Newton Day 2012:
achromatic refracting optics

by Peter H. Mao for Margaret and Owen

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Abstract

For Sir Isaac Newton's 370th birthday, we delve into the a topic that Newton believed was impossible: lens arrangements that do not produce rainbows. Perhaps it was due to this certainty that he produced the first working reflecting telescope – the forebearer of today's scientific telescopes, from the radio dishes that talk to Voyager to the the optical telescopes atop Mauna Kea and to NuSTAR, newly in orbit around the Earth this year.

This Newton Day, our aim is to understand the achromatic doublet that Newton missed out on. We will use prisms to help us understand the achromatic doublet and then we will disassemble the most commonly available, high quality achromatic refracting optics available – camera lenses – and learn what we can about them.

1 Introduction

> "And by this means might Telescopes be brought to sufficient perfection, were it not for the different Refrangibility of several sorts of Rays. But by reason of this different Refrangibility, I do not yet see any other means of improving Telescopes by Refractions alone, than that of increasing their lengths, for which end the late Contrivance of Hugenius seems well accommodated...
>
> Seeing therefore the Improvement of Telescopes of given lengths by Refractions is desperate; I contrived heretofore a Perspective by Reflexion, using instead of an Object-glass a concave Metal."
>
> *OPTICKS Book I, Part I, PROP. VII, Theor. VI*, p. 102.

Sometimes the great man gets it wrong. During Newton's time, refracting telescopes (those with transparent lenses) were the primary tool for astronomy. The main problem with refracting telescopes was chromatic aberration – the separation of colors in the image that affected the spatial resolution of the telescope (ie, the ability to discern fine details in images of other planets). In the above quote, Newton refers to the practice of limiting chromatic aberration in astronomical telescopes by using extremely long focal length optics (up to 64 m!). For comparison, the Subaru telescope's focal length is "only" 15 meters.

In his study of light, Newton proved that white light is composed of all colors and that the dispersion of light by a prism is the same mechanism which produces chromatic aberration in refracting telescopes. Newton solved the chromatic aberration problem in telescopes by producing the first working reflecting telescope (since the reflection angle from mirrors is independent of color). Newton's telescope was hampered by the poor reflectivity of the mirror, but in terms of resolution, it rivaled the behemoths of the day and fit easily on a table top.

Newton did present some ideas in *Opticks* that suggest the possibility for achromatic refracting optics. In the quote above, he is referring to an optic made of glass and water – materials with different indices of refraction. Newton, however, was under the impression that the dispersion by these different media was the same, and so one could not cancel out the chromatic aberration. Also, on page 187 of *Opticks* (see Figure 1, Newton diagrams a setup with two prisms and a lens where dispersed light is recombined into white light (a setup that is reproduced at the Huntington Library and Gardens in their Newton exhibit). The setup is indeed achromatic and refracting, but the presence of the lens in the middle makes it difficult to generalize into an imaging system.

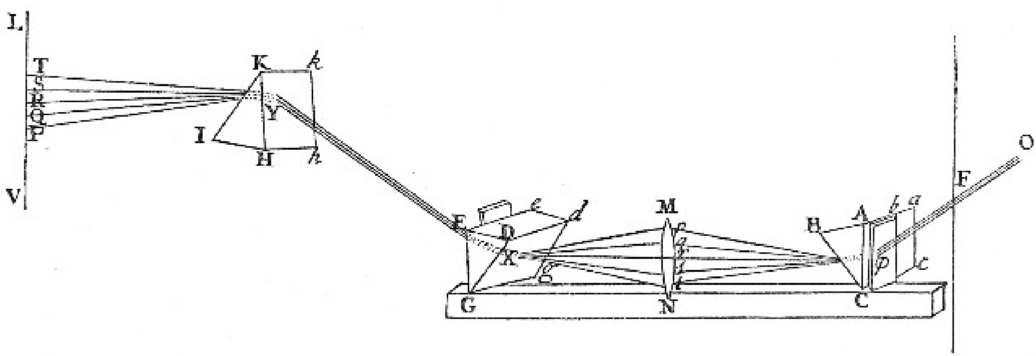


Figure 1: Newton's experimental setup for combining colored light into the white light. (*Opticks*, Part II, Fig. 16)

2 The Achromatic Doublet of Chester Moore Hall

Look up "achromatic doublet" and you will find a positive (convex) low-dispersion lens adjacent to a negative (concave or plano-concave) high-dispersion lens, as in Figure 2. This arrangement was invented by a Chester Moore Hall, a British lawyer with a strong interest in optics. Mr. Hall first proposed this design in 1729, two years after Newton passed away. The story of this invention and the troubles that it brought to the optical trade in London are well documented in Fred Watson's *Stargazer: the life and times of the telescope*.

