Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

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Chapter 1

Limits

1.1 Introduction to Limits

Almost 2,500 years ago, the Greek philosopher Zeno of Elea gifted the world with a set of philosophical paradoxes that provide the foundation for how we will begin our study of calculus. Perhaps the most famous of Zeno's paradoxes is the paradox of Achilles and the Tortoise.

1.1.1 Achilles and the Tortoise

In the paradox of Achilles and the Tortoise, the Greek hero Achilles is in a race with a tortoise. Obviously the tortoise is much slower than Achilles, and so the tortoise gets a 100m head start. The race begins, and while the tortoise moves as quickly as it can, Achilles has the obvious advantage. They both are running at a constant speed, with Achilles running faster. Achilles runs 100m, catching up to the tortoise's starting point.

In the meantime, the tortoise has moved 2 meters. Achilles has almost caught up and passed the tortoise at this point! In a *very* short time, Achilles is able to run the 2 meters to catch up to where the tortoise was. Unfortunately, in that short amount of time, the tortoise has kept on moving, and is farther along by now.

Every time Achilles catches up to where the tortoise was, the tortoise has moved farther along, and Achilles has to keep catching up.

Can Achilles, the paragon of athleticism, ever catch the tortoise?

1.1.2 A Modern Retelling

A college student is excited about having enrolled in their first calculus class. On the first day of class, they head to class. When they enter the hallway with their classroom at the end, they take a breath and excitedly head to class.

In order to get to class, though, they have to travel halfway down the hallway. Almost there.

Now, to go the rest of the way, the student will half to get to the point that is halfway between them and the door. Getting closer.

They're getting excited. Finally, their first calculus class! But to get to the class, they have to reach the point halfway between them and the door.

Their smile starts fading. They repeat the process, and go halfway from their position to the door. They're closer, but not there yet.

If they keep having to reach the new halfway point, and that halfway point is never actually *at* the door, then will they ever get there?

Halfway to the door, then halfway again, closer and closer without ever getting there.

Will the student ever get there, or are they doomed to keep getting closer and closer without ever reaching the door?

1.2 The Definition of the Limit

1.2.1 Defining a Limit

Activity 1.2.1 Close or Not? We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Definition 1.2.1 Limit of a Function. For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the **limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x \to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$. ◊

When we say "around $x = a$", we really just mean on either side of it. We can clarify if we want.

Definition 1.2.2 Left-Sided Limit. For the function $f(x)$ defined at all x-xalues around and less than a, we say that the **left-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but less than, a. We write this as:

$$\lim_{x \to a^-} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^-$. ◊

Definition 1.2.3 Right-Sided Limit. For the function $f(x)$ defined at all x-xalues around and greater than a, we say that the **right-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but greater than, a. We write this as:

$$\lim_{x \to a^+} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^+$. ◊

1.2.2 Approximating Limits Using Our New Definition

Activity 1.2.2 Approximating Limits. For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the graph of the function $f(x)$ below.

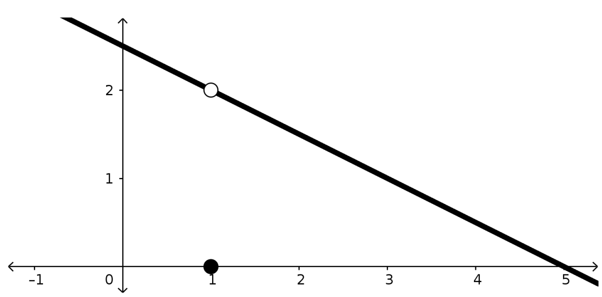


Figure 1.2.4

(b) Approximate $\lim_{x\to 2} g(x)$ using the graph of the function $g(x)$ below.

