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Single-Variable Calculus Topics with Motivating ActivitiesSingle-Variable Differential Calculus Topics with Motivating ActivitiesSingle-Variable Integral Calculus Topics with Motivating ActivitiesAll of the Activities from Each ChapterActivities for Differential Calculus TopicsActivities for Integral Calculus Topics

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Acknowledgements

Disclosure about the Use of AI

This book has been lovingly written by a human.

Me.

Peter Keep.

I have used a lot of different tools, both for inspiration and for actually creating resources for this book. *None* of those tools has involved any form of generative AI.

I could list all of the ways that I think using generative AI in education is, at minimum, problematic. More pointedly, I believe that it is unethical. More broadly, I believe that the use of generative AI for any use-case that I have encountered to be unethical.

In my classes, I try to help students realize the joy and value of working at something and creating something and struggling with something and knowing something. Giving worth to something, even an imperfect thing. Celebrating our accomplishments, even when (especially when?) there is room to grow in those accomplishments. And so I have taken that advice in the creation of this book. I have created a book that is definitely not perfect. I have struggled to write it. There are parts of it that could be (need to be) improved.

But I was the one that created it. I struggled with it. I know it.

I hope that this book can also be a useful tool for others to use, and I have left the copyright to be about as open as possible. Others can take this, use it, can change it, add to it, subtract from it, etc.

In leaving this copyright open for others to change this book, I cannot guarantee that every version of this book is free from the mindless and joyless output from some Large Language Model. But I want to leave this note up in hopes that anyone who *does* inject some output from some generative AI product into this book will take it down. If this note, or some statement similar to it, is not present in the version of the book you are accessing, please be cautious. Find a different calculus textbook to read!

Find something written by a human. Find the words of some other mathematician who tries, maybe imperfectly, to share the ideas of calculus.

Teaching and learning is about humans communicating with each other, and only humans can do that.

Notes for Instructors

Notes for Students

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Chapter 1

Limits

1.0.1 Practice Problems

1. Explain in your own words the meaning of:

$$\lim_{x \to a^-} f(x) = L.$$

2. Explain in your own words the meaning of:

$$\lim_{x \to a^+} f(x) = L.$$

3. Explain in your own words the meaning of:

$$\lim_{x \to a} f(x) = L.$$

4. Say we know that $\lim_{x \to 3^-} f(x) = 2$ and $\lim_{x \to 3^+} f(x) = 2$. What do we know (specifically or in general, if anything) about each of the following?

(a) $f(3)$

Solution. We do not know anything about the value of $f(3)$, including whether or not there is such a vale.

(b) $f(2.999)$

Solution. The value of $f(2.999)$ should be close to the number 2. We do not know exactly what the value is, though.

(c) $f(3.001)$

Solution. The value of $f(3.001)$ should be close to the number 2. We do not know exactly what the value is, though.

(d) $\lim_{x \to 3} f(x)$

Solution. Since both the left and the right sided limits are the same single, real number (2), so is the limit: $\lim_{x \to 3} f(x) = 2$.

5. Which of the following is possible? Explain why or why not, and any other conclusions that we can draw.

(a) For some function $f(x)$, $\lim_{x \to 5} f(x) = 6$ and $f(5) = -3$

Solution. This is possible.

(b) For some function $g(x)$, $\lim_{x\to 4^-} f(x) = -\frac{3}{2}$ and $\lim_{x\to 4^+} g(X) = \frac{4}{7}$.

Solution. This is possible, although it means that $\lim_{x\to 4} g(x)$ does not exist.

(c) For some function $\ell(t)$, $\lim_{t\to\alpha} \ell(t) = 2$ and $\lim_{t\to\alpha^+} \ell(t) = 1$.

Solution. This is not possible, since for $\lim_{t\to\alpha} \ell(t) = 2$ we would need $\lim_{t\to\alpha^+} \ell(t) = 2$ and $\lim_{t\to\alpha^-} \ell(t) = 2$.

(d) For some function $r(\theta)$, $\lim_{\theta\to 0} r(\theta)$ does not exist, $\lim_{\theta\to 0^-} r(\theta) = \pi$, and $\lim_{\theta\to 0^+} r(\theta) = -\frac{\pi}{2}$.

Solution. This is possible.

(e) For some function $j(w)$, $j(4) = \pi$ while $\lim_{w\to 4} j(w)$ does not exist.

Solution. This is possible.

6. Fill in the following tables in order to satisfy the requirements listed. Afterwards, include a sentence or two justifying your choices.

(a) *Requirements:* $\lim_{x\to 1} f(x) = 3$

x	____	0.93	____	1	____	____	1.04
$f(x)$	____	____	____	____	____	____	____

(b) *Requirements:* $\lim_{x\to -5^-} f(x) = 2$, $f(-5) = 6$, and $\lim_{x\to -5} f(x)$ doesn't exist.

x	-5.2	____	____	-5	____	4.98	____
$f(x)$	____	____	____	____	____	____	____

(c) *Requirements:* $f(7)$ does not exist and $\lim_{x\to 7} f(x) = 3$

x	____	____	6.985	7	____	____	7.002
$f(x)$	____	____	____	____	____	____	____

(d) *Requirements:* $\lim_{x\to 0^-} f(x) = \pi$ and $\lim_{x\to 0^+} f(x) = e$.

x	____	-0.14	____	0	____	____	0.5
$f(x)$	____	____	____	____	____	____	____

7. From the following tables, estimate/report each of the requested values. Explain your choices.

(a) *Requested:* $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$, and $f(1)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.4	2.48	2.4998	9	2.5004	2.52	2.8

Solution. *Estimated:*

- $\lim_{x \to 1^-} f(x) = 2.5$
- $\lim_{x \to 1^+} f(x) = 2.5$
- $\lim_{x \to 1} f(x) = 2.5$

Reported: $f(1) = 9$

(b) *Requested:* $\lim_{x \to 8^-} f(x)$, $\lim_{x \to 8^+} f(x)$, $\lim_{x \to 8} f(x)$, and $f(8)$

x	7.9	7.99	7.999	8	7.001	7.01	7.1
$f(x)$	-1.5	-1.9	-1.999	-2	7.0001	7.2	7.5

Solution. *Estimated:*

- $\lim_{x \to 8^-} f(x) = -2$
- $\lim_{x \to 8^+} f(x) = 7$
- $\lim_{x \to 8} f(x)$ doesn't exist

Reported: $f(8) = -2$

(c) *Requested:* $\lim_{x \to \pi^-} f(x)$, $\lim_{x \to \pi^+} f(x)$, $\lim_{x \to \pi} f(x)$, and $f(\pi)$

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$f(x)$	-3	-3	-3	does not exist	-3	-3	-3

Solution. *Estimated:*

- $\lim_{x \to \pi^-} f(x) = -3$
- $\lim_{x \to \pi^+} f(x) = -3$
- $\lim_{x \to \pi} f(x) = -3$

Reported: $f(\pi)$ does not exist

8. For each of the listed requirements, sketch a graph of a function that satisfies each. Afterwards, include a sentence or two justifying you sketch.

(a) *Requirements:* $f(6) = 0$, $\lim_{x \to 6} f(x) = -2$, $\lim_{x \to -2^-} f(x) = 1$, and $\lim_{x \to -2} f(x)$ does not exist.

(b) *Requirements:* $\lim_{\omega \to 0} \rho(\omega) = 8$, $\lim_{\omega \to 2} \rho(\omega) = -2$, and $\rho(2)$ does not exist.

(c) *Requirements:* $\lim_{t \to -3^-} q(t) = 0$, $\lim_{t \to -3^+} q(t) = 4$, $\lim_{t \to -1} q(t) = 9$, and $q(-1) = 9$.

9. From the graph of $f(x)$ below, estimate of each of the requested values. Explain each of your choices.

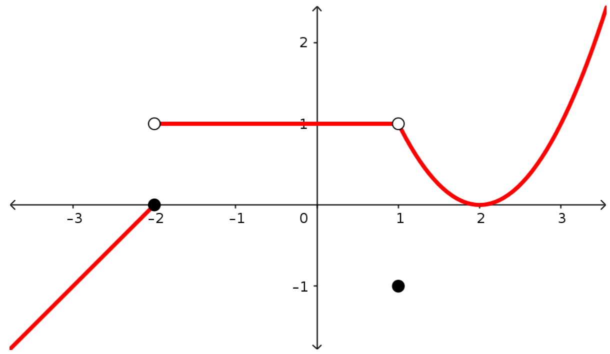


Figure 1.1.18 The function $f(x)$.

