Derivation of the New Keynesian Phillips Curve: In the lecture, we discussed that the problem of a firm choosing its price P_t^* in the current period can be stated as:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathrm{E}_t \left\{ \frac{\Lambda_{t,t+k}}{P_{t+k}} \left(P_t^* Y_{t+k|t} - \mathbb{C}_{t+k}\left(Y_{t+k|t} \right) \right) \right\}$$

$$\text{s.t. } Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k},$$

where Ψ_{t+k} is total costs of sales at price P_t^* in period $t+k$, which depends on labor costs and productivity, and where $Y_{t+k|t}$ is output in period $t+k$ for a firm that last reset its price in t. Furthermore, $\Lambda_{t,t+k}$ is the "stochastic discount factor", equal to the one facing consumers (who own the firms) between periods t and $t+k$:

$$\Lambda_{t,t+k} \equiv \beta^k \left\{ \frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} \right\} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right).$$

Instead of writing up the problem as a Lagrangean, we simply insert for $Y_{t+k|t}$ from the constraint, and obtain the expression on slide 17 in Lecture 2:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathrm{E}_t \left\{ \frac{\Lambda_{t,t+k}}{P_{t+k}} \left(P_t^* \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - \mathbb{C}_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right) \right\}.$$

The first-order condition for the optimal choice of P_t^* reads as:

$$\begin{aligned} 0 \quad = \quad & \sum_{k=0}^{\infty} \theta^k \cdot \\ & \mathrm{E}_t \left\{ \frac{\Lambda_{t,t+k}}{P_{t+k}} \left[\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} + P_t^* (-\varepsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon-1} \frac{C_{t+k}}{P_{t+k}} - \Psi_{t+k|t} (-\varepsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon-1} \frac{C_{t+k}}{P_{t+k}} \right] \right\}, \end{aligned}$$

with $\Psi_{t+k|t} \equiv \mathbb{C}'_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right)$ denoting the derivative of total costs w.r.t. output, i.e. (nominal) marginal costs.

Now let's use the expression for $Y_{t+k|t}$ from the demand function facing each firm, i.e. the constraint, to obtain:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \theta^k \mathrm{E}_t \left\{ \frac{\Lambda_{t,t+k}}{P_{t+k}} \left[Y_{t+k|t} - \varepsilon Y_{t+k|t} + \Psi_{t+k|t} \varepsilon \frac{Y_{t+k|t}}{P_t^*} \right] \right\} \Leftrightarrow && (1) \\ 0 &= \sum_{k=0}^{\infty} \theta^k \mathrm{E}_t \left\{ \Lambda_{t,t+k} \frac{Y_{t+k|t}}{P_{t+k}} \left[1 - \varepsilon + \varepsilon \frac{1}{P_t^*} \Psi_{t+k|t} \right] \right\} \Leftrightarrow \\ 0 &= \sum_{k=0}^{\infty} \theta^k \mathrm{E}_t \left\{ \Lambda_{t,t+k} \frac{Y_{t+k|t}}{P_{t+k}} \left[P_t^* - \mathcal{M} \Psi_{t+k|t} \right] \right\}, && (2) \end{aligned}$$

cf. slide 18, where the last line follows from the fact that the term in square brackets must be zero to satisfy the equality, so that $1-\varepsilon+\varepsilon\frac{1}{P_t^*}\Psi_{t+k|t}=0 \Leftrightarrow P_t^*=\Psi_{t+k|t}\frac{\varepsilon}{\varepsilon-1} \Leftrightarrow P_t^*=\Psi_{t+k|t}\mathcal{M}$, where $\mathcal{M}\equiv\frac{\varepsilon}{\varepsilon-1}$.

The first-order condition indicates that in each period, the firm would ideally like to set its price equal to the desired markup times its nominal marginal cost. The firm anticipates that it may be stuck with its current price for a number of periods into the future, and therefore sets its price today as a weighted average of expected future desired prices (each equal to expected future marginal costs times desired markups). The weights are determined by the probability of still being stuck with the current price each period in the future, the expected output level in each period in the future, and the stochastic discount factor of the representative household, who owns the firm.

