Fourier Series Examples

Introduction | Derivation | Examples | Aperiodicity | Printable

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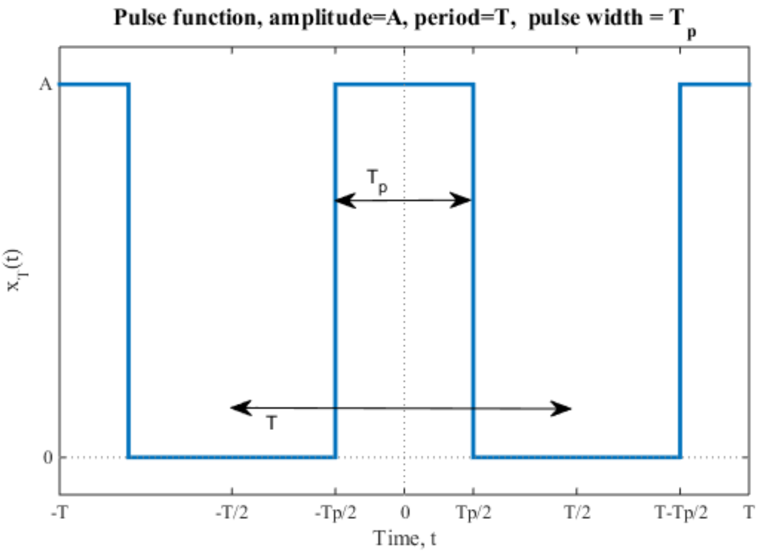
This document derives the Fourier Series coefficients for several functions. The functions shown here are fairly simple, but the concepts extend to more complex functions.

Even Pulse Function (Cosine Series)

Consider the periodic pulse function shown below. It is an even function with period T. The function is a pulse function with amplitude A, and pulse width T_p. The function can be defined over one period (centered around the origin) as:

$$x_T(t) = \begin{cases} A, & |t| \le \frac{T_p}{2} \\ 0, & |t| > \frac{T_p}{2} \end{cases}, \qquad -\frac{T}{2} < t \le \frac{T}{2}$$



Aside: the periodic pulse function

The periodic pulse function can be represented in functional form as $\Pi_T(t/T_p)$. During one period (centered around the origin)

$$\Pi_T(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}, \qquad -\frac{T}{2} < t \le \frac{T}{2}$$

$\Pi_T(t)$ represents a periodic function with period T and pulse width ½. The pulse is scaled in time by T_p in the function $\Pi_T(t/T_p)$ so:

$$\Pi_T\left(\frac{t}{T_p}\right) = \begin{cases} 1, & |t| \le \frac{T_p}{2} \\ 0, & |t| > \frac{T_p}{2} \end{cases}, \qquad -\frac{T}{2} < t \le \frac{T}{2}$$

This can be a bit hard to understand at first, but consider the sine function. The function *sin(x/2)* twice as slow as *sin(x)* (i.e., each oscillation is twice as wide). In the same way $\Pi_T(t/2)$ is twice as wide (i.e., slow) as $\Pi_T(t)$.

The Fourier Series representation is

$$x_T(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

Since the function is even there are only a_n terms.

$$x_T(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t)$$

The average is easily found,

$$a_0 = A\frac{T_p}{T}$$

The other terms follow from

$$a_n = \frac{2}{T}\int_T x_T(t)\cos(n\omega_0 t)dt, \quad n \ne 0$$

Any interval of one period is allowed but the interval from $-T/2$ to $T/2$ is straightforward in this case.

$$a_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt$$

Since $x_T(t)=A$ between $-T_p/2$ to $+T_p/2$ and zero elsewhere the integral simplifies and can be solved

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} A\cos(n\omega_0 t)dt \\ &= \frac{2}{T}\frac{A}{n\omega_0}\sin(n\omega_0 t)\Big|_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} \\ &= \frac{2}{T}\frac{A}{n\omega_0}\left(\sin\left(+n\omega_0\frac{T_p}{2}\right) - \sin\left(-n\omega_0\frac{T_p}{2}\right)\right) \end{aligned}$$

Since *sine* is an odd function, *sin(a)-sin(-a)=2sin(a)*, and using the fact that $\omega_0=2\pi/T$ and

$$a_n = \frac{4}{T}\frac{A}{n\omega_0}\sin\left(n\omega_0\frac{t_p}{2}\right) = 4\frac{A}{n2\pi}\sin\left(n\pi\frac{t_p}{T}\right) = 2\frac{A}{n\pi}\sin\left(n\pi\frac{t_p}{T}\right)$$

This result is further explored in two examples.

Example 1: Special case, Duty Cycle = 50%

Consider the case when the duty cycle is 50% (this means that the function is high 50% of the time, or $T_p=T/2$), $A=1$, and $T=2$. In this case $a_0=average=0.5$ and for $n \ne 0$:

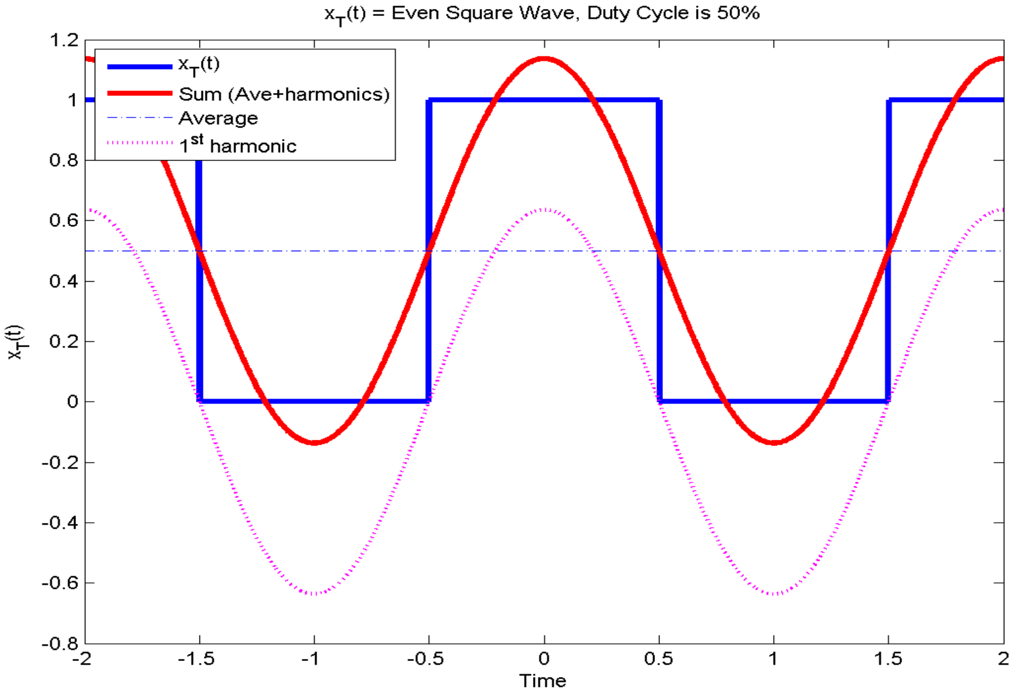
$$a_n = 2\frac{A}{n\pi}\sin\left(n\pi\frac{t_p}{T}\right) = 2\frac{A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ldots \quad \sin\left(\frac{n\pi}{2}\right) = 0,1,0,-1,0,1,\ldots = \begin{cases} -1^{\frac{n-1}{2}}, & n\ odd \\ 0, & n\ even \end{cases}$$