(a) $\lim_{x \to -2^-} f(x)$

Solution. $\lim_{x \to -2^-} f(x) = 0$

(b) $\lim_{x \to -2^+} f(x)$

Solution. $\lim_{x \to -2^+} f(x) = 1$

(c) $\lim_{x \to -2} f(x)$

Solution. $\lim_{x \to -2} f(x)$ doesn't exist, since the left and right sided limits don't match.

(d) $\lim_{x \to 0^-} f(x)$

Solution. $\lim_{x \to 0^-} f(x) = 1$

(e) $\lim_{x \to 0^+} f(x)$

Solution. $\lim_{x \to 0^+} f(x) = 1$

(f) $\lim_{x \to 0} f(x)$

Solution. $\lim_{x \to 0} f(x) = 1$

(g) $\lim_{x \to 1^-} f(x)$

Solution. $\lim_{x \to 1^-} f(x) = 1$

(h) $\lim_{x \to 1^+} f(x)$

Solution. $\lim_{x \to 1^+} f(x) = 1$

(i) $\lim_{x \to 1} f(x)$

Solution. $\lim_{x \to 1} f(x) = 1$

(j) $\lim_{x \to 2^-} f(x)$

Solution. $\lim_{x \to 2^-} f(x) = 0$

(k) $\lim_{x\to 2^+} f(x)$

Solution. $\lim_{x\to 2^+} f(x) = 0$

(l) $\lim_{x\to 2} f(x)$

Solution. $\lim_{x\to 2} f(x) = 0$

1.0.2 Practice Problems

1. Given $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = -2$, evaluate the following limits. If the limit doesn't exist, explain why. Write out a few steps and explanations to justify your work.

(a) $\lim_{x\to 3}\left(6f(x) - \dfrac{g(x)}{3}\right)$

(b) $\lim_{x\to 3}(f(x))^2 g(x)$

(c) $\lim_{x\to 3^-}\left(\dfrac{4g(x)}{f(x)} + 3f(x)\right)$

(d) $\lim_{x\to 3^+}\left(\sqrt{f(x)} + \sqrt[3]{g(x)}\right)$

2. Evaluate each limit. Justify your answers.

(a) $\lim_{x\to 0}(4x^3 - 6x^2 + 7x - 10)$

(b) $\lim_{x\to -2}(9 - 3x + x^2 + 3x^3 + x^4)$

(c) $\lim_{t\to a}\left(9t^2 + 3at - 1\right)$ where a is some real number

(d) $\lim_{s\to 1}\left(\dfrac{5s^2 - 6s + 1}{s^2 - 4}\right)$

(e) $\lim_{t\to 2}\left(\dfrac{4t - 5}{6 + t^2}\right)$

(f) $\lim_{z\to 2}\sqrt{z^2 + 4z - 8}$

3. Evaluate each limit. If the limit does not exist, explain why not.

(a) Let $f(x) = \begin{cases} 3x - 2 & \text{if } x < -1 \\ x^2 + x - 4 & \text{if } x \geq -1 \end{cases}$.

Evaluate $\lim_{x\to -1^-} f(x)$, $\lim_{x\to -1^+} f(x)$, and $\lim_{x\to -1} f(x)$.

(b) Let $g(x) = \begin{cases} \sqrt{6 - x} & \text{if } x < -3 \\ 6 & \text{if } x = -3 \\ \frac{2x+15}{3} & \text{if } x > -3 \end{cases}$.

Evaluate $\lim_{x\to -3^-} g(x)$, $\lim_{x\to -3^+} g(x)$, and $\lim_{x\to -3} g(x)$.

(c) Let $s(t) = \begin{cases} t^2 + 1 & \text{if } t < 1 \\ 1 - t^2 & \text{if } t \geq 1 \end{cases}$.

Evaluate $\lim_{t\to 1^-} s(t)$, $\lim_{t\to 1^+} s(t)$, and $\lim_{t\to 1} s(t)$

(d) Let $r(\theta) = \begin{cases} \sqrt{\theta+2}+\theta^3 & \text{if } \theta < 2 \\ 3\theta+2 & \text{if } \theta \geq 2 \end{cases}$.

Evaluate $\lim_{\theta\to 2^-} r(\theta)$, $\lim_{\theta\to 2^+} r(\theta)$, and $\lim_{\theta\to 2} r(\theta)$.

1.0.3 Practice Problems

1. Explain, in your own words, why Theorem 1.3.3, p. **??** is true.

2. Consider the following limit:

$$\lim_{x\to 3}\left(\frac{2x^2-5x-3}{x^2+5x-24}\right).$$

(a) Confirm that this limit has an indeterminate form.

(b) Evaluate the limit.

(c) When you were evaluating the limit, you likely "cancelled" a factor of $(x-3)$ from the numerator and denominator. Why might you have known, before factoring anything, that $(x-3)$ would be a factor shared in the numerator and denominator?

For instance, how did you know it wasn't going to be $(x-4)$ or $(x+1)$ or something else?

3. Consider the following limit:

$$\lim_{x\to}$$

4. Use the algebra tricks from Assemblage , p. **??** to evaluate each limit.

(a) $\lim_{x\to 2}\left(\frac{x^2-4}{x^2+3x-10}\right)$

(b) $\lim_{x\to -1}\left(\frac{2x^2+x-1}{x+1}\right)$

(c) $\lim_{x\to 4}\left(\frac{4-x}{x-4}\right)$

(d) $\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right)$

(e) $\lim_{x\to 2}\left(\frac{\sqrt{x^2-1}-\sqrt{x+1}}{x-2}\right)$

(f) $\lim_{h\to 0}\left(\frac{\sqrt{a+h}-\sqrt{a}}{h}\right)$ where a is some non-negative real number

(g) $\lim_{t\to 3}\left(\frac{\frac{1}{t}-\frac{1}{3}}{t-3}\right)$

(h) $\lim_{t\to 6}\left(\frac{\frac{t+1}{t-1}-\frac{7}{11-t}}{t-6}\right)$

(i) $\lim_{h\to 0}\left(\frac{\frac{1}{a+h}-\frac{1}{a}}{h}\right)$ where a is some real number

Chapter 2

Derivatives

Chapter 3

Implicit Differentiation

Chapter 4

Applications of Derivatives

Chapter 5

Antiderivatives and Integrals

Chapter 6

Applications of Integrals

6.0.1 Practice Problems

1. Explain the following terms in reference to an object moving along a straight path from time $t = a$ to time $t = b$.

- **(a)** **Position** of the object at time t.
- **(b)** **Displacement** of the object.
- **(c)** **Distance** traveled by the object.
- **(d)** **Velocity** of the object at time t.
- **(e)** **Speed** of the object at time t.

2. Consider the graph of a velocity function, $v(t)$, of some object moving along a line on the time interval $[0, 7]$.

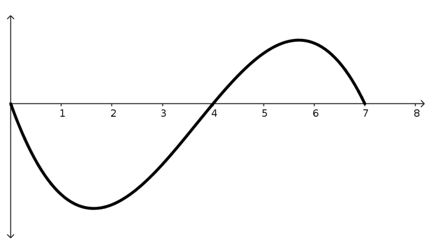


Figure 6.1.7

- **(a)** Do you expect the displacement of the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?
- **(b)** Write two different expressions that represent the total displacement of the object from $t = 0$ to $t = 7$.
- **(c)** Do you expect the distance traveled by the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?
- **(d)** Write two different expressions that represent the total distance traveled by the object from $t = 0$ to $t = 7$.

3. Let's consider an animal running along a straight path with the velocity function:

$$v(t) = \frac{t^4}{10} - t^3 + \frac{27t^2}{10} - \frac{9t}{5}$$

$$= \frac{t}{10}(t-1)(t-3)(t-6)$$

on the time interval $[0, 6]$.

