Newton Day 2013:

Base-$n : n \neq 10$

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Abstract

For Sir Isaac Newton's 371st birthday, at Maggie's request, the topic of the year is mathematics in bases other than base-10. Admittedly, this is only tangentially related to Newton, but the Newton Day activity does involve orbits and spin. In order to generalize numbers to arbitrary bases, we need to understand positional numbering, which in turn, requires understanding exponentiation.

1 Introduction

When we learn to count, we implicitly learn the base-10 system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, etc. We use ten symbols to represent the numbers zero through nine, and ten is represented by a composite of 1 and 0, where the positions of each symbol is significant to the value of the number. We learn to add, subtract, multiply and divide in base-10, and almost always, this is the numbering system we use. But it's not the only numbering system we use, and sometimes it's useful (or just fun) to work in other base systems. Base-10 positional numbering is more commonly known as decimal numbers, Arabic numbers, Hindu numbers, or Hindu-Arabic numbers.

In our exploration of alternate base numbering systems, we will look at a few non-decimal systems in common use today, and we will figure out how to write and evaluate numbers in bases other than base-10. To minimize confusion, in this document, non-decimal numbers will be typeset in the `typewriter font`.

2 Numbering systems

Although the decimal system seems intuitive and obvious, matching our usual biological configuration, the existence of this base-10 positional numbering system has been with us for less than 2000 years. By comparison, writing has been with us for just over 5000 years, so decimal numbers are a relative new-comer to human thought. The decimal system was invented in India some time between the first and fourth centuries (CE). From India, it spread west through the Arab world and east to China over the next several centuries. Leonardo Fibonacci of Pisa, he of the famous integer sequence, is credited with spreading the decimal system from Algeria, where he studied in Bejaia, to Europe in his book *Liber Abaci*, published in 1202.[12] Before decimal numbers came on the scene, there were other numbering systems in use, many of which stay with us to this day.

2.1 Roman numerals

Before Fibonacci brought decimal numbers to Europe, the Roman numeral system held sway. It is absolutely amazing to me that the Romans, well known and respected for their engineering talents (the aqueducts, roads, the ramp to Masada, etc.) used an absolutely daft numbering system. I, V, X, L, C, D, M. The symbols increase by alternating factors of two and five. The positions tell you how to add or subtract from the greatest value. If they had also invented logarithms, there might be some justification for this system, but logs had to wait until John Napier, long after the Roman Empire vanished from the Earth.

Since the Romans had such a lousy numbering system and yet were able to perform great feats of engineering, I can only guess that their achievements were built on a solid foundation of Greek geometry.

Today, Roman numerals live on as ornamentation, but never[3] in mathematics.

2.2 Time and angles: base-7, 12, 60, etc.

Time and angles (and astronomy) are intricately related. Although they are represented by decimal numbers, the commonly used units are not in multiples of ten:

- 60 (arc)seconds in a(n) (arc)minute
- 60 (arc)minutes in an hour or a degree(!)
- 24 hours in a day
- 7 days in a week
- 12 months in a year
- 365 days in a year (except for leap years)
- 360 degrees in a full circle

By comparison, time and angles make the Roman system look logical! Where did this mess come from? Astronomy! Starting from the biggest value, we see there's nothing to be done about the number of days in a year, short of changing the rotation or orbital velocities of the Earth. The degree is related to the number of days in a year, in that it represents the angular change in the position of the sun relative to the stars in a day. This is the only situation where using degrees for angle makes any sense (the rest of it is cultural heritage). But why is a degree one 360th of a full circle? Why not make it one 365th? You'd have to ask the Sumerians (modern day Iraq), who were fond of base-60, presumably for the many ways in which it can be divided evenly. Claudius Ptolemy, the ancient Roman astronomer, divided

[1]Geographically, Hindu-Arabic numbers showed up via the Moors on the Iberian peninsula before Fibonacci, but he is credited with spreading the idea throughout Europe.

[2]Decimal fractions were invented much later later by John Napier.

[3]"Never" is a strong word – someone prove me wrong!

the degree into base-60 fractions: *partes minutae primae*, known today as minutes and *partes minutae secundae*, known today as seconds.[4] Can you guess what one 60th of a second is called?