$$a_n = \begin{cases} 2\frac{A}{n\pi}\left(-1^{\frac{n-1}{2}}\right), & n\ odd \\ 0, & n\ even,\ n \neq 0 \end{cases}$$

The values for a_n are given in the table below. Note: this example was used on the page introducing the Fourier Series. Note also, that in this case a_n (except for *n=0*) is zero for even *n*, and decreases as *1/n* as *n* increases.

n	a_n
0	0.5
1	0.6366
2	0
3	-0.2122
4	0
5	0.1273
6	0
7	-0.0909



Average + 1st harmonic ○ up to 3rd harmonic ○ ...5th harmonic ○ ...7th ○ ...21st

The graph shows the function $x_T(t)$ (blue) and the partial Fourier Sum (from *n=0* to *n=N*) (red)

$$\sum_{n=0}^{N} a_n \cos(\omega_0 t)$$

as well as the highest frequency harmonic, $a_N cos(N\omega_0 t)$ (dotted magenta). Lower frequency harmonics in the summation are thin dotted blue lines (but harmonics with a_n0 are not shown. You can change *n* by clicking the buttons. As before, note:

- As you add sine waves of increasingly higher frequency, the approximation improves.
- The addition of higher frequencies better approximates the rapid changes, or details, (i.e., the discontinuity) of the original function (in this case, the square wave).
- Gibb's overshoot exists on either side of the discontinuity.
- Because of the symmetry of the waveform, only odd harmonics (1, 3, 5, ...) are needed to approximate the function. The reasons for this are discussed below
- The rightmost button shows the sum of all harmonics up to the 21st harmonic, but not all of the individual sinusoids are explicitly shown on the plot. In particular harmonics between 7 and 21 are not shown.

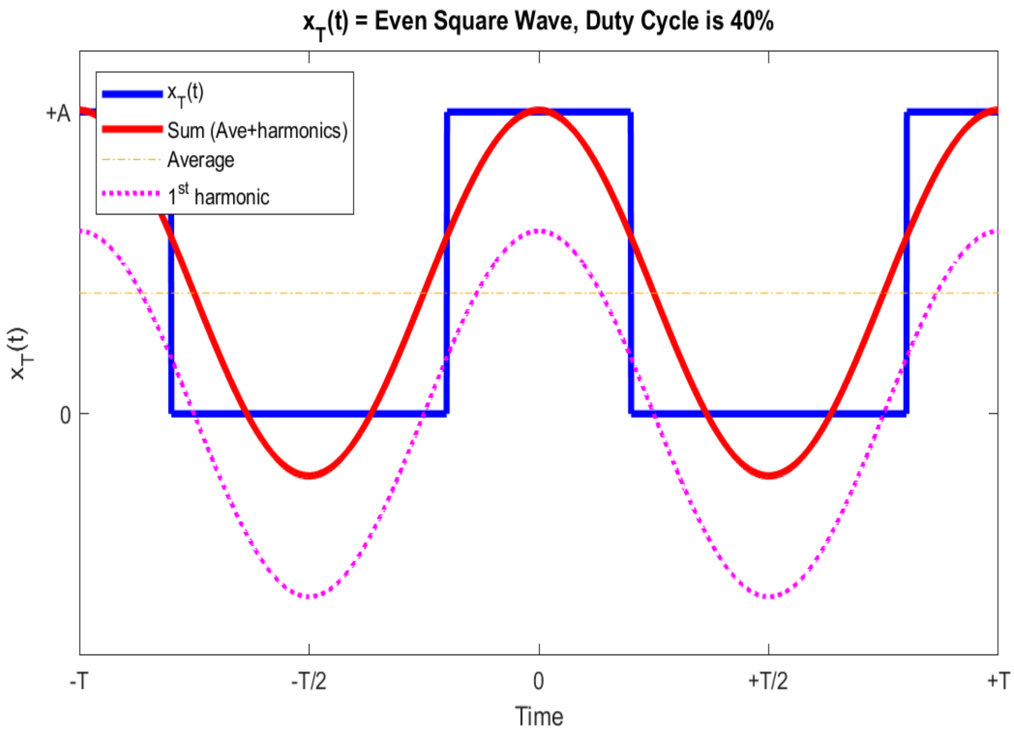
Example 2: Special case, Duty Cycle = 40%

Now consider the case when the duty cycle is 40%, *A=1*, and *T=2*. In this case a_0=*average=0.4* and for *n≠0*:

$$a_n = \frac{4}{T}\frac{A}{n\omega_0}\sin\left(n\omega_0\frac{t_p}{2}\right) = 2\frac{A}{n\pi}\sin(0.4n\pi)$$

The values for a_n are given in the table below (note: this example was used on the previous page).

n	a_n
0	0.4
1	0.6055
2	0.1871
3	-0.1247
4	-0.1514
5	-0.0000
6	0.1009
7	0.0535



◉ Average + 1st harmonic ○ up to 2nd harmonic ○ ...3rd harmonic ○ ...4th ○ ...21st

The graph shows the function $x_T(t)$ (blue) and the partial Fourier Sum (from *n=0* to *n=N*) (red)

$$\sum_{n=0}^{N} a_n \cos(\omega_0 t)$$

Note that because this example is similar to the previous one, the coefficients are similar, but they are no longer equal to zero for n even.

