Discover Calculus II - Activity Book

Activities for Integral Calculus Topics

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Peter Keep
Moraine Valley Community College

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Contents

Chapter 6

Applications of Integrals

6.1 Integrals as Net Change

Activity 6.1.1 Estimating Movement.

We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 6.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

(b) When was the acceleration of the object the greatest? When was it the least?

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

Activity 6.1.2 A Friendly Jogger.

Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

(b) Does this time represent a maximum or minimum velocity for the jogger?

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

Activity 6.1.3 Tracking our Jogger.

Let's revisit our jogger from Activity 6.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

(b) Think back to our description of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.

6.2 Area Between Curves

Activity 6.2.1 Remembering Riemann Sums.

Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

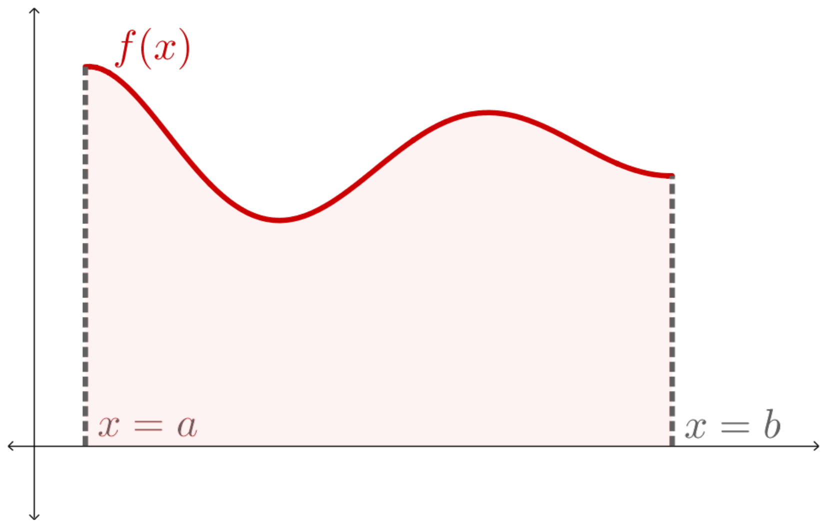


Figure 6.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

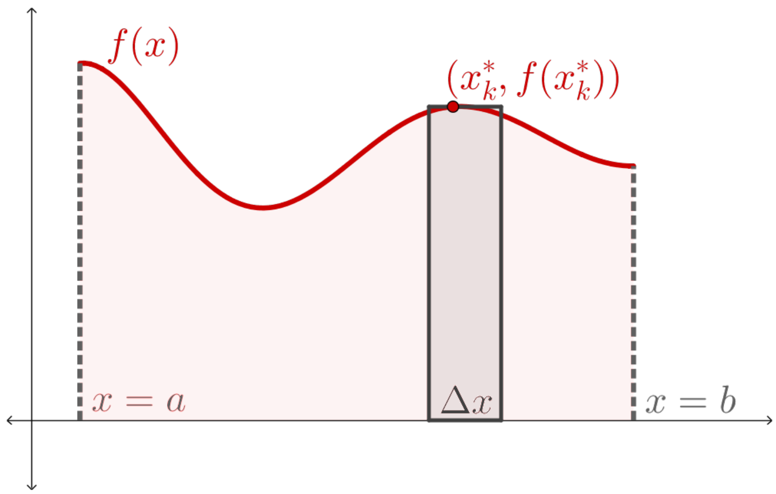


Figure 6.2.2

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

Activity 6.2.2 Area Between Curves.

Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

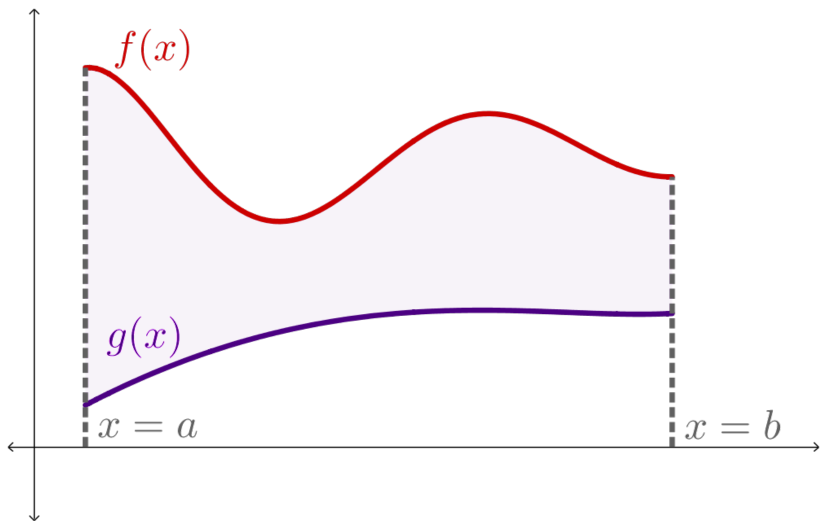


Figure 6.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

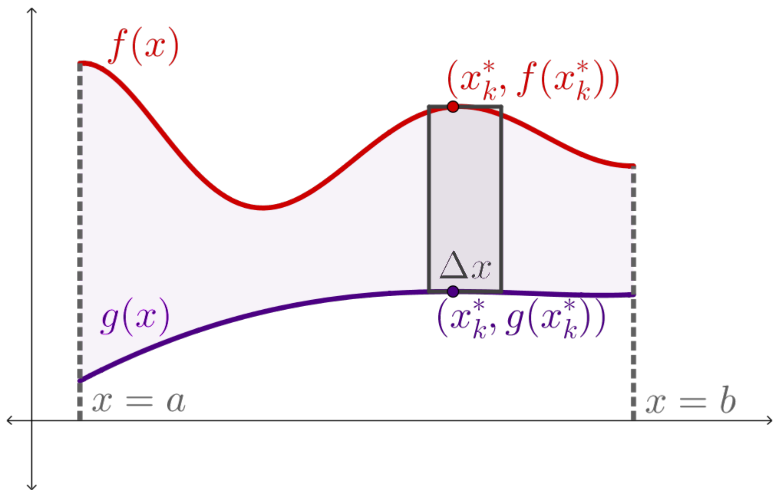


Figure 6.2.4

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Activity 6.2.3 Trying for a Single Integral.

Let's consider the same setup as earlier: the region bounded between two curves, $y = x + 6$ and $y = x^3$, as well as the x-axis (the line $y = 0$). We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

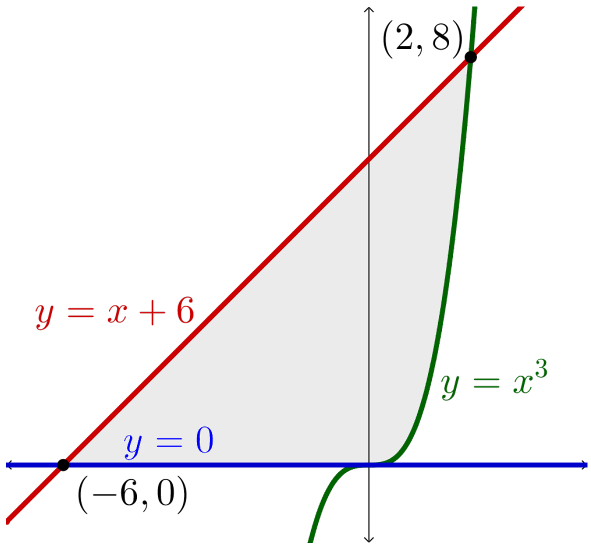


Figure 6.2.5

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g, determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths of each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

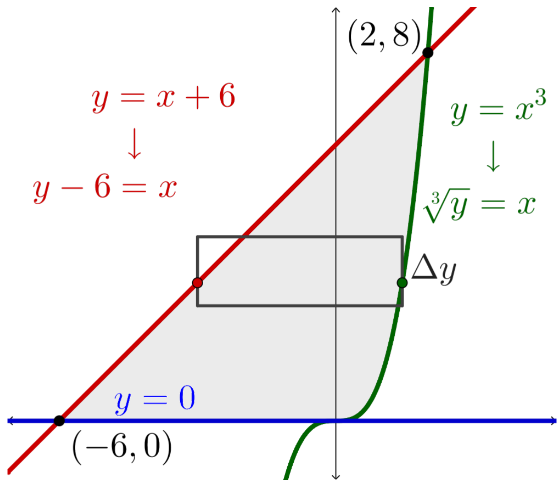


Figure 6.2.6

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

(j) Find A_k, the area of this kth rectangle.

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A.

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

6.3 Volumes of Solids of Revolution

Activity 6.3.1 Carving out a Hole in the Center.

We're going to look at the same solid as in Figure 6.3.2. But this time, when we define the 2-dimensional region that we're going to revolve around the x-axis, we're going to add a lower boundary function, $g(x)$.

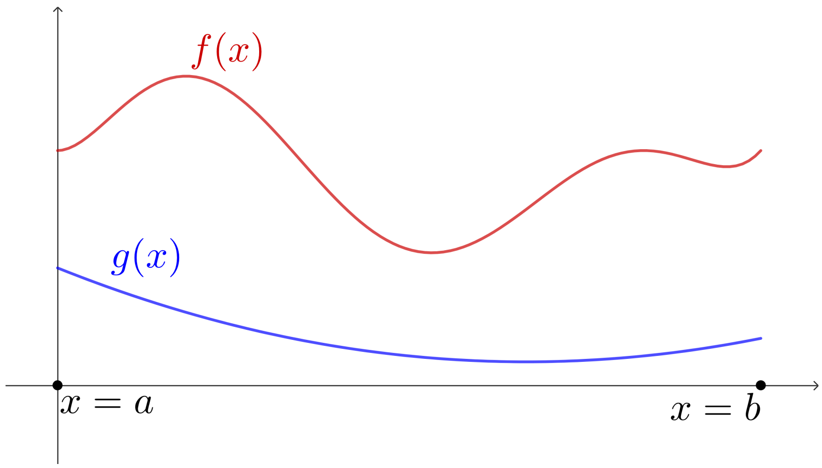


Figure 6.3.1

When we revolve this region around the x-axis, we get the following 3-dimensional solid.

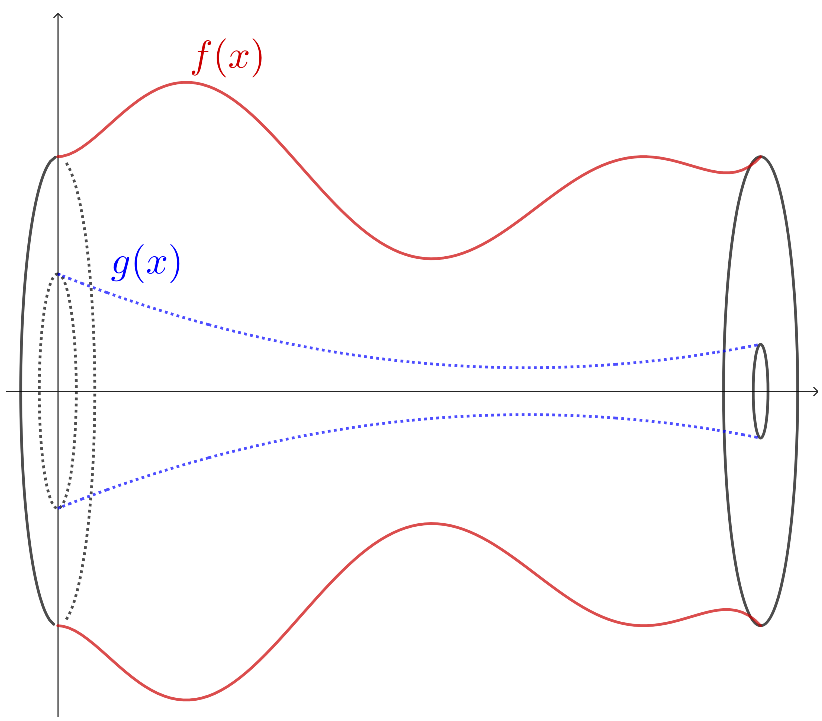


Figure 6.3.2

(a) How is a single generic slice on this solid different than the one in Figure 6.3.2?

