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Single-Variable Calculus Topics with Motivating ActivitiesSingle-Variable Differential Calculus Topics with Motivating ActivitiesSingle-Variable Integral Calculus Topics with Motivating ActivitiesAll of the Activities from Each ChapterActivities for Differential Calculus TopicsActivities for Integral Calculus Topics

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Acknowledgements

Disclosure about the Use of AI

This book has been lovingly written by a human.

Me.

Peter Keep.

I have used a lot of different tools, both for inspiration and for actually creating resources for this book. *None* of those tools has involved any form of generative AI.

I could list all of the ways that I think using generative AI in education is, at minimum, problematic. More pointedly, I believe that it is unethical. More broadly, I believe that the use of generative AI for any use-case that I have encountered to be unethical.

In my classes, I try to help students realize the joy and value of working at something and creating something and struggling with something and knowing something. Giving worth to something, even an imperfect thing. Celebrating our accomplishments, even when (especially when?) there is room to grow in those accomplishments. And so I have taken that advice in the creation of this book. I have created a book that is definitely not perfect. I have struggled to write it. There are parts of it that could be (need to be) improved.

But I was the one that created it. I struggled with it. I know it.

I hope that this book can also be a useful tool for others to use, and I have left the copyright to be about as open as possible. Others can take this, use it, can change it, add to it, subtract from it, etc.

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Find something written by a human. Find the words of some other mathematician who tries, maybe imperfectly, to share the ideas of calculus.

Teaching and learning is about humans communicating with each other, and only humans can do that.

Notes for Instructors

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1.1 The Definition of the Limit

1.1.1 Defining a Limit

Activity 1.1.1 Close or Not? We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

1.1.2 Approximating Limits Using Our New Definition

Activity 1.1.2 Approximating Limits. For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim\limits_{x\to 1} f(x)$ using the graph of the function $f(x)$ below.

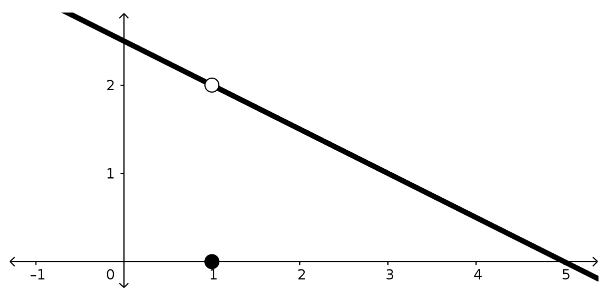


Figure 1.1.5

(b) Approximate $\lim\limits_{x\to 2} g(x)$ using the graph of the function $g(x)$ below.

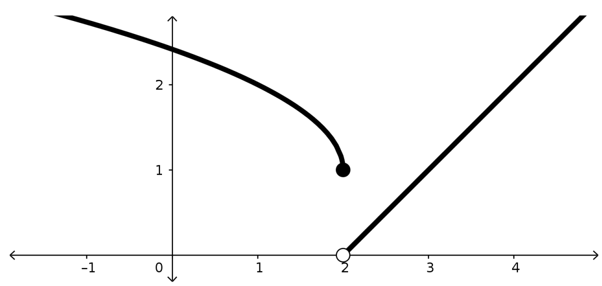


Figure 1.1.6

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim\limits_{x\to -1} h(x)$
- $\lim\limits_{x\to 0} h(x)$
- $\lim\limits_{x\to 2} h(x)$

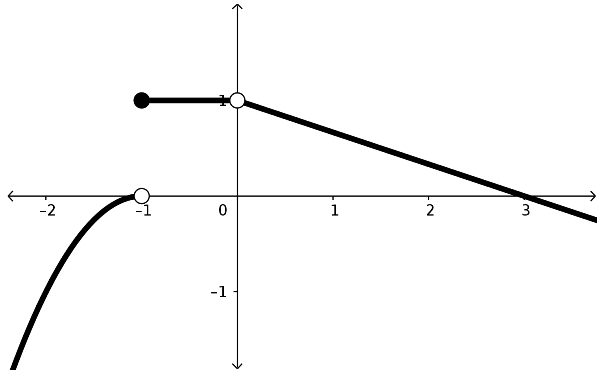


Figure 1.1.7

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.1.3 Approximating Limits Numerically. For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x\to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.1.15

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x\to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.1.16

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x\to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.1.17

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.2 Evaluating Limits

1.2.1 Adding Precision to Our Estimations

Activity 1.2.1 From Estimating to Evaluating Limits (Part 1). Let's consider the following graphs of functions $f(x)$ and $g(x)$.

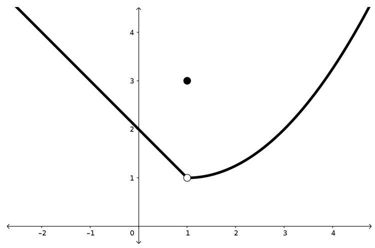


Figure 1.2.1 Graph of the function $f(x)$.

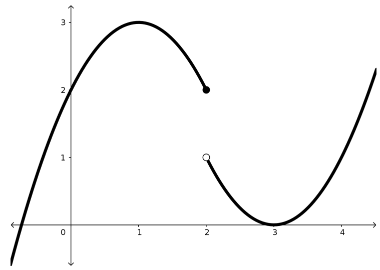


Figure 1.2.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.2.2 From Estimating to Evaluating Limits (Part 2). Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

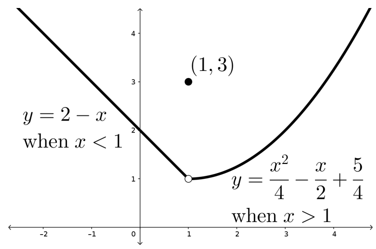


Figure 1.2.3 Graph of the function $f(x)$.

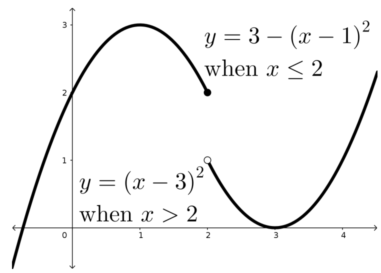


Figure 1.2.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 1^-} f(x)$
- $\lim\limits_{x \to 1^+} f(x)$
- $\lim\limits_{x \to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim\limits_{x \to 2^-} g(x)$
- $\lim\limits_{x \to 2^+} g(x)$
- $\lim\limits_{x \to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

1.2.2 Limit Properties

Activity 1.2.3 Combinations of Functions. We want to remind ourselves how we can combine functions using different operations, and how we might find outputs based on the different combinations. Our goal is to then think about how this might work with limits: how can we summarize the behavior of combinations of functions around some point?

Let's consider some functions $f(x) = x^2 + 3$ and $g(x) = x - \dfrac{1}{x}$. We'll say that the domain of both functions is $(0, \infty)$ for our own convenience.

(a) Let's consider the function $h(x) = f(x) + g(x)$. Describe at least two different ways of finding the value of $h(2)$.

Hint. You might think about writing out a function rule for $h(x)$. But can you also find $h(2)$ without ever writing out a rule for $h(x)$?

Solution.

(a) Since $h(x) = (x^2 + 3) + \left(x - \dfrac{1}{x}\right)$, we can evaluate $h(2)$ by:

$$\begin{aligned} h(2) &= 2^2 + 3 + 2 - \frac{1}{2} \\ &= \frac{17}{2} \end{aligned}$$

(b) Since $h(x) = f(x) + g(x)$, we can evaluate $h(2)$ by:

$$\begin{aligned} f(2) &= 7 \\ g(2) &= \frac{3}{2} \\ h(2) &= f(2) + g(2) \\ &= 7 + \frac{3}{2} \\ &= \frac{17}{2} \end{aligned}$$

(b) If we instead define the function $h(x) = f(x) - g(x)$, how would you describe at least two different ways of finding the value of $h(2)$?

(c) What about a scaled version of one of these functions? If we let $h(x) = 4f(x)$ and $j(x) = \dfrac{g(x)}{3}$, can you describe more than one way to find the value of $h(3)$ and $j(3)$?

(d) You can probably guess where we're going: we're going to define a function that is the product of f and g: $h(x) = f(x) \cdot g(x)$. Describe more than one way of evaluating $h(4)$.

(e) And finally, let's define $h(x) = \dfrac{f(x)}{g(x)}$. Now describe more than one way of finding $h(4)$.

(f) If $h(x) = \dfrac{f(x)}{g(x)}$, then are there any x-values that are in the domain of f and g (the domain is $x > 0$) that $h(x)$ cannot be defined for? Why?

Ok, we can confront this big idea: when we combine functions, we can either evaluate the combination of the functions at some x-value or evaluate each function separately and just combine the answers! Of course, there are some limitations (like when the combination isn't nicely defined because of division by 0 or something else), but this is a good framework to move forward with!

Activity 1.2.4 Limits of Polynomial Functions. We're going to use a combination of properties from Theorem 1.2.5 and Theorem 1.2.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x\to 1} f(x)$. First, use the properties from Theorem 1.2.5 to re-write this limit as 4 different limits added or subtracted together.

Solution.

$$\lim_{x\to 1}(2x^4) - \lim_{x\to 1}(4x^3) + \lim_{x\to 1}\left(\frac{x}{2}\right) - \lim_{x\to 1} 5$$

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.2.6. Evaluate your limits.

Solution.

$$2\left(\lim_{x\to 1} x\right)^4 - 4\left(\lim_{x\to 1} x\right)^3 + \frac{1}{2}\left(\lim_{x\to 1} x\right) - \lim_{x\to 1} 5$$

(c) Based on the definition of a limit (Definition 1.1.1), we normally say that $\lim_{x\to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x\to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x\to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x\to a} p(x)$ is the same value as $p(a)$.

1.3 First Indeterminate Forms

1.3.1 A First Introduction to Indeterminate Forms

Activity 1.3.2

(a) Were going to evaluate $\lim_{x\to 3}\left(\dfrac{x^2-7x+12}{x-3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \dfrac{x^2-7x+12}{x-3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \dfrac{x^2-7x+12}{x-3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3)-4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \dfrac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\displaystyle\lim_{x\to-2}\left(\frac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3-\frac{3}{x+3}}{x^2+2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\displaystyle\lim_{x\to5} f(x) \overset{?}{\to} \frac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

1.3.2 What if There Is No Algebra Trick?

Activity 1.3.3 Let's consider a new limit:

$$\lim_{\theta\to0}\frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

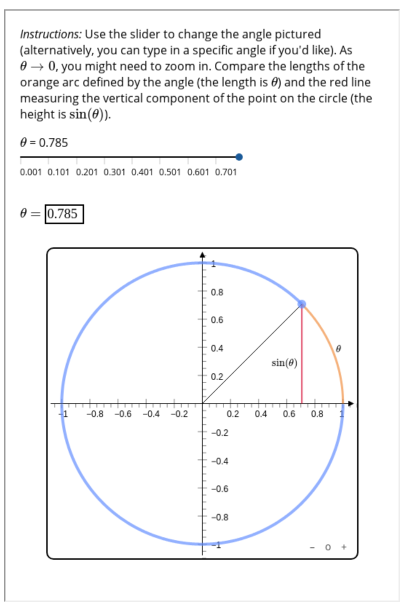
(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

Hint 1. On the unit circle, if we plot some point at an angle of θ, then the coordinates of that point can be represented with trig functions! Which ones?

Hint 2. The length of the curve defining a unit circle is 2π. This also corresponds to the angle we would use to represent moving all the way around the circle. What must the length of the portion of the circle be up to some point at an angle θ?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim\limits_{\theta \to 0} \dfrac{\sin(\theta)}{\theta}$?

1.4 Limits Involving Infinity

1.4.1 Infinite Limits

Activity 1.4.1 What Happens When We Divide by 0? First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.3), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

1.4.2 End Behavior Limits

Activity 1.4.2 What Happens When We Divide by Infinity? Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

1.5 The Squeeze Theorem

Activity 1.5.1 A Weird End Behavior Limit. In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

Hint 1. Start with Theorem 1.2.5 to think about the numerator and denominator separately.

Hint 2. Can you use Theorem 1.4.6 in the denominator?

Hint 3. Is Theorem 1.4.5 relevant?

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\boxed{} \le \sin(x) \le \boxed{}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\boxed{} \le \sin^2(x) \le \boxed{}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\boxed{}}{x^2+1}}_{\text{call this } f(x)} \le \frac{\sin^2(x)}{x^2+1} \le \underbrace{\frac{\boxed{}}{x^2+1}}_{\text{call this } h(x)}$$

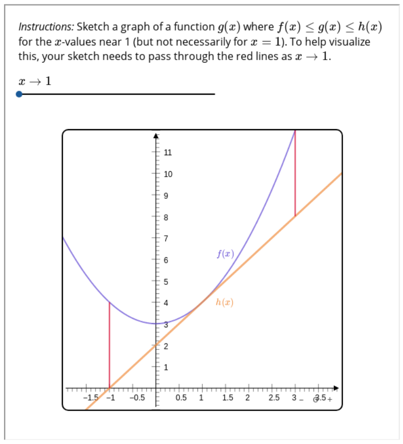
(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim\limits_{x\to\infty}\left(\dfrac{\sin^2(x)}{x^2+1}\right)$?

Activity 1.5.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2+3$ and $h(x) = 2x+2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.



(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x \to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x \to 3} g(x)$ and $\lim_{x \to 0} g(x)$?

Do we know that these limits exist? If they do, do we have information about their values?

2 Derivatives

2.1 Introduction to Derivatives

2.1.1 Defining the Derivative

Activity 2.1.1 Thinking about Slopes. We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ **and...**	**Slope**
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs along side of the curve there instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line. Let's think about a new function, $g(x) = \sqrt{2 - x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

2.1.2 Calculating a Bunch of Slopes at Once

Activity 2.1.3 Calculating a Bunch of Slopes. Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.

Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.3 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$j'(a) = \lim_{x \to a} \left(\frac{j(x) - j(a)}{x - a} \right)$$

$$= \lim_{x\to a}\left(\frac{(x^2-4)-(a^2-4)}{x-a}\right).$$

Remember, you know how this goes! You're going to do the same sorts of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to re-calculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

2.2 Interpreting Derivatives

2.2.1 The Derivative is a Slope

Activity 2.2.1 Interpreting the Derivative as a Slope. In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \dfrac{1}{x^2}$.

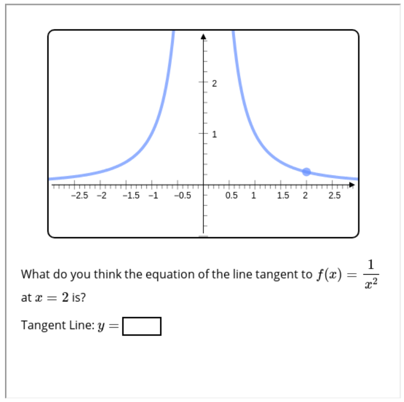
(a) Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?

(b) We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.

Remember that we can write the equation of a line in two ways:

- The equation of a line with slope m that passes through the point $(a, f(a))$ is:
$$y = m(x-a) + f(a).$$
- The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:
$$y = mx + b.$$

Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \dfrac{1}{x^2}$ below to check your equation.



(c) This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.

Use this idea of slope to predict where the y-value of our function will be at 2.01.

Hint. We know that slope is $\dfrac{\Delta y}{\Delta x}$ and we're using the fact that $\dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x}$ for small values of Δx.

Here, we have $\Delta x = 0.01$, so can you use the slope to approximate the corresponding Δy and figure out the new y-value?

(d) Compare this value with $f(2.01) = \dfrac{1}{2.01^2}$. How close was it?

2.2.2 The Derivative is a Rate of Change

Activity 2.2.2 Interpreting the Derivative as a Rate of Change. This is going to somewhat feel redundant, since maybe we know that a slope is really just a rate of change. But hopefully we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \le t \le 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

Hint. What is the rate at which the position (in feet) of the vehicle changes per unit time (in seconds)? What do we call this, and what are the units?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

2.2.4 The Derivative is a Function

Activity 2.2.3 Interpreting the Derivative as a Function. In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.3 Some Early Derivative Rules

2.3.1 Derivatives of Common Functions

Activity 2.3.1 Derivatives of Power Functions. We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.

Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\frac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

Hint. Remember that the graph of $x^2 - 4$ has the same shape as the graph of x^2, but has been shifted down by 4 units. Why does this vertical shift not change the value of the derivative at any x-value?

(b) Find $\frac{d}{dx}\left(x^3\right)$.

Hint. Remember that $(x + \Delta x)^3 = (x + \Delta x)(x + \Delta x)(x + \Delta x)$

(c) Find $\frac{d}{dx}\left(x^4\right)$.

Hint. Remember that $(x + \Delta x)^4 = (x + \Delta x)(x + \Delta x)(x + \Delta x)(x + \Delta x)$

(d) Notice that in these derivative calculations, the main work is in multiplying $(x+\Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\begin{aligned}\frac{d}{dx}\left(x^5\right) &= \lim_{\Delta x \to 0}\left(\frac{(x + \Delta x)^5 - x^5}{\Delta x}\right)\\ &= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right)\\ &= \rightsquigarrow \ldots\end{aligned}$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x+\Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

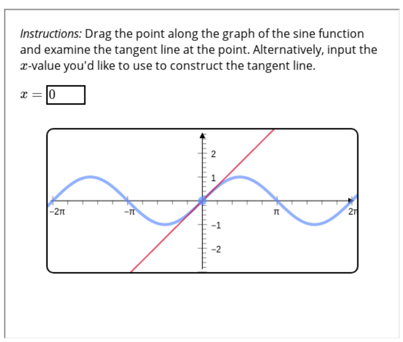
1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this has done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just that whatever terms still have at least one remaining copy of Δx in it "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Activity 2.3.2 Derivatives of Trigonometric Functions. Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing out interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

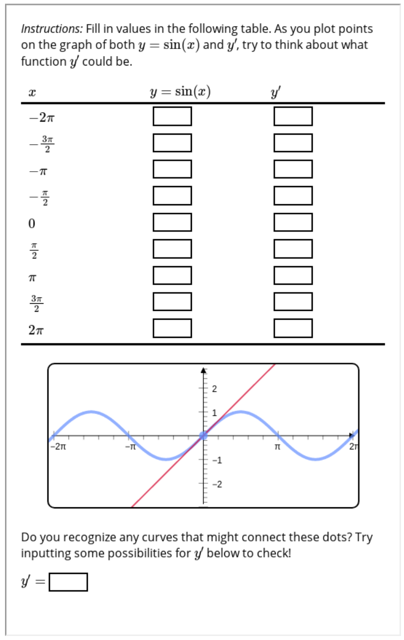
(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

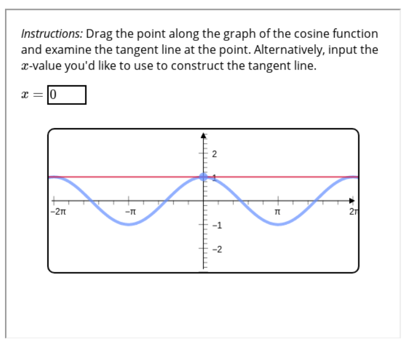
Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



(c) Let's repeat this process using the $y = \cos(x)$ function instead.

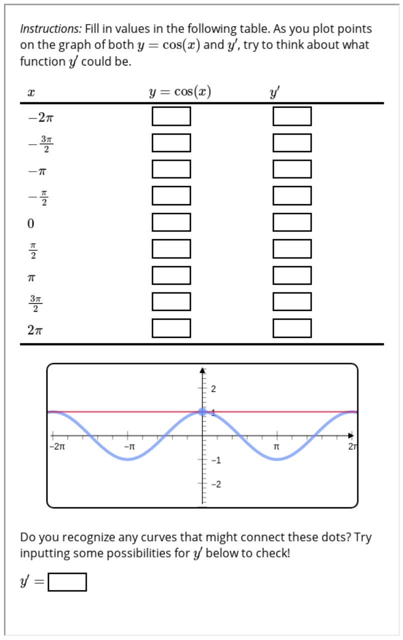
The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

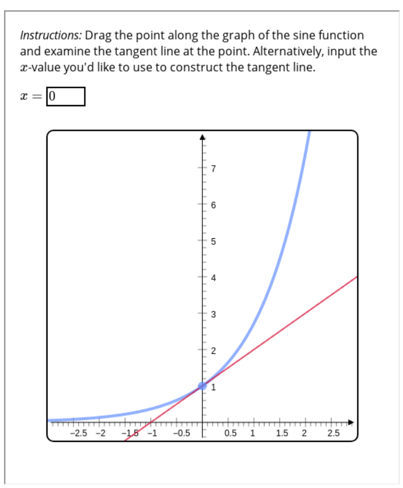
Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



Activity 2.3.3 Derivative of the Exponential Function. We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function in a similar way to use thinking about the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

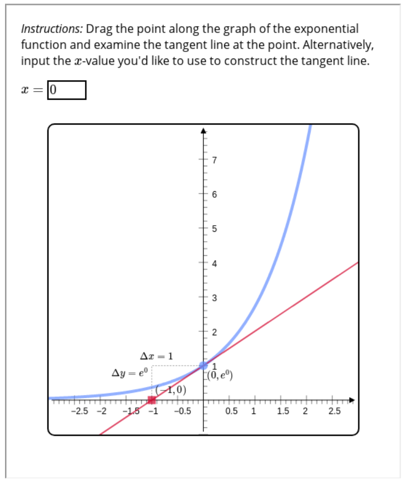
(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. Are there any x-values where the slope is negative? Are there any where the slope is equal to 0? What happens to the slopes as x increases?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!



What information does this reveal about the slopes?

Hint. Especially it might be helpful to think about the slopes and their relationship to the y-value of the point we are building the tangent line at.

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/ equation for y' is then?

2.4 The Product and Quotient Rules

2.4.1 The Product Rule

Activity 2.4.2 Building a Product Rule. Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything.

If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

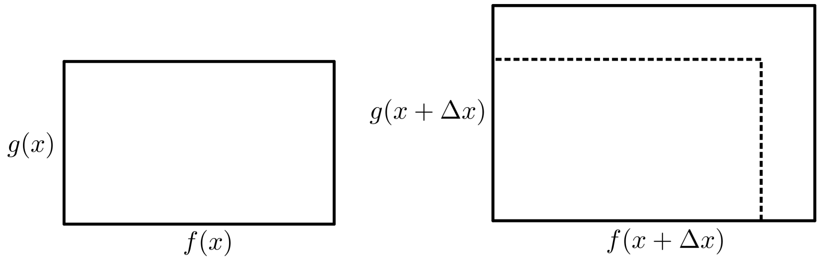


Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.

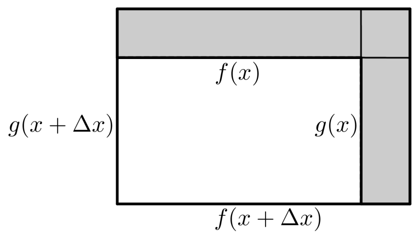


Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$$f(x+\Delta x)g(x+\Delta x)-f(x)g(x) = \boxed{}.$$

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x \to 0} &\left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right) \\ = \lim_{\Delta x \to 0} &\left(\frac{\boxed{}}{\Delta x}\right)\end{aligned}$$

We can split this fraction up into three fractions:

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) = &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\ &+ \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) \\ &+ \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\end{aligned}$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

In the third limit, factor out either $\lim\limits_{\Delta x \to 0} (f(x + \Delta x) - f(x))$ or $\lim\limits_{\Delta x \to 0} (g(x + \Delta x) - g(x))$.

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) = f(x) &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ + g(x) &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ + &\lim_{\Delta x \to 0}\left(\boxed{}\right)\left(\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\right)\end{aligned}$$

(f) Use Definition 2.1.2 The Derivative Function to re-write these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

2.4.2 What about Dividing?

Activity 2.4.3 Constructing a Quotient Rule. We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \dfrac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$$u(x) = \boxed{}$$

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$. Right? We care about finding $f'(x)$! Use what you found for $u'(x)$ and solve for $f'(x)$.

$$f'(x) = \boxed{}$$

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \dfrac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\dfrac{u(x)}{v(x)}$.

$$f'(x) = \boxed{}$$

2.5 The Chain Rule

2.5.1 Composition and Decomposition

Activity 2.5.1 Composition (and Decomposition) Pictionary. This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet-spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

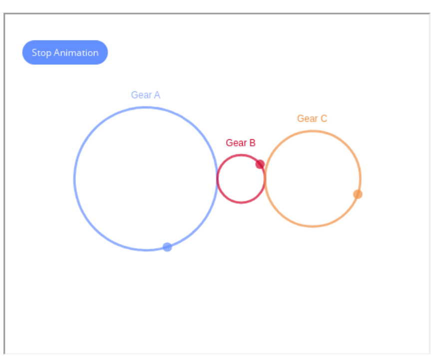
(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that will still give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?

2.5.2 The Chain Rule, Intuitively

Activity 2.5.2 Gears and Chains. Let's think about some gears. We've got three gears, all different sizes. But the gears are linked together, and a small motor works to spin one of the gears. Since the gears are linked, when one gear spins, they all do. But since they are different sizes, they complete a different number of revolutions: the smaller ones spin more times than the larger ones, since they have a smaller circumference.



For our purpose, let's say that Gear A is being driven by the motor.

(a) Let's try to quantify how much "faster" Gear B is spinning compared to Gear A. How many revolutions does Gear B complete in the time it takes Gear A to complete one revolution?

(b) Now quantify the speed of Gear C compared to its neighbor, Gear B. How many revolutions does Gear C complete in the time it takes Gear B to complete one revolution?

(c) Use the above relative "speeds" to compare Gear C and Gear A: how many revolutions does Gear C complete in the time it takes Gear A to complete one revolution?

More importantly, how do we find this?

(d) Now let's translate this into some derivative notation: we've really been finding rates at which one thing changes (the speed of the gear spinning) relative to another's.

Call the speed of Gear B compared to Gear A: $\frac{dB}{dA}$. Now call the speed of Gear C compared to Gear B: $\frac{dC}{dB}$. Come up with a formula to find $\frac{dC}{dA}$.

3 Implicit Differentiation

3.1 Implicit Differentiation

3.1.2 Using a Derivative as an Operator

Activity 3.1.1 Thinking about the Chain Rule.

(a) Explain to someone how (and why) we use the The Chain Rule to find the following derivative:

$$\frac{d}{dx}\left(\sqrt{\sin(x)}\right).$$

(b) Let's say that $f(x) = \sin(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{f(x)}\right).$$

How is this different, or not different, than the previous derivative?

(c) Let's say that we have some other function, $g(x)$. Explain how we find the following derivative:

$$\frac{d}{dx}\left(\sqrt{g(x)}\right).$$

How is this different, or not different, than the previous derivatives?

(d) What is the difference between the following derivatives:

$$\frac{d}{dx}\left(\sqrt{x}\right) \qquad \frac{d}{dx}\left(\sqrt{y}\right) \qquad \frac{d}{dy}\left(\sqrt{y}\right)$$

Hint. When do we need to use the Chain Rule? When do we need to use some linking derivative to connect the function we're looking at with the variable we care about?

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{x}\right) &= \frac{d}{dx}\left(x^{1/2}\right)\\ &= \frac{1}{2}(x^{-1/2})\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

$$\frac{d}{dx}\left(\sqrt{y}\right) = \frac{d}{dy}\left(y^{1/2}\right)\cdot\frac{dy}{dx}$$

$$
\begin{aligned}
&= \frac{1}{2}(y^{-1/2}) \cdot \frac{dy}{dx} \\
&= \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} \\
\text{or } &\frac{y'}{2\sqrt{y}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dy}\left(\sqrt{y}\right) &= \frac{d}{dx}\left(x^{1/2}\right) \\
&= \frac{1}{2}(y^{-1/2}) \\
&= \frac{1}{2\sqrt{y}}
\end{aligned}
$$

Activity 3.1.2 Slopes on a Circle. Visualize the unit circle. Feel free to draw one, or find the picture above. We're going to think about slopes on this circle.

(a) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly horizontal. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(b) Point out the locations on the unit circle where you would expect to see tangent lines that are perfectly vertical. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

(c) Find the point(s) where $x = \frac{1}{2}$. What do you think the value of the derivative, $\frac{dy}{dx}$, would be at these points?

Hint. There are two points to consider here: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

(d) For the unit circle defined by the equation $x^2 + y^2 = 1$, apply the derivative to both sides of this equation to get the following:

$$
\begin{aligned}
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\
\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(1)
\end{aligned}
$$

Carefully consider each of these derivatives (each of the terms). Which of these will you need to apply the Chain Rule for?

(e) Differentiate. Solve for $\frac{dy}{dx}$ or y', whichever notation you decide to use.

Hint 1. Make sure to use the Chain Rule when necessary!

Hint 2. $\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$ or $2yy'$

(f) Go back to the first few questions, and try to answer them again:

(a) Find the locations of any horizontal tangent lines (where $\frac{dy}{dx} = 0$).

(b) Find the locations of any vertical tangent lines (where $\frac{dy}{dx}$ doesn't exist, or where you would divide by 0).

(c) Find the values of $\frac{dy}{dx}$ for the points on the circle where $x = \frac{1}{2}$.

Activity 3.1.3 . Let's consider a new curve:

$$\sin(x) + \sin(y) = x^2y^2.$$

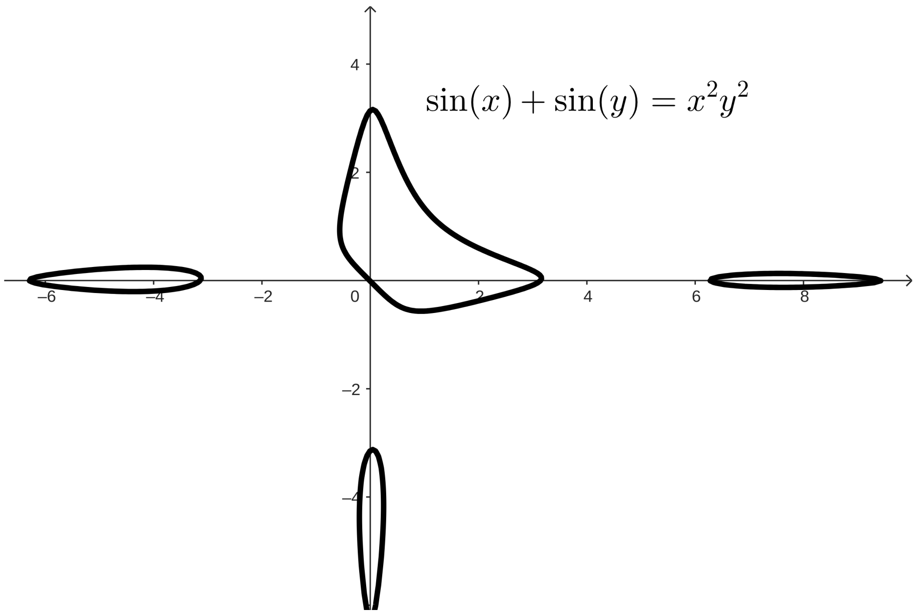


Figure 3.1.4

(a) We are going to find $\frac{dy}{dx}$ or y'. Let's dive into differentiation:

$$\begin{aligned} \frac{d}{dx}\left(\sin(x) + \sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \end{aligned}$$

Think carefully about these derivatives. For each of the three, how will you approach it? What kinds of nuances or rules or strategies will you need to think about? Why?

Hint. Are any of these derivatives involving a variable other than x, the input variable (based on our $\frac{dy}{dx}$ notation, since we are thinking about how y changes *with regard to* x).

Are any of these derivatives involving any other combination of functions? Are there products and/or quotients that we need to think about?

(b) Implement your ideas or strategies from above to differentiate each term.

Hint. We need to apply the Chain Rule to $\frac{d}{dx}\left(\sin(y)\right)$ and then we need to apply the Product Rule $\frac{d}{dx}\left(x^2y^2\right)$. Notice that when we find the derivative of y^2 for the Product Rule, we need to use the Chain Rule!