The 24 hour day is a bit more arbitrary. That comes out of ancient Egypt, where base-12 (also conveniently divisible in many ways) was favored. During the New Kingdom (1550-1077 BCE), Egyptians kept track of a set of 24 stars (decans) to keep time over night. For the daytime, they used the sun dial. Prior to Ptolemy, Greek astronomers melded the Egyptian 24 hour day with the Babylonian/Sumerian sexagesimal (base-60) system to split hours into base-60 divisions.[5] Although the French tried to do something about this after their first Revolution, with their decimal clock, the Egyptian/Sumerian/Babylonian/Greek system of time is still with us. As with the number of days in a year and the degree, the 24 hour day and its sexagesimal divisions into minutes and seconds also has an angular counterpart: astronomical right ascension (RA). RA is essentially a longitudinal position on the sky, where the circle is divided into 24 hours. Time zones are similar to RA in that they are a (rough) measure of longitudinal angle relative to the Greenwich Observatory in the United Kingdom. For example, here in Los Angeles, during standard time, we are 8 hours behind GMT, so in longitude, we are about 1/3 of the way around the Earth from the United Kingdom.

2.2.1 An aside on sidereal time

Astronomers locate objects on the celestial sphere using right ascension (RA) and declination (dec) (the equatorial coordinate system). RA is given in hours, minutes and seconds, and corresponds to the longitude of the object on the sky. Declination is given in *degrees* minutes and seconds, and corresponds to latitude on the sky. An object is directly overhead if its declination matches the observer's latitude *and* its RA matches the observer's sidereal time. In general, you can use the declination, along with your latitude on Earth to determine the maximum elevation of an astronomical object.[6] In geographic coordinates, a degree latitude is always a degree, but a degree in longitude is $\cos(\text{lat})$, so a degree longitude is only a true degree on the equator and it is diminished by the cosine of the latitude elsewhere. Likewise, in equatorial coordinates, an hour in RA is $15\cos(\text{dec})$ degrees, so a minute in RA is $15\cos(\text{dec})$ minutes and a second in RA is $15\cos(\text{dec})$ minutes. Seems crazy, doesn't it? It is practical, however, if you keep time using a sidereal clock. For example, if your sidereal clock says it is 05:30:00, and you want to observe an object at RA=06:30:00, you know that it will be on the meridian (the arc intersecting Polaris and zenith) in one hour, independent of the declination.

The sidereal clock differs from a solar clock in that a solar clock tracks rotation of Earth relative to the sun, while the sidereal clock tracks the rotation of Earth relative to the stars. The sidereal system does, however, ignore the precession of the Earth's equinox[7], so the coordinate system is forever drifting, which is why astronomical coordinates always are qualified with their epoch (J2000, for example). Newton writes that he accounted for the precession of the equinoxes, which Chandrasekhar disputes,[8] but we will leave that topic for another year.

[4] Claudius Ptolemy, *Almagest*, ca. 105 CE.

[5] See Charles Siefe's *Zero: The Biography of a Dangerous Idea*.

[6] ...left as an exercise for you!

[7] In practical terms, the Earth's equinox defines the direction of North on the celestial sphere.

[8] S. Chandrasekhar, *Newton's Principia for the Common Reader*, pg. 475.

Ignoring precession, a sidereal day is about 23 hours, 56 minutes and 4 seconds, so sidereal clocks run $1 + \frac{1}{365.25} \approx 1.0027$ times faster than "normal" clocks. This means that a sidereal year has 366.25 days, so a sidereal clock only looks "right" in late September/early October. The exact day of year when sidereal and local time match up depends on your location in your time zone and whether or not daylight saving time is in effect.[9] On the other hand, we always know that when the sidereal clock reads 05:34:32, the Crab Nebula is directly overhead.[10]

The point of this Herodotean digression into sidereal time is that I always *try* to incorporate some activity into Newton Day, and this year's activity explores the relationship between the number of sidereal days and the number of solar days.

2.2.2 Newton Day activity: sidereal vs. solar days

Materials:

- 1 lamp to represent the Sun
- 1 flashlight to represent a bright extra-solar astronomical object (star, galaxy, favorite black hole, etc.)
- 43086.75 star (or galaxy) icon stickers per child
- 43084.75 sun icon stickers per child
- Children who want to get dizzy

First, set up your universe (see Figure 1). In true Copernican fashion, put the Sun in the middle and lay out the orbit for your planet (the kid who wants to get dizzy). Place the extra-solar object far outside you solar system, and invite the planet to orbit the Sun. As the planet orbits the sun give it a "sun" sticker each time it faces the sun, and give it a "star" sticker each time it faces the extra-solar object.