Even Square Wave (Exploiting Symmetry)

In problems with even and odd functions, we can exploit the inherent symmetry to simplify the integral. We will exploit other symmetries later. Consider the problem above. We have an expression for a_n, *n≠0*

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt$$

If $x_T(t)$ is even, then the product $x_T(t)\cdot\cos(n\cdot\omega_0 t)$ is even (the product of two even functions is even). We can then use the fact that for an even function, *e(t)*,

$$\int_{-a}^{+a} e(t)dt = 2\int_{0}^{a} e(t)dt$$

so

$$a_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} x_T\,(t)\cos(n\omega_0 t)dt$$

which generates the same answer as before. This will often be simpler to evaluate than the original integral because one of the limits of integration is zero.

Even Square Wave (Exponential Series)

Consider, again, the pulse function. We can also represent $x_T(t)$ by the Exponential Fourier Series

$$x_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

We find the c_n

$$c_n = \int_{T} x_T(t)e^{-jn\omega_0 t}dt$$

As before the integral is from $-T/2$ to $+T/2$ and make use of the facts that the function is constant for $|t|<T_p/2$ and zero elsewhere, and the $T\cdot\omega_0=2*\cdot\pi$.

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} Ae^{-jn\omega_0 t}dt$$

$$= \frac{A}{-Tjn\omega_0}e^{-jn\omega_0 t}\Big|_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} = \frac{A}{-j2\pi n}\left(e^{-jn\omega_0\frac{T_p}{2}} - e^{+jn\omega_0\frac{T_p}{2}}\right)$$

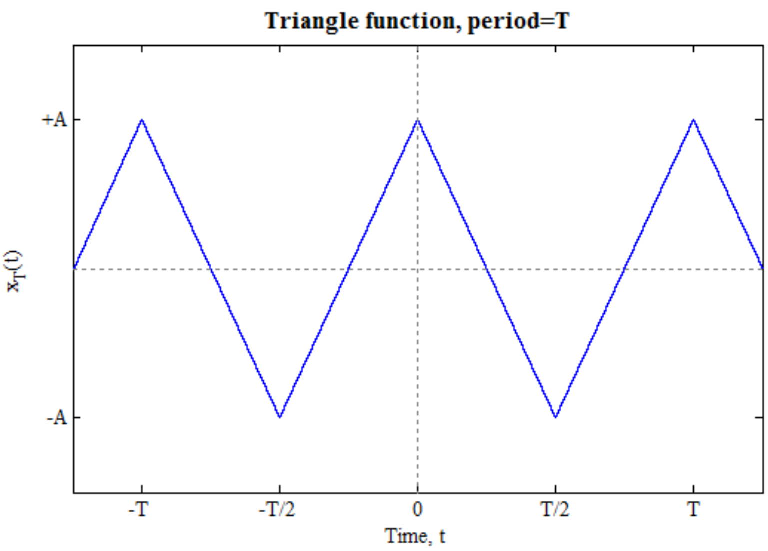
Euler's identities dictate that $e^{+j\theta}-e^{-j\theta}=2j\sin(\theta)$ so $e^{-j\theta}-e^{+j\theta}=-2j\sin(\theta)$.
and

$$c_n = \frac{-2jA}{-j2\pi n}\sin\left(n\omega_0\frac{T_p}{2}\right) = \frac{A}{\pi n}\sin\left(n\omega_0\frac{T_p}{2}\right)$$

Note that, as expected, $c_0=a_0$ and $c_n=a_n/2$, $(n\neq 0)$ (since this is an even function $b_n=0$).

Even Triangle Wave (Cosine Series)

Consider the triangle wave



The average value (i.e., the 0^{th} Fourier Series Coefficients) is $a_0=0$. For $n>0$ other coefficients the even symmetry of the function is exploited to give

$$a_n = \frac{2}{T}\int_T x_T(t)\cos(n\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt = \frac{4}{T}\int_0^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt$$

Between $t=0$ and $t=T/2$ the function is defined by $x_T(t)=A-4At/T$ so

$$a_n = \frac{4}{T}\int_0^{+\frac{T}{2}}\left(A-\frac{4A}{T}t\right)\cos(n\omega_0 t)dt = \frac{4A}{T}\left(\int_0^{+\frac{T}{2}}\cos(n\omega_0 t)dt - \frac{4}{T}\int_0^{+\frac{T}{2}} t\cos(n\omega_0 t)dt\right)$$

Perform the integrations (either by hand using integration by parts, or with a table of integrals, or by computer) and use the fact that $\omega_0 \cdot T=2\cdot\pi$

$$a_n = \frac{4A}{T}\left(\frac{T\sin(\pi n)}{2\pi n} + \frac{4}{T}\frac{T^2\left(2\sin\left(\frac{\pi n}{2}\right)^2 - \pi n\sin(\pi n)\right)}{4\pi^2 n^2}\right)$$

Since $\sin(\pi\cdot n)=0$ this simplifies to

$$a_n = \frac{4A}{T}\frac{4}{T}\frac{T^2 2\sin\left(\frac{\pi n}{2}\right)^2}{4\pi^2 n^2} = \frac{8A\sin\left(\frac{\pi n}{2}\right)^2}{\pi^2 n^2}$$

This answer is correct, but noting that

$$n = 0,1,2,3,4,5,6,7,\ldots \qquad \sin\left(\frac{\pi n}{2}\right)^2 = 0,1,0,1,0,1,0,\ldots = \frac{1-(-1)^n}{2}$$