(a) What is the total displacement of the animal on the time interval $[0, 1]$?

(b) What is the total displacement of the animal on the time interval $[1, 3]$?

(c) What is the total displacement of the animal on the time interval $[3, 6]$?

(d) What is the total displacement of the animal on the time interval $[0, 6]$?

(e) What is the total distance traveled by the animal on the time interval $[0, 6]$?

(f) Write a short summary of the animal's movement, including notes about direction, speed, and where the animal travels.

4. Consider an object with velocity function $v(t) = t^2 - 4t + 2$ on the interval $[0, 100]$ with the initial position $s(0) = 3$.

(a) Determine the position function, $s(t)$, for $0 \leq t \leq 100$ using the Future Position of an Object, p. **??**.

(b) Determine the position function, $s(t)$, for $0 \leq t \leq 100$ using the Solving Initial Value Problems, p. **??** strategy.

(c) Compare the results from both methods. Explain why these are equivalent.

5. Consider an object with an acceleration function $a(t) = t + \sin(2\pi t)$ for $t \geq 0$ with $v(0) = 5$.

(a) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Future Position of an Object, p. **??**.

(b) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Solving Initial Value Problems, p. **??** strategy.

(c) Can you obtain the position function, $s(t)$? Explain why or why not, based on the information given.

6. During a brake test for a heavy truck, the truck decelerates from an initial velocity of 88 ft/s with the acceleration function $a(t) = -17$ ft/s². Assume that the initial position of the truck is $s(0) = 0$.

(a) Find the velocity function for the truck.

(b) When does the truck stop? In this situation, the truck won't have a negative velocity (since it's just braking and not eventually going in reverse). What time interval is the velocity function relevant on?

(c) What is the total displacement of the truck on this time interval?

(d) Safety standards say that for a truck like this, it needs to be able to stop (from a speed of 88ft/s) in, at most, 200 feet.

Do we need to make changes to the braking mechanism, in order to have the acceleration function change? If so, what does the acceleration need to be (assuming it is constant and we are just replacing it with a new negative number)?

6.0.2 Practice Problems

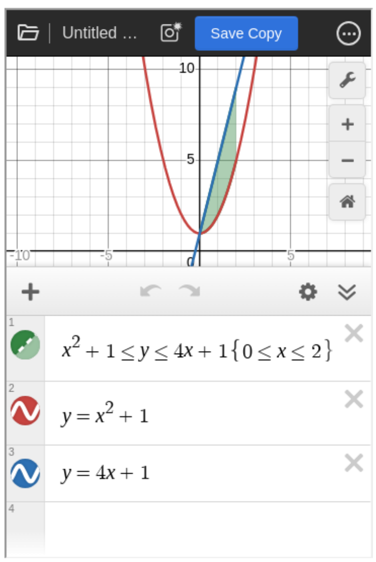
1. Explain how we use the "slice and sum" method to build an integral formula for the area bounded between curves. Give some details, enough to make sure you understand how the Riemann sums are constructed and how they turn into our integral formula.

2. What are some changes/considerations that we need to make when we decide to set up our integral in terms of y instead of x?

3. Set up (and practice evaluating) an integral expression representing the area of each of the regions described below.

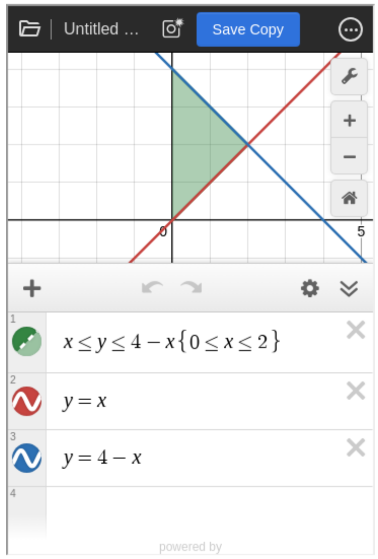
(a) The region bounded by the curves $y = x^2+1$ and $y = 4x+1$ between $x = 0$ and $x = 2$.

Hint.



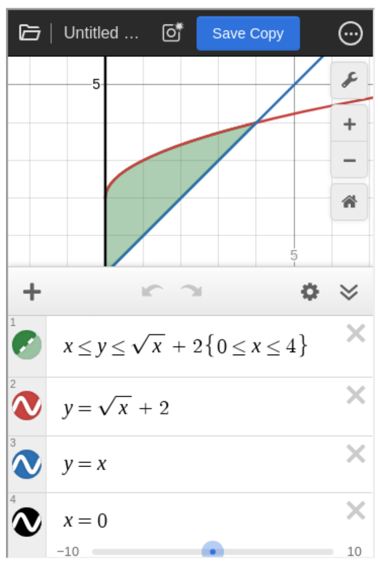
(b) The region bounded by the curves $y = x$ and $y = 4 - x$ between $x = 0$ and $x = 2$

Hint.



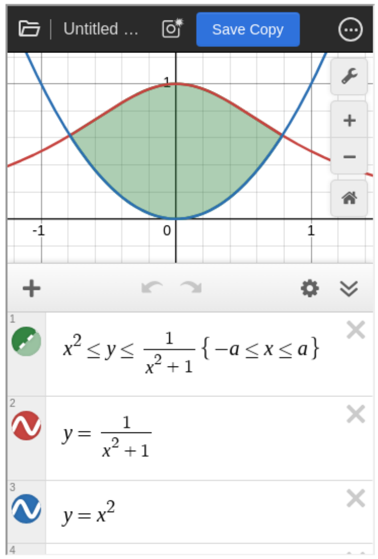
(c) The region bounded by the curves $y = \sqrt{x} + 2$ and $y = x$ and the line $x = 0$.

Hint.



(d) The region bounded by the curves $y = \dfrac{2}{x^2 + 1}$ and $y = x^2$.

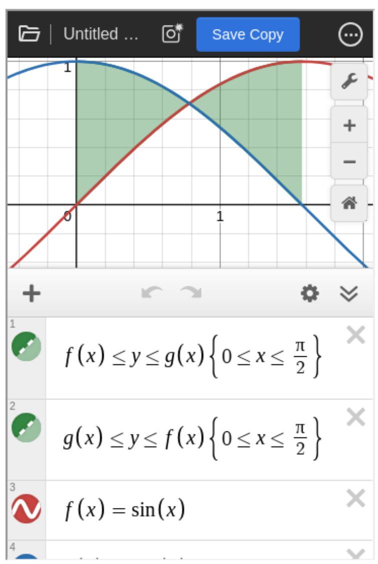
Hint.



4. Set up and evaluate an integral representing the area of each of the regions described below. Explain whether you chose to integrate with respect to x or y, and why you made that choice.

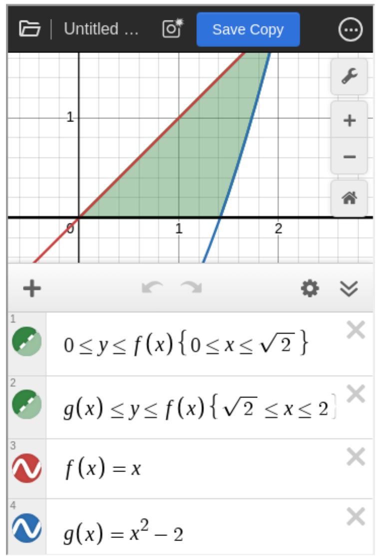
(a) The region bounded by the curves $y = \sin(x)$ and $y = \cos(x)$ and the line $y = 0$ between $x = 0$ and $x = \frac{\pi}{2}$.

Hint.



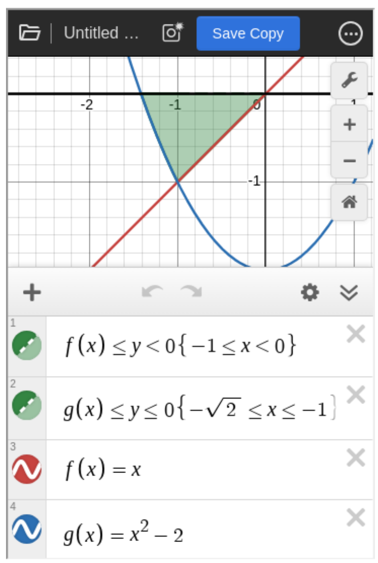
(b) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the first quadrant.

