Oceanography on a Rotating Earth

Newton Day Paper, 2021

by Randy Patton

Introduction The earth is covered in fluids. Not only the wet kind, like oceans and lakes, but also the atmosphere which, though it is a gas, also obeys the physical laws of fluid dynamics. Studying these fluids on large-scales requires including a factor we don't often think about; the earth's rotation. Though this rotation doesn't seem to affect us in our daily lives, it is very important when describing weather and ocean currents and the climate in general.

The major oceanic currents are shown in Figure 1. Also, the following link is an animation showing the currents in motion:

Ocean currents animation – NASA https://www.nasa.gov/topics/earth/features/perpetual-ocean.html

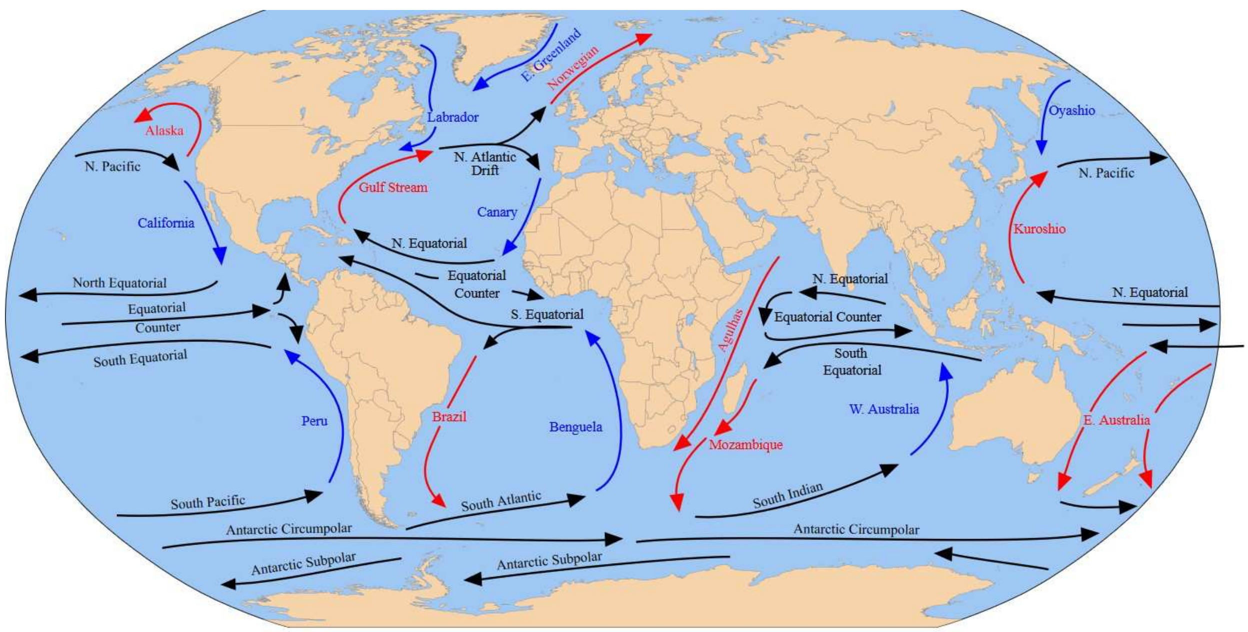


Figure 1 – Major ocean currents

This paper will focus mainly on how the earth's rotation affects large-scale phenomena in the ocean such as currents (Gulf Stream), and weather patterns (El Niño in the Pacific).

The Coriolis effect

When water is flowing along in the (rotating) ocean, there appears to be a gentle but "mysterious" force pushing it to the right (in the Northern hemisphere). In fact, any object moving in a supposedly straight line will also be deflected slightly. We don't notice this effect on the small scales we normally deal with (bathtub drains included), but on the large-scales of 100s of kilometers associated with ocean currents, it can make a big difference. This sideways force is called the Coriolis force and is due entirely to the rotation of the earth.

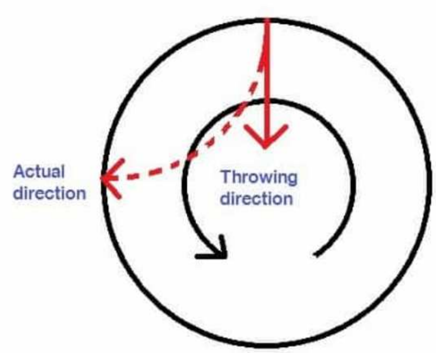
There actually is a way to experience this force in a neighborhood park; the merry-go-round (although they may not be around anymore!). Imagine sitting on a spinning merry-go-round, looking at the center post, while holding a basketball. Say you are rotating counter-clockwise when looking down from above. You are now in a rotating reference frame where everything *inside* the merry-go-round seems fixed, since it is spinning along with you. In fact, you can hold the basketball out in your hand and it will stay in your hand just like if you were sitting on the stationary ground. The force pushing you back against the railing is the well-known *centrifugal* force, like that experienced in a car that is turning. While this force is due to the rotation, it is not the Coriolis force but rather a so-called body force, like gravity, which, by the way, has been holding you down on the spinning platform the whole while.

