Newton Day 2015:
Gyroscope magic

by Peter H. Mao for Margaret and Owen

December 25, 2015

Abstract

For Sir Isaac Newton's 373rd birthday, I built a simple gyroscope with which we will explore some rigid body dynamics. Herein, I calculate the angular velocity required to support uniform precession and apply it to my gyroscope. I estimate the required velocity at the edge of the wheel to be 30 m/s (close to 70 miles/hr).

1 Acknowledgement

Many thanks to Mike Roy for allowing me access to the Caltech Chemistry machine shop. Sadly, the Physics department closed its machine shop earlier this year. I am extremely grateful that Mike allowed me to use one of his lathes to make the axle for the gyroscope.

2 Magic

Consider the following situations:

I. Take a toy gyroscope. Spin up the wheel with the string and set the axis horizontally on the stand. If you've spun up the wheel enough, the gyroscope will stably turn about its pivot on the stand, but will not fall to the ground, even though the center of mass is not directly over the stand. Try the same experiment with a pencil. Why doesn't the gyroscope fall?

II. A pendulum released from a given height will travel in a vertical plane and return (nearly) to its point of release. What happens if the bob of the pendulum is spinning? How fast does the bob need to spin for the pendulum to behave like the toy gyroscope?

3 Excuses and dead-ends

I originally envisioned the theme of this year's Newton Day celebration to center on a simplified model of a gyroscope, which would make the seemingly bizarre dynamics of real gyroscopes more intuitive. So I thought of the wheel and the axle as a collection of point masses, subject to the constraint that the wheel and axle form two rigid objects. Then I dealt with the external force by requiring that the angular acceleration of each point mass be identical. In the end, after all this trouble, I just ended up back at $\tau = \mathbf{r} \times \mathbf{F} = d\mathbf{L}/dt$, which is the standard approach given for the gyroscope in any modern textbook.[1]

So for now, all I can tell you is what you can find anywhere: that angular momentum and torque are vector quantities, and that the component of torque that is parallel (or anti-parallel) to the angular momentum changes the magnitude, but not direction of the angular momentum, while the perpendicular component of torque changes the direction, but not the magnitude of the angular momentum. The conceptual difficulty with torque and angular momentum is that their directions are always perpendicular to the more familiar force and momentum vectors.

Now, having conceded that understanding torque τ and angular momentum $\mathbf{L}$ is the best way to understand the gyroscope, I began to wonder: (1) When did physicists start using vectors? and (2) What did Newton know about rigid body dynamics? The next two topics are entirely tangential to the main topic of the day, but having spent the last two days in these rabbit-holes, I just wanted to share a bit of the fun.

3.1 Vectors in physics

> "Even Prof. Willard Gibbs must be ranked as one of the retarders of Quaternion progress, in virtue of his pamphlet on *Vector Analysis*; a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassman."
>
> Peter Guthrie Tait, *Treatise on Quaternions*, 1890.

> "I think it is practically certain that there is no chance whatever for Quaternions as a practical system of mathematics for the use of physicists...
>
> "It is to Prof. Tait's devotion to his master [Hamilton] that we should look for the reason of the little progress made in the last 20 years in spreading vector-analysis...
>
> "The quaternionic calm and peace have been disturbed. There is confusion in the quaternionic citadel; alarms and excursions, and hurling of stones and pouring of boiling water upon the invading host. What else is the meaning of his letter, and more especially of the concluding paragraph? But the worm may turn; and turn the tables."
>
> Oliver Heaviside, *Nature* vol 47, pp 533-534, 1893.

When did the modern vector formalism with its "dot" and "cross" products enter physics? Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) did not make use of vectors in this way. Leonhard Euler (1707-1783) gave us $e^{i\theta}$, which is one of my favorite tools for two dimensional work, but it is not easy to extend the complex numbers into the third dimension, or higher. On October 16, 1843, after more than 15 years of getting nowhere, William Rowan Hamilton (Tait's master, in Heaviside's words) discovered the quaternions, and with it, a system for manipulating (rotating, translating, multiplying and adding) Cartesian triplets.

