Newton Day 2019:
Surface waves

by Peter H. Mao for Owen and Margaret
with contributions from Roger Smith and Randy Patton

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Abstract

For Sir Isaac Newton's 377th birthday we will explore surface waves. Surface waves (the water-air interface in bathtubs, lakes and oceans) are an excellent example of dispersive waves that we encounter in everyday life. For this year, I've pulled together stories and contributions from friends that have intimate knowledge of surface waves in different contexts.

1 Preface

This year's topic was suggested late last year by Chaz Morantz, one of my colleagues at JPL with whom I worked on Prime Focus Spectrograph. I recalled that surface waves was one of the topics covered in the classes "Applications of Classical Physics" (Ph136) and "Order of magnitude physics" (Ph101) that I took in graduate school in the mid 90's. Looking at the course materials from those classes for inspiration, I found that the best exposition for surface waves had already been written, in Sanjoy Mahajan's 1998 Caltech Physics PhD thesis. Mahajan uses dimensional analysis to conjure the functional form of the dispersion relationship (frequency vs. wavelength) of surface waves from deep to shallow water and from surface tension-dominated to gravity-dominated restoring forces. Since I can do no better with the exposition, this year, I requested contributions from friends with intimate knowledge and experience with surface waves.

We'll start with Roger Smith's account of his eventful return from a field expedition in Maningrida in the remote Northern Territory of Australia. I first heard this fantastic story over lunch a few years ago. The prominence of surface waves in his telling stuck with me, so I asked if he could write it up as a contribution to this year's Newton Day paper.

After Roger's story, I'll recap the basic results on surface waves with a minimum of exposition, since you are best off reading Mahajan's work. From there, I'll try to make some sense of surfing from the bits of knowledge that Emad Hashim, Ryan Randall, Chaz Morantz, and Coleman Richdale saw fit to divulge.

We'll finish with Randy's write-up on the measurement of coastal water depths using sythetic aperture radar imagery. The method he describes is a direct application of the surface wave dispersion relationship explained in Mahajan's thesis to planning for a propsed extension of Camp Schwab in Okinawa, Japan.

2 The return trip from Maningrida by Roger Smith

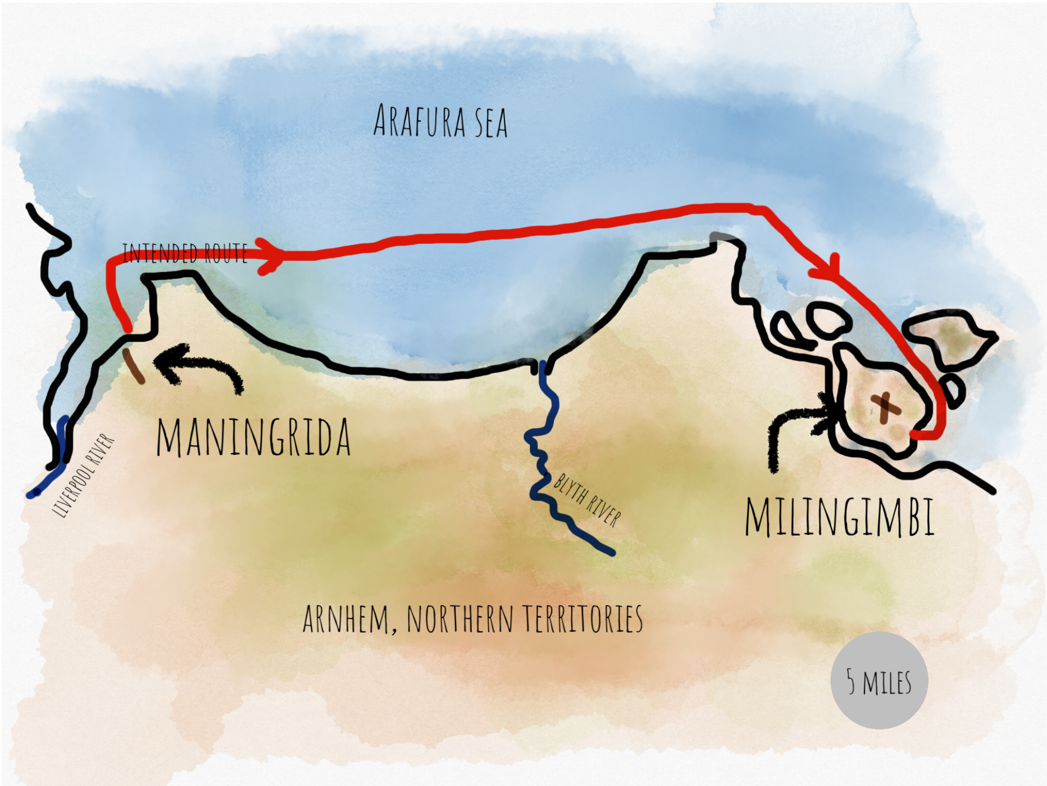

Figure 1:

It was 1974. Six of us set out from Maningrida to meet our colleagues on the island of Milingimbi on the coast of northern Australia. The river boats, aluminum dinghies really, each with one 40 HP outboard motor were ill equipped to move quickly in the choppy Arafura sea which separates the tropical wetlands of Arnhem Land from New Guinea. Four of us were students from Sydney University who needed a ride home after the Darwin Airport (and most of Darwin itself) – about 200 miles of impassable dirt tracks to the west – had been destroyed by Cyclone Tracy.[1] We were to meet the university's four seat Cessna on Milingimbi where the comparatively rocky runway would allow a plane to land since Maningrida was still too muddy.

We set out not that long after lunch, but the going was slow on the choppy ocean. Though I was only 19, I had spent my teens racing small skiffs on Sydney harbour and had been out on the open ocean in larger yachts a few times. I might have had the most boating experience in the group – probably more than the professor of geology who was to be our pilot home and who was pressing to leave in spite of the foreboding afternoon storm clouds. The biologist

[1] December 21–26, 1974.

who led the crocodile research (capture and release) program we had been working for over the last two months was a man of considerable talent but too much confidence to foresee events to come. I was too junior to be taken seriously.

