Newton Day 2010

"Why doesn't the Moon fall out of the sky?"

by Peter H. Mao for Margaret and Owen

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Abstract

For Sir Isaac Newton's 368th birthday, we will discuss a question posed by Margaret on one of our long commutes home.

1 Preface

Since Maggie started daycare about two years ago, I've been occasionally borrowing books from her daycare, the Child Educational Center (CEC), in order to stay informed about this wonderful and rapidly changing person that Deirdre and I have brought into our lives. One of the books that I found very interesting is by David Elkind, and bears the rather alarming title *Miseducation: Preschoolers at Risk*. One of the takeaway messages from that book is that giving a technically correct answer to a young child's question is not always the best thing to do. On the topic of children's questions, Elkind writes:

> "If we respond appropriately to the children's questions, we provide them with the sense that the effort and anxiety involved in taking the intellectual and social initiative are worthwhile. We thus provide children with the foundation for taking the initiative as older children or as adults."

Elkind goes on to say that giving preschoolers a scientifically correct answer may present them with information that does not result in any meaningful understanding of the underlying concepts.

During one of our long drives home from the CEC, Maggie asked my "Why doesn't the Moon fall [out of the sky]?" Not fully prepared for this question, I gave her the stock answer – it *is* falling towards the Earth, but it misses. To which she replied, "Oh, . . . and that's the same reason that the Sun doesn't fall out of the sky!" Well, not exactly, but a nice bit of reductionism nonetheless (if you're into that kind of thing). I realized, a few days after this conversation, that I should have given her more of an opportunity to explore her own ideas about the Moon and falling. After talking to Maggie's teachers, I attempted to revisit the question with her class.

2 Group time at CEC

One of the regular activities at CEC is "group time," which may happen just before lunch and/or around 4 PM, depending on whether or not the children are interested in participating. Sometimes, the children are asked to some open-ended question and their responses are transcribed by a teacher. I asked her Flying Squirrel teachers if I could pose Maggie's question to the rest of her class. I scheduled my discussion for the pre-lunch time slot on Nov 30, 2010 because the Moon would be visible in the sky at that time. Here is what the kids had to say:

Question 1: Why doesn't the Moon fall down?

Chloe: "The Moon doesn't fall down – ever."

Julian: "Moon goes down very slowly so the sun can come up and it can be morning time."

Alison: "Sometimes, the Moon and sun are both there."

Anastasia: "The Moon never falls down."

Faith: [inaudible]

Eric: "It's supposed to be around the Earth."

Melanie: "Because the Earth spins . . . [trails off]"

Jocelyn: "The Moon doesn't fall down because it rotates."

Maggie: "Because it's stuck to the sky."

Chloe: "Because the Moon is a sticker."

Melanie and Jocelyn were onto something! In hindsight, this would have been the time to introduce pseudopotentials – perhaps by spinning around with a ball on a string. I wasn't so well prepared, so I asked a follow-up question:

Question 2: Are there other things that don't fall?

Eric: "The Sun."

Chloe: "The clouds."

Alison: "The sky."

Ann: "The ground doesn't fall down."

Hayden: "The clouds fall down and that's my answer."

Maggie: "The sky doesn't fall out of the sky!"

Simone: "The sky of Wall-E's Earth doesn't fall down."

After we collected responses from all children willing to say something, the teachers asked me present my answer to the first question. I can't say that my answer made much of an impression, but hopefully I did no harm in the process.

Elkind states that four to five year olds are mainly interested in the purpose of things, rather than their inner workings, but I am still unable to figure out how to rephrase Maggie's question as a "purpose" question.

3 What if the Moon *did* fall down?

Anyways, enough of David Elkind – we are celebrating Sir Isaac Newton! During my discussion with the Flying Squirrels, I thought of, but did not pose, the question: If the Moon did fall to Earth, how long would that take?

3.1 Upper limit

A quick upper limit can be estimated by assuming that the radius of the Earth is much smaller than the distance to the Moon, and knowing the orbital period of the Moon (about 28 days). In this case, one is assuming that the falling Moon falls toward Earth with constant acceleration where the constant is taken at the Earth-Moon distance. The quick estimate is a quarter of the orbital period, 7 days.

