

Hint 2. The function $f(x)$ has one critical point at $(0, 0)$. Why isn't there a critical point at $x = 3$? What is happening there instead?

Hint 3.

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
f'					
f					

Answer. The domain of $f(x)$ is $(-\infty, 3) \cup (3, \infty)$ due to the vertical asymptote at $x = 3$. The only critical point is at $(0, 0)$. The table below shows where f is increasing and decreasing, as well as any local maximums or minimums.

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
f'	$\ominus$	0	$\oplus$		$\ominus$
f	$\searrow$ decreasing	$(0, 0)$ local min	$\nearrow$ increasing	asymptote	$\searrow$ decreasing

(b) Consider the function

$$g(x) = \sqrt{x} - x + 1.$$

Find the domain of g, the critical points of g, and then the intervals where g is increasing/decreasing. Then, classify any critical points local maximums/minimums if necessary.

Hint 1. $g'(x) = \dfrac{1}{2\sqrt{x}} - 1$

Hint 2. Notice that, by our definition of critical points, both $(0, 1)$ and $\left(\frac{1}{4}, \frac{3}{4}\right)$ are critical points.

Hint 3.

x	0	$\left(0, \frac{1}{4}\right)$	$\frac{1}{4}$	$\left(\frac{1}{4}, \infty\right)$
g'				
g				

Answer. The domain of $g(x)$ is $[0, \infty)$. There are two critical points: one at $(0, 1)$ and another at $\left(\frac{1}{4}, \frac{3}{4}\right)$. The table below shows where g is increasing and decreasing, as well as any local maximums or minimums.

x	0	$\left(0, \frac{1}{4}\right)$	$\frac{1}{4}$	$\left(\frac{1}{4}, \infty\right)$
g'	DNE	$\ominus$	0	$\oplus$
g	$(0, 1)$ local max	$\searrow$ decreasing	$\left(\frac{1}{4}, \frac{3}{4}\right)$ local min	$\nearrow$ increasing

□

So we have two things to notice:

1. When we have some gap or missing spot in the domain of a function, that can still divide up the intervals where our function is increasing or decreasing! We should notice, though, that since this isn't actually a *point* on the curve of our function, it won't be a critical point and so we have to interpret it differently: we can't use the First Derivative Test!

2. An ending point of an interval is a location where the derivative cannot exist! We could define a *one-sided derivative* (similar to how we defined one-sided continuity in Definition 1.7.1), but for now we'll just say that the derivative doesn't exist, and call those ending points critical points. That means that depending on the direction that a function goes away (or leading up to) that ending point, we can classify it as a local maximum or minimum.

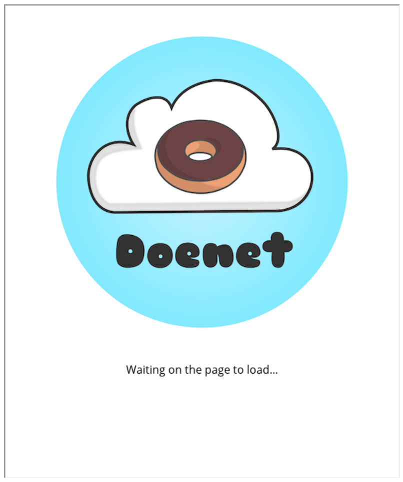
Lastly, just a couple of notes: in these little tables or charts (sometimes called **sign charts**, since they are showing the signs of the derivative), we'll use some shorthand notation. In Example 4.2.7, we used "DNE" to mean that a derivative "does not exist" at a point. Similarly, we used to represent the vertical asymptote at that x-value (so that we didn't accidentally think it was a local maximum or minimum based on the signs of the derivative around it).

Moving forward, we'll use this same kind of analysis to think about how the derivative might be changing on these intervals. This rate of change of the slopes, the **second derivative**, will be a useful tool for gathering more information about how a function might be acting.

4.3 Concavity

Activity 4.3.1 Compare These Curves.