To understand the achromatic doublet, I turned to Born and Wolf's *Principles of Optics*[4], which was almost (but not completely) impenetrable, and A. Walther's *The Ray and Wave Theory of Lenses*[3], which I highly recommend for his discussion of Gaussian optics. Unfortunately, I can't improve upon Walther's pedagogy, so I won't attempt to trace out the path of understanding from Snell's Law (first published by Ibn Sahl of Baghdad in 984) to Fermat's principle to Gaussian optics to Abbe's number and the achromatic doublet. Here, I'll give

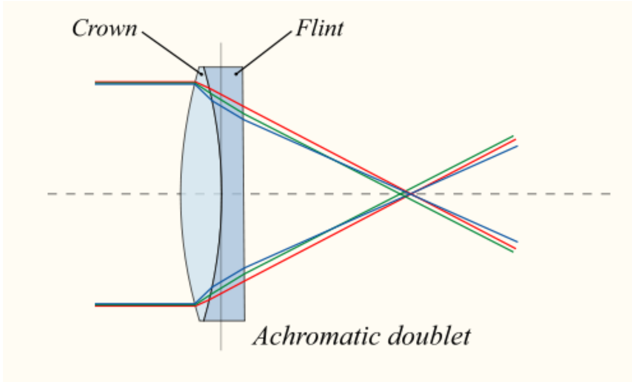


Figure 2: Achromatic doublet (aka achromat). (courtesy of Wikimedia Commons)

you the solution for a thin, achromatic doublet, and then we'll go into the prism example.

First, however, a few bits of terminology need to be defined:

1. The focal length is one of the major parameters of a lens. A convex lens (like a magnifying glass) has a positive focal length, and concave lens (like my glasses) has a negative focal length.

2. The "optical power" (P) of a lens is the inverse focal length. The unit for optical power is the diopter, which has units of m^{-1}.

3. The "dispersive power" (Δ) of a transparent medium (like glass or water or even air) is a measure of the change in its refractive index with color. This is a dimensionless quantity and is the inverse of the Abbe number[1]. This number gives you the fractional change in the optical power over the optical band.

If we want an achromatic doublet with optical power P, Gauss tells us that the power of a pair of thin lenses is simply their sum, $P_1 + P_2$. Skipping over all of Gaussian optics (as promised), to the penultimate step, we have the requirement that

$$\Delta_1 P_1 + \Delta_2 P_2 = 0, \tag{2.1}$$

so the optical powers of the convex and concave elements are

$$P_1 = \frac{\Delta_2}{\Delta_2 - \Delta_1} P \tag{2.2}$$

$$P_2 = \frac{-\Delta_1}{\Delta_2 - \Delta_1} P. \tag{2.3}$$

Going through Born and Wolf and Walther, I can believe these equations, but I felt a certain lack of physical insight, hence the prism example.

[1]The Abbe number for a medium is $\nu = \frac{n_d - 1}{n_F - n_C} = \frac{1}{\Delta}$. See Section 6 for more details

3 Understanding the achromatic doublet through prisms

A prism isn't so different from a lens in that both change the direction of travel of light that passes through them. You can also think of a lens as a collection of annular prisms. In fact, in a Fresnel lens *is* a concentric array of annular prisms. The main advantage of thinking about prisms is that their faces are flat, so the math is much easier. The big disadvantage is, of course, that they do not produce images with light.

3.1 Deflection of light by a prism

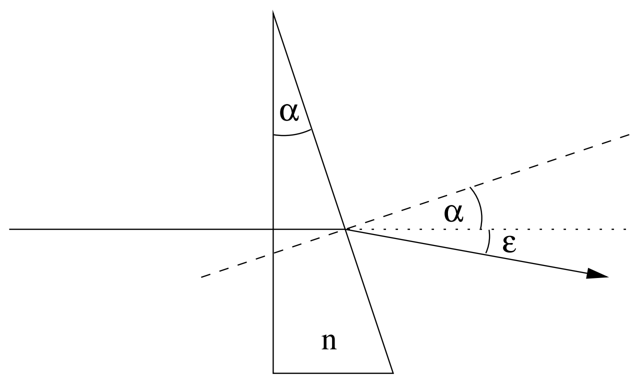


Figure 3: Beam deflection by a prism.