(b) Find a formula for the area of the face of the cross-sectional slice.

(c) Use the slice-and-sum process to create an integral expression representing the volume of this solid.

Activity 6.3.2 Volumes by Disks/Washers.

Consider the region bounded between the curves $y = 4 + 2x - x^2$ and $y = \dfrac{4}{x+1}$. We will create a 3-dimensional solid by revolving this region around the x-axis.

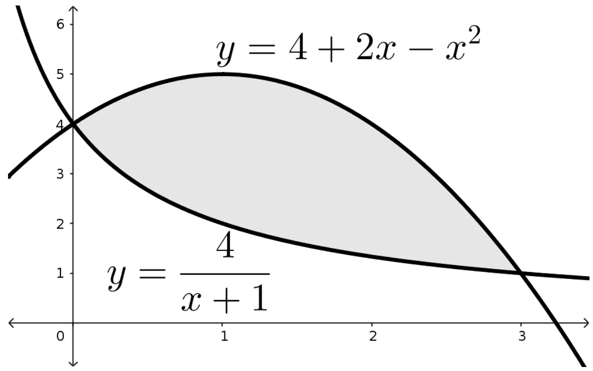


Figure 6.3.3

(a) Visualize the solid you'll create when you revolve this region around the x-axis.

(b) Draw a single rectangle in your region, standing perpendicular to the x-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of the solid.

(f) Can you describe how you would antidifferentiate and evaluate this integral?

Activity 6.3.3 Another Volume.

Now let's consider another region: this time, the one bounded between the curves $y = x$ and $y = 3\sqrt{x}$. We will, again, create a 3-dimensional solid by revolving this region around the y-axis.

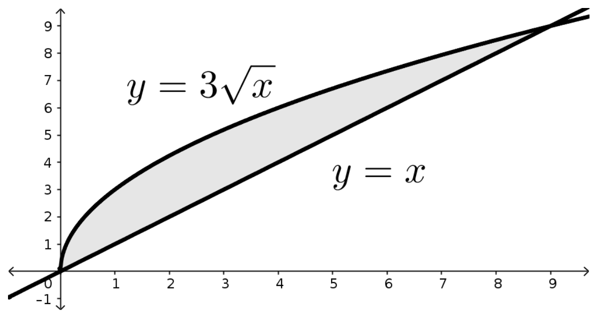


Figure 6.3.4

(a) Visualize the solid you'll create when you revolve this region around the y-axis.

(b) Draw a single rectangle in your region standing perpendicular to the y-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up the integral representing the volume of this solid.

Activity 6.3.4 Volume by Shells.

Let's consider the region bounded by the curves $y = x^3$ and $y - x + 6$ as well as the line $y = 0$. You might remember this region from Activity 6.2.3. This time, we'll create a 3-dimensional solid by revolving the region around the x-axis

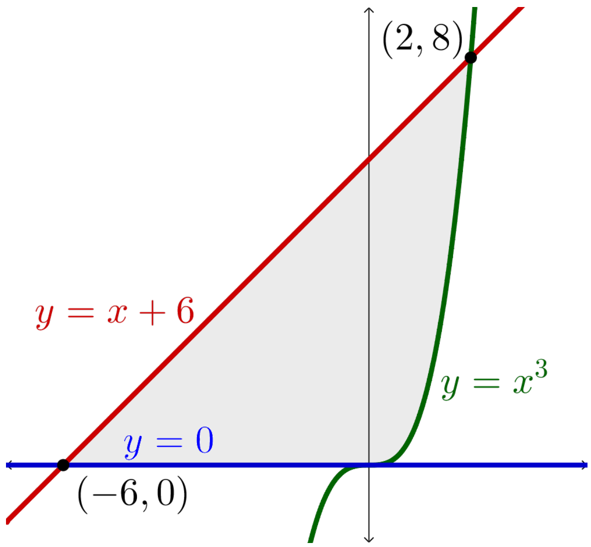


Figure 6.3.5

(a) Sketch one or two rectangles that are *perpendicular* to the x-axis. Then set up an integral expression to find the volume of the solid using them.

(b) Now, draw a single rectangle in the region that is *parallel* to the axis of revolution. Use this rectangle to visualize the kth slice of this 3-dimensional solid. What does that single rectangle create when it is revolved around the x-axis?

(c) Set up the integral expression representing the volume of the solid.

(d) Confirm that your volumes are the same, no matter your approach to setting it up.

6.4 More Volumes: Shifting the Axis of Revolution

Activity 6.4.1 What Changes (in the Washer Method) with a New Axis?

Let's revisit Activity 6.3.2 Volumes by Disks/Washers, and ask some more follow-up questions. First, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = -3$.

(a) What changes, if any, do you have to make to the rectangle you drew in Activity 6.3.2?

(b) What changes, if any, do you have to make to the area of the "face" kth washer?

(c) What changes, if any, do you have to make to the eventual volume integral for this solid?

Activity 6.4.2 What Changes (in the Shell Method) with a New Axis?

Let's revisit Activity 6.3.4 Volume by Shells, and ask some more follow-up questions about the shell method. Again, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = 9$.

(a) What changes, if any, do you have to make to the rectangle you drew in Activity 6.3.4?

(b) What changes, if any, do you have to make to the area of the rectangle formed by ”unrolling” up kth cylinder?

(c) What changes, if any, do you have to make to the eventual volume integral for this solid?

Activity 6.4.3 More Shifted Axes.

We're going to spend some time constructing *several* different volume integrals in this activity. We'll consider the same region each time, but make changes to the axis of revolution. For each, we'll want to think about what kind of method we're using (disks/washers or shells) and how the different axis of revolution gets implemented into our volume integral formulas.

Let's consider the region bounded by the curves $y = \cos(x) + 3$ and $y = \dfrac{x}{2}$ between $x = 0$ and $x = 2\pi$.

(a) Let's start with revolving this around the x-axis and thinking about the solid formed. While you set up your volume integral, think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

(b) Now, let's create a different solid by revolving this region around the y-axis. Set up a volume integral, and continue to think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

(c) We'll start shifting our axis of revolution now. We'll revolve the same region around the horizontal line $y = -1$ to create a solid. Set up an integral expression to calculate the volume.

(d) Now, revolve the region around the line $y = 5$ to create a solid of revolution, and write down the integral representing the volume.

(e) Let's change things up. Revolve the region around the vertical line $x = -1$ to create a new solid. Set up an integral representing the volume of that solid.

(f) We'll do one more solid. Let's revolve this region around the line $x = 7$. Set up an integral representing the volume.

6.5 Arc Length and Surface Area

Activity 6.5.1 Measuring Distance.

(a) Consider the following right-triangle with the normal names of side lengths.

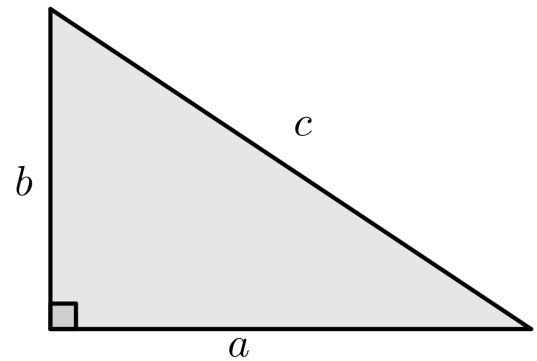


Figure 6.5.1

How do we use the Pythagorean Theorem to find the length of c?

(b) Consider the two points (x_1, y_1) and (x_2, y_2) below.

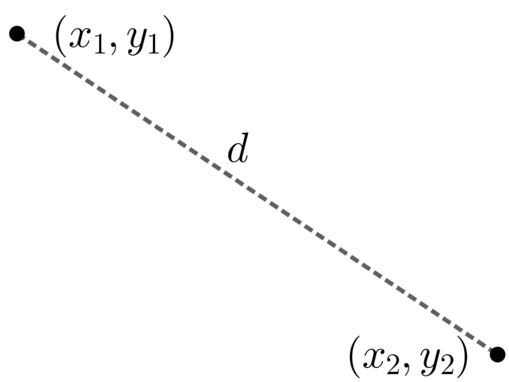


Figure 6.5.2

How do we use the distance formula to find the length of the line connecting the two points, d?

(c) How are these two things the same? How are they different?

6.6 Other Applications of Integrals

Chapter 7

Techniques for Antidifferentiation

7.1 Improper Integrals

Activity 7.1.1 Remembering a Theme so Far.

(a) Let's say that we want to find what the y-values of some function $f(x)$ are when the x-values are "infinitely close to" some value, $x = a$. Since there is no single x-value that is "infinitely close to" a that we can evaluate $f(x)$ at, we need to do something else. How do we do this?

(b) Let's say that we want to find the rate of change of some function instantaneously at a point with $x = a$. We can't find a rate of change unless we have two points, since we need to find some differences in the outputs and inputs. How do we do this?

(c) Suppose you want to find the total area, covered by an infinite number of infinitely thin rectangles. You have a formula for finding the dimensions and areas for some finite number of rectangles, but how do we get an infinite number of them?

(d) Can you find the common calculus theme in each of these scenarios?

Activity 7.1.2 Remembering the Fundamental Theorem of Calculus.

We want to think about generalizing our notion of integrals a bit. So in this activity, section, we're going to think about some of the requirements for the Fundamental Theorem of Calculus and try to loosen them up a bit to see what happens. We'll try to construct meaningful approaches to these situations that fit our overall goals of calculating area under a curve.
This practice, in general, is a really good and common mathematical process: taking some result and playing with the requirements or assumptions to see what else can happen. So it might feel like we're just fiddling with the "What if?" questions, but what we're actually doing is good mathematics!

(a) What does the Fundamental Theorem of Calculus say about evaluating the definite integral $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$?

(b) What do we need to be true about our setup, our function, etc. for us to be able to apply this technique to evaluate $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$?

We are going to introduce the idea of "Improper Integrals" as kind-of-but-not-quite definite integrals that we can evaluate. They are going to violate the requirements for the Fundamental Theorem of Calculus, but we'll work to salvage them in meaningful ways.

Activity 7.1.3 Approximating Improper Integrals.

In this activity, we're going to look at two improper integrals:

1. $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\, dx$
2. $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$

(a) First, let's just clarify to ourselves what it means for an integral to be improper. Why are each of these integrals improper? Be specific!