Solution.

$$\begin{aligned} \frac{d}{dx}\left(\sin(x) + \sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right) &= \frac{d}{dx}\left(x^2y^2\right) \\ \cos(x) + \cos(y) \cdot \frac{dy}{dx} &= 2xy^2 + 2x^2y \cdot \frac{dy}{dx} \\ \text{or } \cos(x) + y'\cos(y) &= 2xy^2 + 2x^2yy' \end{aligned}$$

(c) Now we need to solve for $\frac{dy}{dx}$ or y', whichever you are using. While this equation can look complicated, we can notice something about the "location" of $\frac{dy}{dx}$ or y' in our equation.

Why do we always know that $\frac{dy}{dx}$ or y' will be *multiplied* on a term whenever it shows up?

(d) Now that we are confident that we will *always* know that we are multiplying this derivative, we can employ a consistent strategy:

(a) Rearrange our equation so that every term with a $\frac{dy}{dx}$ or y' is on one side, and everything without is on the other.

(b) Now we are guaranteed that $\frac{dy}{dx}$ or y' is a common factor: factor it out.

(c) Now we can solve for $\frac{dy}{dx}$ or y' by dividing!

Solve for $\frac{dy}{dx}$ or y' in your equation!

(e) Build the equation of a line that lays tangent to the curve at the origin. Does the value of $\frac{dy}{dx}$ at $(0, 0)$ look reasonable to you?

3.2 Derivatives of Inverse Functions

3.2.1 Wielding Implicit Differentiation

Activity 3.2.1 Building the Derivative of the Logarithm. We're going to accomplish two things here:

1. By the end of this activity, we'll have a nice way of thinking about $\frac{d}{dx}(\ln(x))$, and we will now be able to differentiate functions involving logarithms!

2. Throughout this activity, we're going to develop a way of approaching derivatives of inverse functions more generally. Then we can apply this framework to other functions!

Let's think about this logarithmic function!

(a) We have stated (a couple of times now) how we define the log function:

$$y = e^x \longleftrightarrow x = \ln(y).$$

This arrow goes both directions: the log function is the inverse of the exponential, but the exponential is the inverse of the log function!

Can you re-write the relationship $y = \ln(x)$ using its inverse (the exponential)?

(b) For your inverted $y = \ln(x)$ from above (it should be $x =$), apply a derivative operator to both sides, and use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Where do we have to use Chain Rule?

(c) A weird thing that we can notice is that when we use implicit differentiation, it is common to end up with our derivative written in terms of both x and y variables. This makes sense for our earlier examples: we needed specific coordinates of the point on the circle, for instance, to find the slope there.

But if $y = \ln(x)$, we want $\frac{dy}{dx}$ or y' to be a function of x:

$$f(x) = \ln(x) \longrightarrow f'(x) = \qquad .$$

Your derivative is written with only y values.

Since $y = \ln(x)$, replace any instance of y with the log function. What do you have left?

(d) Remember that $y = \ln(x)$. Substitute this into your equation for $\frac{dy}{dx}$. Can you write this in a pretty simplistic way?

Hint. Remember that $e^{\ln(x)} = x$, since these functions are inverses of each other!

(e) Before we go much further, we should be a bit careful. What is the domain of this derivative?

What are the values of x where $\frac{d}{dx}(\ln(x))$ makes sense to think about?

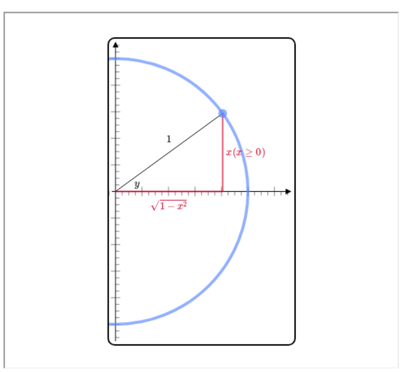
3.2.2 Derivatives of the Inverse Trigonometric Functions

Activity 3.2.2 Finding the Derivative of the Inverse Sine Function. We're going to do the same trick, except that there will be a couple of small differences due to thinking specifically about trigonometric functions.

Let's think about the function $y = \sin^{-1}(x)$. We know that this is equivalent to $x = \sin(y)$ (for y-values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

(a) Move the point around the portion of the unit circle in the graph below. Convince yourself that:

- $\sin(y) = x$
- $\sin(y) \geq 0$ when $0 \leq y \leq \frac{\pi}{2}$
- $\sin(y) < 0$ when $-\frac{\pi}{2} \leq y < 0$



What is $\cos(y)$ in this figure? Does the sign change depending on the value of y?

(b) Use implicit differentiation and the equation $x = \sin(y)$ to find $\frac{dy}{dx}$ or y'

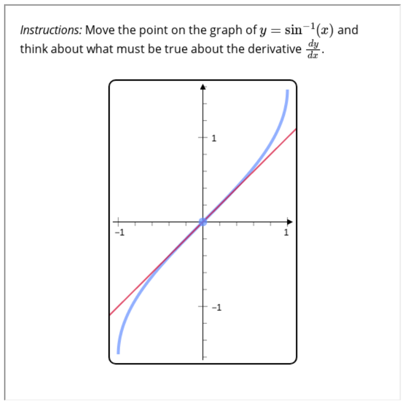
(c) If you still have your derivative written in terms of y, make sure to write $\cos(y)$ in terms of x!

(d) Let's think about the domain of this derivative: what x-values make sense to think about?

Think about this both in terms of what x-values reasonably fit into your formula of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$ as well as the domain of the inverse sine function in general.

(e) Notice that in the denominator of $\frac{d}{dx}\left(\sin^{-1}(x)\right)$, you have a square root. Based on that information (and the visual above), what do you expect to be true about the sign of the derivative of the inverse sine function?

Confirm this by playing with the graph of $y = \sin^{-1}(x)$ below.



(f) Investigate the behavior of $\frac{dy}{dx}$ at the end-points of the function: at $x = -1$ and $x = 1$. What do the slopes look like they're doing, graphically?

How does this work when you look at the function you built above? What happens when you try to find $\left.\frac{dy}{dx}\right|_{x=-1}$ or $\left.\frac{dy}{dx}\right|_{x=1}$?

Activity 3.2.3 Building the Derivatives for Inverse Tangent and Secant.

(a) Consider the triangle representing the case when $y = \tan^{-1}(x)$.

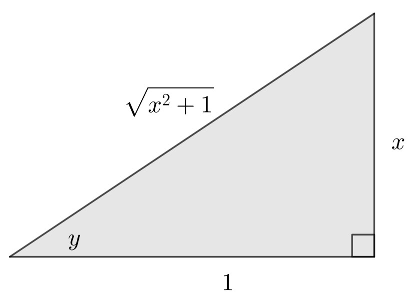


Figure 3.2.2

For $x = \tan(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ based on the triangle above, but we will refer back to the version with the $\sec(y)$ in it later.

(b) Consider the triangle representing the case when $y = \sec^{-1}(x)$.

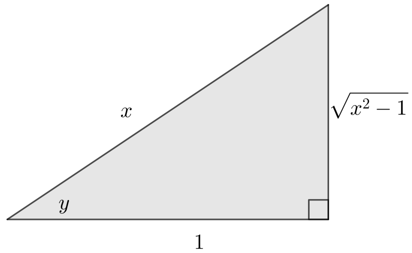
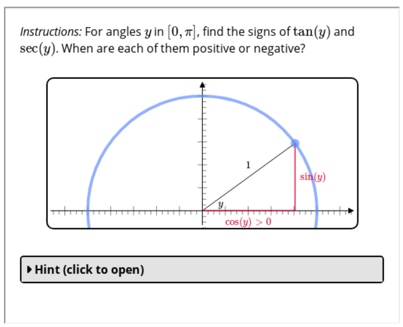


Figure 3.2.3

For $x = \sec(y)$, find $\frac{dy}{dx}$ using implicit differentiation. Find an appropriate expression for $\sec(y)$ and $\tan(y)$ based on the triangle above, but we will refer back to the version with the functions of y in it later.

(c) Here's a graph of just the unit circle for angles $[0, \pi]$. We are choosing to focus on this region, since these are the angles that the inverse tangent and inverse secant functions will return to us. We want to investigate the signs of $\tan(y)$ and $\sec(y)$.

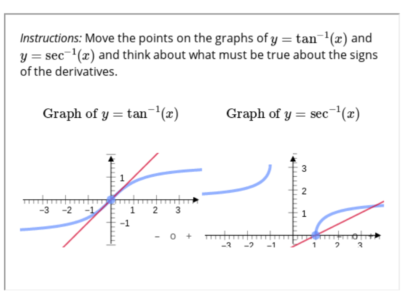


(d) Go back to our derivative expressions for both the inverse tangent and inverse secant functions. What do we know about the signs of these derivatives?

Hint. Notice that in $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2(y)}$, we know that the derivative must be positive. Even when $\sec(y) < 0$, we are squaring it.

In $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{\sec(y)\tan(y)}$, we know that the derivative must also always be positive. Whenever $\sec(y) < 0$, we have $\tan(y) < 0$, and so the product must be positive.

(e) Confirm your idea about the sign of the derivatives by investigating the graphs of each function.

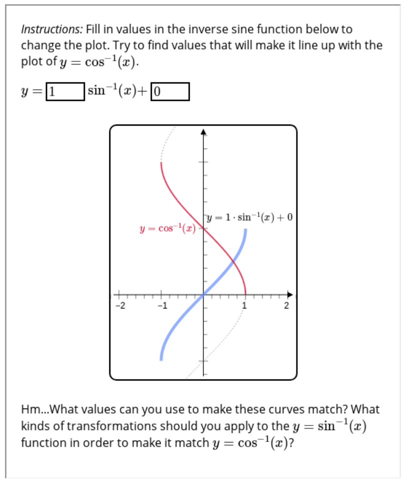


(f) How do we need to write these derivatives, when we write them in terms of x to account for the sign of the derivative?

Hint. Use an absolute value in the formula for $\frac{d}{dx}\left(\sec^{-1}(x)\right)$!

Activity 3.2.4 Connecting These Inverse Functions. We're going to look at a graph of $y = \cos^{-1}(x)$, but we're specifically going to try to compare it to the graph of $y = \sin^{-1}(x)$. We'll use some graphical transformations to make these functions match up, and then later we'll think about derivatives.

(a) Ok, consider the graph of $y = \cos^{-1}(x)$ and a transformed version of the inverse sine function. Apply some graphical transformations to make these match!



(b) It might be fun to think about another reason that this connection between $\sin^{-1}(x)$ and $\cos^{-1}(x)$ exists.

Consider this triangle:

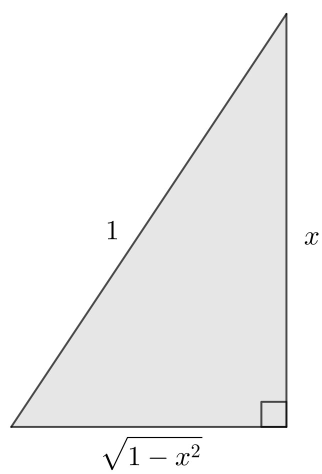


Figure 3.2.4

We're going to think about these inverse trigonometric functions as angles: let $\alpha = \cos^{-1}(x)$ and $\beta = \sin^{-1}(x)$. We can re-write these as:

$$\cos(\alpha) = x$$
$$\sin(\beta) = x.$$

Can you fill in your triangle using this information?

Why does $\alpha + \beta = \frac{\pi}{2}$? Convince yourself that this is what we did with the graphical transformations above, as well.

(c) Use this equation above, re-writing $\cos^{-1}(x)$ as some expression involving the inverse sine function, and then find

$$\frac{d}{dx}\left(\cos^{-1}(x)\right).$$

Hint.

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{d}{dx}\left(-\sin^{-1}(x) + \frac{\pi}{2}\right)$$

We could repeat this task to try to connect the graph of $y = \tan^{-1}(x)$ with $y = \cot^{-1}(x)$ as well as the graph of $y = \sec^{-1}(x)$ with $y = \csc^{-1}(x)$, but we can hopefully see what will happen. In each case, we have the same kind of connection that we saw in the triangle, since these are cofunctions of each other!

We can summarize by believing that:

$$\begin{aligned}
\frac{d}{dx}\left(\cos^{-1}(x)\right) &= -\frac{d}{dx}\left(\sin^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}\\
\frac{d}{dx}\left(\cot^{-1}(x)\right) &= -\frac{d}{dx}\left(\tan^{-1}(x)\right) = -\frac{1}{x^2+1}\\
\frac{d}{dx}\left(\csc^{-1}(x)\right) &= -\frac{d}{dx}\left(\sec^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}$$

3.3 Logarithmic Differentiation

3.3.1 Logs Are Friends!

Activity 3.3.1 Returning to the Power Rule. Back in Section 2.3 we built an explanation for why $\frac{d}{dx}(x^n) = nx^{n-1}$ that relied on thinking about exponents as repeated multiplication: it relied on n being some positive integer. We said, at the time, that the Power Rule generalizes and works for *any* integer, but did so without explanation.

Let's consider $y = x^n$ where n is just some real number without any other restrictions.

(a) Apply a logarithm to both sides of this equation:

$$\ln(y) = \ln(x^n)$$

Now use one of the Properties of Logarithms to re-write this equation.

(b) Use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Remember that when you solve for $\frac{dy}{dx}$ or y', you might have some y-variables in your derivative. Replace them with $y = x^n$.

(c) Explain to yourself why this is equivalent to the Power Rule that we built so long ago:

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(d) Let's get weird. What if we have a not-quite-power function? Where the thing in the exponent isn't simply a number, but another variable?

Let's use the same technique to think about $y = x^x$ and its derivative. First, though, confirm that this is not a power function (and so we cannot use the Power Rule to find the derivative) and is also not an exponential function (and so the derivative isn't itself or itself scaled by a log).

(e) Now apply a log to both sides:

$$\ln(y) = \ln(x^x).$$

Re-write this using the same log property as before, and then use implicit differentiation to find $\frac{dy}{dx}$ or y'.

Hint. Don't forget that in order to find $\frac{dy}{dx} x\ln(x)$, we need to use the Product Rule.

(f) Explain to yourself why logs are friends, especially when trying to differentiate functions in the form of $y = (f(x))^{g(x)}$.

3.3.2 Wow, So Friendly!

Activity 3.3.2 Logarithmic Differentiation with Products and Quotients. Let's say we've got some function that has products and quotients in it. Like, a lot. Consider the function:

$$y = \frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}.$$

(a) Work out a general strategy for how you would find y' directly. Where would you have to use Quotient Rule? What are the pieces? Where would you have to use Product Rule? What are the pieces? Where would you have to use Chain Rule? What are the pieces?

To be clear: do not actually differentiate this. Just look at it in horror and try to outline a plan that some other fool would use.

Click on the "Solution" below to see what the fool did.

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right)\\ &= \frac{(x+1)^7(x+5)^3\frac{d}{dx}\left(\overbrace{(x-4)^2\sqrt{3x+1}}^{\text{Product Rule}}\right) - (x-4)^2\sqrt{3x+1}\,\frac{d}{dx}\left(\overbrace{(x+1)^7(x+5)^3}^{\text{Product Rule}}\right)}{((x+1)^7(x+5)^3)^2}\\ &= \frac{(x+1)^7(x+5)^3\left(2(x-4)\sqrt{3x+1} + \frac{3(x-4)^2}{2\sqrt{3x+1}}\right) - (x-4)^2\sqrt{3x+1}\left(7(x+1)^6(x+5)^3 + 3(x -\right.}{(x+1)^{14}(x+5)^6}\end{aligned}$$

What now? Can we "simplify" this somehow? Maybe, but I am *not* doing any more of this!

(b) Let's instead use logarithmic differentiation. First, apply the log to both sides to get:

$$\ln(y) = \ln\left(\frac{(x-4)^2\sqrt{3x+1}}{(x+1)^7(x+5)^3}\right).$$

Since this function is just a bunch of products of things with exponents all put into some big quotient, we can use our log properties to re-write this!

(c) We should have:

$$\ln(y) = 3\ln(x-4) + \frac{1}{2}\ln(3x+1) - 7\ln(x+1) - 3\ln(x+5).$$

Confirm this.

(d) Now differentiate both sides! You'll have to use some Chain Rule (but not a lot)! Refer back to Fact 3.3.1 for help here.

(e) Solve for $\frac{dy}{dx}$ or y'.

(f) While this is not a *nice* looking expression for the derivative, spend some time confirming that this was a nicer *process* then differentiating directly. This is because logs are friends.

4 Applications of Derivatives

4.2 Increasing and Decreasing Functions

4.2.2 Direction of a Function (and Where it Changes)

Activity 4.2.2 Comparing Critical Points. Let's think about four different functions:

- $f(x) = 4 + 3x - x^2$
- $g(x) = \sqrt[3]{x+1} + 1 + x$
- $h(x) = (x-4)^{2/3}$
- $j(x) = 1 - x^3 - x^5$

Our goal is to find the critical points on the interval $(-\infty, \infty)$ and then to try to figure out if these critical points are local maximums or local minimums or just points that the function increases or decreases through.

(a) To start, we're going to be finding critical points. Without looking at a picture of the graph of the function, find the derivative.

Are there any x-values (in the domain of the function) where the derivative doesn't exist? We are normally looking for things like division by 0 here, but we could be finding more than that. Check out When Does a Derivative Not Exist? to remind yourself if needed.

Are there any x-values (in the domain of the function) where the derivative is 0?

(b) Now that we have the critical points for the function, let's think about where the derivative might be positive and negative. These will correspond to the direction of a function, based on Theorem 4.1.5 Sign of the Derivative and Direction of a Function.

Write out the intervals of x-values around and between your list of critical points. For each interval, what is the sign of the derivative? What do these signs mean about the direction of your function?

(c) Without looking at the graph of your function:

- What changes about how your function increases up to or decreases down to a critical point based on whether the derivative was 0 or the derivative didn't exist?
- Does your function change direction at a critical point? What will that look like, whether it does or does not change direction?

(d) Give a basic sketch of your graph. It might be helpful to find the y-values for any critical points you've got. Then you can sketch your function increasing/decreasing in the intervals between these points.

In your sketch, include enough detail to tell whether the derivative is 0 or doesn't exist at each critical point.

(e) Compare your sketch to the actual graph of the function (you can find all of the graphs in the hint).

Hint.

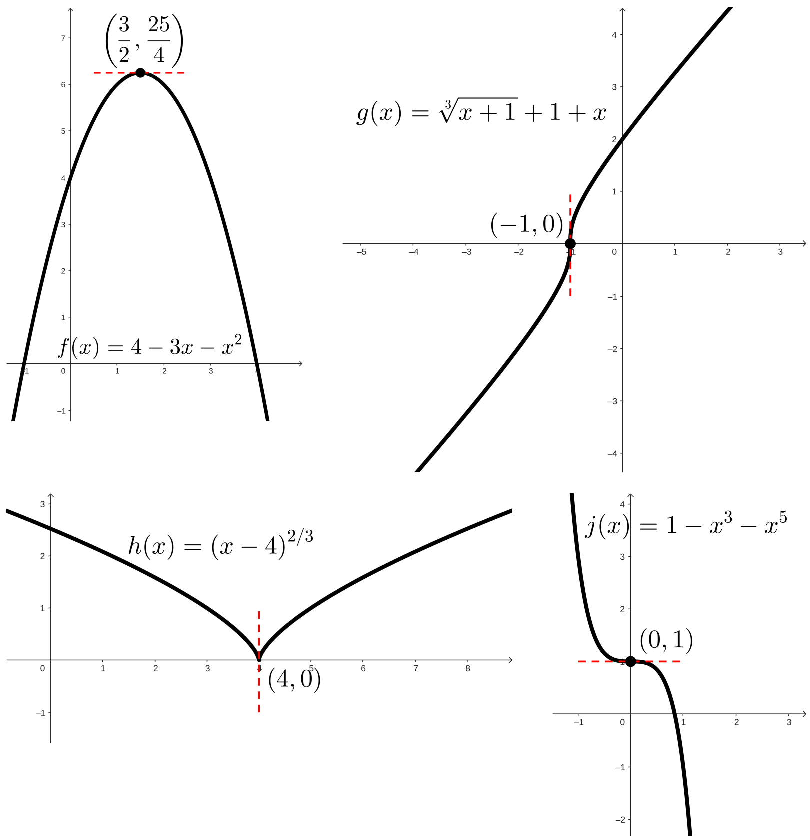
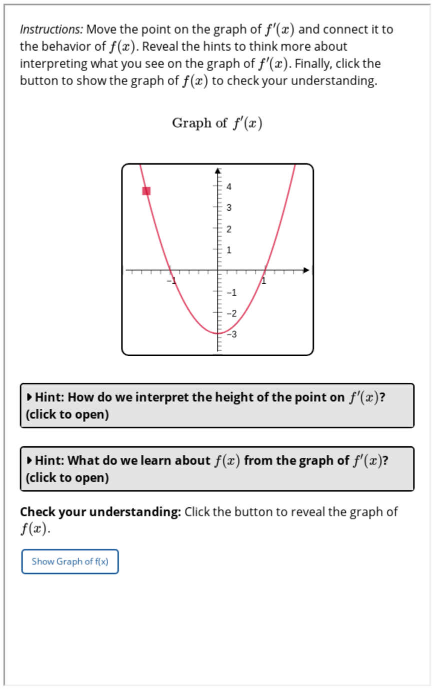


Figure 4.2.5

4.2.3 Using the Graph of the First Derivative

Activity 4.2.3 First Derivative Test Graphically. Let's focus on looking at a picture of a derivative, $f'(x)$, and trying to collect information about the function $f(x)$. This is what we've done already, except that we've done it by thinking about the representation of $f'(x)$ as a function rule written out with algebraic symbols. Here we'll focus on connecting all of that to the picture of the graphs.

For all of the following questions, refer to the plot below. You can add information with the hints whenever you need to. Don't reveal the picture of $f(x)$ until you're really ready to check what you know.



(a) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is increasing.

(b) Based on the graph of $f'(x)$, estimate the interval(s) of x-values where $f(x)$ is decreasing.

(c) Find the x-values of the critical points of $f(x)$. Once you've estimated these, classify them as local maximums, local minimums, or neither. Explain your reasoning.

(d) What do you think the graph of $f(x)$ looks like? Do your best to sketch it or explain it before revealing it!

(e) Why could we estimate the x-values of the critical numbers of $f(x)$, but not find the actual coordinates? How come we can't find the y-value based on looking at the graph of $f'(x)$?

4.3 Concavity

4.3.1 Defining the Curvature of a Curve

Activity 4.3.2 Finding a Function's Concavity. We're going to consider a pretty complicated function to work with, and employ a strategy similar to what we did with the first derivative:

- Our goal is to find the sign of $f''(x)$ on different intervals and where that sign *changes*.
- To do' this, we'll find the places that $f''(x) = 0$ or where $f''(x)$ doesn't exist. These are the critical points of $f'(x)$.
- From there, we can build a little sign chart, where we split up the x-values based on the domain of f and the critical numbers of f'. Then we can

attach each interval of x-values with a sign of the second derivative, f'', on that interval.

- We'll interpret these signs. For any intervals where $f''(x) > 0$, we know that $f'(x)$ must be increasing and so $f(x)$ is concave up. Similarly, for any intervals where $f''(x) < 0$, we know that $f'(x)$ must be decreasing and so $f(x)$ is concave down.

Let's consider the function

$$f(x) = \ln(x^2+1) - \frac{x^2}{2}.$$

(a) Find the first derivative, $f'(x)$, and use some algebra to confirm that it is really:

$$f'(x) = -\frac{x(x+1)(x-1)}{x^2+1}.$$

While we have this first derivative, we *could* find the critical points of $f(x)$ and then classify those critical points using the First Derivative Test.

For our goal of finding the intervals where $f(x)$ is concave up and concave down, and then finding the inflection points of f, let's move on.

(b) Find the second derivative, $f''(x)$, and confirm that this is really:

$$f''(x) = -\frac{x^4+4x^2-1}{(x^2+1)^2}.$$

(c) Our goal is to find the x-values where $f''(x) = 0$ or where $f''(x)$ doesn't exist.

Note that in order to find where $f''(x) = 0$, we are really looking at the x-values that make the numerator 0:

$$x^4 + 4x^2 - 1 = 0.$$

Then, to find where $f''(x)$ doesn't exist, we are finding the x-values that make the denominator 0:

$$(x^2+1)^2 = 0.$$

Solve these equations.

Hint. Here are two strategies for solving $x^4 + 4x^2 - 1 = 0$:

(a) Write some sort of substitution where $u = x^2$

$$\begin{aligned}(x^2)^2 + 4(x^2) - 1 &= 0\\ u^2 + 4u + 1 &= 0\end{aligned}$$

Now solve this for u using the quadratic formula. Note that in the end, the two values you get will be possibilities for $u = x^2$. Make sure you get your answer to be in terms of x!

(b) A similar technique (really the same thing) is to "complete the square" and write your equation this way:

$$(x^2+2)^2 - 5 = 0.$$

Now solve for x in a natural way.

Something to note here is that $x^2 + 2 > 0$, so $x^2 + 2 = \sqrt{5}$ (and notably not $-\sqrt{5}$).

Answer. You should get that the critical points of $f'(x)$ are at $x = -\sqrt{\sqrt{5}-2}$ and $x = \sqrt{\sqrt{5}-2}$.

(d) You have two critical points of $f'(x)$ (let's just call them x_1 and x_2). These are possible inflection points of $f(x)$, but we need to check to see if the concavity changes at these points.

Fill in the following sign chart and interpret.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''					
f					

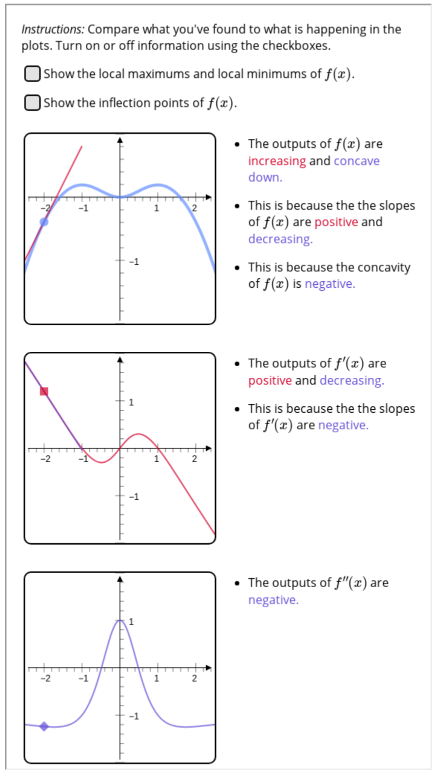
Answer.

x	$(-\infty, x_1)$	x_1	(x_1, x_2)	x_2	(x_2, ∞)
f''	$\ominus$	0	$\oplus$	0	$\ominus$
f	$\frown$ concave down	inflection point	$\smile$ concave up	inflection point	$\frown$ concave down

Let's confirm what we've just calculated graphically. Below, we have three graphs:

1. $f(x) = \ln(x^2+1) - \dfrac{x^2}{2}$
2. $f'(x) = -\dfrac{x(x+1)(x-1)}{x^2+1}$
3. $f''(x) = -\dfrac{x^4+4x^2-1}{(x^2+1)^2}$

Move the point on any graph and make sure the statements about signs, directions, and concavity match what you found! You should notice that signs of the first and second derivative change at the local maximums/minimums and the inflection points that we found.



4.5 Optimization

4.5.2 Balancing Volume and Surface Area

Activity 4.5.1 Constructing a Can. A typical can of pop is 355 ml, and has around 15 ml of headspace (air). This results in a can that can hold approximately 23 cubic inches of volume.

Let's say we want to construct this can in the most efficient way, where we use the least amount of material. How could we do that? What are the dimensions of the can?

(a) First, let's think of our can and try to translate this to some geometric shape with variable names. Collect as much information as we can about this setup! What is the shape? What are the names of the dimensions?

Solution. Our can is probably a cylinder! For a cylinder, we can define it based on a radius and height, and we normally call them r and h. Both of these will be measured in inches.

(b) What is the actual measurement that we are trying to optimize? Are we finding a maximum or a minimum?

Solution. We're trying to minimize the amount of material used to create the can: this should match up with surface area!

The formula for the surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$.

(c) What other information about the can do we know? How do we translate this into a constraint, or a connection between our variables?

Solution. We're told that the volume is 23 cubic inches! The volume formula for a cylinder is $V = \pi r^2 h$, so we have a constraint:

$$23 = \pi r^2 h.$$

(d) What kinds of values can our variables take on? Is there a smallest value for either? A largest? Are there other limitations to these?

Solution. It's hopefully obvious that we need positive dimensions to make sense: we can't have a can with a negative radius or a negative height.

Notice, also, that neither dimension can be equal to 0: if one of them were 0, then the volume would also be 0, which doesn't work with our constraint!

We can also notice that as one dimension gets close to 0, think about the radius for example, then the other dimension, the height in this case, would need to get *really* large to compensate and keep the volume fixed.

All of this to say that our intervals are:

$$r > 0$$
$$h > 0$$

(e) Now we need to set up a calculus statement. This part mostly relies on us finding a way to build a single-variable function defining the surface area. Build that function, and then write down the calculus statement.

Hint. Can you solve the constraint, $23 = \pi r^2 h$ for one of the variables? Can you then use that replace or substitute out one of the variables in our surface area formula to create a function, $A(h)$ or $A(r)$?

Solution. We have two options here, depending on which variable you isolated in the constraint and then substituted. These are the two ways of isolating for a variable in the constraint:

(a) $23 = \pi r^2 h \longrightarrow h = \dfrac{23}{\pi r^2}$

(b) $23 = \pi r^2 h \longrightarrow r = \sqrt[3]{\dfrac{23}{\pi h}}$

(a) We want to find the minimum of $A(r) = 2\pi r + \dfrac{46}{r}$ on $(0, \infty)$.

(b) We want to find the minimum of $A(h) = 2\pi \sqrt[3]{\dfrac{23}{\pi h}} + \dfrac{46}{h}$ on $(0, \infty)$

Personally, I think the first option is probably easier to work with.

(f) Do some calculus to find the global maximum or minimum, and solve the optimization problem.

Hint. Find a derivative, then find critical points, and test them using with the First Derivative Test or the Second Derivative Test for Local Maximums or Local Minimums. Make sure you find the other dimension after you've confirmed that you found the minimum of the A function!

Solution. Let's look at the version with $A(r) = 2\pi r^2 + \dfrac{46}{r}$ on $(0, \infty)$.

$$A'(r) = 4\pi r - \frac{46}{r^2}$$

$$\begin{aligned}
\text{Solve } A'(r) &= 0 \\
0 &= 4\pi r - \frac{46}{r^2} \\
\frac{46}{r^2} &= 4\pi r \\
\frac{1}{r^2} &= \frac{2\pi r}{23} \\
r^2 &= \frac{23}{2\pi r} \\
r^3 &= \frac{23}{2\pi} \\
r &= \sqrt[3]{\frac{23}{2\pi}}
\end{aligned}$$

So we have a critical point on $A(r)$ at $r = \sqrt[3]{\frac{23}{2\pi}}$. Now we need to test it to ensure that it is a minimum (since that's what we're trying to find). We have two options to do this: the First Derivative Test, or the Second Derivative Test.

(a) *FDT:*

r	$\left(0, \sqrt[3]{\frac{23}{2\pi}}\right)$	$\sqrt[3]{\frac{23}{2\pi}}$	$\left(\sqrt[3]{\frac{23}{2\pi}}, \infty\right)$
A'	$\ominus$	0	$\oplus$
A	$\searrow$ decreasing	local min	$\nearrow$ increasing

Since this is the only turning point, then we know that there is a global minimum of A at $r = \sqrt[3]{\frac{23}{2\pi}}$.

(b) *SDT:*

$$A''(r) = 4\pi + \frac{92}{r^3}$$

Hang on one second: notice that no matter what, since $r > 0$, then we can see that $A''(r) > 0$. We don't even need to evaluate this second derivative at the critical point to know that the function is concave up here!