The first relationship that we have to establish is the angular deflection of a beam of light by a prism. In Figure 3, I have diagrammed the simplest possible setup. We have light entering from the left (as is always the case in optics) incident at right angles (*normal*) to the face of the prism. I chose this geometry so that I wouldn't have to muck up the equations with the angle of the beam inside the prism. The prism has an apex angle, $alpha$, and an index of refraction n. We'll use 1.0 for the refractive index of air (the medium outside the prism). The incidence angle of the beam with the right-hand side of the prism is α, so by Snell/Ibn Sahl's Law, we have

$$n \sin \alpha = 1.0 \sin(\alpha + \epsilon). \tag{3.1}$$

In the small angle approximation, where we use $\sin x \approx x$ (and we keep in mind that the fractional error in this approximation is $\sim x^2/6$) this simplifies to

$$n\alpha = \alpha + \epsilon. \tag{3.2}$$

The deflection angle of the light is, therefore,

$$\epsilon = (n - 1)\alpha. \tag{3.3}$$

This is directly analogous to the relationship in Gaussian optics between optical power and surface curvature.

Now, you may have think I cheated a bit by setting up the prism so that the incident light is normal to the first surface, but as long as the small angle approximation is valid (I'd say out to 0.1 or maybe even 0.2 radians)[2] this result holds even if we rotate the prism.

Actually, I did cheat a little bit. I left out some minus-signs to keep the above discussion cleaner. If I was careful, I'd define rays that point up (going from left to right) as positive and rays that point down as negative. If we keep this convention, then

$$\epsilon = (1 - n)\alpha. \tag{3.4}$$

Since > 1 for optical light, we have our result that ϵ is negative. Although this negative sign does not affect the next section, I thought it best to clear the air ahead of time.

3.2 Deflection of light by two prisms – the achromatic prism doublet

What happens if we have two prisms? I assume that you've either checked me on my claim that the small angle deflection through a prism (with a small apex angle!) is generally valid, or you've derived it yourself. With two prisms, the deflection is simply the sum of the individual deflections:

$$\epsilon_{\text{total}} = \epsilon_1 + \epsilon_2. \tag{3.5}$$

Keep in mind that for an upside-down prism (apex angle pointing down), the apex angle is negative ($\alpha < 0$).

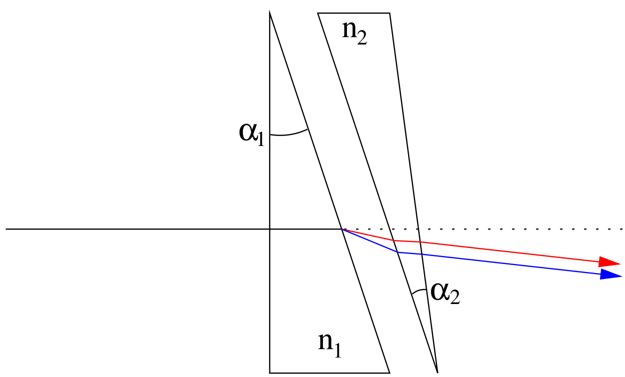


Figure 4: Achromatic prism doublet. In this two-prism system, $\alpha_1 > 0$ and $\alpha_2 < 0$. The refractive indices, n_1 and n_2 are dependent on the color of the light, so the blue and red rays are deflected to different angles. This setup is achromatic because the blue and red rays exit the system at the same angle.

You know, from playing with prisms, that we can make "rainbows" by putting them in sunlight. The "rainbow" that you see (chromatic angular dispersion) is the source of chromatic aberration in old-school refracting telescopes, like Galileo's. Can we make a pair of prisms that don't make rainbows? We need to come up with a geometry and a set of materials that will make the exit angles for different colors identical. Except for at the very

[2] 5 to 10 degrees, if you insist, but you really should be thinking in radians!

edges of the prism, it's ok for the rays to be displaced from each other, as long as they are parallel.

The angular dispersion from a single prism is due entirely to the differential refractive index across the visible spectrum:

$$d\epsilon = -dn\alpha. \tag{3.6}$$

The angular dispersion for the system of two prisms is

$$\begin{aligned} d\epsilon_{\text{total}} &= d\epsilon_1 + d\epsilon_2 \\ &= -dn_1\alpha_1 - dn_2\alpha_2. \end{aligned} \tag{3.7}$$

For the exit rays to be parallel, we want $d\epsilon_{\text{total}} = 0$. This leaves us with

$$dn_1\alpha_1 + dn_2\alpha_2 = 0 \tag{3.8}$$

(see how the negative signs didn't matter?). Let us further define the quantity $\alpha_{\text{total}} = \alpha_1 + \alpha_2$. Then we can write α_1 and α_2 in terms of α_{total}:

$$\alpha_1 = \frac{dn_2}{dn_2 - dn_1}\alpha_{\text{total}} \tag{3.9}$$

$$\alpha_1 = \frac{-dn_1}{dn_2 - dn_1}\alpha_{\text{total}}. \tag{3.10}$$

Do these equations remind you of the equations for the achromatic doublet? It's close, but not exactly the same. I'll leave it to you to derive the exact analogy, so you see where dispersive power comes in. Remember that the prism's deflection angle is directly analogous to optical power in a lens.