Hint.



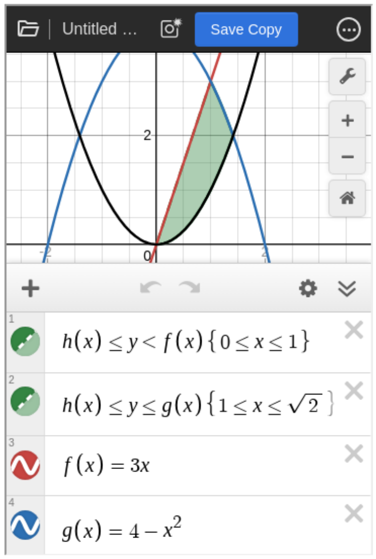
(c) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the third quadrant.

Hint.



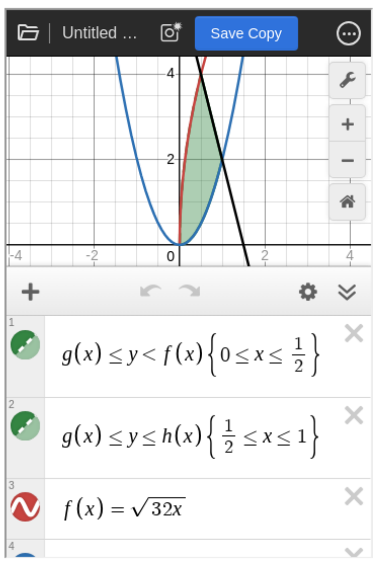
(d) The region bounded by the curves $y = 3x$, $y = 4 - x^2$, and $y = x^2$ in the first quadrant.

Hint.



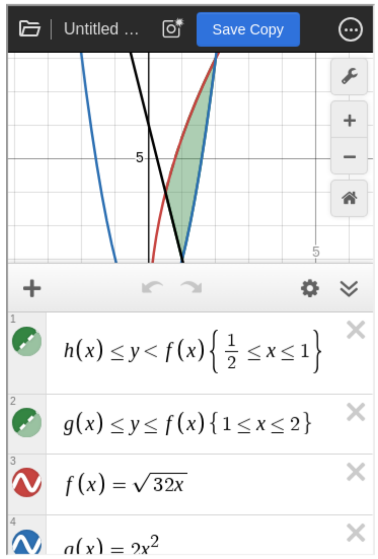
(e) The region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



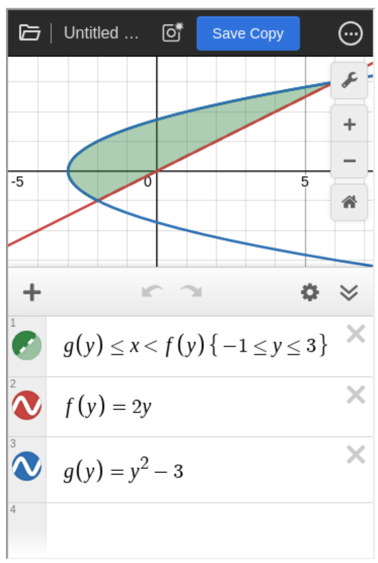
(f) The *other* region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



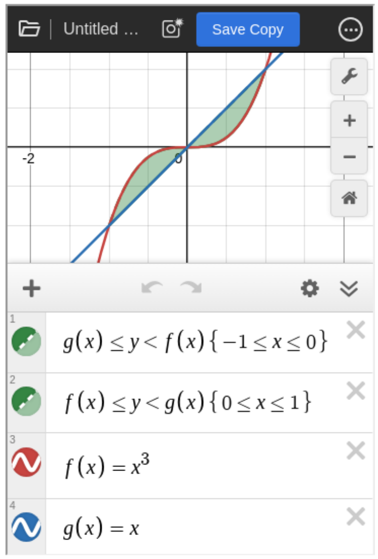
(g) The region bounded by the curves $x = 2y$ and $x = y^2 - 3$.

Hint.



(h) The region(s) bounded by the curves $y = x^3$ and $y = x$.

Hint.



6.0.3 Practice Problems

1. We say that the volume of a solid can be thought of as $\displaystyle\int_{x=a}^{x=b} A(x)\ dx$ where $A(x)$ is a function describing the cross-sectional area of our solid at an x-value between $x = a$ and $x = b$. Explain how this integral formula gets built, referencing the slice-and-sum (Riemann sum) method.

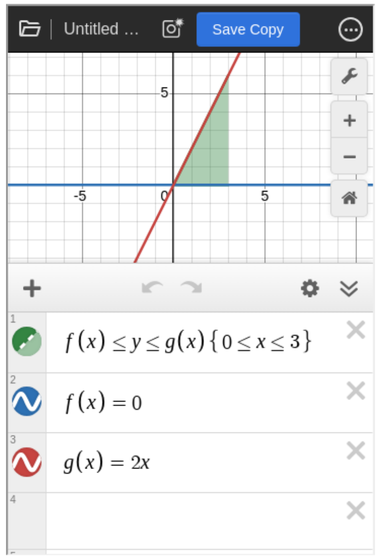
2. Explain the differences and similarities between the disk and washer methods for finding volumes of solids of revolution.

3. When do we integrate with regard to x (using a dx in our integral and writing our functions with x-value inputs) and when do we integrate with regard to y (using a dy in our integral and writing our functions with y-value inputs) when we're finding volumes using disks and washers? How do we know?

4. For each of the solids described below, set up an integral using the *disk/washer method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

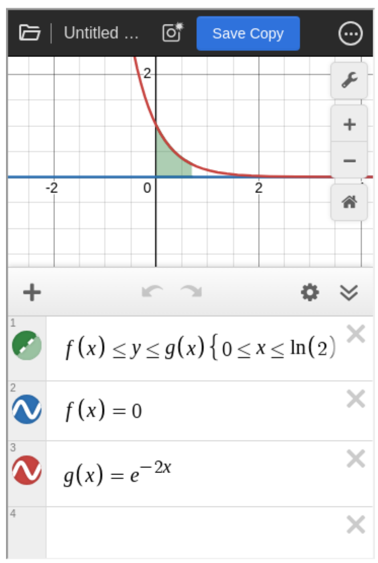
(a) The region bounded by the curve $y = 2x$ and the lines $y = 0$ and $x = 3$, revolved around the x-axis.

Hint.



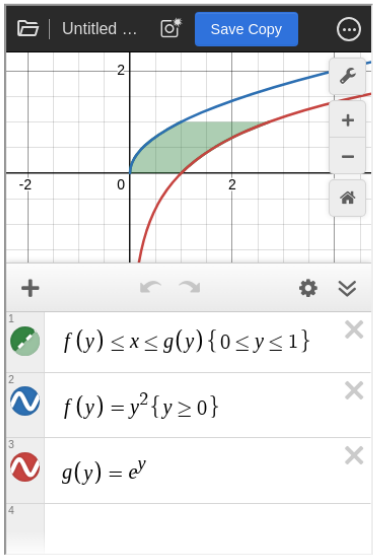
(b) The region bounded by the curve $y = e^{-2x}$ and the x-axis between $x = 0$ and $x = \ln(2)$, revolved around the x-axis.

Hint.



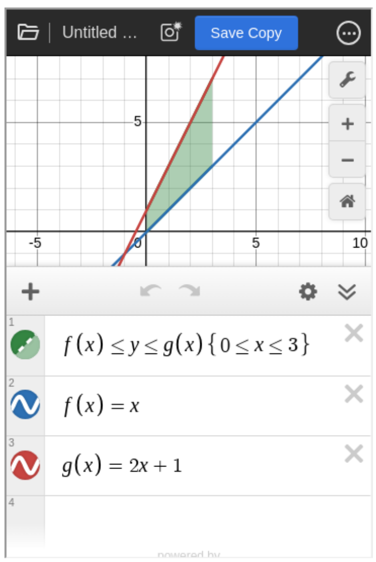
(c) The region bounded by the curves $y = \ln(x)$ and $y = \sqrt{x}$ between $y = 0$ and $y = 1$, revolved around the y-axis.