So then we have to have a local minimum, and since it's the only turning point, a global minimum of A at $r = \sqrt[3]{\frac{23}{2\pi}}$.

So we found our minimum! Now we just need to find the other dimension, h.

$$\begin{aligned}
h &= \frac{23}{\pi r^2} && \text{This is the constraint.} \\
h &= \frac{23}{\pi\left(\sqrt[3]{\frac{23}{2\pi}}\right)^2} \\
h &= \left(\frac{23}{\pi}\right)\left(\frac{2\pi}{23}\right)^{2/3} \\
h &= 2^{2/3}\left(\frac{23}{\pi}\right)^{1/3}
\end{aligned}$$

$$h2\sqrt[3]{\frac{23}{2\pi}}$$

This is a lot of algebra, when we could have just plopped these values into a calculator and noticed the same, interesting thing.

(g) What is the relationship between r and h, here? How do they compare? What about the height and diameter of our can?

Is this relationship noticeable in a typical can of pop?

Hint. The height is twice the radius, or the height is equal to the diameter!

We might call this a "square-cylinder", since the "width" and the height of the cylinder are equal.

Do you ever see "square-cylinder" cans?

4.6 Linear Approximations

4.6.1 Linearly Approximating a Function

Activity 4.6.1 Approximating an Exponential Function. Let's consider the function $f(x) = e^x$. We're going to build the linear approximation, $L(x)$, but we also want to focus on understanding what the "center" is, and how we think about accuracy of our estimations.

(a) We first need to find a "good" center for our linear approximation. We have two real requirements here:

(a) We need the center to be some x-value that will be "close" to the inputs we're most interested in. We know that $L(x) \approx f(x)$ for x-values "near" the center, so we should keep this in mind. We don't have a specific input that we're interested in (we are not specifically focused on estimating $f(7.35)$ for instance), so we don't need to worry about this for now.

(b) We are going to need to evaluate the function and its derivative at the center: we use $f'(a)$ to find the slope and $f(a)$ to find a y-value for a point on the line. We'd like to choose a center that will make evaluating these functions reasonable, if we can!

We are going to choose a center of 0: why?

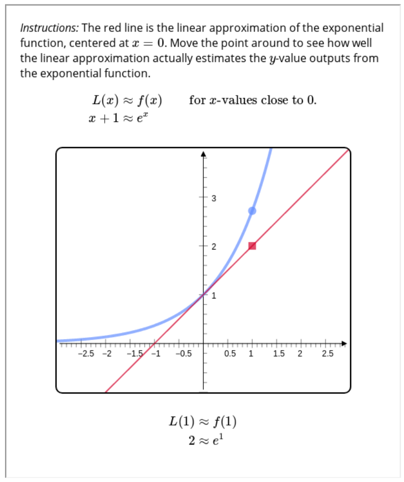
(b) Build a linear approximation of $f(x) = e^x$ centered at $x = 0$.

Hint. Build a line with a slope of $f'(0)$ that goes through the point $(0, f(0))$.

(c) Use your linear approximation to estimate the value of $\sqrt[10]{e} = e^{\frac{1}{10}}$.

Hint. Since $L(x) \approx f(x)$ for x-values near 0, we can say that $L(1/10) \approx f(1/10)$. So you can evaluate your linear approximation function at $x = \frac{1}{10}$.

(d) Let's visualize this approximation a bit:



Are you confident in your approximation of $\sqrt[10]{e}$? Would you be more or less confident in an approximation of $\frac{1}{e}$? Why?

(e) Is your estimate of $\sqrt[10]{e}$ too big or too small? How can you tell, without even calculating the actual value of $\sqrt[10]{e}$?

How can you tell that *every* estimate that you get out of *any* linear approximation of e^x (no matter what the center is) is going to be too small?

Hint. It might be helpful to think about how the function moves away from the tangent line: how do the slopes of e^x change? Can you link this to ideas of concavity?

Activity 4.6.2 Approximating some Values. Pick one of the following values to approximate:

- $\sin(-0.023)$
- $\ln(2)$
- $\sqrt{8}$
- $\sqrt[3]{10}$

Throughout the rest of this activity, use your value to build a linear approximation of some relevant function and estimate the value you chose.

(a) To build a linear approximation of some function at some center, we need two things:

(a) A function.

(b) A center.

What function will you be using for $f(x)$? Why that one?

Hint. Your value should look like the output of some function after you plug in some x-value. What function?

(b) What center are you choosing? Why that one?

Hint. Remember that we want some input for your function that is both close to the input you'd like to estimate your function at and also a reasonable one to know the value of your function and its derivative.

(c) Build your linear approximation at your center! You should end up with an actual linear function. It might be helpful to plot this linear function and your actual function to confirm that you have actually built a tangent line.

(d) Use your linear approximation function to estimate your value! Report the estimate, and comment on the accuracy of your estimate. Without calculating the actual value, can you tell if this is close or not? Do you have an overestimate or underestimate?

Hint. Think about issues relating to the distance from the center, the concavity of the function, and even the rate at which the slopes change away as we move away from the center.

4.6.2 Approximating Zeros of a Function

Activity 4.6.3 Walking in the Footsteps of Ancient Mathematicians. Let's travel all the way back to the first (or maybe second) century AD and recreate Heron's method to approximate the value of $\sqrt{2}$. We'll develop this using modern calculus, and simple linear approximation.

We're going to re-frame the problem, and instead we're going to try to use a linear approximation of $f(x) = x^2 - 2$ to approximate the x-value where $f(x) = 0$. We know enough about quadratic functions to know that there are two values: $x = -\sqrt{2}$ and $x = \sqrt{2}$.

(a) We're going to build a linear approximation of $f(x) = x^2 - 2$, and we need a reasonable center. Honestly, any integer will work, since we can evaluate f and f' really easily, but we want to find one that is close to $\sqrt{2}$. Let's center our approximation at $x = 2$.

Find $f'(x)$, and then construct the linear approximation:

$$L(x) = f'(2)(x-2) + f(2).$$

Solution. Since $f'(x) = 2x$, we have $f'(2) = 4$ and $f(2) = 2$. So then we end up with the following for our linear approximation:

$$L(x) = 4(x-2) + 2.$$

(b) Now we know that $L(x) \approx f(x)$ for x-values near our center, $x = 2$. What if we estimate the x-value where $f(x) = 0$ by solving $L(x) = 0$ instead? Since $L(x) \approx f(x)$, the x-value where $L(x) = 0$ should make $f(x)$ pretty close to 0 at least.

Solve $L(x) = 0$.

Solution.

$$\begin{aligned}0 &= 4(x-2)+2\\ -24(x-2)&\\ -\frac{2}{4}x-2&\\ 2-\frac{1}{2}x&\\ x &= \frac{3}{2}\end{aligned}$$

(c) Ok, this might be kind of close to the value of $\sqrt{2}$, right? Let's visualize this.

Hm...so this isn't that good of an approximation yet. We can check this by looking at the actual value of our function at $x = \frac{3}{2}$ and seeing if it's close to 0.

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 2$$
$$= \frac{9}{4} - 2$$
$$\frac{1}{4}$$

This...isn't that close to 0.

So let's try this again. This time, though, let's center our *new* linear approximation at $x = \frac{3}{2}$.

Solution.

$$L(x) f'\left(\frac{3}{2}\right)\left(x - \frac{3}{2}\right) + f\left(\frac{3}{2}\right)$$
$$= 3\left(x - \frac{3}{2}\right) + \frac{1}{4}$$

(d) Now set this *new* linear approximation equal to 0 and solve $L(x) = 0$ to estimate the solution to $f(x) = 0$.

Solution.

$$0 = 3\left(x - \frac{3}{2}\right) + \frac{1}{4}$$
$$-\frac{1}{4} = 3\left(x - \frac{3}{2}\right)$$
$$-\frac{1}{12} = x - \frac{3}{2}$$
$$\frac{3}{2} - \frac{1}{12}$$
$$x = \frac{17}{12}$$

(e) We can keep repeating this process, and that's exactly what the mathematicians we talked about discovered.

Say we've build a linear approximation at some x-value (we'll call it x_{old}).

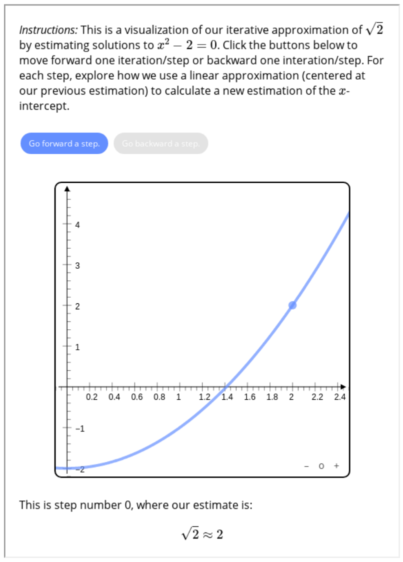
$$L(x) = f'(x_{\text{old}})(x - x_{\text{old}}) + f(x_{\text{old}}).$$

Set this equal to 0 and solve.

Solution.

$$0 = f'(x_{\text{old}})(x - x_{\text{old}}) + f(x_{\text{old}})$$
$$-f(x_{\text{old}}) = f'(x_{\text{old}})(x - x_{\text{old}})$$
$$-\frac{f(x_{\text{old}})}{f'(x_{\text{old}})} x - x_{\text{old}}$$
$$x = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

(f) Let's visualize these calculations.



Something kind of strange happens in the last two steps. Why does the value of our estimation not change? What happens to our estimate?

4.7 L'Hôpital's Rule

4.7.2 L'Hôpital's Rule

Activity 4.7.1 Building L'Hôpital's Rule. We're going to take a closer look at the indeterminate form, $\frac{0}{0}$, and use our new ideas of linear approximation to think about how these types of things work.

We're going to be working with the following limit:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are differentiable at $x = a$ (since we're going to want to build linear approximations of them).

(a) Write out the linear approximations for both $f(x)$ and $g(x)$, both centered at $x = a$. We'll call them $L_f(x)$ and $L_g(x)$.

Hint. We're just using the formula for Linear Approximation of a Function, but with $f(x)$ for one of them and $g(x)$ in the other.

(b) Describe how well or how poorly these linear approximations estimate the values from our functions $f(x)$ and $g(x)$? What happens to these approximations as we get close to the center $x = a$? What happens in the limit as $x \to a$?

Hint. You can revisit the local linearity visualization from Section 4.6 to see what happens, in general, with a linear approximation of a function as we zoom in on the center.

(c) Let's re-write our limit. We can replace $f(x)$ with our formula for its linear approximation, $L_f(x)$ and replace $g(x)$ with its linear approximation,

$L_g(x)$:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left(\frac{\qquad\qquad}{\qquad\qquad} \right)$$

(d) Up until now, we have not thought about indeterminate forms at all. Let's start now.

If this limit is a $\frac{0}{0}$ indeterminate form, then that means that $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$.

Since our functions are, by definition, differentiable at $x = a$, then they also have to be continuous at $x = a$. What does this mean about the values of $f(a)$ and $g(a)$?

Hint. Take a look back at our definition of function being Continuous at a Point. How does the function value relate to the limit? What does that mean in our case?

(e) Use this new information about the values of $f(a)$ and $g(a)$ to revisit the limit. We re-wrote $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ by replacing each function with its linear approximation. What happens with the algebra when we know this information about $f(a)$ and $g(a)$?

Hint. We re-wrote

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left(\frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)} \right)$$

and also know that in this case (due to the $\frac{0}{0}$ indeterminate form) that $f(a) = 0$ and $g(a) = 0$.

So we have a really nice result here! In the $\frac{0}{0}$ indeterminate form, we can replace the ratio of the y-values from our functions with the ratio of slopes (coming from the first derivatives) of our functions.

In general, we'll put a step in between, where we find $f'(x)$ and $g'(x)$ first before trying to evaluate these derivatives at $x = a$.

5 Antiderivatives and Integrals

5.1 Antiderivatives and Indefinite Integrals

5.1.2 Initial Value Problems

Activity 5.1.2 A File Sorting Speed Test. A computer program is trying to sort a group of computer files based on their size. The program isn't very efficient, and the time that it takes to sort the files increases when it tries to sort more files.

The time that it takes, measured in seconds, based on the total, cumulative size of the files g, measured in gigabytes, is modeled by a function $T(g)$. We don't know the function, but we do know that the time increases at an instantaneous rate of $0.0001g$ seconds when the total size, g increases slightly.

(a) We can build a function for $T'(g)$. What is it?

Solution. $T'(g) = 0.0001g$

(b) Find all of the possibilities for the function modeling the time, T, that it takes the computer program to sort files with a total size of g.

Hint. We are looking for the family of antiderivatives of $T'(g)$.

(c) What does your constant C represent, here? You can interpret it graphically, interpret it by thinking about derivatives, but you should also interpret it in terms of the time that it takes this program to sort these files by size.

(d) Let's say that we feed some number of files totaling up to 1.4GB in size into this program. It takes 0.24 seconds to sort the files by size.

Find the function, $T(g)$, that models the how quickly this program sorts these files.

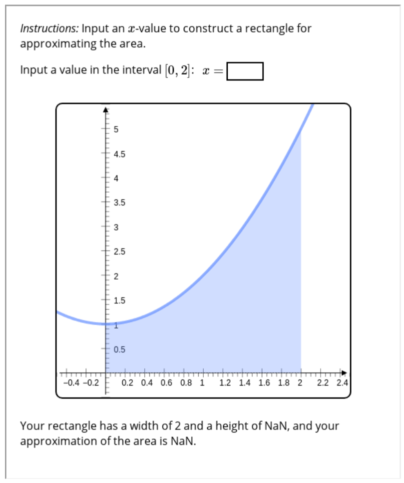
5.2 Riemann Sums and Area Approximations

5.2.1 Rectangular Approximations

Activity 5.2.2 Approximating the Area using Rectangles. We're going to stick with the function $g(x) = x^2 + 1$ on the interval $[0, 2]$, and keep thinking about the area bounded by the curve and the x-axis on this interval. We're going to approximate the area in a couple of different tries, each one more accurate than the one before. By the end of this activity, we'll have a pretty good process built!

(a) Let's start with approximating this region with a single rectangle. We're going to define the rectangle by picking some x-value in the interval $[0, 2]$. Then, we'll use the point at that x-value to define the height of our rectangle.

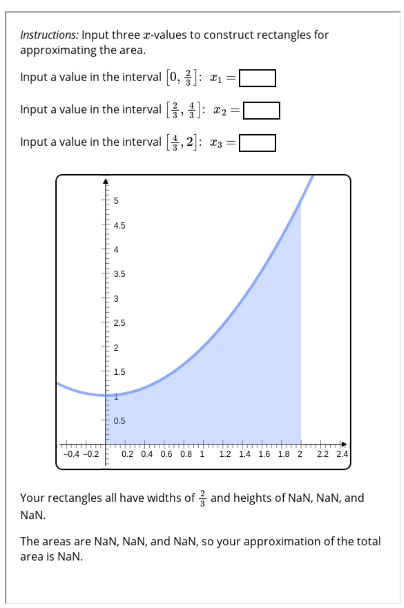
Essentially, we are picking a single point on the our function on the interval and our approximation is pretending that the single point we picked is representative of the whole function on the interval.



(b) Can you try re-picking an x-value, and trying to find one that gets you an area approximation that is pretty good?

(c) We're going to use more rectangles. Let's jump up to 3 rectangles. If we split up the interval between $x = 0$ and $x = 2$ into 3 rectangles, we can make them all the same width, and pick an x-value that we can use to get a representative point for each of the 3 rectangles.

We'll need to pick 3 x-values this time.



(d) Can you try re-picking your x-values, and trying to find one that gets you an area approximation that is pretty good?

(e) Let's scale this up a bit. Pick a good number for your number of rectangles. We'll call this value n.

(If you're working in a classroom, maybe it would be good to pick the number of groups or the number of students, or some other number between 10 and 20 or something like that.)

For your value n, we're going to divide up the interval between $x = 0$ and $x = 2$ into n pieces. These will be the intervals that we pick from to get our rectangles. What are the subintervals? What are the widths of each subinterval (and then the widths of the rectangles)? Call this with Δx.

(f) For each subinterval, pick an x-value in the subinterval to represent it.

(g) Evaluate the function $f(x) = x^2 + 1$ at each of the x-values you picked. These are the heights of your rectangles!

(h) Find the areas of each rectangle by multiplying the height of each rectangle by Δx, the width of each rectangle.

(i) Add these areas up to get a total approximation of the actual area!

What do you think: is it worth fiddling with what x-value to pick from each subinterval to try to get a better approximation? If n is large, do you think it matters how we pick the x-values from each subinterval?

5.3 The Definite Integral

5.3.2 Signed Area

Activity 5.3.1 Weird Areas. Let's think about a simple linear function, $f(x) = 4-3x$. We'll both approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 0$ to $x = 3$:

$$\int_{x=0}^{x=3} (4 - 3x)\, dx$$

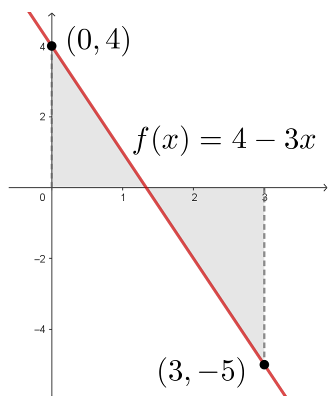


Figure 5.3.3

(a) Explain why we do not need to think about Riemann sums in order for us to calculate the shaded in area. How would you calculate this without using calculus?

Calculate the area!

Hint. Are there some shapes that you recognize? What are the dimensions of these shapes?

(b) Let's approximate this area using a Riemann sum. Calculate L_3, the Left Riemann sum with $n = 3$ rectangles.

Hint. You're going to divide up the interval from $x = 0$ to $x = 3$ into 3 subintervals: $[0, 1]$, $[1, 2]$, and $[2, 3]$. Note that $\Delta x = 1$.

Then you're picking the left-most x-value from each subinterval ($x_1^* = 0$, $x_2^* = 1$, and $x_3^* = 2$) to plug into $f(x)$ in order to find the heights of your rectangles.

(c) Let's approximate this area a second time, but with a different selection strategy for our x-values. Calculate R_3, the Right Riemann sum with $n = 3$ rectangles.

Hint. You're still going to divide up the interval from $x = 0$ to $x = 3$ into 3 subintervals: $[0, 1]$, $[1, 2]$, and $[2, 3]$. We still have $\Delta x = 1$.

Then you're picking the right-most x-value from each subinterval ($x_1^* = 1$, $x_2^* = 2$, and $x_3^* = 3$) to plug into $f(x)$ in order to find the heights of your rectangles.

(d) Compare your answers for L_3 and R_3. They should be *very* different. Why? What is happening that makes R_3 specifically such a weird value?

(e) Do you need to go back and re-calculate the area geometrically (from the first part of this activity)? Explain why your answer for $\displaystyle\int_{x=0}^{x=3} (4-3x)\ dx$ *should* be negative, based on the Riemann sums we calculated.

Hint. Did you account for "negative" area in the second trianglular region in this integral?

(f) Find a value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x)\ dx = 0.$$

(g) Find a *different* value $x = b$ such that:

$$\int_{x=0}^{x=b} (4 - 3x)\ dx = 0.$$

Is there a second way of making this area 0?

Hint. Depending on what you did earlier, you might have to find some ending x-value that "balances" the area above and below the x-axis. If you already did this, then you might have to find an ending x-value that collapses this shape down to a 1-dimensional shape with no area.

Activity 5.3.2 Weird Areas - Part 2. We're going to think about the same region, kind of.

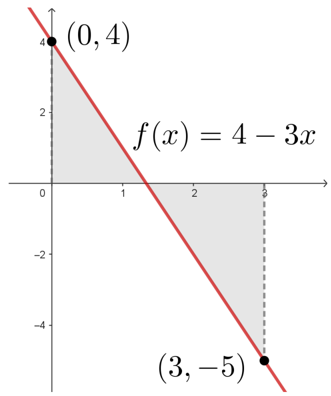


Figure 5.3.4

Let's think about the same linear function, $f(x) = 4 - 3x$, but this time we'll approximate and evaluate the area bounded between $f(x)$ and the x-axis from $x = 3$ to $x = 0$:

$$\int_{x=3}^{x=0} (4 - 3x)\, dx$$

(a) Use geometry to calculate the area. Compare this to the result from Activity 5.3.1.

(b) Let's approximate this using a Riemann sum. Calculate M_3, the Midpoint Riemann sum with $n = 3$ rectangles.

Hint. You're going to divide up the interval from $x = 3$ to $x = 0$ into 3 subintervals: $[3, 2]$, $[2, 1]$, and $[1, 2]$.

Then you're picking the middle x-value from each subinterval ($x_1^* = \frac{1}{2}$, $x_2^* = \frac{3}{2}$, and $x_3^* = 52$) to plug into $f(x)$ in order to find the heights of your rectangles.

If $\Delta x = \frac{b-a}{n}$ and $a = 3$, $b = 0$, and $n = 3$, then what value do we use for the widths, Δx?

(c) Do you need to go back and re-calculate the area geometrically (from the first part of this activity)? Explain why your answer for $\int_{x=3}^{x=0} (4-3x)\, dx$ *should* be positive, based on the Riemann sums we calculated.

5.4 The Fundamental Theorem of Calculus

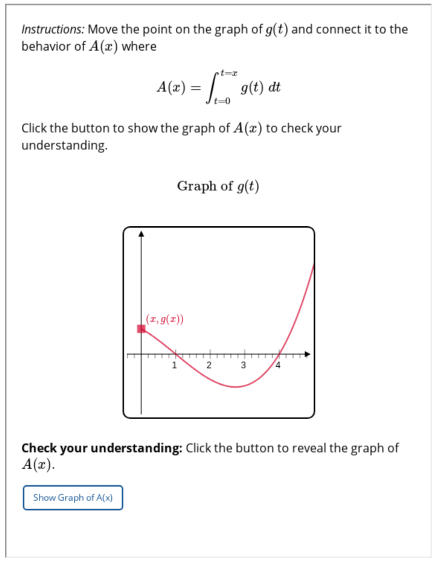
5.4.1 Areas and Antiderivatives

Activity 5.4.2 Interpreting Area. First, we're going to define a bit of a weird function. Sometimes it's called the **Area function**:

$$A(x) = \int_{t=0}^{t=x} g(t)\, dt.$$

This is a strange function, because we're defining the function as an integral of another function. Specifically, note that the *input* for our area function $A(x)$ is the ending limit of integration: we're calculating the signed area "under" the curve of $g(t)$ from $t = 0$ up to some variable ending point $t = x$.

We can visualize this function by looking at the areas we create as we change x. For now, get used to just seeing the area "under" g when we move the point around. The areas themselves are the outputs of the function $A(x)$.



Now we can think about this area function, and try to connect it to the graph of $g(t)$.

(a) List the intervals on which $A(x)$ is increasing. What about decreasing?

(b) Find the x-values of any local maximums and/or local minimums of $A(x)$.

(c) List the intervals on which $A(x)$ is concave up. What about concave down?

(d) Find the x-values of any inflection points of $A(x)$.

(e) Compare your answers here to your answers about the behavior of $f(x)$ based on the (same) graph of $f'(x)$ in Activity 5.4.1.

What does this mean about the connection between areas and derivatives, or areas and antiderivatives?

5.4.2 Evaluating Definite Integrals

Activity 5.4.3 Evaluating Areas and Antiderivatives. In this short activity, we'll just collect information about antiderivatives and this new area function,

$$A(x) = \int_{t=a}^{t=x} f(t)\ dt$$

for a function $f(t)$ that is continuous on the interval $a \leq t \leq x$.

For our purposes in this activity, let's say that $f(x) = x + \cos(x)$.

(a) From the Fundamental Theorem of Calculus (Part 1), we know that $A(x)$ is an antiderivative of $f(x)$, since $A'(x) = f(x)$.

Write out the function $A(x)$, and then name/write out one *other* antiderivative of $f(x)$, some $F(x)$.

Hint. $A(x) = \displaystyle\int_{t=a}^{t=x} t + \cos(t)\ dt$, and for $F(x)$ you can use your antidifferentiation rules from Section 5.1 Antiderivatives and Indefinite Integrals.

(b) We know that all of the antiderivatives of a function are connected to each other.

Describe the connection between $A(x)$ and your $F(x)$.

Hint. This is the result that we proved in Theorem 4.1.7 and used to define a **family of antiderivatives** in Definition 5.1.1.

(c) What is the value of $A(a)$? What is the value of $F(a)$? How are they different from each other?

Hint.

$$A(a) = \int_{t=a}^{t=a} f(t)\ dt = 0$$

For $F(a)$, you can evaluate your antiderivative at $x = a$. The important part is thinking about how these two values are different from each other.

(d) What is the value of $A(b)$? What is the value of $F(b)$? How are they different from each other?

Hint.

$$A(b) = \int_{t=a}^{t=b} f(t)\ dt$$

For $F(a)$, you can evaluate your antiderivative at $x = b$. The important part is thinking about how these two values are different from each other. Is the difference between these values the same, or different from the difference between $A(a)$ and $F(a)$?

(e) What about the differences: $A(b) - A(a)$ compared to $F(b) - F(a)$?

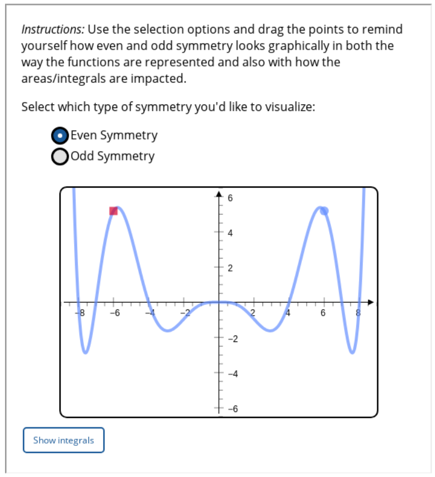
Hint. It is worth noting that:

$$\begin{aligned} A(b) - A(a) &= A(b) - 0 \\ &= \int_{t=a}^{t=b} f(t)\ dt \end{aligned}$$

5.5 More Results about Definite Integrals

5.5.1 Symmetry

Activity 5.5.1 Symmetry in Functions and Integrals. First, let's take a moment to remind ourselves (or see for the first time) what two types of "symmetry" we'll be considering. We call them "even" and "odd" symmetry, but sometimes we think of them as a "reflective" symmetry and a "rotational" symmetry in the graphs of our functions.



(a) Convince yourself that you know what we mean when we say that a function is **even symmetric** on an interval if $f(-x) = f(x)$ on the interval.

Similarly, convince yourself that you know what we mean when we say that a function is **odd symmetric** on an interval if $f(-x) = -f(x)$ on the interval.

Hint. Look at the relationship between the points on the graphs when you select the different symmetries: How do their x-values relate to each other? How do their y-values relate to each other?

(b) Now let's think about areas. Before we visualize too much, let's start with a small question: How does the height of a function impact the area defined by a definite integral? It should be helpful to think about Riemann sums and areas of rectangles here.

The important question then, is how does a function being even or odd symmetric tell us information about areas defined by definite integrals of that function?

Hint. If we know that for an even symmetric function, there are some heights/y-values that are the same, then we know that there are some areas/integrals that should also be the same. Which ones?

If we know that for an odd symmetric function, there are some heights/y-values that are opposite, then we know that there are some areas/integrals that should also be opposite. Which ones?

Activity 5.5.2 Connecting Symmetric Integrals. We're going to do some sketching here, and I want you to be clear about something: your sketches can be absolutely terrible. It's ok! They just need to embody the kind of symmetry we're talking about. You will probably sketch something and notice that your areas aren't to scale (or maybe even the wrong sign!), and that's fine.

It might be helpful to practice sketching graphs accurately, but don't worry if that part is a struggle.

(a) Sketch a function $f(x)$ with the following properties:

- $f(x)$ is *even symmetric* on the interval $[-6, 6]$
- $\displaystyle\int_{x=0}^{x=6} f(x)\,dx = 4$

- $\displaystyle\int_{x=-6}^{x=-2} f(x)\,dx = -1$

(b) Find the values of the following integrals:

- $\displaystyle\int_{x=0}^{x=2} f(x)\,dx$
- $\displaystyle\int_{x=-6}^{x=6} f(x)\,dx$

Hint. Since $f(x)$ is even symmetric, what are the two other integrals that we know about? How can we use those to help us find these two?

Answer.

- $\displaystyle\int_{x=0}^{x=2} f(x)\,dx = 5$
- $\displaystyle\int_{x=-6}^{x=6} f(x)\,dx = 8$

(c) Sketch a function $g(x)$ with the following properties:

- $g(x)$ is *odd symmetric* on the interval $[-9, 9]$
- $\displaystyle\int_{x=0}^{x=4} g(x)\,dx = 5$
- $\displaystyle\int_{x=-9}^{x=0} g(x)\,dx = 2$

(d) Find the values of the following integrals:

- $\displaystyle\int_{x=-9}^{x=-4} g(x)\,dx$
- $\displaystyle\int_{x=-4}^{x=9} g(x)\,dx$

Hint. Since $g(x)$ is even symmetric, what are the two other integrals that we know about? How can we use those to help us find these two?

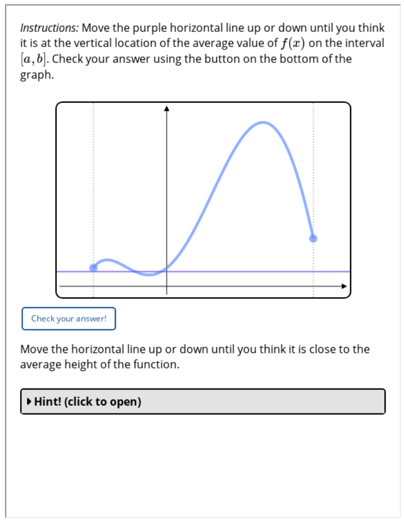
Answer.

- $\displaystyle\int_{x=-9}^{x=-4} g(x)\,dx = 7$
- $\displaystyle\int_{x=-4}^{x=9} g(x)\,dx = -2$

5.5.2 Average Value of a Function

Activity 5.5.3 Visualizing the Average Height of a Function. We are going to build a formula to find the "average height" or "average value" of a function $f(x)$ on the interval $[a, b]$. We're going to look at a function and try to find the average height. Along the way, we'll think a bit about areas!

(a) Consider the following function. Find the average height of the function on the interval pictured!



(b) How does the area ”under” the curve $f(x)$ on the interval compare to the area of the rectangle formed by the average height line?

(c) How do you define the two areas?

Hint. One of these is the area under $f(x)$ from $x = a$ to $x = b$, which we can use calculus for!

The other is the area of a rectangle with a height (the average height of $f(x)$) and a width (the width of the interval).

(d) Set up an equation connecting the two areas, and solve for the average height of $f(x)$.

Hint. If area $=$ height $\times$ width, then doesn’t it make sense that height $= \frac{\text{area}}{\text{width}}$? How, then, do we find average height by dividing an area and a width?

5.6 Introduction to u-Substitution

5.6.1 Undoing the Chain Rule

Activity 5.6.2 Picking the Pieces of a Substitution. We’re going to look at three integrals. Instead of working through them one-at-a-time, we’ll look at all three simultaneously, where we can practice identifying, substituting, and antidifferentiating all at the same time.

(a) Let’s consider these three integrals:

- $\displaystyle\int \frac{3x^2+1}{(x^3+x-2)^2}\,dx$
- $\displaystyle\int \cos(x)\sqrt{\sin(x)}\,dx$
- $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$

For each of these integrals, identify the substitution: define u as some function of x.

Hint. Can you find the composition in each one? If you notice the function-derivative pairing instead, then you can think about making u the "function" portion!

Answer.