We took the shortest route from the headland at the mouth of the Liverpool River to the northern point 30 miles East which we had to round before heading 15 miles south to Milingimbi Island. This took us more than 6 miles out to sea, so the low-lying land fell out of view.

Soon we were too busy to notice. The wind steadily increased. This was no 20-minute squall caused by one of the many thunderstorms which marched across the vast ocean. The sky grew dark and steely grey and the winds built and built. We were nosing hard into the ever rising rollers. The boat I was in carried the 44 gallon fuel drum standing vertically (not sure why) and lashed down tightly; the extra weight made us slower than the other boat. Charles, the professor/pilot of the other boat, forged well ahead and we lost sight of him increasingly often as one or both of us disappeared into the now deep troughs between the endless rollers. After a long time he turned back to look for us and we transferred a passenger to his boat to even the load and went forward keeping closer together.

Soon, the winds blew hard and waves grew alarmingly steep. If you examine the satellite map (made years later) you can see a light area of sea bed almost certainly due to silt build up from the Blyth River. Something I knew then from the first-year physics class that my fellow student, Powell, and I had just completed, was that wavelength depends on depth. As the waves move from deep water to shallow they get steeper and wavelength gets shorter. Waves are moving slower and "piling up," I guess. The crests were breaking along great lengths with a few feet of foam on top and the peaks were as a high as houses – probably 15 feet. This was moving from an adventure to a looming catastrophe but we were 2/3 of the distance or more to the far point where we could turn and run with the seas which were pushing us towards the coast and shallower water. That way lay shipwreck in one of the least accessible places on the planet.

We pushed seaward but confronted with breaking waves we could not penetrate we had to run along the faces of the waves picking up impressive speed in spite of our heavy load. With each opportunity to head towards deeper water where the waves would be less steep, we would power up the face of the waves, but we were frequently forced to run with the waves until we could find a gap in the foamy crests. Every time we surfed down the face we were speeding towards the coast. It was still 5 or 6 miles away, but that direction, took us into the shallower water with more dangerous conditions. We were barely holding course on average. The wind now blew so hard that it tore sheets of spray off the surface of the water and stung our faces so that we had to turn away to let our rain gear protect us from the impact. Communication was only possible by shouting into one's hands cupped over the ears of the listener. I had been in winds like this before but only briefly and only near land.

With each wave we crested we pushed towards deeper water with increasing urgency now using the velocity gained running with the waves to accelerate up the faces and launch ourselves over the crest. Soon we learned to crouch with legs bent as the boat went slightly airborne and then splat down hard as it landed. These were river boats with flatter bottoms designed to plane over smooth surfaces at speed and not vee hulled to cushion the landing, so it was a rough ride.

Rough not only for us, but also for the fuel tank. After many such launches over crests the fuel line slid between the drum and the deck and the drum sliced it open on a hard landing. Thankfully, it did not catch fire but the fuel-starved engine soon quit, so we had no way to steer the bow into the next foaming crest, and the broadside swamped the boat. Graham, the crocodile biologist and trapper, knew what to do – he quickly found a replacement hose, struggled frantically to connect both ends, primed the line, and tried to restart the engine. Meanwhile I ditched the food from the cooler and used it as a gigantic bucket to bail the boat and was making quite good headway as Graham got the still swamped boat pointed towards the sea. Soon, I had the boat bailed out and we were back in business.

Well, we would have been, but amid the howling wind, our colleagues had been so concerned by our plight that they forgot to keep bow to sea and took a roller over the stern. So now we were afloat and they were swamped but upright. We tossed a line that they tied off and we succeeded in pulling their bow around to face the waves. Flooded with water, they were a dead weight – the first big roller that passed beneath us pushed us seaward as it carried them back. The line strained, parted at their end, and snapped back like an elastic band. Falling into the water, the loose line instantly wound tightly around our propeller. Disabled once again, the next wave swamped us and capsized the other boat. The following wave did the same for us.

It was too heavy to right. The students from the other boat swam across to us as we rigged ropes across the bottom of the upturned hull so we could sit on the hull and hold on as waves broke over us. The last light faded as we clung to the turtled boat. Mercifully, the water was warm, but we had no desire to swim in it for longer than necessary, knowing that there were plenty of creatures that might kill you: Tiger sharks, hammer heads, saltwater crocodiles and box jelly fish, to name a few. We were too preoccupied to notice the fading light and after a while, in the darkness, the wind abated. In time the seas grew calmer.

Hours wore on and while one of the students was silent and petrified with fear, my friend Powell and I talked with the biologist "strategizing" as if thinking clearly would offer a solution. Was there a current? Probably, though after some time the sea became impossibly smooth and the sky now clear threw reflected starlight across the water. Were we traveling nearer the coast or nearer New Guinea?

A long time went by and we fell mostly to silent thinking, dozing but unable to lie down.

Glug . . . glug.

Glug . . . glug.

Since being a small child playing in rowing boats I had always been mesmerized by the magical sound or water slapping gently on the boat hulls. My parents would rent a liveaboard motor boat at least once a year for a family vacation and moor in quiet inlets of the Hawkesbury River and Broken Bay, surrounded by national parks where silence was only broken by the calls of magpies, crows, and cockatoos. By night, I would listen to the rhythm of the *slap slap* of the small waves rippling across the water. Later, sailing would be all about the rhythm of bouncing over the waves in the harbor. With all that experience on the water, I developed an instinct, a feeling for whats normal.

So long into the night I had a funny feeling something was odd and couldnt quite identify what it was at first. Then I realized.

Glug-glug . . . glug-glug . . . glug-glug.

The tiny waves on the glassy sea were speaking to me. They were faster.