[1] I have heard rumors that Feynman had a very nice explanation, but it is not recorded in *The Feynman Lectures on Physics*.

In the 1880's J. Willard Gibbs (1839-1903) (of thermodynamics fame) and Oliver Heaviside (1850-1925) (of much less fame than he deserves) independently developed the modern vector formalism out of quaternions. Heaviside's motivation was to understand James Clerk Maxwell's (1831-1879) electrodynamics and, in fact, it was he who cast Maxwell's equations in their modern form. The exchanges between Tait and Heaviside are legendarily mean-spirited, but by 1910, the vector formalism of Gibbs and Heaviside had won out over quaternions.

Given the history of vector analysis, the modern pedagogical explanation for the gyroscope cannot be more than $\sim$ 125 years old. It would be interesting to see how rigid body dynamics were handled in physics education before Gibbs and Heaviside, but I did not have time this year to delve into that subject.

Regarding the quaternions, their banishment from the standard physics curriculum was so complete that I knew nothing about them until Kristin Madsen introduced them to me when we were building NuSTAR, via B.K.P. Horn's classic paper on absolute orientation.[9]

3.2 Newtonian rigid body dynamics

> "...and the total annual precession, arising from the united forces of both, will be 49'',48''' the quantity of which motion agrees with the phenomena."
>
> Newton, *P*rincipia Mathematica, Book III, Prop 39.

> "Altogether, I am convinced that Newton himself could not have really believed that he had quantitatively accounted for the precession of the equinoxes."
>
> S. Chandrasekhar, *Newton's Principia for the Common Reader*, 1995.

The precession of the the Earth's polar axis relative to the fixed stars was known since long before Newton, and the rate of precession was known to be about 50'' per year (corresponding to a 26,000 year period). In Book III of Principia Mathematica, Newton applies his mechanics to the precession of the Earth's polar axis ("the equinoxes").

Chandrasekhar points out that Newton "cooked" his numbers quite a bit to arrive at the known result, but gives him credit for the key concepts of moment of inertia, angular momentum and torque. Chandrasekhar also credits Newton with "the notion of circulation some two hundred years before Lord Kelvin to whom it is commonly credited."[[2], p 471]

In trying to sort out the circulation business, I came across some papers by Geoffrey J. Dobson, a historian of science, in which the author disputes the validity of Chandrasekhar's attributions of Newton's contributions. In essence, Dobson claims that rigid body dynamics was not properly sorted out until long after Newton.

For lack of time, I have to leave this controversy where it stands.

4 Nuts and bolts

How fast does the wheel in Figure 1 need to spin so that it precesses rather than pendulates? Recall that[2] the angular velocity of precession Ω is related to the geometry and state of the

[2]This is secret code for "I will neither prove nor derive the following."