I prefer to leave these equation as-is, because the mechanical (apex angle) and optical (refractive index) properties of the prism are cleanly separated.

3.3 Numerical simulations of prisms

It's not too difficult to write a program that will give you the exit angle of a beam of light through an arbitrary arrangement of prisms. I'll restrict myself to two dimensions here, but it's not too difficult to extend this to three dimensions and allow the interfaces to take on arbitrary angles.[3] For now, though, lets only consider planar interfaces that are perpendicular to the page.

Figure 5 shows a ray coming in at an arbitrary angle θ_1 to an interface tilted at a different arbitrary angle (α_1). To the left of the first interface, the refractive index is n_1 and the the right, it is n_2. Snell/Ibn Sahl and a tiny bit of geometry tells us that the exit angle is related to the other physical quantities by

$$n_2 \sin(\theta_2 - \alpha_1) = n_1 \sin(\theta_1 - \alpha_1). \tag{3.11}$$

Note that this is the Snell's Law equation you know and love, but with the system rotated by α_1. We can generalize this to the i^{th} interface by replacing 1 with i and 2 with $i+1$. The

[3]The only tricky part would be to express Snell/Ibn Sahl's Law in quaternions, then the calculation proceeds in exactly the same way as outlined here.

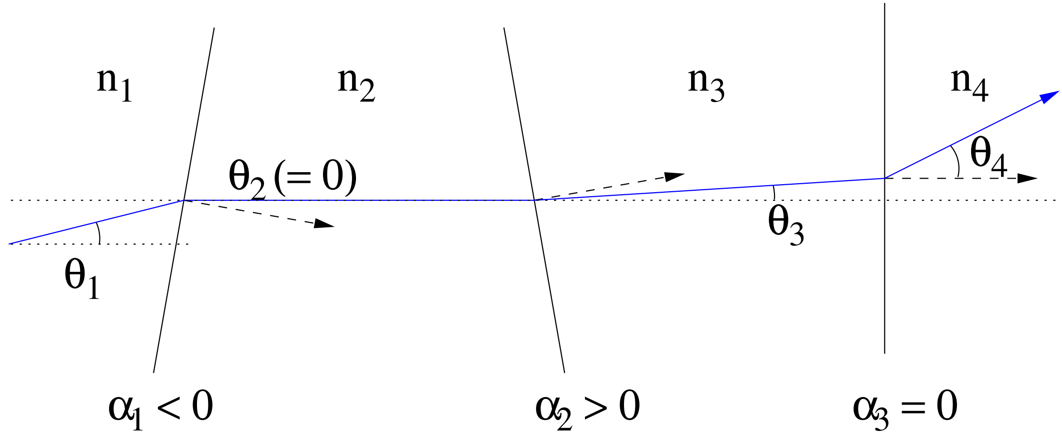


Figure 5: Deflection of rays through planar interfaces. At each interface, we use Snell/Ibn Sahl's Law to calculate the angle of the beam to the right of the interface. Positive angles have positive slope, and negative angles have negative slope. The angle of an interface is defined by the right-pointing normal to the surface, shown as a dashed-line vector. The horizontal dotted line is the reference axis (angle = 0).

angle following the i^{th} interface is exactly

$$\theta_{i+1} = \arcsin\left(\frac{n_i}{n_{i+1}}\sin(\theta_i - \alpha_i)\right) + \alpha_i \tag{3.12}$$

or, in the small angle approximation,

$$\theta_{i+1} = \frac{n_i}{n_{i+1}}(\theta_i - \alpha_i) + \alpha_i. \tag{3.13}$$

This is tedious to calculate by hand, but easy to code. Your inputs are

1. θ_1, the incident angle on the first surface
2. $\alpha_1, \alpha_2, \ldots, \alpha_m$, the angles of the m interfaces
3. $n_1, n_2, \ldots, n_m, n_{m+1}$, the index of refraction before each of the m interfaces and the index of refraction after the last interface,

and you `for` loop over all of the interfaces. In Figure 6, I use this code to (over) estimate the refractive index vs. color for the "45-45-90" prism that Maggie bought for me last year (for Newton Day, of course!).