SHALLOW WATER !!!

Could it be? I felt it necessary to explain to my fellow castaways that I was not mad but was “testing a theory,” and jumped overboard to make my point.

Yes, only chest deep. Even better it was a lovely soft sandy bottom. A beach nearby, maybe. Suddenly they were all in the water dragging the boat a few feet to be sure not to loose it in the inky darkness but soon it was clear the direction of the sloping bottom. In no time we stood on dry land. More accurately we stumbled up the beach. I found a slight depression that shielded me from the breeze. In moments I was asleep from exhaustion.

The sun rose to reveal blue skies and an endless white beach as beautiful as any on earth. It was strewn with numerous seashells nearly as a large as my head, just lying about as if no man had ever come here before.

Figure 2: Bailer shell. Credit: Didier Descouens [CC BY-SA 3.0].

So what happened? Powell and I continued to dream up solutions to the most immediate problems. While our colleagues looked for any useful flotsam from our “shipwreck,” we took some of the Bailer shells which made excellent tools for scooping sand. Finding a depression behind a low dune we dug to the water table which we found to be only about 5 feet down and were relieved that our hypothesis was correct – that the rainwater seeping into the ground was flowing towards the ocean and would be fresh in spite of its proximity to the sea. With water to drink, we survived fine for the 36 hours during which the crew of the plane flew along the coast to look for us. The next day a boat arrived from Milingimbi crammed with locals eager to see what trouble the white people had gotten themselves into this time.

Several of them swam ashore through the large surf which had been absent when we arrived. They showed us the way overland taking a shortcut to the sheltered side of the promontory, where the fishing boat met us and ferried us all back to the “mission” on Milingimbi. Some kind souls there offered us a sandwich and drink, and within the hour Powell and I climbed aboard the Cessna, with Charles taking the pilot seat and one of the university staff returning with us as copilot. Flying into the night was eerie as there was no light on the ground. We flew by stars and compass hoping for the lights of Tennant Creek to appear over the horizon. (GPS had not been invented yet and we were out of range of radio beacons.) At one point we thought we had found the light of the only building on the entire map sheet. It was labeled "homestead," but it turned out to be the headlights of truck on a lonely road.

Eventually we found Tennant Creek, a big relief given the alternative. We piled into a motel room and took a shower but climbed back out into shorts and salty tee shirts. We had no changes of clothes and no wallets. The pilot had a credit card. Crossing Australia from North to south the next day, we made refueling stops at Oodnadatta and Broken hill. Each landing, I watch the wobble in the rear right wheel which I described to Charles. "We'll fix it when we get home," he declared. This from the man who led us to sea in river boats, with a storm brewing.

When we eventually made it to Bankstown Airport in Western Sydney, it took some time to find the air strip among the many rows of lights, but after several discussions with the control tower who kept the airport open for us past the midnight curfew, we lined up to land. As we rolled down the run way, the wheel wobbled wildly and then seized. The plane skidded and spun around coming to halt on the side of the runway facing in the wrong direction.

I'd had enough adventures for the day. Climbing out of the cockpit in the glare of the runway lights, in my head, I could hear the Pink Floyd soundtrack with crashing aircraft[2]. I turned back and declared, "Thank you, Charles, but I'll walk from here." Then I set off to walk down the line of those runway lights and back to the hanger in what was now a silent airport, closed for the night.

Someone from the biology department was there and drove us home. At 3 AM, I rang the doorbell of my parents' house, with quite a story to tell.

They had gone away on vacation.

[2] "On the Run" from Dark Side of the Moon, 3:02 minutes in . . . listen to it on Spotify!

3 Making waves

Can you go surfing on a lake? Generally, no. Not even on really big ones, like Lake Michigan. How about bigger bodies of water? The Caspian Sea? The Black Sea? The Mediterranean? The Gulf of Mexico? The Carribean? The Mediterranean may be the lower limit on water body size, but it is not known primarily for its surf spots. The Carribean islands certainly have well known surf locations, but they are almost all on the Atlantic side of those islands. Comparing the big oceans, the Pacific Ocean generally appears to be more surfer-friendly than the Atlantic or Indian Oceans. Size appears to matter, but why?

The waves that we encounter on beaches start from storms over the ocean, like the one Roger got caught in back in 1974. If you watch a lake surface on a gusty day, you will see that the wind imparts short, choppy ripples onto the water's surface. The same interaction is taking place in storms over the ocean; however, these storms cover large areas of water ("fetch") and can last for several days (or weeks) until they make landfall and dissipate. If the wind direction is steady for more than a day[3], the phase velocity of the waves will begin to approach the mean wind speed, thus creating the long swells that surfers look for. Larger bodies of water can support larger storms that last longer because the storms don't encounter land as quickly as they do on smaller bodies of water.

Surfers generally look for long (15-20 s) period waves. Waves need to have a critical steepness in order to be surfed, and for a given steepness, longer waves can support taller waves. Large bodies of water are also more amenable to sorting out long waves from the short ones for surfers, by giving the storm-generated waves a long enough run from the disturbance site to the beach for the long waves to get ahead of the shorter waves.