Now comes the Coriolis part, which is fairly straightforward. If you throw the basketball straight across the platform, you will notice that it seems to curve around to the right instead of going straight. However, for an observer fixed relative to the ground and looking down at the whole scene, the ball will actually be seen to be traveling in a perfectly straight line. You are the one who is curving around. This simple process is demonstrated in this video link:

https://www.youtube.com/watch?v=okaxKzoyMK0

The important points here are:

- the Coriolis "force" arises in rotating reference frames
- it only acts on moving bodies
- it is not a real force since nothing is pushing the moving body to make it curve



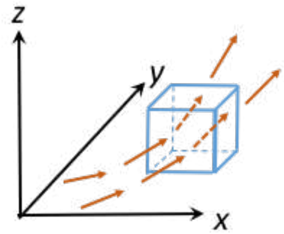
Physics in a rotating reference frame

The motion of fluids under the influence of various forces is described by the aptly named *equations of motion*. These equations incorporate Newton's Second Law which states:

Law 2. *A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.*

Note that this law can be stated more simply[1] as *Force = Mass × acceleration* or *F = ma*.

For fluids, a "body" is considered to be a small parcel of fluid that rides along with all of its neighbors as they flow around in 3-dimensions. To describe this flow, the equations of motion are applied in a small cubical volume, fixed with respect to the fluid flow, and through which the fluid flows. The forces affecting the fluid flow like gravity, pressure and friction are expressed in this 3-d reference frame.



There is a problem, however, when the reference frame is rotating. Due to the Coriolis effect, a body's trajectory appears to change even though there are no real forces acting on it. This violates Newton's First Law of Motion, or law of inertia:

Law 1. *A body continues in its state of rest, or in uniform motion in a straight line, unless acted upon by a force.*

A rotating reference frame is thus a *non-inertial* reference frame and requires the introduction of a "fictitious" or apparent force to account for the Coriolis effect. The magnitude of this effect is proportional to how fast the reference frame is rotating and the speed of the body moving in it. This apparent force can be expressed[2] as a frequency times a velocity, or

$F_{\text{Coriolis}} \sim f\,\mathbf{v}$

where the parameter f represents the frequency of rotation and the vector $\mathbf{v}$ is the 3-d speed and direction of the flow. Keep in mind that F_{Coriolis} acts perpendicularly to the velocity $\mathbf{v}$.

For a non-inertial reference frame fixed to the (rotating) earth, the equations of motion are represented by a balance of terms:

acceleration = gravity + pressure gradient + Coriolis + friction

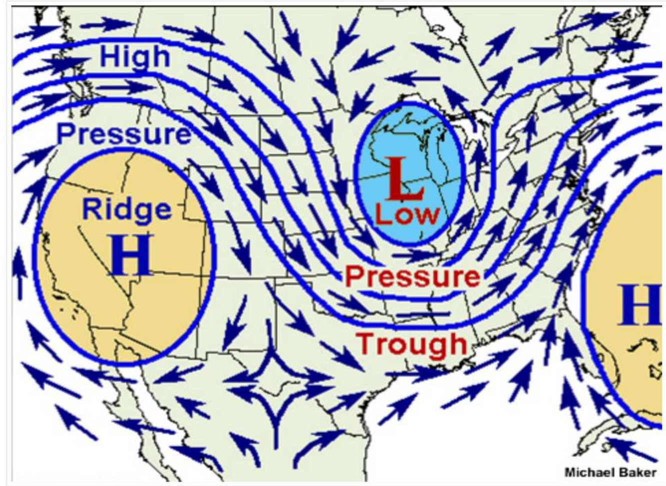
We can simplify this equation for our present purpose by keeping only the terms important for large-scale flows. If the flows are steady (not changing in time), they are not accelerating[3] and so the **acceleration** term on the left is zero. Also, large ocean currents are generally horizontal so that the **gravity** term, which affects motions in the vertical, can also be neglected. Finally, out in the middle of the deep ocean, the friction between the currents and physical boundaries is also expected to be negligible, so **friction** → 0. On the other hand, a difference in pressure between points in the ocean represents a horizontal force that cannot be neglected.

The remaining terms represent a balance between the pressure gradient and the Coriolis force:

$$\Delta p \sim f\,\mathbf{v}.$$

[1] This assumes that the mass is constant in time.
[2] The actual units and vector nature of this expression are alluded to using the ~ notation.
[3] Technically, they are not accelerating in the direction of the flow.