Hint.



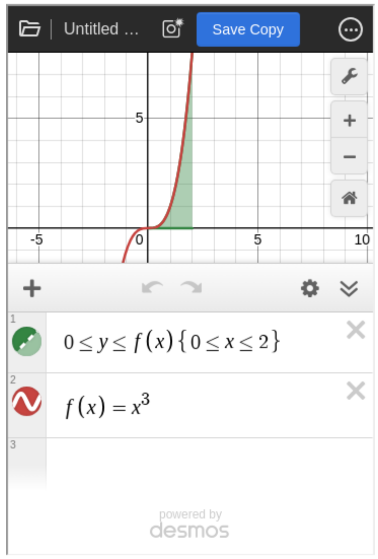
(d) The region bounded by the curves $y = 2x + 1$ and $y = x$ between $x = 0$ and $x = 3$, revolved around the x-axis.

Hint.



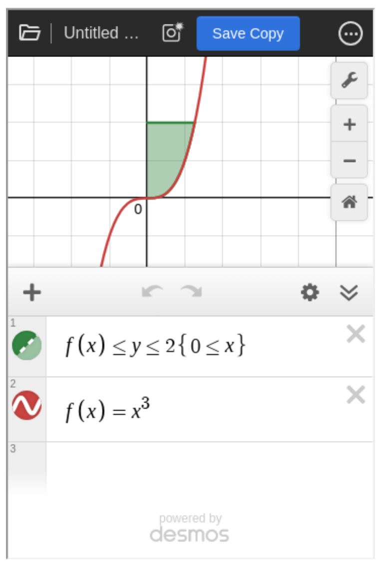
(e) The region bounded by the curve $y = x^3$, the x-axis, and the line $x = 2$, revolved around the y-axis.

Hint.



(f) The region bounded by the curve $y = x^3$ and the y-axis between $y = 0$ and $y = 2$, revolved around the y-axis.

Hint.



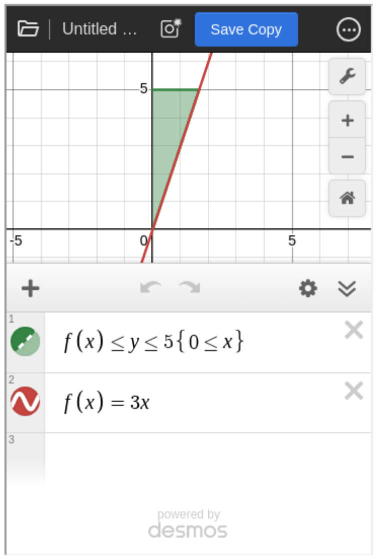
5. Explain where the pieces of the shell formula come from. How is this different than using disks/washers?

6. Say we're revolving a region around the x-axis to create a solid. Using the disk/washer method, we will integrate with respect to x. Using the shell method, we integrate with respect to y. Explain the difference, and why this difference occurs.

7. For each of the solids described below, set up an integral using the *shell method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

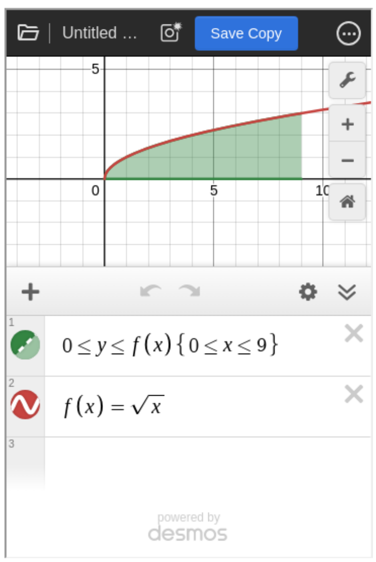
(a) The region bounded by the curve $y = 3x$ and the lines $x = 0$ and $y = 5$, revolved around the y-axis.

Hint.



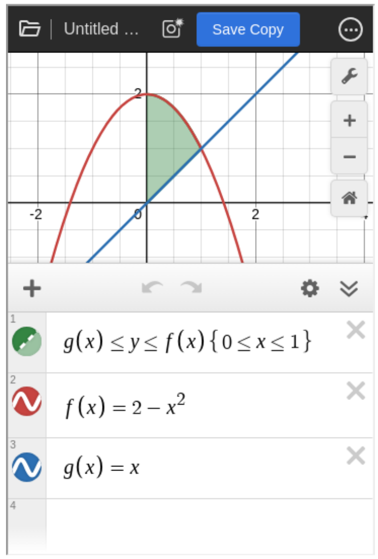
(b) The region bounded by the curve $y = \sqrt{x}$ and the x-axis between $x = 0$ and $x = 9$, revolved around the x-axis.

Hint.



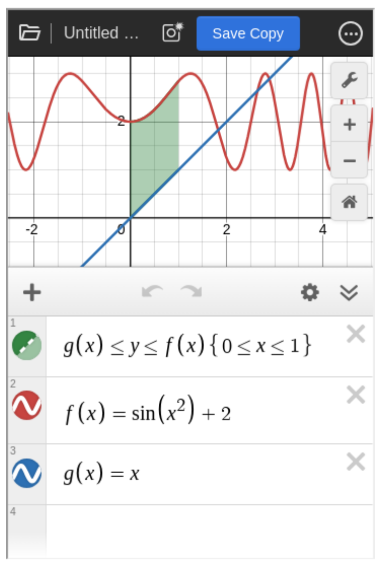
(c) The region bounded by the curves $y = 2 - x^2$ and $y = x$ and the line $x = 0$ revolved around the y-axis.

Hint.



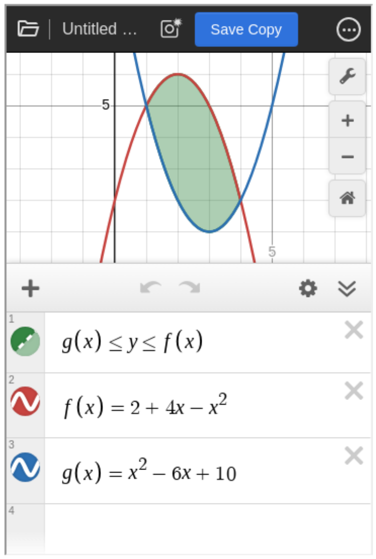
(d) The region bounded by the curves $y = sin(x^2) + 2$ and $y = x$ from $x = 0$ to $x = 1$, revolved around the y-axis.

Hint.



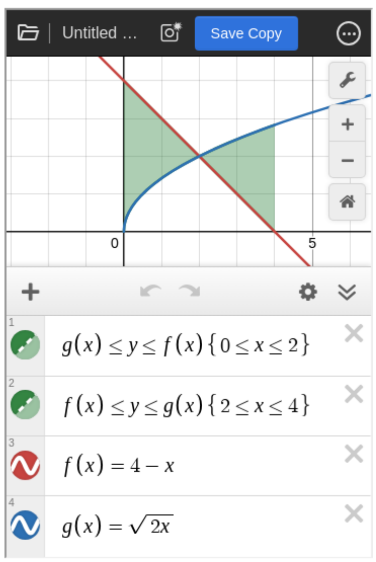
(e) The region bounded by the curves $y = x^2 - 6x + 10$ and $y = 2 + 4x - x^2$ revolved around the y-axis.

Hint.



(f) The region bounded by the curves $y = \sqrt{2x}$ and $y = 4 - x$ and the x-axis between $x = 0$ and $x = 4$, revolved around the x-axis.

Hint.



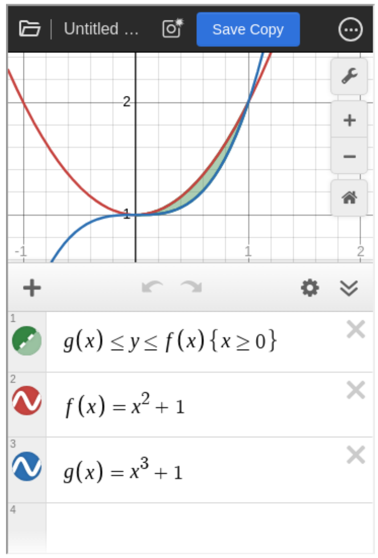
8. Pick at least 2 integrals from Exercise 4, p. 23 to re-write using shells instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

9. Pick at least 2 integrals from Exercise 7, p. 26 to re-write using disks/washers instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

10. For each of the following solids, set up an integral expression using either the disk/washer method or the shell method. You don't need to evaluate them, but you should do some careful thinking about how you set these up, especially as you choose between methods and what variable you are integrating with.