Scenarios to explore:

1. Earth: Ask your planet to spin in the same sense that it is rotating around the Sun (prograde spin). Ask the planet to spin around 366.25 times for trip around the Sun.
2. Mercury: Ask your planet to turn slowly as it orbits the sun so that it makes 1.5 turns per orbit. This is the famous 3:2 resonance of Mercury's orbit.
3. Venus: Ask your planet to spin in the opposite sense to its rotation around the sun (retrograde spin). Like Mercury, Venus spins slowly, so only make 2 solar days per year for Venus (1.92, if you're picky.)
4. Uranus: To mimic Uranus, your planet must lie on the ground with his/her head pointed in the same direction relative to the extra-solar object throughout its orbit and spin around 42718 times per orbit. Make sure to leave extra time to explore this scenario. For extra points, ask the planet to keep its head about 1/7th of its height above its feet and spin in the retrograde sense.

[9] In Los Angeles, this happens on October 3 or 4, depending on the year.

[10] More precisely, it is on the meridian at that time of day

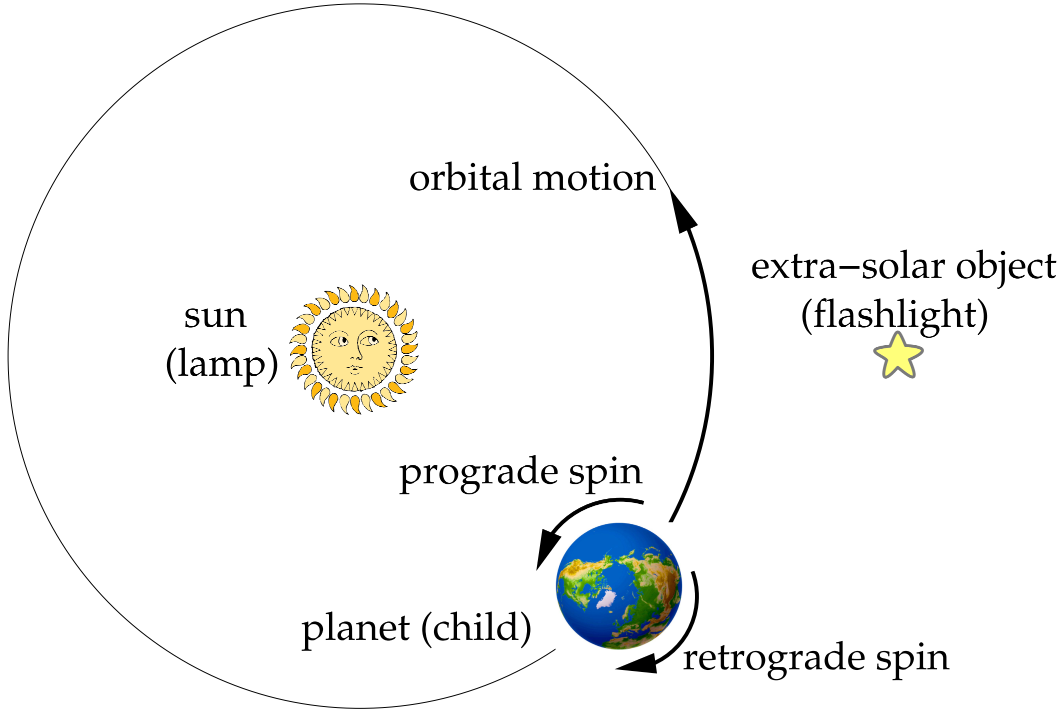

Figure 1: Simplified universe in your living room, viewed from above.

5. No spin: After the Uranus experiment, give your planet a break by asking it to orbit without spin. How many solar days per year in this case?

6. Tidal lock: Last one. Ask your planet to orbit the Sun while keeping the sun to its left. How many days per year? How many sidereal days?

Bonus question: Before we return to base-n numbering systems, at what time of year is the Crab on the meridian at 9:00 PM?