Geostrophy

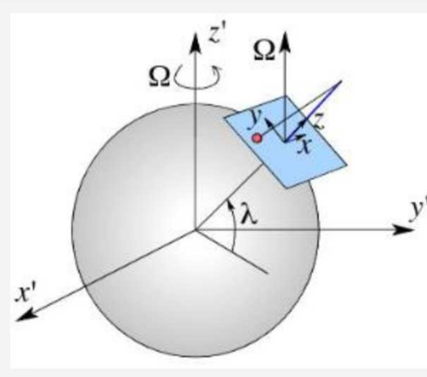


The last equation is the celebrated *geostrophic balance*. It is used in oceanography and meteorology to describe steady-state[4] flow fields. The figure to the right shows the geostrophic flow field produced in the atmosphere by hypothetical areas of high and low pressure over the US.

An important feature in this map are the *streamlines* showing the path of the fluid flow. In geostrophic flow, streamlines correspond to lines of constant pressure known as *isobars*. In going from a high pressure area to a low area, you cross over lines of decreasing levels of pressure. This high-to-low pressure *gradient* tries to push the air towards the low. However, the Coriolis force diverts the flow to the right so that, in a steady state, it flows at right angles to the pressure gradients and thus along the isobars. The yellow lines in the image of the weatherman and his map are the isobars/streamlines that are so useful in describing the weather.

Latitude Variation of the Coriolis Force

The equations of motion we have studied so far apply to a simple merry-go-round at a spot on the rotating earth. But there is more to the story. On the earth, the effective rotation actually varies with latitude. This means that the Coriolis force will vary in the north-south direction on the large scales we are interested in (100s-1000s of km). This introduces additional flow features.



The Rotating coordinate system {x,y,z} is non-inertial since Earth is rotating. As a result, a Coriolis force is added when working in this frame of reference.

Consider a spinning globe. The *effective* rotation of a point on its surface varies from a maximum at the poles, where our merry-go-round rotates at exactly the same rate as the earth, to zero at the equator. This is a little tricky to envision, but give this a try:

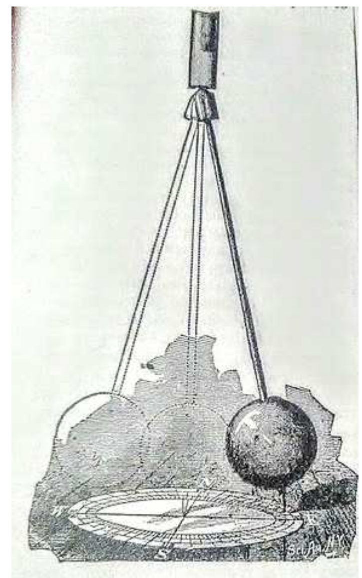
> Imagine standing still on the North Pole and looking out straight ahead at the horizon (i.e., parallel to the ground), at the fixed stars (the northern star Polaris will be directly

[4] "Steady-state" means the non-geostrophic components of the flow field can adjust fast enough to maintain the geostrophic balance.

overhead, by the way, but you are not looking up!). As you look out, you will see the stars appear to rotate past your field of vision, finally returning to their initial configuration after one day. You will have clearly rotated 360^o (or 2π radians) with respect to a fixed reference frame, i.e., the stars in the plane perpendicular the earth's axis of rotation.

Now go to any spot on the equator and face north. Again, look out straight ahead at the field of stars; you can focus on Polaris (which will be very low in the sky). As the earth rotates, Polaris will remain fixed in your field of view (or indeed any star which, like Polaris, lies on an imaginary line extending out along the earth's axis). This shows that you have not rotated with respect to a fixed reference frame.

This variation of rotation speed with latitude was demonstrated in a clever way by the French physicist Jean Foucault, who first used it in 1851 to demonstrate the rotation of the earth. He built a pendulum that was precise and massive enough to overcome friction effects over long time periods (> 1 day). The pendulum swings in a plane that is fixed compared to the stars. As the earth rotates around, the pendulum appears to rotate when viewed by a person standing on the earth. There are many Foucault pendulums around the world, including one in LA at the Griffiths Observatory.



Foucault's first pendulum

At different latitudes, it takes the pendulum different times to complete a rotation. The formula is

Foucault pendulum period = *1 day/sin(θ)*

where θ is the latitude. For LA, the period is about 43 hours. The following link shows how a pendulum rotates completely in one day at the North Pole but doesn't rotate at all at the equator:

https://upload.wikimedia.org/wikipedia/commons/f/ff/FoucaultMultiAnima.gif

This variation in the effective rotation with latitude gives rise to some interesting oceanic phenomena.

Western Boundary Currents

The large-scale circulation of water in the oceans is driven by the earth's global wind field, shown schematically in the following figure. These winds consist of dominating westerly[5] winds

[5] Wind direction has historically been designated by where the wind is coming *from*; thus a westerly wind comes from the west and blows to the east.

at latitudes between 30 and 60 degrees in the northern and southern hemispheres (the *Westerlies*) and dominating easterly winds in the tropical/subtropical zone (the *Easterlies* or *Trade winds*). Note also the convergence of winds at the equator towards the west (Intertropical Convergence Zone).