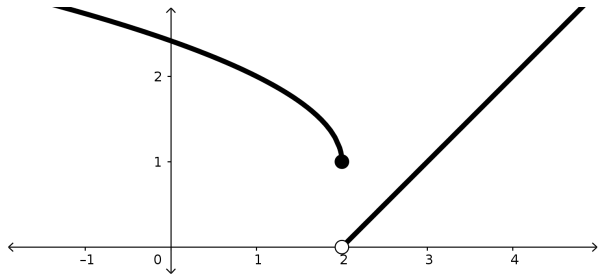


Figure 1.2.5

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x\to -1} h(x)$
- $\lim_{x\to 0} h(x)$
- $\lim_{x\to 2} h(x)$

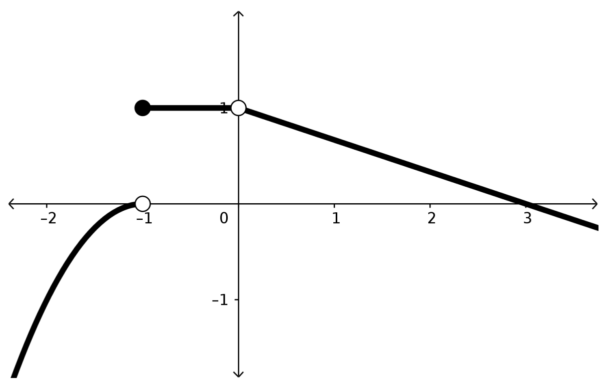


Figure 1.2.6

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.2.3 Approximating Limits Numerically. For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x\to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.2.7

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.2.8

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.2.9

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.3 Evaluating Limits

1.3.1 Adding Precision to Our Estimations

Activity 1.3.1 From Estimating to Evaluating Limits (Part 1). Let's consider the following graphs of functions $f(x)$ and $g(x)$.

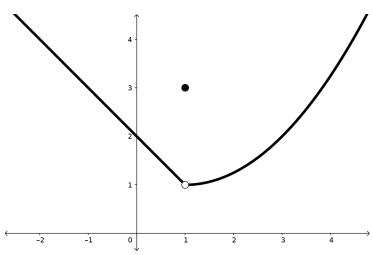


Figure 1.3.1 Graph of the function $f(x)$.

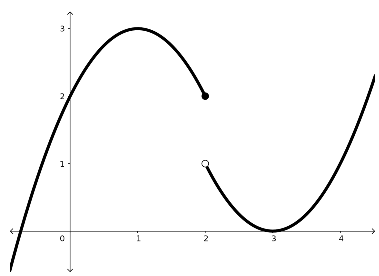


Figure 1.3.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$
- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 2^-} g(x)$

- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.3.2 From Estimating to Evaluating Limits (Part 2). Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

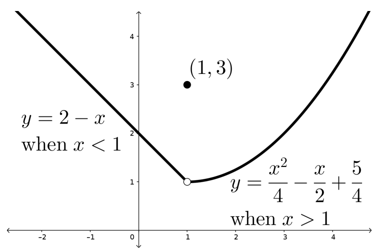


Figure 1.3.3 Graph of the function $f(x)$.

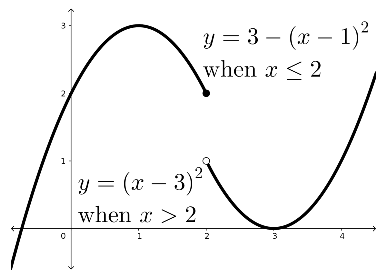


Figure 1.3.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

1.3.2 Limit Properties

Theorem 1.3.5 Combinations of Limits. *If $f(x)$ and $g(x)$ are two functions defined at x-values around, but maybe not at, $x = a$ and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then we can evaluate limits of combinations of these functions.*

1. Sums: *The limit of the sum of $f(x)$ and $g(x)$ is the sum of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

2. Differences: *The limit of a difference of $f(x)$ and $g(x)$ is the difference of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$$

4. Products: *The limit of a product of $f(x)$ and $g(x)$ is the product of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

5. Quotients: *The limit of a quotient of $f(x)$ and $g(x)$ is the quotient of the limits of $f(x)$ and $g(x)$ (provided that you do not divide by 0):*

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)} \qquad (\textit{if } \lim_{x\to a} g(x) \neq 0)$$

Theorem 1.3.6 Limits of Two Basic Functions. *Let a be some real number.*

1. Limit of a Constant Function: *If k is some real number constant, then:*

$$\lim_{x\to a} k = k$$

2. Limit of the Identity Function:

$$\lim_{x\to a} x = a$$

Activity 1.3.3 Limits of Polynomial Functions. We're going to use a combination of properties from Theorem 1.3.5 and Theorem 1.3.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x\to 1} f(x)$. First, use the properties from Theorem 1.3.5 to re-write this limit as 4 different limits added or subtracted together.