(b) Let's focus on $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\, dx$ first. We're going to look at the slightly different integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $t > 2$, then our function is continuous and bounded on $[2, t]$, and so we can evaluate this integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$.

Find and antiderivative, $F(x)$.

(c) Now we're going to evaluate some areas for different values of t. Use your antiderivative $F(x)$ from above!

- Let's start with making $t = 99$. So we're going to evaluate:

$$\int_{x=2}^{x=99} \frac{1}{(x+1)^2}\, dx = F(99) - F(2)$$

- Now let $t = 999$. Evaluate:

$$\int_{x=2}^{x=999} \frac{1}{(x+1)^2}\, dx = F(999) - F(2)$$

- Now let $t = 9999$. Evaluate:

$$\int_{x=2}^{x=9999} \frac{1}{(x+1)^2}\, dx = F(9999) - F(2)$$

(d) Based on what you found, what do you *think* is happening when $t \to \infty$ to the definite integral

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\, dx = F(t) - F(2)?$$

(e) Ok, we're going to switch our focus to the other improper integral, $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\, dx$. again, we'll look at a slightly different integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx.$$

As long as t is some real number with $-1 < t < 2$, then our function is continuous and bounded on $[t, 2]$, and so we can evaluate this integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$. We can use the same antiderivative as before!

We're going to evaluate this intergal for different values of t again, but this time we'll use values that are close to -1, but slightly bigger, since we want to be in the interval $[-1, 2]$.

- Let's start with making $t = -\frac{9}{10}$. So we're going to evaluate:

$$\int_{x=-9/10}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{9}{10}\right)$$

- Now let $t = -\frac{99}{100}$. Evaluate:

$$\int_{x=-99/100}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{99}{100}\right)$$

- Now let $t = -\frac{999}{1000}$. Evaluate:

$$\int_{x=-999/1000}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F\left(-\frac{999}{1000}\right)$$

(f) Based on what you found, what do you *think* is happening when $t \to -1^+$ to the definite integral

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\, dx = F(2) - F(t)?$$

We can think about putting this a bit more generally into limit notation, but we'll get to this later.

7.2 More on u-Substitution

Activity 7.2.1 Recapping u-Substitution.

We're going to consider a few integrals, and work through each of the questions for all integrals.

1. $\displaystyle\int \left(\frac{1}{\sqrt{4-3x}}\right)\, dx$
2. $\displaystyle\int x^2(5+x^3)^7\, dx$
3. $\displaystyle\int \left(\frac{\sin(x)}{\cos(x)}\right)\, dx$
4. $\displaystyle\int \left(xe^{-x^2}\right)\, dx$

(a) For each integral, explain why u-substitution is a good choice. How can you tell, just by looking at the integral, that this strategy will be a reasonable thing to try?

(b) For each integral, explain your choice of u, and what that means for how we define du.

(c) For each integral, is your definition of du present in the integrand function? How do you go about making this substitution when the integrand function isn't set up perfectly?

(d) Finish the substitution and integration, and substitute back to the original variable.

Activity 7.2.2 Turn Around Problems.

The two integrals that we're going to look at are "just" some u-substitution problems, but I like to call integrals like these **turn-around** problems. We'll see why!

(a) Consider the integral:

$$\int x\sqrt[3]{x+5}\,dx.$$

First, explain why u-substitution is reasonable here.

(b) Identify du for your chosen u-substitution. When you substitute, you should notice that there are some extra bits in this integrand function that have not been assigned to be translated over to be written in terms of u. Which parts?

(c) We need to think about how to write x in terms of u. Luckily, we already have everything we need! We have defined a link between the x variable and the u variable. We defined it as u being written as some function of x, but can we "turn around" that link to write x in terms of u?

(d) Substitute the integral to be fully written in terms of u.

(e) Before antidifferentiating, compare this integral with the original one. Specifically thinking about how we might multiply, describe the differences between the integrals with regard to composition and rewriting our integrand.

Then, go ahead and use this nicely rewritten version to antidifferentiate and substitute back to x.

(f) Apply this same strategy to the following integral:

$$\int \frac{x}{x+5}\,dx.$$

This integral might be a bit trickier to find the composition in order to identify the u-substitution! Give some things a try!

(g) Compare your integral in terms of x with the substituted version, in terms of u. Why was the second one so much easier to think about or rewrite?

7.3 Manipulating Integrands

Activity 7.3.1 A Negative Exponent.

Let's think about this integral:

$$\int \frac{1}{1+e^{-x}}\,dx.$$

(a) Is there any composition in this integral? Pick it out, and either explain or show that using this to guide your substitution will not be helpful.

(b) What does e^{-x} mean? What does $\frac{1}{e^{-x}}$ mean?

(c) Rewrite the integral, specifically focusing on the negative exponent. You should find that the function looks worse! How can you clean that up?

(d) Why is this new integral set up so much better for the purpose of u-substitution? How could we tell this just by looking at the initial integral?

Activity 7.3.2 Integrating a Rational Function Three Ways.

We're going to think about the integral:

$$\int \left(\frac{x^2+3x-1}{x-1}\right)\, dx.$$

Let's find 3 different ways of integrating this. This is kind of misleading, since we're actually going to look at 2, since we've already used u-substitution to integrate this in Example 7.2.1.

(a) Let's just notice some things about this rational function.

- Are there any vertical asymptotes? How do you know where to find them?
- Are there any horizontal asymptotes? How do you know that there *aren't*?
- When you zoom really far out on the graph of this function, it looks like a different kind of function. What kind of function? Why is that?

(b) Now we're going to rewrite the function itself: $\dfrac{x^2+3x-1}{x-1}$ means we're dividing (x^2+3x-1) by $(x-1)$. So let's do the division!

$$x-1 \;\overline{\big)\; x^2 \quad +3x \quad -1}$$

(c) Rewrite your integral using this new version of the function. Notice that we haven't done any calculus or antidifferentiating yet. Explain why this new version of this integrand function is easier to antidifferentiate. What do you get?

(d) Let's approach this integral differently. We said earlier that this function is really an "almost" linear function in disguise: when we divide the quadratic numerator by a linear denominator, we expect a linear function to be left over. In the long division, we saw this happen! We ended with a linear function and some remainder.

Let's try to uncover this linear function. If we're looking to find what linear functions multiply together to get (x^2+3x-1), then we can try factoring!

$$\frac{x^2+3x-1}{x-1} = \frac{(\qquad\qquad)(\qquad\qquad)}{x-1}$$

In order for this factoring to be useful, we want to be able to "cancel" out the $x-1$ factor in the denominator. We're really only interested in what linear factor will multiply by $(x-1)$ to get (x^2+3x-1).

$$\frac{x^2+3x-1}{x-1} = \frac{(x-1)(\qquad\qquad)}{x-1}$$

First, explain why there is no linear function factor that accomplishes this.

(e) What if we're able to "almost" factor this?

If there *was* a linear factor that multiplied by $(x-1)$ to get (x^2+3x-1), then the linear portions would multiply together to get x^2. What does this mean about the first linear term of our factor?

(f) What does the constant term of our missing factor need to be? We are hoping that whatever it is can multiply by x (from $x-1$) and combine with the $-x$ (from the constant -1 multiplied by x in our missing factor) to match the $3x$ in x^2+3x-1.

What is it?

(g) Note that we have *not* factored $(x^2 + 3x - 1)$! We *almost* did: we found two factors:

$$(x-1)(\qquad\qquad) = x^2 + 3x + \qquad.$$

How far off is the actual polynomial that we are working with, $x^2 + 3x - 1$?

Write $x^2 + 3x - 1$ as your two factors, plus or minus some remainder.

(h) You should get the same thing that we got from using long division! Great! The rest of the integral will work the same.

Before we end, though, compare this antiderivative to the one we got in Example 7.2.1. It's different. Why? Is this a problem?

Activity 7.3.3 Comparing Two Very Similar Integrals.

We're going to compare these two integrals:

$$\int \frac{x+2}{x^2+4x+5}\,dx \qquad\qquad \int \frac{2}{x^2+4x+5}\,dx$$

(a) Describe why $u = x^2+4x+5$ is such a useful choice for the first integral, but not for the second. How do the differences in these two integrals influence this substitution, even though the denominators are the same?

(b) Why would it be useful to have a *linear* substitution rule (instead of the *quadratic* one that we picked) for the second integral? Why would that match the structure of the numerator better?

Go ahead and integrate the first integral.

(c) We're going to write the denominator, x^2+4x+5 in a different way, in order to get a linear function composed into something familiar.

Complete the square for this polynomial: that is, find some linear factor $(x+k)$ and a real number b such that $(x+h)^2+b = x^2+4x+3$. This should feel familiar, since we already haves tried to force polynomials to factor cleverly in Activity 7.3.2.

(d) There is an intuitive substitution to pick, since we now have more obvious composition. Pick it. What kind of integral do we end up with and how do we antidifferentiate? Complete this problem!

7.4 Integration By Parts

Activity 7.4.1 Discovering the Integration by Parts Formula.

The product rule for derivatives says that:

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = \boxed{} + \boxed{}.$$

We know that we intend to "undo" the product rule, so let's try to reframe the product rule from a rule about derivatives to a rule about antiderivatives.

(a) Antidifferentiate the product rule by antidifferentiating each side of the equation.

$$\begin{aligned} \int \left(\frac{d}{dx}(u \cdot v) \right) dx &= \int \boxed{} + \boxed{}\, dx \\ \boxed{} &= \int \boxed{}\, dx + \int \boxed{}\, dx \end{aligned}$$

(b) On the right side, we have two integrals. Since each of them has a product of functions (one function and a derivative of another), we can isolate one of them in this equation and create a formula for how to antidifferentiate a product of functions! Solve for $\int uv'\, dx$.

(c) Look back at this formula for $\int uv'\, dx$. Explain how this is really the product rule for derivatives (without just undoing all of the steps we have just done).

Activity 7.4.2 Picking the Parts for Integration by Parts.

Let's consider the integral:

$$\int x \sin(x)\, dx.$$

We'll investigate how to set up the integration by parts formula with the different choices for the parts.

(a) We'll start with selecting $u = x$ and $dv = \sin(x)\, dx$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= x & v &= \underline{\qquad} \\ du &= \underline{\qquad} & dv &= \sin(x)\, dx \end{aligned}$$

(b) Now, set up the integration by parts formula using your labeled pieces. Notice that the integration by parts formula gives us another integral. Don't worry about antidifferentiating this yet, let's just set up the pieces.

(c) Let's swap the pieces and try the setup with $u = \sin(x)$ and $dv = x\, dx$. Fill in the following with the rest of the pieces:

$$\begin{aligned} u &= \sin(x) & v &= \underline{\qquad} \\ du &= \underline{\qquad} & dv &= x\, dx \end{aligned}$$

(d) Now, set up the integration by parts formula using this setup.

(e) Compare the two results we have. Which setup do you think will be easier to move forward with? Why?

(f) Finalize your work with the setup you have chosen to find $\int x \sin(x)\, dx$.