yields an even simpler result

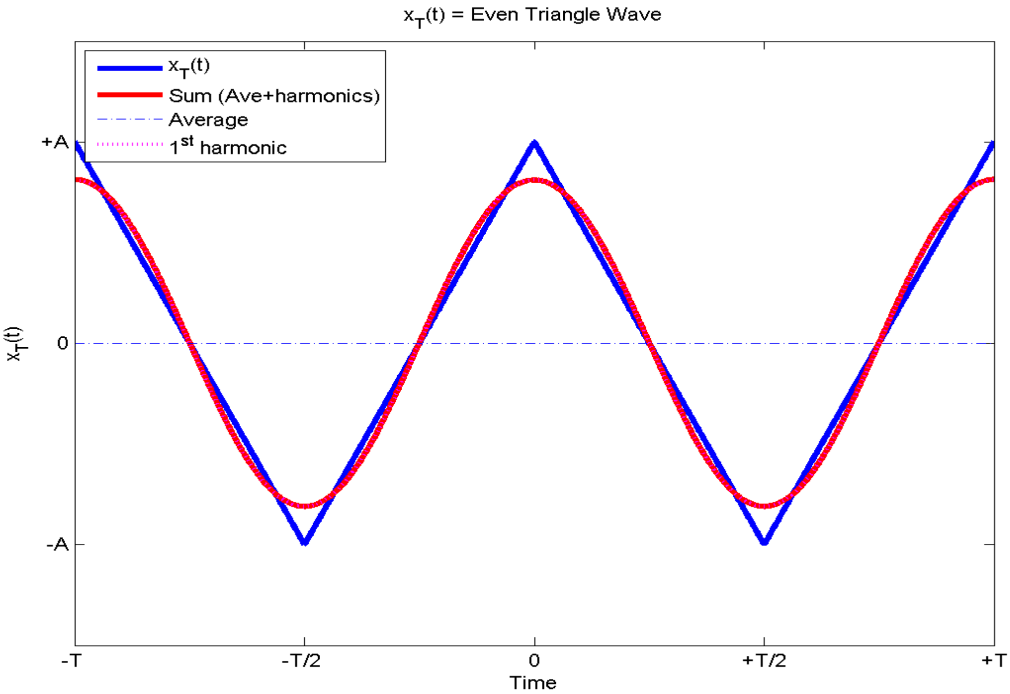
$$a_n = \begin{cases} 4A\frac{1-(-1)^n}{\pi^2 n^2}, & \text{nodd} \\ 0, & \text{neven} \end{cases}$$

Example 3: Triangle wave

If $x_T(t)$ is a triangle wave with $A=1$, the values for a_n are given in the table below (note: this example was used on the previous page).

n	a_n
0	0
1	0.8106
2	0
3	0.0901

4	0
5	0.0324
6	0
7	0.0165



Average + 1st harmonic   up to 3rd harmonic   ...5th harmonic   ...7th ...9th

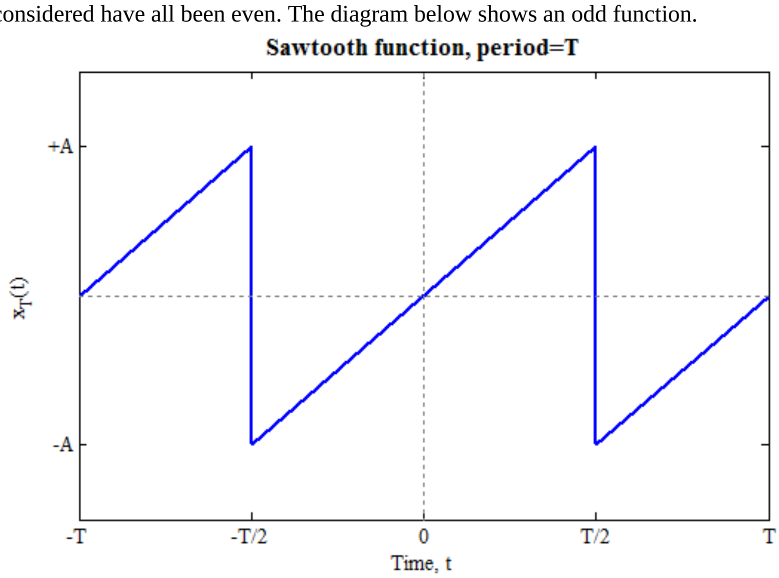
Note: this is similar, but not identical, to the triangle wave seen earlier.

Note:

- As you add sine waves of increasingly higher frequency, the approximation gets better and better, and these higher frequencies better approximate the details, (i.e., the change in slope) in the original function.
- The amplitudes of the harmonics for this example drop off much more rapidly (in this case they go as $1/n^2$ (which is faster than the $1/n$ decay seen in the pulse function Fourier Series (above)). Conceptually, this occurs because the triangle wave looks much more like the 1st harmonic, so the contributions of the higher harmonics are less. Even with only the 1st few harmonics we have a very good approximation to the original function.
- There is no discontinuity, so no Gibb's overshoot.
- As before, only odd harmonics (1, 3, 5, ...) are needed to approximate the function; this is because of the symmetry of the function.

Odd Function (Sawtooth Wave)

Thus far, the functions considered have all been even. The diagram below shows an odd function.



In this case, a Fourier Sine Series is appropriate

$$x_T(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad b_n = \frac{2}{T}\int_T x_T(t)\sin(n\omega_0 t)dt$$

It is easiest to integrate from *-T/2* to *+T/2*. Over this interval $x_T(t) = 2At/T$.

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \frac{2At}{T}\sin(n\omega_0 t)dt$$

Performing the integration (and using the fact that $\omega_0 \cdot T = 2\cdot\pi$) the integral yields

$$b_n = \frac{2}{T}\frac{AT(\sin(\pi n) - \pi n\cos(\pi n))}{\pi^2 n^2}$$

Using two simplification, $\sin(\pi \cdot n)=0$ and $\cos(\pi \cdot n)=(-1)^n$ gives