- For $\int \frac{3x^2+1}{(x^3+x-2)^2}\,dx$, let $u = x^3+x-2$.
- For $\int \cos(x)\sqrt{\sin(x)}\,dx$, let $u = \sin(x)$.
- For $\int \frac{(\ln(x))^3}{x}\,dx$, let $u = \ln(x)$.

(b) For each substitution, define $du = u'\,dx$.

Answer.

- For $u = x^3+x-1$, $du = (3x^2+1)\,dx$.
- For $u = \sin(x)$, $du = \cos(x)\,dx$.
- For $u = \ln(x)$, $du = \frac{1}{x}\,dx$.

(c) For each integral, use your substitution (for both u and the differential du) to re-write the integral.

Answer.

- $\int \frac{3x^2+1}{(x^3+x-2)^2}\,dx = \int \frac{1}{u^2}\,du$
- $\int \cos(x)\sqrt{\sin(x)}\,dx = \int \sqrt{u}\,du$
- $\int \frac{(\ln(x))^3}{x}\,dx = \int u^3\,du$

(d) Antidifferentiate each integral, and then use your substitution to write each integral back in terms of x.

Answer.

- $$\begin{aligned}\int \frac{1}{u^2}\,du &= -\frac{1}{u} + C\\ &= -\frac{1}{x^3+x-2} + C\end{aligned}$$
- $$\begin{aligned}\int \sqrt{u}\,du &= \frac{2u^{3/2}}{3} + C\\ &= \frac{2(\sin(x))^{3/2}}{3} + C\end{aligned}$$
- $$\begin{aligned}\int u^3\,du &= \frac{u^4}{4} + C\\ &= \frac{(\ln(x))^4}{4} + C\end{aligned}$$

Activity 5.6.3 Compare Two Integrals. Let's compare two integrals, and use this to build a more general strategy for performing u-substitution.

(a) Consider the following integral:

$$\int -4x^3 \sec^2(1-x^4)\,dx$$

Select and justify a choice for u.

Hint. Where is the composition? Do you see a function-derivative pair?

(b) Perform the u-substitution and antidifferentiate, and then substitute back to write your antiderivative in terms of x.

Solution. We'll let $u = 1 - x^4$, and so $du = -4x^3\,dx$.

$$\begin{aligned}\int -4x^3 \sec^2(1-x^4)\,dx &= \int \sec^2(\underbrace{1-x^4}_{u})\underbrace{(-4x^3)\,dx}_{du}\\ &= \int \sec^2(u)\,du\\ &= \tan(u) + C\\ &= \tan(1-x^4) + C\end{aligned}$$

(c) Compare that integral to this one:

$$\int x^3 \sec^2(1-x^4)\,dx$$

What is different about this new integral? What has remained the same? How does that impact your choice for u, or *does* it impact your choice for u?

Hint. Notice that nothing has changed about the "inside" function!

(d) Has that changed what du should be?

Hint. Remember that $du = u'\,dx$. If we didn't change our selection of u, then shouldn't du remain the same as well?

(e) Ok so we've noticed an issue here. There are *plenty* of good ways of solving this problem, where du doesn't "show up" perfectly in our integral. In this case, we have that we're missing a necessary coefficient. We have the x^3 part, but we are missing the -4.

Try to re-write our integral with a -4 coefficient in there. We'll do that by multiplying the integrand function by 1, disguised as $\frac{-4}{-4}$ or $\left(-\frac{1}{4}\right)(-4)$.

Answer.

$$\begin{aligned}\int x^3 \sec^2(1-x^4)\,dx &= \int \left(-\frac{1}{4}\right)(-4)x^3 \sec^2(1-x^4)\,dx\\ &= -\frac{1}{4}\int -4x^3 \sec^2(1-x^4)\,dx\end{aligned}$$

(f) Now we can use the same u-substitution as before, and integrate in a similar way! Notice, though, that we will retain the coefficient of $-\frac{1}{4}$.

(This should be reasonable: our integral is $-\frac{1}{4}$ of the original one, since our coefficient was 1 to the original's -4.)

Go ahead and integrate!

Answer.

$$\int x^3 \sec^2(1 - x^4)\, dx = -\frac{1}{4} \tan(1 - x^4) + C$$

6 Applications of Integrals

6.1 Integrals as Net Change

6.1.1 Estimating Movement

Activity 6.1.1 Estimating Movement. We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 6.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

Hint. How do we interpret the different values of velocity? How do we interpret the sign of velocity? What about how velocity changes from one of the 30-second time points to the next?

(b) When was the acceleration of the object the greatest? When was it the least?

Hint. You can decide how to interpret the "least" acceleration: it is either where the acceleration is closes to 0, or it is the most negative value of the acceleration. These are interpreted differently, but it's a bit ambiguous what we might mean when we say "least acceleration."

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

Hint. How do we use velocity and some time interval to estimate the distance traveled? How do we estimate/assume the velocity on each 30-second time interval?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

Hint. How do we think about (or ignore) the direction of the object? Why is this important here?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

Hint. Can we limit the time intervals that we use to calculate the object's displacement? How do we use displacement and a starting point to find an ending point?

6.1.2 Position, Velocity, and Acceleration

Activity 6.1.2 A Friendly Jogger. Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

Hint. Solve $a(t) = v'(t) = 0$.

(b) Does this time represent a maximum or minimum velocity for the jogger?

Hint. You can use the First Derivative Test or the Second Derivative Test here!

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

6.1.3 Displacement, Distance, and Speed

Activity 6.1.3 Tracking our Jogger. Let's revisit our jogger from Activity 6.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

Hint. Set up and evaluate a definite integral here, using the velocity function.

(b) Think back to our descroption of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.

Hint. Remember that we're really calculating:

$$\left|\int_{t=0}^{t=c} v(t)\ dt\right| + \left|\int_{t=c}^{t=3} v(t)\ dt\right|$$

6.1.5 Practice Problems

6.1.5.1. Explain the following terms in reference to an object moving along a straight path from time $t = a$ to time $t = b$.

(a) **Position** of the object at time t.

(b) **Displacement** of the object.

(c) **Distance** traveled by the object.

(d) **Velocity** of the object at time t.

(e) **Speed** of the object at time t.

6.1.5.2. Consider the graph of a velocity function, $v(t)$, of some object moving along a line on the time interval $[0, 7]$.

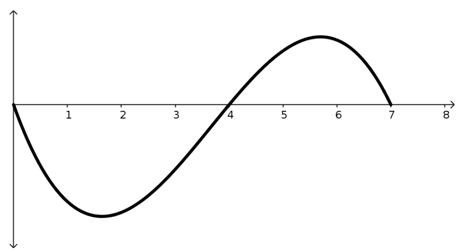


Figure 6.1.7

(a) Do you expect the displacement of the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?

(b) Write two different expressions that represent the total displacement of the object from $t = 0$ to $t = 7$.

(c) Do you expect the distance traveled by the object from $t = 0$ to $t = 7$ to be positive, negative, or 0?

(d) Write two different expressions that represent the total distance traveled by the object from $t = 0$ to $t = 7$.

6.1.5.3. Let's consider an animal running along a straight path with the velocity function:

$$\begin{aligned} v(t) &= \frac{t^4}{10} - t^3 + \frac{27t^2}{10} - \frac{9t}{5} \\ &= \frac{t}{10}(t-1)(t-3)(t-6) \end{aligned}$$

on the time interval $[0, 6]$.

(a) What is the total displacement of the animal on the time interval $[0, 1]$?

(b) What is the total displacement of the animal on the time interval $[1, 3]$?

(c) What is the total displacement of the animal on the time interval $[3, 6]$?

(d) What is the total displacement of the animal on the time interval $[0, 6]$?

(e) What is the total distance traveled by the animal on the time interval $[0, 6]$?

(f) Write a short summary of the animal's movement, including notes about direction, speed, and where the animal travels.

6.1.5.4. Consider an object with velocity function $v(t) = t^2 - 4t + 2$ on the interval $[0, 100]$ with the initial position $s(0) = 3$.

(a) Determine the position function, $s(t)$, for $0 \leq t \leq 100$ using the Future Position of an Object.

(b) Determine the position function, $s(t)$, for $0 \leq t \leq 100$ using the Solving Initial Value Problems strategy.

(c) Compare the results from both methods. Explain why these are equivalent.

6.1.5.5. Consider an object with an acceleration function $a(t) = t + \sin(2\pi t)$ for $t \geq 0$ with $v(0) = 5$.

(a) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Future Position of an Object.

(b) Determine the velocity function, $v(t)$, for $t \geq 0$ using the Solving Initial Value Problems strategy.

(c) Can you obtain the position function, $s(t)$? Explain why or why not, based on the information given.

6.1.5.6. During a brake test for a heavy truck, the truck decelerates from an initial velocity of 88 ft/s with the acceleration function $a(t) = -17$ ft/s². Assume that the initial position of the truck is $s(0) = 0$.

(a) Find the velocity function for the truck.

(b) When does the truck stop? In this situation, the truck won't have a negative velocity (since it's just braking and not eventually going in reverse). What time interval is the velocity function relevant on?

(c) What is the total displacement of the truck on this time interval?

(d) Safety standards say that for a truck like this, it needs to be able to stop (from a speed of 88ft/s) in, at most, 200 feet.

Do we need to make changes to the braking mechanism, in order to have the acceleration function change? If so, what does the acceleration need to be (assuming it is constant and we are just replacing it with a new negative number)?

6.2 Area Between Curves

6.2.1 Remembering Riemann Sums

Activity 6.2.1 Remembering Riemann Sums. Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

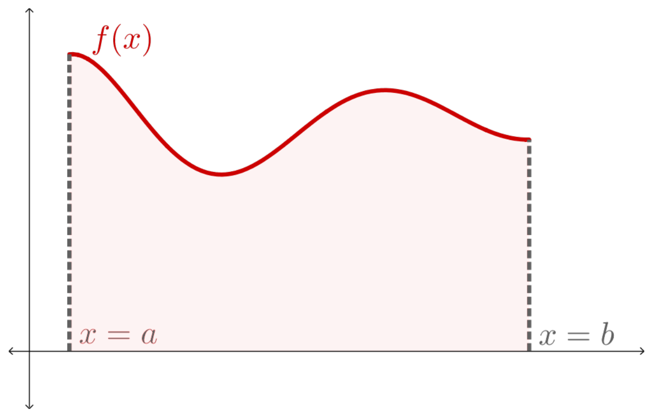


Figure 6.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

Hint. These points are just general ones, and you don't have to come up with actual numbers for the x-values or the corresponding y-values. Instead, just draw them in on the curve, somewhere in each of the subintervals.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

Hint. You won't have any numbers to calculate here, really: instead, see if you can calculate the widths by thinking about the total width of your interval. Then calculate the heights by thinking about the points you created.

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

Hint. Try to think about the accuracy of your area approximation by looking at it visually. Are there any places where your approximation looks far away from the actual area we're thinking about?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

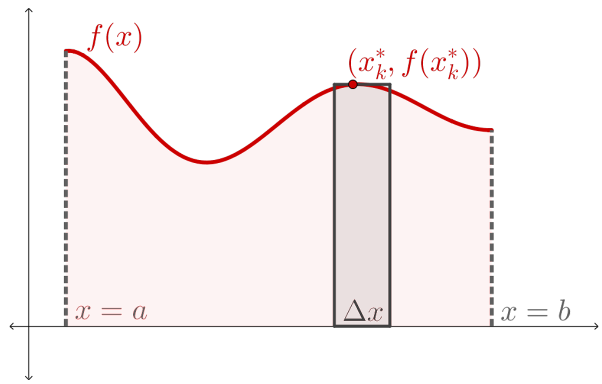


Figure 6.2.2

What are the dimensions of this kth rectangle?

Answer. Height: $f(x_k^*)$

Width: Δx

(g) Find A_k, the area of this kth rectangle.

Answer. $A_k = f(x_k^*)\Delta x$

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

Answer. $A \approx \displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

Answer.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= \int_{x=a}^{x=b} f(x)\, dx \end{aligned}$$

6.2.2 Building an Integral Formula for the Area Between Curves

Activity 6.2.2 Area Between Curves. Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

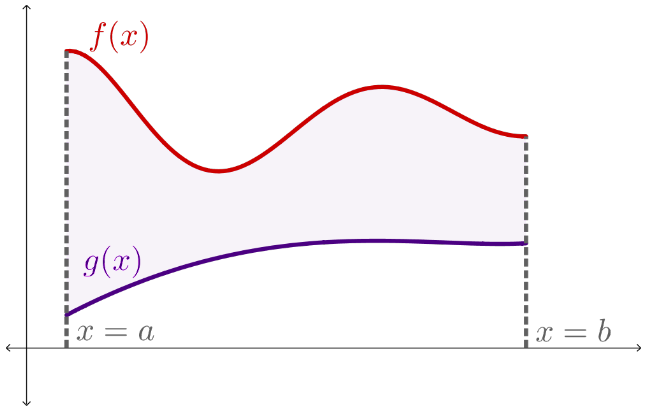


Figure 6.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

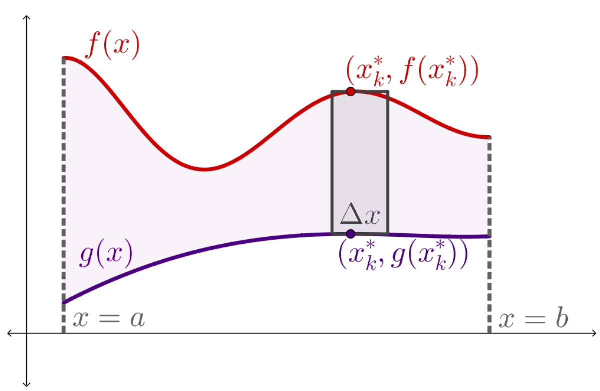


Figure 6.2.4

What are the dimensions of this kth rectangle?

Answer. Height: $f(x_k^*) - g(x_k^*)$

Width: Δx

(g) Find A_k, the area of this kth rectangle.

Answer. $A_k = (f(x_k^*) - g(x_k^*))\,\Delta x$

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

Answer. $A \approx \displaystyle\sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\,\Delta x$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Answer.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\,\Delta x \\ &= \int_{x=a}^{x=b} (f(x) - g(x))\ dx \end{aligned}$$

6.2.3 Changing Perspective

Activity 6.2.3 Trying for a Single Integral. Let's consider the same setup as earlier: the region bounded between two curves, $y = x + 6$ and $y = x^3$, as well as the x-axis (the line $y = 0$). We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

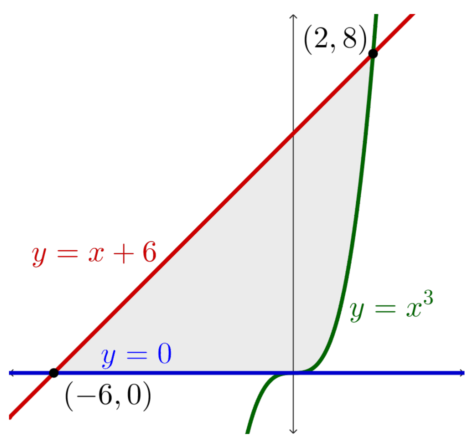


Figure 6.2.7

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

Hint. $\Delta y = \dfrac{8 - 0}{4}$

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

Hint. You're really just putting your y-values into the equation $y = x + 6$ and solving for x. Or you can solve for $f^{-1}(y)$ in general, by solving for x while leaving y as a variable.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths for each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

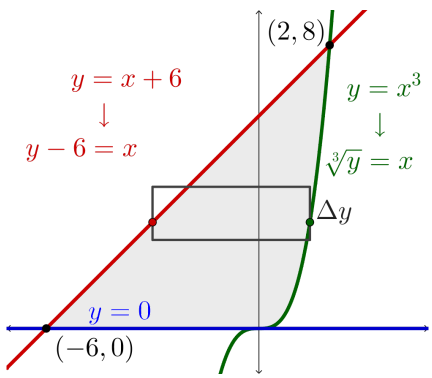


Figure 6.2.8

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

Answer. Height: Δy

Width: $\sqrt[3]{y_k^*} - (y_k^* - 6)$

(j) Find A_k, the area of this kth rectangle.

Answer. $A_k = \left(\sqrt[3]{y_k^*} - (y_k^* - 6)\right) \Delta y$

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

Answer. $A \approx \displaystyle\sum_{k=1}^{n} \left(\sqrt[3]{y_k^*} - (y_k^* - 6)\right) \Delta y$

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Answer.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt[3]{y_k^*} - (y_k^* - 6) \right) \Delta y \\ &= \int_{y=0}^{y=8} \left(\sqrt[3]{y} - (y-6) \right)\, dy \end{aligned}$$

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

Solution.

$$\begin{aligned} \int_{y=0}^{y=8} \left(\sqrt[3]{y} - (y-6) \right)\, dy &= \int_{y=0}^{y=8} \left(y^{1/3} - y + 6 \right)\, dy \\ &= 28 \end{aligned}$$

6.2.4 Practice Problems

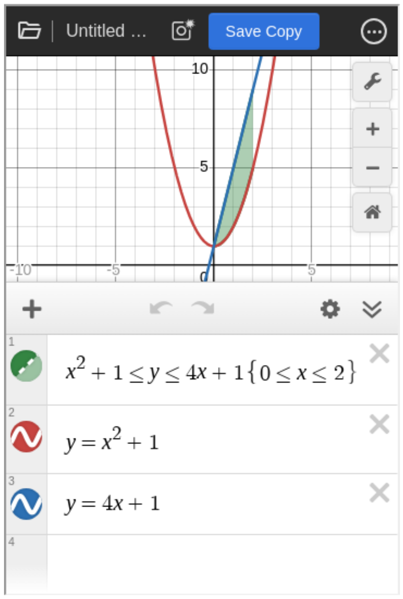
6.2.4.1. Explain how we use the "slice and sum" method to build an integral formula for the area bounded between curves. Give some details, enough to make sure you understand how the Riemann sums are constructed and how they turn into our integral formula.

6.2.4.2. What are some changes/considerations that we need to make when we decide to set up our integral in terms of y instead of x?

6.2.4.3. Set up (and practice evaluating) an integral expression representing the area of each of the regions described below.

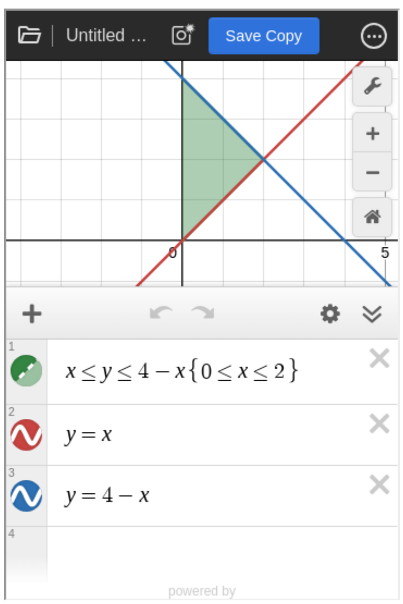
(a) The region bounded by the curves $y = x^2 + 1$ and $y = 4x + 1$ between $x = 0$ and $x = 2$.

Hint.



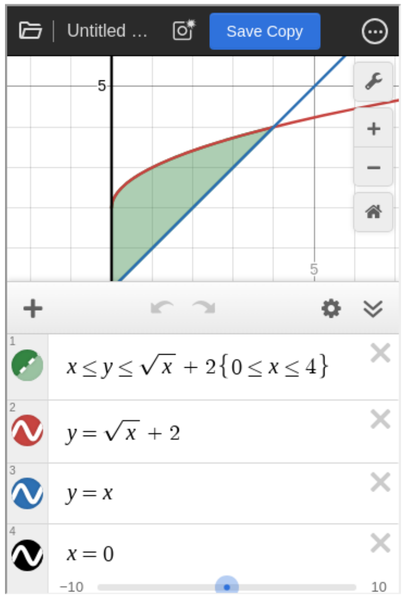
(b) The region bounded by the curves $y = x$ and $y = 4 - x$ between $x = 0$ and $x = 2$

Hint.



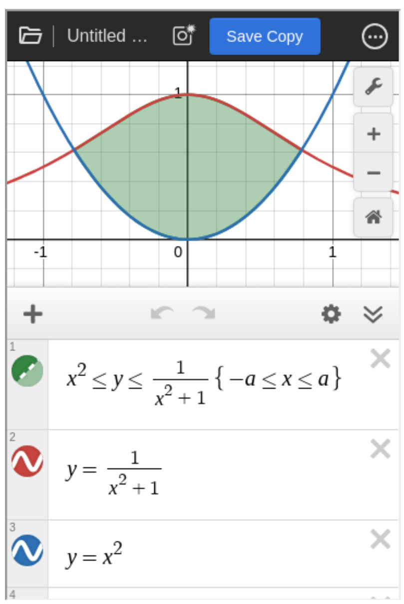
(c) The region bounded by the curves $y = \sqrt{x} + 2$ and $y = x$ and the line $x = 0$.

Hint.



(d) The region bounded by the curves $y = \dfrac{2}{x^2+1}$ and $y = x^2$.

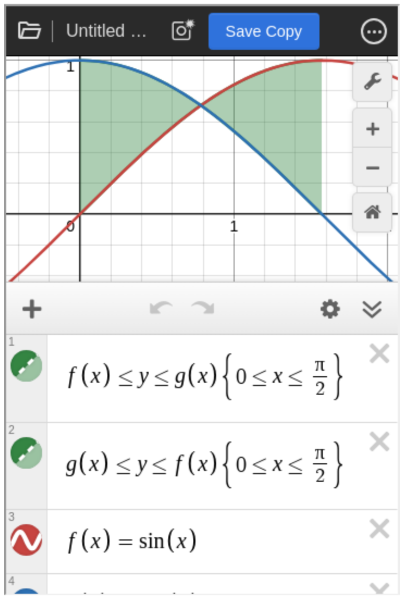
Hint.



6.2.4.4. Set up and evaluate an integral representing the area of each of the regions described below. Explain whether you chose to integrate with respect to x or y, and why you made that choice.

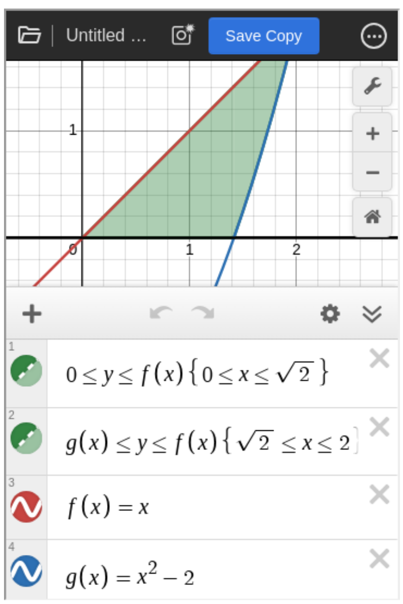
(a) The region bounded by the curves $y = \sin(x)$ and $y = \cos(x)$ and the line $y = 0$ between $x = 0$ and $x = \frac{\pi}{2}$.

Hint.



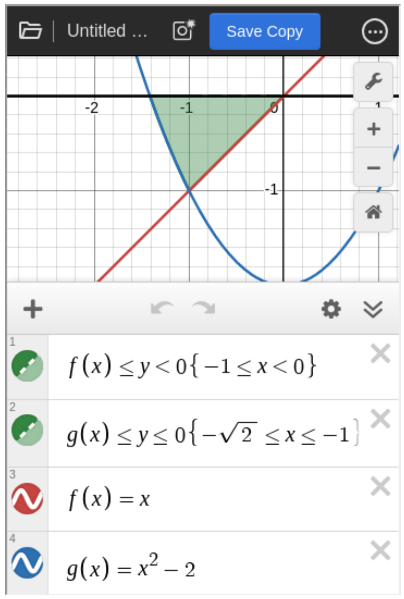
(b) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the first quadrant.

Hint.



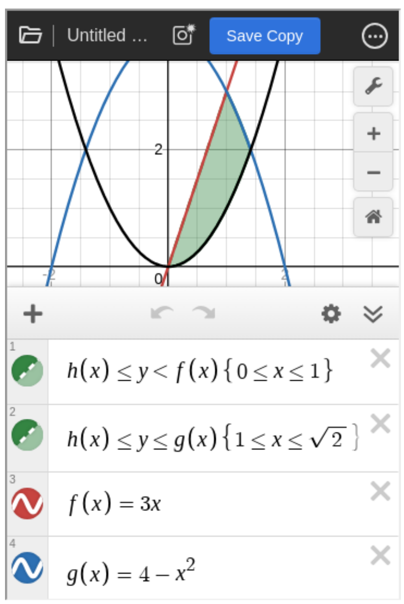
(c) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the third quadrant.

Hint.



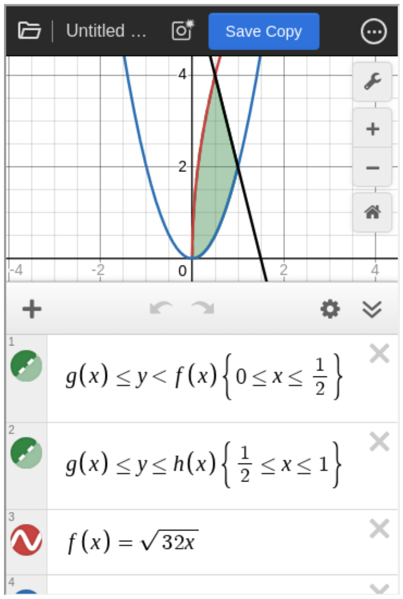
(d) The region bounded by the curves $y = 3x$, $y = 4 - x^2$, and $y = x^2$ in the first quadrant.

Hint.



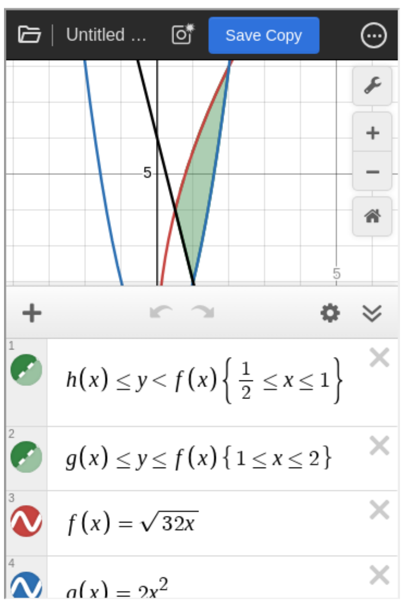
(e) The region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



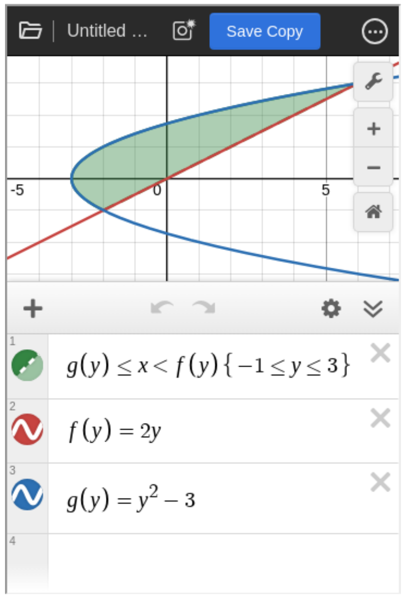
(f) The *other* region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



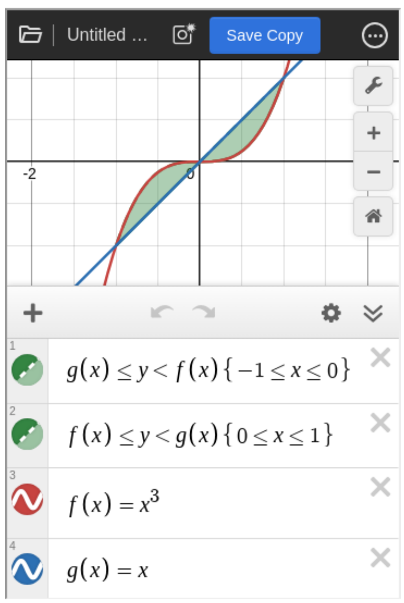
(g) The region bounded by the curves $x = 2y$ and $x = y^2 - 3$.

Hint.



(h) The region(s) bounded by the curves $y = x^3$ and $y = x$.

Hint.



6.3 Volumes of Solids of Revolution

6.3.2 Solids of Revolution

Activity 6.3.1 Carving out a Hole in the Center. We're going to look at the same solid as in Figure 6.3.2. But this time, when we define the 2-dimensional region that we're going to revolve around the x-axis, we're going to add a lower boundary function, $g(x)$.

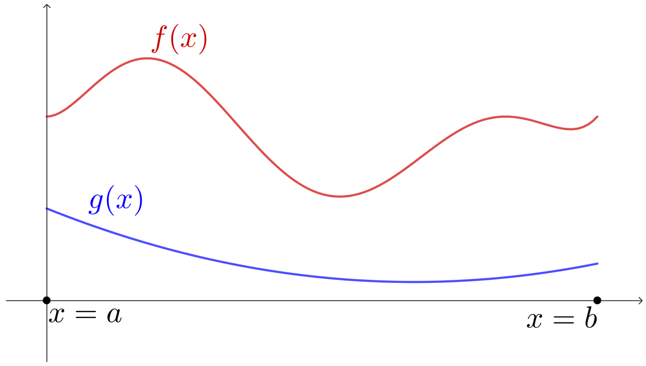


Figure 6.3.7

When we revolve this region around the x-axis, we get the following 3-dimensional solid.

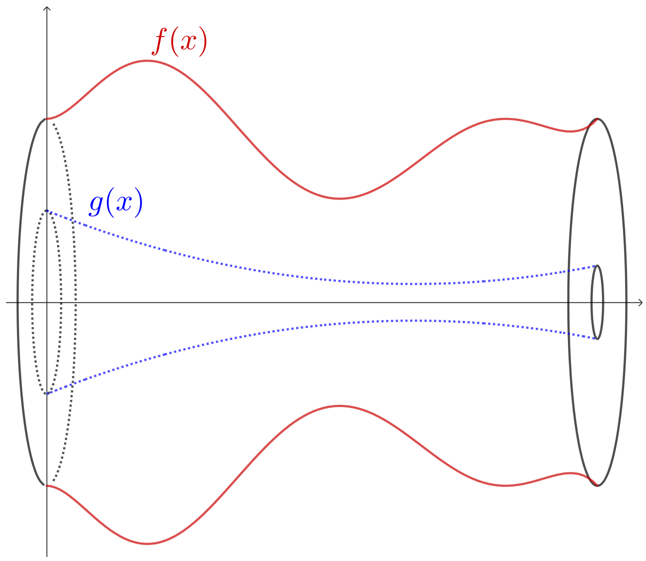


Figure 6.3.8

(a) How is a single generic slice on this solid different than the one in Figure 6.3.2?

Hint 1. Here is the rectangle that will define that slice!

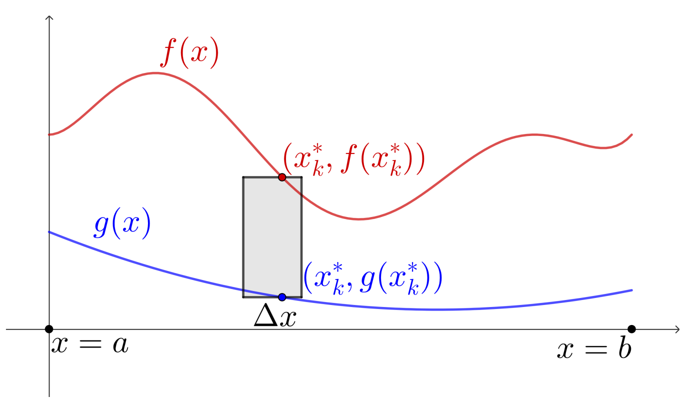


Figure 6.3.9

Hint 2. Here is the slice formed when the rectangle revolves around the axis!