2.3 Binary and hexadecimal

Binary and hexadecimal are numbering systems commonly used with computers. As you know, at their core, today's computers only know about `1` and `0`, so, deep in their innards, they count in binary. A digit in computer-speak is a "bit" and eight bits make a "byte." A "nibble" is four bits (half a byte). With a nibble, you can count from 0 to 15, so one nibble corresponds to one hexadecimal digit (`0`, `1`, `2`, `3`, `4`, `5`, `6`, `7`, `8`, `9`, `a`, `b`, `c`, `d`, `e`, `f`). A byte can have 256 (16^2) values, and is commonly represented by a pair of hexadecimal digits. For example, this year, I am `29` in hexadecimal, but the jig is up next month, when I turn `2a`. In binary, I'll be `101010`, which is a number of some significance for hitchhikers.[11]

[11] Tip of the hat to Pablo Ortiz, who pointed this out to me several years ago.

2.4 Talking math with aliens

After millennia of floundering with bizarre number systems, in the past 1000 years, the decimal system has finally taken hold in almost all human cultures. What will we do when we meet aliens who have eleven fingers on a hand? What if we need to have a mathematical conversation with birds (base-4, 6 or 8), insects (base-6), spiders (octal), octopi (octal) or fish (?!?)? We would, of course, need to learn how to work in their native numbering system. (Unless they are more advanced than us and are willing to accommodate our preference for decimal numbers). To understand these alternate base numbering systems, we'll need to understand our base-10 positional numbering system and then generalize upon it.

3 Positional numbering

Positional numbering means that the position of the digit in the written number signifies its magnitude in the value of the number. For example, in early grade school, you formally learned (or will learn) that there is a "ones" place, a "tens" place, a "hundreds" place, etc. for a given written number. For example, in decimal, "25" indicates that there are two 10's and five 1's. In hexadecimal, however, "25" indicates there are two 16's and five 1's (decimal 37). Twenty-five consists of one 16 and nine 1's, so in hexadecimal, it is written as `19`.

To understand and generalize these concepts, we need to take a short detour into addition, multiplication and exponentiation.

3.1 Addition

The word for addition is "and" or "plus." One and one equals two, two plus three equals five. Symbolically, we write this as:

$$1 + 1 = 2 \tag{3.1}$$

$$2 + 3 = 5 \tag{3.2}$$

$$a + b = c. \tag{3.3}$$

There is a very important number in addition, and its name is zero (0). Zero was discovered in India between 1600 and 2000 years ago, and also by the Maya (or their predecessors, the Olmecs) in about the same time frame.[12] Zero is the **identity** element of addition. Anything and zero is itself. Very important! Don't forget!

3.2 Multiplication

The word for multiplication is either "by" or "times." Four by eight (a standard American size for plywood in feet) makes thirty-two. Five times five is twenty-five. Symbolically, we write multiplication with $\times$, $\cdot$, or nothing at all:

$$4 \times 8 = 32 \tag{3.4}$$

$$5 \times 5 = 25 \tag{3.5}$$

$$a \times b = a \cdot b = ab = c. \tag{3.6}$$

[12] This is a drastically simplified version of the story of zero. See Robert Kaplan's or Charles Siefe's books on the history of zero for more details.

What *is* multiplication? It is repeated addition. How is two times three equal to six? Usually, we learn this as:

$$2 \times 3 = 2 + 2 + 2. \tag{3.7}$$

If I insert addition's identity into the definition of multiplication, the answer doesn't change, but then we write two times three as:

$$2 \times 3 = 0 + 2 + 2 + 2. \tag{3.8}$$

Now consider the following multiplications:

$$2 \times 3 = 0 + 2 + 2 + 2 \tag{3.9}$$
$$2 \times 2 = 0 + 2 + 2 \tag{3.10}$$
$$2 \times 1 = 0 + 2 \tag{3.11}$$
$$2 \times 0 = 0. \tag{3.12}$$

Not that this is a huge intellectual leap, but I hope this convinces you that $2 \times 0 = 0$. In fact, anything times zero is zero. Maybe you can attribute this to the fact that zero is the identity element of addition, and that multiplication is repeated addition to the identity of addition.