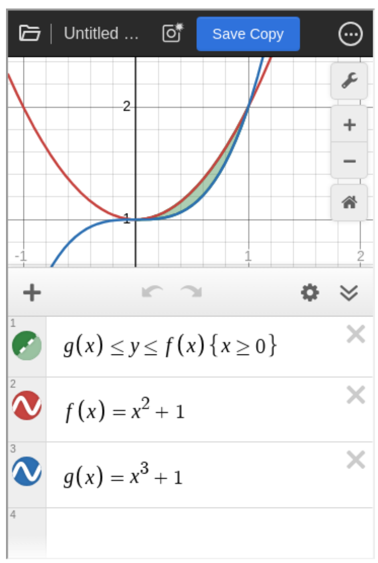
(a) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the x-axis.

Hint.



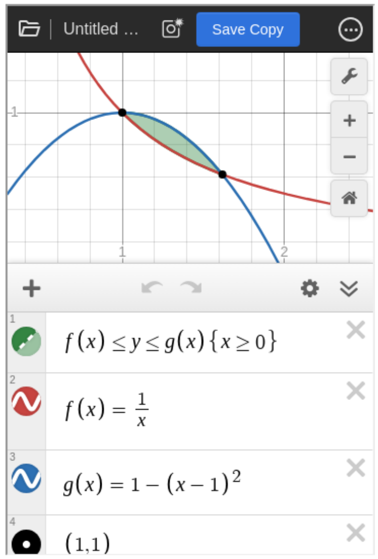
(b) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the y-axis.

Hint.



(c) The region bounded by the curves $y = \frac{1}{x}$ and $y = 1 - (x-1)^2$ in the first quadrant, revolved around the x-axis.

Hint.



6.0.4 Practice Problems

1. Consider the integral formula for computing volumes of a solid of revolution using the disk/washer method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius represented using the function output from the curve(s) defining the region?

2. Consider the integral formula for computing volumes of a solid of revolution using the shell method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius not represented using the function output from the curve(s) defining the region?

3. For each of the solids described below, set up an integral expression using disks/washers representing the volume of the solid.

(a) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = 2$.

(b) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = -1$.

(c) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = -2$.

(d) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = 3$.

4. For each of the solids described below, set up an integral expression using shells representing the volume of the solid.

(a) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = 2$.

(b) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = -1$.

(c) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = -2$.

(d) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = 3$.

Chapter 7

Techniques for Antidifferentiation

7.0.1 Practice Problems

1. Explain what it means for an integral to be improper. What kinds of issues are we looking at?

2. Give an example of an integral that is improper due to an unbounded or infinite interval of integration (infinite width).

3. Give an example of an integral that is improper due to an unbounded integrand (infinite height).

4. What does it mean for an improper integral to ”converge?” How does this connect with limits?

5. What does it mean for an improper integral to ”diverge?” How does this connect with limits?

6. Why do we need to use limits to evaluate improper integrals?

7. For each of the following improper integrals:

- Explain why the integral is improper. Be specific, and point out the issues in detail.
- Set up the integral using the correct limit notation.
- Antidifferentiate and evaluate the limit.
- Explain whether the integral converges or diverges.

(a) $\displaystyle\int_{x=0}^{\infty} \frac{1}{\sqrt{x+1}}\, dx$

(b) $\displaystyle\int_{x=0}^{\infty} e^{-2x}\, dx$

(c) $\displaystyle\int_{x=-1}^{x=3} \frac{1}{x+1}\, dx$

(d) $\displaystyle\int_{-\infty}^{x=0} \sqrt{e^x}\, dx$

(e) $\displaystyle\int_{x=2}^{x=8} \frac{5}{(x-2)^3}\, dx$

(f) $\displaystyle\int_{x=1}^{x=12} \frac{dx}{\sqrt[5]{12-x}}$

8. One of the big ideas in probability is that for a curve that defines a probability density function, the area under the curve needs to be 1. What value of k makes the function $\dfrac{kx}{(x^2+3)^{5/4}}$ a valid probability distribution on the interval $[0, \infty)$?

9. Let's consider the integral $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$. This is a difficult integral to evaluate!

(a) First, compare $\sqrt{x^2+1}$ to $\sqrt{x^2}$ using an inequality: which one is bigger?

(b) Second, use this inequality to compare the function $\dfrac{\sqrt{x^2+1}}{x^2}$ to $\dfrac{1}{x}$ for $x > 0$: which one is bigger? Again, use your inequality from above to help!

(c) Now compare $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ to $\displaystyle\int_{x=1}^{\infty} \frac{1}{x}\,dx$. Which one is bigger?

(d) Explain how we can use this result to make a conclusion about whether our integral, $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ converges or diverges.

7.0.2 Practice Problems

1. Use polynomial division or some clever factoring to re-write and find the following indefinite integrals or evaluate the following definite integrals.

(a) $\displaystyle\int \left(\frac{x+4}{x-3}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x^2+4}{x-4}\right)\,dx$

(c) $\displaystyle\int \left(\frac{t^2+t+6}{t^2+1}\right)\,dt$

(d) $\displaystyle\int_{x=2}^{x=4} \left(\frac{x^3+1}{x-1}\right)\,dx$

(e) $\displaystyle\int_{x=0}^{x=1} \left(\frac{x^4+1}{x^2+1}\right)\,dx$

2. Complete the square in order to find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x^2-2x+10}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x}{x^2+4x+8}\right)\,dx$

(c) $\displaystyle\int \left(\frac{2x}{x^4+6x^2+10}\right)\,dx$

3. Find the following indefinite integrals.

(a) $\int \left(\frac{1}{x^{-1}+1} \right)\, dx$

(b) $\int \left(\frac{\sin(\theta)+\tan(\theta)}{\cos^2(\theta)} \right)\, d\theta$

(c) $\int \left(\frac{1-x}{1-\sqrt{x}} \right)\, dx$

(d) $\int \left(\frac{1}{1-\sin^2(\theta)} \right)\, d\theta$

(e) $\int \left(\frac{x^{2/3}-x^3}{x^{1/4}} \right)\, dx$

(f) $\int \left(\frac{4+x}{\sqrt{1-x^2}} \right)\, dx$

7.0.3 Practice Problems

1. Explain how we build the Integration by Parts formula, as well as what the purpose of this integration strategy is.

2. How do you choose options for u and dv? What are some good strategies to think about?

3. Let's say that you make a choice for u and dv and begin working through the Integration by Parts strategy. How can you tell if you've made a poor choice for your parts? Can you *always* tell?

4. Integrate the following.

(a) $\int 3x\sin(x)\, dx$

(b) $\int 5xe^x\, dx$

(c) $\int x^2e^{-x}\, dx$

(d) $\int x^2\ln(x)\, dx$

(e) $\int x^2\cos(x)\, dx$

(f) $\int x^3e^{-x}\, dx$

(g) $\int x\sin(x)\cos(x)\, dx$

(h) $\int e^x\sin(x)$

(i) $\int \sin^{-1}(x)\, dx$

(j) $\int \tan^{-1}(x)\, dx$

5. Evaluate the following definite integrals.

(a) $\displaystyle\int_{x=1}^{x=e} x\ln(x)\ dx$

(b) $\displaystyle\int_{x=0}^{x=\pi/4} x\cos(2x)\ dx$

(c) $\displaystyle\int_{x=0}^{x=\ln(5)} xe^x\ dx$

6. In this problem, we'll consider the integral $\displaystyle\int \sin^2(x)\ dx$. We'll integrate this in two different ways!