Another application of this code is to design a polycarbonate prism that will cancel out the dispersion in Figure 8a. Acrylic has optical properties close to crown glass, the convex element of an achromatic doublet, and polycarbonate has optical properties close to that of flint glass, the concave element of the doublet. Using the refractive indices for acrylic and polycarbonate from RefractiveIndex.info (see Table 1), I made a polycarbonate prism with 0.3596 radians between the faces for Newton Day (see Figure 7). Figure 8b demonstrates the expected dispersion cancellation in the positive-angle beam and enhanced dispersion in the negative-angle beam.

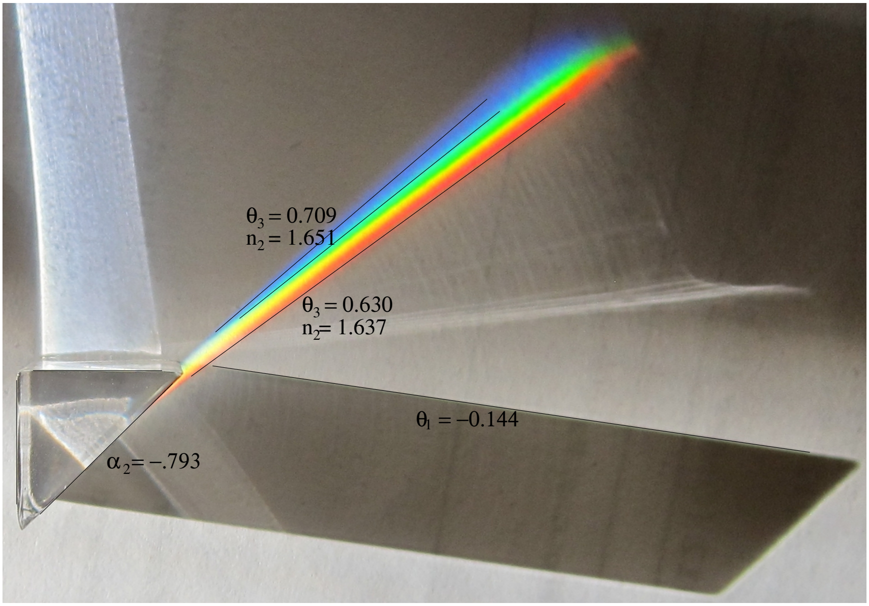


Figure 6: Dispersion by a 45-45-90 prism. The refractive indices were calculated with the code described in this section, but the values are overestimates because the sunlight is not parallel to the plane of the paper, so the exit angles are all rotated on the vertical axis. A full 3-D treatment of the problem should give values much closer to the published values for acrylic. Quiz: this photograph was taken on Dec 15 at 8:30 AM in Santa Monica. What is the height to hypotenuse ratio for this prism?

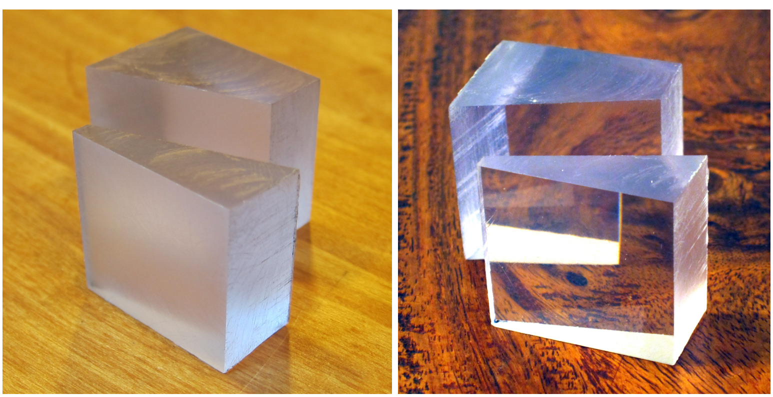
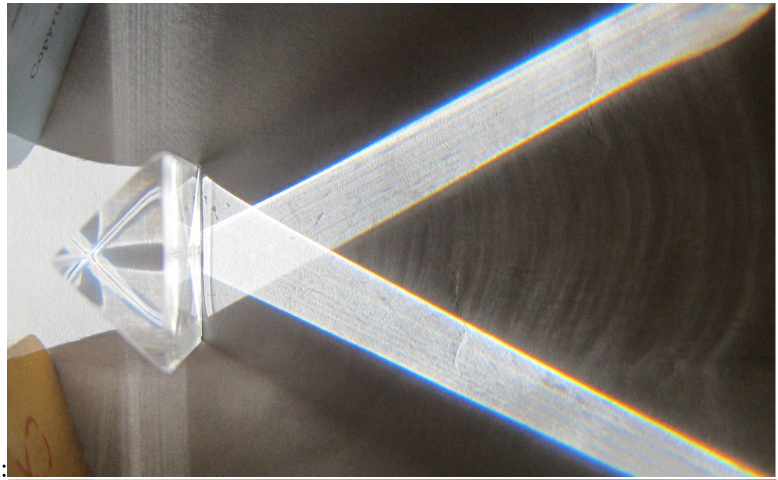
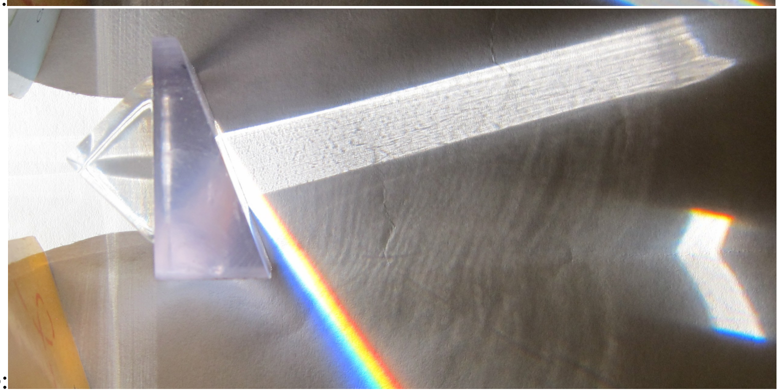