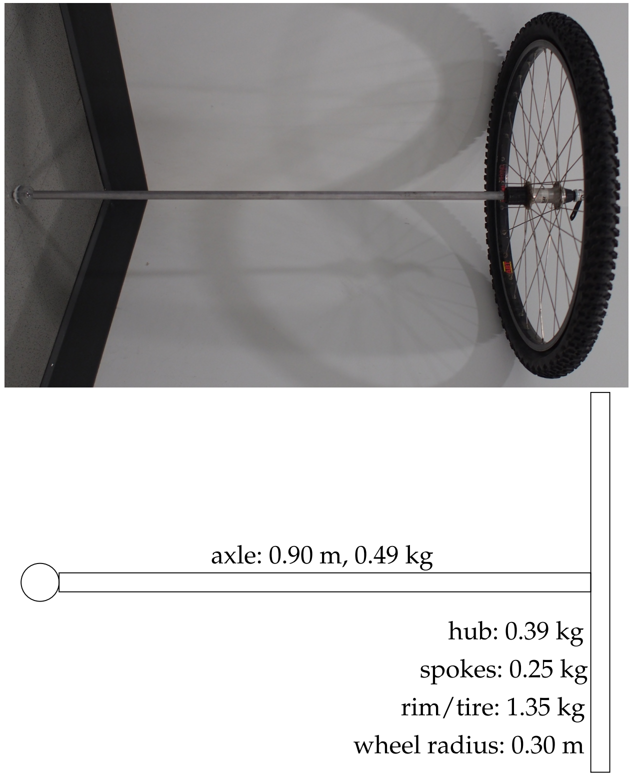

Figure 1: The gyroscope I built for Newton Day and its key physical parameters.

gyroscope (its mass M, the location of its center of mass r_{cm}, the moment of inertia of the wheel I_0, and rotational velocity of the wheel ω) by

$$\Omega = \frac{Mgr_{\text{cm}}}{I_0\omega}. \tag{4.1}$$

Check any physics text for the derivation.

When you let go of the gyroscope, it will not precess right away. The translational kinetic energy of precession has to come from somewhere, and it's not the spinning wheel. The center of mass of the gyroscope will fall through a height of Δh, and the reduction in potential energy will go into the translational kinetic energy of precession. If we define the position of the pivot as our origin, and the angle off vertical as ϕ, then the height of the center of mass is related to the angle off vertical by

$$h = r_{\text{cm}} \cos\phi \tag{4.2}$$

and a change in the height is related to a change in the angle by

$$\Delta h = -r_{\text{cm}} \sin\phi \Delta\phi. \tag{4.3}$$

Now we can estimate how much (in angle) the gyroscope will fall as it begins precessing by equating the loss in potential energy to the gain in translational kinetic energy:

$$
\begin{aligned}
-\cancel{M}g\Delta h &= \frac{1}{2}\cancel{M}(r_{\mathrm{cm},\perp}\Omega)^2 && (4.4)\\
g\cancel{r_{\mathrm{cm}}}\,\cancel{\sin\phi}\,\Delta\phi &= \frac{1}{2}r_{\mathrm{cm}}^{\cancel{2}}\sin^{\cancel{2}}\phi\left(\frac{Mgr_{\mathrm{cm}}}{I_0\omega}\right)^2 \\
\Delta\phi &= \frac{gr_{\mathrm{cm}}^3\sin\phi}{2(I_0/M)^2\omega^2} && (4.5)
\end{aligned}
$$

With a simple re-arrangement, we can also use this equation to estimate how fast the wheel needs to spin in order to precess.

$$
\omega^2 = \frac{gr_{\mathrm{cm}}^3\sin\phi}{2(I_0/M)^2\Delta\phi} \tag{4.6}
$$

The physical parameters we need to put numbers to our problem are the distance of the center of mass to the pivot and the moment of inertia of the wheel. I get $r_{\mathrm{cm}} \approx 0.8$ m and $I_0/M \approx 0.05\mathrm{m}^2$. Do you get comparable numbers?

Let's figure out how fast the wheel has to spin so that for $\sin\phi = 1$, $\Delta\phi = 0.1$.

$$
\begin{aligned}
\omega^2 &\approx \frac{10\mathrm{m/s}^2 \times (0.8\mathrm{m})^3 \times 1}{2\times(0.05\mathrm{m}^2)^2\times 0.1} && (4.7)\\
&\approx 10^4 \\
\omega &\approx 100 \text{ radians per second} && (4.8)
\end{aligned}
$$

My estimate of the required angular velocity is about 70 miles per hour at the rim of the wheel. Spinning up the gyro with a bicycle wheel-to-wheel, we may get to about 30 mph, turning the crank by hand. If, instead, we use the bicycle wheel to turn the free-hub body, then we will have $\sim 20\times$ mechanical advantage over the wheel-to-wheel situation, so we may yet see precession from my gyroscope on Newton Day. On paper, with a $20\times$ mechanical advantage, we can spin the wheel up to 600 miles per hour, but safety concerns will preclude that.

