Response to Referees and Summary of Changes

We thank the referees for careful reading of our manuscript and providing valuable feedback. Below, we address each suggestion, and refer to the corresponding changes we made to the manuscript.

Reply to Referee A

*1) Regarding the implementation in Yb ensembles, the authors show optimal numerical results based on Rydberg blockade due to Rydberg-Rydberg interaction of atoms excited to principal quantum number of about 120. I would like the authors to discuss the level spacing between neighboring Rydberg levels and given the Rabi frequency and duration of the control pulses what is the probability of exciting multiple Rydberg levels (generating Rydberg wavepackets)? If this probability is not negligible, one needs to consider errors due to such an effect.*

We agree with the referee, that it is important to that this error is negligible. We calculated the level spacing between the nearest neighboring Rydberg levels, and found it (2 Pi \* 4 GHz) to be much bigger than the Rabi-frequency (~ 2 Pi \* 8.6 MHz). We added a short explanation about this to the beginning of section VI.A in SI.

*2) The authors mention that ionization rate with expected trapping field intensity for Rydberg levels of about 100 is negligible. It is worth quantifying such a rate for Rydberg level of 120.*

We evaluated the photoionization rate for Yb(I) 6s120p(1S0), and compared it with the natural decay rate of this level. We changed the last sentence of section V.A in the Supplementary to “Furthermore, the photoionization rate from the n = 120 Rydberg level, in a trapping field with 10^4 W/cm^2 intensity, is about five times smaller (gamma\_PI ~ 110 s^{-1}), than the natural lifetime (gamma ~ 540 s^{-1}).” This effect can be practically eliminated by modulating the trapping light (i.e. switching it off when atoms are excited into Rydberg states.) This is noted in the first paragraph on page four of the main text.

*3) The authors refer to the case with the messenger atom in the main text and in the supplementary material as an approach that would suffer from a smaller set of error sources. Even though this argument is true, the authors use the same error estimations to make this conclusion. I believe error terms that involve averaging over <1/\Delta> needs to be evaluated separately for the messenger atom case, as the current averaging approach takes every two atoms in a 2D or 3D configuration, whereas for messenger atom this averaging should consider a fixed position for one of the atoms (the messenger atom).*

We added a section to the supplementary (now section XI in SI), where we evaluate <1/Delta\_{12}^2> for the case when the messenger atom is placed close to the boundary of the cloud. The resulting error estimate is a factor of two larger than the previous estimate. This, however, does not change the total error significantly because it has a relatively small contribution in both cases. We added a paragraph to the end of section I in SI, explaining the result.

*4) In the supplementary material, section III C, in estimating $\epsilon\_5$, the authors include the collective enhancement factor of $n$ in the spontaneous emission rate. Given that the collective excitation in |e> is generated using pulses number II and III as labeled in figure 2 (b), one might need to consider the phase matching condition in collective enhancement of the photon emission rate for |e> to |g> transition. The total wavevector for this process is $\Delta K = K\_{II} - K\_{III} - K\_{IV}$. For co-propagating fields it is reasonable to assume this will have no impact on the collective spontaneous emission. However, given that $K\_{IV}$ is assumed to be enhanced by a cavity, achieving $\Delta K = 0$ may not be trivial. Therefore, a correction factor of order $Sinc^2(\Delta K \times n^{1/D} \times \lambda\_{magic})$ might be necessary, where $D = {2,3}$ for 2 or 3 dimensional ensembles.*

We agree with the referee, that precise description of the phase matching condition is important for future work in this subject. We added a paragraph about the considerations of the phase matching condition in the presence of an optical cavity to the end of section IV.E in SI.

*5) The previous point also raises the question that how the authors imagine different control fields to be applied in their scheme. A discussion on the details of different orientations of the applied fields and their implications is useful.*

We agree, to aid future implementation of our ideas, we added a figure (now Fig 1.) to section IV.E of SI, which illustrates the orientation of the coherent driving fields with respect to the optical cavity field.

*6) The authors present the stability gain by comparing their result to the standard approach, where the stability is set by the total number atoms and coherence time of the local oscillator. I would like the authors to compare their results to the case using a spin-squeezed ensemble in Ref. [10].*

Comparing our results to the standard quantum limit, as a benchmark, has the advantage of being easily comparable to other results that also compare themselves to SQL. In case of the results reported in Ref [10], they report a 70-fold increase in accuracy of phase measurement, which ideally would translate to the same enhancement in clock stability. We found a 12-fold enhancement in our analysis, compared to SQL. We want to emphasize that the atom numbers employed in our entanglement scheme are significantly smaller than what is used in ref [10]. To make this clear and the comparison easier, we changed the sentence about Ref [10] in the introduction to “Significant noise reduction has recently been demonstrated with spin-squeezed states in a single ensemble of atoms in [10], which reported a 70-fold enhancement of phase measurement accuracy beyond the standard quantum limit, relying on much larger number of atoms.”

*7) In section III E, the authors neglect fiber and coupling loss. This needs to be justified, maybe with reference to experimental values or a fair theoretical estimate. As the entanglement generation between separate clocks relies on single photon transmission, the authors should discuss the effect of inefficiency due to fiber or coupling loss. If this can be circumvented via a heralding BSM, how does a limited interference visibility affect the fidelity?*

We added three paragraphs describing the limitations of our scheme arising from photon loss errors. We derived typical maximal distances, for which the photon propagation loss is not significantly larger than the inherent probabilistic “loss” of the two-photon scheme. We report results for both optical fiber links between terrestrial labs and free-space optical links between satellites, in section IV. C. of the supplementary.