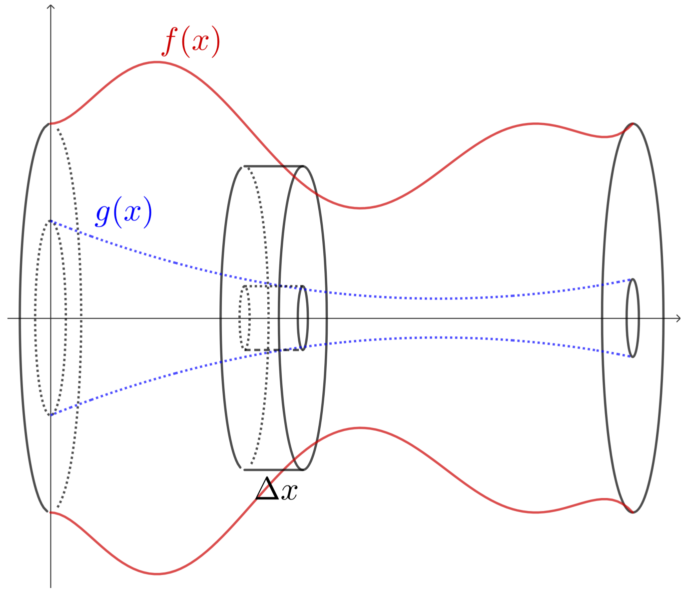


Figure 6.3.10

(b) Find a formula for the area of the face of the cross-secional slice.

Hint. Here's a picture of the face of the slice!

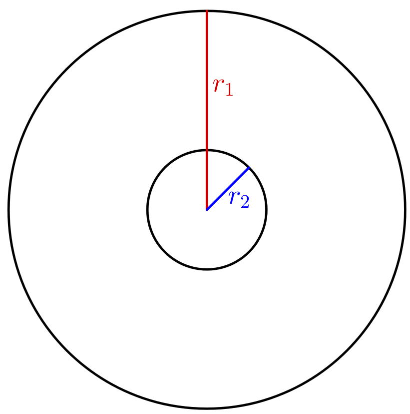


Figure 6.3.11

Solution.

$$A = \pi r_1^2 - \pi r_2^2$$
$$A(x_k^*) = \pi\left(f(x_k^*)\right)^2 - \pi\left(g(x_k^*)\right)^2$$

(c) Use the slice-and-sum process to create an integral expression representing the volume of this solid.

Solution.

$$\begin{aligned} V_k &= \left(\pi\left(f(x_k^*)\right)^2 - \pi\left(g(x_k^*)\right)\right)\Delta x \\ &= \pi\left((f(x_k^*))^2 - (g(x_k^*))^2\right)\Delta x \\ V &\approx \sum_{k=1}^{n} \pi\left((f(x_k^*))^2 - (g(x_k^*))^2\right)\Delta x \end{aligned}$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi \left((f(x_k^*))^2 - (g(x_k^*))^2\right) \Delta x$$
$$= \int_{x=a}^{x=b} \pi \left((f(x))^2 - (g(x))^2\right)\, dx$$

Activity 6.3.2 Volumes by Disks/Washers. Consider the region bounded between the curves $y = 4 + 2x - x^2$ and $y = \dfrac{4}{x+1}$. Will will create a 3-dimensional solid by revolving this region around the x-axis.

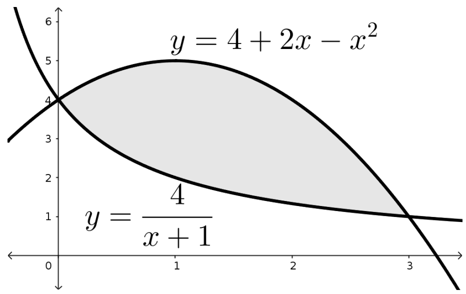


Figure 6.3.13

(a) Visualize the solid you'll create when you revolve this region around the x-axis.

(b) Draw a single rectangle in your region, standing perpendicular to the x-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

Hint. It should be a circle within a circle like in Figure 6.3.11, but can you be more detailed with labeling the radii?

(d) Find the area of the face of the kth slice.

Hint. Note that this is a 2-dimensional shape, and we're just finding the area of it.

Answer. $A(x_k^*) = \pi\left((4 + 2x_k^* - (x_k^*)^2)^2 - \left(\frac{4}{x_k^*+1}\right)^2\right)$

(e) Set up the integral representing the volume of the solid.

Hint. The sum is going to be adding up all of the areas multiplied by Δx. What will that look like in the integral?

Solution.

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi \left((4 + 2x_k^* - (x_k^*)^2)^2 - \left(\frac{4}{x_k^*+1}\right)^2\right) \Delta x$$
$$= \pi \int_{x=0}^{x=3} \left((4 + 2x - x^2)^2 - \left(\frac{4}{x+1}\right)^2\right)\, dx$$

(f) Can you describe how you would antidifferentiate and evaluate this integral?

Hint. What happens when you square these functions? What kinds of strategies will you use for the types of functions you're left with?

Solution. The first function, the quadratic, will be annoying to square. We'll end up with some big degree 4 polynomial, though, and antidifferentiating will be easy, since we can use the power rule.

The second function squared will give us $\frac{4}{(x+1)^2}$. We can use a u-substitution here with $u = x+1$. Then we have a negative exponent and we can use more power rule!

Activity 6.3.3 Another Volume. Now lets consider another region: this time, the one bounded between the curves $y = x$ and $y = 3\sqrt{x}$. We will, again, create a 3-dimensional solid by revolving this region around the y-axis.

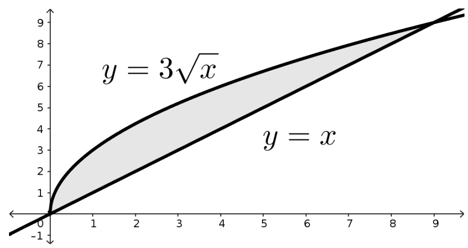


Figure 6.3.14

(a) Visualize the solid you'll create when you revolve this region around the y-axis.

(b) Draw a single rectangle in your region standing perpendicular to the y-axis.

Hint. Notice that your rectangle is sitting on its side now! This will change some things for us in a familiar way!

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

Hint. It should be a circle within a circle like in Figure 6.3.11, but can you be more detailed with labeling the radii?

(d) Find the area of the face of the kth slice.

Hint. Note that this is a 2-dimensional shape, and we're just finding the area of it. You'll also notice that the radii are measuring a horizontal distance in terms of a differing height, so you'll want to express these as functions of y.

Solution. The outer radius comes from the function $y = x$, but we'll invert it to be $x = y$.

The inner radius comes from the function $y = 3\sqrt{x}$, but we'll invert it be written as $x = \left(\frac{y}{3}\right)^2$ or $x = \frac{y^2}{9}$.

$$A(y_k^*) = \pi\left((y_k^*)^2 - \left(\frac{(y_k^*)^2}{9}\right)^2\right)$$

(e) Set up the integral representing the volume of this solid.

Solution.

$$V = \lim_{n\to\infty}\sum_{k=1}^{n}\pi\left((y_k^*)^2 - \left(\frac{(y_k^*)^2}{9}\right)^2\right)\Delta y$$

$$= \pi \int_{y=0}^{y=9} y^2 - \frac{y^4}{81}\, dy$$

6.3.3 Re-Orienting our Rectangles

Activity 6.3.4 Volume by Shells. Let's consider the region bounded by the curves $y = x^3$ and $y - x + 6$ as well as the line $y = 0$. You might remember this region from Activity 6.2.3. This time, we'll create a 3-dimensional solid by revolving the region around the x-axis

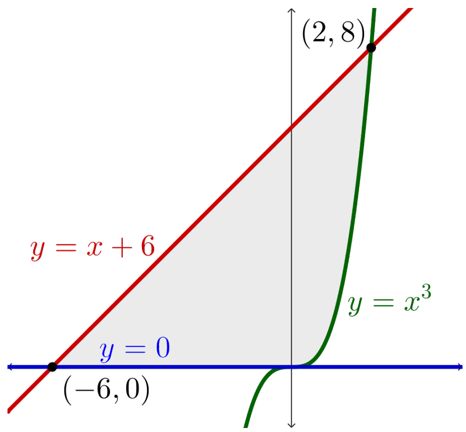


Figure 6.3.20

(a) Sketch one or two rectangles that are *perpendicular* to the x-axis. Then set up an integral expression to find the volume of the solid using them.

Hint. Note that in this context, we're actually using disks and washers. Also note that the bottom of the rectangles are bounded by $y = 0$ from $x = -6$ to $x = 0$ and then switches to being bounded by $y = x^3$ from $x = 0$ to $x = 2$.

Answer.

$$V = \pi \int_{x=-6}^{x=0} (x+6)^2\, dx + \pi \int_{x=0}^{x=2} (x+6)^2 - (x^3)^2\, dx$$

(b) Now draw a single rectangle in the region that is *parallel* to the axis of revolution. Use this rectangle to visualize the kth slice of this 3-dimensional solid. What does that single rectangle create when it is revolved around the x-axis?

Hint. This won't create a disk or washer! We'll have to change variables, and try to see how we can create a shell.

(c) Set up the integral expression representing the volume of the solid.

Answer.

$$V = 2\pi \int_{y=0}^{y=8} y(\sqrt[3]{y} - (y-6))\, dy$$

(d) Confirm that your volumes are the same, no matter your approach to setting it up.

6.3.4 Practice Problems

6.3.4.1. We say that the volume of a solid can be thought of as $\displaystyle\int_{x=a}^{x=b} A(x)\,dx$ where $A(x)$ is a function describing the cross-sectional area of our solid at an x-value between $x = a$ and $x = b$. Explain how this integral formula gets built, referencing the slice-and-sum (Riemann sum) method.

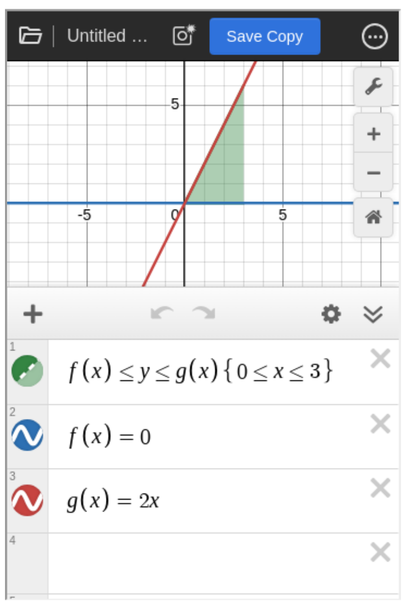
6.3.4.2. Explain the differences and similarities between the disk and washer methods for finding volumes of solids of revolution.

6.3.4.3. When do we integrate with regard to x (using a dx in our integral and writing our functions with x-value inputs) and when do we integrate with regard to y (using a dy in our integral and writing our functions with y-value inputs) when we're finding volumes using disks and washers? How do we know?

6.3.4.4. For each of the solids described below, set up an integral using the *disk/washer method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

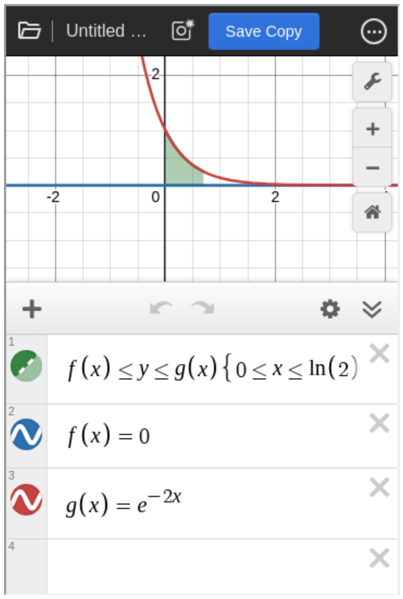
(a) The region bounded by the curve $y = 2x$ and the lines $y = 0$ and $x = 3$, revolved around the x-axis.

Hint.



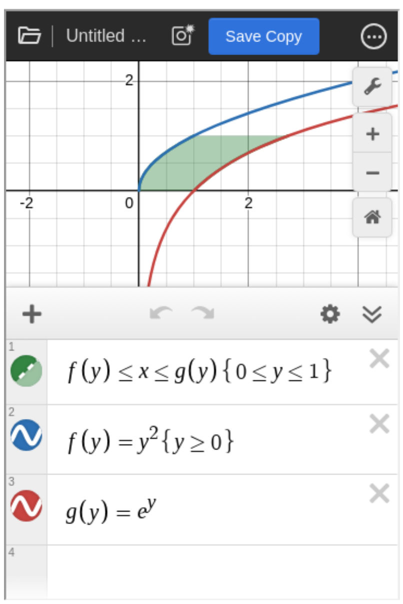
(b) The region bounded by the curve $y = e^{-2x}$ and the x-axis between $x = 0$ and $x = \ln(2)$, revolved around the x-axis.

Hint.



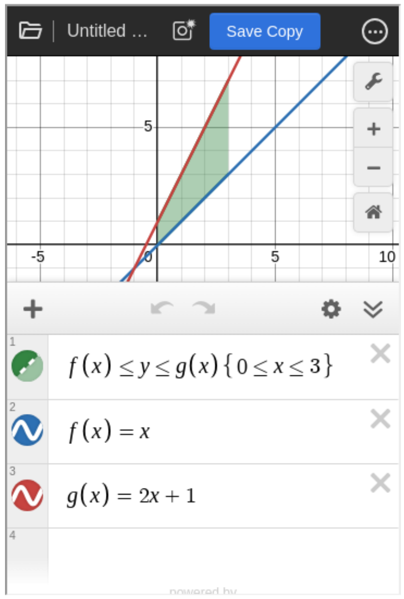
(c) The region bounded by the curves $y = \ln(x)$ and $y = \sqrt{x}$ between $y = 0$ and $y = 1$, revolved around the y-axis.

Hint.



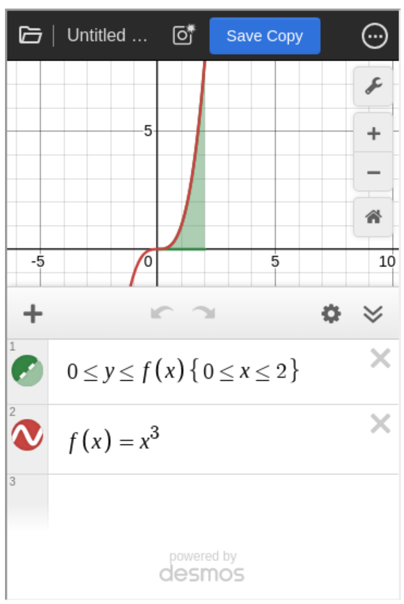
(d) The region bounded by the curves $y = 2x + 1$ and $y = x$ between $x = 0$ and $x = 3$, revolved around the x-axis.

Hint.



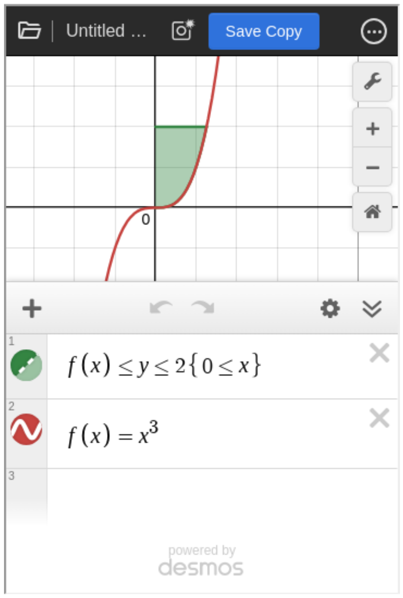
(e) The region bounded by the curve $y = x^3$, the x-axis, and the line $x = 2$, revolved around the y-axis.

Hint.



(f) The region bounded by the curve $y = x^3$ and the y-axis between $y = 0$ and $y = 2$, revolved around the y-axis.

Hint.



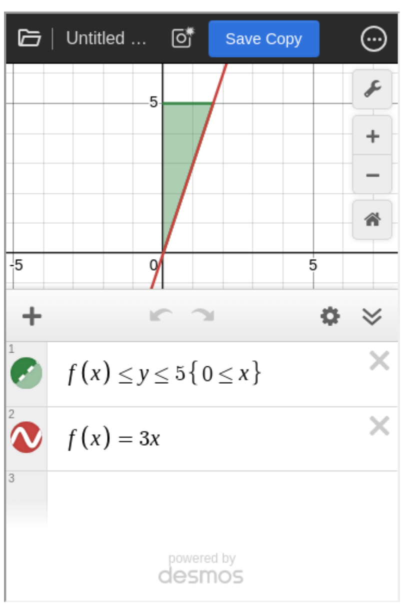
6.3.4.5. Explain where the pieces of the shell formula come from. How is this different than using disks/washers?

6.3.4.6. Say we're revolving a region around the x-axis to create a solid. Using the disk/washer method, we will integrate with respect to x. Using the shell method, we integrate with respect to y. Explain the difference, and why this difference occurs.

6.3.4.7. For each of the solids described below, set up an integral using the *shell method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

(a) The region bounded by the curve $y = 3x$ and the lines $x = 0$ and $y = 5$, revolved around the y-axis.

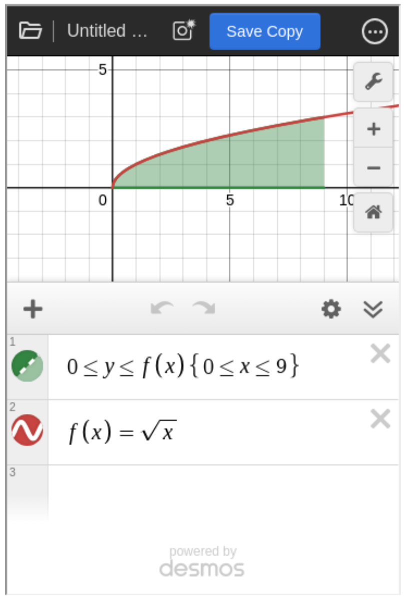
Hint.



(b) The region bounded by the curve $y = \sqrt{x}$ and the x-axis between $x = 0$

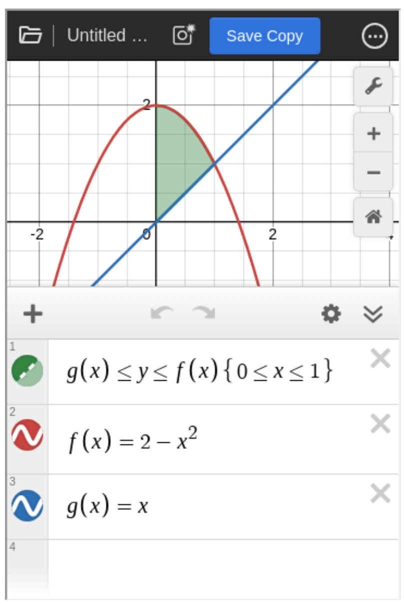
and $x = 9$, revolved around the x-axis.

Hint.



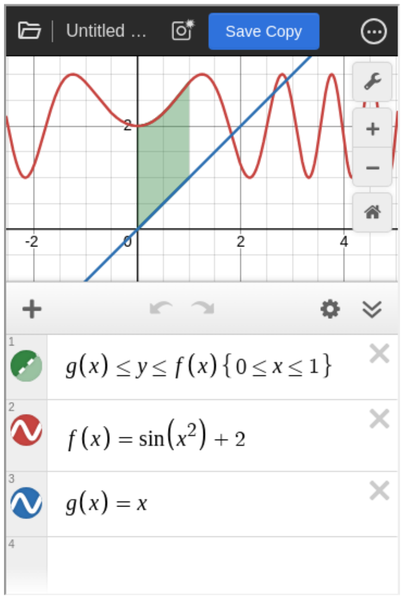
(c) The region bounded by the curves $y = 2 - x^2$ and $y = x$ and the line $x = 0$ revolved around the y-axis.

Hint.



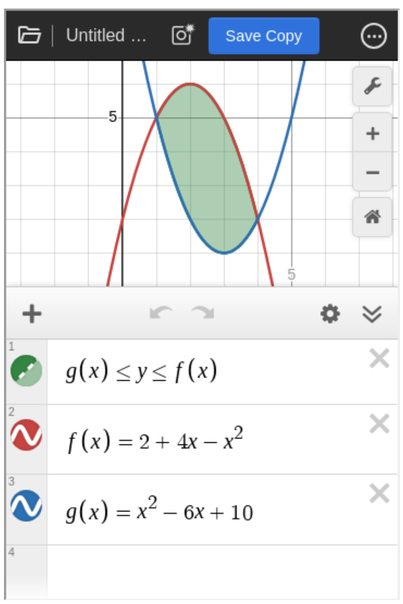
(d) The region bounded by the curves $y = sin(x^2) + 2$ and $y = x$ from $x = 0$ to $x = 1$, revolved around the y-axis.

Hint.



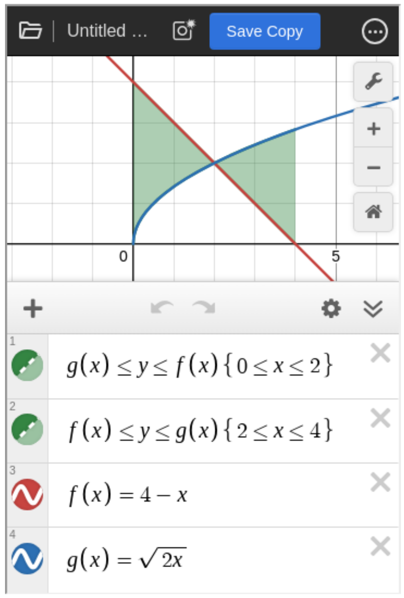
(e) The region bounded by the curves $y = x^2 - 6x + 10$ and $y = 2 + 4x - x^2$ revolved around the y-axis.

Hint.



(f) The region bounded by the curves $y = \sqrt{2x}$ and $y = 4 - x$ and the x-axis between $x = 0$ and $x = 4$, revolved around the x-axis.

Hint.



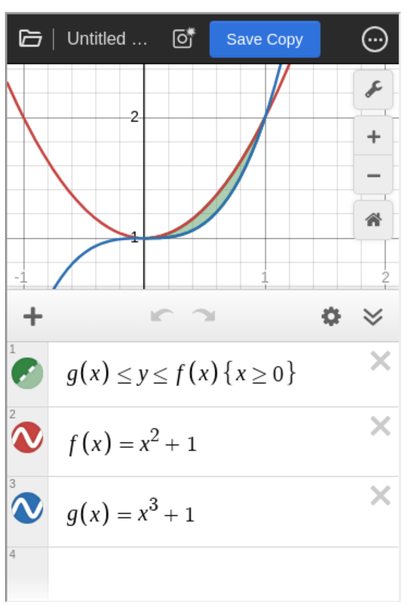
6.3.4.8. Pick at least 2 integrals from Exercise 4 to re-write using shells instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

6.3.4.9. Pick at least 2 integrals from Exercise 7 to re-write using disks/washers instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

6.3.4.10. For each of the following solids, set up an integral expression using either the disk/washer method or the shell method. You don't need to evaluate them, but you should do some careful thinking about how you set these up, especially as you choose between methods and what variable you are integrating with.

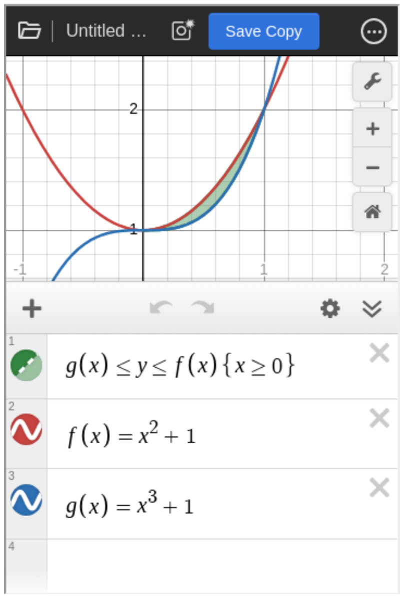
(a) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the x-axis.

Hint.



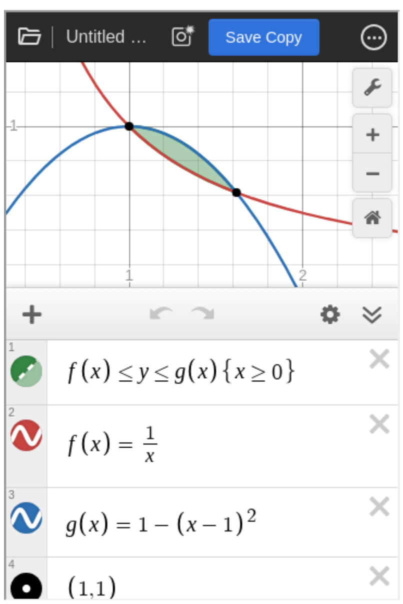
(b) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the y-axis.

Hint.



(c) The region bounded by the curves $y = \frac{1}{x}$ and $y = 1 - (x-1)^2$ in the first quadrant, revolved around the x-axis.

Hint.



6.4 More Volumes: Shifting the Axis of Revolution

6.4.1 What Changes?

Activity 6.4.1 What Changes (in the Washer Method) with a New Axis? Let's revisit Activity 6.3.2 Volumes by Disks/Washers, and ask some more follow-up questions. First, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = -3$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.2?

(b) What changes, if anything, do you have to make to the area of the "face" kth washer?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

Activity 6.4.2 What Changes (in the Shell Method) with a New Axis? Let's revisit Activity 6.3.4 Volume by Shells, and ask some more follow-up questions about the shell method. Again, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = 9$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.4?

(b) What changes, if anything, do you have to make to the area of the rectangle formed by "unrolling" up kth cylinder?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

6.4.2 Formalizing These Changes in the Washers and Shells

Activity 6.4.3 More Shifted Axes. We're going to spend some time constructing *several* different volume integrals in this activity. We'll consider the same region each time, but make changes to the axis of revolution. For each, we'll want to think about what kind of method we're using (disks/washers or shells) and how the different axis of revolution gets implemented into our volume integral formulas.

Let's consider the region bounded by the curves $y = \cos(x) + 3$ and $y = \dfrac{x}{2}$ between $x = 0$ and $x = 2\pi$.

(a) Let's start with revolving this around the x-axis and thinking about the solid formed. While you set up your volume integral, think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

Hint. Note that in this region, we definitely want to use rectangles that stand up vertically. That means that they'll have a very small width, Δx, and sit perpendicular to the axis of revolution.

(b) Now let's create a different solid by revolving this region around the y-axis. Set up a volume integral, and continue to think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

Hint. We still will use the same tall rectangle with a small Δx side length, but this time it will be parallel to our axis of revolution.

(c) We'll start shifting our axis of revolution now. We'll revolve the same region around the horizontal line $y = -1$ to create a solid. Set up an integral expression to calculate the volume.

Hint 1. Note that we're still using the same rectangle (perpendicular to this horizontal axis), and so still integrating with regard to x, and using the washer method.

Hint 2. Since in the washer method our function outputs represent the radii, we need to re-measure the distance from our curves to the axis of revolution to find each circle's radius in the washer formula. For a y-value on each curve, how do we find the vertical distance down to the line $y = -1$?

(d) Now revolve the region around the line $y = 5$ to create a solid of revolution, and write down the integral representing the volume.

Hint. Note, now, that the y-value of the axis of revolution is larger than all of the y-values on the curves, meaning that to measure the distance from the axis of revolution to the curves, we might measure them in the opposite direction. Also, which curve is further away from the axis of revolution, representing the larger/outer radius?

(e) Let's change things up. Revolve the region around the vertical line $x = -1$ to create a new solid. Set up an integral representing the volume of that solid.

Hint 1. Note that the same rectangle that we used before is standing parallel to our axis of revolution. We're going to change methodology, and use the shell method!

Hint 2. Normally we use the input variable (x in this case) to measure the radius from the rectangles at different x-value to the axis of revolution, the y-axis. Now, though, we're not looking at the distance from x-values to $x = 0$. We're looking to find the radius, the distance from x-values in this region to $x = -1$.

(f) We'll do one more solid. Let's revolve this region around the line $x = 7$. Set up an integral representing the volume.

Hint. Note that this time, the axis of revolution's x-value is larger than all of the x-values in our region. So when we subtract to measure the radius, we need to subtract from $x = 7$ down to the varying x-values in the region.

6.4.3 Practice Problems

6.4.3.1. Consider the integral formula for computing volumes of a solid of revolution using the disk/washer method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius represented using the function output from the curve(s) defining the region?

6.4.3.2. Consider the integral formula for computing volumes of a solid of revolution using the shell method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius not represented using the function output from the curve(s) defining the region?

6.4.3.3. For each of the solids described below, set up an integral expression using disks/washers representing the volume of the solid.

(a) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = 2$.

(b) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = -1$.

(c) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = -2$.

(d) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = 3$.

6.4.3.4. For each of the solids described below, set up an integral expression using shells representing the volume of the solid.

(a) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = 2$.

(b) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = -1$.

(c) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = -2$.

(d) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = 3$.

6.5 Arc Length and Surface Area

6.5.1 Integrals for Evaluating the Length of a Curve

Activity 6.5.1 Measuring Distance.

(a) Consider the following right-triangle with the normal names of side lengths.

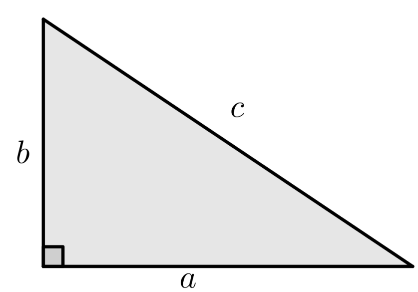


Figure 6.5.1

How do we use the Pythagorean Theorem to find the length of c?

(b) Consider the two points (x_1, y_1) and (x_2, y_2) below.

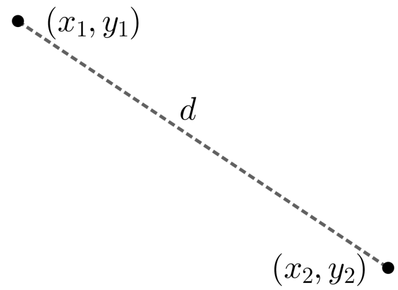


Figure 6.5.2

How do we use the distance formula to find the length of the line connecting the two point, d?

(c) How are these two things the same? How are they different?

7 Techniques for Antidifferentiation

7.1 Improper Integrals

7.1.1 Improprt Integrals

Activity 7.1.3 Approximating Improper Integrals. In this activity, we're going to look at two improper integrals:

1. $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\,dx$
2. $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\,dx$

(a) First, let's just clarify to ourselves what it means for an integral to be improper. Why are each of these integrals improper? Be specific!

(b) Let's focus on $\displaystyle\int_{x=2}^{\infty} \frac{1}{(x+1)^2}\,dx$ first. We're going to look at the slightly different integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\,dx.$$

As long as t is some real number with $t > 2$, then our function is continuous and bounded on $[2, t]$, and so we can evaluate this integral:

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\,dx = F(t) - F(2)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$.

Find and antiderivative, $F(x)$.

Hint 1. You'll need to use u-substitution, with $u = (x+1)$ and $du = dx$.

Hint 2. You should be thinking about finding an antiderivative using an indefinite integral: $\displaystyle\int \frac{1}{u^2}\,du = \int u^{-2}\,du$.

(c) Now we're going to evaluate some areas for different values of t. Use your antiderivative $F(x)$ from above!

- Let's start with making $t = 99$. So we're going to evaluate:

$$\int_{x=2}^{x=99} \frac{1}{(x+1)^2}\,dx = F(99) - F(2)$$

- Now let $t = 999$. Evaluate:

$$\int_{x=2}^{x=999} \frac{1}{(x+1)^2}\,dx = F(999) - F(2)$$

- Now let $t = 9999$. Evaluate:

$$\int_{x=2}^{x=9999} \frac{1}{(x+1)^2}\,dx = F(9999) - F(2)$$

(d) Based on what you found, what do you *think* is happening when $t \to \infty$ to the definite integral

$$\int_{x=2}^{x=t} \frac{1}{(x+1)^2}\,dx = F(t) - F(2)?$$

(e) Ok, we're going to switch our focus to the other improper integral, $\displaystyle\int_{x=-1}^{x=2} \frac{1}{(x+1)^2}\,dx$. again, we'll look at a slightly different integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\,dx.$$

As long as t is some real number with $-1 < t < 2$, then our function is continuous and bounded on $[t, 2]$, and so we can evaluate this integral:

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\,dx = F(2) - F(t)$$

where $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{(x+1)^2}$. We can just use the same antiderivative as before!