(a) Consider the two curves pictured below. Compare and contrast them. What characteristics of these functions are the same? What characteristics of these functions are different?



(b) Explain the similarities you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(c) Explain the differences you found by only talking about the slopes of each function (the values of $f'(x)$ and $g'(x)$).

(d) Make a conjecture about the *rate of change* of both f' and g'. We'll call these things **second derivative** functions, $f''(x)$ and $g''(x)$.

4.3.1 Defining the Curvature of a Curve

Definition 4.3.1 Concavity and Inflection Points. We say that a function if **concave up** on an interval (a, b) if $f'(x)$ is increasing on the interval. If $f'(x)$ is decreasing on the interval, then we say that $f(x)$ is **concave down**.

We say that a point $(c, f(c))$ is an **inflection point** if it is a point at which f changes concavity (from concave up to concave down or vice versa). ◊

Note that when we think about a function being concave up or down on some interval, we can think about this in different ways. Curvature can sometimes be hard to recognize visually, but one of the things we can see from the visual above is the interaction between the tangent line and the curve:

- If the function is concave up on some interval, then the tangent line sits below the function at every point on that interval. The function curves *upward* away from the tangent line. Sometimes people will say that the curvature is concave ”up, like a cup.”

- If the function is concave down on some interval, then the tangent line sits above the function at every point on that interval. The function curves *downward* away from the tangent line. Sometimes people will say that the curvature is concave ”down, like a frown.”

So we have some visual ways of thinking about these different types of curvature. Annoyingly, though, it is still relatively difficult to pinpoint the exact (or even close) location of an inflection point. We might be able to pretty easily point at the locations of local maximums and local minimums on a graph of a function, but it can be hard to see the exact point at which a graph of a function changes from concave up to down or vice versa. We'll focus on finding them algebraically first, but then we'll think a bit more about the graphs of functions later.

Activity 4.3.2 Finding a Function's Concavity. We're going to consider a pretty complicated function to work with, and employ a strategy similar to what we did with the first derivative:

- Our goal is to find the sign of $f''(x)$ on different intervals and where that sign *changes*.
- To do' this, we'll find the places that $f''(x) = 0$ or where $f''(x)$ doesn't exist. These are the critical points of $f'(x)$.
- From there, we can build a little sign chart, where we split up the x-values based on the domain of f and the critical numbers of f'. Then we can attach each interval of x-values with a sign of the second derivative, f'', on that interval.
- We'll interpret these signs. For any intervals where $f''(x) > 0$, we know that $f'(x)$ must be increasing and so $f(x)$ is concave up. Similarly, for any intervals where $f''(x) < 0$, we know that $f'(x)$ must be decreasing and so $f(x)$ is concave down.

Let's consider the function

$$f(x) = \ln(x^2+1) - \frac{x^2}{2}.$$

(a) Find the first derivative, $f'(x)$, and use some algebra to confirm that it is really:

$$f'(x) = -\frac{x(x+1)(x-1)}{x^2+1}.$$

While we have this first derivative, we *could* find the critical points of $f(x)$ and then classify those critical points using the First Derivative Test.

For our goal of finding the intervals where $f(x)$ is concave up and concave down, and then finding the inflection points of f, let's move on.

(b) Find the second derivative, $f''(x)$, and confirm that this is really:

$$f''(x) = -\frac{x^4+4x^2-1}{(x^2+1)^2}.$$

(c) Our goal is to find the x-values where $f''(x) = 0$ or where $f''(x)$ doesn't exist.

Note that in order to find where $f''(x) = 0$, we are really looking at the x-values that make the numerator 0:

$$x^4 + 4x^2 - 1 = 0.$$

Then, to find where $f''(x)$ doesn't exist, we are finding the x-values that make the denominator 0:

$$(x^2+1)^2 = 0.$$

Solve these equations.

Hint. Here are two strategies for solving $x^4 + 4x^2 - 1 = 0$:

(a) Write some sort of substitution where $u = x^2$

$$(x^2)^2 + 4(x^2) - 1 = 0$$
$$u^2 + 4u + 1 = 0$$

Now solve this for u using the quadratic formula. Note that in the end, the two values you get will be possibilities for $u = x^2$. Make sure you get your answer to be in terms of x!