What made things so much better when we chose $u = x$ compared to $dv = x\, dx$? We know that the new integral from our integration by parts formula will be built from the new pieces, the derivative we find from u and the antiderivative we pick from dv. So when we differentiate $u = x$, we get a constant, compared to antidifferentiating $dv = x\, dx$ and getting another power function, but with a larger exponent. We know this will be combined with a $\cos(x)$ function no matter what (since the derivative and antiderivatives of $\sin(x)$ only will differ in their sign). So picking the version that gets that second integral to be built from a trig function and a constant is going to be much nicer than a trig function and a power function. It was nice to pick x to be the piece that we found the derivative of!

Activity 7.4.3 Picking the Parts for Integration by Parts.

This time we'll look at a very similar integral:

$$\int x \ln(x)\, dx.$$

Again, we'll set this up two different ways and compare them.

(a) We'll start with selecting $u = x$ and $dv = \ln(x)\, dx$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = x & v = \underline{\hspace{2cm}} \\ du = \underline{\hspace{2cm}} & dv = \ln(x)\, dx \end{array}$$

(b) Ok, so here we *have to* swap the pieces and try the setup with $u = \ln(x)$ and $dv = x\, dx$, since we only know how to differentiate $\ln(x)$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = \ln(x) & v = \underline{\hspace{2cm}} \\ du = \underline{\hspace{2cm}} & dv = x\, dx \end{array}$$

(c) Now set up the integration by parts formula using this setup.

(d) Why was it fine for us to antidifferentiate x in this example, but not in Activity 7.4.2?

(e) Finish this work to find $\int x \ln(x)\, dx$.

So here, we didn't actually get much choice. We couldn't pick $u = x$ in order to differentiate it (and get a constant to multiply into our second integral) since we don't know how to antidifferentiate $\ln(x)$ (yet: once we know how, it might be fun to come back to this problem and try it again with the parts flipped). But we can also notice that it ended up being fine to antidifferentiate x: the increased power from our power rule didn't really matter much when we combined it with the derivative of the logarithm, since the derivative of the log is *also a power function*! So we were able to combine those easily and actually integrate that second integral.

Activity 7.4.4 Squared Trig Functions.

Let's look at two integrals. We'll talk about both at the same time, since they're similar.

$$\int \sin^2(x)\,dx \qquad\qquad \int \cos^2(x)\,dx$$

(a) What does it mean to "square" a trig function? Write these integrals in a different way, where the meaning of the "squared" exponent is more clear. What do you notice about the structure of these integrals, the operation in the integrand function? What does this mean about our choice of integration technique?

(b) If you were to use integration by parts on these integrals, does your choice of u and dv even matter here? Why not?

(c) Apply the integration by parts formula to each. What do you notice?

(d) Instead of applying another round of integration by parts to the resulting integral, use the Pythagorean identities to rewrite these integrals:

$$\sin^2(x) = 1 - \cos^2(x)$$
$$\cos^2(x) = 1 - \sin^2(x)$$

(e) You should notice that, in your equation for the integration of $\sin^2(x)\,dx$, you have another copy of $\int \sin^2(x)\,dx$. Similarly, in your equation for the integration of $\cos^2(x)\,dx$, you have another copy of $\int \cos^2(x)\,dx$.

Replace these integrals with a variable, like I (for "integral"). Can you "solve" for this variable (integral)?

7.5 Integrating Powers of Trigonometric Functions

Activity 7.5.1 Compare and Contrast.

Let's do a quick comparison of two integrals, keeping the above examples in mind. Consider these two integrals:

$$\int \sin^4(x)\cos(x)\ dx \qquad\qquad \int \sin^4(x)\cos^3(x)\ dx$$

(a) Consider the first integral, $\displaystyle\int \sin^4(x)\cos(x)\ dx$. Think about and set up a good technique for antidifferentiating. Without actually solving the integral, explain why this technique will work.

(b) Now consider the second integral, $\displaystyle\int \sin^4(x)\cos^3(x)\ dx$. Does the same integration strategy work here? What happens when you apply the same thing?

(c) We know that $\sin(x)$ and $\cos(x)$ are related to each other through derivatives (each is the derivative of the other, up to a negative). Is there some other connection that we have between these functions? We might especially notice that we have a $\cos^2(x)$ left over in our integral. Can we write this in terms of $\sin(x)$, so that we can write it in terms of u?

(d) Why would this strategy not have worked if we were looking at the integrals $\int \sin^4(x)\cos^2(x)\ dx$ or $\displaystyle\int \sin^4(x)\cos^4(x)\ dx$? What, specifically, did we need in order to use this combination of substitution and trigonometric identity to solve the integral?

Activity 7.5.2 Compare and Contrast (Again).

We're going to do another Compare and Contrast, but this time we're only going to consider one integral:

$$\int \sec^4(x)\tan^3(x)\,dx.$$

We're going to employ another strategy, similar to the one for Integrating Powers of Sine and Cosine.

(a) Before you start thinking about this integral, let's build the relevant version of the Pythagorean Identity that we'll use. Our standard version of this is:

$$\sin^2(x)+\cos^2(x)=1.$$

Since we want a version that connects $\tan(x)$, which also is written as $\frac{\sin(x)}{\cos(x)}$, with $\sec(x)$, or $\frac{1}{\cos(x)}$, let's divide everything in the Pythagorean Identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{}+\frac{\cos^2(x)}{}=\frac{1}{}$$

$$\underline{}+\underline{}=\underline{}$$

(b) Now start with the integral. We're going to use two different processes here, two different u-substitutions. First, set $u=\tan(x)$. Complete the substitution and solve the integral.

(c) Now try the integral again, this time using $u=\sec(x)$ as your substitution.

(d) For each of these integrals, why were the exponents set up *just right* for u-substitution each time? How does the structure of the derivatives of each function play into this?

(e) Which substitution would be best for the integral $\int \sec^4(x)\tan^4(x)\,dx$. Why?

(f) Which substitution would be best for the integral $\int \sec^3(x)\tan^3(x)\,dx$. Why?

7.6 Trigonometric Substitution

Activity 7.6.1 Difference of Squares.

Consider the integral:

$$\int \sqrt{1-x^2}\, dx.$$

(a) First, convince yourself that a normal u-substitution will not be an effective strategy for integration in this case. Why not?

(b) Second, convince yourself that $\sqrt{1-x^2} \neq \sqrt{1} - \sqrt{x^2}$. Why can we not distribute roots across sums and differences like this? When *can* we "distribute" roots across multiple things?

(c) Our goal, then, is to utilize a substitution (using trigonometric functions) to somehow transform this difference of squared terms under the square root into a single product of squared things under the square root.

Which trigonometric identities from our list of them above utilize differences of thing squared, and equate them to a single term?

Can you use the order of the subtraction to help guide which substitution we should use?

(d) When we do a variable substitution in an integral, we are not only finding a way of transforming x to be in terms of some other variable (in this case, θ). We also need to transform the differential, dx. Based on your substitution of $x = T(\theta)$, what is dx?

(e) Perform your substitution! Use your substitution $x = T(\theta)$ and $dx = T'(\theta)\, d\theta$. Note that we have picked this substitution with a very specific goal: we are hoping to notice a Pythagorean Identity.

After you have performed your substitution, apply the relevant Pythagorean Identity to the **radicand**: the bit of our function underneath the radical or root. What integral are we left with (in terms of θ)?

This new integral is something we can antidifferentiate now! We already have done this one in Activity 7.4.4 Squared Trig Functions. So we can end up with:

$$\begin{aligned}\int \sqrt{1-x^2}\, dx &= \int \cos^2(\theta)\, d\theta \\ &= \frac{\theta + \sin(\theta)\cos(\theta)}{2} + C\end{aligned}$$

It is up to us, now, to translate this antiderivative family to be written in terms of x. We can utilize our substitution to do this, but let's first think about how this variable substitution works a bit more.

Activity 7.6.2 Trig Substitution Geoemtry.

We're going to consider three triangles, and we're going to fill in side lengths. In each of these, we'll assume that the lengths x and a are real numbers and are positive.

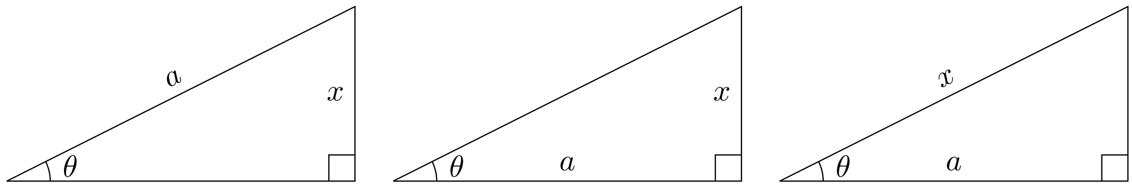


Figure 7.6.1 Three triangles to guide our trigonometric substitutions.

(a) Use the Pythagorean theorem to label the missing side length in each of the three triangles.

(b) For each triangle, explain how you can tell which side length represents the hypotenuse when you see the lengths x, a, and then the missing lengths you found above: $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 + a^2}$.

(c) For each triangle, find a trigonometric function of θ that connects lengths x and a to each other.

Solve each for x to reveal the relevant substitution.

(d) For each substitution, find the corresponding substitution for the differential, dx.

Activity 7.6.3 Practicing Trigonometric Substitution.

Let's look at three integrals, and practice the kind of thinking we'll need to use to apply trigonometric substitution to them.

1. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$
2. $\displaystyle\int \frac{2}{(4-x^2)^{3/2}}\,dx$
3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

For each integral, do the following:

(a) Identify the term (or terms) that signify that trigonometric substitution might be a reasonable strategy.

(b) Use that portion of the integral to compare three side lengths of a triangle. Which one is the largest (and so must represent the length of the hypotenuse)?

(c) Construct the triangle, label an angle θ, and use a trigonometric function to connect the two single-term side lengths. (Feel free to change the angle you label in order to use the sine, secant, or tangent functions instead of their co-functions).

(d) Define your substitution (for both x and the differential dx), and identify the Pythagorean Identity that will be relevant for the integral.

(e) Substitute and antidifferentiate!

(f) Use your triangle to substitute your antiderivative back in terms of x.

7.7 Partial Fractions

Activity 7.7.1 Comparing Rational Integrands.

We're going to compare three integrals:

$$\int \frac{2}{x^2+4x+5}\,dx$$

$$\int \frac{2}{x^2+4x+3}\,dx$$

$$\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right)\,dx$$

(a) Start with the first integral:

$$\int \frac{2}{x^2+4x+5}\,dx.$$

How would you approach integrating this?

(b) Try the same tactic on the second integral:

$$\int \frac{2}{x^2+4x+3}\,dx.$$

You don't need to complete this integral, but think about how you might proceed.

(c) Think about the third integral:

$$\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right)\,dx.$$

How would you integrate this?

(d) The third integral is unique from the other two in that it is has two terms. Let's combine them together to see how we could write this integral to compare it more closely to the other two.

Subtract $\frac{1}{x+1} - \frac{1}{x+3}$ using common denominators and compare your rewritten integral to the other two.

(e) Which of these integrals and/or representations of an integral is easiest to work with? Which one is most annoying to work with? Why?

Activity 7.7.2 First Examples of Partial Fractions.

(a) Consider the integral:

$$\int \frac{6x-16}{x^2-4x+3}\,dx.$$

First, confirm that this would be *very* annoying to try to use u-substitution on, even though we have a linear numerator and quadratic denominator.

