Newton Day 2007
Celebrating 365 years of Sir Isaac Newton

by Peter Mao for Margaret

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Abstract

Sir Isaac Newton, the greatest mathematical physicist of the past 14 billion years (or so), was born in 1642 on the day the Brits of the time celebrated the birth of Jesus, who, by the way, was also not born on December 25. This year is Maggie's second Newton day and for this occasion, we will ruminate on the nature of the tides. As you will see, tides are simple in the abstract but very complicated in the details.

1 Forward

When I decided that this year's Newtonian Celebration would focus on the tides, I wanted to give a historical summary and a quick mathematical treatment on the subject. I read parts of Newton's *Principia*, David Edgar Cartwright's *Tides: a Scientific History*, and the Wikipedia entry on tides. Completeness on this topic is much more daunting of a task than I originally anticipated; therefore, only as needed will I give some cursory mathematical treatment on the physics of the problem.

2 $\pi\rho o\tau o$-Newton

Knowledge of the tides is crucially important to sailors, lest they find themselves on the rocks. Although the basic phenomena of tides was well known for millennia before Newton, the underlying physical mechanisms that generate tides remained a mystery. For those who do not live near large bodies of water, the remainder of this section summarizes what most coastal and sea-faring people knew.

The tides rise and fall roughly twice a day. *Roughly*, because the time of high or low water is later on each successive day. *Twice*, because there isn't a concise English word for $1.935\ldots$. How much later in the day the high or low water event occurs is highly variable (by a factor of two or more), but every lunar month, the time of the tide cycles by a full Earth day, give or take some fraction of an hour.

Throughout the lunar cycle, the height of the high and low water marks each day also varies. High and low tide are at their most extreme near Full and New Moon, and at their least extreme near the Half Moon phases. These phenomena are referred to, respectively, as Spring and Neap tides. And they suggest to proto-Newtonians that the Sun may also have something to do with the tides.

3 Newton

Before Newton, many people tried to explain the tides, including some arguably clever ones like Galileo and Descartes, with his erroneous vortex theory of mechanics. Newton's explanation of the tides is a direct consequence of his discovery of the universal gravitational attraction between all massive objects. Newton's Law of Gravitation resulted from his application of his famous relationship $\vec{F} = m\vec{a}$[1] to Kepler's Three

[1] The *only* equation a student studying classical mechanics for the first time needs to remember.

