Connection between Root Test and Ratio Test for Series in the Calculus Course

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ABSTRACT. The Root Test and the Ratio Test for series $\sum_{n=0}^{\infty} a_n$ are usually discussed and proved independently in a Calculus course. This creates in students an impression that if one of these two tests is inconclusive because the radius ρ is 1, then there is a chance that the second test can be useful. We suggest a new approach to the presentation of these two tests, which eliminates this confusion. Suppose in the Ratio Test $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \rho$. We show that this implies that the Root Test provides the same value of ρ in the limit of the root ($\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$.), and the limit of the root can be used for verification of the convergence of the series. This proof of the Root Test is easier than the standard proof of the Root Test and explains the relation between the two Tests.

1. Introduction

In the study of infinite series' convergence tests, we usually discuss the Root Test and the Ratio Test. Namely, let us consider the series

$$\sum_{n=0}^{\infty} a_n.$$

Define

$$\rho_{\text{ratio}} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

and

$$\rho_{\text{root}} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

The Ratio Test states that if $\rho_{\text{ratio}} < 1$, the series is convergent. If $\rho_{\text{ratio}} > 1$, the series is divergent. If $\rho_{\text{ratio}} = 1$, the Ratio Test is inconclusive.

Similar result can be formulated for the Root Test. If $\rho_{\text{root}} < 1$, the series is convergent. If $\rho_{\text{root}} > 1$, the series is divergent. If $\rho_{\text{root}} = 1$, the Root Test is inconclusive.

Each of the two tests is valuable for applications. Some series are easier to test with the Root Test, and the Ratio Test is more suitable for the others.

The convergence, divergence and inconclusiveness of power series for the Root Test and for the Ratio Test are usually proved independently.

Some textbooks ([**S**]) contain a note saying that if one the two tests has $\rho = 1$ and, thus, it is inconclusive, then the other test will also be inconclusive. This statement is not justified and does not explain the connection between the Ratio Test and the Root Test.

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There are many examples ([**HWT**], [**C-U**]) showing that the limit in the Root Test may exist while the limit in Ratio Test does not. The Root Test verifies more series than the Ratio Test. Namely, if $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$ exists, then $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ exists as well and equal to the first limit.

Below we will show that the Root Test can be derived from the Ratio Test. The benefits of this proof are two-fold. First, the new proof, where the Root Test is derived from the Ratio Test is simpler than the standard proof of the Root Test. Second, this proof shows the connection between the Root Test and the Ratio Test. In particular, it explains why inconclusiveness of one Test (when $\rho = 1$) will imply inconclusiveness of the other Test. This will prevent attempts of students to verify convergence with another Test, when the first test has $\rho = 1$.

2. Proof of the Root Test

Theorem 2.1 (The Ratio Test). *Let $\sum a_n$ be a series and suppose that*

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \rho_{ratio}$$

Then, the series converges absolutely if $\rho_{ratio} < 1$; the series diverges if $\rho_{ratio} > 1$ or ρ_{ratio} is infinite; the test is inconclusive if $\rho_{ratio} = 1$.

For the proof see, for example, [**HWT**].

Next, we will derive the Root Test from the Ratio Test.

Corollary 2.2 (The Root Test). *Let $\sum a_n$ be a series and the following limit exists (finite or infinite):*

$$\rho_{ratio} = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}.$$

Define

$$\rho_{root} = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

Then, ρ_{root} exists and

$$\rho_{root} = \rho_{ratio}.$$

The Root Test, thus, shows that the series converges absolutely if $\rho_{\text{root}} < 1$; the series diverges if $\rho_{\text{root}} > 1$ or ρ_{root} is infinite; the test is inconclusive if $\rho_{\text{root}} = 1$.

Before proving this result, it is also easy to illustrate to students some intuitive arguments of the proof.

Proof. Suppose there exists a finite limit: $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \rho_{\text{ratio}}$. For simplicity, we will write ρ for ρ_{ratio}.

By the definition of limit, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\rho - \epsilon < \frac{|a_{n+1}|}{|a_n|} < \rho + \epsilon.$$

In particular,

$$\rho - \epsilon < \frac{|a_{N+1}|}{|a_N|} < \rho + \epsilon.$$

Then,

$$(\rho-\epsilon)\cdot|a_N| < |a_{N+1}| < |a_N|\cdot(\rho+\epsilon)$$

and

$$(\rho-\epsilon)^k\cdot|a_N| < |a_{N+k}| < |a_N|\cdot(\rho+\epsilon)^k$$

Taking the $(N+k)$-th root, we obtain the estimate:

$$\sqrt[N+k]{(\rho-\epsilon)^k}\cdot\sqrt[N+k]{|a_N|} < \sqrt[N+k]{|a_{N+k}|} < \sqrt[N+k]{|a_N|}\cdot\sqrt[N+k]{(\rho+\epsilon)^k}.$$

We can take the limit as $k \to \infty$ in the last inequality. The following limit is easy to calculate:

$$\lim_{k\to\infty} \sqrt[N+k]{(\rho \pm \epsilon)^k} \cdot \sqrt[N+k]{|a_N|} = (\rho \pm \epsilon) \cdot 1 = \rho \pm \epsilon$$

and consequently

$$\rho - \epsilon < \lim_{k\to\infty} \sqrt[N+k]{|a_{N+k}|} < \rho + \epsilon$$

for any positive ϵ. This implies that

$$\rho_{\text{root}} = \lim_{k\to\infty} \sqrt[N+k]{|a_{N+k}|} = \rho = \rho_{\text{ratio}}.$$

If $\rho_{\text{ratio}} = \infty$, then for every $M > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$M < \frac{|a_{n+1}|}{|a_n|}$$

Arguments similar to the above yield the estimate:

$$M^k \cdot |a_N| < |a_{N+k}|.$$

Taking the $(N + k)$-th root, we obtain

$$\sqrt[N+k]{M^k} \cdot \sqrt[N+k]{|a_N|} < \sqrt[N+k]{|a_{N+k}|}.$$

Then,

$$\lim_{k\to\infty} \sqrt[N+k]{M^k} \cdot \sqrt[N+k]{|a_N|} < \lim_{k\to\infty} \sqrt[N+k]{|a_{N+k}|},$$

i.e.,

$$M < \lim_{k\to\infty} \sqrt[N+k]{|a_{N+k}|}$$

for every $M > 0$. This means that $\rho_{\text{root}} = \lim_{k\to\infty} \sqrt[N+k]{|a_{N+k}|} = \infty$. □

The Corollary 2.2 immediately implies

COROLLARY 2.3. *The Root Test and the Ratio Test can be inconclusive only simultaneously. More precisely,*

- *If $\rho_{ratio} = 1$, then $\rho_{root} = 1$.*
- *If $\rho_{root} = 1$, then either $\rho_{ratio} = 1$ or $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ does not exist.*

References

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