We're going to evaluate this intergal for different values of t again, but this time we'll use values that are close to -1, but slightly bigger, since we want to be in the interval $[-1, 2]$.

- Let's start with making $t = -\frac{9}{10}$. So we're going to evaluate:

$$\int_{x=-9/10}^{x=2} \frac{1}{(x+1)^2}\,dx = F(2) - F\left(-\frac{9}{10}\right)$$

- Now let $t = -\frac{99}{100}$. Evaluate:

$$\int_{x=-99/100}^{x=2} \frac{1}{(x+1)^2}\,dx = F(2) - F\left(-\frac{99}{100}\right)$$

- Now let $t = -\frac{999}{1000}$. Evaluate:

$$\int_{x=-999/1000}^{x=2} \frac{1}{(x+1)^2}\,dx = F(2) - F\left(-\frac{999}{1000}\right)$$

(f) Based on what you found, what do you *think* is happening when $t \to -1^+$ to the definite integral

$$\int_{x=t}^{x=2} \frac{1}{(x+1)^2}\,dx = F(2) - F(t)?$$

We can think about putting this a bit more generally into limit notation, but we'll get to this later.

7.1.4 Practice Problems

7.1.4.1. Explain what it means for an integral to be improper. What kinds of issues are we looking at?

7.1.4.2. Give an example of an integral that is improper due to an unbounded or infinite interval of integration (infinite width).

7.1.4.3. Give an example of an integral that is improper due to an unbounded integrand (infinite height).

7.1.4.4. What does it mean for an improper integral to "converge?" How does this connect with limits?

7.1.4.5. What does it mean for an improper integral to "diverge?" How does this connect with limits?

7.1.4.6. Why do we need to use limits to evaluate improper integrals?

7.1.4.7. For each of the following improper integrals:

- Explain why the integral is improper. Be specific, and point out the issues in detail.
- Set up the integral using the correct limit notation.
- Antidifferentiate and evaluate the limit.
- Explain whether the integral converges or diverges.

(a) $\displaystyle\int_{x=0}^{\infty} \frac{1}{\sqrt{x+1}}\, dx$

(b) $\displaystyle\int_{x=0}^{\infty} e^{-2x}\, dx$

(c) $\displaystyle\int_{x=-1}^{x=3} \frac{1}{x+1}\, dx$

(d) $\displaystyle\int_{-\infty}^{x=0} \sqrt{e^x}\, dx$

(e) $\displaystyle\int_{x=2}^{x=8} \frac{5}{(x-2)^3}\, dx$

(f) $\displaystyle\int_{x=1}^{x=12} \frac{dx}{\sqrt[5]{12-x}}$

7.1.4.8. One of the big ideas in probability is that for a curve that defines a probability density function, the area under the curve needs to be 1. What value of k makes the function $\dfrac{kx}{(x^2+3)^{5/4}}$ a valid probability distribution on the interval $[0,\infty)$?

7.1.4.9. Let's consider the integral $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\, dx$. This is a difficult integral to evaluate!

(a) First, compare $\sqrt{x^2+1}$ to $\sqrt{x^2}$ using an inequality: which one is bigger?

(b) Second, use this inequality to compare the function $\dfrac{\sqrt{x^2+1}}{x^2}$ to $\dfrac{1}{x}$ for

$x > 0$: which one is bigger? Again, use your inequality from above to help!

(c) Now compare $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ to $\displaystyle\int_{x=1}^{\infty} \frac{1}{x}\,dx$. Which one is bigger?

(d) Explain how we can use this result to make a conclusion about whether our integral, $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\,dx$ converges or diverges.

7.2 More on u-Substitution

7.2.1 Variable Substitution in Integrals

Activity 7.2.2 Turn Around Problems. The two integrals that we're going to look at are "just" some u-substitution problems, but I like to call integrals like these **turn-around** problems. We'll see why!

(a) Consider the integral:

$$\int x\sqrt[3]{x+5}\,dx.$$

First, explain why u-substitution is reasonable here.

Hint. Do you see composition? A function inside of something?

(b) Identify du for your chosen u-substitution. When you substitute, you should notice that there are some extra bits in this integrand function that have not been assigned to be translated over to be written in terms of u. Which parts?

Solution. If we let $u = x+5$, then $du = dx$.

$$\int x\sqrt[3]{x+5}\,dx = \int {\color{red}x}\sqrt[3]{u}\,du$$

(c) We need to think about how to write x in terms of u. Luckily, we already have everything we need! We have defined a link between the x variable and the u variable. We defined it as u being written as some function of x, but can we "turn around" that link to write x in terms of u?

Hint. We let $u = x+5$. Solve this for x

(d) Substitute the integral to be fully written in terms of u.

Solution.

$$u = x+5 \longrightarrow x = u-5$$
$$du = dx$$
$$\begin{aligned}\int x\sqrt[3]{x+5}\,dx &= \int (u-5)\sqrt[3]{u}\,du\\ &= \int (u-5)u^{1/3}\,du\end{aligned}$$

(e) Before antidifferentiating, compare this integral with the original one. Specifically thinking about how we might multiply, describe the differences between the integrals with regard to composition and re-writing our integrand.

Then, go ahead and use this nicely re-written version to antidifferentiate and substitute back to x.

Solution.

$$\begin{aligned}\int x\sqrt[3]{x+5}\,dx &= \int (u-5)u^{1/3}\,du\\ &= \int u^{4/3} - 5^{1/3}\,du\\ &= \frac{3u^{7/3}}{7} - \frac{15u^{4/3}}{4} + C\\ &= \frac{3(x+5)^{7/3}}{7} - \frac{15(x+5)^{4/3}}{4} + C\end{aligned}$$

(f) Apply this same strategy to the following integral:

$$\int \frac{x}{x+5}\,dx.$$

This integral might be a bit trickier to find the composition in order to identify the u-substitution! Give some things a try!

Solution.

$$\begin{aligned}u = x+5 &\longrightarrow x = u-5\\ du &= dx\\ \int \frac{x}{x+5}\,dx &= \int \frac{u-5}{u}\,du\\ &= \int \frac{u}{u} - \frac{5}{u}\,du\\ &\int 1 - \frac{5}{u}\,du\\ &= u - 5\ln|u| + C\\ &= (x+5) - 5\ln|x+5| + C\end{aligned}$$

Since this antiderivative has an extra constant (the 5 being added to x), we can write a smaller version of this by combining the 5 with the constant of integration:

$$\int \frac{x}{x+5}\,dx = x - 5\ln|x+5| + C.$$

(g) Compare your integral in terms of x with the substituted version, in terms of u. Why was the second one so much easier to think about or re-write?

7.3 Manipulating Integrands

7.3.1 Rewriting the Integrand

Activity 7.3.1 A Negative Exponent. Let's think about this integral:

$$\int \frac{1}{1+e^{-x}}\,dx.$$

(a) Is there any composition in this integral? Pick it out, and either explain or show that using this to guide your substitution will not be helpful.

Hint. Notice that $-x$ is composed inside of the exponential function. Try a substitution with $u = -x$.

(b) What does e^{-x} mean? What does $\frac{1}{e^{-x}}$ mean?

(c) Re-write the integral, specifically focusing on the negative exponent. You should find that the function looks worse! How can you clean that up?

Hint 1. Re-write e^{-x} as $\frac{1}{e^x}$, giving you:

$$\int \frac{1}{1 + \frac{1}{e^x}}\, dx.$$

Hint 2. Either add the fractions in the denominator or multiply the whole fraction by $\frac{e^x}{e^x}$.

Solution. You should have an integral that looks like:

$$\int \frac{e^x}{e^x + 1}\, dx.$$

(d) Why is this new integral set up so much better for the purpose of u-substitution? How could we tell this just by looking at the initial integral?

7.3.2 Antidifferentiating Rational Functions

Activity 7.3.2 Integrating a Rational Function Three Ways. We're going to think about the integral:

$$\int \left(\frac{x^2 + 3x - 1}{x - 1}\right) dx.$$

Let's find 3 different ways of integrating this. This is kind of misleading, since we're actually going to look at 2, since we've already used u-substitution to integrate this in Example 7.2.1.

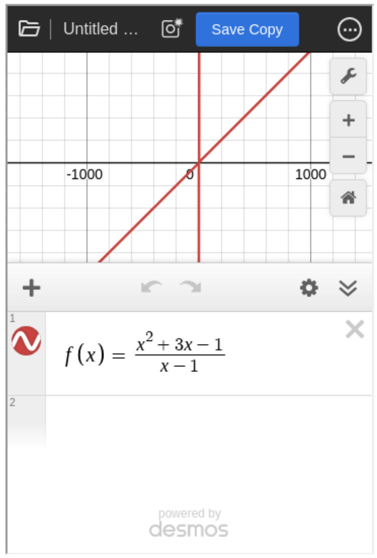
(a) Let's just notice some things about this rational function.

- Are there any vertical asymptotes? How do you know where to find them?
- Are there any horizontal asymptotes? How do you know that there *aren't*?
- When you zoom really far out on the graph of this function, it looks like a different kind of function. What kind of function? Why is that?

Solution.

- There's a vertical asymptote at $x = 1$, since that's where we'd find a limit with the form $\frac{\#}{0}$, like in Subsection 1.4.1 Infinite Limits.
- There aren't any horizontal asymptotes! We know that because the degree is larger in the numerator, like in Subsection 1.4.2 End Behavior Limits.

- This graph looks linear:



This is because the numerator is one degree larger than the denominator. When we divide these two functions, we expect to end up with some sort of linear function and a remainder that approachs 0 as $x \to \pm\infty$.

(b) Now we're going to re-write the function itself: $\dfrac{x^2+3x-1}{x-1}$ means we're dividing (x^2+3x-1) by $(x-1)$. So let's do the division!

$$x-1 \enclose{longdiv}{x^2 \quad +3x \quad -1}$$

Solution.

$$\begin{array}{rlll} & x & +4 & +\frac{3}{x-1} \\ x-1 \,\big) & x^2 \quad +3x & -1 & \\ - & (x^2 \quad -x) & \downarrow & \\ \hline & \qquad\;\; 4x & -1 & \\ & - \quad (4x & -4) & \\ \hline & & 3 & \end{array}$$

(c) Re-write your integral using this new version of the function. Notice that we haven't done any calculus or antidifferentiating yet. Explain why this new version of this integrand function is easier to antidifferentiate. What do you get?

Solution.

$$\begin{aligned} \int \left(\frac{x^2+3x-1}{x-1}\right) dx &= \int x+4+\frac{3}{x-1}\,dx \\ &= \frac{x^2}{2}+4x+3\ln|x-1|+C \end{aligned}$$

(d) Let's approach this integral differently. We said earlier that this function is really an "almost" linear function in disguise: when we divide the quadratic numerator by a linear denominator, we expect a linear function to be left over. In the long division, we saw this happen! We ended with a linear function and some remainder.

Let's try to uncover this linear function. If we're looking to find what linear functions multiply together to get $(x^2 + 3x - 1)$, then we can try factoring!

$$\frac{x^2 + 3x - 1}{x - 1} = \frac{(\quad\quad)(\quad\quad)}{x - 1}$$

In order for this factoring to be useful, we want to be able to "cancel" out the $x - 1$ factor in the denominator. We're really only interested in what linear factor will multiply by $(x - 1)$ to get $(x^2 + 3x - 1)$.

$$\frac{x^2 + 3x - 1}{x - 1} = \frac{(x - 1)(\quad\quad)}{x - 1}$$

First, explain why there is no linear function factor that accomplishes this.

(e) What if were able to "almost" factor this?

If there *was* a linear factor that multiplied by $(x - 1)$ to get $(x^2 + 3x - 1)$, then the linear portions would multiply together to get x^2. What does this mean about the first linear term of our factor?

Solution. We need $(x - 1)(x + \quad) = (x^2 + 3x - 1)$ since $x(x) = x^2$.

(f) What does the constant term of our missing factor need to be? We are hoping that whatever it is can multiply by x (from $x - 1$) and combine with the $-x$ (from the constant -1 multiplied by x in our missing factor) to match the $3x$ in $x^2 + 3x - 1$.

What is it?

Solution. We need to find a where when we multiply $(x - 1)(x + a)$ we end up with $ax - x = 3x$. It should be clear that $a = 4$.

Our missing factor is $(x - 4)$.

(g) Note that we have *not* factored $(x^2 + 3x - 1)$! We *almost* did: we found two factors:

$$(x - 1)(\quad\quad) = x^2 + 3x + \quad.$$

How far off is the actual polynomial that we are working with, $x^2 + 3x - 1$?

Write $x^2 + 3x - 1$ as your two factors plus or minus some remainder.

Hint. Find the constant term where:

$$x^2 + 3x - 1 = (x - 1)(x + 4) + \quad.$$

Solution. Since $(x - 1)(x + 4) = x^2 + 3x - 4$, we need to add 3 to get $x^2 + 3x - 1$.

$$x^2 + 3x - 1 = \overbrace{(x - 1)(x + 4)}^{x^2+3x-4} + 3$$

(h) You should get the same thing that we got from using long division! Great! The rest of the integral will work the same.

Before we end, though, compare this antiderivative to the one we got in Example 7.2.1. It's different. Why? Is this a problem?

Hint. It's only off by a constant! Show this by expanding all of the multiplication in

$$\frac{(x-1)^2}{2} + 5(x-1) + 3\ln|x-1|.$$

Activity 7.3.3 Comparing Two Very Similar Integrals. We're going to compare these two integrals:

$$\int \frac{x+2}{x^2+4x+5}\,dx \qquad\qquad \int \frac{2}{x^2+4x+5}\,dx$$

(a) Describe why $u = x^2+4x+5$ is such a useful choice for the first integral, but not for the second. How do the differences in these two integrals influence this substitution, even though the denominators are the same?

Hint. If $u = x^2+4x+5$, then $du = (2x+4)\,dx$ for both integrals. Why is this good for one integral but not the other?

(b) Why would it be useful to have a *linear* substitution rule (instead of the *quadratic* one that we picked) for the second integral? Why would that match the structure of the numerator better?

Go ahead and integrate the first integral.

Solution.

$$u = x^2+4x+5 \quad du = (2x+4)\,dx$$

$$\begin{aligned}\int \frac{x+2}{x^2+4x+5}\,dx \\ &= \frac{1}{2}\int \frac{2(x+2)}{x^2+4x+5}\,dx \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^2+4x+5| + C\end{aligned}$$

(c) We're going to write the denominator, x^2+4x+5 in a different way, in order to get a linear function composed into something familiar.

Complete the square for this polynomial: that is, find some linear factor $(x+k)$ and a real number b such that $(x+h)^2+b = x^2+4x+3$. This should feel familiar, since we have already tried to force polynomials to factor cleverly in Activity 7.3.2.

Hint. We want to find a constant term so that $(x+\quad)^2$ gives us $x^2+4x+\quad$. Then we can compare the quadratic to x^2+4x+5 to see how far off it is!

Solution. We can use $(x+2)^2$ since $(x+2)^2 = x^2+4x+4$. Then we can write:

$$x^2+4x+5 = \overbrace{(x+2)^2}^{x^2+4x+4} + 1.$$

(d) There is an intuitive substitution to pick, since we now have more obvious composition. Pick it. What kind of integral do we end up with and how do we antidifferentiate? Complete this problem!

Solution.

$$\int \frac{4}{x^2+4x+5}\,dx = \int \frac{4}{(x+4)^2+1}\,dx$$

$$u = x+4 \quad du = dx$$

$$\begin{aligned}\int \frac{4}{(x+4)^2+1}\,dx &= \int \frac{4}{u^2+1}\,du \\ &= 4\tan^{-1}(u)+C \\ &= 4\tan^{-1}(x+4)+C\end{aligned}$$

7.3.3 Practice Problems

7.3.3.1. Use polynomial division or some clever factoring to re-write and find the following indefinite integrals or evaluate the following definite integrals.

(a) $\displaystyle\int \left(\frac{x+4}{x-3}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x^2+4}{x-4}\right)\,dx$

(c) $\displaystyle\int \left(\frac{t^2+t+6}{t^2+1}\right)\,dt$

(d) $\displaystyle\int_{x=2}^{x=4} \left(\frac{x^3+1}{x-1}\right)\,dx$

(e) $\displaystyle\int_{x=0}^{x=1} \left(\frac{x^4+1}{x^2+1}\right)\,dx$

7.3.3.2. Complete the square in order to find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x^2-2x+10}\right)\,dx$

(b) $\displaystyle\int \left(\frac{x}{x^2+4x+8}\right)\,dx$

(c) $\displaystyle\int \left(\frac{2x}{x^4+6x^2+10}\right)\,dx$

7.3.3.3. Find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x^{-1}+1}\right)\,dx$

(b) $\displaystyle\int \left(\frac{\sin(\theta)+\tan(\theta)}{\cos^2(\theta)}\right)\,d\theta$

(c) $\displaystyle\int \left(\frac{1-x}{1-\sqrt{x}}\right)\,dx$

(d) $\displaystyle\int \left(\frac{1}{1-\sin^2(\theta)}\right)\,d\theta$

(e) $\int \left(\frac{x^{2/3} - x^3}{x^{1/4}}\right)\, dx$

(f) $\int \left(\frac{4 + x}{\sqrt{1 - x^2}}\right)\, dx$

7.4 Integration By Parts

7.4.1 Discovering the Integration by Parts Formula

Activity 7.4.1 Discovering the Integration by Parts Formula. The product rule for derivatives says that:

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = \underline{\hspace{3cm}} + \underline{\hspace{3cm}}.$$

We know that we intend to "undo" the product rule, so let's try to re-frame the product rule from a rule about derivatives to a rule about antiderivatives.

(a) Antidifferentiate the product rule by antidifferentiating each side of the equation.

$$\begin{aligned} \int \left(\frac{d}{dx}(u \cdot v)\right)\, dx &= \int \underline{\hspace{3cm}} + \underline{\hspace{3cm}} dx \\ \underline{\hspace{3cm}} &= \int \underline{\hspace{3cm}} dx + \int \underline{\hspace{3cm}} dx \end{aligned}$$

Hint. Note that on the left side of this equation you're antidifferentiating a derivative. What will that give you? Then, on the right side, we're just splitting up the terms of the product rule into two different integrals.

(b) On the right side, we have two integrals. Since each of them has a product of functions (one function and a derivative of another), we can isolate one of them in this equation and create a formula for how to antidifferentiate a product of functions! Solve for $\int uv'\, dx$.

(c) Look back at this formula for $\int uv'\, dx$. Explain how this is really the product rule for derivatives (without just undo-ing all of the steps we have just done).

What made it so useful to pick $u = x$ instead of $dv = x\ dx$ in this case? Since we know that we are going to get another integral, one that specifically has the new derivative and new antiderivative that we find from the parts we picked, we noticed that differentiating the function x was much nicer than antidifferentiating it: we get a constant that we multiply by the trig function in this new integral, instead of a power function with an even bigger exponent. We can also notice that when it comes to the trig function, it doesn't really matter if we differentiate it or antidifferentiate it. In both cases, we get a $cos(x)$ in our new integral, with the only difference being whether it is positive or negative.

7.4.2 Intuition for Selecting the Parts

Activity 7.4.2 Picking the Parts for Integration by Parts. Let's consider the integral:

$$\int x\sin(x)\ dx.$$

We'll investigate how to set up the integration by parts formula with the different choices for the parts.

(a) We'll start with selecting $u = x$ and $dv = \sin(x)\ dx$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = x & v = \underline{\qquad\qquad} \\ du = \underline{\qquad\qquad} & dv = \sin(x)\ dx \end{array}$$

(b) Now set up the integration by parts formula using your labeled pieces. Notice that the integration by parts formula gives us another integral. Don't worry about antidifferentiating this yet, let's just set the pieces up.

Hint. $\int u\ dv = uv - \int v\ du$

(c) Let's swap the pieces and try the setup with $u = \sin(x)$ and $dv = x\ dx$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = \sin(x) & v = \underline{\qquad\qquad} \\ du = \underline{\qquad\qquad} & dv = x\ dx \end{array}$$

(d) Now set up the integration by parts formula using this setup.

Hint. $\int u\ dv = uv - \int v\ du$

(e) Compare the two results we have. Which setup do you think will be easier to move forward with? Why?

Hint. When we say we need to keep moving forward with our setup, what we mean is that we have another integral to antidifferentiate. Which one will be easier to work with: $\int(-\cos(x))\ dx$ or $\int\left(\frac{x^2}{2}\cos(x)\right)$?

(f) Finalize your work with the setup you have chosen to find $\int x\sin(x)\ dx$.

What made things so much better when we chose $u = x$ compared to $dv = x\ dx$? We know that the new integral from our integration by parts formula will be built from the new pieces, the derivative we find from u and the antiderivative we pick from dv. So when we differentiate $u = x$, we get a constant, compared to antidifferentiating $dv = x\ dx$ and getting another power function, but with a larger exponent. We know this will be combined with a $cos(x)$ function no matter what (since the derivative and antiderivatives of $sin(x)$ will only differ in their sign). So picking the version that gets that second integral to be built from a trig function and a constant is going to be much nicer than a trig function and a power function. It was nice to pick x to be the piece that we found the derivative of!

Activity 7.4.3 Picking the Parts for Integration by Parts. This time we'll look at a very similar integral:

$$\int x \ln(x)\, dx.$$

Again, we'll set this up two different ways and compsare them.

(a) We'll start with selecting $u = x$ and $dv = \ln(x)\, dx$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = x & v = \boxed{} \\ du = \boxed{} & dv = \ln(x)\, dx \end{array}$$

Hint. You're not forgetting how to antidifferentiate $\ln(x)$. This is just something we don't know yet!

(b) Ok, so here we *have to* swap the pieces and try the setup with $u = \ln(x)$ and $dv = x\, dx$, since we only know how to differentiate $\ln(x)$. Fill in the following with the rest of the pieces:

$$\begin{array}{ll} u = \ln(x) & v = \boxed{} \\ du = \boxed{} & dv = x\, dx \end{array}$$

(c) Now set up the integration by parts formula using this setup.

Hint. $\int u\, dv = uv - \int v\, du$

(d) Why was it fine for us to antidifferentiate x in this example, but not in Activity 7.4.2?

(e) Finish this work to find $\int x \ln(x)\, dx$.

Hint. Notice that $\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) = \frac{x}{2}$.

So here, we didn't actually get much choice. We couldn't pick $u = x$ in order to differentiate it (and get a constant to multiply into our second integral) since we don't know how to antidifferentiate $\ln(x)$ (yet: once we know how, it might be fun to come back to this problem and try it again with the parts flipped). But we can also notice that it ended up being fine to antidifferentiate x: the increased power from our power rule didn't really matter much when we combined it with the derivative of the logarithm, since the derivative of the log is *also a power function*! So we were able to combine those easily and actually integrate that second integral.

7.4.4 Solving for the Integral

Activity 7.4.4 Squared Trig Functions. Let's look at two integrals. We'll talk about both at the same time, since they're similar.

$$\int \sin^2(x)\, dx \qquad\qquad \int \cos^2(x)\, dx$$

(a) What does it mean to "square" a trig function? Write these integrals in a different way, where the meaning of the "squared" exponent is more clear. What do you notice about the structure of these integrals, the operation in the integrand function? What does this mean about our choice of integration technique?

Hint.

$$\int \sin^2(x)\ dx = \int \sin(x)\sin(x)\ dx$$
$$\int \cos^2(x)\ dx = \int \cos(x)\cos(x)\ dx$$

(b) If you were to use integration by parts on these integrals, does your choice of u and dv even matter here? Why not?

Hint. Is there a difference in the two functions being multiplied?

(c) Apply the integration by parts formula to each. What do you notice?

Solution. For the integral $\int \sin^2(x)\ dx$:

$$\begin{aligned} u &= \sin(x) & v &= -\cos(x) \\ du &= \cos(x)\ dx & dv &= \sin(x)\ dx \end{aligned}$$
$$\int \sin^2(x)\ dx = -\sin(x)\cos(x) + \int \cos^2(x)\ dx$$

For the integral $\int \cos^2(x)\ dx$:

$$\begin{aligned} u &= \cos(x) & v &= \sin(x) \\ du &= -\sin(x)\ dx & dv &= \cos(x)\ dx \end{aligned}$$
$$\int \cos^2(x)\ dx = \sin(x)\cos(x) + \int \sin^2(x)\ dx$$

(d) Instead of applying another round of integration by parts to the resulting integral, use the Pythagorean identities to re-write these integrals:

$$\sin^2(x) = 1 - \cos^2(x)$$
$$\cos^2(x) = 1 - \sin^2(x)$$

(e) You should notice that in your equation for the integration of $\sin^2(x)\ dx$, you have another copy of $\int \sin^2(x)\ dx$. Similarly, in your equation for the integration of $\cos^2(x)\ dx$, you have another copy of $\int \cos^2(x)\ dx$.

Replace these integrals with a variable, like I (for "integral"). Can you "solve" for this variable (integral)?

Solution. For $\int \sin^2(x)\ dx$:

$$\begin{aligned} \int \sin^2(x)\ dx &= -\sin(x)\cos(x) + \int 1 - \sin^2(x)\ dx \\ I &= -\sin(x)\cos(x) + \int 1\ dx - \underbrace{\int \sin^2(x)\ dx}_{I} \\ I &= -\sin(x)\cos(x) + x - I \\ 2I &= -\sin(x)\cos(x) + x \\ I &= \frac{-\sin(x)\cos(x) + x}{2} + C \end{aligned}$$

So we end up with:

$$\int \sin^2(x)\ dx = \frac{x - \sin(x)\cos(x)}{2} + C.$$

For $\int \cos^2(x)\ dx$:

$$\begin{aligned}
\int \cos^2(x)\ dx &= \sin(x)\cos(x) + \int 1 - \cos^2(x)\ dx \\
I &= \sin(x)\cos(x) + \int 1\ dx - \underbrace{\int \cos^2(x)\ dx}_{I} \\
I &= \sin(x)\cos(x) + x - I \\
2I &= \sin(x)\cos(x) + x \\
I &= \frac{\sin(x)\cos(x) + x}{2} + C
\end{aligned}$$

So we end up with:

$$\int \cos^2(x)\ dx = \frac{x + \sin(x)\cos(x)}{2} + C.$$

7.4.5 Practice Problems

7.4.5.1. Explain how we build the Integration by Parts formula, as well as what the purpose of this integration strategy is.

7.4.5.2. How do you choose options for u and dv? What are some good strategies to think about?

7.4.5.3. Let's say that you make a choice for u and dv and begin working through the Integration by Parts strategy. How can you tell if you've made a poor choice for your parts? Can you *always* tell?

7.4.5.4. Integrate the following.

(a) $\displaystyle\int 3x\sin(x)\ dx$

(b) $\displaystyle\int 5xe^x\ dx$

(c) $\displaystyle\int x^2e^{-x}\ dx$

(d) $\displaystyle\int x^2\ln(x)\ dx$

(e) $\displaystyle\int x^2\cos(x)\ dx$

(f) $\displaystyle\int x^3e^{-x}\ dx$

(g) $\displaystyle\int x\sin(x)\cos(x)\ dx$

(h) $\displaystyle\int e^x\sin(x)$

(i) $\displaystyle\int \sin^{-1}(x)\ dx$

(j) $\displaystyle\int \tan^{-1}(x)\ dx$

7.4.5.5. Evaluate the following definite integrals.

(a) $\displaystyle\int_{x=1}^{x=e} x\ln(x)\ dx$

(b) $\displaystyle\int_{x=0}^{x=\pi/4} x\cos(2x)\ dx$

(c) $\displaystyle\int_{x=0}^{x=\ln(5)} xe^x\ dx$

7.4.5.6. In this problem, we'll consider the integral $\displaystyle\int \sin^2(x)\ dx$. We'll integrate this in two different ways!

(a) We know that:

$$\int \sin^2(x)\ dx = \int \sin(x)\sin(x)\ dx.$$

Use the Integration by Parts strategy, and especially note that you can solve for the integral (Subsection 7.4.4 Solving for the Integral).

(b) We can use a trigonometric identity to re-write the integral:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}.$$

So we have:

$$\int \sin^2(x)\ dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\ dx.$$

Use u-substitution.

(c) Were your answers the same or different? Should they be the same? Why or why not? Are they connected somehow?

Hint. They might be different, but they should only be different by at most a constant.

7.4.5.7. For these next problems, we'll use $x = u^2$ and $dx = 2u\ du$ to substitute into the integral as written. Then use Integration by Parts.

(a) $\displaystyle\int \sin(\sqrt{x})\ dx$

(b) $\displaystyle\int e^{\sqrt{x}}\ dx$

7.5 Integrating Powers of Trigonometric Functions

7.5.1 Building a Strategy for Powers of Sines and Cosines

Activity 7.5.1 Compare and Contrast. Let's do a quick comparison of two integrals, keeping the above examples in mind. Consider these two integrals:

$$\int \sin^4(x)\cos(x)\ dx \qquad\qquad \int \sin^4(x)\cos^3(x)\ dx$$

(a) Consider the first integral, $\displaystyle\int \sin^4(x)\cos(x)\ dx$. Think about and set up a good technique for antidifferentiating. Without actually solving the integral, explain why this technique will work.

Hint. It might be helpful to notice that $\sin^4(x)$ can be re-written as $(\sin(x))^4$. Does this help reveal something important about the structure of this integrand?

(b) Now consider the second integral, $\displaystyle\int \sin^4(x)\cos^3(x)\ dx$. Does the same integration strategy work here? What happens when you apply the same thing?

Hint. Let $u = \sin(x)$ again, and $du = \cos(x)\ dx$. What happens with the cosine functions? How many are "left" after applying our substitution?

(c) We know that $\sin(x)$ and $\cos(x)$ are related to each other through derivatives (each is the derivative of the other, up to a negative). Is there some other connection that we have between these functions? We might especially notice that we have a $\cos^2(x)$ left over in our integral. Can we write this in terms of $\sin(x)$, so that we can write it in terms of u?

Hint. We have a trigonometric identity (the Pythagorean Identity):

$$\sin^2(x) + \cos^2(x) = 1.$$

(d) Why would this strategy not have worked if we were looking at the integrals $\int \sin^4(x)\cos^2(x)\ dx$ or $\displaystyle\int \sin^4(x)\cos^4(x)\ dx$? What, specifically, did we need in order to use this combination of substitution and trigonometric identity to solve the integral?

7.5.2 Building a Strategy for Powers of Secants and Tangents

Activity 7.5.2 Compare and Contrast (Again). We're going to do another Compare and Contrast, but this time we're only going to consider one integral:

$$\int \sec^4(x)\tan^3(x)\ dx.$$

We're going to employ another strategy, similar to the one for Integrating Powers of Sine and Cosine.

(a) Before you start thinking about this integral, let's build the relevant version of the Pythagorean Identity that we'll use. Our standard version of this is:

$$\sin^2(x) + \cos^2(x) = 1.$$

Since we want a version that connects $\tan(x)$, which is also written as $\frac{\sin(x)}{\cos(x)}$, with $\sec(x)$, or $\frac{1}{\cos(x)}$, let's divide everything in the Pythagorean Identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{\qquad\qquad} + \frac{\cos^2(x)}{\qquad\qquad} = \frac{1}{\qquad\qquad}$$

$$\qquad\qquad + \qquad\qquad = \qquad\qquad$$

Solution.

$$\tan^2(x) + 1 = \sec^2(x)$$

(b) Now start with the integral. We're going to use two different processes here, two different u-substitutions. First, set $u = \tan(x)$. Complete the substitution and solve the integral.