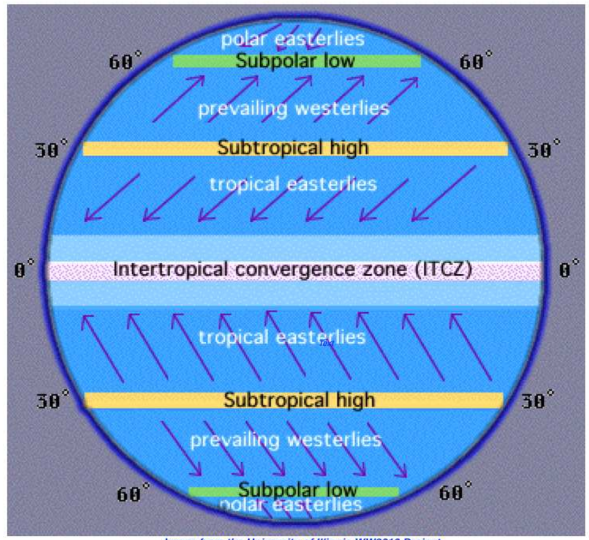
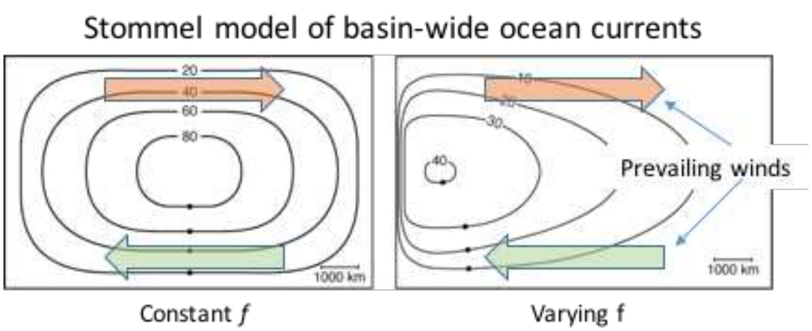
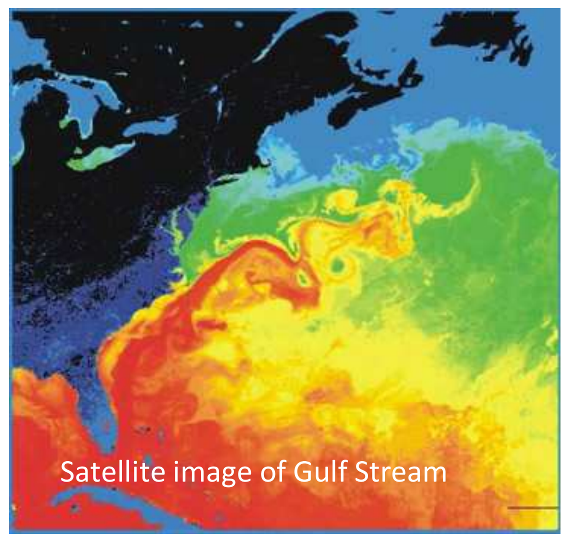


Image from the University of Illinois WW2010 Project.

At mid-latitudes, these winds give rise to basin-wide circulating *gyres* in the Atlantic, Pacific and Indian oceans*;* see Figure 1 (at the beginning this paper).

A notable feature of oceanic circulation is the existence of strong currents along the western boundaries of continents (see Figure 1). These include the Gulf Stream off of the eastern coast of the US (western side of the Atlantic Ocean; see figure), the Kuroshio off of Japan and the Agulhas Current off the south-east coast of Africa. These strong boundary currents are a direct result of the variation in the Coriolis parameter f with latitude, as was shown in a famous paper[6] by Henry Stommel in 1948. In this paper, he showed the result of a simplified model of an ocean basin under these prevailing winds with and without a varying f (see figure).





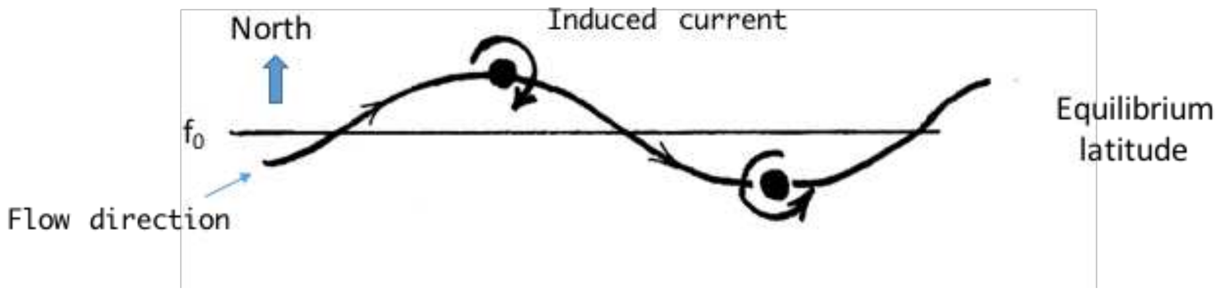
<u>Planetary Waves: Conservation of Vorticity</u>