Stay tuned.

4.1 Bonus question

Suppose I didn't give you Equation 4.1. Could you still come up with an estimate for the angular velocity required for precession?

A Simple gyroscope model, Chris Mindas

The primary difficulty in forming a simple or intuitive model of a gyroscope is that the constraint equations are very difficult to handle (outside of Lagrangian mechanics, which eventually leads to the Euler equations of motion, which are neither simple nor intuitive). Therefore, in order to build a simple model, you need to relax the constraints. Two approaches come to mind, a coarse discretization of time and a "flexible disk" model.

A.1 Discrete time model

Consider the simplest possible top which consists of a single mass mounted on an axle (see the fairly horrible drawing below labeled figure 2). This top has no symmetry, and as a result the equations of motion will be complicated. In order to create a simple picture, consider a model where the mass, when viewed straight down the axle, can only be in one of four possible positions (Up, Down, Left, or Right).

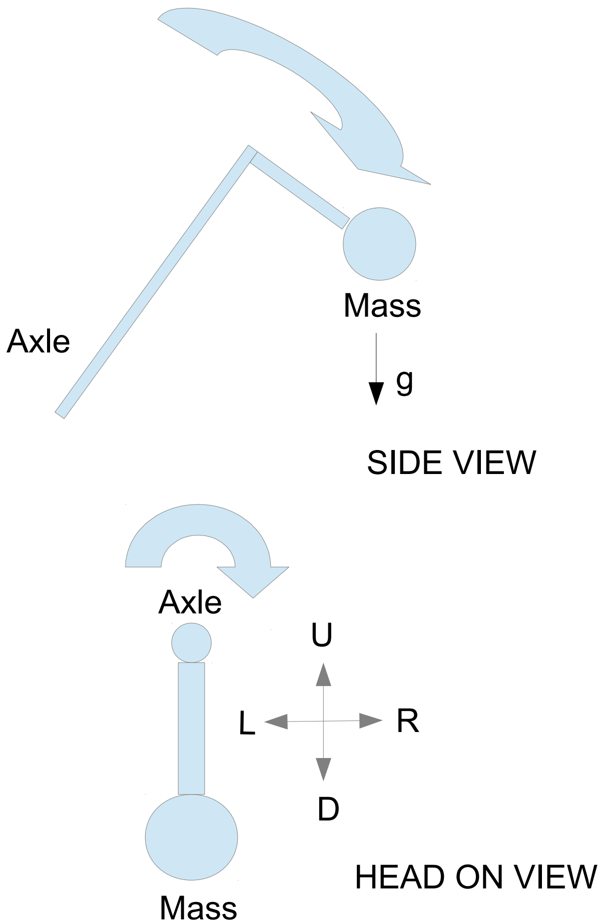

Figure 2: Simple drawing showing the top described in Method A.

At $t = 0$, the mass is in the D position. The top begins falling due to gravity and the mass acquires a component of velocity along the D direction. At $t = 1$, the mass is in the L position. The velocity which the mass acquired at $t = 0$ now points along the L direction (since the mass rotated $90°$ around its axle). This causes the top to precess to the left. At $t = 2$, the mass is in the U position. The velocity points in the U direction and causes the top to rise counteracting the fall it experienced at $t = 0$. Gravity points in the opposite direction to the velocity of the mass, which is now reduced to zero. At $t = 3$, the mass is in the R position, and the velocity of the mass is zero.

Of course, you can also start with a small upward velocity at $t = 0$ to make $t = 1$ and $t = 3$ symmetric (in this case, the top precesses to the left at both $t = 1$ and $t = 3$). I ignored downward velocity components acquired at $t = 1$ and $t = 3$ (which either slow down or speed up the rotation rate of the mass about the axle).