(a) We know that:

$$\int \sin^2(x)\ dx = \int \sin(x)\sin(x)\ dx.$$

Use the Integration by Parts strategy, and especially note that you can solve for the integral (Subsection 7.4.4 Solving for the Integral, p. **??**).

(b) We can use a trigonometric identity to re-write the integral:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}.$$

So we have:

$$\int \sin^2(x)\ dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\ dx.$$

Use u-substitution.

(c) Were your answers the same or different? Should they be the same? Why or why not? Are they connected somehow?

Hint. They might be different, but they should only be different by at most a constant.

7. For these next problems, we'll use $x = u^2$ and $dx = 2u\ du$ to substitute into the integral as written. Then use Integration by Parts.

(a) $\displaystyle\int \sin(\sqrt{x})\ dx$

(b) $\displaystyle\int e^{\sqrt{x}}\ dx$

7.0.4 Practice Problems

1. For an integral $\displaystyle\int \sin^a(x)\cos^b(x)\ dx$, how do you know whether to use $u = \sin(x)$ or $u = \cos(x)$ as the substitution?

2. For an integral $\displaystyle\int \tan^a(x)\sec^b(x)\ dx$, how do you know whether to use $u = \tan(x)$ or $u = \sec(x)$ as the substitution?

3. Integrate the following.

(a) $\displaystyle\int \sin^3(x)\cos^2(x)\; dx$

(b) $\displaystyle\int \sin^2(x)\cos^3(x)\; dx$

(c) $\displaystyle\int \sin^3(x)\cos^3(x)\; dx$

(d) $\displaystyle\int \tan^4(x)\sec^4(x)\; dx$

(e) $\displaystyle\int \tan^3(x)\sec^3(x)\; dx$

(f) $\displaystyle\int \tan^3(x)\sec^4(x)\; dx$

4. Integrate the following.

(a) $\displaystyle\int \sin^{3/4}(x)\cos^5(x)\; dx$

(b) $\displaystyle\int \tan^5(x)\sec^{-1/2}(x)\; dx$

(c) $\displaystyle\int \sin^{3/4}(x)\cos^5(x)\; dx$

(d) $\displaystyle\int \tan^5(x)\sec^{-1/2}(x)\; dx$

5. Consider the integral $\displaystyle\int \sin^2(x)\; dx$.

(a) Use the trigonometric identity:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

Hint. Check out Subsection 7.4.4 Solving for the Integral, p. **??**

(c) Which of these techniques do you think was easier to implement and use? Why is that?

6. Consider the integral $\displaystyle\int \cos^4(x)\; dx$.

(a) Use the trigonometric identity:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

Hint. Try picking $u = \cos^3(x)$ and $dv = \cos(x)\; dx$.

(c) Which of these techniques did you prefer? Why?

7. Integrate the following integrals.

(a) $\displaystyle\int \tan^2(x)\, dx$

Hint. Use a Pythagorean Identity to convert this to be written in terms of secant functions.

(b) $\displaystyle\int \sec^3(x)\, dx$

Hint. Integration by parts works well here, and it's helpful to know the derivative of $\sec(x)$ and an antiderivative of $\sec^2(x)$.

(c) $\displaystyle\int \tan^5(x)\, dx$

Hint. You can technically use either $u = \sec(x)$ or $u = \tan(x)$ here.

(d) $\displaystyle\int \sin^5(x)\, dx$

7.0.5 Practice Problems

1. Explain how trigonometric substitution helps to convert sums or differences of squares to products of squares. Why is this helpful? When is it helpful?

2. Draw a right triangle with $\sqrt{x^2-4}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{x^2-4}$?

3. Draw a right triangle with $\sqrt{4-x^2}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{4-x^2}$?

4. Draw a right triangle with $\sqrt{x^2+4}$ as one of the hypotenuse. What are the lengths of the other two sides? What would be an appropriate substitution for an integral containing $\sqrt{x^2+4}$?

5. Integrate the following using an appropriate trigonometric substitution.

(a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\, dx$

(b) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\, dx$

(c) $\displaystyle\int \frac{1}{(9x^2+1)^{3/2}}\, dx$

(d) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\, dx$

(e) $\displaystyle\int \sqrt{49-x^2}\, dx$

(f) $\displaystyle\int \frac{1}{x(x^2-1)^{3/2}}\, dx$ (for $x > 1$)

(g) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(h) $\displaystyle\int \frac{x^2}{(x^2+81)^2}\,dx$

6. Complete the square and then integrate.

(a) $\displaystyle\int \frac{1}{x^2-8x+62}\,dx$

(b) $\displaystyle\int \frac{x^2-8x+16}{(-x^2+8x+9)^{3/2}}\,dx$

7.0.6 Practice Problems

1. Why do we use partial fraction decomposition on some integrals of rational functions? Give an example and explain why it is helpful in your example.

2. For each rational function described, write out the corresponding partial fraction forms.

(a) $\dfrac{p(x)}{(x-4)(x+2)(x-1)}$ where $p(x)$ is some polynomial with degree less than 3.

(b) $\dfrac{p(x)}{(x+1)^2(3x-5)^3}$ where $p(x)$ is some polynomial with degree less than 5.

(c) $\dfrac{p(x)}{(x^2+1)(x^2+2x+5)}$ where $p(x)$ is some polynomial with degree less than 4.

(d) $\dfrac{p(x)}{x^4-1}$ where $p(x)$ is some polynomial with degree less than 4.

Hint. There's some factoring to be done here! Note that $x^4-1=(x^2-1)(x^2+1)$ and then we can factor $x^2-1=(x-1)(x+1)$.

3. Consider the following integral, with the partial fraction forms written out:

$$\int \frac{x^3+6x^2-x}{(x-2)(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x-2}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\right)\,dx.$$

(a) Write an equation connecting the numerators.

(b) Find (and use) a specific x-value to input into the equation to solve for A.

Hint. Use $x=2$, and notice what happens to the rest of the terms.

(c) Find (and use) a specific x-value to input into the equation to solve for B.

Hint. Use $x=-1$, and notice what happens to the rest of the terms.

(d) Why can you not use this strategy to solve for coefficients C or D?

(e) Find the cubic terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for C.

(f) Find the constant terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for D.

(g) Integrate!

4. Explain why partial fractions is not an appropriate technique for the following integral:

$$\int \frac{x^2+x}{x^2-x+1}\, dx.$$

How should we approach this integral, instead?

Hint. Note the degree in the numerator compared to the denominator!

5. Integrate the following.

(a) $\displaystyle\int \frac{2}{(x-1)(x+3)}\, dx$

(b) $\displaystyle\int \frac{4x+1}{(x-4)(x+5)}\, dx$

(c) $\displaystyle\int \frac{2x^2-15x+32}{x(x^2-8x^2+64)}$

(d) $\displaystyle\int \frac{1}{(x+2)(x-2)}\, dx$

(e) $\displaystyle\int \frac{20x}{(x-1)(x^2+4x+5)}\, dx$

(f) $\displaystyle\int \frac{x^2}{(x-2)^3}\, dx$

6. In the problems we are looking at in this section, we're limiting ourselves to, at most, irreducible quadratic factors in the denominator. In problems with simple linear factors, repeated linear factors, or irreducible quadratic factors, what types of antiderivative functions do you expect to see? Explain.

7. For each of the following integrals, we will do some preliminary work before using partial fractions to integrate. Really, we'll perform a specific u-substitution that will give us some resulting integral to use partial fractions on.

(a) $\displaystyle\int \frac{4e^{2x}}{(e^{2x}+3)(e^{2x}-5)}\, dx$ where we use $u = e^{2x}$.

Hint. If $u = e^{2x}$ then $du = 2e^{2x}\, dx$ and our resulting integral looks like:

$$\int \frac{2}{(u+3)(u-5)}\, du.$$

(b) $\displaystyle\int \sqrt{e^x+1}\, dx$ where we use $u = \sqrt{e^x+1}$.

Hint. If $u = \sqrt{e^x+1}$ then $du = \dfrac{e^x}{2\sqrt{e^x+1}}\, dx$ and the resulting integral is:

$$\int \frac{2u^2}{u^2-1}\, du = \int 2 + \frac{2}{u^2-1}\, du.$$

(c) $\displaystyle\int \frac{\sqrt{x}+3}{\sqrt{x}(x-1)}\, dx$ where we use $u = \sqrt{x}$.