The *vorticity* of a fluid parcel is a measure of the amount of spin it has. A good example is a *vortex*, like the swirl of fluid down a drain or a hurricane. In our rotating earth reference frame, the vorticity can be split into two components, one representing the swirling motion as seen in the rotating reference frame (relative vorticity) and the other due to just the earth's rotation (planetary vorticity):

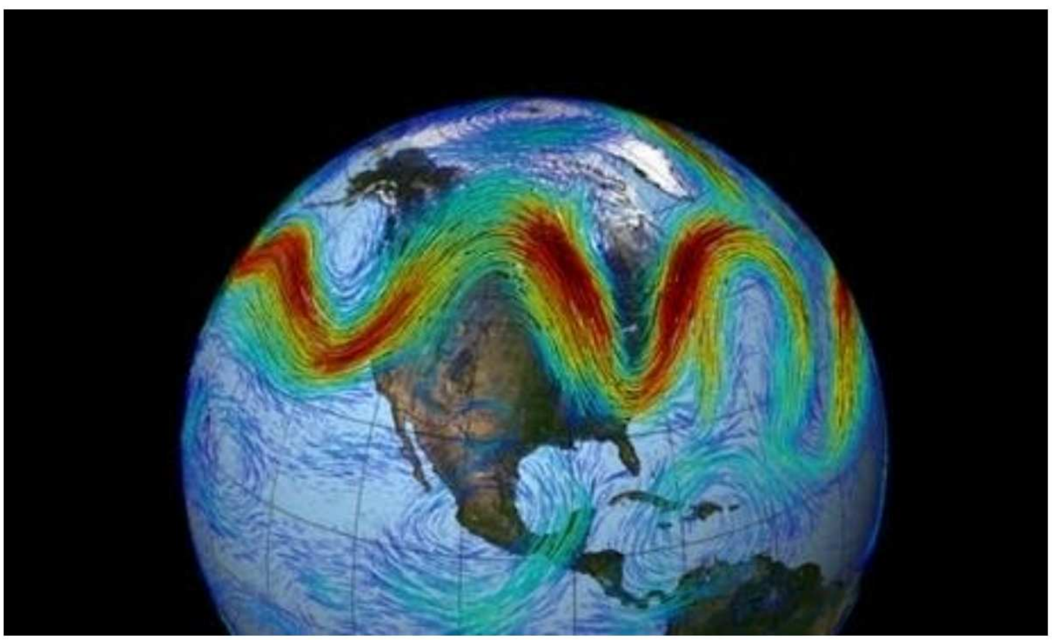
Total vorticity = relative + planetary.

[6] *The westward intensification of wind-driven ocean currents,* H. Stommel, Eos Volume 29, April 1948

The variation of planetary vorticity with latitude means that water moving in the north-south direction will intrude into areas with a different planetary vorticity. The reaction of the fluid is to *conserve* its vorticity by inducing a flow field to counteract this change in its surroundings. The above equation shows that when the fluid experiences an increase in the planetary vorticity, it must decrease its relative vorticity in order for the total vorticity to remain constant. Thus, a parcel of fluid moving northward develops clockwise spin. This affects fluid to the east and west causing the parcel to turn back southward. When it overshoots its equilibrium latitude (because of its inertia), it develops a circulation that moves it northward again (see figure).



This oscillatory behavior is reminiscent of physical systems which have a restoring force such as a pendulum (gravity) and a mass on a spring (Hooke's Law). In the present case, the restoring force is due to varying vorticity. These oscillations in space and time give rise to *planetary waves*. The image below, showing a rendition of atmospheric Rossby waves interacting with the jet stream, gives an idea of the scale of the waves.

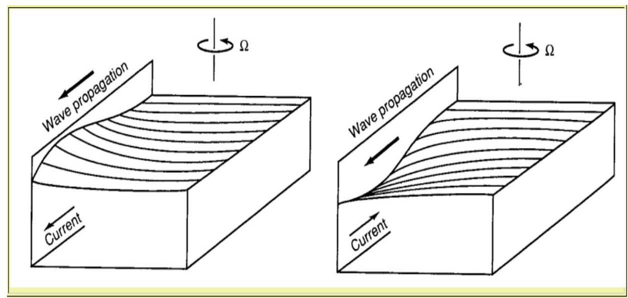


The jet stream that circles Earth's north pole travels west to east. But when the jet stream interacts with a Rossby wave, as shown here, the winds can wander far north and south, bringing frigid air to normally mild southern states. Credit: NASA Goddard Space Flight Center

Oceanic Rossby waves always travel to the west in both the northern or southern hemisphere. Because the *change* in the planetary vorticity is greater near the equator than further north[7], they travel faster near the equator than further north. Nonetheless, they are very slow, taking many months to cross the Pacific near the equator and many years at higher latitudes. Rossby waves have very small vertical displacement at the surface, on the order of 10 cm, and very long wavelengths (100s-1000s of km), so they do not look like a giant surface (gravity) waves we are familiar with. However, they do deflect the subsurface layers substantially, inducing 3-d flows that affect the surface expression of the wave (eg., temperature).