4 Terminology and basic surface wave results

Waves are periodic both in time and space. When we discuss waves, we describe the temporal characteristics by period (T), frequency (f or ν) or angular frequency (ω), and we describe the spatial characteristics by wavelength (λ) or wavenumber (k). Of these terms, angular frequency and wavenumber are the least familiar to people outside of technical fields. These two terms exist primarily because they make wave equations mathematically "clean" – a wave described by

$$\eta(x, t) = A\cos(\omega t - kx) \tag{4.1}$$

has no messy 2π's running around to lose track of when ploughing through algebraic manipulations. So when you encounter these terms, remember that ω is just like frequency, but in units of radians per unit time (instead of cycles), and think of k as the spatial frequency, but in units of radians per unit length (and not cycles per unit length). The relationship among the temporal terms are

$$f = \frac{\omega}{2\pi} = \frac{1}{T}, \tag{4.2}$$

[3]In *Wind Waves: Their generation and propagation on the ocean surface*, Blair Kinsman estimates that it takes about 42 hours for storm winds to fully develop the long waves that match the average wind speed.

and for the spatial terms, we have

$$\lambda = \frac{2\pi}{k}. \tag{4.3}$$

Surface waves are generally travelling waves, so we also need terms to decribe the velocity of these waves. Referring back to Equation 4.1, the peaks of this function occur whenever the phase, $\omega t - kx$, is an integer multiple of 2π. As time advances, the peaks move in the $+x$ direction with velocity

$$v_\phi = \frac{\omega}{k}. \tag{4.4}$$

Naturally, we call this velocity the "phase velocity." This is the relevant velocity when one is trying to catch a wave. The other velocity that comes up in discussions on waves is the group velocity. This is the velocity at which energy from the wave is propagated across the surface of the water. A clear explanation of group velocity is given in Section I.48-4 of the Feynman Lectures, and a more detailed exposition can be found in Lighthill's *Waves in Fluids*. Deferring to the exposition in those texts, the group velocity is calculated from ω and k as

$$v_g = \frac{d\omega}{dk}. \tag{4.5}$$

For nondispersive waves, such as light in vacuum, $\omega \propto k$, so group and phase velocity are identical. For dispersive waves, such as surface waves in deep water, the relationship between ω and k is a bit more interesting. If you have not already done so, now is a good time to download Sanjoy Mahajan's 1998 Caltech Physics PhD thesis and read through Chapter 6. There you will find his excellent derivation of the deep and shallow gravity-restored surface wave dispersion relationships from dimensional analysis. For reference, the deep water dispersion relationship is

$$\omega = \sqrt{gk}, \tag{4.6}$$

where g is the acceleration due to gravity at the surface of the Earth; the shallow water relationship is

$$\omega = \sqrt{gh}k, \tag{4.7}$$

where h is the depth of the water. As Mahajan points out, when we speak of "deep" and "shallow," it is relative to the wavelength of the wave under scrutiny. The product hk is our guide for whether we are in the deep ($hk >> 1$) or shallow ($hk << 1$) regime.

In Figure 3, I've plotted the phase velocities of several fixed-period waves at depths ranging from ankle-deep to the mean depth of the ocean. Many interesting features of surface waves are encoded on this figure. In deep water, long waves, be it period or wavelength, travel faster. Another way to read the deep water dispersion relationship (Equation 4.6) is that the wavelength quadruples for each doubling in the period. Because long waves travel faster, when there is a storm on the other side of the Pacific, the first waves to reach us in Santa Monica are the longest waves.

As the waves enter shallow water (to the left in the figure), their velocity drops. In this regime, the velocity is governed by the depth of the water rather than the length of the

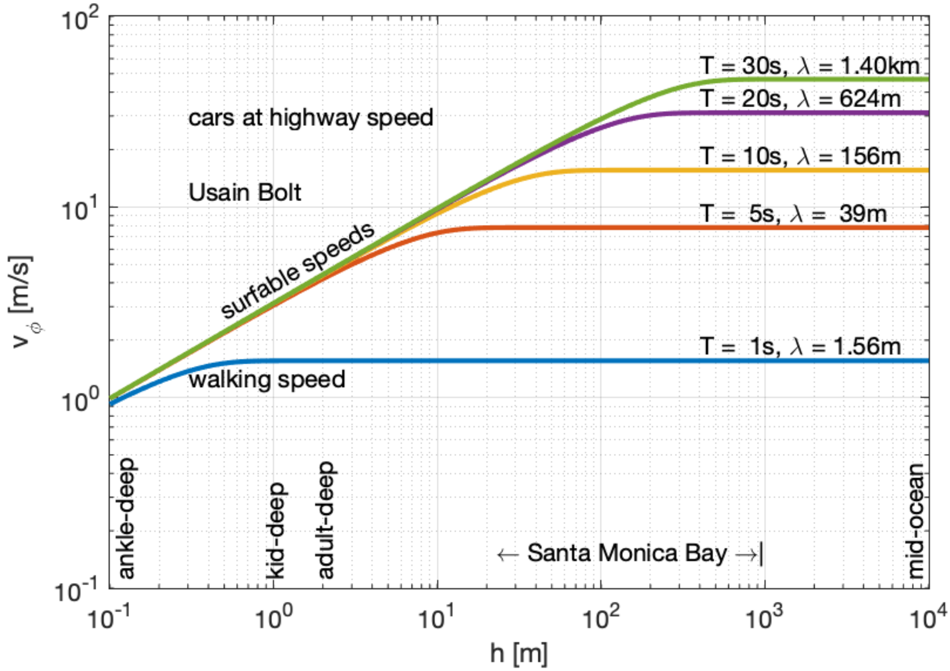

Figure 3: Phase velocity (v_ϕ) of 1-30 second period waves over water depths (h) from 10 cm (ankle-deep) to 10 km (mean ocean depth). The wavelength (λ) for each case is the deep-water wavelength. As the waves approach shore, their wavelength remains proportional to their phase velocity.

waves. For longer waves, the drop in velocity occurs sooner, in deeper waters. For the shortest waves indicated in the figure (with 1 s period), the velocity does not drop from its deep water speed until it reaches knee-deep water. You've probably noticed that waves always approach the beach nearly head on, no matter where you are (see Figure 1 in Randy Patton's section). The slowing of the wave in shallower water causes the wavefronts to turn in the direction of the gradient – classical refraction at work. In the same way, seamounts act like convergent lenses and underwater canyons act like divergent lenses on the waves that are long enough to "see" them.