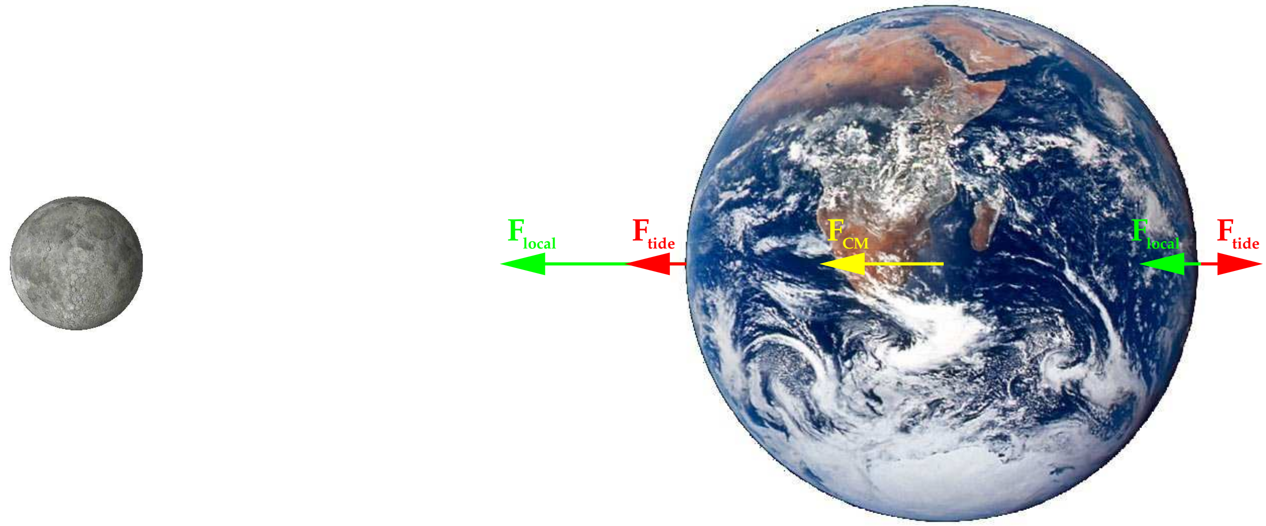

Figure 1: Schematic of gravitational forces and their tidal components due the Moon on the Earth. The local gravitational force due to theh moon is shown in green, the force at the Earth's center of mass is in yellow, and the resulting tidal forces are in red. Vector magnitudes are grossly exaggerated. Apollo 17 "Blue Marble" image courtesy NASA Johnson Space Center, Moon image courtesy of NASA/Goddard Space Flight Center Scientific Visualization Studio.

Correct Laws of planetary motion.[2] The Law of Gravitation states that

$$\vec{F}_{1,2} = -\frac{Gm_1m_2}{r^2}\hat{r}, \tag{1}$$

i.e., the force of attraction between two objects of mass m_1 and m_2 is proportional to the product of their masses and inversely proportional to the square of their separation.[3]

Because the gravitational attraction toward a given body falls off as $1/r^2$, the Moon's gravitational force on an object on the Earth's surface is stronger when the object is on side of the Earth facing the Moon than when it is on the side facing away from the Moon (see Figure 1). Mathematically, we express the tidal component of the gravitational force on an object as

$$\vec{F}_{\text{tide}} = \vec{F}_{\text{local}} - \vec{F}_{\text{CM}}. \tag{2}$$

In the Earth-Moon system, $\vec{F}_{\text{local}}$ is the gravitational attraction between the Moon and a test particle on the surface of the Earth and $\vec{F}_{\text{CM}}$ is the gravitational attraction between the Moon and the test particle if it was located at the Earth's center of mass. When the Moon is directly overhead, the tidal force per unit mass is

$$\frac{\vec{F}_{\text{tide}}}{m} = GM_{☾}\left(\frac{1}{(r_{☾} - R_{\text{E}})^2} - \frac{1}{r_{☾}^2}\right)\hat{R} \tag{3}$$

and when it is directly underfoot, the tidal force per unit mass is

$$\frac{\vec{F}_{\text{tide}}}{m} = GM_{☾}\left(\frac{-1}{(r_{☾} + R_{\text{E}})^2} + \frac{1}{r_{☾}^2}\right)\hat{R}. \tag{4}$$

[2]This, like all historical references contained herein should not be taken as Scripture (pun intended), it's just how it's organized in my head.

[3]G is the gravitational constant, m_i is the mass of object i, r is the distance between the objects, and the negative sign indicates that the force is attractive.

In the previous two equations, $M_☾$ is the mass of the Moon, $r_☾$ is the distance between the Earth and the Moon, R_{E} is the radius of the Earth, and $\hat{R}$ is the radial unit vector originating from the Earth's center of mass. To first order, these equations both reduce to

$$\frac{\vec{F}_{\text{tide}}}{m} = GM_☾ \frac{2R_{\text{E}}}{r_☾^3}\hat{R}. \tag{5}$$

Surprise, surprise!! See how the radial component is positive whether the moon is overhead or underfoot? Using Newton's laws and a first order Taylor expansion, we've shown that there's a semidiurnal (twice a day) upward force on the surface of the Earth due to the Moon's gravity! It is more difficult (but possible) to show that when the moon is rising or setting, the tidal force is, to first order again, radially inward. Newton interpreted these forces as the cause of the semidiurnal high and low tides and tried to extend the theory to the phasing of the tide relative to the Moon's position and even to the Earth/Moon mass ratio.

So that's basically how we get tides, but the details are much more complicated than Newton got around to figuring out.