Hint. Here, $du = \sec^2(x)\ dx$. We'll also use $\sec^2(x) = \tan^2(x) + 1$.

Solution.

$$\begin{aligned}
\int \sec^4(x)\tan^3(x)\ dx &= \int \underbrace{\sec^2(x)}_{\tan^2(x)+1}\ \tan^3(x)\sec^2(x)\ dx \\
&= \int (u^2+1)u^3\ du \\
&= \int u^5 + u^3\ du \\
&= \frac{u^6}{6} + \frac{u^4}{4} + C \\
&= \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C
\end{aligned}$$

(c) Now try the integral again, this time using $u = \sec(x)$ as your substitution.

Hint. Now $du = \sec(x)\tan(x)\ dx$, and we'll use the same Pythagorean identity, just re-written as $\tan^2(x) = \sec^2(x) - 1$.

Solution.

$$\begin{aligned}
\int \sec^4(x)\tan^3(x)\ dx &= \int \sec^3(x)\ \underbrace{\tan^2(x)}_{\sec^2(x)-1}\ \sec(x)\tan(x)\ dx \\
&= \int u^3(u^2-1)\ du \\
&= \int u^5 - u^3\ du \\
&= \frac{u^6}{6} - \frac{u^4}{4} + C \\
&= \frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + C
\end{aligned}$$

(d) For each of these integrals, why were the exponents set up *just right* for u-substitution each time? How does the structure of the derivatives of each function play into this?

Hint. Notice we had an even exponent on the $\sec(x)$ function, but an odd exponent on the $\tan(x)$ function.

(e) Which substitution would be best for the integral $\int \sec^4(x)\tan^4(x)\ dx$. Why?

(f) Which substitution would be best for the integral $\int \sec^3(x)\tan^3(x)\ dx$. Why?

7.5.3 Practice Problems

7.5.3.1. For an integral $\int \sin^a(x)\cos^b(x)\ dx$, how do you know whether to use $u = \sin(x)$ or $u = \cos(x)$ as the substitution?

7.5.3.2. For an integral $\int \tan^a(x)\sec^b(x)\ dx$, how do you know whether to use $u = \tan(x)$ or $u = \sec(x)$ as the substitution?

7.5.3.3. Integrate the following.

(a) $\int \sin^3(x)\cos^2(x)\ dx$

(b) $\int \sin^2(x)\cos^3(x)\ dx$

(c) $\int \sin^3(x)\cos^3(x)\ dx$

(d) $\int \tan^4(x)\sec^4(x)\ dx$

(e) $\int \tan^3(x)\sec^3(x)\ dx$

(f) $\int \tan^3(x)\sec^4(x)\ dx$

7.5.3.4. Integrate the following.

(a) $\int \sin^{3/4}(x)\cos^5(x)\ dx$

(b) $\int \tan^5(x)\sec^{-1/2}(x)\ dx$

(c) $\int \sin^{3/4}(x)\cos^5(x)\ dx$

(d) $\int \tan^5(x)\sec^{-1/2}(x)\ dx$

7.5.3.5. Consider the integral $\int \sin^2(x)\ dx$.

(a) Use the trigonometric identity:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

Hint. Check out Subsection 7.4.4 Solving for the Integral

(c) Which of these techniques do you think was easier to implement and use? Why is that?

7.5.3.6. Consider the integral $\int \cos^4(x)\, dx$.

(a) Use the trigonometric identity:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

to integrate.

(b) Use integration by parts to integrate.

Hint. Try picking $u = \cos^3(x)$ and $dv = \cos(x)\, dx$.

(c) Which of these techniques did you prefer? Why?

7.5.3.7. Integrate the following integrals.

(a) $\int \tan^2(x)\, dx$

Hint. Use a Pythagorean Identity to convert this to be written in terms of secant functions.

(b) $\int \sec^3(x)\, dx$

Hint. Integration by parts works well here, and it's helpful to know the derivative of $\sec(x)$ and an antiderivative of $\sec^2(x)$.

(c) $\int \tan^5(x)\, dx$

Hint. You can technically use either $u = \sec(x)$ or $u = \tan(x)$ here.

(d) $\int \sin^5(x)\, dx$

7.6 Trigonometric Substitution

7.6.1 Another Type of Variable Substitution

Activity 7.6.2 Trig Substitution Geoemtry. We're going to consider three triangles, and we're going to fill in side lengths. In each of these, we'll assume that the lengths x and a are real numbers and are positive.

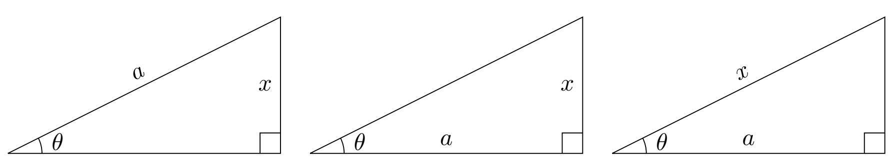


Figure 7.6.2 Three triangles to guide our trigonometric substitutions.

(a) Use the Pythagorean theorem to label the missing side length in each of the three triangles.

(b) For each triangle, explain how you can tell which side length represents the hypotenuse when you see the lengths x, a, and then the missing lengths you found above: $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 + a^2}$.

Hint. We know that the hypotenuse is the longest side length in a triangle. Just based on the square root length, how can you tell which length is longest?

Solution. If one of the side lengths is $\sqrt{x^2 - a^2}$, then we know that $x > a$ (otherwise the square root is a non-real number). We also know that $x > \sqrt{x^2 - a^2}$ (because $a > 0$). This means that x is the length of the hypotenuse.

If one of the side lengths is $\sqrt{a^2 - x^2}$, then we know that $a > x$ (otherwise the square root is a non-real number). We also know that $a > \sqrt{a^2 - x^2}$ (because $x > 0$). This means that a is the length of the hypotenuse.

If one of the side lengths is $\sqrt{x^2 + a^2}$, then we know that $\sqrt{x^2 + a^2} > a$ and $\sqrt{x^2 + a^2} > x$ (because $x, a > 0$). This means that $\sqrt{x^2 + a^2}$ is the length of the hypotenuse.

(c) For each triangle, find a trigonometric function of θ that connects lengths x and a to each other.

Solve each for x to reveal the relevant substitution.

(d) For each substitution, find the corresponding substitution for the differential, dx.

Activity 7.6.3 Practicing Trigonometric Substitution. Let's look at three integrals, and practice the kind of thinking we'll need to use to apply trigonometric substitution to them.

1. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x}\, dx$
2. $\displaystyle\int \frac{2}{(4 - x^2)^{3/2}}\, dx$
3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 + 1}}\, dx$

For each integral, do the following:

(a) Identify the term (or terms) that signify that trigonometric substitution might be a reasonable strategy.

Hint. In each case, you're looking for some sum or difference of squared terms, normally (but not always) nested inside of some square root. It also might be nice to re-write the second integral in order to notice the root:

$$\int \frac{2}{(4 - x^2)^{3/2}}\, dx = \int \frac{2}{\left(\sqrt{4 - x^2}\right)^3}\, dx.$$

(b) Use that portion of the integral to compare three side lengths of a triangle. Which one is the largest (and so must represent the length of the hypotenuse)?

Hint.

(a) For the side lengths x, 3, and $\sqrt{x^2 - 9}$, which must be the largest? Think about $\sqrt{x^2 - 9}$ and what we can learn from it.

(b) For the side lengths x, 2, and $\sqrt{4-x^2}$, which must be the largest? Think about $\sqrt{4-x^2}$ and what we can learn from it.

(c) For the side lengths x, 1, and $\sqrt{x^2+1}$, which must be the largest? Think about $\sqrt{x^2+1}$ and what we can learn from it.

Solution.

(a) We know that $x > 3$, since $x^2 - 9 > 0$. Similarly, we know that $x > \sqrt{x^2-9}$. So x has to be the length of the hypotenuse.

(b) We know that $2 > x$, since $4 - x^2 > 0$. Similarly, we know that $2 > \sqrt{4-x^2}$. So 2 has to be the length of the hypotenuse.

(c) We know that $\sqrt{x^2+1} > x$, since we are adding 1 to x^2 under the square root. This is also the reason that $\sqrt{x^2+1} > 1$ (we are adding x^2 under the root). So $\sqrt{x^2+1}$ has to be the length of the hypotenuse.

(c) Construct the triangle, label an angle θ, and use a trigonometric function to connect the two single-term side lengths. (Feel free to change the angle you label in order to use the sine, secant, or tangent functions instead of their co-functions).

Solution.

(a) For the integral $\int \frac{\sqrt{x^2-9}}{x}\,dx$, we have the following triangle.

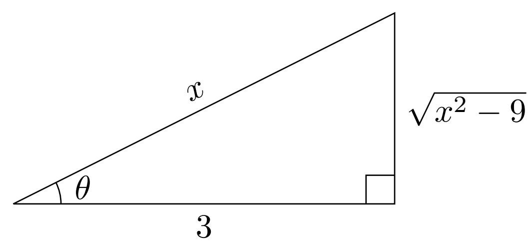


Figure 7.6.8

This gives us $\cos(\theta) = \frac{3}{x}$ or, equivalently, $\sec(\theta) = \frac{x}{3}$.

(b) For the integral $\int \frac{2}{(4-x^2)^{3/2}}\,dx$, we have the following triangle.

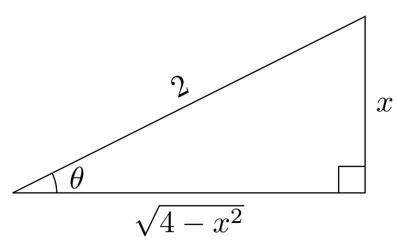


Figure 7.6.9

This gives us $\sin(\theta) = \frac{x}{2}$.

(c) For the integral $\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$, we have the following triangle.

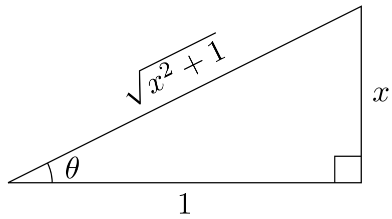


Figure 7.6.10

This gives us $\tan(\theta) = \dfrac{x}{1}$ or, equivalently, $\tan(\theta) = x$.

(d) Define your substitution (for both x and the differential dx), and identify the Pythagorean identity that will be relevant for the integral.

Solution.

(a) We will use $x = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)\ d\theta$.
Then, we can expect to find the Pythagorean identity $9\sec^2(\theta) - 9 = 9\tan^2(\theta)$.

(b) We will use $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)\ d\theta$.
Then, we can expect to find the Pythagorean identity $4 - 4\sin^2(\theta) = 4\cos^2(\theta)$.

(c) We will use $x = \tan(\theta)$ and $dx = \sec^2(\theta)\ d\theta$.
Then, we can expect to find the Pythagorean identity $\tan^2(\theta) + 1 = \sec^2(\theta)$.

(e) Substitute and antidifferentiate!

Hint.

(a) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\ dx = \int \frac{\sqrt{9\sec^2(\theta)-9}}{3\sec(\theta)}\ (3\sec(\theta)\tan(\theta))\ d\theta$
Also note that $\sqrt{9\sec^2(\theta)-9} = \sqrt{9\tan^2(\theta)}$.

(b) $\displaystyle\int \frac{2}{(4-x^2)^{3/2}}\ dx = \int \frac{2}{\left(\sqrt{4-4\sin^2(\theta)}\right)^3}\ (2\cos(\theta))\ d\theta$

Also note that $\sqrt{4-4\sin^2(\theta)} = \sqrt{4\cos^2(\theta)}$.

(c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\ dx = \int \frac{1}{\tan^2(\theta)\sqrt{\tan^2(\theta)+1}}\ \left(\sec^2(\theta)\right)\ d\theta$
Also note that $\sqrt{\tan^2(\theta)+1} = \sqrt{\sec^2(\theta)}$.

Solution.

(a) Using $x = 3\sec(\theta)$, we get:

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\ dx &= \int \frac{\sqrt{9\sec^2(\theta)-9}}{3\sec(\theta)}\ (3\sec(\theta)\tan(\theta))\ d\theta \\
&= \int \frac{3\sqrt{9\tan^2(\theta)}\sec(theta)\tan(\theta)}{3\sec(\theta)}\ d\theta \\
&= 3\int \tan^2(\theta)\ d\theta \\
& 3\int \sec^2(\theta) - 1\ d\theta
\end{aligned}$$

$$3\left(\tan(\theta)-\theta\right)+C$$
$$3\tan(\theta)-3\theta+C$$

(b) Using $x=2\sin(\theta)$, we get:

$$\begin{aligned}\int \frac{2}{(4-x^2)^{3/2}}\,dx &= \int \frac{2}{\left(\sqrt{4-4\sin^2(\theta)}\right)^3}\left(2\cos(\theta)\right)\,d\theta\\ &= \int \frac{4\cos(\theta)}{\left(\sqrt{4\cos^2(\theta)}\right)^3}\,d\theta\\ &= \frac{1}{2}\int \frac{1}{\cos^2(\theta)}\,d\theta\\ &= \frac{1}{2}\int \sec^2(\theta)\,d\theta\\ &= \frac{1}{2}\tan(\theta)+C\end{aligned}$$

(c) Using $x=\tan(\theta)$, we get:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+1}}\,dx &= \int \frac{1}{\tan^2(\theta)\sqrt{\tan^2(\theta)+1}}\left(\sec^2(\theta)\right)\,d\theta\\ &= \int \frac{\sec^2(\theta)}{\tan^2(\theta)\sqrt{\sec^2(\theta)}}\,d\theta\\ &= \int \frac{\sec(\theta)}{\tan^2(\theta)}\\ &= \int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta\\ &= \int \csc(\theta)\cot(\theta)\,d\theta\\ &= -\csc(\theta)+C\end{aligned}$$

(f) Use your triangle to substitute your antiderivative back in terms of x.

Solution.

(a) From the triangle, we get $\tan(\theta)=\dfrac{\sqrt{x^2-9}}{3}$. Then, since $\sec(\theta)=\dfrac{x}{3}$, we can use $\theta=\sec^{-1}\left(\dfrac{x}{3}\right)$. This gives us:

$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x}\,dx &= 3\int \sec^2(\theta)-1\,d\theta\\ &= 3\tan(\theta)-3\theta+C\\ &= \sqrt{x^2-9}-3\sec^{-1}\left(\frac{x}{3}\right)+C\end{aligned}$$

(b) From the triangle, we get $\tan(\theta)=\dfrac{x}{\sqrt{4-x^2}}$. This gives us:

$$\begin{aligned}\int \frac{2}{(4-x^2)^{3/2}}\,dx &= \frac{1}{2}\int \sec^2(\theta)\,d\theta\\ &= \frac{1}{2}\tan(\theta)+C\\ &= \frac{x}{2\sqrt{4-x^2}}+C\end{aligned}$$

(c) From the triangle, we get $\csc(\theta) = \dfrac{\sqrt{x^2+1}}{x}$. This gives us:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+1}}\,dx &= \int \csc(\theta)\cot(\theta)\,d\theta \\ &= -\csc(\theta) + C \\ &= -\frac{\sqrt{x^2+1}}{x} + C\end{aligned}$$

7.6.2 Practice Problems

7.6.2.1. Explain how trigonometric substitution helps to convert sums or differences of squares to products of squares. Why is this helpful? When is it helpful?

7.6.2.2. Draw a right triangle with $\sqrt{x^2-4}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{x^2-4}$?

7.6.2.3. Draw a right triangle with $\sqrt{4-x^2}$ as one of the non-hypotenuse side lengths. What is the length of the hypotenuse? What about the other side length? What would be an appropriate substitution for an integral containing $\sqrt{4-x^2}$?

7.6.2.4. Draw a right triangle with $\sqrt{x^2+4}$ as one of the hypotenuse. What are the lengths of the other two sides? What would be an appropriate substitution for an integral containing $\sqrt{x^2+4}$?

7.6.2.5. Integrate the following using an appropriate trigonometric substitution.

(a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

(b) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

(c) $\displaystyle\int \frac{1}{(9x^2+1)^{3/2}}\,dx$

(d) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

(e) $\displaystyle\int \sqrt{49-x^2}\,dx$

(f) $\displaystyle\int \frac{1}{x(x^2-1)^{3/2}}\,dx$ (for $x > 1$)

(g) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(h) $\displaystyle\int \frac{x^2}{(x^2+81)^2}\,dx$

7.6.2.6. Complete the square and then integrate.

(a) $\displaystyle\int \frac{1}{x^2-8x+62}\,dx$

(b) $\displaystyle\int \frac{x^2-8x+16}{(-x^2+8x+9)^{3/2}}\,dx$

7.7 Partial Fractions

7.7.2 How?

Activity 7.7.2 First Examples of Partial Fractions.

(a) Consider the integral:

$$\int \frac{6x-16}{x^2-4x+3}\,dx.$$

First, confirm that this would be *very* annoying to try to use u-substitution on, even though we have a linear numerator and quadratic denominator.

(b) Notice that the denominator can be factored:

$$\int \frac{6x-16}{(x-3)(x-1)}\,dx.$$

If this integrand function were a sum of two "smaller" rational functions, what would their denominators be? What do we know about their numerators?

Hint. The denominators would need to multiply to get $(x-3)(x-1)$, and each numerator would have to have a degree smaller than its respective denominator.

Solution. We might have something like $\dfrac{\square}{x-3} + \dfrac{\square}{x-1} = \dfrac{6x-16}{(x-3)(x-1)}$, where the numerators are constant terms (since their degree must be smaller than 1).

(c) Use some variables (it's typical to use capital letters like A, B, C, etc.) to represent the numerators, and then add the partial fractions together. What do you get? How does this rational function compare to $\dfrac{6x-16}{(x-3)(x-1)}$?

Hint. We want to try to add up $\dfrac{A}{x-3}+\dfrac{B}{x-1}$ and compare the result to $\dfrac{6x-16}{(x-3)(x-1)}$.

Solution.

$$\begin{aligned}
\frac{A}{x-3}+\frac{B}{x-1} &= \frac{A}{x-3}\left(\frac{x-1}{x-1}\right)+\frac{B}{x-1}\left(\frac{x-3}{x-3}\right)\\
&= \frac{A(x-1)}{(x-3)(x-1)}+\frac{B(x-3)}{(x-3)(x-1)}\\
&= \frac{A(x-1)+B(x-3)}{(x-3)(x-1)}
\end{aligned}$$

When we compare this to $\dfrac{6x-16}{(x-3)(x-1)}$, we can see that the denominators are the same, and so, if these are equal to each other, the numerators must be as well.

(d) Set up an equation connecting the numerators, and solve for your unknown variables. What are the two fractions that added together to get $\dfrac{6x-16}{(x-3)(x-1)}$?

Solution.

$$\begin{aligned} A(x-1)+B(x-3) &= 6x-16 \\ (A+B)x-(A+3B) &= 6x-16 \end{aligned}$$

This means that $A+B=6$ and $A+3B=16$. There are a variety of ways to solve this, but we can say that $A=6-B$, and so then

$$\begin{aligned} A+3B &= 16 \\ (6-B)+3B &= 16 \\ 2B &= 10 \\ B &= 5 \end{aligned}$$

Since $A=6-B$, we have $A=6-5=1$.

$$\frac{6x-16}{(x-3)(x-1)} = \frac{1}{x-3}+\frac{5}{x-1}$$

(e) Antidifferentiate to solve the integral $\displaystyle\int \frac{6x-16}{(x-3)(x-1)}\,dx$.

Solution.

$$\begin{aligned} \int \frac{6x-16}{(x-3)(x-1)}\,dx &= \int \frac{1}{x-3}+\frac{5}{x-1}\,dx \\ &= \ln|x-3|+5\ln|x-1|+C \end{aligned}$$

(f) Let's do the same thing with a new integral:

$$\int \frac{3x^2-2x+3}{x(x^2+1)}\,dx.$$

What are the partial fraction forms? What kinds of numerators should we expect to see? Use variables to represent these.

Solution. We expect to see partial fractions $\dfrac{\quad}{x}$ and $\dfrac{\quad}{x^2+1}$. We know the numerator over x needs to be constant, but we could have a linear numerator over x^2+1. So our partial fractions are:

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}.$$

(g) Add the partial fractions together and set up an equation for the numerators to solve. What are the two partial fractions after you solve for the unknown coefficients?

Hint. We want to find values for A, B, and C where

$$\frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{3x^2 - 2x + 3}{x(x^2 + 1)}.$$

Use common denominators to add

$$\frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

and then set the resulting numerator equal to $3x^2 - 2x + 3$ and try to solve.

Solution.

$$\begin{aligned}\frac{A}{x} + \frac{Bx + C}{x^2 + 1} &= \frac{A}{x}\left(\frac{x^2 + 1}{x^2 + 1}\right) + \frac{Bx + C}{x^2 + 1}\left(\frac{x}{x}\right) \\ &= \frac{A(x^2 + 1) + (Bx + C)(x)}{x(x^2 + 1)}\end{aligned}$$

Then we can set the numerators equal to get:

$$\begin{gathered}A(x^2 + 1) + (Bx + C)(x) = 3x^2 - 2x + 3 \\ (A + B)x^2 + Cx + A3x^2 - 2x + 3\end{gathered}$$

So now we can see that since $Cx = -2x$, we have $C = -2$. We can also look at the constant terms and see that $A = 3$. Now we know that $(A + B)x^2 = 3x^2$, but since $A = 3$ we have $B = 0$.

So we have $\displaystyle\int \frac{3x^2 - 2x + 3}{x(x^2 + 1)}\, dx = \int \frac{3}{x} - \frac{2}{x^2 + 1}\, dx.$

(h) Antidifferentiate and solve the integral $\displaystyle\int \frac{3x^2 - 2x + 3}{x(x^2 + 1)}\, dx.$

Solution.

$$\begin{aligned}\int \frac{3x^2 - 2x + 3}{x(x^2 + 1)}\, dx &= \int \frac{3}{x} - \frac{2}{x^2 + 1}\, dx \\ &= 2\ln|x| - 2\tan^{-1}(x) + C\end{aligned}$$

7.7.3 More Specific Strategies

7.7.3.3 Partial Fraction Type: Repeated Linear Factors

Activity 7.7.3 Fiddling with Repeated Factors. Let's sit with the following integral:

$$\int \frac{3x + 5}{x^2 + 2x + 1}\, dx.$$

Before we start, we can think about how annoying it would be to try to start with a u-substitution where $u = x^2 + 2x + 1$.

(a) Factor the denominator! What's different about the factors in this denominator compared to the ones in Activity 7.7.2?

Answer. $x^2 + 2x + 1 = (x + 1)(x + 1)$

(b) Why can't we use these two factors to create two partial fractions with Simple Linear Factors?

(c) Ok, instead, we're going to do some algebra that is reminiscent of what we have done before in Section 7.3.

Can you write the numerator, $3x + 5$, as some coefficient on the factor $(x + 1)$ with some constant "remainder?"

$$3x + 5 = \underline{\qquad}(x + 1) + \underline{\qquad}$$

Answer.

$$3x + 5 = 3(x + 1) + 2$$

(d) Why is this re-forming of the numerator useful? What does that do, when we write it over the factored denominator? Why did we choose $(x + 1)$ as the factor to use for our re-writing?

Feel free to show why this is helpful!

Hint. Split the fraction up across the sum in the numerator!

Solution.

$$\begin{aligned}\frac{3x+5}{(x+1)^2} &= \frac{3(x+1)+2}{(x+1)^2}\\ &= \frac{3(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}\\ &= \frac{3}{x+1} + \frac{2}{(x+1)^2}\end{aligned}$$

(e) Integrate $\displaystyle\int \frac{3x+5}{x^2+2x+1}\,dx$ using your clever re-writing. Explain why this is a friendlier form.

Solution. We'll use our re-written integral:

$$\int \frac{3x+5}{x^2+2x+1}\,dx = \int \frac{3}{x+1} + \frac{2}{(x+1)^2}\,dx.$$

Now, we can split this into two integrals to deal with separately. For both, we can use the u-substitution $u = x + 1$.

$$\begin{aligned}\int \frac{3}{x+1} + \frac{2}{(x+1)^2}\,dx &= \int \frac{3}{u}\,du + \int \frac{2}{u^2}\,du\\ &= 3\ln|u| - \frac{2}{u} + C\\ &= 3\ln|x+1| - \frac{2}{x+1} + C\end{aligned}$$

This is something we can do, algebraically, for every fraction with a "repeated" factor like this. But, more importantly, we can incorporate this idea into how we think about partial fractions.

7.7.4 Practice Problems

7.7.4.1. Why do we use partial fraction decomposition on some integrals of rational functions? Give an example and explain why it is helpful in your example.

7.7.4.2. For each rational function described, write out the corresponding partial fraction forms.

(a) $\dfrac{p(x)}{(x-4)(x+2)(x-1)}$ where $p(x)$ is some polynomial with degree less than 3.

(b) $\dfrac{p(x)}{(x+1)^2(3x-5)^3}$ where $p(x)$ is some polynomial with degree less than 5.

(c) $\dfrac{p(x)}{(x^2+1)(x^2+2x+5)}$ where $p(x)$ is some polynomial with degree less than 4.

(d) $\dfrac{p(x)}{x^4-1}$ where $p(x)$ is some polynomial with degree less than 4.

Hint. There's some factoring to be done here! Note that $x^4-1=(x^2-1)(x^2+1)$ and then we can factor $x^2-1=(x-1)(x+1)$.

7.7.4.3. Consider the following integral, with the partial fraction forms written out:

$$\int \frac{x^3+6x^2-x}{(x-2)(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x-2}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\right)\,dx.$$

(a) Write an equation connecting the numerators.

(b) Find (and use) a specific x-value to input into the equation to solve for A.

Hint. Use $x=2$, and notice what happens to the rest of the terms.

(c) Find (and use) a specific x-value to input into the equation to solve for B.

Hint. Use $x=-1$, and notice what happens to the rest of the terms.

(d) Why can you not use this strategy to solve for coefficients C or D?

(e) Find the cubic terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for C.

(f) Find the constant terms (you will need to do some multiplication) on both sides of your equation. Use these to solve for D.

(g) Integrate!

7.7.4.4. Explain why partial fractions is not an appropriate technique for the following integral:

$$\int \frac{x^2+x}{x^2-x+1}\,dx.$$

How should we approach this integral, instead?

Hint. Note the degree in the numerator compared to the denominator!

7.7.4.5. Integrate the following.

(a) $\displaystyle\int \frac{2}{(x-1)(x+3)}\,dx$

(b) $\displaystyle\int \frac{4x+1}{(x-4)(x+5)}\,dx$

(c) $\displaystyle\int \frac{2x^2-15x+32}{x(x^2-8x^2+64)}$

(d) $\displaystyle\int \frac{1}{(x+2)(x-2)}\,dx$

(e) $\displaystyle\int \frac{20x}{(x-1)(x^2+4x+5)}\,dx$

(f) $\displaystyle\int \frac{x^2}{(x-2)^3}\,dx$

7.7.4.6. In the problems we are looking at in this section, we're limiting ourselves to, at most, irreducible quadratic factors in the denominator. In problems with simple linear factors, repeated linear factors, or irreducible quadratic factors, what types of antiderivative functions do you expect to see? Explain.

7.7.4.7. For each of the following integrals, we will do some preliminary work before using partial fractions to integrate. Really, we'll perform a specific u-substitution that will give us some resulting integral to use partial fractions on.

(a) $\displaystyle\int \frac{4e^{2x}}{(e^{2x}+3)(e^{2x}-5)}\,dx$ where we use $u=e^{2x}$.

Hint. If $u=e^{2x}$ then $du=2e^{2x}\,dx$ and our resulting integral looks like:

$$\int \frac{2}{(u+3)(u-5)}\,du.$$

(b) $\displaystyle\int \sqrt{e^x+1}\,dx$ where we use $u=\sqrt{e^x+1}$.

Hint. If $u=\sqrt{e^x+1}$ then $du=\dfrac{e^x}{2\sqrt{e^x+1}}\,dx$ and the resulting integral is:

$$\int \frac{2u^2}{u^2-1}\,du=\int 2+\frac{2}{u^2-1}\,du.$$

(c) $\displaystyle\int \frac{\sqrt{x}+3}{\sqrt{x}(x-1)}\,dx$ where we use $u=\sqrt{x}$.

Hint. If $u=\sqrt{x}$ then $du=\dfrac{1}{2\sqrt{x}}\,dx$ and the resulting integral is:

$$\int \frac{2u+6}{u^2-1}\,du.$$

8 Infinite Series

8.1 Introduction to Infinite Sequences

8.1.1 Sequences as Functions

Activity 8.1.1 Building our First Sequences. We might already have some familiarity with sequences. Here, we'll focus less on some of the detailed mechanics and just think about these sequences as functions.

(a) Describe a sequence of numbers where you use a consistent rule/function to build each term (each number) based only on the *previous term* in the sequence. You will need to decide on some first term to start your sequence.

(b) Describe a different sequence of numbers using the same rule to generate new terms/numbers from the previous one. What do you need to do to make these two sequences different from each other?

(c) Describe a new sequence of numbers where you use a consistent rule/function to build each term based on its position in the sequence (i.e. the first term will be some rule/function based on the input 1, the second will be based on 2, you'll use 3 to get the third term, etc.). We will call the position of each term in the sequence the *index*.

(d) Describe another, new, sequence of numbers where you use a consistent rule/function to build each term based on its index. This time, make the terms get smaller in size as the index increases.

Activity 8.1.2 Returning to our First Sequences. Let's return back to the four sequences we created in Activity 8.1.1.

(a) For each of the sequences, how are we going to define them? Explicit formulas? Recursion relations? How do you know?

(b) Now, for each sequence, define the sequence formally using either an explicit formula or recursion relation, whichever matches with how you described the sequence in Activity 8.1.1.

Activity 8.1.3 Describing These Sequences. Let's look at the sequences from Example 8.1.3. Go through the following tasks for each sequence.

(a) What do you think each sequence is "counting towards" (if anything)?

Hint. If you're not sure, maybe you need to write out a few more terms! You can also change how you write the numbers themselves: in some cases, fractions might be helpful, but in others it might be useful to write the numbers in decimal form. Maybe you'll approximate values of the sine or exponential functions, or maybe you'll leave them as $\sin(2)$ or e^3.

(b) Can you show that the sequence is counting towards what you think it is with a limit (or show that it's not counting towards anything)?

Hint. Some of these limits, as $k \to \infty$, will be tricky to work with! When might you want to use The Squeeze Theorem? When might you want to use L'Hôpital's Rule?

Activity 8.1.4 Write the Sequence Rules. We'll look at some sequences by writing out the first handful of terms. From there, our goal is to write out more terms and eventually define each sequence fully.

For each sequence, write an explicit formula and a recursion relation to define the sequence. You can choose whether to start your index at $k = 0$ or $k = 1$.

(a) $\{a_k\} = \left\{4, \frac{2}{3}, \frac{1}{9}, \frac{1}{54}, \ldots\right\}$

Hint 1. It might be helpful to write these numbers using a common denominator! Or at least some of the numbers. Alternatively, you can try a common *numerator* (which is very fun to do, since we normally don't do that).

Hint 2. If you are recursively multiplying by a number each time, what will that look like in the explicit formula? How do we represent repeated multiplication?

(b) $\{b_k\} = \left\{\frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, \ldots,\right\}$

Hint 1. You can re-write these fractions! Have any of them been "reduced?"

Hint 2. Re-write $\frac{2}{5}$ and $\frac{3}{13}$ by scaling the numerator and denominator by 2. Can you find a formula for the numerator and denominator separately? This one is *very* difficult to find a recursion relation for, so feel free to only write it explicitly!

(c) $\{c_k\} = \left\{\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, \ldots\right\}$

Hint 1. This one will definitely be helpful to re-write so that all fractions have a common denominator.

Hint 2. If you are recursively adding something, how does that show up in the explicit formula? How do we repeatedly add?

(d) What kinds of connections do you notice between the explicit formulas and the recursion relations for these sequences?