Both the group and phase velocity can be extracted from the dispersion relations in Equations 4.7 and 4.6. In shallow water, the group and phase velocities are identical ($\sqrt{gh}$), while in deep water, the phase velocity ($\omega/k = \sqrt{g/k}$) is twice as fast as the group velocity ($d\omega/dk = \frac{1}{2}\sqrt{g/k}$). An observable consequence of the deep-water dispersion relationship is that if you try to follow the crest of a deep-water wave, eventually it will disappear off the front of its swell, but another crest will develop behind it. A good place to see this is from the palisades in Santa Monica to the north of the pier.

The basic lesson on surface waves is that longer waves move faster in deep water and that all waves move at the same speed in shallow water at the common speed of $\sqrt{gh}$.

5 Surfer lore

Let's see what surfer lore we can explain with our understanding of small-amplitude gravity waves.

5.1 The incoming tide has stronger waves than the outgoing tide

I heard the claim that waves on the incoming tide are bigger than ones on the outgoing tide from both Emad and the JPL surfers. Through the summer months, Emad takes his family to the beach at the south end of Santa Monica on a weekly basis to teach his kids how to surf (his daughter, Iwa, is in the same grade as Owen). This is highly credible information because small kids are highly sensitive wave-amplitude detectors, and his kids often find the incoming tide too strong to surf in, while the outgoing tide is just right for them. The problem, however, is that the hand-waving explanation that the tide is adding or subtracting energy from the waves doesn't make sense. The tides are on an 11 to 12 hour period, so the incoming tide takes an average of 6 hours to come in, while the waves are arriving at a frequency of several per minute. The time scales are just too far apart!

I thought about tackling this problem, but fortunately, Coleman found a reference that seems to explain the phenomenon[5]. This turns out to be well beyond the scope of small amplitude gravity waves. The paper is by M.A. Davidson, T.J. O'Hare and K.J. George from the University of Plymouth in the UK. Using almost eight years worth of tide and wave data at Perranporth, Cornwall, UK, they found that the wave height is maximal "1 hour and 6 minutes before high tide," confirming, at least at Perranporth, that objective data supports this bit of surfer lore. Considering a multitude of possible causes for "tidal modulation of incident wave heights," they conclude that "contratidal flow" due to the receding tide dissipates the energy of incoming waves.

5.2 Time between sets is proportional to deep water swell amplitude

Waves come in sets. It's an observational fact that all surfers know. What surfers call "sets," physicists call "groups." The set moves at the group velocity, while the waves that are easily discerned move at the phase velocity. The physical basis for sets is additive interference, as developed in Feynman[4], but real ocean waves are certainly the result of more than two frequencies of waves.

Questions like, "How many waves do you typically see in a set?" and "Which is the best wave to surf?" will not get exact answers. For the first question, I've gotten reponses that range from "a few" to "some," and for the second question, the general consensus seems to be to get one of the early waves. So I was surprised to hear from Chaz that one of the relations he uses is that time between arrival of sets in minutes is approximately equal to the amplitude of the deep bouy measurements in feet (peak-to-peak, or half that, he didn't say). His claim is that when the waves are really big, you have to wait a long time between the good sets (and the sea is relatively calm between them!). From a physics standpoint, I have no idea what to make of this relationship. The only thing to do is to go out gather data.

5.3 Use the product of deep water amplitude and period to determine "good" surfing conditions

Here is another tidbit from Chaz. He mentioned that when he's checking the surf conditions, he is looking for the product of amplitude (A) and period (T) to be "large." Clearly both factors matter, but why take just the product? The naive physics view is that the energy in the wave scales like A^2, so shouldn't he be looking at A^2T instead of AT?

Let's take a more careful look at the energy in a wave. Since these are gravity waves, we know that gravity ($g[\mathrm{L}/\mathrm{T}^2]$)[4] and density ($\rho[\mathrm{M}/\mathrm{L}^3]$) will factor into the expression. The remaining length scales are the wavelength (λ) and amplitude (A) of the wave. Amplitude figures into the potential energy of a fluid element at the crest of a wave – mgA, which when integrated over the height of the wave, gives $\frac{1}{2}mgA^2$. Therefore, we expect an expression like $g\rho A^2$ to be related to the energy of the wave. Looking at the units of this quantity, we find

$$\left[\frac{\mathrm{L}}{\mathrm{T}^2}\right] \times \left[\frac{\mathrm{M}}{\mathrm{L}^3}\right] \times \left[\mathrm{L}^2\right] = \left[\frac{\mathrm{M}}{\mathrm{T}^2}\right]. \tag{5.1}$$

Energy has units $[\mathrm{ML}^2/\mathrm{T}^2]$, so we recognize g^2 as an energy per unit area, i.e., the surface energy density of the wave. If we then multiply λ into the surface energy density, we find that $g\rho A^2\lambda$ is a measure of the energy per unit transverse length (width) of one cycle of the wave. Just as Equation 4.6 gives us $\omega^2 \propto k$, using Equations 4.2 and 4.3, we have the deep-water relationship $T^2 \propto \lambda$. Putting it all together, we find that the energy per unit width of one wave cycle is proportional to $(AT)^2$. Coming back to Chaz's hueristic, we interpret it as the square root of the energy per unit width of each wave cycle.

5.4 In the surf zone, amplitude is a decent gauge for depth

Emad tells his kids that the height of the waves in the area where they surf is approximately the depth of the water in that location. I didn't get a chance to find out if he meant the mean depth or the minimum depth. I'm not sure that this is generally true – it may be particular to the area where they usually surf, where the bottom has a gentle slope for at least 30 to 40 meters off the beach. Or it may be that if the wave is surfable, then the combination of factors (steepness, amplitude and depth) conspire to make it generally true.