4 $\mu\epsilon\tau\alpha$-Newton

Even though Newton hit upon the root physical cause of the tides, it was clear that the simple model discussed above does not explain all of the phenomena. In 1738, the French *Académie Royale des Sciences* offered up a prize for the best essay on the tides. Two years later, they presented the prize to four scientists: Colin Maclaurin, Leonard Euler, Daniel Bernoulli, and Antoinne Cavalleri. Maclaurin rigorously proved that the equipotential due to the tidal force is a prolate spheroid. Euler proved that the vertical displacement of the tides was due to the shear tidal stress (the horizontal component of the force), not the vertical forces that are mentioned in the previous section. Bernoulli included the tidal effect of the sun and greatly improved tide predictions. Cavalleri, the least deserving, was stuck in the past, expounding Descartes' vortex theory, for which the French still held out some hope.

In the late 18th century, Pierre Simon, Marquis de Laplace, developed a hydrodynamic theory of the tides, in contrast to Newton's quasi-static approach. In his work on the tides, Laplace included the effect of the Earth's rotation on the motion of the tides. Gaspard-Gustave Coriolis got his name attached to this "effect," but the honor really belongs to Laplace. Laplace's tidal equations still form the basis by which modern tide predictions are made.

5 Other tidbits about tides

Tides produce deformations and in a frictionful world, deformations lead to energy dissipation in the form of heat. Just as the Moon induces tides on the Earth, the Earth also induces tidal deformation on the Moon. The tidal force of the Earth on the Moon slowed down the Moon's rotation such that the same side is always facing the Earth. This is referred to as *tidal locking*. The Moon is still in the process of slowing down the Earth, so some day, the Earth will be tidally locked to the Moon, and one will only be able to see the moon from one half of the planet. The philosopher Immanuel Kant was the first person on record to suggest that the Moon might be slowing down the spin of the Earth due to tidal friction.

Despite tidal dissipation, the Earth-Moon system is still required by law to preserve angular momentum. From our perspective, the Moon lengthening the day on Earth, but from the Moon's perspective, the spin of the Earth is shortening the lunar month. As the Moon's orbital speed increases, it moves further from the Earth. Laser ranging measurements have shown that the Moon is receding by about 3.8 cm/year.[4]

Because the Moon really is slowing down the Earth's rotation, the mean tidal bulge lies somewhere east of the terrestrial longitude where the Moon is at (upper or lower) transit. High tide, therefore, should occur sometime after the lunar transit. Newton states that high tide should occur 3 hours after transit (Book 3,

[4]These measurements are possible because during the Apollo 11 Mission, Buzz Aldrin left a big reflector on the Moon.

Prop. 24, Theorem 19) if it were only due to the Moon. Observation of the tide either at Santa Monica or even at Santa Cruz will show that this is not definitively the case. At Santa Monica, the high tide appears two to six hours *before* lunar transit! At Santa Cruz, the tide comes in one to four hours before lunar transit.

In the 19th century William Whewell compiled TONS of tidal data and produced maps of *cotidal* lines in the oceans. A cotidal line is a locus of points in the ocean where high (or low) tide occurs at a given universal time. Whewell's map of the North Sea led him to propose the existence of "points of no-tide" where the cotidal lines intersect. This concept was doubted at the time, but eventually Whewell's ideas were proven to be true and they are now known as *amphidromic* points. Cotidal lines rotate about the amphidromic points semidiurnally, which explains why the tide can lead the moon in some places.

6 Activities

Santa Cruz: Go to Santa Cruz pier at 9:45 AM to watch the tide come in (at 10:10). Note that the moon is nowhere to be seen! Return to the Pier at 5:00 PM to watch the tide go out (at 5:24). Moonrise is about 1.5 hours after the predicted low tide.

Los Angeles: Go to the Santa Monica pier at 8:45 AM to watch the tide com in (9:15 AM) and return at 4:30 PM to witness low tide (4:41 PM). Moonrise is at 6:51 PM, more than two hours after the predicted low tide.