8.2 Introduction to Infinite Series

8.2.1 Partial Sums

Activity 8.2.1 How Do We Think About Infinite Series? Let's consider the following sequence:

$$\left\{\frac{9}{10^k}\right\}_{k=1}^{\infty}$$

(a) White out the first 5 terms of the sequence.

(b) What does this sequence converge to? Show this with a limit!

(c) Now we'll construct a new sequence, this time by adding things up. We're going to be working with the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{k=n} \left(\frac{9}{10^k}\right).$$

Write out the first five terms of this sequence: S_1, S_2, S_3, S_4, S_5.

(d) Can you come up with an explicit formula for S_n?

(e) Does $\{S_n\}$ converge or diverge? Use a limit to find what it converges to!

(f) What do you think this means for the infinite series $\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right)$? Does the infinite series converge or diverge?

This is hopefully a nice little introduction to how we'll think about infinite series: we'll consider, instead, the sequence of sums where we add up more and more terms. This is also a nice first example, because we really just showed that

$$0.999... = 1$$

since

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right) &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ... \\ &= 0.9 + 0.09 + 0.009 + ... \\ &1.\end{aligned}$$

But more importantly, we now have a good strategy for thinking about infinite series as sequences of *partial sums.*

8.3 The Divergence Test and the Harmonic Series

8.3.1 The Relationship Between a Sequence and Series

Activity 8.3.1 Investigating the Harmonic Series.

(a) Write out the first several terms of the harmonic series, terms from $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$. Write however many you need to get a feel for how the terms work.

(b) Can you find out how many terms you would have to go "into" the series before the term was less than 0.00000001?

Hint. When is $\frac{1}{k} < \frac{1}{10^8}$?

(c) Can you do this same kind of thing, no matter how small? For instance, how many terms would you have to go into the series before the term was less than some real number ε where $\varepsilon > 0$?

Hint. When is $\frac{1}{k} < \varepsilon$?

(d) Remind/explain/convince yourself that what we've really done is show that $\lim_{k\to\infty} \frac{1}{k} = 0$. This isn't a new or terribly interesting fact, but make sure that you understand why the argument above shows this.

(e) Let's do something very similar, but with $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}_{n=1}^{\infty}$, the sequence of partial sums, instead. Write out the first few partial sums. There's no specific number that you *need* to write, but make sure to write enough partial sums to get a feel for how the partial sums work.

(f) Can you find out how many terms you need to add up until the partial sum is larger than 1?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 1.$$

(g) Can you find out how many terms you need to add up until the partial sum is larger than 5?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 5.$$

Solution.

$$\sum_{k=1}^{83} \frac{1}{k} \approx 5.00207...$$

This is the first partial sum greater than 5.

(h) Can you find out how many terms you need to add up until the partial sum is larger than 10?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 10.$$

This will be absolutely awful to try calculating by hand! Use some piece of technology!

Solution.

$$\sum_{k=1}^{12367} \frac{1}{k} \approx 10.000043...$$

This is the first partial sum greater than 10.

(i) Do you think that for any positive number S, we can always find some partial sum $\sum_{k=1}^{n} \frac{1}{k} > S$? What do you think this would mean about

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}?$$

To actually show that for any $S > 0$ we could always find an $n > 1$ where

$$\sum_{k=1}^{n} \frac{1}{k} > S$$

is an extremely difficult task! We will show that the Harmonic Series diverges in a different way, but for now I want us to notice these contradictory results: we have a series whose terms get small, but whose partial sums do not seem to converge.

We have $\frac{1}{k} \to 0$ but it seems like $\lim_{n\to\infty} S_n$ does not exist. Is this behavior special to the Harmonic Series? Is this something we should make note of? Is there some other connection between the terms of a series and the behavior of the partial sums of the series that we need to note?

8.4 The Integral Test

8.4.2 The Integral Test

Activity 8.4.1 Integrals and Infinite Series. We're going to work with a graph of a continuous function, and we're going to start with a couple of conditions:

1. Our function will be continuous wherever it's defined.
2. Our function will be decreasing on its domain.
3. All of the function outputs will be positive.

Let's not worry about picking a specific function for this, but we will visualize a graph of one that meets these three requirements.

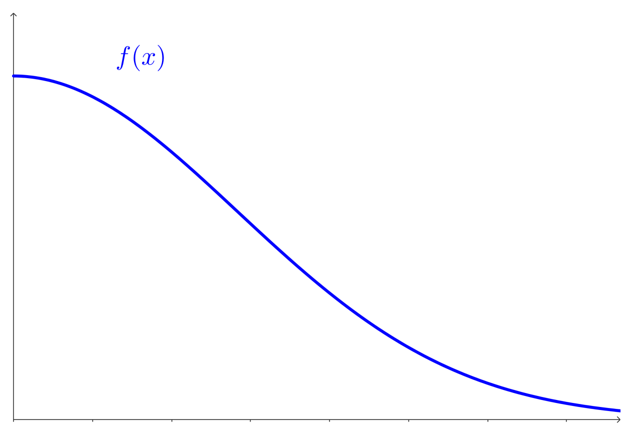


Figure 8.4.2

We can then visualize the sequence of terms, $a_k = f(k)$ for $k = 0, 1, 2, ...,$.

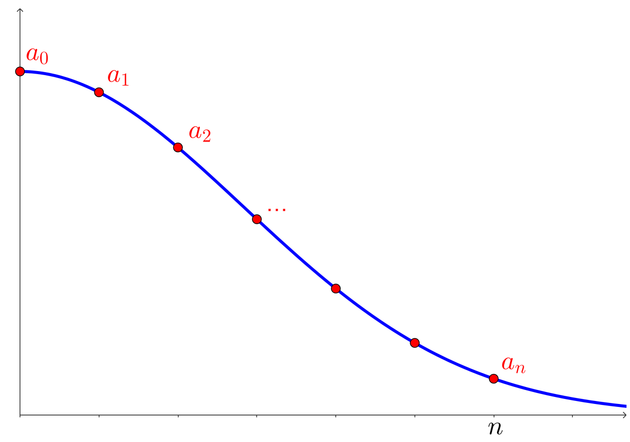


Figure 8.4.3

(a) How does the partial sum, $\sum_{k=0}^{n} a_k$ compare to the Riemann sum for $f(x)$ from $x = 0$ to $x = n$ with n rectangles?

Hint. It might help to visualize the Left Riemann sum!

Solution. Hopefully we can see that the partial sum, $\sum_{k=0}^{n} a_k$ is the exact same thing as the left Riemann sum!

(b) We're going to visualize the accumulation of $f(x)$ from $x = 0$ to $x = n$ by thinking about the integral:

$$\int_{x=0}^{x=n} f(x)\, dx.$$

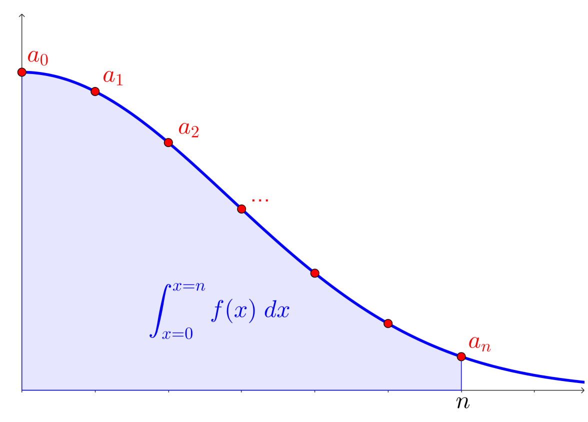


Figure 8.4.4

How does this area compare to the Riemann sum you thought of above? Compare them with an inequality and make sure you can explain why this has to be true.

Hint. Here's a picture of the left Riemann sum!

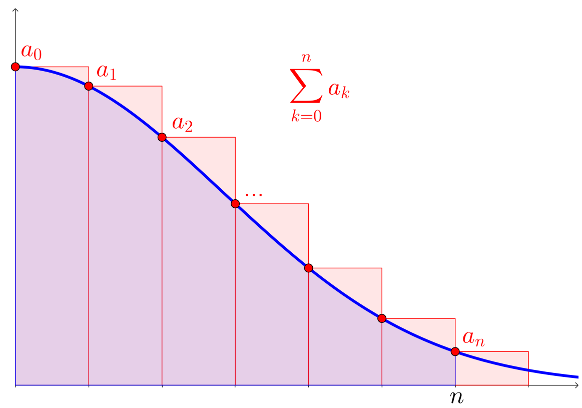


Figure 8.4.5

Solution. Since $\sum_{k=0}^{n} a_k$ is a left Riemann sum for $f(x)$, and since $f(x)$ is decreasing, then we know that each rectangle is formed from the highest point on each subinterval. That means that each rectangle's area overestimates the area under the curve on that subinterval. Note, also, that since this is a left Riemann sum, the nth rectangle is hanging past the end of the definite integral. This means that:

$$\sum_{k=0}^{n} a_k > \int_{x=0}^{x=n} f(x)\ dx.$$

(c) Remove the first term of the series, a_0, and instead think of the sum $\sum_{k=1}^{n} a_k$. Can you still think of this as a Riemann sum to approximate the area from the integral $\int_{x=0}^{x=n} f(x)\ dx$?

How does this new Riemann sum compare to the area formed by the integral? Compare them with an inequality and make sure you can explain why this has to be true.

Hint. Notice, now, that we need a_1 to be the representative height for the first rectangle. How does that change the formation of the rectangles?

Solution. We can form a Right Riemann sum! Note that we don't "overhang" the interval anymore.

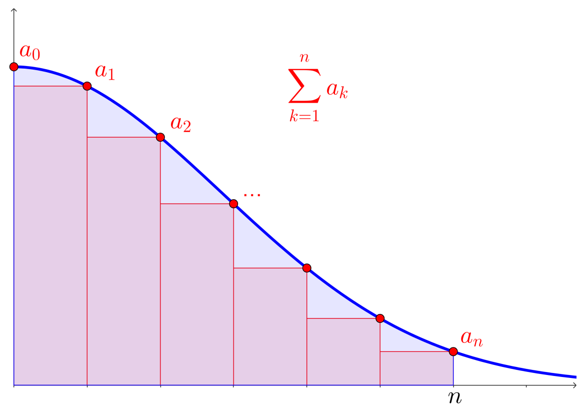


Figure 8.4.6

Note, now, that we are using the lowest point on each subinterval to form the rectangle. This means that:

$$\sum_{k=1}^{n} a_k < \int_{x=0}^{x=n} f(x)\,dx.$$

(d) We have thought about two sums, and we can connect them:

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k.$$

Use the sums to bound the integral:

$$\rule{4cm}{0pt} < \int_{x=0}^{x=n} f(x)\,dx < \rule{4cm}{0pt}$$

Solution.

$$\sum_{k=1}^{n} a_k < \int_{x=0}^{x=n} f(x)\,dx < \sum_{k=0}^{n} a_k$$
$$\left(\sum_{k=0}^{n} a_k\right) - a_0 < \int_{x=0}^{x=n} f(x)\,dx < \sum_{k=0}^{n} a_k$$

(e) Similarly, use the integral to bound the sum:

$$\rule{4cm}{0pt} < \sum_{k=0}^{n} a_k < \rule{4cm}{0pt}$$

Solution.

$$\int_{x=0}^{x=n} f(x)\,dx < \sum_{k=0}^{n} a_k < a_0 + \int_{x=0}^{x=n} f(x)\,dx$$

These bounds are going to be super useful! Discovering them is the main task for finding the connections between improper integrals and infinite series. These inequalities might seem kind of strange at first, but we're going to apply a limit to everything as $n \to \infty$, and then think about our definitions of convergence (Definition 7.1.4 and Definition 8.2.2).

8.5 Alternating Series and Conditional Convergence

8.5.1 Defining Alternating Series, and the Main Result

Activity 8.5.2 Approximating an Alternating Series. Let's look, again, at the picture of the partial sums of an alternating series in Figure 8.5.2. We're going to assume that the series converges, which means that:

- $\lim_{n\to\infty} S_n$ exists.
- $\lim_{n\to\infty} a_n = 0$.

Let's add to our figure.

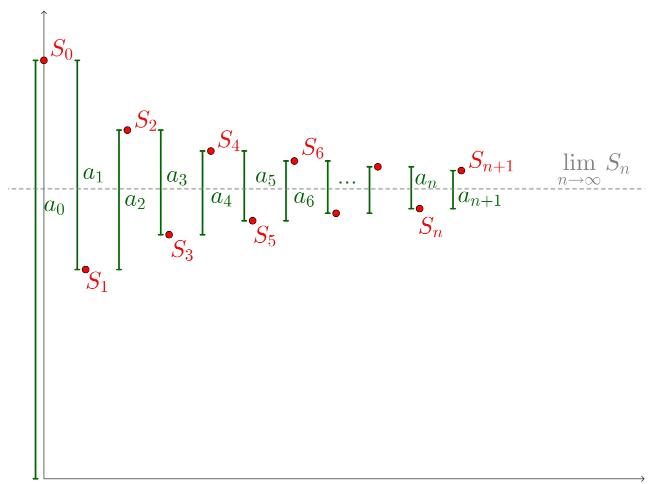


Figure 8.5.4

(a) Why are the even-indexed partial sums sitting above the odd-indexed partial sums?

Hint. We started, in this case, with a positive term a_0, and then the rest of the signs alternate. How does this change the partial values of partial sums?

(b) Why are the even-indexed partial sums sitting above the horizontal line, $\lim_{n\to\infty} S_n$?

Hint. We know that the sizes of the terms, $|a_n|$, decreases. This means that the sequence $\{S_0, S_2, S_4, ...\}$ is decreasing.

What does that mean about how these partial sums relate to the limit of partial sums?

(c) Why are the odd-indexed partial sums sitting below the horizontal line, $\lim_{n\to\infty} S_n$?

Hint. We know that the sizes of the terms, $|a_n|$, decreases. This means that the sequence $\{S_1, S_3, S_5, ...\}$ is increasing.

What does that mean about how these partial sums relate to the limit of partial sums?

(d) If we were trying to approximate the value of $\lim_{n\to\infty} S_n$, how can we use the partial sums to build an interval that approximates the value?

Solution. Since all of the even-indexed partial sums are upper bounds of any approximation and the odd-indexed partial sums are lower bounds.

This means that any pair of partial sums serves as an approximation:

$$S_1 < \lim_{n\to\infty} S_n < S_0$$
$$S_{2n+1} < \lim_{n\to\infty} S_n < S_{2n}$$

8.5.2 Convergence, More Carefully

Activity 8.5.3 The Alternating Harmonic Series Converges.

(a)

Activity 8.5.4 The Alternating Harmonic Series Converges (Again).

(a)

8.6 Common Series Types

8.6.3 Recapping Our Mathematical Objects

Activity 8.6.1 (Im)Possible Combinations. When we have thought about infinite series, we have thought about three different mathematical objects: the sequence of terms of the series, the sequence of partial sums of the series, and the infinite series itself. As a reminder, if we had an infinite series

$$\sum_{k=1}^{\infty} a_k$$

we can say that:

- $\{a_k\}_{k=1}^{\infty}$ is the sequence of terms of the series
- $S_n = \sum_{k=1}^{n} a_k$ is a partial sum and $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series

For each of these three objects—the terms, the partial sums, and the series—we have some notion of what it means for that object to converge or diverge.

Consider the following table of all of the different combinations of convergence and divergence of the three objects. For each combination, decide whether this combination is possible or impossible. If it is possible, give an example of an infinite series whose terms, partial sums, and the series itself converge/diverge appropriately. If it is impossible, give an explanation of why.

Table 8.6.5 (Im)Possible Combinations

$\{a_k\}_{k=1}^{\infty}$	$\{S_n\}_{n=1}^{\infty}$	$\sum_{k=1}^{\infty} a_k$	**(Im)Possible?**	**Example or Explanation**
Converges	Converges	Converges		
Converges	Converges	Diverges		
Converges	Diverges	Converges		
Converges	Diverges	Diverges		
Diverges	Converges	Converges		
Diverges	Converges	Diverges		
Diverges	Diverges	Converges		
Diverges	Diverges	Diverges		

Hint 1. We can think back to some results or definitions that connect pairs of these objects. Can you think of any result or definition that connects an infinite series and a sequence of partial sums? What about a result or definition that connects the sequence of terms with the infinite series?

Hint 2. Look back at Definition 8.2.2 and Theorem 8.3.3.

Solution.

Table 8.6.6 (Im)Possible Combinations

(Im)Possible?	**Example or Explanation**
Possible	Any converging series serves as an example.
Impossible	The sequence of partial sums and the infinite series are the same object, and so must behave in
Impossible	The sequence of partial sums and the infinite series are the same object, and so must behave in
Possible	
Impossible	If the infinite series converges, then the sequence of terms must converge to 0. Theorem 8.3.3
Impossible	The sequence of partial sums and the infinite series are the same object, and so must behave in
Impossible	Both of the reasons, Definition 8.2.2 and Theorem 8.3.3 apply here!
Possible	

8.7 Comparison Tests

8.7.1 Comparing Partial Sums

Activity 8.7.1 Comparisons to Bound Partial Sums. This activity is mostly going to be thinking about proof mechanisms, and so it might be helpful to review Activity 8.4.1 Integrals and Infinite Series. If you want to see more, then the proof of Integral Test will provide some further details on why the inequalities we built were useful.

(a) In the Integral Test, how did we guarantee that our sequence of partial sums was monotonic?

(b) How did we know that, as long as the corresponding integral converged, then our sequence of partial sums was bounded?

(c) How did we know that, as long as our integral diverged, then our sequence of partial sums had to diverge as well?

(d) What happens if we swap out the integral we're connecting our series to with a different series?

For these inequalities to be useful, what do we need to be true?

Hint 1. For our sequence of partial sums to be bounded, we originally had that our partial sums were smaller than an expression involving an integral. Then, those integrals eventually converged to something, and so our sequence of partial sums was bounded by a value.

Can we swap out an integral for a different partial sum? What has to happen in the limit to guarantee that our series converges?

Hint 2. Similarly, when we showed that our partial sums were always greater than something involving an integral, then we were able to show that if that integral diverged, then our partial sums had no choice but to diverge to infinity as well.

What kind of partial sum do you need to change out the integral for?

8.7.2 (Un)Helpful Comparisons

Activity 8.7.2 (Un)Helpful Comparisons. We're going to consider a handful of infinite series, here:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(3^k)}$
2. $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k+1}}{k-2}$

(a) Pick a series that is reasonable to use as a comparison for each of the series listed. Remember, we want:

- A series that is recognizable (probably a Geometric Series or a p-Series), so that we know the behavior of it: we need to know whether the series we're comparing to converges or diverges!
- A series that is similar enough to the series we're working with that we can construct an inequality comparing the term structure. It can be hard to compare functions that are seemingly unrelated to each other!
- A series that has terms that are either larger or smaller than our series, depending on whether we are trying to show that our series converges or diverges.

Hint 1. It can be useful to think about what the dominating pieces of each term structure are. What parts of the function will be most important, especially as the index, k, gets larger?

Hint 2.

(a) In the denominator, think about k contrasted with 3^k. Which of these will be more influential in determining the value of the term when k gets large?

(b) In the numerator, think about k contrasted with $+1$ under the square root. Which of these will be more influential? In the denominator, contrast k and -2. Which of these will be more influential? What does this fraction look like when we just consider the numerator and denominator's most dominant pieces?

Solution.

(a) Let's link the following two series using a comparison:

$$\sum_{k=1}^{\infty} \frac{2}{k(3^k)} \sim \sum_{k=1}^{\infty} \frac{2}{3^k}.$$

We can notice that $\sum_{k=1}^{\infty} \frac{2}{3^k}$ is a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3}$. It converges!
(Specifically, this series converges to 1.)

(b) Let's link the following two series using a comparison:

$$\sum_{k=3}^{\infty} \frac{\sqrt{k+1}}{k-2} \sim \sum_{k=3}^{\infty} \frac{\sqrt{k}}{k} = \sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}.$$

Note that $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}$ is a diverging p-series, since $p = \frac{1}{2}$.

(b) Build the comparison from the series we start with to the one you picked. What kinds of conclusions can you make?

Hint.

(a) When we compare $\frac{2}{k(3^k)}$ with $\frac{2}{3^k}$, we can start with the denominator. What does multiplying by k do to the value of 3^k? It might help to write some of the values out for $k = 1, 2, 3, \ldots$

(b) When we compare $\frac{\sqrt{k+1}}{k-2}$ with $\frac{\sqrt{k}}{k}$, we can think of the numerator and denominator separately. What does adding 1 under the square root do to the value of $\sqrt{k}$? What does subtracting 2 in the denominator do to k? How do these impact the fraction?

Solution.

(a) Since $k \geq 1$, we can say that

$$k(3^k) \geq 3^k \text{ for } k = 1, 2, 3, \ldots$$

This means that

$$\frac{1}{k(3^k)} \leq \frac{1}{3^k} \text{ for } k = 1, 2, 3, \ldots$$

since a big denominator leads to a small fraction. Then, we can scale both by a factor of 2:

$$\frac{2}{k(3^k)} \leq \frac{2}{3^k} \text{ for } k = 1, 2, 3, \ldots$$

Since $\sum_{k=1}^{\infty} \frac{2}{3^k}$ is a converging geometric series (since $r = \frac{1}{3}$ and so $|r| < 1$), then we can say that $\sum_{k=1}^{\infty} \frac{2}{k(3^k)}$ must also converge.

(b) We can see that $k-2<k$. Then, when we think about reciprocals, we see that

$$\frac{1}{k-2} > \frac{1}{k} \text{ for } k>2$$

since a small denominator leads to a big fraction. Then, we can multiply both by $\sqrt{k}$:

$$\frac{\sqrt{k}}{k-2} > \frac{\sqrt{k}}{k}.$$

Finally, since $k+1>k$, we know that $\sqrt{k+1} > \sqrt{k}$.
This means that:

$$\frac{\sqrt{k+1}}{k-2} > \frac{\sqrt{k}}{k-2} > \frac{\sqrt{k}}{k}.$$

So, we have that $\dfrac{\sqrt{k+1}}{k-2} > \dfrac{1}{\sqrt{k}}$ for $k=3,4,5,...$

Since $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}$ is a diverging p-series (since $p=\frac{1}{2}\leq 1$), then $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k+1}}{k-2}$ must also diverge.

(c) We're going to change the series we're considering to two slightly different series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{3^k}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k-1}}{k+2}$

How do these small changes impact the inequalities you built?

Hint.

(a) We moved k from the denominator to the numerator. If the numerator is the thing getting bigger, how does that impact the size of this fraction in the same comparison as before?

(b) We changed the signs: we're subtracting in the numerator and adding in the denominator. How does that impact the size of this fraction in the same comparison as before?

Solution.

(a) Since $k\geq 1$, we know that $2k\geq 2$. This means that

$$\frac{2k}{3^k} \geq \frac{2}{3^k} \text{ for } k=1,2,3,...$$

(b) We have $k+2>k$, and so:

$$\frac{1}{k+2} < \frac{1}{k} \text{ for } k=3,4,5...$$

This means that:

$$\frac{\sqrt{k}}{k+2} < \frac{1}{\sqrt{k}} \text{ for } k=3,4,5...$$

Finally, since $\sqrt{k-1} < \sqrt{k}$, we have:

$$\frac{\sqrt{k-1}}{k+2} < \frac{1}{\sqrt{k}} \text{ for } k = 3, 4, 5...$$

(d) How do these changes in the inequalities change the conclusions we can draw from the Direct Comparison Test?

Solution. In both cases, the Direct Comparison Test is inconclusive. This means that we cannot conclude that either of these series converges or diverges: we don't have enough information to justify any claim we might make.

(e) What do you *think* is happening with these series: do you think that these small changes are enough to change the behavior of the series (i.e. whether it converges or diverges)?

8.7.3 Limit Comparison

Activity 8.7.3 Ratios for Comparison. Let's start with some functions: we'll consider $f(x)$ and $g(x)$ as two functions that are continuous when $x \geq 0$ with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

All of this is so that we can think about $\dfrac{f(x)}{g(x)}$ and know that we have an indeterminate form. We could put the requirement of differentiability on these functions (so that we could think about L'Hôpital's Rule), but we don't need to do that.

We're going to now consider the limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

(a) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $g(x) \to 0$ with a faster growth rate than $f(x)$ does? In this case, we might say that:

$$f(x) >> g(x).$$

Hint. What does it normally look like when a fraction of numbers has a very small denominator compared to the numerator?

(b) What would the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like if $f(x) \to 0$ with a faster growth rate than $g(x)$ does? In this case, we might say that:

$$g(x) >> f(x).$$

Hint. What does it normally look like when a fraction of numbers has a very large denominator compared to the numerator?

(c) If the functions $f(x)$ and $g(x)$ eventually act equivalently, then what does the limit $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

Hint 1. When we say that these two functions act equivalently, we might mean that while they both approach 0 in the limit, they do it in the same way, or with identical growth rates.

Hint 2. What does it normally look like when a fraction of numbers is something over an equivalent thing?

(d) If the function $f(x)$ eventually acts like some scaled version of $g(x)$, then what does the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ look like?

Hint. We can think that $f(x) \to L(g(x))$ as $x \to \infty$ for some real number L.

8.8 The Ratio and Root Tests

8.8.1 Eventually Geometric-ish

Activity 8.8.1 Reminder about Geometric Series. We are going to build some convergence tests that try to link some infinite series to the family of geometric series and show that even though a series is *not* geometric, it might act enough like one to be considered ”eventually geometric-ish.”

But first, what does it mean for a series to be a geometric series?

(a) Describe a defining characteristic of a geometric series. What makes it geometric?

(b) Can you describe this characteristic in another way? For instance, if you described a geometric series using a characteristic about the Explicit Formula, can you describe the same characteristic in the context of the Recursion Relation instead? Or vice versa?

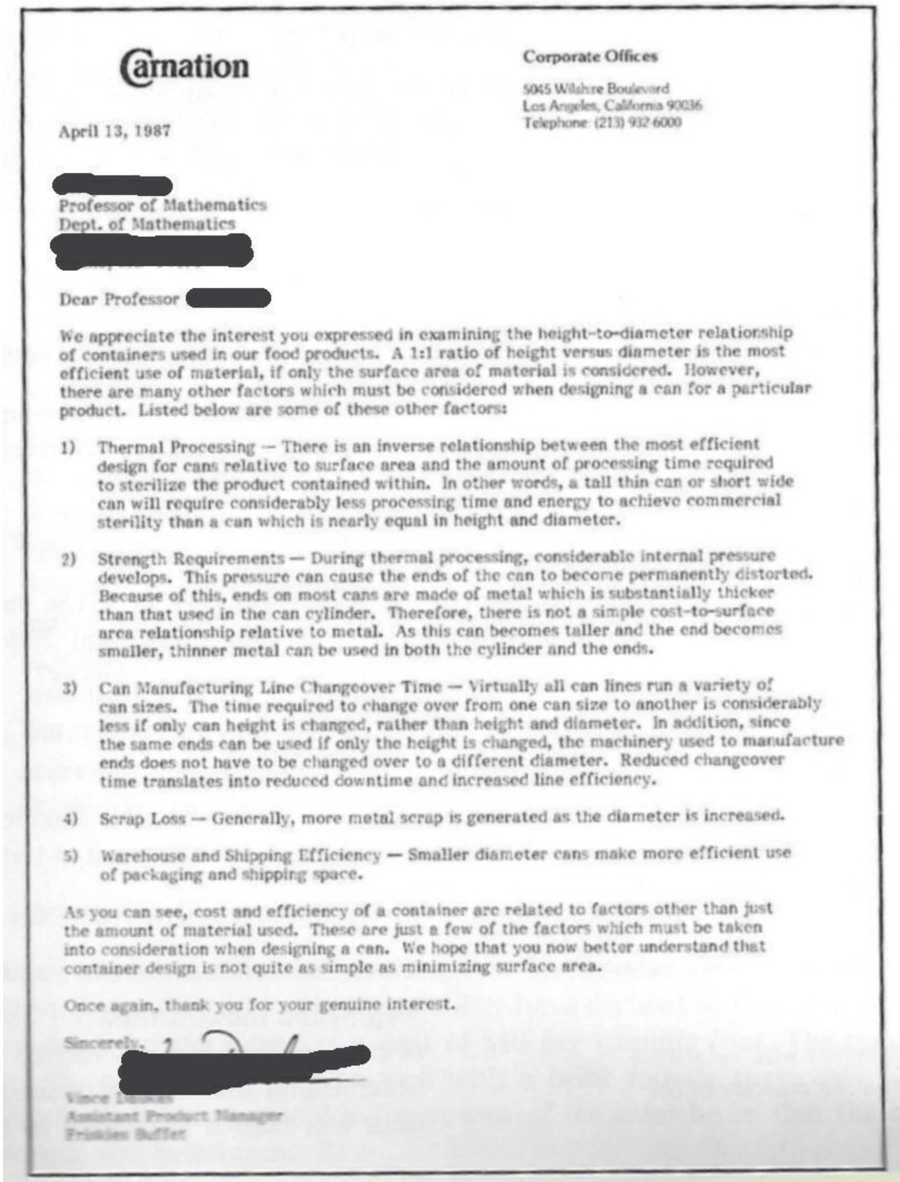
Hint 1. What kinds of functions do we see in the formula for the terms of a geometric series?

Hint 2. How do you describe how you might get from one term in a geometric series to the next one?

(c) Write out a generalized and simplified form of the term a_k of a geometric series explicitly and recursively. In each case, solve for r, the ratio between terms.

Appendix A

Carnation Letter



Carnation

Corporate Offices

5045 Wilshire Boulevard
Los Angeles, California 90036
Telephone (213) 932 6000

April 13, 1987

[redacted]
Professor of Mathematics
Dept. of Mathematics
[redacted]

Dear Professor [redacted]

We appreciate the interest you expressed in examining the height-to-diameter relationship of containers used in our food products. A 1:1 ratio of height versus diameter is the most efficient use of material, if only the surface area of material is considered. However, there are many other factors which must be considered when designing a can for a particular product. Listed below are some of these other factors:

1) Thermal Processing — There is an inverse relationship between the most efficient design for cans relative to surface area and the amount of processing time required to sterilize the product contained within. In other words, a tall thin can or short wide can will require considerably less processing time and energy to achieve commercial sterility than a can which is nearly equal in height and diameter.

2) Strength Requirements — During thermal processing, considerable internal pressure develops. This pressure can cause the ends of the can to become permanently distorted. Because of this, ends on most cans are made of metal which is substantially thicker than that used in the can cylinder. Therefore, there is not a simple cost-to-surface area relationship relative to metal. As this can becomes taller and the end becomes smaller, thinner metal can be used in both the cylinder and the ends.

3) Can Manufacturing Line Changeover Time — Virtually all can lines run a variety of can sizes. The time required to change over from one can size to another is considerably less if only can height is changed, rather than height and diameter. In addition, since the same ends can be used if only the height is changed, the machinery used to manufacture ends does not have to be changed over to a different diameter. Reduced changeover time translates into reduced downtime and increased line efficiency.

4) Scrap Loss — Generally, more metal scrap is generated as the diameter is increased.

5) Warehouse and Shipping Efficiency — Smaller diameter cans make more efficient use of packaging and shipping space.

As you can see, cost and efficiency of a container are related to factors other than just the amount of material used. These are just a few of the factors which must be taken into consideration when designing a can. We hope that you now better understand that container design is not quite as simple as minimizing surface area.

Once again, thank you for your genuine interest.

Sincerely,

[redacted]

Vince [redacted]
Assistant Product Manager
Friskies Buffet

Figure A.0.1 Response letter from Carnation.

Full Text of the Carnation Letter. April 13, 1987

[REDACTED]
Professor of Mathematics
Dept. of Mathematics
[REDACTED]

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Once again, thank you for your genuine interest.

Sincerely,
[REDACTED]
Vince [Illegible]
Assistant Product Manager
Friskies Buffet

Colophon

This book was authored in PreTeXt.