$$b_n = -\frac{2A}{\pi n}(-1)^n$$

Aside: using symmetry

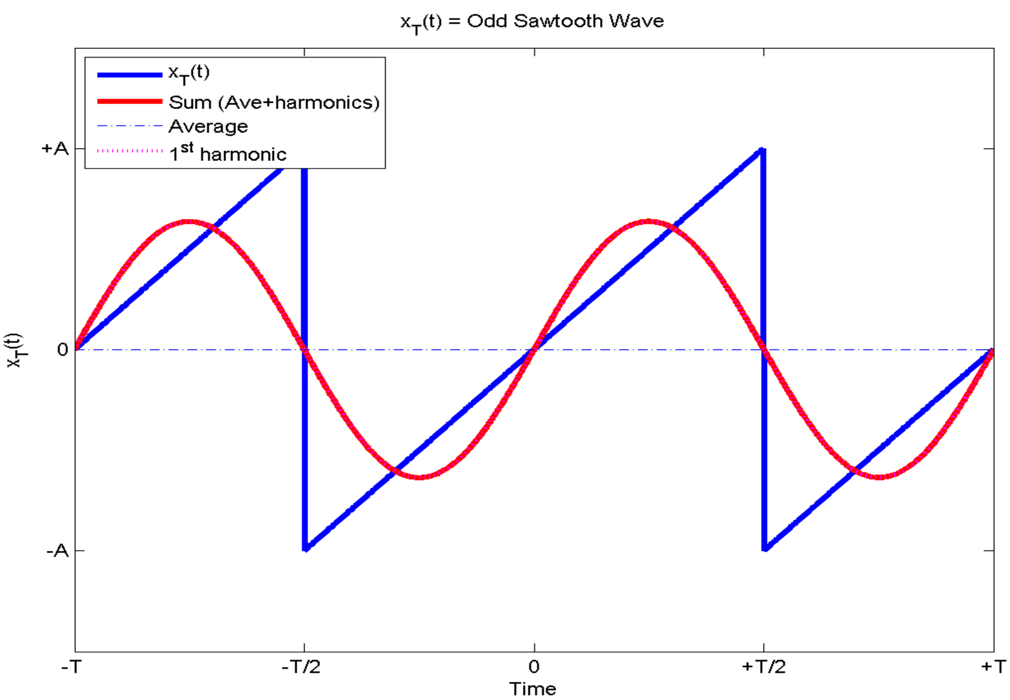
In this case since $x_T(t)$ is odd and is multiplied by another odd function ($\sin(n\cdot\omega_0 t)$), their product is even and the integral can be rewritten as:

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt = b_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt$$

Example 4: Odd Sawtooth Wave

If $x_T(t)$ is a sawtooth wave with $A=1$, the values for b_n are given in the table below

n	b_n
1	0.6366
2	-0.3183
3	0.2122
4	-0.1592
5	0.1273
6	-0.1061
7	0.0909



◉ Average + 1st harmonic ○ up to 2nd harmonic ○ ...3rd ○ ...4th ○ ...5th ○ ...20th

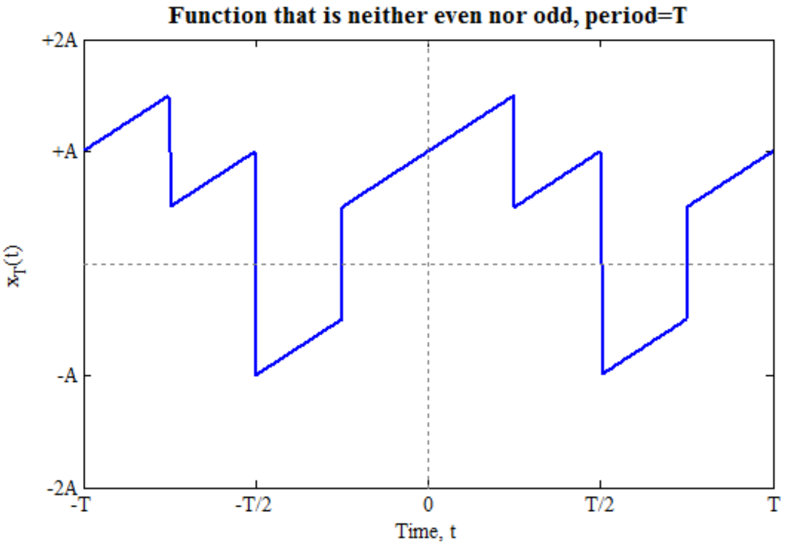
Note: this is similar, but not identical, to the sawtooth wave seen earlier.

Note:

- As you add sine waves of increasingly higher frequency, the approximation gets better and better, and these higher frequencies better approximate the details, (i.e., the change in slope) in the original function.
- Since this function doesn't look as much like a sinusoid as the triangle wave, the coefficients decrease less rapidly (as $1/n$ instead of $1/n^2$
- There is Gibb's overshoot caused by the discontinuity.

Functions that are neither even nor odd

So far, all of the functions considered have been either even or odd, but most functions are neither. This presents no conceptual difficult, but may require more integrations. For example if the function $x_T(t)$ looks like the one below



Since this has no obvious symmetries, a simple Sine or Cosine Series does not suffice. For the Trigonometric Fourier Series, this requires three integrals

$$x_T(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$a_0 = \frac{2}{T} \int_T x_T(t)\, dt$$

$$a_n = \frac{2}{T} \int_T x_T(t) \cos(n\omega_0 t) dt, \quad n \neq 0$$

$$b_n = \frac{2}{T} \int_T x_T(t) \sin(n\omega_0 t) dt$$

However, an exponential series requires only a single integral

$$x_T(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int_T x_T(t)\, e^{-jn\omega_0 t} dt$$

For this reason, among others, the Exponential Fourier Series is often easier to work with, though it lacks the straightforward visualization afforded by the Trigonometric Fourier Series.

Example 5: Neither Even nor Odd

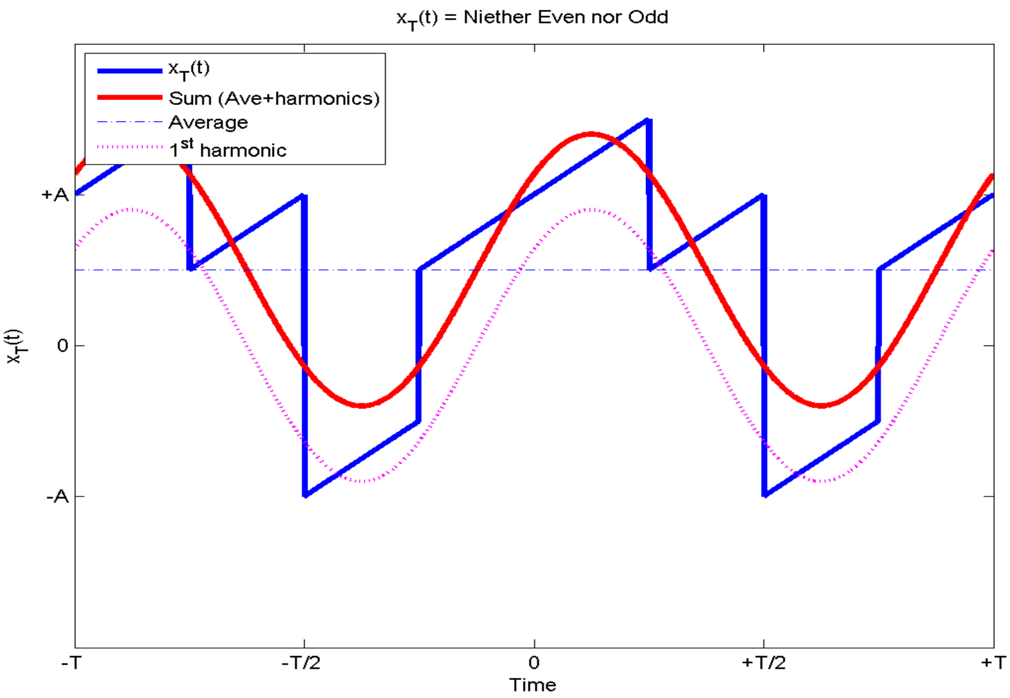
In this case, but not in general, we can easily find the Fourier Series coefficients by realizing that this function is just the sum of the square wave (with 50% duty cycle) and the sawtooth so

