Trigonometric Derivatives Made Easy
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CLASSROOM CAPSULES

Classroom Capsules are short (1–3 page) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and submitted through Editorial Manager.

Trigonometric Derivatives Made Easy

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Most students who have taken calculus will remember (some with possible trepidation?) the "special" trigonometric limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0. \tag{1}$$

These typically arise as necessary ingredients in the most common proofs of the rules for differentiating the basic trigonometric functions,

$$(\sin x)' = \cos x \quad \text{and} \quad (\cos x)' = -\sin x. \tag{2}$$

The usual proof of the first derivative in (2) using the limit definition of the derivative [**1**] involves expanding $\sin(x + h)$ using an angle sum formula, factoring the resulting expression, and applying both of the limits in (1). Students are usually required to demonstrate their understanding of these "special" limits by computing related limits such as

$$\lim_{x\to 0}\frac{\sin(3\sin(2\sin x))}{x}.$$

Nevertheless, the importance of such limits is easily misplaced; their role is purely auxiliary and indeed (1) is nothing more than the special case of (2) when $x = 0$. Direct proofs of (1) take about a page or more and are usually banished to an appendix. Moreover, the insight to be gained from (1)—that x must be measured in radians for (1) and (2) to be correct as stated—is easily buried in the details.

The purpose of this note is to identify a particularly simple proof of (2) that does not require the limits (1). As an added bonus, the argument makes clear why x must be measured in radians.

To begin, we recall the definition of the two basic trigonometric functions: $\cos t$ and $\sin t$ are, respectively, the x- and y-coordinates of the point on the unit circle with angle t measured from the positive x-axis. Thus, the curve $\gamma(t) = (\cos t, \sin t)$ traverses the unit circle counterclockwise. Recall that the length of an arc on the unit circle is equal to the angle it subtends (when measured in radians). Hence, when t is measured in radians, $\gamma(t)$ moves at unit speed.

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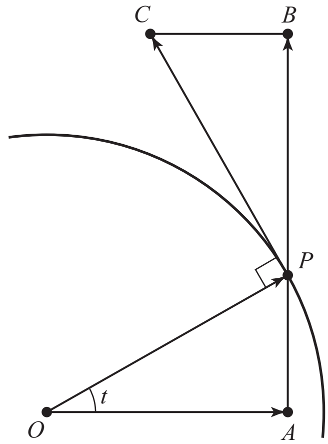

Figure 1. The point $P = (\cos t, \sin t)$. The "position triangle" $\triangle OPA$ and "velocity triangle" $\triangle PCB$ are congruent when t is measured in radians.

Now fix $t \in \mathbb{R}$. For simplicity, we may assume $t \in [0, \pi/2]$. Let $O = (0, 0)$, $P = \gamma(t)$, and $A = (\cos t, 0)$. Define points B and C so that $\overrightarrow{PC}$ is the velocity vector of γ at P, $\overrightarrow{AP}$ is parallel to $\overrightarrow{PB}$, and $\angle PBC = \pi/2$; see Figure 1.

Since $\overrightarrow{PC}$ is tangent to the circle at P, $\angle OPC = \pi/2$. Since $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are parallel, $\angle AOP = \angle BPC$ and $\angle OPA = \angle PCB$. Hence, triangles OPA and PCB are similar.

If t is measured in radians, then $|\overrightarrow{PC}| = 1$. Since $|\overrightarrow{OP}| = 1$, triangles OPA and PCB are actually congruent.

Since $\overrightarrow{PC}$ is the instantaneous velocity of γ at P and P is in the first quadrant, its respective x and y-components, $\overrightarrow{BC}$ and $\overrightarrow{PB}$, satisfy

$$(\cos t)' = -|\overrightarrow{BC}| \quad \text{and} \quad (\sin t)' = |\overrightarrow{PB}|.$$

Combining this with the congruence of the position and velocity triangles, we have

$$(\cos t)' = -|\overrightarrow{AP}| = -\sin t \quad \text{and} \quad (\sin t)' = |\overrightarrow{OA}| = \cos t,$$

which proves (2).

Summary. We give geometric proofs of the basic trigonometric derivative formulas, avoiding the trigonometric limit formulas required in the usual limit definition proofs.

Reference

1. J. Stewart, *Calculus*. Eighth ed. Cengage, Boston, 2016.