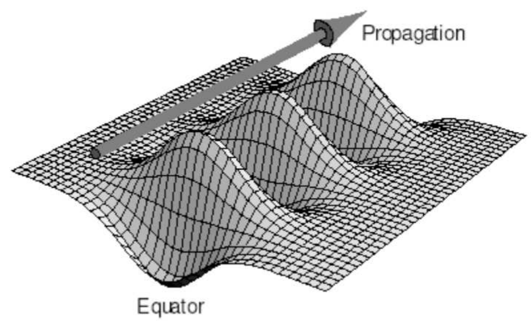
Equatorial waves and El Niño

On the earth-fixed reference frame, the effective rotation changes from the northern to southern hemispheres, going to zero at the equator. This switching of the sign of the Coriolis parameter at the equator creates a waveguide that supports a very different type of planetary wave; the *equatorial Kelvin wave*.



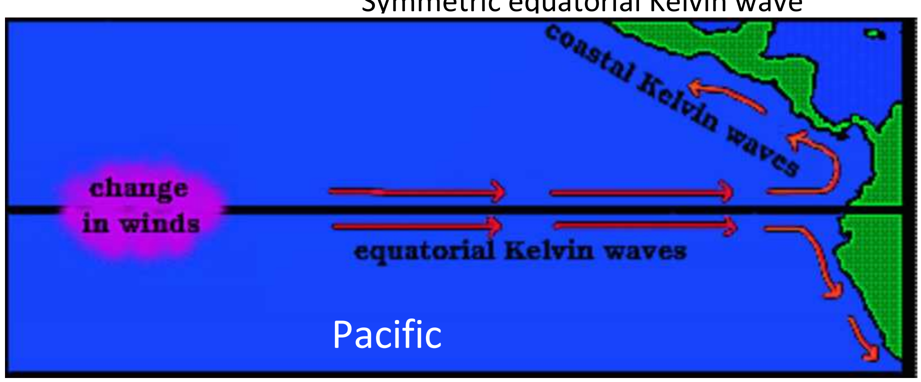
Coastal Kelvin wave in northern hemisphere

Kelvin waves also come in a *coastal* variety. Their unique feature is that they need a boundary, such as a coastline, to "lean against". At northern mid-latitudes, they propagate with the coast on their right (see figure) and on their left in the southern hemisphere. Propagating waves induce current flows along the coast which transport water vertically. As these waves pass by, they alternately raise and lower the subsurface layers. Coastal Kelvin waves are also fast, covering over 100 km in a day.



Symmetric equatorial Kelvin wave

On the Equator, of course, there is no boundary. However, the change in sign of the planetary vorticity allows Kelvin waves on each side of the equator to lean against *each other*. This creates a symmetric wave form that travels *east* along the equator (see figure). These Equatorial Kelvin waves also travel relatively fast, on the order of 1 meter/sec or approximately 100 km/day. They thus can cover an ocean basin in a matter of weeks.



A remarkable feature of Equatorial Kelvin waves is that they turn into Coastal Kelvin waves when they hit the coast, as depicted in the figure. When the wind field

[7] Planetary vorticity varies as the *sine* of the latitude which means its *change* with latitude goes as the *cosine*, which is 1 at the equator and goes to zero at the poles.

changes significantly along the equator, the ocean responds by organizing itself into an Equatorial Kelvin wave that proceeds to travel east. When it hits the coast, it splits up into north and south traveling Coastal Kelvin waves.

As shown earlier, the winds near the equator are easterly and on average tend to pile water up along the western boundary (Asia). Every few years or so, an El Niño event occurs where these winds slacken appreciably. The gravitational potential energy of the piled up water is then released. This pulse of energy is carried eastward along the equator via equatorial Kelvin waves. As they pass, the waves deepen the subsurface layers. When they hit the eastern boundary, they divide up and propagate as coastal Kelvin waves up along Central America and down along South America.

The following link shows short animation of sea surface height anomalies associated with eastward Kelvin wave propagation in the Pacific.

https://www.youtube.com/watch?v=NNn4V52fCbw

Here is another one showing multi-year satellite data. Notice how fast the eastward propagating Kelvin waves moves across the Pacific and impinge on the eastern boundary.