n	a_n	b_n
0	0.5	----
1	0.6366	0.6366
2	0	-0.3183
3	-0.2122	0.2122
4	0	-0.1592
5	0.1273	0.1273
6	0	-0.1061
7	-0.0909	0.0909

From the relationship between the Trigonometric and Exponential Fourier Series

$$c_0 = a_0 \; and \; c_n = \frac{a_n}{2} - j\frac{b_n}{2} for n \neq 0, \quad with \; c_{-n} = c_n^*$$

n	c_n
-7	-0.0455 + 0.0455j
-6	-0.0531j
-5	0.0637 + 0.0637j
-4	-0.0796j
-3	-0.1061 + 0.1061j
-2	-0.1592j
-1	0.3183 + 0.3183j
0	0.5
1	0.3183 - 0.3183j
2	0.1592j
3	-0.1061 - 0.1061j
4	0.0796j
5	0.0637 - 0.0637j
6	0.0531j
7	-0.0455 - 0.0455j



◉ Average + 1st harmonic ○ up to 2nd harmonic ○ ...3rd ○ ...4th ○ ...5th ○ ...20th

Note: this is similar, but not identical, to the sawtooth wave seen earlier.

Note:

- As you add sine waves of increasingly higher frequency, the approximation gets better and better, and these higher frequencies better approximate the details, (i.e., the change in slope) in the original function.
- There is Gibb's overshoot caused by the discontinuities.

Effect of Function Symmetry on Coefficients

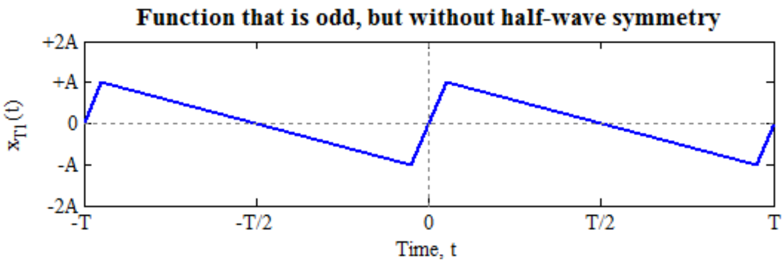
If the function $x_T(t)$ has certain symmetries, we can simplify the calculation of the coefficients.

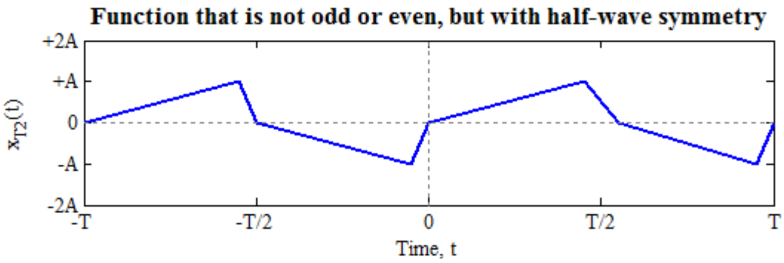
Symmetry Trigonometric Series and Symmetry

Symmetry	Simplification
$x_T(t)$ is even	$a_0 = average$ $a_n = \frac{4}{T}\int_0^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt, \quad n \neq 0$ $b_n = 0$
$x_T(t)$ is odd	$a_n = 0$ $b_n = \frac{4}{T}\int_0^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt$
$x_T(t)$ has *half-wave symmetry* A function can have half-wave symmetry without being either even or odd.	$a_n = b_n = 0, \quad n\ even$ $a_n = \frac{4}{T}\int_0^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt, \quad n\ odd$ $b_n = \frac{4}{T}\int_0^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt, \quad n\ odd$

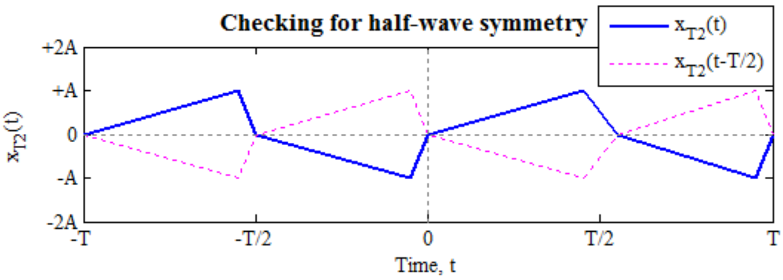
The first two symmetries are were discussed previously in the discussions of the pulse function ($x_T(t)$ is even) and the sawtooth wave ($x_T(t)$ is odd).

Half-wave symmetry is depicted the diagram below.