Equation 3.11 points us to the very important number of multiplication, one (1). One has been with us for a long long time. We likely knew one even before we developed writing. One is the oldest of the mathematics' five great numbers[13] and it is the **identity** element of multiplication. Anything times one is itself. Very important! Don't forget!

3.3 Exponentiation

Exponentiation has a phrase instead of a word, and its phrase is "to the . . . power." Two to the fourth power is sixteen. Five to the fourth power is six hundred twenty-five. Symbolically, we write these equations as:

$$2^4 = 16 \tag{3.13}$$
$$5^4 = 625 \tag{3.14}$$
$$a^b = c. \tag{3.15}$$

When the second (superscripted) number, also known as the exponent, is two, we call it the "square" of the first number ("two squared is four") and when the exponent is three, we call it the "cube" of the first number ("two cubed is eight"). To my knowledge, we don't have names for powers of exponentiation higher than 3.

What *is* exponentiation? It is repeated multiplication. In keeping with the relationship between addition and multiplication, let us define exponentiation as repeated multiplication against the identity of multiplication, 1. For example, here is two to the fourth power:

$$2^4 = 1 \times 2 \times 2 \times 2 \times 2 = 16. \tag{3.16}$$

The expression, 2^4 is evaluated by multiplying the identity (1) by two, four times. Where you add in multiplication, you multiply in exponentiation. Where you insert the identity of

[13] In order of discovery, the five great numbers are $1, \pi, 0, i$ and e.

addition in to multiplication, you insert the identity of multiplication into exponentiation. Let's see what happens as we decrease the powers of 2 from 4 to 0:

$$2^4 = 1 \times 2 \times 2 \times 2 \times 2 \tag{3.17}$$
$$2^3 = 1 \times 2 \times 2 \times 2 \tag{3.18}$$
$$2^2 = 1 \times 2 \times 2 \tag{3.19}$$
$$2^1 = 1 \times 2 \tag{3.20}$$
$$2^0 = 1. \tag{3.21}$$

I hope that, for the uninitiated, $2^0 = 1$ is as unremarkable a result as multiplication's $2 \times 0 = 0$. What else could it be? In fact, anything to the zeroth power is one. Even $0^0 = 1$![14] For the initiated (myself included), exponentiation has traditionally been taught simply as repeated multiplication, and argument that the zeroth power of a number is 1 requires going into negative exponents. By thinking of exponentiation as repeated multiplication *against the identity of multiplication*, the evaluation of the zeroth power becomes "intuitively obvious."

3.4 The decimal numbering system, generalized

In the decimal numbering system, the position of the digit in the number tells you its "power of ten." For example, in the number 3243, the left-most "3" is in the thousands place, the "2" is in the hundreds place, the "4" is in the tens place, and right-most "3" is in the ones place. Mathematically this is written as:

$$3243 = 3 \times 1000 + 2 \times 100 + 4 \times 10 + 3 \times 1. \tag{3.22}$$

Using exponentiation, this number can also be written as:

$$3243 = 3 \times 10^3 + 2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0. \tag{3.23}$$

We call the decimal system a base-10 system because, as we read from right to left, the digits are multiplied by successively larger integer powers of 10. For integers, the positional values start at 10^0. It is interesting to note that in Arabic, writing goes from right to left, but in Latin languages, it goes from left to right. Thanks to Fibonacci, we write our numbers in the same way they are written in Arabic. The difference, however, is that we read from the largest (most significant) digit to the smallest, while in Arabic, the number is read from the smallest (least significant) digit to the largest.

To work in a different base numbering system, we replace the 10's in equations like Equation 3.23 with the value of the base, b. For example, in hexadecimal ($b = 16$), the first three places (from right to left) are the ones, the sixteens and the two hundred fifty-sixes. What is decimal 3243 translated into hexadecimal?

$$3243 = 12 \times 16^2 + 10 \times 16^1 + 11 \times 16^0. \tag{3.24}$$

Hexadecimal uses the letters `a` through `f` for ten through fifteen, so we find that decimal 3243 is `cab` in hexadecimal.

[14] Ha ha, pun intended! But, seriously, if you don't believe me, check out Su, Francis E., et al. "Zero to the Zero Power." Math Fun Facts.