Waves grow in amplitude as they approach the beach. We can get a handle on how this happens by using the energy density expression we found in the previous section. Assuming that the wave approaching shore is not losing any energy, the quantity $A^2\lambda$ remains constant as the wave moves from deeper to shallower water. As the wave slows down, the wavelength decreases accordingly. Frequency does not change because (as Randy explains) the deep water wates act as a forcing function on the shallow water waves. As long as the wave does not break at the top or churn the sand at the bottom, the energy in each cycle remains constant, so as $\lambda \propto \sqrt{h}$ decreases, A^2 increases proportionally.

In order to fully suss out Emad's surf zone depth estimate, I'd have to figure out the conditions under which waves break, but there's no time for that before the holiday begins! In any case, this topic is a nice segue to Randy's contribution.

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[4] M, L and T here are the dimensional analysis units of mass, length, and time, respectively. I'm putting these in standard Roman text, especially to distinguish time "T" from period T.

6 Estimating Coastal Water Depths Using Satellite-derived Ocean Wave Data by Randy Patton

This short note describes a novel approach to estimating the depth of a lagoon off the island of Okinawa. The ultimate goal was to determine the volume of earth needed to fill in the lagoon in order to make an airfield for a nearby US Military base (Camp Schwab). The sponsoring agency (a large construction company) wanted to investigate the feasibility of obtaining nearshore ocean depth estimates using space-based remote sensing data in the hopes of providing a more timely and efficient survey methodology.

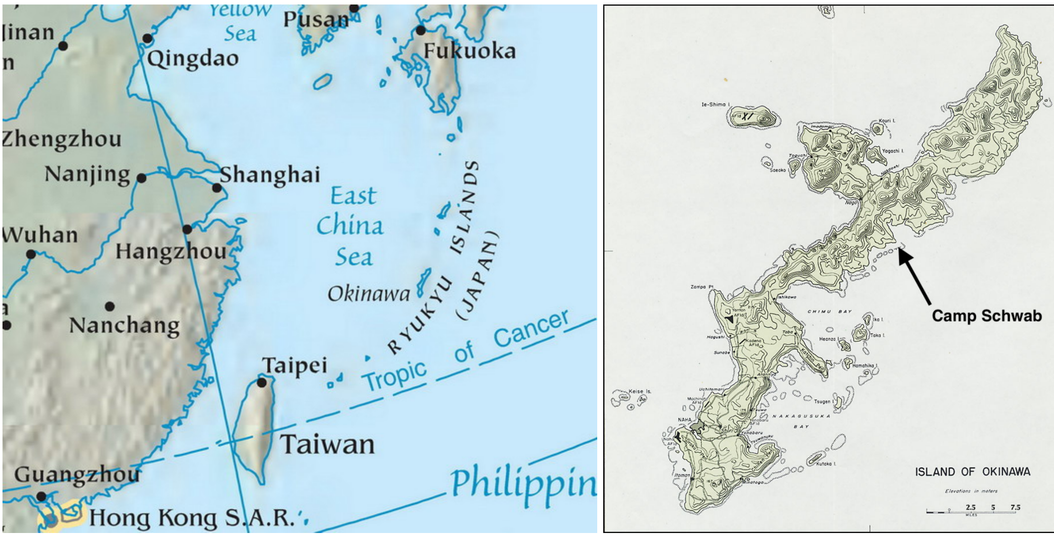

Figure 4: Location of Okinawa in the East China Sea.

Figure 5: Champ Schwab and early construction in lagoon. The proposed land-fill is the boxed-in area. Presently the project is on hold.

The approach was to use high resolution Synthetic Aperture Radar (SAR) imagery obtained by the RADARSAT satellite (provided by the Canadian government). An example of this type of data is shown in Figure 6 for an area off the coast of Northern California (Point Reyes).

Figure 6: Synthetic Aperture Radar (SAR) image of the ocean and coast near Point Reyes, CA.

As can be seen in the image, the wavelengths of the large ocean swell waves can be readily discerned. The changing of these wavelengths as they shoal (i.e., encounter shallower depths) will provide the information needed to estimate depth, as described below.

6.1 Small amplitude ocean wave theory

The mathematical description of waves in the ocean comes from solving the so-called equations of motion for fluids, which are a full-blown version of Newtons basic law F = ma. Luckily for the present application, various reasonable approximations can be made that greatly simplify the equations. These include assuming that the waves have small amplitude compared to their wavelength so that various nonlinear effect can be neglected. At the time and space scales we are interested in, we can also safely ignore rotational (Coriolis), surface tension (capillary) and compressibility (sound) wave solutions. We are left with waves whose only restoring force is gravity, i.e., gravity waves.

The pared down equations still require that boundary conditions be applied, in particular that the vertical water movement induced by the waves goes to zero at the ocean bottom, i.e., water cant go through a solid bottom. Also, in this simplified scenario, the amplitude of the vertical displacement of the water induced by the waves can be assumed to fall off exponentially with depth.

At the surface, a dynamic boundary condition must be used in which the wave height solution is required to match a sinusoidal profile such as

$$\eta(x,t) \sim \sin(\omega t - kx) \tag{6.1}$$

where η is the wave height, ω the frequency and k the wavenumber. We can write the latter two parameters using the more familiar quantities wavelength, λ, and period, T, as $\omega = 2\pi/T$ and $k = 2\pi/\lambda$.

In order to satisfy the boundary conditions, a special relationship between the above parameters is required. This is known as a *dispersion relation* and is given by (g = Earth's gravity)

$$\omega = \sqrt{kg\tanh(kH)}. \tag{6.2}$$

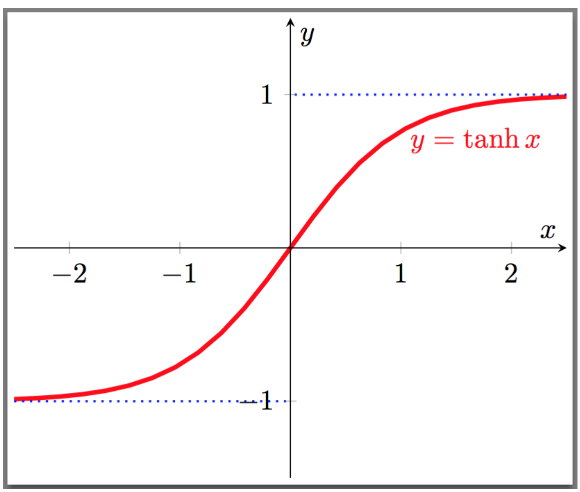

Figure 7: Hyperbolic tangent.