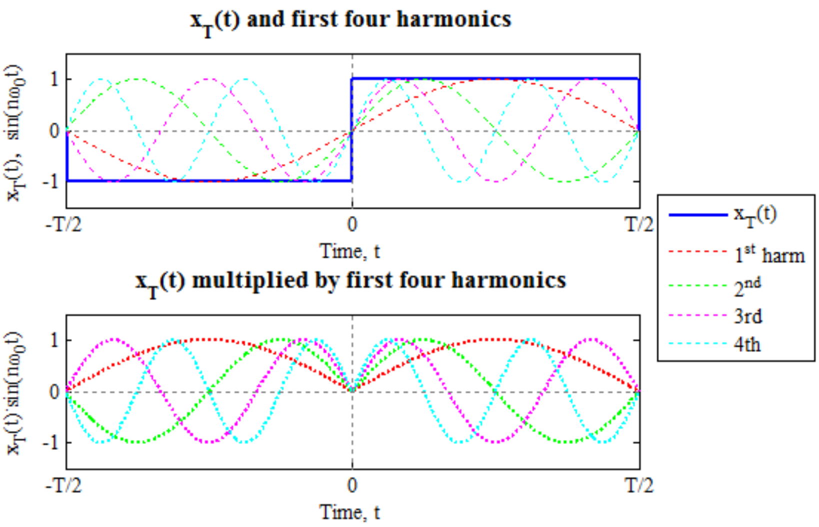




The top function, $x_{T1}(t)$, is odd ($x_{T1}(t)$=-$x_{T1}(-t)$), but does not have half-wave symmetry. The bottom function, $x_{T2}(t)$ is nether even nor odd, but since $x_{T2}(t)$=-$x_{T2}(t-T/2)$, it has halfwave symmetry. To visualize this imagine shifting the function by half a period ($T/2$); for half-wave symmetry the shifted function should be the mirror image of the original function (about the horizontal axis) as shown below



The reason the coefficients of the even harmonics are zero can be understood in the context of the diagram below. The top graph shows a function, $x_T(t)$ with half-wave symmetry along with the first four harmonics of the Fourier Series (only sines are needed because $x_T(t)$ is odd). The bottom graph shows the harmonics multiplied by $x_T(t)$.



Now imagine integrating the product terms from $-T/2$ to $+T/2$. The odd terms (from the 1st (red) and 3rd (magenta) harmonics) will have a positive result (because they are above zero more than they are below zero). The even terms (green and cyan) will integrate to zero (because they are equally above and below zero). Though this is a simple example, the concept applies for more complicated functions, and for higher harmonics.

The only function discussed with half-wave symmetry was the triangle wave and indeed the coefficients with even indices are equal to zero (as are all of the b_n terms because of the even symmetry). The square wave with 50% duty cycle would have half wave symmetry if it were centered around zero (i.e., centered on the horizontal axis). In that case the a_0 term would be zero and we have already shown that all the terms with even indices are zero, as expected.

Simplifications can also be made based on *quarter-wave symmetry*, but these are not discussed here.

Exponential Series and Symmetry

Since the coefficients c_n of the Exponential Fourier Series are related to the Trigonometric Series by

$$c_0 = a_0$$
$$c_n = \frac{a_n}{2} - j\frac{b_n}{2} for\ n \neq 0$$
$$c_{-n} = c_n^*$$

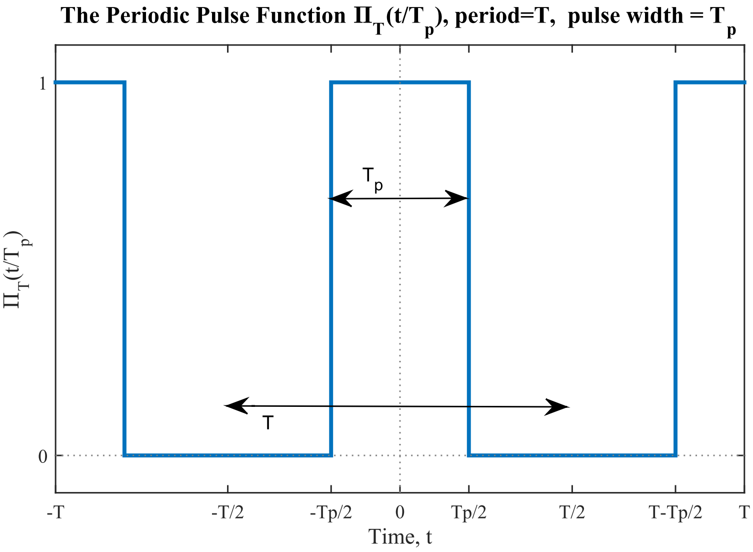
(assuming $x_T(t)$ is real) we can use the symmetry properties of the Trigonometric Series to find a_n and b_n and hence c_n.

However, in addition, the coefficients of c_n contain some symmetries of their own. In particular,

- The magnitude of the c_n terms are even with respect to n: $|c_{-n}|=|c_n|$.
- The angle of the c_n terms are odd with respect to n: $\angle c_{-n}=-\angle c_n$.
- The real part of c_n is even ($Re(c_{-n}) = Re(c_n)$) and the imaginary part is odd ($Im(c_{-n}) =-Im(c_n)$)
- If $x_T(t)$ is even, then $b_n=0$ and c_n is even and real.
- If $x_T(t)$ is odd, then $a_n=0$ and c_n is odd and imaginary.

Some Comments about the Pulse Function

Let's examine the Fourier Series representation of the periodic rectangular pulse function, $\Pi_T(t/T_p)$, more carefully.



Since the function is even, we expect the coefficients of the Exponential Fourier Series to be real and even(from symmetry properties). Furthermore, we have already calculated the coefficients of the Trigonometric Series, and could easily calculate those of the Exponential Series. However, let us do it from first principles. The Exponential Fourier Series coefficients are given by

$$\Pi_t\left(\frac{t}{T_p}\right)=\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$
$$with$$
$$c_n=\frac{1}{T}\int_T \Pi_t\left(\frac{t}{T_p}\right)e^{-jn\omega_0 t}dt$$

We can change the limits of integration to $-T_p/2$ and $+T_p/2$ (since the function is zero elsewhere) and proceed (the function is one in that interval, so we can drop it). We also make use of the fact the $\omega_0=2\pi/T$ and Euler's identity for sine.

$$\begin{aligned}
c_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \Pi_t\left(\frac{t}{T_p}\right)e^{-jn\omega_0 t}dt=\frac{1}{T}\int_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} e^{-jn\omega_0 t}dt \\
&= \frac{1}{-jn\omega_0 T}\left(e^{-jn\omega_0\frac{T_p}{2}}-e^{+jn\omega_0\frac{T_p}{2}}\right)=\frac{1}{-jn2\pi}\left(e^{-j\frac{n\pi T_p}{T}}-e^{+j\frac{n\pi T_p}{T}}\right) \qquad \omega_0=\frac{2\pi}{T} \\
&= \frac{1}{-jn2\pi}2j\sin\left(-\frac{n\pi T_p}{T}\right)=\frac{1}{n\pi}\sin\left(\frac{n\pi T_p}{T}\right) \\
&= \frac{T_p}{T}\frac{\sin\left(\frac{n\pi T_p}{T}\right)}{\frac{n\pi T_p}{T}}
\end{aligned}$$