Next, rewrite (2) as:

$$0=\sum_{k=0}^{\infty}\theta^k \mathrm{E}_t\left\{\Lambda_{t,t+k}\frac{Y_{t+k|t}}{P_{t+k}}\left[\frac{P_t^*}{P_{t-1}}-\mathcal{M}MC_{t+k|t}\frac{P_{t+k}}{P_{t-1}}\right]\right\}, \tag{3}$$

where $MC_{t+k|t}\equiv\frac{\Psi_{t+k|t}}{P_{t+k}}$ denotes real marginal costs. We now take a first-order Taylor expansion of this expression around a zero-inflation steady state, which is characterized by:

$$\frac{P_t^*}{P_{t-1}}=1,\quad \Pi_{t-1,t+k}=1,\quad Y_{t+k|t}=Y,\quad MC=\mathcal{M}^{-1},\quad \Lambda_{t,t+k}=\beta^k.$$

From this, we note that the term in square brackets in (3) equals zero in steady state. Hence, in our Taylor expansion, we can ignore the steady state terms, as well as the terms containing the derivatives of the variables outside the square brackets ($\Lambda_{t,t+k}$, $Y_{t+k|t}$, and P_{t+k}), since they will be zero (as they will be multiplied by the square bracket evaluated in steady state, which is zero). We then obtain:

$$\begin{aligned}
0 \quad \simeq \quad & 0+\sum_{k=0}^{\infty}\theta^k\beta^k\frac{Y}{P}\frac{1}{P}\left(P_t^*-P\right)-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{Y}{P}\frac{P^*}{P^2}\left(P_{t-1}-P\right)\\
& -\mathrm{E}_t\sum_{k=0}^{\infty}\theta^k\beta^k\frac{Y}{P}\mathcal{M}\frac{P}{P}\left(MC_{t+k|t}-MC\right)-\mathrm{E}_t\sum_{k=0}^{\infty}\theta^k\beta^k\frac{Y}{P}\mathcal{M}MC\frac{1}{P}\left(P_{t+k}-P\right)\\
& +\sum_{k=0}^{\infty}\theta^k\beta^k\frac{Y}{P}\mathcal{M}MC\frac{P}{P^2}\left(P_{t-1}-P\right),
\end{aligned}$$

which we can rewrite as:

$$\begin{aligned}
0 \quad &= \quad \frac{Y}{P}\mathrm{E}_t\sum_{k=0}^{\infty}(\theta\beta)^k\left(p_t^*-p_{t-1}-\widehat{mc}_{t+k|t}-p_{t+k}+p_{t-1}\right)\Leftrightarrow\\
0 \quad &= \quad \mathrm{E}_t\sum_{k=0}^{\infty}(\theta\beta)^k\left(p_t^*-\widehat{mc}_{t+k|t}-p_{t+k}\right),
\end{aligned}$$

where $p_t^* \equiv \frac{P_t^*-P}{P}$ and $\widehat{mc}_{t+k|t} \equiv \frac{MC_{t+k|t}-MC}{MC} = \left(MC_{t+k|t} - MC\right)\mathcal{M} = mc_{t+k|t} + \log\mathcal{M} = mc_{t+k|t} + \mu$, and where we have ignored $\frac{Y}{P}$ since it is never zero (so the rest needs to be zero). We further get:

$$
\begin{aligned}
\sum_{k=0}^{\infty} (\beta\theta)^k p_t^* &= \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\widehat{mc}_{t+k|t} + p_{t+k}\right\} \Leftrightarrow \\
\frac{p_t^*}{1-\beta\theta} &= \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\widehat{mc}_{t+k|t} + p_{t+k}\right\} \Leftrightarrow \\
p_t^* &= (1-\beta\theta)\,\mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{mc_{t+k|t} + \mu + p_{t+k}\right\} \Leftrightarrow \\
p_t^* &= \mu + (1-\beta\theta)\,\mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\psi_{t+k|t}\right\}, \qquad (4)
\end{aligned}
$$

where $\psi_{t+k|t} = mc_{t+k|t} + p_{t+k}$ is the log of the nominal marginal cost. This is eq. (11) in Galí (see also slide 20). As we discussed in class, this states that the optimal price depends on current and expected future real marginal costs as well as the expected future aggregate price level.