Hint. If $u = \sqrt{x}$ then $du = \dfrac{1}{2\sqrt{x}}\, dx$ and the resulting integral is:

$$\int \frac{2u+6}{u^2-1}\, du.$$

Chapter 8

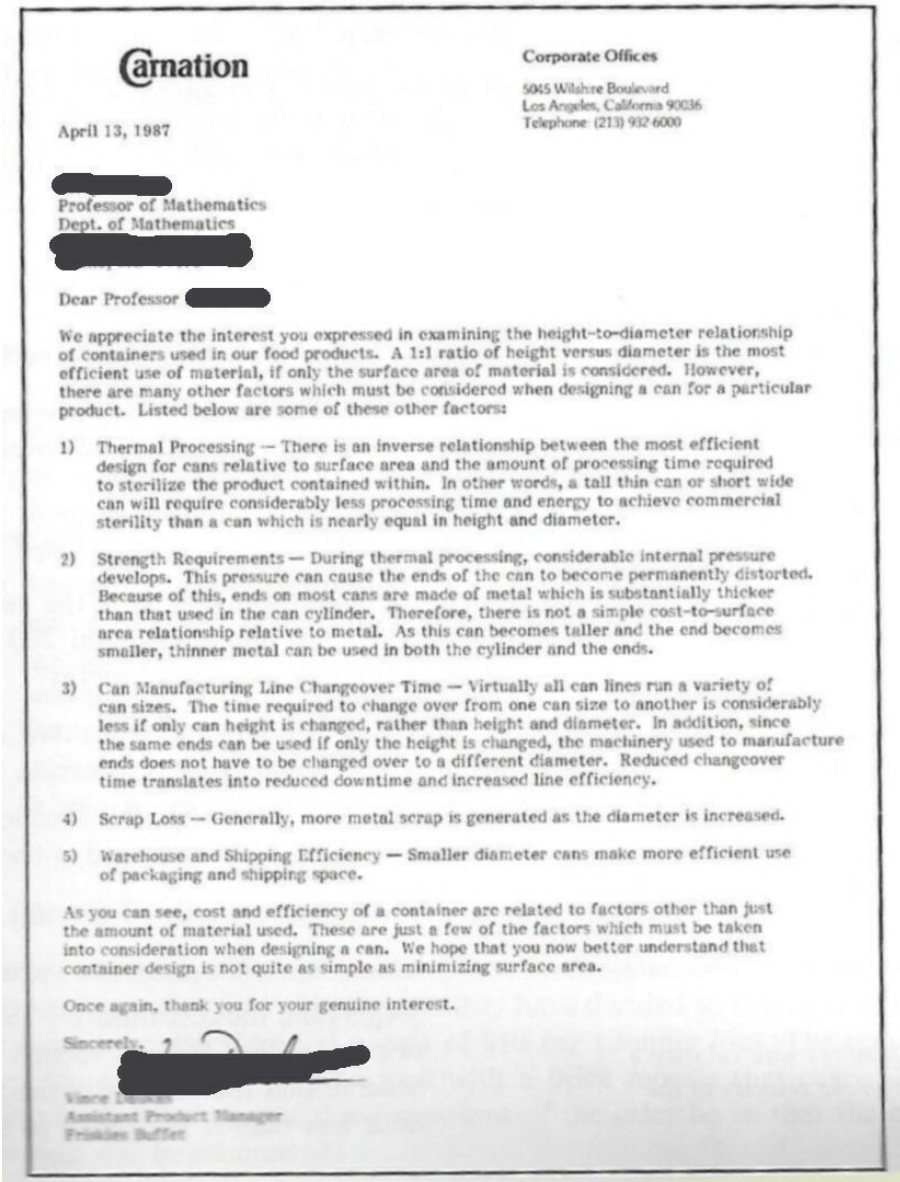
Infinite Series

Chapter 9

Power Series

Appendix A

Carnation Letter



Carnation

Corporate Offices

5045 Wilshire Boulevard
Los Angeles, California 90036
Telephone (213) 932 6000

April 13, 1987

[redacted]
Professor of Mathematics
Dept. of Mathematics
[redacted]

Dear Professor [redacted]

We appreciate the interest you expressed in examining the height-to-diameter relationship of containers used in our food products. A 1:1 ratio of height versus diameter is the most efficient use of material, if only the surface area of material is considered. However, there are many other factors which must be considered when designing a can for a particular product. Listed below are some of these other factors:

1) Thermal Processing — There is an inverse relationship between the most efficient design for cans relative to surface area and the amount of processing time required to sterilize the product contained within. In other words, a tall thin can or short wide can will require considerably less processing time and energy to achieve commercial sterility than a can which is nearly equal in height and diameter.

2) Strength Requirements — During thermal processing, considerable internal pressure develops. This pressure can cause the ends of the can to become permanently distorted. Because of this, ends on most cans are made of metal which is substantially thicker than that used in the can cylinder. Therefore, there is not a simple cost-to-surface area relationship relative to metal. As this can becomes taller and the end becomes smaller, thinner metal can be used in both the cylinder and the ends.

3) Can Manufacturing Line Changeover Time — Virtually all can lines run a variety of can sizes. The time required to change over from one can size to another is considerably less if only can height is changed, rather than height and diameter. In addition, since the same ends can be used if only the height is changed, the machinery used to manufacture ends does not have to be changed over to a different diameter. Reduced changeover time translates into reduced downtime and increased line efficiency.

4) Scrap Loss — Generally, more metal scrap is generated as the diameter is increased.

5) Warehouse and Shipping Efficiency — Smaller diameter cans make more efficient use of packaging and shipping space.

As you can see, cost and efficiency of a container are related to factors other than just the amount of material used. These are just a few of the factors which must be taken into consideration when designing a can. We hope that you now better understand that container design is not quite as simple as minimizing surface area.

Once again, thank you for your genuine interest.

Sincerely,

[redacted]

Vince [illegible]
Assistant Product Manager
Friskies Buffet

Figure A.0.1 Response letter from Carnation.

Full Text of the Carnation Letter. April 13, 1987

[REDACTED]
Professor of Mathematics
Dept. of Mathematics
[REDACTED]

Dear Professor [REDACTED],

We appreciate the interest you expressed in examining the height-to-diameter relationship of containers used in our food products. A 1:1 ratio of height versus diameter is the most efficient use of material, if only the surface area of material is considered. However, there are many other factors which must be considered when designing a can for a particular product. Listed below are some of these other factors:

1) Thermal Processing — There is an inverse relationship between the most efficient design for cans relative to surface area and the amount of processing time required to sterilize the product contained within. In other words, a tall thin can or short wide can will require considerably less processing time and energy to achieve commercial sterility than a can which is nearly equal in height and diameter.

2) Strength Requirements — During thermal processing, considerable internal pressure develops. This pressure can cause the ends of the can to become permanently distorted. Because of this, ends on most cans are made of metal which is substantially thicker than that used in the can cylinder. Therefore, there is not a simple cost-to-surface area relationship relative to metal. As this can becomes taller and the end becomes smaller, thinner metal can be used in both the cylinder and the ends.

3) Can Manufacturing Line Changeover Time — Virtually all can lines run a variety of can sizes. The time required to change over from one can size to another is considerably less if only can height is changed, rather than height and diameter. In addition. since the same ends can be used if only the height is changed, the machinery used to manufacture ends does not have to be changed over to a different diameter. Reduced changeover time translates into reduced downtime and increased line efficiency.

4) Scrap Loss — Generally, more metal scrap is generated as the diameter is increased.

5) Warehouse and Shipping Efficiency — Smaller diameter cans make more efficient use of packaging and shipping space.

As you can see, cost and efficiency of a container are related to factors other than just the amount of material used. These are just a few of the factors which must be taken into consideration when designing a can. We hope that you now better understand that container design is not quite as simple as minimizing surface area.

Once again, thank you for your genuine interest.

Sincerely,
[REDACTED]
Vince [Illegible]
Assistant Product Manager
Friskies Buffet

Colophon

This book was authored in PreTeXt.