(b) Notice that the denominator can be factored:

$$\int \frac{6x-16}{(x-3)(x-1)}\,dx.$$

If this integrand function were a sum of two "smaller" rational functions, what would their denominators be? What do we know about their numerators?

(c) Use some variables (it's typical to use capital letters like A, B, C, etc.) to represent the numerators, and then add the partial fractions together. What do you get? How does this rational function compare to $\dfrac{6x-16}{(x-3)(x-1)}$?

(d) Set up an equation connecting the numerators, and solve for your unknown variables. What are the two fractions that added together to get $\dfrac{6x-16}{(x-3)(x-1)}$?

(e) Antidifferentiate to solve the integral $\displaystyle\int \frac{6x-16}{(x-3)(x-1)}\,dx$.

(f) Let's do the same thing with a new integral:

$$\int \frac{3x^2-2x+3}{x(x^2+1)}\,dx.$$

What are the partial fraction forms? What kinds of numerators should we expect to see? Use variables to represent these.

(g) Add the partial fractions together and set up an equation for the numerators to solve. What are the two partial fractions after you solve for the unknown coefficients?

(h) Antidifferentiate and solve the integral $\displaystyle\int \frac{3x^2-2x+3}{x(x^2+1)}\,dx$.

Chapter 8

Infinite Series

8.1 Introduction to Infinite Sequences

Activity 8.1.1 Building our First Sequences.

We might already have some familiarity with sequences. Here, we'll focus less on some of the detailed mechanics and just think about these sequences as functions.

(a) Describe a sequence of numbers where you use a consistent rule/function to build each term (each number) based only on the *previous term* in the sequence. You will need to decide on some first term to start your sequence.

(b) Describe a different sequence of numbers using the same rule to generate new terms/numbers from the previous one. What do you need to do to make these two sequences different from each other?

(c) Describe a new sequence of numbers where you use a consistent rule/function to build each term based on its position in the sequence (i.e. the first term will be some rule/function based on the input 1, the second will be based on 2, you'll use 3 to get the third term, etc.). We will call the position of each term in the sequence the *index*.

(d) Describe another, new, sequence of numbers where you use a consistent rule/function to build each term based on its index. This time, make the terms get smaller in size as the index increases.

Activity 8.1.2 Returning to our First Sequences.

Let's return back to the four sequences we created in Activity 8.1.1.

(a) For each of the sequences, how are we going to define them? Explicit formulas? Recursion relations? How do you know?

(b) Now, for each sequence, define the sequence formally using either an explicit formula or recursion relation, whichever matches with how you described the sequence in Activity 8.1.1.

Activity 8.1.3 Describing These Sequences.

Let's look at the sequences from Example 8.1.3. Go through the following tasks for each sequence.

(a) What do you think each sequence is "counting towards" (if anything)?

(b) Can you show that the sequence is counting towards what you think it is with a limit (or show that it's not counting towards anything)?

Activity 8.1.4 Write the Sequence Rules.

We'll look at some sequences by writing out the first handful of terms. From there, our goal is to write out more terms and eventually define each sequence fully.
For each sequence, write an explicit formula and a recursion relation to define the sequence. You can choose whether to start your index at $k = 0$ or $k = 1$.

(a) $\{a_k\} = \left\{4, \frac{2}{3}, \frac{1}{9}, \frac{1}{54}, ...\right\}$

(b) $\{b_k\} = \left\{\frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, ...,\right\}$

(c) $\{c_k\} = \left\{\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, ...\right\}$

(d) What kinds of connections do you notice between the explicit formulas and the recursion relations for these sequences?

8.2 Introduction to Infinite Series

Activity 8.2.1 How Do We Think About Infinite Series?

Let's consider the following sequence:

$$\left\{\frac{9}{10^k}\right\}_{k=1}^{\infty}$$

(a) White out the first 5 terms of the sequence.

(b) What does this sequence converge to? Show this with a limit!

(c) Now we'll construct a new sequence, this time by adding things up. We're going to be working with the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{k=n}\left(\frac{9}{10^k}\right).$$

Write out the first five terms of this sequence: S_1, S_2, S_3, S_4, S_5.

(d) Can you come up with an explicit formula for S_n?

(e) Does $\{S_n\}$ converge or diverge? Use a limit to find what it converges to!

(f) What do you think this means for the infinite series $\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right)$? Does the infinite series converge or diverge?

This is hopefully a nice little introduction to how we'll think about infinite series: we'll consider, instead, the sequence of sums where we add up more and more terms. This is also a nice first example, because we really just showed that

$$0.999... = 1$$

since

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right) &= \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+...\\ &= 0.9+0.09+0.009+...\\ 1.\end{aligned}$$

But more importantly, we now have a good strategy for thinking about infinite series as sequences of *partial sums*.

8.3 The Divergence Test and the Harmonic Series

Activity 8.3.1 Investigating the Harmonic Series.

(a) Write out the first several terms of the harmonic series, terms from $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$. Write however many you need to get a feel for how the terms work.

(b) Can you find out how many terms you would have to go "into" the series before the term was less than 0.00000001?

(c) Can you do this same kind of thing, no matter how small? For instance, how many terms would you have to go into the series before the term was less than some real number ε where $\varepsilon > 0$?

(d) Remind/explain/convince yourself that what we've really done is show that $\lim_{k\to\infty} \frac{1}{k} = 0$. This isn't a new or terribly interesting fact, but make sure that you understand why the argument above shows this.

(e) Let's do something very similar, but with $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}_{n=1}^{\infty}$, the sequence of partial sums, instead. Write out the first few partial sums. There's no specific number that you *need* to write, but make sure to write enough partial sums to get a feel for how the partial sums work.

(f) Can you find out how many terms you need to add up until the partial sum is larger than 1?

(g) Can you find out how many terms you need to add up until the partial sum is larger than 5?

(h) Can you find out how many terms you need to add up until the partial sum is larger than 10?

(i) Do you think that for any positive number S, we can always find some partial sum $\sum_{k=1}^{n} \frac{1}{k} > S$? What do you think this would mean about

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}?$$

To actually show that for any $S > 0$ we could always find an $n > 1$ where

$$\sum_{k=1}^{n} \frac{1}{k} > S$$

is an extremely difficult task! We will show that the Harmonic Series diverges in a different way, but for now I want us to notice these contradictory results: we have a series whose terms get small, but whose partial sums do not seem to converge.

We have $\frac{1}{k} \to 0$ but it seems like $\lim_{n\to\infty} S_n$ does not exist. Is this behavior special to the Harmonic Series? Is this something we should make note of? Is there some other connection between the terms of a series and the behavior of the partial sums of the series that we need to note?

8.4 The Integral Test

Activity 8.4.1 Integrals and Infinite Series.

We're going to work with a graph of a continuous function, and we're going to start with a couple of conditions:

1. Our function will be continuous wherever it's defined.
2. Our function will be decreasing on its domain.
3. All of the function outputs will be positive.

Let's not worry about picking a specific function for this, but we will visualize a graph of one that meets these three requirements.

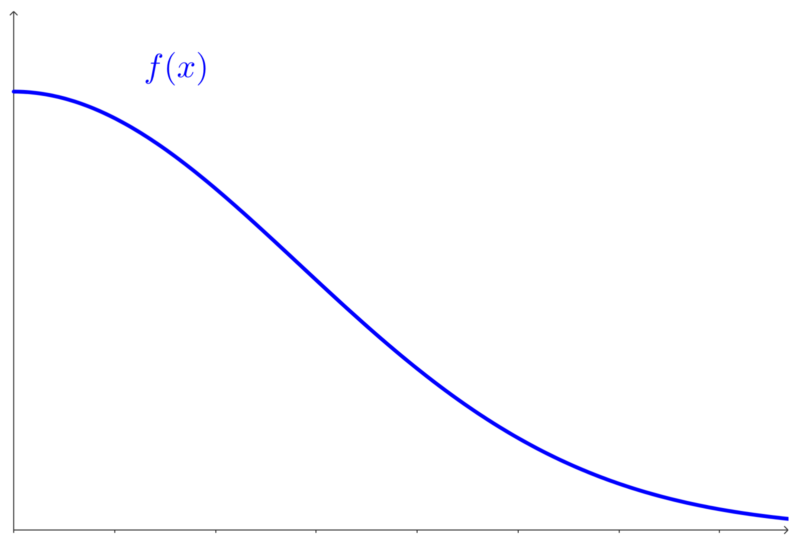


Figure 8.4.1

We can then visualize the sequence of terms, $a_k = f(k)$ for $k = 0, 1, 2, \ldots,$.

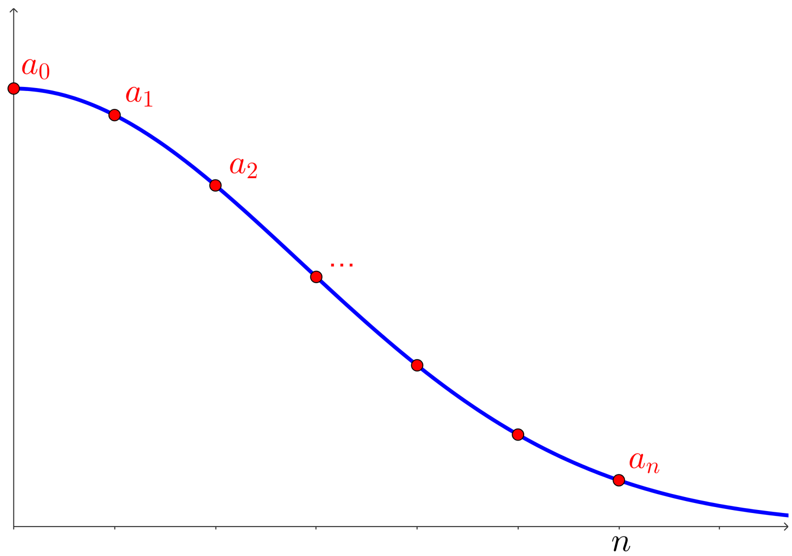


Figure 8.4.2

(a) How does the partial sum, $\sum_{k=0}^{n} a_k$ compare to the Riemann sum for $f(x)$ from $x = 0$ to $x = n$ with n rectangles?

(b) We're going to visualize the accumulation of $f(x)$ from $x = 0$ to $x = n$ by thinking about the integral:

$$\int_{x=0}^{x=n} f(x)\, dx.$$

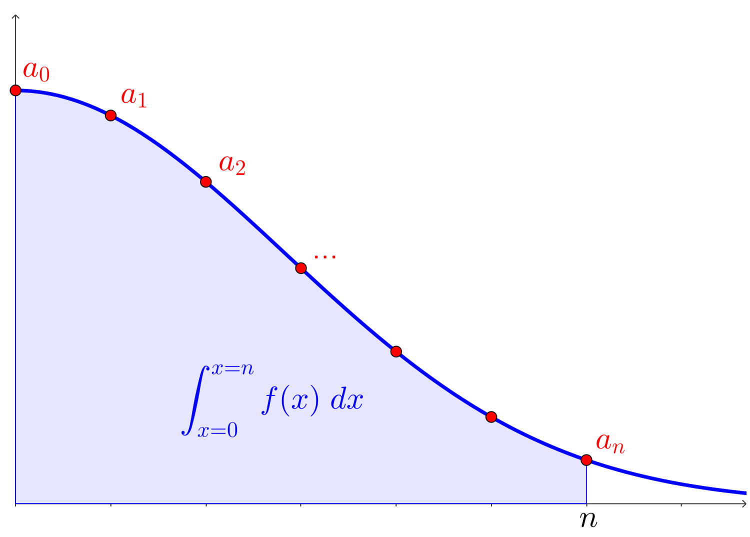


Figure 8.4.3

How does this area compare to the Riemann sum you thought of above? Compare them with an inequality and make sure you can explain why this has to be true.