6.2 Estimating depth from the dispersion relation

The frequency of waves in this simple model is a function of the wavenumber, k, and depth, H. In particular, we will take advantage of the hyperbolic tangents dependence on the argument kH at very large and very small values (Figure 7). Large values of kH occur when the wavelength $\lambda = 2\pi/k$ is small compared to the depth. This is the deep water case, i.e., the waves are too short to feel the bottom. Correspondingly, $\tanh(kH)$ asymptotes to the constant 1 for large kH so that

$$\omega = \sqrt{kg}. \qquad \textit{Deep water case} \tag{6.3}$$

On the other hand, shallow areas with relatively long waves result in small values of kH and the $\tanh(kH)$ function can be approximated by the line $y = kH$. The dispersion relation then becomes

$$\omega = k\sqrt{gH}. \qquad \textit{Shallow water case} \tag{6.4}$$

We thus have two (overlapping) regimes, one in which the wavelength is sensitive to the depth (shallow water case) and one where the wavelength depends only on the frequency of the wave (deep water case).

Figure 8 shows curves for the full dispersion relation (gray) as well as the shallow (orange) and deep cases (blue) vs. wavenumber. Notice the transition from the deep water regime (high wave number, short wavelengths on the right) to the shallow regime on the left (long wavelengths).

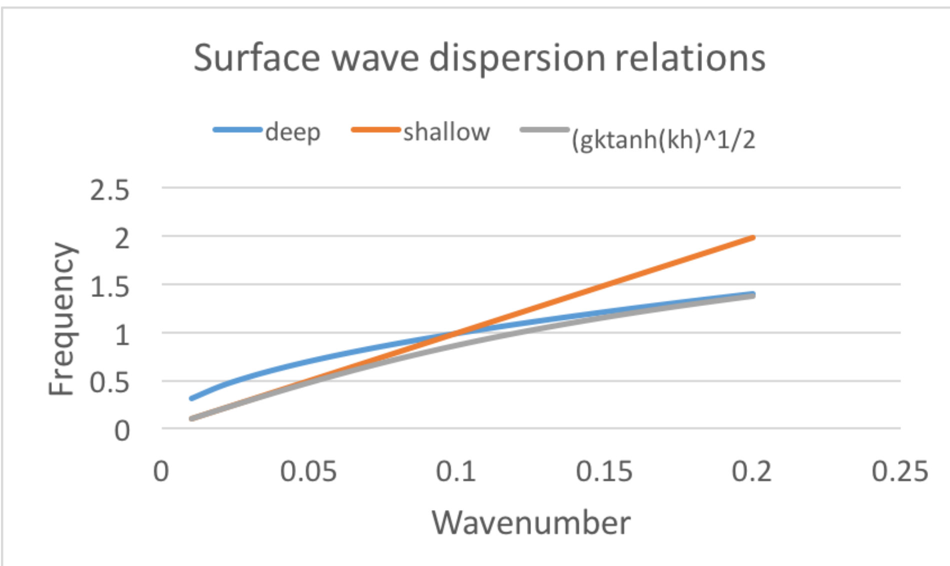

Figure 8:

The stormy North Pacific generates plenty of low frequency waves that impact the island. These open water swell waves have long wavelengths (> 100 meters), making them readily visible in SAR imagery. The water depth offshore of Okinawa also increases rapidly in the south-east direction, increasing to over 1000 m within 20-40 km. This is deep enough to make this region a deep water regime even for long swell waves.

It is also fortuitous that the low frequency waves are very narrowband (i.e., they appear nearly sinusoidal), which allows a single wavenumber to be used to approximate the wave field. The narrow band nature of the waves arises from the fact that waves of different frequencies travel at different speeds and thus separate or disperse themselves as they travel. The faster, low frequency, long wavelength wave groups also attenuate less (decrease in amplitude) as they travel away from their area of generation. Thus, Okinawa sees long, low frequency sinusoidal waves from big faraway storms.

As the incoming deep water waves approach the coast they transition to the shallow water case. A key feature is that the frequency of the waves in the shallow regime stays the same as in the deep water regime. This fixed frequency, determined using the measured deep water wavelengths, can then be used in the shallow water dispersion relation to solve for the depth H. In fact, by equating the dispersion relations, it is found that[5]

$$H = \frac{\lambda_{\text{shallow}}^2}{\lambda_{\text{deep}}} \tag{6.5}$$

A plot of this relation for a fixed deep water wavelength is shown in Figure 9.

