Newton Day 2008
Celebrating 366 years of Sir Isaac Newton

by Peter Mao for Margaret

final pre-Holiday version

Abstract

Sir Isaac Newton, the most clever mathematical physicist of the Holocene, was born on December 25, 1642, January 4, 1643 or 11/14, 4339, depending on your calendar of choice. The Brits of the time used the first one. This year is Maggie's third Newton Day and for this occasion, we will return to the pendulum. This time, however, the test mass is active and can make the pendulum swing way beyond the small angle regime. Yes, Maggie has taken a shine to the playground swing!

1 Introduction

A child on a playground swing serves as an interesting example of a forced oscillator. The standard forces oscillator problems presented in elementary physics classes are temporally sinusoidal and one typically calculates the amplitude and phase response of a damped oscillator subjected to the periodic forcing function.

The playground swing is an entirely different problem. The amplitude of the motion is typically way beyond the small angles where it looks like simple harmonic motion, and the forcing function dynamically adjusts to the changing period of oscillation. Furthermore, the forcing function is typically only active over one half of each cycle.

This year, we shall consider how it is possible for a child on a swing to gain altitude without resorting to external forces.

2 The forcing function

Consider a child on a swing, pulled back from the vertical such that when released, she will swing in the forward direction. The basic swing-forcing technique calls for the child to lean back on the way down and sit back up at the end of the swing. Usually one only performs the pumping motion in the forward half of the cycle and relaxes upright on the return trip. With each successive cycle, the child swings higher and higher with no external impetus necessary.

With some parental direction, Maggie has learned this motion well enough to sustain her swinging indefinitely. Since she deems it necessary to play on the swing upon entry and before exiting her daycare, this knowledge has allowed her to extend the transition time for drop-off and pick-up from said daycare. A little knowledge is, indeed, a dangerous thing.

3 Theory of swinging

Maggie can only swing higher because she is mechanically imparting energy into the swing. Where does this happen? She starts out sitting on the swing, which is pulled back so that the chain makes an angle θ_i with respect to the vertical (see Figure 1). The only other relevant quantity in this problem is the location of her center of mass, which determines the initial length, r_i, of the pendulum.

As she drops, she leans back, lowering her center of mass and effectively increasing the length of the pendulum to $r_i + \Delta r$. Assuming she has leaned back as far as she will go when her center of mass passes through $\theta = 0$, we can easily calculate her velocity at that point. But we don't need to – she hasn't put any energy in to the system yet, so her center of mass will rise to exactly the height from which she started (neglecting frictional losses). Now for the sleight of hand: because she has made the pendulum longer, though she rises to her starting height, the maximum angle that the swing makes with the vertical, θ_f, is larger than θ_i. By pulling herself back into a sitting position, Maggie raises her center of mass above her initial height and can enjoy the ride back to the greater swing amplitude that she had requested! *Voila!*