Answer.

$$\lim_{x\to 1}(2x^4) - \lim_{x\to 1}(4x^3) + \lim_{x\to 1}\left(\frac{x}{2}\right) - \lim_{x\to 1} 5$$

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.3.6. Evaluate your limits.

Hint.

$$2\left(\lim_{x\to 1} x\right)^4 - 4\left(\lim_{x\to 1} x\right)^3 + \frac{1}{2}\left(\lim_{x\to 1} x\right) - \lim_{x\to 1} 5$$

(c) Based on the definition of a limit (Definition 1.2.1), we normally say that $\lim_{x\to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x\to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x\to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x\to a} p(x)$ is the same value as $p(a)$.

Theorem 1.3.7 Limits of Polynomials. *If $p(x)$ is a polynomial function and a is some real number, then:*

$$\lim_{x\to a} p(x) = p(a)$$

1.4 First Indeterminate Forms

Activity 1.4.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x\to 1} f(x)$.

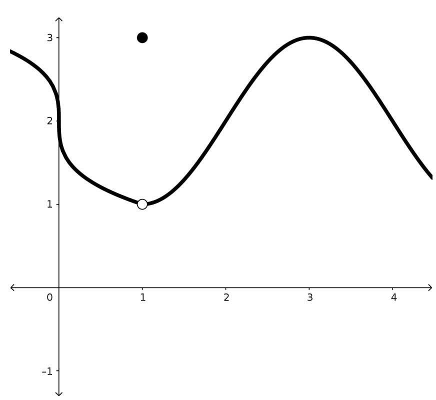


Figure 1.4.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate

$\lim_{x\to 1} g(x)$.

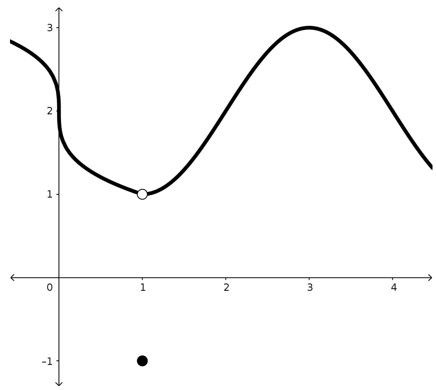


Figure 1.4.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{x\to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{x\to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Theorem 1.4.3 Limits of (Slightly) Different Functions. *If $f(x)$ and $g(x)$ are two functions defined at x-values around a (but maybe not at $x = a$ itself) with $f(x) = g(x)$ for the x-values around a but with $f(a) \neq g(a)$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, if the limits exist.*

1.4.1 A First Introduction to Indeterminate Forms

Definition 1.4.4 Indeterminate Form. We say that a limit has an **indeterminate form** if the general structure of the limit could take on any different value, or not exist, depending on the specific circumstances.

For instance, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that the limit $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right)$ has an indeterminate form. We typically denote this using the informal symbol $\frac{0}{0}$, as in:

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) \overset{?}{\to} \frac{0}{0}.$$

◊

Activity 1.4.2

(a) Were going to evaluate $\lim_{x \to 3} \left(\frac{x^2 - 7x + 12}{x - 3} \right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \frac{x^2 - 7x + 12}{x - 3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \frac{x^2 - 7x + 12}{x - 3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x \to 1} \left(\frac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4} \right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \frac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2 + 3} + 2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2 + 3} - 2)(\sqrt{x^2 + 3} + 2) = (x^2 + 3) - 4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \frac{\sqrt{x^2 + 3} - 2}{x^2 - 5x + 4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x \to -2} \left(\frac{3 - \frac{3}{x+3}}{x^2 + 2x} \right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \frac{3 - \frac{3}{x+3}}{x^2 + 2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \frac{3 - \frac{3}{x+3}}{x^2 + 2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim_{x \to 5} f(x) \overset{?}{\to} \frac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

1.4.2 What if There Is No Algebra Trick?

We've seen some nice examples above where we were able to use some algebra to manipulate functions in such was as to force some shared factor in the numerator and denominator into revealing itself. From there, we were able to apply Theorem 1.4.3 and swap out our problematic function with a new one, knowing that the limit would be the same.