Why not use $28/2\pi$?

3.2 Lower limit

A quick lower limit on the time-to-fall is to assume that the Moon undergoes constant acceleration of g (980 cm/s^2), which is what we experience on a daily basis. Taking two integrals of $\ddot{x} = g$ we get the familiar equation $x = \frac{1}{2}gt^2$. The Earth-Moon distance is $r_☾ = 4.0 \times 10^{10}$ cm, making the falling time $\sqrt{80 \times 10^6}$, which is about 9000 seconds or 2.5 hours.

Is there a better way to get a quick lower limit?

3.3 Direct calculation

To calculate the time for the Moon (or something on it) to fall directly to Earth, recall that the classical (Newtonian) gravitational potential between two massive spheres is

$$V = -\frac{G_N m_1 m_2}{r}$$

where G_N is the Newtonian gravitational constant (6.67×10^{-8} cm^2/g/s^2), m_1 and m_2 denote the masses and r is the distance between the centers of the objects. Of course, one could integrate the force along the path, but that method creates undue hardships.

Assuming zero initial velocity, the radial velocity at any radial distance is related to the difference in potential energy by

$$\frac{1}{2}mv^2 = G_N m_\oplus m(\frac{1}{r_☾} - \frac{1}{r})$$

or

$$v = -\sqrt{2G_N m_\oplus \left(\frac{r_☾ - r}{r_☾ r}\right)}$$

where $m_\oplus$ is the mass of the Earth (6.0×10^{27} g) and the negative sign is a reminder that the velocity is radially inward.

The time to fall is, therefore

$$\Delta t = \int_{r_☾}^{R_\oplus} \frac{dr}{v(r)}$$
$$= -\sqrt{\frac{r_☾}{2G_N m_\oplus}} \int_{r_☾}^{R_\oplus} \sqrt{\frac{r}{r_☾ - r}} dr$$

where $R_\oplus$ is the radius of the Earth ($\frac{4\times10^4}{2\pi}$ km, or 6.4×10^8 cm). Someday, Mag and Owen will have to learn how to solve integrals like this one, but their old man just typed `integral sqrt(r/(R-r)) dr` into Wolfram α and got...

$$\int_{r_☾}^{R_\oplus} \sqrt{\frac{r}{r_☾ - r}} dr = \left(\sqrt{r(r_☾ - r)} - r_☾ \arctan\sqrt{\frac{r}{r_☾ - r}}\right)\Bigg|_{r_☾}^{R_\oplus}$$
$$= \sqrt{R_\oplus(r_☾ - R_\oplus)} + r_☾\left(\frac{\pi}{2} - \arctan\sqrt{\frac{R_\oplus}{r_☾ - R_\oplus}}\right).$$

Plugging the solution to the integral back into the expression for Δt, I find that the zero orbital angular momentum Moon hits the Earth in 5.1 days.

As a historical note (thanks to my office-mate, Hiromasa Miyasaka, for pointing this out), Buzz Aldrin, Jr. and Neil Armstrong's return trip from the Moon took 2.5 days. How can you account for the difference?

4 Addendum/Celebration of Kepler's 439th birthday

David W. Hogg wrote back with a very elegant solution to the time-to-fall question:

> "BTW, the simplest calculation is as follows:
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> "The radial orbit that touches Earth surface and the Moon's orbit has semi-major axis $a_{Moon}/2$. Use Kepler's law and note that the transfer orbit is one half (peri out to ap only)."

Kepler's [third] law states that the square of the orbital period is proportional to the cube of the semi-major axis. Using the notation of the previous sections, we have

$$(2\Delta t)^2 \propto \left(\frac{r_☾}{2}\right)^3.$$

Denoting the lunar period (aka "month") as $T_☾$, we can turn the above proportionality into an equality by substituting $r_☾^3$ with $T_☾^2$. We are thus left with

$$\Delta t = \frac{T_☾}{4\sqrt{2}} \approx 5 \text{ days}.$$

Note that Hogg's Keplerian solution agrees exactly with the result of Section 3.3 if $R_\oplus$ is replaced with 0.