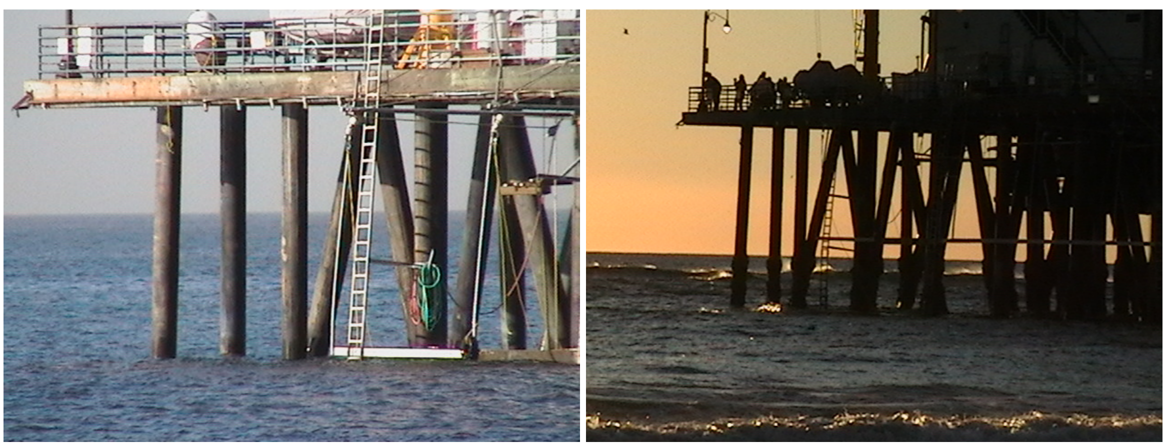

Figure 2: Santa Monica Pier near morning high tide (9:23) and afternoon low tide (16:25) on Newton's Birthday. Note the level of the water relative to the ladder.

Question 1: Is the tide "spring" or "neap?" What the phase of the moon?[5]

Question 2: What is wrong with Figure 1?[6]

Bonus question: What is the rate of energy transfer from the Earth to the Moon due to tidal dissipation?

7 Conclusion

We experience the gravity of the Earth on a daily basis: I forget to tie my laces; I trip and fall. I drop a cup; it falls to the ground and breaks. Tides are a result of the same force that makes the dropped cup fall to the ground, but the beauty of the tides is that they are the result of gravitational attraction by the Moon and

[5]A: This is a spring tide, as the moon is nearly full.

[6]A: The image of the moon shows the earth-facing side of the moon, so the moon's orientation is off by 90° in longitude.

Figure 3: Santa Monica Beach, from the pier looking north near morning high tide (9:29) and afternoon low tide (16:52) on Newton's Birthday. For reference, note that the trashbarrels on the far right are placed just beyond the high water mark.

Figure 4: Moonrise (19:19) seen from the roof at N34°8′31.4″, W118°7′37.2″. Mountains to the NE of Pasadena block the view of the moon for the first 25 minutes.

the Sun on our surroundings. Tides are easy to observe if one lives near the coast, and the connection of the tides with the Sun and Moon can be discerned with at least half a month of observations. We have Newton to thank for making the connection between that which makes apples fall on one's head and that which makes the seas rise and fall.

8 Sources

- Sir Isaac Newton, *Philosophiæ Naturalis Principia Mathematica*
- David Edgar Cartwright, *Tides: A Scientific History*
- Hannes Alfvén and Gustav Arrhenius, *Evolution of the Solar System*, Chapter 9 (`http://history.nasa.gov/SP-345/ch9.htm`)
- `http://en.wikipedia.org/wiki/Tide`

- http://www.tidesonline.com
- http://tbone.biol.sc.edu/tide/

9 Afterword

I didn't get to spend this Newton Day with Mag and DSA. Deirdre had promised her Dad that he'd get to see the grand daughter on one of the major holidays (T-day being the other). I stayed behind in LA to be in the lab to get the first solar oxygen measurement with MegaSIMS.

Figure 5: (10:44 or 11:44) Mag and Deirdre at a beach near Santa Cruz, CA. The uncertainty in time is due to the setting on the camera – I can't remember if it was left on PDT at this time.