# Higher-categorical Strucures and Type Theories: DRAFT!

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# Contents

CHAPTER 1

# Introduction and background

## 1. Homotopy Type Theory: an introduction

1.1. Similar to the introduction to my previous paper: a quick, accessible intro to the higher-categorical view of identity types.

1.2. Refer to appendices for full background on globular higher cats & on DTT. However: include a *rough* introduction to the higher cats, & a *full* introduction to (& discussion of) identity types.

## 2. Survey of the field

2.1. Goals! What we're working towards in the short-term (eg sound and complete semantics, good analysis of categorical properties of $\Pi$, $\Sigma$-types, etc.

2.2. What's actually been done! Some models, a few structures, syntactic analysis in dimension 2, applications as independence results...

Lots of references should go in this, of course!

## 3. Outline of the present work

3.1. Overall structure (composition of 2-cells along bounding 0-cell), reason therefor (aim of analysing type theories in the well-understood quasi-categorical setting).

3.2. Overview of the "universal-algebraic aspects" setup: technical, dry, but necessary!

3.3. Results of the "syntactic structures" section.

3.4. Results of the "homotopical constructions" section.

## 4. Outlook: visions of a higher-categorical foundation

4.1. Write up some of what's currently just in folklore, the $n$-lab, the categories list, boozy nights out with the gang, etc. :

Voevodsky's model(s) + axiom; the type theorists' OTT etc.; notions of "the same"; "category theory without equality", etc.

## 5. Acknowledgements

(Should go before this chapter, or here at end of it?)

— Steve! Krzys/Chris. Other HTT'ers: Michael, Richard, Benno, Chris. Pittsburgh PL crowd: Bob H, Dan L, Noam Z. Also in Pittsburgh: Kohei, Henrik, James C, Rick S, Peter A, Dana. Chicago group: Mike, Emily, Claire, Daniel.

Nottingham: Thorsten and his merry men. Elsewhere on-topic: Martin H, Andrej, Pierre-Louis C., Paul-Andr M?, Thomas F?. Off-topic: Yimu, orchestras, parents!

(To do: ask people's permission for this??)

CHAPTER 2

# Universal-algebraic aspects

Possibly fold this chapter into the next??

## 1. A category of Type Theories

1.1. If poss, use elegant Fiore et al "second-order…" as framework for this! Explain: this is *much more literal to the syntax* than Cartmell, categories with attributes etc; but on the other hand, much more categorically tractable than more ad hoc presentations of the syntax.

Mention possible tie-in with formalisation in Agda?? (*)

## 2. Internal algebras for operads

2.1. Give fuller account of what I rush through in my previous paper: show correspondence btn different notions of algebras for an operad! (a) models of ess. alg. (Lawvere) theory (poss with extra structure: "P-maps"); (b) Batanin: monoidal globular categories (as used + nicely expounded in [GvdB]); (c) Leinster: (weak) T-structured categories.

Lovely rarely-cited WEBER paper gives source for most of this! Possibly even *everything* I need is there, in which case possibly move this section to appendix, and fold the first part of this chapter into the next chapter???

CHAPTER 3

# Globular structures from type theory

## 1. The fundamental weak omega-groupoid of a type

Update of my prev paper (+ more detailed comparison with Richard + Benno): type gives internal weak omega-groupoid in the classifying category.

First,

## 2. The classifying weak omega-category of a type theory

CHAPTER 4

# Homotopical constructions from globular higher categories

(NON)SECTION AIM: Motivate constructing the simplicial nerve, and hence the model structure. TODO: cut down, too discursive?? Maybe move this to end of previous section, or even to introduction, and here give an overview of simplicial structures / methods?

0.1. The great power of classifying categories in 1-categorical logic come from [‘depends on’ ?] an analysis of the logical constructors and rules in categorical terms: substitution as pullbacks, Π- and Σ-types as adjoints, and so on. So, a natural first impulse is to try to analyse the universal properties of the type constructors within $\mathbb{C}\lessdot_\omega(\mathcal{T})$, which we would expect to be weak-higher-categorical analogues of the usual logical structure.

Unfortunately, the theory of logical structure on globular higher-categories is not yet well-understood. Of course, we can hope that the developing dictionary with type theory will help understand how such structure should behave! However, there is an alternative model of higher categories for which the relevant theory is already much further advanced: Joyal’s quasi-categories. Quasi-categories are not a fully general theory of higher categories: they only model so-called (∞,1)-categories, in which all cells above dimension 1 are (weakly) invertible. However, as we have seen, the classifying categories of type theories are of this form; so quasi-categories seem potentially excellently-suited for our desired analysis, if only we can give quasi-category models for $Clw(\mathcal{T})$!