(b) A similar technique (really the same thing) is to "complete the square" and write your equation this way:

$$(x^2 + 2)^2 - 5 = 0.$$

Now solve for x in a natural way.
Something to note here is that $x^2 + 2 > 0$, so $x^2 + 2 = \sqrt{5}$ (and notably not $-\sqrt{5}$).

Answer. You should get that the critical points of $f'(x)$ are at $x = -\sqrt{\sqrt{5} - 2}$ and $x = \sqrt{\sqrt{5} - 2}$.

(d) You have two critical points of $f'(x)$ (let's just call them x_1 and x_2). These are possible inflection points of $f(x)$, but we need to check to see if the concavity changes at these points.

Fill in the following sign chart and interpret.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''					
f					

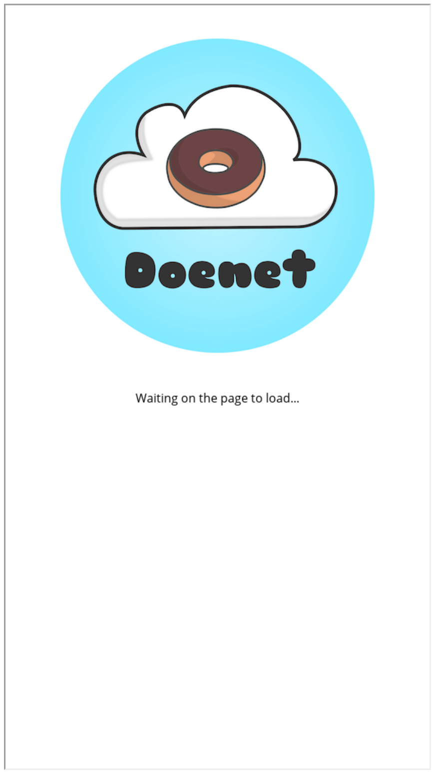
Answer.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''	$\ominus$	0	$\oplus$	0	$\ominus$
f	$\frown$ concave down	inflection point	$\smile$ concave up	inflection point	$\frown$ concave down

Let's confirm what we've just calculated graphically. Below, we have three graphs:

1. $f(x) = \ln(x^2 + 1) - \dfrac{x^2}{2}$
2. $f'(x) = -\dfrac{x(x+1)(x-1)}{x^2 + 1}$
3. $f''(x) = -\dfrac{x^4 + 4x^2 - 1}{(x^2 + 1)^2}$

Move the point on any graph and make sure the statements about signs, directions, and concavity match what you found! You should notice that signs of the first and second derivative change at the local maximums/minimums and the inflection points that we found.



Let's circle back to the definition of Concavity and Inflection Points and think about these from the perspective of $f'(x)$.

We can notice that, by the definition, any inflection point is a point at which $f(x)$ changes concavity, and so is a point where $f'(x)$ changes direction. That means we are looking at local maximums or local minimums of $f'(x)$ (as long as they're not at the end points of some domain)! Similarly, these are points at which $f''(x)$ changes sign, and so we are thinking about the x-intercepts of these second derivatives (or other kinds of locations where the second derivative could change signs).

Example 4.3.2 Let's look at a few more examples, but this time they can be ones with derivatives that are a bit easier to work with.

(a) Consider the function $f(x) = \ln(x^2 + 1)$. Find the intervals where f is concave up, the intervals where it is concave down, and then find the locations of any inflection points.

(b) Consider the function $g(x) = e^{-x^2}$. Find the intervals where g is concave up, the intervals where it is concave down, and then find the locations of any inflection points.

□

At this point, we have three different functions that we are juggling: a function $f(x)$ (or whatever name we give it), the first derivative $f'(x)$, and the second derivative $f''(x)$. We'll want to keep in mind the role of each of these.

- $f(x)$ tells us the height, the y-value, of the function at some point.