But what if we can't do that? What if the specific structure of the function seems *resistent* somehow to our attempts at wielding algebra?

This happens a lot, and we'll investigate some more of those types of limits in Section **??**. For now, though, let's look at a very famous limit and reason our way through the indeterminate form.

Activity 1.4.3 Let's consider a new limit:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \frac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

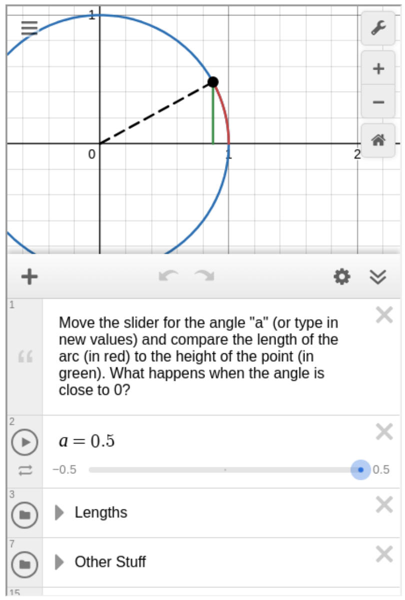
(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

Hint 1. On the unit circle, if we plot some point at an angle of θ, then the coordinates of that point can be represented with trig functions! Which ones?

Hint 2. The length of the curve defining a unit circle is 2π. This also corresponds to the angle we would use to represent moving all the way around the circle. What must the length of the portion of the circle be up to some point at an angle θ?

(d) In the following graph, use the slider for a to change the angle of the point and compare the arc length and the height. What happens when the angle is close to 0? You will probably have to zoom in!



(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{0}$?

1.5 Limits Involving Infinity

Two types of limits involving infinity. In both cases, we'll mostly just consider what happens when we divide by small things and what happens when we divide by big things. We can summarize this here, though:

Fractions with small denominators are big, and fractions with big denominators are small.

1.5.1 Infinite Limits

Activity 1.5.1 What Happens When We Divide by 0? First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.4), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?

- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Definition 1.5.1 Infinite Limit. We say that a function $f(x)$ has an **infinite limit** at a if $f(x)$ is arbitrarily large (positive or negative) when x is sufficiently close to, but not equal to, $x = a$.

We would then say, depending on the sign of the values of $f(x)$, that:

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad\qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

If the sign of both one-sided limits are the same, we can say that $\lim_{x \to a} f(x) = \pm\infty$ (depending on the sign), but it is helpful to note that, by the definition of the Limit of a Function, this limit does not exist, since $f(x)$ is not arbitrarily close to a single real number. ◊

Theorem 1.5.2 Dividing by 0 in a Limit. *If* $f(x) = \dfrac{g(x)}{h(x)}$ *with* $\lim_{x \to a} g(x) \neq 0$ *and* $\lim_{x \to a} h(x) = 0$*, then* $f(x)$ *has an Infinite Limit at* a*. We will often denote this behavior as:*

$$\lim_{x \to a} f(x) \xrightarrow{?} \frac{\#}{0}$$

where # is meant to be some shorthand representation of a non-zero limit in the numerator (often, but not necessarily, some real number).

Evaluating Infinite Limits.

Once we know that $\lim_{x \to a} f(x) \xrightarrow{?} \dfrac{\#}{0}$, we know a bunch of information right away!

- This limit doesn't exist.
- The function $f(x)$ has a vertical asymptote at $x = a$, causing these unbounded y-values near $x = a$.
- The one sided limits *must* be either ∞ or $-\infty$.

- We only need to focus on the sign of the one sided limits! And signs of products and quotients are easy to follow.

So a pretty typical process is to factor as much as we can, and check the sign of each factor (in a numerator or denominator) as $x \to a^-$ and $x \to a^+$. From there, we can find the sign of $f(x)$ in both of those cases, which will tell us the one-sided limit.

Example 1.5.3 For each function, find the relevant one-sided limits at the input-value mentioned. If you can use a two-sided limit statement to discuss the behavior of the function around this input-value, then do so.