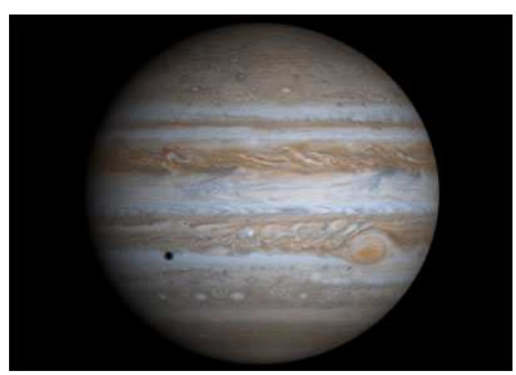
https://www.youtube.com/watch?v=cCxQUqnaQnA

Normally, off the coast of Peru, upwelling brings nutrient-rich deep water up to the sun-lit surface layers. This boosts the growth of phytoplankton, which are at the base of the food chain, which sustains a robust fishing industry. During an El Niño, after the Kelvin waves hit the coast, the southern branch travels down the Peruvian coast and suppresses the normally occurring coastal upwelling, devastating the fisheries. Since this seemed to occur during the Christmas season, it was dubbed an "El Niño", after the Christ child. This an example of long-distance "teleconnections", where the collapse of fisheries off of Peru is caused by a slackening of winds over the equatorial Pacific many 1000s of kilometers away.

Summary

This paper has described the basic physics of motions in a rotating reference frame, how they apply to the rotating earth and the effects on large-scale ocean flows.

A selection of examples showed how the Coriolis effect associated with the earth's rotation controls the dynamics of large ocean currents such as the Gulf Stream and supports planetary waves on scales of 1000s of kilometers that distribute energy over entire ocean basins.



Obviously, there are many more aspects of geophysical fluid dynamics to explore. The dynamics described here also apply to the other planets, especially the gas giants like Jupiter.

Newton Day 2021:
Coriolis, here and out there

by Randy Patton and Peter H. Mao
for Owen and Margaret

December 25, 2021

Abstract

For Sir Isaac Newton's 379th birthday, the main event is Randy Patton's overview of the Coriolis force on terrestrial fluid flows. My meager contribution is some notes on making a table-top demonstration of the stability of the off-axis Lagrange points (L4 and L5).

1 Preface

This year, I was expecting to be empty handed for Newton Day. "Circumstances of life" in the summer and autumn left me with little free mental capacity to find a topic, read up on it and compactify it for the holidy. Several months ago, I solicited (begged for) write-ups from people who read technical books with me,[1] and once again, Randy Patton took up the challenge. Please enjoy his exposition on Coriolis effects on terrestrial fluid flows, "Oceanography on a Rotating Earth."

My part in Newton Day was inspired by Owen. A few weeks ago, over breakfast, he mentioned that one of his classmates gave a presentation on the Webb Telescope. We reminisced about the time we got to see the behemoth at Northrop Grumman's facility,[2] and then he surprised me when told me it was going to be placed at one of the Earth-Sun Lagrange points. I asked him how many Lagrange points there are in the Earth-Sun system, and without missing a beat, he told me, "Five." After picking myself up off the floor, I gave him some garbage information on L3. Later in the day, he schooled me on L3-L5. So that day, I figured my part would have something to do with Lagrange.

The basic overview of the Lagrange points is very nicely covered in NASA's outreach page on Lagrange points, which also links to an excellent mathematical treatment of the topic by Neil J. Cornish. My offering for this year are some notes on making a tabletop demonstration of the stability of L4 and L5, which requires the Coriolis force.

[1] The MIT Club of Southern California Techinical Book Club, which accepts all comers, whether or not they are affilliated with MIT.

[2] A few years ago, Jim Hall, another one of my book club compatriots, allowed us a peek into the giant Northrop Grumman high bay where they were integrating the Webb.

2 Coin Funnels and potential fields

If you've been to a science museum, you've probably seen the funnel-shaped coin collector, where you roll a coin along the lip of the funnel and wait three hours for the coin to scrub off enough kinetic energy to fall into the hole at the bottom of the funnel. In addition to collecting money for the museum, those funnels are a demonstration of dynamics in a central force field. The funnel is a model of the gravitational potential of the Sun, and the coins represent planets orbiting the Sun. Thankfully, the model is not entirely accurate – the frictional drag due to air and contact with the funnel are much larger than any dissipative forces on planets in our solar system, so we won't be falling into the Sun any time soon.

The force due to a potential points in the steepest "downhill" direction and is proportional to the spatial derivative of the potential. In the physical demonstration model, the force (due to the Earth's gravity) also points in the direction of steepest descent, but its magnitude, rather than being proportional to the slope of the surface (ie, the trigonometric tangent), is $mg\sin\theta$ along the surface. Furthermore, the demonstration model is supposed to be modelling only the transverse motions. For slope $\tan\theta$, the transverse force is $mg\sin\theta\cos\theta$ and the vertical force is $mg\sin^2\theta$. This means that the maximum transverse acceleration is only $g/2$, occurring when the surface angle is $\pi/4$ (45 degrees); if we want the model to maintain fidelity with the physical system, we have to keep the slopes well below 1.