When you are working in a given base, whether it is binary, decimal, hexadecimal, sexagesimal, etc., multiplying any number by the base is very, very easy. All you have to do is tack a zero onto the end! For computers, multiplying by 2 (the base of the binary system) simply involves shifting the bits one space to the left and popping a zero into the lowest position! Can you convince yourself that in binary, `101101` + `101101` = `1011010`? You can also rewrite this addition problem as a multiplication problem: `101101` × `10`.

In other words, in any given base, multiplying by the base is easy because the base is always represented by `10`:

$$\texttt{10} = 1 \times b^1 + 0 \times b^0 = b. \tag{3.25}$$

3.5 Example: base-3

After binary, the next easiest base to work in is base-3. The addition and multiplication tables for base-3 are shown in Table 1. All the rules for addition, subtraction, multiplication and division that you've learned for the decimal system also apply to base-3, except that here, only the digits 0, 1 and 2 exist! Written in base-3, the decimal numbers 3, 9 and 27 are written as `10`, `100` and `1000`.

Table 1: Addition and multiplication tables for base-3

+	0	1	2	10
0	0	1	2	10
1	1	2	10	11
2	2	10	11	12
10	10	12	12	20

×	0	1	2	10
0	0	0	0	0
1	0	1	2	10
2	0	2	11	20
10	0	10	20	100

Baseball uses base-3 to record the fractional part of the number of innings pitched by a pitcher. Three outs make an inning, so if a starting pitcher make two outs in the 6th inning before being relieved, his number of innings pitched is 5.`2`. Since there are 3 strikes to an out, would a pitcher pulled after 2 outs and 2 strikes have pitched 5.`22` innings?

3.6 Exercise: base-36

With the ten Hindu-Arabic digits and the twenty-six letters in our alphabet, we can define a base-36 numbering system where `a` through `z` represent the numbers ten through thirty-five. Can you solve this problem?:

```
      d e i r d r e
          p e t e r
        m a g g i e
+           o w e n
-------------------
```

You get bonus points for converting this number to base-3. Although it might be useful to have 18 fingers on a hand, I don't think I would like having to memorize the base-36 multiplication table!

4 Conclusion

The theme of Newton Day this year has been to understand how to work in non-decimal numbering systems. I hope that I have explained this well enough, so that it is conceptually easy.

For my own benefit, I have three take-away lessons from this year's holiday preparations:

1. Mathematics really is the universal language. When I check the time of day, I am connected to the cultures of ancient Egypt and ancient Sumeria through ancient Babylon and Greece. I can't read cuneiform or hieroglyphics, but the 24 hour day with its divisions by factors of 60 puts me in contact with those civilizations and the flow of history from then until now.

2. The decimal system is, in an historical sense, new. It seems so obvious to count by tens, but the Hindu-Arabic decimal system was invented less than 2000 years ago and was introduced to mainstream European culture less than 1000 years ago. Without this invention, it is difficult to image that the European Scientific Revolution, of which Newton was an integral part, could have taken place.

3. $x^0 = 1$. When I learned this mathematical fact, I was given some plausibility arguments, but essentially, I took to it on faith. When I was faced with explaining this to Maggie and Owen, I was horrified at the prospect of asking them to take it on faith. I remember Gian Carlo-Rota saying that one of the roles of mathematicians is to simplify proofs so that their truth becomes obvious. I hope that the method for demonstrating $x^0 = 1$ that I have presented follows in that tradition.

5 Resources

1. Subrahmanyan Chandrasekhar, *Newton's Principia for the Common Reader*, Oxford University Press, 1995. This is much easier to digest than reading *Principia* in Latin or even the Motte translation into English.

2. Charles Siefe, *Zero: The Biography of a Dangerous Idea*, Viking, 2000. A fast, accessible read on the history of zero.

3. Robert Kaplan, *The Nothing that Is: A Natural History of Zero*, Oxford University Press, 2000. A slightly more technical history of zero, also pitched to the general public.

4. Francis Su et al., "Zero to the zero power," Math Fun Facts. I woke up one morning thinking, "Oh no! If 0^0 is not 1, I'm in deep trouble!" Thank you, Francis!

5. R. E. Grimm, "The Autobiography of Leonardo Pisano", *The Fibonacci Quarterly*, 11(1), 1973.

6. Wikipedia. One of the best non-primary sources of information in the world.

7. http://openclipart.org. Clip art for the sidereal time diagram.