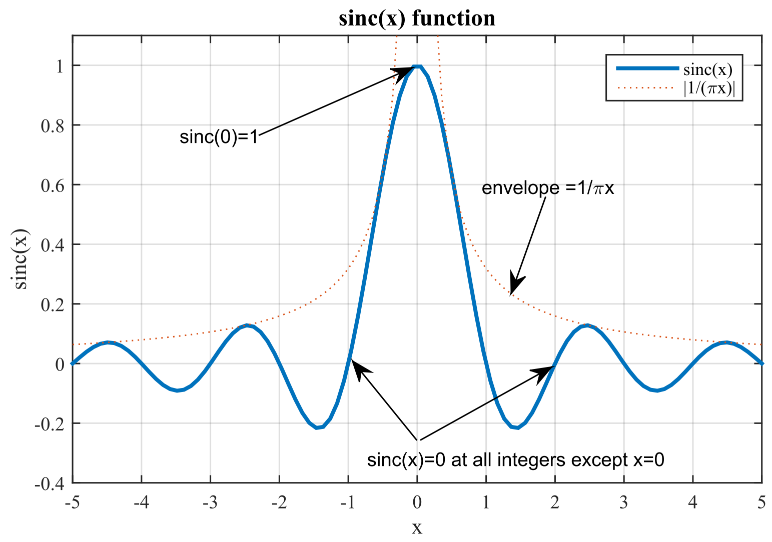
The last step in the derivation is performed so we can use the *sinc()* function (pronounced like "*sink*"). This function comes up often in Fourier Analysis.

$$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

With this definition the coefficients simplify to

$$c_n=\frac{T_p}{T}\mathrm{sinc}\left(\frac{nT_p}{T}\right)$$

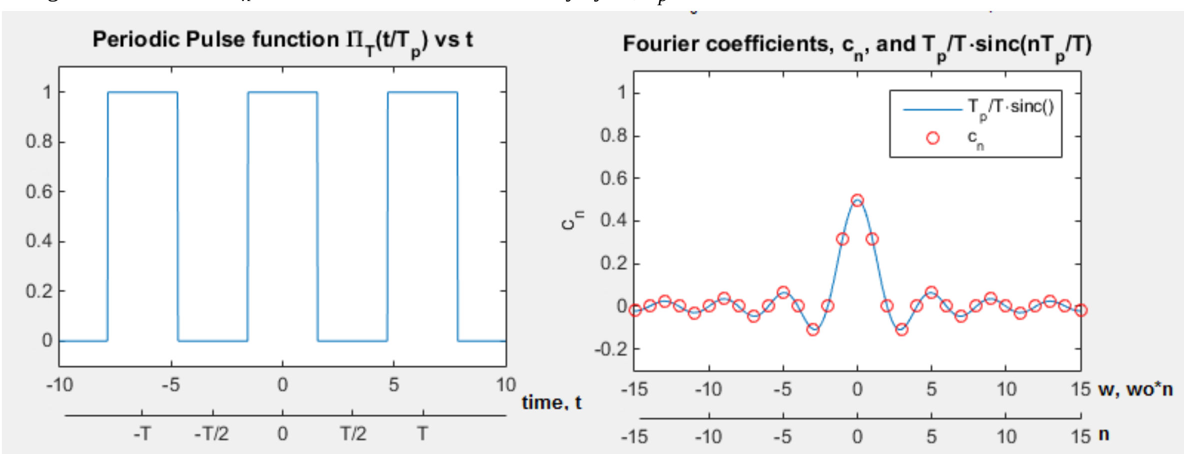
Aside: the "sinc()" function



The sinc function has several important features:

- *sinc(x)=0* for all integer values of *x* except at *x=0* where *sinc(0)=1*. This is because *sin(π·n)=0* for all integer values of n. However at *n=0* we have *sin(π·n)/(π·n)* which is zero divided by zero, but by L'Hôpital's rule get a value of 1.
- The first zeros away from the origin occur when *x=±1*.
- The function decays with an envelope of *1/(π·x)* as *x* moves from the origin. This is because the *sin()* function has successive maxima with an amplitude of 1, and the sin function is divided by *π·x*.

The diagram below shows c_n vs n for several values of the duty cycle, T_p/T.



○ Duty cycle = 0.1 ○ ...0.25 ◉ ...0.5 ○ ...0.75 ○ ...0.9

The graph on the left shows the time domain function. If you hit the middle button, you will see a square wave with a duty cycle of 0.5 (i.e., it is high 50% of the time). The period of the square wave is $T=2\cdot\pi$;. The graph on the right shown the values of c_n vs n as red circles vs n (the lower of the two horizontal axes; ignore the top axis for now). The blue line goes through the horizontal axis whenever the argument of the *sinc()* function, $n\cdot T_p/T$ is an integer (except when $n=0$.). In particular the first crossing of the horizontal axis is given by $n\cdot T_p/T=1$ or $n=T/T_p$ (note this is not an integer values of T_p). There are several important features to note as T_p is varied.

- As T_p decreases (along with the duty cycle, T_p/T), so does the value of c_0. This is to be expected because c_0 is just the average value of the function and this will decrease as the pulse width does.
- As T_p decreases, the "width" of the *sinc()* function broadens. This tells us that as the function becomes more localized in time (i.e., narrower) it becomes less localized in frequency (broader). In other words, if a function happens very rapidly in time, the signal must contain high frequency coefficients to enable the rapid change.
- Let us call the "width" of the *sinc()* function the width of the main lobe (i.e., between the first two zero crossings around $\omega=0$), $\Delta n=2\cdot T/T_p$. If we call the width of the pulse $\Delta t=t_p$ then

$$(\Delta n)\cdot(\Delta t)=\left(2\frac{T}{T_p}\right)\cdot(T_p)=2T=\text{constant}$$

This tells us explicitly that the product of width in frequency (i.e., Δn) multiplied by the width in time (Δt) is constant - if one is doubled, the other is halved. Or - as one gets more localized in time, it is less localized in frequency. We will discuss this more later.

References