Figure 7: Polycarbonate prisms designed to cancel the dispersion of visible light passing through a 45-45-90 acrylic prism. Left image is taken after machining and sanding the polycarbonate to 600-grit. Right image shows the prisms after polishing with 1 μm alumina.



a:



b:

Figure 8: **a:** Notice the subtle chromatic dispersion at the edges of the refracted beams of sunlight. **b:** By placing the polycarbonate $\alpha = 0.3596$ prism after the acrylic prism, we get the expected dispersion cancellation in the positive-angle beam and enhanced dispersion in the negative-angle beam. In the lower right, the sideways chevron is direct sunlight through the polycarbonate prism. Notice that its chromatic dispersion is visibly much stronger than the acrylic's dispersion.

Table 1: Refractive indices for acrylic and polycarbonate. Values are from Mikhail Polyanskiy's RefractiveIndex.info website[7].

	n_F 486.1 nm	n_C 656.3 nm
Acrylic	1.4973	1.48796
Polycarbonate	1.5994	1.57817

4 Camera lenses

Camera lenses are a common example of achromatic refractive optics. Photographers tend to be picky about the quality of their equipment, so it's not too difficult to acquire inexpensive less-than-perfect (but good enough for our purposes) lenses. I was able to get my hands on a Ricoh 55 mm screw-mount lens and a pair of Olympus 50 mm lenses. Bad lens lots come up frequently on Ebay, and I found that the price for lots with obsolete mounts (like Olympus M and Canon FD) are much, much cheaper than lots with currently usable mounts (Nikon, Pentax, and Canon EF). An adjustable spanner wrench (I bought one from S.K. Grimes in Rhode Island) is essential disassembling lenses.

4.1 Ricoh 55 mm screw-mount

Figure 9: The Ricoh lens is a Cooke triplet design. (Diagram courtesy of Wikimedia Commons). The bottom image shows the aperture body and the three lens elements of the Ricoh lens. Each lens is bonded to its metal ring.

The Ricoh was not a very popular lens – I couldn't find any information on it on the web. I disassembled this lens prior to Newton Day and found that it had only three elements.

Dan Reiley immediately identified this as a Cooke triplet (see Figure 9). In addition to being achromatic, the Cooke triplet's claim to fame is that it has zero low order field curvature (Petzval sum). It was patented in 1893, but the first use of this arrangement was by Peter Dollond in an astronomical telescope in 1763[4]. To the great consternation of others in the trade, the Dollond family patented and commercialized Halls achromatic doublet. According to Watson, this triplet, brought him back into the good graces of the British optical trade of the time.

I noticed that the front two elements of a Cooke triplet form a Galilean telescope, and indeed, I find that removing the back element leaves me with a very weak telescope (approximately 1.3X magnification). It's almost certainly has chromatic aberration in this configuration, but I can't see it.

4.2 Olympus M F.Zuiko

Figure 10: The Olympus lens is a double Gauss design. The majority of fixed focal length SLR camera lenses are variants of the double Gauss design. (Diagram courtesy of Wikimedia Commons)

The Olympus is a much more well known lens. It is readily available on Ebay for about \$40.00 and is one of a large assortment of "Double Gauss" lenses. For fixed focal length lenses, the double Gauss design apparently dominated the camera market for the latter half of the 20th century. The diagram of the Olympus was taken from the Wikipedia page on double Gauss lenses [9] which has the diagrams for 75 different camera lenses marketed from 1936 to 2010.

