Area of a Circle via the Second Fundamental Theorem of Calculus

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We present an intriguing application of the second fundamental theorem of calculus (FTC) to prove that the area of a quarter circle of unit radius, expressed by the integral

$$\int_0^1 \sqrt{1-t^2}\,dt, \tag{1}$$

is $\pi/4$.

Of course this result is easily obtained by making the substitution $t = \sin\theta$ and using the double angle formula for cosine to evaluate the subsequent integral. However, this is something that a student of calculus usually learns in a second calculus course under the guise of "integration techniques," whereas the present method is based on results learned earlier in the sequence.

Consider the function

$$F(x) = \int_{\cos x}^{\sin x} \sqrt{1-t^2}\,dt.$$

The 2nd FTC together with the chain rule yields

$$F'(x) = \sqrt{1-\sin^2 x}\cdot\cos x - \sqrt{1-\cos^2 x}\cdot(-\sin x) = \cos^2 x + \sin^2 x = 1.$$

Thus F has the form

$$F(x) = x + c.$$

Setting $x = 0$ and denoting the integral in (1) by A, we have

$$c = \int_1^0 \sqrt{1-t^2}\,dt = -A.$$

Hence

$$F\left(\frac{\pi}{2}\right) = A = \frac{\pi}{2} - A,$$

which yields $A = \pi/4$ as desired.

The above argument occurred to the author while designing a bonus question for a Calculus I final at the University of North Florida. While this simple calculation must have been observed before, the author does not recall having come across it in any of the textbooks he has used and therefore felt it worthwhile to present here.

Summary. We provide an easy argument, using the second fundamental theorem of calculus, for computing the area of the unit circle.

http://dx.doi.org/10.4169/college.math.j.46.4.299
MSC: 26A06