(c) Remove the first term of the series, a_0, and instead think of the sum $\sum_{k=1}^{n} a_k$. Can you still think of this as a Riemann sum to approximate the area from the integral $\int_{x=0}^{x=n} f(x)\, dx$?

How does this new Riemann sum compare to the area formed by the integral? Compare them with an inequality and make sure you can explain why this has to be true.

(d) We have thought about two sums, and we can connect them:

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k.$$

Use the sums to bound the integral:

$$\underline{\qquad\qquad} < \int_{x=0}^{x=n} f(x)\, dx < \underline{\qquad\qquad}$$

(e) Similarly, use the integral to bound the sum:

$$\underline{\qquad\qquad} < \sum_{k=0}^{n} a_k < \underline{\qquad\qquad}$$

These bounds are going to be super useful! Discovering them is the main task for finding the connections between improper integrals and infinite series. These inequalities might seem kind of strange at first, but we're going to apply a limit to everything as $n \to \infty$, and then think about our definitions of convergence (Definition 7.1.4 and Definition 8.2.2).

8.5 Alternating Series and Conditional Convergence

Activity 8.5.1 Which is More Likely to Converge?

We're going to try to think about what might be different when we analyze an alternating series compared to a series with only positive (or non-negative) terms.
Let's say that $\{a_k\}_{k=0}^{\infty}$ is some sequence of positive real numbers. Now let's consider the two series:

$$\sum_{k=0}^{\infty} a_k \qquad \text{vs} \qquad \sum_{k=0}^{\infty} (-1)^k a_k$$

(a) Let's first consider the sequences of terms: $\{a_k\}$ compared with $\{(-1)^k a_k\}$. Is either of these more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(b) Now let's think of the partial sums:

$$\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty} \qquad \text{vs} \qquad \left\{\sum_{k=0}^{n} (-1)^k a_k\right\}_{n=0}^{\infty}$$

Is either of these sequences more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

(c) Now make a conjecture about which infinite series is more likely to converge:

$$\sum_{k=0}^{\infty} a_k \qquad \text{vs} \qquad \sum_{k=0}^{\infty} (-1)^k a_k$$

Remember that $a_k > 0$ for $k = 0, 1, 2, \ldots$, so the only differences are the changes in sign.

Activity 8.5.2 Approximating an Alternating Series.

Let's look, again, at the picture of the partial sums of an alternating series in Figure 8.5.2. We're going to assume that the series converges, which means that:

- $\lim_{n\to\infty} S_n$ exists.
- $\lim_{n\to\infty} a_n = 0$.

Let's add to our figure.

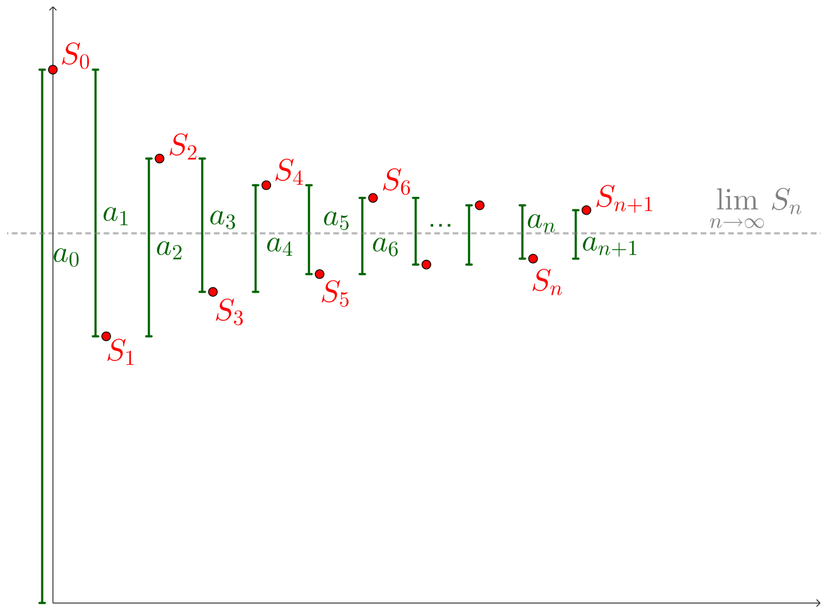


Figure 8.5.1

(a) Why are the even-indexed partial sums sitting above the odd-indexed partial sums?

(b) Why are the even-indexed partial sums sitting above the horizontal line, $\lim_{n\to\infty} S_n$?

(c) Why are the odd-indexed partial sums sitting below the horizontal line, $\lim_{n\to\infty} S_n$?

(d) If we were trying to approximate the value of $\lim_{n\to\infty} S_n$, how can we use the partial sums to build an interval that approximates the value?

Activity 8.5.3 The Alternating Harmonic Series Converges.

Let's consider the alternating harmonic series, as written below:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + -...$$

(a) First, confirm that this series does converge, using the Alternating Series Test.

(b) Find $S_6 = \sum_{k=1}^{k=6} \frac{(-1)^{k+1}}{k}$.

(c) Use Theorem 8.5.5 Approximations of Alternating Series to create a bound on this estimation. Report an interval that you know that the actual value that the alternating harmonic series converges to is in.

(d) Use technology to find S_{1000}. Compare this value to $\ln(2)$. Would it surprise you for someone to claim that the alternating harmonic series converges to $\ln(2)$?

Activity 8.5.4 The Alternating Harmonic Series Converges (Again).

Now let's consider a new, rearrranged, version of the alternating harmonic series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$$

We can write this in summation notation as:

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \right).$$

(a) First, confirm that all of the terms from the alternating harmonic series will eventually show up in this series. Convince yourself that we are truly adding (and subtracting) all of the same values with just a different order.

(b) Does this new series converge? Check, using the Alternating Series Test.

(c) Add up the first few terms of this series to find the value of a partial sum. You can choose how many terms you add. Does it look like it will also converge to $\ln(2)$?

(d) Use technology to add up many terms of this series. Can you convince yourself what this series converges to?

8.6 Common Series Types

Activity 8.6.1 Building a Convergence Formula for Geometric Series.

We're going to think of constructing two different ways of thinking about how much area of of a circle has been shaded. We can pretend we have a circle with area that is 1, where the radius is $r = \sqrt{\frac{1}{\pi}}$, giving

$$A = \pi \left(\sqrt{\frac{1}{\pi}}\right)^2 = 1.$$

Then we can describe the areas we're looking at as almost a percentages of the total area.

(a) We are going to split our circle into two parts, with r amount of the area left unshaded and so $1 - r$ area shaded. We'll shade in some angular sector.

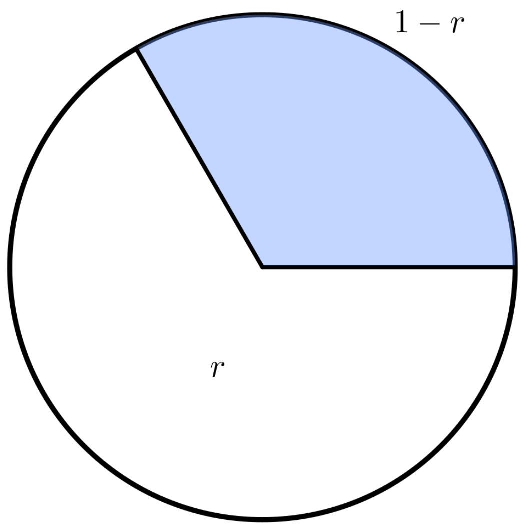


Figure 8.6.1

This part is easy: how much of the area is shaded?

(b) This next step will set the stage for how we think about this problem now: we're going to divide the remaining white area up into the same proportional pieces: we'll shade in a ratio of $1 - r$ of the remaining white space and leave a ratio of r of the white space unshaded.

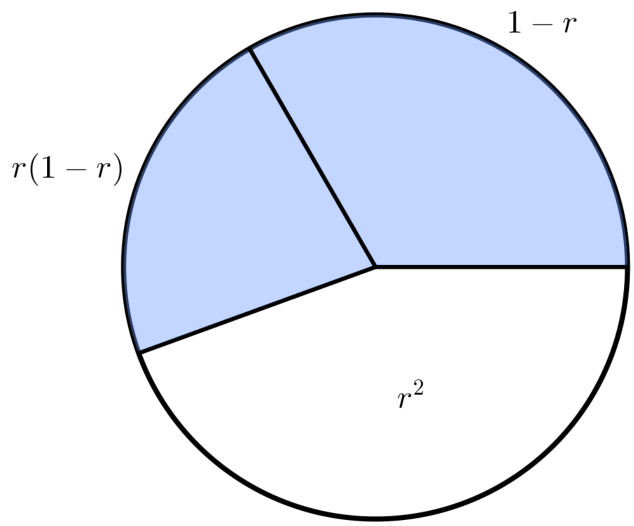


Figure 8.6.2

Can you describe two ways of calculating the total amount of shaded area?

(c) We'll repeat the process: shade in more, where the ratio of shaded area to unshaded area is $1-r$ to r.

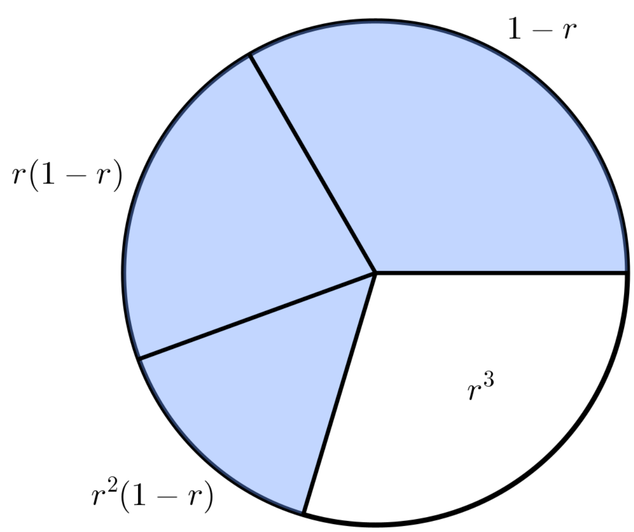


Figure 8.6.3

Can you describe two ways of calculating the total amount of shaded area?

(d) Now we're going to repeat this process until we've done it a total of $n+1$ times.