*8) It will be informative, if the authors provide a timescale for a successful generation of a global GHZ state in their scheme for the total number or atoms/ensembles/clocks in the example discussed in the paper.*

We added a new section (now VIII in SI), where we analyze the expected amount of time to set up a globally entangled GHZ state on between 10 clocks, where the neighbors are connected by 5km-long optical fibers. We find that it takes 1.7us to establish all required links, which is the bottleneck in terms of time.

*9) In the calculation regarding precision or stability of the quantum*

*network of clocks, is there a hidden assumption on the local*

*oscillators at spatially separate clocks to be phase-locked?*

Yes, local oscillators of the clocks have to be phase locked prior to entangling the atoms. We added a clarification to the introduction: “… network of atomic clocks can result in substantial boost of the overall precision if multiple clocks are phase locked and connected by quantum entanglement.”

*Minor suggestions/corrections:*

*1) I suggest the authors to add M to the figure 1 to clarify number of ensembles in each clock.*

We added labels “M ensembles” to figure 1.

*2) In the first paragraph in page 2, description of the 4th step is not clear.*

To improve the explanation of step 4 we changed the following:

- We modified the following sentence

“, which promotes any population in s to r\_2, which then blocks the path via r\_1.” to

“This promotes any population in s to r\_2, which then blocks the path g ? r\_1 ? f.”

- We moved the lower indices inside the kets in Eq. 4, so that the description in following text is easier to follow.

- We changed

“measurement of n\_{s\_k} ? m in {0,1}” to

“measurement of n\_{s\_k}, yielding m in {0,1}”

- We replaced the arrow in “n\_{s\_k} ? 0” and “n\_{s\_k} ? 1” with equal signs.

- We moved the “k” index inside the ket in the expression of the GHZ state, to match with the convention used in Eq. (4).

*3) In the description right after Eq. 1, it might be good to mention that these are "collective" spin-wave states. In general, it could be useful to have distinguishable symbols for single atom states and collective states of atoms in an ensemble.*

We added the sentence “The kets, |n\_f>, |n\_s> for n in {0,1} stand for collective spin waves being excited by n quanta.” to the end of the paragraph of Eq 1.

Furthermore, to make the distinction between single-atom and collective states, we changed the symbol for the ground state from “|0>” to “|0\_f 0\_s>”. This way, it is clear that if a letter appears alone inside the ket, say |f>, it refers to a single atom state, while if it appears as subscript to a number, say |0\_f>, then the ket stands for a collective state. We remind the reader of this convention right after Eq. 5.

*4) The discussion leading to Eq. 2 deserves to be slightly expanded to include more details. One could show the results of one-time application of the conditional photon emission as an intermediate step.*

We added the sentence “This particular sequence results in emitting a single photon (from e ? g transition) provided that the level s is empty, i.e. |0\_s>|vacuum> ? |0\_s>|1 photon>.” to illustrate the immediate effect of applying the pulse sequence once.

*5) A minor typo in the Page 3, right column, paragraph starting with "In practice, ...". The sentence "In such a case, the messenger atom can BE used" is missing "BE".*

Corrected typo. The sentence now reads “In such a case, the messenger atom can be used…”.

*6) Page 4, below figure 5, the sentence starting with "Collective enhancement and phase matching of the ..." needs a reference to the supplementary material.*

Added “(See Supplementary for details.)” after the sentence in question.

*7) In the supplementary material, section 1, before Eq. 2, [pi]\_{f,r1} --> [\pi]\_{f,r1}*

Corrected typo. Now it reads “[\pi]\_{f,r1}”, in the Supplementary.

*8) In the supplementary material, section III C, "the spontaneous emission lifetime of the |e> --> |f> transition" should be "|e> --> |g> transition"?*

Corrected typo. Now it reads "|e> -->|g> transitions", in the Supplementary.

*9) In Eq. 22 in the supplementary material, I suggest to use a different symbol for the cavity finesse.*

We changed the symbol for cavity finesse from “f” to “I”, in and after Eq 22, in the Supplementary.

Reply to Referee B

*- I found the title not specific enough. Ref. 6 that introduces the general concept was already entitled “A quantum network of clocks”. I would suggest changing the title to explicitly stating that this paper is providing a recipe to eventually implement this concept. It could be something like “Quantum network of atom clocks: a possible implementation with neutral atoms”…*

We changed to title to the more specific “Quantum network of atom clocks: a possible implementation with neutral atoms”.

*- The error per atom depends on the lattice geometry. This is indicated in the caption of Table 1 where the reader is sent to the SI for the 2D case. It would be useful to stress this point in the main text by adding a short sentence about this dependency.*

We added “Overall fidelity turns out to depend on the lattice geometry; it is the highest for 3D optical lattice.” to the end of paragraph 2 on page 4.

*- In the SI, errors are evaluated in the ideal case. As this paper can be considered as a road map, it would be useful to state the current state of the art for the different steps and to address the general question: from an experimental point of view, how far we are from realizing even a rudimentary version of this network where a boost of the overall precision can be unambiguously demonstrated?*

Following the referee’s suggestion, we looked up fidelity values achieved by different experimental setups using photons to entangle remote ensembles and Rydberg interaction to mediate coherent coupling between distant atoms. Currently, these fidelities are too low to demonstrate overall accuracy gain. We added a section (IX in SI), where we identify these steps of our protocol and compare experimentally demonstrated fidelities for them. We note that improving the experimental fidelity of preparing entangled states of Rydberg atoms is currently a very active effort in the experimental AMO community.