The double Gauss design has a rich history and is technically very interesting. I won't get into it here, but for those interested in this design, I recommend the papers by Walter Mandler [6] (who designed many famous Leitz lenses) and Jan Hoogland [5] listed in the bibliography.

We should find a cemented doublet in the back half of the Olympus lens. If the lens is not bonded to metal, we may try to separate the elements of the doublet in order to measure their

[4]See F. Watson, *Stargazer*, p. 154

densities independently. Newton notes in *Opticks* that refractive index scales with density – an observation that is backed up by the theory of dispersion[5].

5 Acknowledgments

I almost got in over my head this year! Thanks first to Dan Reiley and Roger Smith, my Caltech Optical Observatories colleagues with whom I am working on the Prime Focus Spectrograph for the Subaru telescope. Dan is an optics expert/geek who just joined Caltech in August of this year. Dan has been instrumental in helping me to understand many of the thornier issues of optics design.

Roger is a long time Caltecher originally from Australia. Because both astronomy instrumentation and Australia have small populations, the probability that Roger would know Fred Watson is very high. That, in fact, turns out to be the case. While I was stuck on the achromatic prism example, Roger pointed out to me the very important insight that in ray tracing, **parallel rays are identical.** I had been hung up for a few weeks because of the divergence out of the first prism.[6]

Thanks also go out to the Caltech Physics machine shop, where I cut the polycarbonate prism, and to Prof. George Rossman, for giving me access to his polishing facility and for his guidance in the art of polishing.

Thanks to my sister, Julie, who gave me Fred Watson's book – the primary source for this year's paper. This is the book that initially pointed me in the direction of achromatic optics.

I credit Roger Cicala for the idea to disassemble camera lenses. Earlier this year, there was a report of a photographer at Yellowstone whose equipment (rented from Mr. Cicala's company, LensRentals) was attacked by a grizzly bear. That report led me to LensRentals.com and Mr. Cicala's entertaining and informative blog posts. There is a whole section entitled "Teardowns and Disassemblies," [11] and there are also some nice articles on lens genealogy.[12]

A big hug goes to Margaret Adams-Mao (Figure 11). She gave me the acrylic prism that shows up in Figures 6 and 8 because "I know that you love science." So true...

[5] See Born and Wolf, Section 2.3.4

[6] This is an example of the difference between knowing and understanding: from my graduate work, I was intimately familiar with Bragg scattering, and in the canonical picture of Bragg scattering, the interfering beams are parallel but displaced from one another. I knew what Roger told me before he mention it, but I didn't understand it until he told me!

Figure 11: Maggie and Owen.

6 Appendix: Gaussian Optics

This section is just meant to be a road map from the basic results of Gaussian optics to the achromatic doublet design equation. I've included it for completeness, but the development of this topic in Walther is much better.

Gaussian optics is optics in the small angle regime. It is used to develop a first order understanding of a system of lenses and is the basis for the achromatic doublet design equation, Equation (2.1). Please refer to Figure 12 in this discussion.

As in all optics diagrams, we have light moving from left to right. Figure 12 shows a system with two surfaces (in Gaussian optics, all surfaces are spherical sections) and three regions. This could be a glass lens in air, where $n_1 = n_3 = 1$ and $n_2 > 1$. The thickness of the element is t and the radius of curvature of the two surfaces are r_1 and r_2. The first surface has a positive radius because the center of curvature is behind (to the right of) the surface, and the second surface is negative because its center of curvature is in front of the surface.

For a single surface, the optical power, P, is

$$
\begin{aligned}
P_i &= \frac{n_{i+1} - n_i}{r_i} \\
&= (n_{i+1} - n_i)k_i, \qquad (6.1)
\end{aligned}
$$