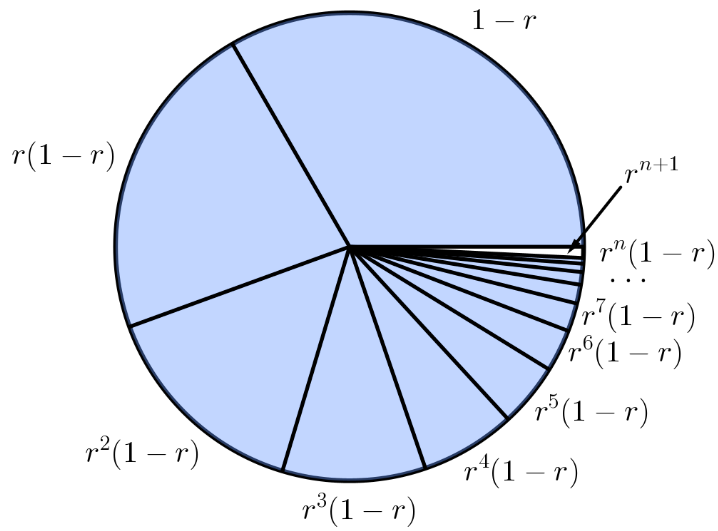


Figure 8.6.4

Can you describe two ways of calculating the total amount of shaded area?

(e) In the limit as $n \to \infty$, how much of the area is shaded in?

Notice that $(1-r)$ is likely a common factor in one of your ways of calculating this area. Convince yourself, then, that:

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{n} r^k &= \lim_{n\to\infty}(1+r+r^2+...+r^n)\\ &= \frac{1}{1-r}\end{aligned}$$

Activity 8.6.2 (Im)Possible Combinations.

When we have thought about infinite series, we have thought about three different mathematical objects: the sequence of terms of the series, the sequence of partial sums of the series, and the infinite series itself. As a reminder, if we had an infinite series

$$\sum_{k=1}^{\infty} a_k$$

we can say that:

- $\{a_k\}_{k=1}^{\infty}$ is the sequence of terms of the series
- $S_n = \sum_{k=1}^{n} a_k$ is a partial sum and $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series

For each of these three objects—the terms, the partial sums, and the series—we have some notion of what it means for that object to converge or diverge.

Consider the following table of all of the different combinations of convergence and divergence of the three objects. For each combination, decide whether this combination is possible or impossible. If it is possible, give an example of an infinite series whose terms, partial sums, and the series itself converge/diverge appropriately. If it is impossible, give an explanation of why.

Table 8.6.5 (Im)Possible Combinations

$\{a_k\}_{k=1}^{\infty}$	$\{S_n\}_{n=1}^{\infty}$	$\sum_{k=1}^{\infty} a_k$	**(Im)Possible?**	**Example or Explanation**
Converges	Converges	Converges		
Converges	Converges	Diverges		
Converges	Diverges	Converges		
Converges	Diverges	Diverges		
Diverges	Converges	Converges		
Diverges	Converges	Diverges		
Diverges	Diverges	Converges		
Diverges	Diverges	Diverges		

8.7 Comparison Tests

Activity 8.7.1 Comparisons to Bound Partial Sums.

This activity is mostly going to be thinking about proof mechanisms, and so it might be helpful to review Activity 8.4.1 Integrals and Infinite Series. If you want to see more, then the proof of Integral Test will provide some further details on why the inequalities we built were useful.

(a) In the Integral Test, how did we guarantee that our sequence of partial sums was monotonic?

(b) How did we know that, as long as the corresponding integral converged, then our sequence of partial sums was bounded?

(c) How did we know that, as long as our integral diverged, then our sequence of partial sums had to diverge as well?

(d) What happens if we swap out the integral we're connecting our series to with a different series?

For these inequalities to be useful, what do we need to be true?

Activity 8.7.2 (Un)Helpful Comparisons.

We're going to consider a handful of infinite series, here:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(3^k)}$
2. $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k+1}}{k-2}$

(a) Pick a series that is reasonable to use as a comparison for each of the series listed. Remember, we want:

- A series that is recognizable (probably a Geometric Series or a p-Series), so that we know the behavior of it: we need to know whether the series we're comparing to converges or diverges!
- A series that is similar enough to the series we're working with that we can construct an inequality comparing the term structure. It can be hard to compare functions that are seemingly unrelated to each other!
- A series that has terms that are either larger or smaller than our series, depending on whether we are trying to show that our series converges or diverges.

(b) Build the comparison from the series we start with to the one you picked. What kinds of conclusions can you make?

(c) We're going to change the series we're considering to two slightly different series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{3^k}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k-1}}{k+2}$

How do these small changes impact the inequalities you built?

(d) How do these changes in the inequalities change the conclusions we can draw from the Direct Comparison Test?

(e) What do you *think* is happening with these series: do you think that these small changes are enough to change the behavior of the series (i.e. whether it converges or diverges)?

Activity 8.7.3 Ratios for Comparison.

Let's start with some functions: we'll consider $f(x)$ and $g(x)$ as two functions that are continuous when $x \geq 0$ with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

All of this is so that we can think about $\dfrac{f(x)}{g(x)}$ and know that we have an indeterminate form. We could put the requirement of differentiability on these functions (so that we could think about L'Hôpital's Rule), but we don't need to do that.

We're going to now consider the limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

(a) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $g(x) \to 0$ with a faster growth rate than $f(x)$ does? In this case, we might say that:

$$f(x) >> g(x).$$

(b) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $f(x) \to 0$ with a faster growth rate than $g(x)$ does? In this case, we might say that:

$$g(x) >> f(x).$$

(c) If the functions $f(x)$ and $g(x)$ eventually act equivalently, then what does the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

(d) If the function $f(x)$ eventually acts like some scaled version of $g(x)$, then what does the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

8.8 The Ratio and Root Tests

Activity 8.8.1 Reminder about Geometric Series.

We are going to build some convergence tests that try to link some infinite series to the family of geometric series and show that even though a series is *not* geometric, it might act enough like one to be considered "eventually geometric-ish."

But first, what does it mean for a series to be a geometric series?

(a) Describe a defining characteristic of a geometric series. What makes it geometric?

(b) Can you describe this characteristic in another way? For instance, if you described a geometric series using a characteristic about the Explicit Formula, can you describe the same characteristic in the context of the Recursion Relation instead? Or vice versa?

(c) Write out a generalized and simplified form of the term a_k of a geometric series explicitly and recursively. In each case, solve for r, the ratio between terms.

Activity 8.8.2 When Are These Tests Useful?

We're going to look at a couple of small examples where we can rewrite some expressions into friendlier forms, and try to connect these rewriting strategies to the Ratio and Root Tests.

(a) Rewrite the following expression into a friendlier form. Explain why this new form is friendlier.

$$\sqrt[k]{\frac{2^{k+1}}{7^{3k}}}$$

(b) Rewrite the following expression into a friendlier form. Explain why this new form is friendlier.

$$\sqrt[k]{\frac{(k+1)^k}{4^{2k+1}}}$$

(c) Rewrite the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{(5^{k+2})(6^{k-3})}{(5^{k-1})(6^{k+1})}$$

(d) Rewrite the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{103!}{99!}$$

(e) Rewrite the following expression into a friendlier form. Explain why this new form is friendlier.

$$\frac{(2k+4)!}{(2k+2)!}$$

(f) Why do you think the Ratio Test especially will be useful for series whose terms include factorials and exponentials?

Why do you think the Root Test will be useful for series whose terms include exponentials and functions raised to functions (of k)?

Chapter 9

Power Series

9.1 Polynomial Approximations of Functions

Activity 9.1.1 Build a Polynomial.

We're going to use the formula in Definition 9.1.1 to construct two different polynomials that approximate two different approximations. Then, we'll use them to approximate things!

(a) We're going to start with approximating the function $f(x) = \sin(x)$ centered at $x = 0$. Let's choose to look at a 5th degree polynomial.

This means we'll need to find the first five derivatives of $f(x) = \sin(x)$. Then, we'll evaluate our function and the five derivatives at the center. After that, we can divide by the relevant factorial in order to create the coefficients of our polynomial.

Fill out the following chart to produce these coefficients.

Table 9.1.1 Coefficients for Polynomial Approximation

k	$f^{(k)}(x)$	$f^{(k)}(a)$	$\frac{f^{(k)}(a)}{k!}$
$k=0$	$f(x) = \sin(x)$	$f(0) =$	
$k=1$	$f'(x) =$	$f'(0) =$	
$k=2$	$f''(x) =$	$f''(0) =$	
$k=3$	$f'''(x) =$	$f'''(0) =$	
$k=4$	$f^{(4)}(x) =$	$f^{(4)}(0) =$	
$k=5$	$f^{(5)}(x) =$	$f^{(5)}(0) =$	

(b) Now we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x - a)^k$ for $k = 0, 1, ..., 5$. These (added together) will form your polynomial, $p_5(x)$.

(c) Approximate $f(1) = \sin(1)$ using your polynomial.

(d) Let's repeat this for another function. Let's build a 5th degree polynomial approximation of $g(x) = e^x$ centered at $x = 0$. We can construct the coefficients in the same way.

Table 9.1.2 Coefficients for Polynomial Approximation

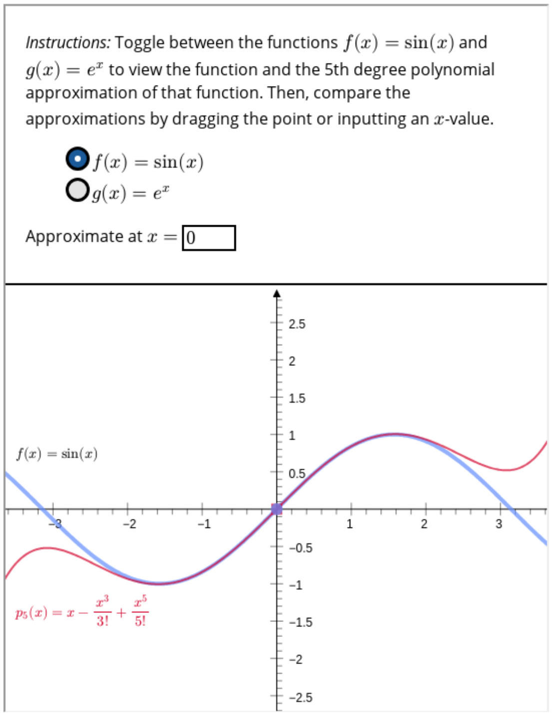
k	$g^{(k)}(x)$		$g^{(k)}(a)$		$\frac{g^{(k)}(a)}{k!}$
$k=0$	$g(x)=e^x$		$g(0)=$		
$k=1$	$g'(x)=$		$g'(0)=$		
$k=2$	$g''(x)=$		$g''(0)=$		
$k=3$	$g'''(x)=$		$g'''(0)=$		
$k=4$	$g^{(4)}(x)=$		$g^{(4)}(0)=$		
$k=5$	$g^{(5)}(x)=$		$g^{(5)}(0)=$		

(e) And now, again, we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x-a)^k$ for $k=0,1,...,5$. These (added together) will form your polynomial, $p_5(x)$.

(f) Approximate $g(-3)=\dfrac{1}{e^3}$ using your polynomial.

Activity 9.1.2 How Good Are Our Approximations?

We're going to think more carefully about our approximations of $\sin(1)$ and e^{-3} from Activity 9.1.1. In order for us to do this, let's visualize the function and the 5th degree polynomial for it.



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(a) How good of a job did the polynomial approximation do when approximating $\sin(1)$? How can you tell, visually?