(a) $\left(\dfrac{2x^2-5x+1}{x^2+8x+16}\right)$ and $x=-4$

(b) $\left(\dfrac{4x^2-x^5}{x^2-4x+3}\right)$ and $x=1$

(c) $\sec(\theta)$ and $\theta = \dfrac{\pi}{2}$

□

1.5.2 End Behavior Limits

Activity 1.5.2 What Happens When We Divide by Infinity? Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

Definition 1.5.4 Limit at Infinity. If $f(x)$ is defined for all large and positive x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently large, then we say:

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ is defined for all large and negative x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently negative, then we say:

$$\lim_{x\to-\infty} f(x) = L.$$

In the case that $f(x)$ has a **limit at infinity** that exists, then we say $f(x)$ has a horizontal asymptote at $y = L$.

Lastly, if $f(x)$ is defined for all large and positive (or negative) x-values and $f(x)$ gets arbitrarily large and positive (or negative) when x gets sufficiently large (or negative), then we could say:

$$\lim_{x\to-\infty} f(x) = \pm\infty \text{ or } \lim_{x\to\infty} f(x) = \pm\infty.$$

$\Diamond$

Because the primary focus for limits at infinity is the end behavior of a function, we will often refer to these limits as **end behavior limits**.

Theorem 1.5.5 End Behavior of Reciprocal Power Functions. *If p is a positive real number, then:*

$$\lim_{x\to\infty} \left(\frac{1}{x^p}\right) = 0 \text{ and } \lim_{x\to-\infty} \left(\frac{1}{x^p}\right) = 0.$$

Theorem 1.5.6 Polynomial End Behavior Limits. *For some polynomial function:*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

with n a positive integer (the degree) and all of the coefficients $a_0, a_1, ..., a_n$ real numbers (with $a_n \neq 0$), then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

That is, the leading term (the term with the highest exponent) defines the end behavior for the whole polynomial function.

Proof. Consider the polynomial function:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

where n is some integer and a_k is a real number for $k = 0, 1, 2, ..., n$. For simplicity, we will consider only the limit as $x \to \infty$, but we could easily repeat

this exact proof for the case where $x \to -\infty$.
Before we consider this limit, we can factor out x^n, the variable with the highest exponent:

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 \\ &= x^n\left(\frac{a^n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + ... + \frac{a_2x^2}{x^n} + \frac{a_1x}{x^n} + \frac{a_0}{x^n}\right) \\ &= x^n\left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

Now consider the limit of this product:

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \lim_{x\to\infty} x^n\left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

We can see that in the second limit, we have a single constant term, a_n, followed by reciprocal power functions. Then, due to Theorem 1.5.5, we know that the second limit will by a_n, since the reciprocal power functions will all approach 0.

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n + 0 + ... + 0 + 0 + 0) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n) \\ &= \lim_{x\to\infty} a_n x^n \end{aligned}$$

And so $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ as we claimed. ■

Example 1.5.7 For each function, find the limits as $x \to \infty$ and $x \to -\infty$.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$

(c) $\dfrac{|x|}{3x}$

□

Activity 1.5.3 Matching the Limits.

(a) We're going to look at four graphs of functions, as well as a list of limit statements. Match the limit statements with the graphs that match that behavior. Note that is is possible for a limit to be relevant on more than one graph.

(b) Now consider these four function definitions. Using your knowledge of limits, as well as the matching you've already done, match the definitions of these four functions with the graphs that go with them, and then also the limits that are relevant. (These limits will already be matched with the graphs, so you don't need to do further work here).

1.6 The Squeeze Theorem

Activity 1.6.1 A Weird End Behavior Limit. In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

Hint 1. Start with Theorem 1.3.5 to think about the numerator and denominator separately.

Hint 2. Can you use Theorem 1.5.6 in the denominator?

Hint 3. Is Theorem 1.5.5 relevant?

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\boxed{} \le \sin(x) \le \boxed{}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\boxed{} \le \sin^2(x) \le \boxed{}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim_{x\to\infty}\left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\boxed{}}{x^2+1}}_{\text{call this } f(x)} \le \frac{\sin^2(x)}{x^2+1} \le \underbrace{\frac{\boxed{}}{x^2+1}}_{\text{call this } h(x)}$$