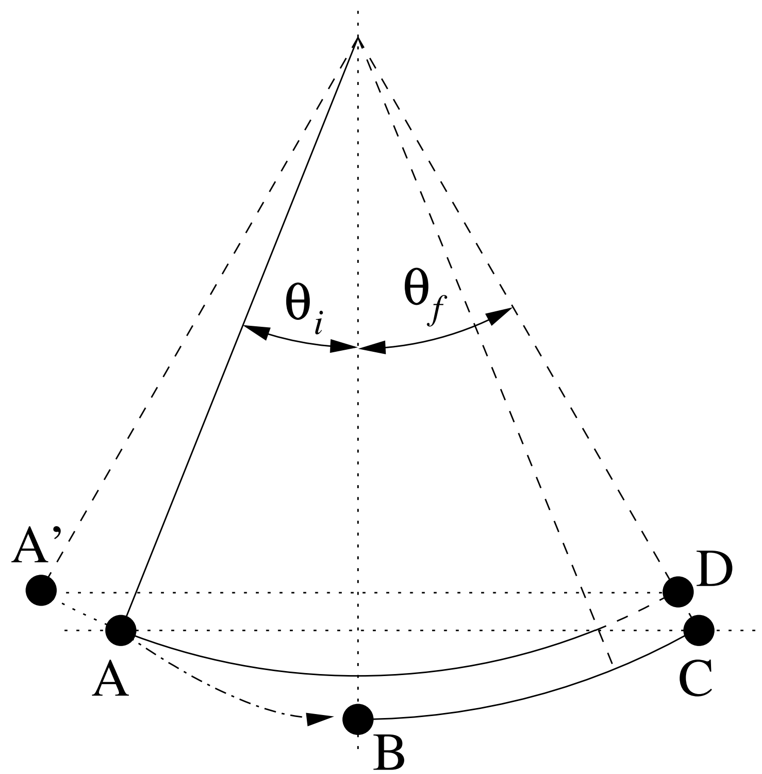

Figure 1: Starting at **A**, sitting upright, the swinger leans back on the way down to **B** where she attains maximum velocity. Conservation of energy dictates that she will swing back such that her center of mass reaches the same height as point **A**. In the leaned-back configuration, she will reach **C**, where she can pull herself into an upright position, raising her center of mass to **D**. She coasts back to **A′** and repeats the process.

4 Bonus points

1. Without external forces, is it possible to swing oneself all the way around?
2. At large angles, the chain goes slack. Is there critical angle, above which the chain slackens?
3. Is it possible to reduce the swing amplitude by running this method backwards? If so, where does the energy go?

5 An unsatisfying return to the period of the pendulum

Recall, from Newton's 364th birthday the derivations for the period of a pendulum (in the small amplitude limit): $T = 2\pi\sqrt{r/g}$ where g is the acceleration due to gravity at the surface of the Earth and r is the distance from the pivot to the pendulum's center of mass. Also recall that we determined that simple harmonic motion under-estimates the true period of a pendulum. Since the potential function for the pendulum is easy ($V(\theta) = mgr(1-\cos\theta)$), let's try to solve for the period.

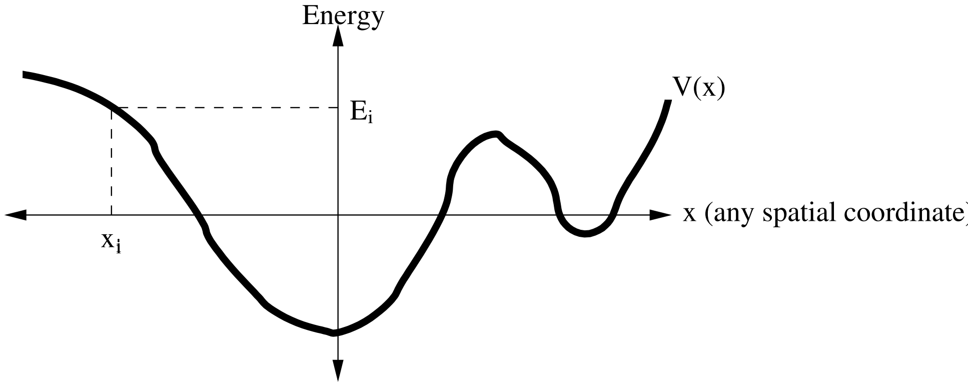

Figure 2: An arbitrary potential in one dimension.

In an arbitrary one-dimensional potential $V(x)$ (see Figure 2), one can derive the oscillation period from conservation of energy. Let E_i be the initial energy of the object and assume that the object is at rest at x_i. Then, since the total energy is the sum of kinetic and potential energies, the velocity is

$$v^2 = \frac{2}{m}(E_i - V(x)).$$

Since $dt = dx/v$ we integrate to find that

$$T = \sqrt{2m}\int_{x_i}^{x_f} \frac{dx}{\sqrt{E_i - V(x)}}$$

where x_f is the maximum excursion of the object.

Applying this to the pendulum and exploiting the symmetry of its potential, we find the exact (but unfortunately open) expression for the pendulum period:

$$T(\theta_i) = 2\sqrt{\frac{2}{g}}\int_0^{\theta_i} \frac{r d\theta}{\sqrt{\cos\theta - \cos\theta_i}}.$$

Sedona Price[1] informed me that this is a Jacobi elliptic integral (of the first kind). Casting the integral into the canonical elliptic integral form

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

is rather involved and explained nicely on the MathWorld website.[2]

[1]Caltech Sophomore who did a SURF with SRL last summer.

[2]Weisstein, Eric W. "Elliptic Integral of the First Kind." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/EllipticIntegraloftheFirstKind.html`