(b) How good of a job did the polynomial approximation do when approximating e^{-3}? How can you tell, visually?

(c) How does the relationship between the "center" and the x-value that we're approximating at impact the accuracy of our approximation?

(d) How do you think you could make these approximations better (without changing the center)?

Activity 9.1.3 Partial Sums of What?

Let's revisit our 5th degree polynomial approximations from Activity 9.1.1.

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

These approximations work well for x-values that are close to 0, but we will not be more formal than that.

(a) Make a conjecture about what the 7th degree polynomial approximations are for each of these functions.

What about the 15th degree?

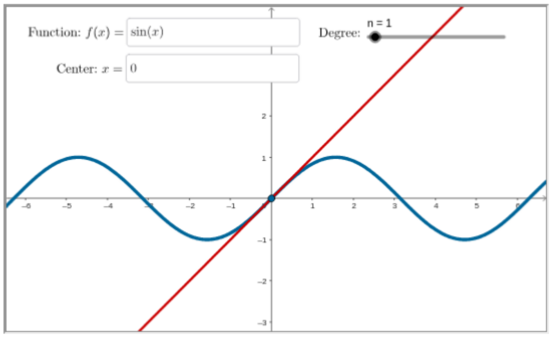
(b) Make a conjecture about what the general formula would be for these terms. If you were to write these out using summation notation, what would they look like?

(c) Why does the polynomial approximation for the sine function only have odd-exponent terms?

(d) Make a conjecture about what a polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$ would be.

Activity 9.1.4 How Do These Polynomials Converge?

We're going to end here by thinking about these polynomials as some partial sum from an infinite series. If there is an infinite series, we should be prepared to think about convergence!
We're going to think about convergence in the same way that we have already: as an end behavior limit of the partial sums. So let's spend our time investigating this end behavior by visualizing polynomial approximations as the degree increases.



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(a) What happens to the polynomial approximation of $\sin(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(b) Does this behavior change if we centered our approximation elsewhere?

(c) What happens to the polynomial approximation of e^x centered at $x = 0$ as the degree $n \to \infty$?

(d) Does this behavior change if we centered our approximation elsewhere?

(e) What happens to the polynomial approximation of $\cos(x)$ centered at $x = 0$ as the degree $n \to \infty$?

(f) Does this behavior change if we centered our approximation elsewhere?

(g) What happens to the polynomial approximation of $\ln(x)$ centered at $x = 2$ as the degree $n \to \infty$?

(h) Describe the difference in what you're seeing with the log function compared to the other functions we've thought about. Describe how the polynomial approximations converge: do they converge to the log function? How? More importantly, *where*?

(i) Does this behavior change if we centered our approximation elsewhere?

9.2 Power Series Convergence

Activity 9.2.1 Polynomial Division.

We're going to do some fiddling with polynomials, and hopefully use this as a bridge to connect how we think of polynomials and power series with how we think about our traditional infinite series and the notions of convergence that we've already built.

(a) We're going to factor some polynomials, but we might end up using some division. First, we'll confirm some factors that we already know.

$$x^2 - 1 = (x-1)(x+1)$$

We'll confirm this by using division.

$$x-1 \,\overline{\big)\, x^2 \qquad -1}$$

(b) Now let's factor $x^3 - 1$. If the factors for this polynomial isn't as familiar, it might be helpful to know that $(x-1)$ is also a factor of $x^3 - 1$.

$$x-1 \,\overline{\big)\, x^3 \qquad\qquad -1}$$

(c) Let's try another one. Complete the following division.

$$x-1 \,\overline{\big)\, x^4 \qquad\qquad\quad -1}$$

(d) Can you generalize this? Find the formula for $\dfrac{x^n-1}{x-1}$ for some positive integer n.

$$x-1 \,\overline{\big)\, x^n \qquad\qquad\quad -1}$$

(e) Now that we have good evidence that

$$\sum_{k=0}^{\infty} x^k = \frac{x^n-1}{x-1},$$

We can apply a limit as $n \to \infty$.

$$\begin{aligned}\sum_{k=0}^{\infty} x^k &= \lim_{n\to\infty} \sum_{k=0}^{n} x^k \\ &= \lim_{n\to\infty} \frac{x^n-1}{x-1}\end{aligned}$$

For what values of x will this limit exist?

(f) Revisit Theorem 8.6.6 Geometric Series Convergence Criteria. Is there any difference for what we've just done compared to this result that we already know?

Activity 9.2.2 Some Power Series and their Convergence.

Let's consider a couple of power series and apply some convergence tests to them in order for us to find out how it might converge.

(a) Consider the power series:

$$\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2}.$$

In order for us to apply the Ratio Test, we'll actually need to consider the positive-term version:

$$\sum_{k=1}^{\infty} \frac{|x-1|^k}{k^2}.$$

Apply the Ratio Test. What do you get in the limit of the ratio between terms?

(b) What kind of result from the Ratio Test guarantees convergence for the series? What are the x-values that guarantee convergence?

(c) The Ratio Test is inconclusive when the limit is equal to 1. What x-values does this happen at? Consider the power series evaluated at each of these x-values. Do these series converge or diverge?

(d) Consider the power series:

$$\sum_{k=1}^{\infty} \frac{(x-1)^k}{\sqrt{k}}.$$

Find the interval of x-values for which this series converges and test the end points of the interval in the same way as earlier. If it differs, explain why.

9.3 How to Build Taylor Series

Activity 9.3.1 Constructing a Taylor Series Directly.

Let's work on this! To be honest, the goal here is to realize how difficult this can be.
Let's consider the function $f(x) = \tan(x)$. We're going to build a Taylor polynomial approximating $f(x)$ centered at $x = 0$.

(a) We're going to build a 4th degree polynomial approximating $f(x) = \tan(x)$. In order to do this, we'll need to find the coefficients for the 5 terms.

Table 9.3.1 Coefficients for Polynomial Approximation

k	$f^{(k)}(x)$		$f^{(k)}(a)$		$\frac{f^{(k)}(a)}{k!}$
$k=0$	$f(x) = \tan(x)$		$f(0) =$		
$k=1$	$f'(x) =$		$f'(0) =$		
$k=2$	$f''(x) =$		$f''(0) =$		
$k=3$	$f'''(x) =$		$f'''(0) =$		
$k=4$	$f^{(4)}(x) =$		$f^{(4)}(0) =$		
$k=5$	$f^{(5)}(x) =$		$f^{(5)}(0) =$		

(b) Now we can use these coefficients to construct the polynomial! These coefficients should all be on power functions in the form $(x - a)^k$ for $k = 0, 1, ..., 5$. These (added together) will form your polynomial, $p_5(x)$.

(c) Can you find a general explicit formula for the terms? If you're having a hard time with this, what could you do to try to make this easier? Why is this difficult?

We're going to stop there. This series is famously difficult to work with. We won't generate the whole Taylor series this way, and this strategy really doesn't work well for functions like this (where the derivatives are pretty annoying to find).

Activity 9.3.2 Connecting to Another Series.

Let's start with some "known" series.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{on } (-1,1)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{on } (-\infty,\infty)$$

We're going to build a Taylor series for a logarithmic function:

$$y = \ln(1+x).$$

(a) In order to start, we're going to find a Taylor series for $g(x) = \dfrac{1}{1+x}$. Which of the "known" series above is this most similar to? How is it different?

(b) Find a Taylor series for $g(x) = \dfrac{1}{1+x}$ by using Theorem 9.2.4. Can you connect this series to the known series by multiplying or composing something?

(c) Notice that the function we're trying to get a Taylor series for, $y = \ln(1+x)$, is an antiderivative of $g(x) = \dfrac{1}{1+x}$. Antidifferentiate the series you found above, using Theorem 9.2.5 Differentiating and Integrating Power Series. We know that at $x = 0$, $y = 0$, so use this information to find specific constant that we need when we antidifferentiate.

(d) What is the interval of convergence for this new series? How does that relate to the interval of convergence of the "known" series you started with?

Activity 9.3.3 Connecting to Yet Another Series.

Here, again, are some *updated* "known" series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{on } (-1,1)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{on } (-\infty,\infty)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} \qquad \text{on } (-1,1]$$

We're going to build a Taylor series for a logarithmic function:

$$y = \tan^{-1}(x).$$

(a) In order to start, we're going to find a Taylor series for $g(x) = \frac{1}{1+x^2}$. Which of the "known" series above is this most similar to? How is it different?

(b) Find a Taylor series for $g(x) = \frac{1}{1+x^2}$ by using Theorem 9.2.4. Can you connect this series to the known series by multiplying or composing something?

(c) Notice that the function we're trying to get a Taylor series for, $y = \tan^{-1}(x)$, is an antiderivative of $g(x) = \frac{1}{1+x^2}$. Antidifferentiate the series you found above, using Theorem 9.2.5 Differentiating and Integrating Power Series. We know that at $x = 0$, $y = 0$, so use this information to find specific constant that we need when we antidifferentiate.

(d) What is the interval of convergence for this new series? How does that relate to the interval of convergence of the "known" series you started with?

9.4 How to Use Taylor Series

Activity 9.4.1 Approximating π and Other Values.

Let's pick a couple of values that are just based on functions with Known Taylor Series and approximate them.

(a) If we note that $\tan^{-1}(1) = \dfrac{\pi}{4}$, then we can say that $\pi = 4\tan^{-1}(1)$.

Find an infinite series that converges to $\tan^{-1}(1)$, and use that to construct a series that converges to π.

(b) If we note that $\dfrac{1}{e} = e^{-1}$, then find an infinite series that converges to $\dfrac{1}{e}$.

(c) Find an infinite series that converges to $\sin(1)$.

(d) Note that each of these three series are alternating series! We can approximate them using a partial sum, and then we can get an error bound for that partial sum by looking at the size of the next term in the infinite series (Theorem 8.5.5 Approximations of Alternating Series).

Approximate the value of each infinite series using a partial sum with the same number of terms. You can pick the number of terms you use. Then, compare the margin of error. Which approximations are most/least accurate? Why do you think that is?

Activity 9.4.2 Integrating using Taylor Series.

Let's consider the definite integral:

$$\int_{x=0}^{x=1} \sin(x^2)\, dx.$$

We know this function is continuous everywhere, and so it is continuous on the interval $[0,1]$. So the Fundamental Theorem of Calculus applies. All we need to do is find an antiderivative, evaluate it at the end-points of the interval, and subtract.
Great! Easy!
Except that we can't write out the family of antiderivatives for this function.
So let's convert over to the Taylor series context and solve our problem there.

(a) Create a Taylor series for the function $f(x) = \sin(x^2)$.

(b) Find the interval of convergence for this Taylor series. Specifically, we want to know if this series converges to $\sin(x^2)$ on the interval $[0,1]$, since that's the interval we're integrating over.

(c) If $f(x) = \sin(x^2)$, find a Taylor series representation of $F(x)$, an antiderivative of $f(x) = \sin(x^2)$.

(d) Evaluate this antidervative at the endpoints and subtract:

$$F(1) - F(0).$$

(e) Use Theorem 8.5.5 Approximations of Alternating Series to approximate the value of $\displaystyle\int_{x=0}^{x=1} \sin(x^2)\, dx$ with a maximum error of 10^{-5}.