From this simple model (and from the lack of symmetry), it appears that this top will not have a stable precession point. It will nutate and the angular velocity of the mass around its axle will change as it precesses.

A.2 Flexible disk model

Consider a flat disk which is made out of a flexible material (like rubber) with an axle attached to its center point. In this case, the top is perfectly symmetric and we have relaxed the constraints by using a non-rigid material.

At some point, we start the disk spinning and release the top. Initially, gravity causes the top begin falling. What happens to the points on the edge of the disk? Inertia will tend to keep the points on the edge in the initial plane of the disk. However, since the axle is tipping over, the plane of the disk begins to rotate (at least near the center where the axle is attached). As a result, the edges of the disk bend.

Because the disk is rotating, the bending is asymmetric. One side bends towards the central pivot, and the other side bends away. The restoring force generates a kick which causes the top to start precessing.

This is really hard to see without a drawing (see figure 3). Unfortunately, it is also really hard to see by looking at this bad drawing as well. In figure 3, the horizontal disk represents the initial state of the rubber disk. If the disk wasn't attached to an axle, angular rotation would carry the red and blue dots clockwise and nothing else would happen.

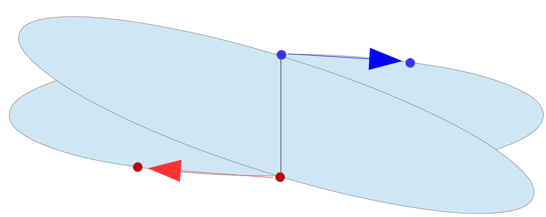

Figure 3: Inertia of the flexible disk.

What happens if the disk is attached to an axle? If the disk were rigid, it would move into the rotated position shown in the figure. Since it is elastic, the center of the disk will move into the rotated position due to the force exerted by the axle. At first, the edges of the disk simply stay in the horizontal plane causing the disk to bend. From the drawing, we see that the blue point is bent away from the axle while the red point is bent towards the axle. The force exerted by the asymmetric bend is the precession force.

Now that the top is precessing, we can use the same argument (applied along the precession direction) to show that the bottom of the disk bends away from the axle, while the top of the disk bends towards the axle. This generates a force which opposes gravity and prevents the top from falling. At this point, the top is precessing around the central pivot, but remains at a constant height above the ground.

References

[1] John C. Baez, The Octonions, 2001.

[2] Subrahmanyan Chandrasekhar, *Newton's Principia for the Common Reader*, Oxford University Press, 1995.

[3] Michael J. Crowe, *A History of Vector Analysis*, Dover, 1994.

[4] Goeffrey J. Dobson, *Arch. Hist. Exact Sci.*, Vol 53, pp 125–145, 1998.

[5] Goeffrey J. Dobson, *Arch. Hist. Exact Sci.*, Vol 53, pp 577–597, 1999.

[6] Goeffrey J. Dobson, *Arch. Hist. Exact Sci.*, Vol 55, pp 345–363, 2001.

[7] Richard Feynman, Robert Leighton and Matthew Sands, The Feynman Lectures on Physics Vol I, Chapter 20-3, 2013.

[8] Oliver Heaviside, Nature vol 47, pp 533-534, 1893

[9] Berthold K. P. Horn, *J. Opt. Soc. Am. A*, Vol 4, pp 629–642, 1987.

[10] D. Kleppner and R. Kolenkow, An Introduction to Mechanics, McGraw Hill, 1973. [3]

[11] Paul J. Nahin, Oliver Heaviside: Sage in Solitude, IEEE Press, 1988.

[12] Isaac Newton, Principia Mathematica, translated by B. Motte, 1729.

[3]Despite my initial misgivings about this book when I used it in my freshman year of college, I have come to deeply appreciate the clarity and completeness of its exposition on the basics of mechanics, especially rigid body mechanics.

Figure 4: Maggie & Owen, June 2015.