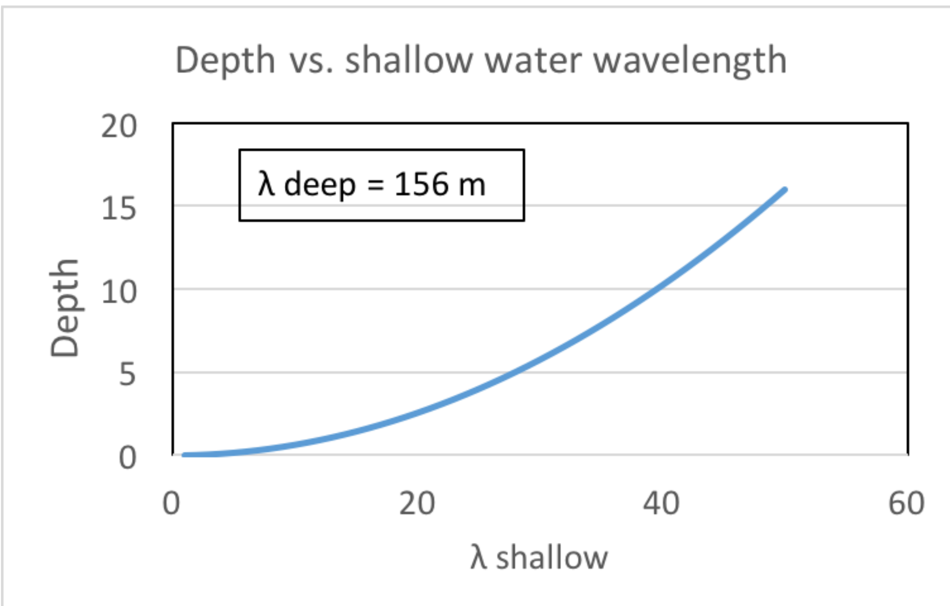

Figure 9:

Thus, the procedure for measuring near shore depths is to first measure the wave length imaged by the SAR in both the far offshore region, λ_{deep}, and in the near shore region, λ_{shallow}, and then use the relation above to estimate the depth. A typical period for swell waves is 10 seconds which corresponds to a 156 meter wavelength. This translates to a 3 meter depth for a shallow water wavelength of 22 meters, which is close to the measurements we found for our study.

In summary, there are several advantageous aspects of this approach:

[5]Rather than separating the dispersion relation into two regimes, the full relation could still be used to solve for depth using straightforward numerical techniques (eg. Newtons Method).

- The deep and shallow regions are simultaneously within the image area of the SAR.
- The SAR resolution is good enough to image the shorter shallow water wavelengths.
- Swell waves are narrow band and thus provide a well-defined "forcing" function.

References

[1] Sanjoy Sondhi Mahajan, *Order of Magnitude Physics: A Textbook with Applications to the Retinal Rod and the Density of Prime Numbers*. Dissertation (Ph.D.), California Institute of Technology, 1998. (doi:10.7907/8MJZ-A984)

Mahajan has written many fantastic books, including *Street Fighting Mathematics* and *The Art of Insight in Science and Engineering*. Chapter 6 of his thesis is a prerequisite for this year's paper.

[2] James Lighthill, *Waves in Fluids*, Cambridge University Press, 1978.

Lighthill held the Lucasian Professorship at Cambridge, much like someone else we know about. This book has detailed expositions on all forms of waves in fluids.

[3] Blair Kinsman, *Wind Waves: Their Generation and Propagation on the Ocean Surface*, Prentice-Hall, 1965. (reprinted by Dover Books)

Randy recommended this book to me. It's fantastic. I love the introduction. There are things he writes about that are broadly applicable to experimental physics.

[4] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics*, Addison-Wesley, 1963.

A classic. With permission from Caltech, in 2013, Michael A. Gottlieb and Rudolf Pfeiffer made the entire text of this book available online. It might be better than the print version.

[5] M.A. Davidson, T.J. O'Hare and K.J. George, "Tidal Modulation of incident wave heights: Fact or Fiction," *Reef Journal*, Vol. 1 (2009) pp 16–32. http://www.thereefjournal.com/

7 Activities

Newton day always includes activities. This year, we spent some time before Newton Day in Corallitos and watched the waves roll in after a big storm. On Newton Day, we'll head down to the palisades in Santa Monica, North of the pier to watch the deep water waves come in to shore.

Another possibility is to find some still water (either Marina del Rey or a local park) and make some waves to see if we can observe the deep-water dispersion characteristics of gravity waves. Things to look for are:

1. Long waves disappear off the leading edge of the disturbance.
2. Short waves appear from the trailing edge and move to the leading edge, getting longer as they go.
3. The width of the disturbance grows in time.

These effects are difficult to see when small pebbles are dropped in water because of the minimum phase velocity of gravity/surface-tension waves explained in both Feynman [4] and Mahajan [1].

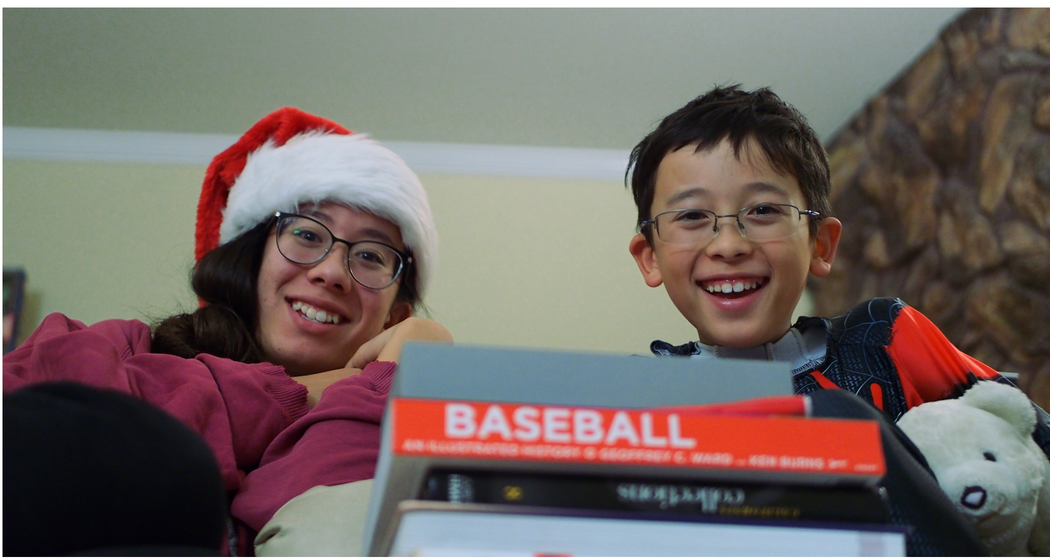

Figure 10: Maggie & Owen.