0.2. In other words, we would like to construct a functor

$$\mathbb{C}\lessdot_\omega^{qcat} \colon \mathbf{Th} \to \mathbf{QCat}.$$

TODO: Hmm, this doesn’t work if the reader doesn’t know yet that quascateogries are simplicial things! Work out how to re-organise to fit that in nicely.

There are two obvious options. Firstly, we could construct $\mathbb{C}\lessdot_\omega^{qcat}(\mathcal{T})$ directly from the theory $\mathcal{T}$. [TODO: Ask Michael whether/how much to mention simplicial type theory.] However, Id-types as they stand are inescapably globular; there seems no obvious way to extract simplicial sets from the theory as cleanly and directly as one can extract globular sets. (The intriguing approach of re-axiomatising Id-types to be “naturally simplicial in shape” has, however, been considered by Warren and Gambino [**?**].)

It thus seems natural to take a different approach: to construct $\mathbb{C}\lessdot_\omega^{qcat}$ in two steps, composing $\mathbb{C}\lessdot_\omega$ from the previous section with a functor

$$\mathcal{N}^{qcat} \colon P\text{-}\mathbf{Alg} \to \mathbf{QCat}$$

giving the "quasi-category nerve" of a globular weak $n$-category. This has the added payoff that such a functor would be of independent interest, since the comparison between globular and simplicial higher categories is as yet little-understood in the fully weak case.

0.3. In Section 3, we will thus construct several candidate nerve functors. Constructing simplicial objects is straightforward; the hard part is proving the requisite horn filling conditions to show that they are quasi-categories.

It is for this that we construct, in Sections and , a Quillen model structure on categories of globular higher categories (under certain extra hypotheses). The computational tools provided by such a structure provide precisely what we need to show that the horns arising in our nerve constructions can be filled, and hence that the nerves are indeed quasi-categories.

In fact, we construct an *algebraic* model structure in the sense of [**?**]. This is a Quillen model structure in which both weak factorisation systems are NWS's and there is moreover a comparison map connecting the two. While we will not need any of the extra power of an algebraic model structure, the algebraicity comes almost for free given the form of our proof.

## 1. Cellular algebraic model structures

SECTION AIM: The general construction of an algebraic model structure from a collection of generating cells. (TODO: read/ask around in case I've missed where something closer to this has already been done; work out terminology for this general construction.)

1.1. Recall what a AWFS and AMS are. (Or put this in appendix?)

1.2. We start by recalling from [**?**], [**?**] the construction of an AWFS on **str-$n$-Cat** whose right maps are precisely the contractible maps.

...do it by the Garner small-object argument! Algebraic freeness shows the right maps are what we think. And Garner shows that this is also the adjunction with computads, fwiw (maybe leave this out if not needed).

Point out how this comes from the map $D(\mathrm{ob}\,\mathbb{G}) \to \mathbb{G} \to \widehat{\mathbb{G}} \to$ **wk-$n$-Cat** of cells/boundaries.

1.3. In fact, the remainder of the construction of the AMS (thouh not the proof that it is one) can be given entirely in terms of this set of generating cofibrations; and, indeed, $L'$ categories will give another example. Thus for the remainder of this section, we will fix a category $\mathcal{E}$ that admits the small object argument [TODO: define this here or elsewhere!], a set $\mathcal{I}$, and a functor $\mathcal{I} \to \mathcal{E}$.

1.4. Some experimenting: 22 to 22 to 22 to 2 to 2.

1.5. Another para just to see how it gets numbered!

## 2. An algebraic model structure on globular higher categories?

2.1. Discuss instantiating the theorem of the previous section to (a) L'-categories, (b) P-algebras; prove as many of the lemmas as possible!

## 3. Simplicial nerves of globular higher categories

3.1. Give aim; different NWFS for different nerves; proof with model structure that these give nerves!

3.2. Prove from model strux that these really do give quasi-categories.

TODO: read up Dugger references properly.

CHAPTER 5

# Globular structures from type theory

## 1. The fundamental weak omega-groupoid of a type

Update of my prev paper (+ more detailed comparison with Richard + Benno): type gives internal weak omega-groupoid in the classifying category.

First,

## 2. The classifying weak omega-category of a type theory

CHAPTER 6

# Background: globular higher category theory

## 1. Strict higher categories

1.1. Define **str-$n$-Cat** and **str-$\omega$-Cat** by enrichment.

1.2. Analyse $T$: pasting diagrams, Batanin trees, familial representability.

## 2. Weak higher categories

2.1. Contractibility.

CHAPTER 7

# Background: "Homotopical" higher categories

## 1. Algebraic Model Structures

1.1. A natural weak factorisation system is dots

1.2. **2. Quasi-categories**

APPENDIX A

# Globular structures from type theory

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