We have derived the following relationship (eq. (15) in Galí) between marginal costs (general) in period $t+k$ and marginal costs in period $t+k$ *for a firm that last changed its price in period* t as:

$$
mc_{t+k|t} = mc_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right).
$$

Now subtract the steady state value of mc_t, denoted by $-\mu$, to get:

$$
\widehat{mc}_{t+k|t} = \widehat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right),
$$

which we can insert in (a re-written version of) equation (4) to get:

$$
\begin{aligned}
p_t^* &= \mu + (1-\beta\theta)\,\mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\psi_{t+k|t}\right\} \Leftrightarrow \\
p_t^* &= (1-\beta\theta)\,\mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\widehat{mc}_{t+k|t} + p_{t+k}\right\} \Leftrightarrow \\
p_t^* &= (1-\beta\theta)\,\mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \left\{\widehat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right) + p_{t+k}\right\}.
\end{aligned}
$$

This is the expression presented on slide 26. The "tedious algebra" now follows, as we can rewrite the above expression as:

$$\begin{aligned}
p_t^* &= -\frac{\alpha\varepsilon}{1-\alpha}\frac{1-\beta\theta}{1-\beta\theta}p_t^* + (1-\beta\theta)\,\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\widehat{mc}_{t+k} + \frac{1-\alpha+\alpha\varepsilon}{1-\alpha}p_{t+k}\right\} \Leftrightarrow \\
\frac{1-\alpha+\alpha\varepsilon}{1-\alpha}p_t^* &= (1-\beta\theta)\,\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\widehat{mc}_{t+k} + \frac{1-\alpha+\alpha\varepsilon}{1-\alpha}p_{t+k}\right\} \Leftrightarrow \\
p_t^* &= (1-\beta\theta)\,\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}\widehat{mc}_{t+k} + p_{t+k}\right\}.
\end{aligned}$$

Now define $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$, and subtract again p_{t-1} on both sides:

$$\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\,\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\Theta\widehat{mc}_{t+k} + p_{t+k} - p_{t-1}\right\} \Leftrightarrow \\
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} + (1-\beta\theta)\,\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{p_{t+k} - p_{t-1}\right\} \Leftrightarrow \\
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} \\
&\quad + (1-\beta\theta)\,\mathrm{E}_t\left[\left\{p_t - p_{t-1}\right\} + \beta\theta\left\{p_{t+1} - p_{t-1}\right\} + (\beta\theta)^2\left\{p_{t+2} - p_{t-1}\right\} + ...\right],
\end{aligned}$$

which we can rewrite as:

$$\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} + (1-\beta\theta)\left\{p_t - p_{t-1}\right\} \\
&\quad + (1-\beta\theta)\,\mathrm{E}_t\left[\beta\theta\left\{p_{t+1} - p_t + p_t - p_{t-1}\right\} + (\beta\theta)^2\left\{p_{t+2} - p_{t+1} + p_{t+1} - p_t + p_t - p_{t-1}\right\} + ...\right]
\end{aligned}$$

$$\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} \\
&\quad + (1-\beta\theta)\,\mathrm{E}_t\left[\pi_t + \beta\theta\left\{\pi_{t+1} + \pi_t\right\} + (\beta\theta)^2\left\{\pi_{t+2} + \pi_{t+1} + \pi_t\right\} + ...\right]
\end{aligned}$$

$$\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} \\
&\quad +\mathrm{E}_t\left[\pi_t + \beta\theta\left\{\pi_{t+1} + \pi_t\right\} + (\beta\theta)^2\left\{\pi_{t+2} + \pi_{t+1} + \pi_t\right\} + ...\right] \\
&\quad -\beta\theta\mathrm{E}_t\left[\pi_t + \beta\theta\left\{\pi_{t+1} + \pi_t\right\} + (\beta\theta)^2\left\{\pi_{t+2} + \pi_{t+1} + \pi_t\right\} + ...\right]
\end{aligned}$$

$$\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta\mathrm{E}_t\sum_{k=0}^{\infty}(\beta\theta)^k\,\widehat{mc}_{t+k} \\
&\quad +\mathrm{E}_t\left[\pi_t + \beta\theta\pi_{t+1} + (\beta\theta)^2\pi_{t+2} + ...\right]
\end{aligned}$$

$$p_t^* - p_{t-1} = (1-\beta\theta)\,\Theta \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \,\widehat{mc}_{t+k} + \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \,\pi_{t+k}. \tag{5}$$