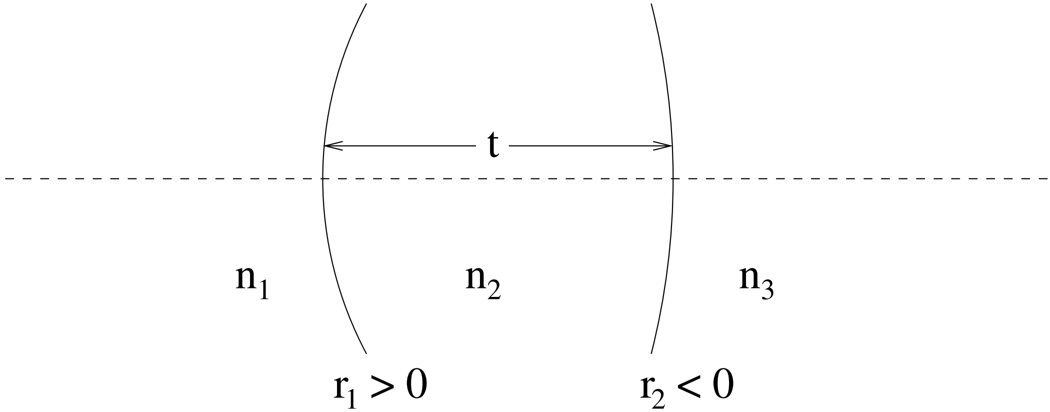


Figure 12: Gaussian optics: This diagram shows a system with three regions separated by two interfaces. For a glass lens in air, $n_1 = n_2 \approx 1$ and n_2 is the refractive index of the glass. The radius of curvature of the front and back interfaces are r_1 and r_2, respectively, and the thickness of the central region (ie, the lens) is t.

where I've introduced the curvature $k_i = 1/r_i$. Note that this means that both surfaces of a (bi)convex lens, like a magnifying glass, have positive optical power, even though the curvature of the back surface is negative (see Equation (6.4) below). The optical power for a pair of surfaces is

$$P = P_1 + P_2 - \frac{t}{n_2} P_1 P_2. \tag{6.2}$$

In the "thin lens" approximation, $t = 0$, so

$$P = P_1 + P_2. \tag{6.3}$$

Equations 6.1 – 6.3 are the major results of Gaussian optics. Not so bad, right? From here, we can derive the achromatic doublet.

For a thin lens in air ($n_1 = n_3 = 1; n_2 = n$),

$$\begin{aligned} P &= (n-1)k_1 + (1-n)k_2 \\ &= (n-1)(k_1 - k_2). \end{aligned} \tag{6.4}$$

Taking a differential of the Equation (6.4), we find the relationship between change in optical power with change in refractive index:

$$dP = dn(k_1 - k_2) \tag{6.5}$$

The dispersive power Δ comes from considering this differential with at the wavelengths of the Fraunhofer F ($\lambda = 486.1$ nm), d ($\lambda = 587.6$ nm) and C ($\lambda = 656.3$ nm) lines:

$$P_d = (n_d - 1)(k_{1,d} - k_{2,d}) \tag{6.6}$$

$$(P_F - P_C) = (n_F - n_C)(k_{1,d} - k_{2,d}) \tag{6.7}$$
$$= \frac{n_F - n_C}{n_d - 1} P_d \tag{6.8}$$
$$= \Delta_d P_d. \tag{6.9}$$

I'll rearrange the above equation a bit:

$$\Delta = \frac{n_F - n_C}{n_d - 1} = \frac{P_F - P_C}{P_d} = \frac{dP_d}{P_d}. \tag{6.10}$$

Now (I hope) you can see that physically, the dispersive power is the fractional change in the optical power of the lens over the visible spectrum.

The thin lens equation for summing powers of lenses, Equation (6.3), is the same as that for summing powers of surfaces,[7] so the differential power of a crown-flint cemented pair of lenses is

$$dP_{\text{doublet}} = dP_{\text{crown}} + dP_{\text{flint}}$$
$$= \Delta_{\text{crown}} P_{\text{crown}} + \Delta_{\text{flint}} P_{\text{flint}}. \tag{6.11}$$

Setting $dP_{\text{doublet}} = 0$ then gives you the achromatic doublet design equation of Section 2, Equation (2.1).

References

[1] Isaac Newton, *Opticks*, Project Gutenberg #33504, 1730.

[2] Fred Watson, *Stargazer: the life and times of the telescope*. Da Capo Press, Cambridge, 2004.

[3] A. Walther, *The Ray and Wave Theory of Lenses*. Cambridge University Press, 1995.

[4] Max Born and Emil Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light, Sixth Edition*. Pergamon Press, 1980.

[5] Jan Hoogland, "Synthesis Of The Double Gauss Lens," Proc. SPIE 0399, Optical System Design, Analysis, and Production, 204 (October 26, 1983); doi:10.1117/12.935432.

[6] Walter Mandler, "Design Of Basic Double Gauss Lenses," Proc. SPIE. 0237, 1980 Intl Lens Design Conf 222 (September 16, 1980) doi: 10.1117/12.959089.

[7] RefractiveIndex.info website.

[8] Wikipedia: Prism (optics).

[9] Wikipedia: Double Gauss lens

[10] Wikipedia: Cooke triplet

[11] LensRentals.com disassembly blog

[12] LensRentals.com: Lens Genealogy, Part 1

[7] This is just the associative property of addition: $a + b + c + d = (a + b) + (c + d)$.