Finally, take out the $k=0$ terms, and rewrite as:

$$\begin{aligned} p_t^* - p_{t-1} &= (1-\beta\theta)\,\Theta \mathrm{E}_t \sum_{k=1}^{\infty} (\beta\theta)^k \,\widehat{mc}_{t+k} + \mathrm{E}_t \sum_{k=1}^{\infty} (\beta\theta)^k \,\pi_{t+k} \\ &\quad + (1-\beta\theta)\,\Theta\widehat{mc}_t + \pi_t \end{aligned}$$

$$p_t^* - p_{t-1} = \beta\theta \underbrace{\left[(1-\beta\theta)\,\Theta \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \,\widehat{mc}_{t+k+1} + \mathrm{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \,\pi_{t+k+1}\right]}_{=\mathrm{E}_t\left(p_{t+1}^* - p_t\right),\ \textit{according to } (5)} + (1-\beta\theta)\,\Theta\widehat{mc}_t + \pi_t$$

$$p_t^* - p_{t-1} = \beta\theta \mathrm{E}_t \left(p_{t+1}^* - p_t\right) + (1-\beta\theta)\,\Theta\widehat{mc}_t + \pi_t.$$

In this way, we can conveniently write out the sums and rewrite the equation as a first order difference equation in $(p_t^* - p_{t-1})$. Now we simply apply the definition of inflation from eq. (8) p. 55 in Galí; $\pi_t = (1-\theta)\,(p_t^* - p_{t-1})$, and obtain:

$$\begin{aligned} \frac{\pi_t}{(1-\theta)} &= \beta\theta \mathrm{E}_t \frac{\pi_{t+1}}{(1-\theta)} + (1-\beta\theta)\,\Theta\widehat{mc}_t + \pi_t \Leftrightarrow \\ \pi_t \frac{\theta}{(1-\theta)} &= \beta\theta \mathrm{E}_t \frac{\pi_{t+1}}{(1-\theta)} + (1-\beta\theta)\,\Theta\widehat{mc}_t \Leftrightarrow \\ \pi_t &= \beta \mathrm{E}_t \pi_{t+1} + \frac{(1-\beta\theta)\,(1-\theta)}{\theta}\Theta\widehat{mc}_t \Leftrightarrow \\ \pi_t &= \beta \mathrm{E}_t \pi_{t+1} + \lambda\widehat{mc}_t, \quad \lambda \equiv \frac{(1-\beta\theta)\,(1-\theta)}{\theta}\Theta. \end{aligned} \tag{6}$$

This is eq. (17), p. 60, as $\widehat{mc}_t = -\widehat{\mu}_t$ (cf. slide 21). As noted in the book, the parameter λ is decreasing in the Calvo parameter θ. Recall that θ is the fraction of firms *not* allowed to change their price each period. If this fraction is high, it means that only few firms are able to adjust their price in response to an economic shock that affects marginal costs, for example a technology shock. In that case, the shock has only a minor impact on the rate of inflation, as most firms simply keep their prices fixed. As the fraction of firms able to adjust their price increases, i.e., θ decreases, the effect on the inflation rate of a given change in marginal costs, i.e., λ, increases.

The final step is to combine (6) with the link between marginal costs and the output gap to obtain an expression that contains only inflation and the output gap. In the lecture, we saw that we could write marginal costs (in deviation from its steady state) as:

$$\widehat{mc}_t = mc_t - mc = \frac{\sigma\,(1-\alpha) + \varphi + \alpha}{1-\alpha}\,(y_t - y_t^n)\,,$$

where we then defined $\widetilde{y}_t \equiv y_t - y_t^n$ as the output gap, so as to obtain

$$\widehat{mc}_t = \frac{\sigma(1-\alpha) + \varphi + \alpha}{1-\alpha} \widetilde{y}_t.$$

Inserting this in (6) yields:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \lambda \frac{\sigma(1-\alpha) + \varphi + \alpha}{1-\alpha} \widetilde{y}_t \Leftrightarrow$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa \widetilde{y}_t, \quad \kappa \equiv \lambda \left(\sigma + \frac{\varphi + \alpha}{1-\alpha} \right). \tag{7}$$

We then finally arrived at the New-Keynesian Phillips Curve, i.e. eq. (22) in the book (see also slide 34).