3 Lagrange points, the rotating frame effective potential

Figure 1 illustrates the relative locations of the five Lagrange points in the Earth-Sun rotating frame along with equipotential contours of the effective potential and the downhill directions of that effective potential. The effective potential shown in the figure incorporates three components of the physical system: the gravity of the Sun $-GM_{\text{Sun}}/r$, the gravity of the Earth $-GM_{\text{Earth}}/r$, and the potential of the centrifugal force $-\frac{1}{2}\omega^2r^2$. The Coriolis force is velocity-dependent and has non-zero curl, so we can't represent it with a scalar potential. In this respect, it is similar to the Larmor force in electromagnetism. Also, just as the Larmor force does no work on charged particles, the Coriolis force does no work on particles under its influence.

Note that the rotating-frame effective potential illustrated here is a different point of view from the fixes angular momentum effective potential that is encountered in most physics texts (Golstein, Landau & Lifshitz, etc.). In that view, because angular momentum, rather than rotation rate is held fixed, the centrifugal potential has the form $+1/r^2$, which gives us the nice harmonic well at the radial distance corresponding to circular orbits. Physically, they are both correct, but they emphasize different aspects of orbital systems.

If you try to intepret the rotating frame potential as an inertial system, you would expect any objects outside of the radius of L4 to roll off to an infinite distance. This makes no sense, since we know of many objects in our solar system outside of the orbit of L4 (Mars, asteroid belt, and the gas giants, to name a few). This is where Coriolis matters. Much like a high pressure system, anything moving in this (counterclockwise) rotating frame will be "pushed" in the starboard direction by the Coriolis force. As they move in this coordinate system, objects may cross equipotentials, but the motion will be oscillatory about some potential. With some frictional dissipation, the objects would eventually settle into tracks along the equipotentials.

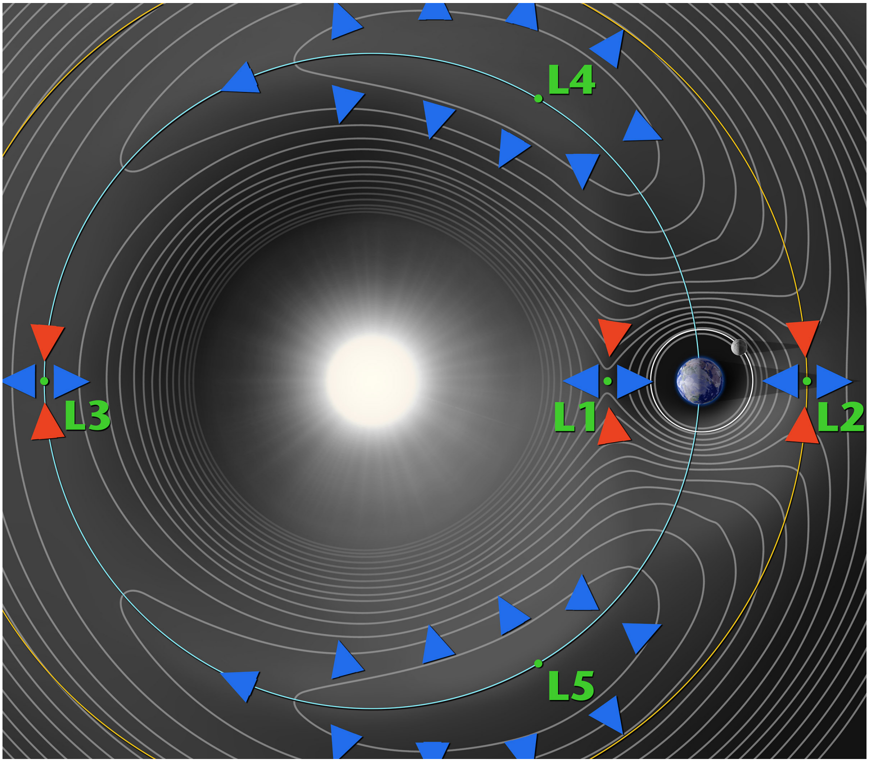


Figure 1: Lagrange points and effective potential of the Earth-Sun system. Earth-Sun-L4 and Earth-Sun-L5 each form equilateral triangles. Credit: NASA

4 Considerations for a table-top demo

Building an Earth-Sun Lagrange point model would rather challenging because the feature heights of the Lagrange points scale with the Earth-Sun mass ratio ($\sim 3 \times 10^{-6}$). The surface itself would only build in the $-1/r$ "gravitational" potentials of the Earth and Sun (the other features being emergent with the rotating system), but the smootheness of the surfaces would need to be on the order of the mass ratio.

An easier system to build would be one with two equal (model) masses. In this case, by symmetry, we maximize the potential difference bewteen L4 (= L5) and L1. The equipotential countours for a rotating model and the topographic contours of the model surface are shown in Figure 2. If we scale the model so that 1 unit of length is 15 cm, then by a quick back-of-the-envelope calculation,[3] I get a rotation rate of around 30 RPM to balance forces near "L4"– pretty close to turntable size and speed. Too bad I don't have easy access to a CNC any more...

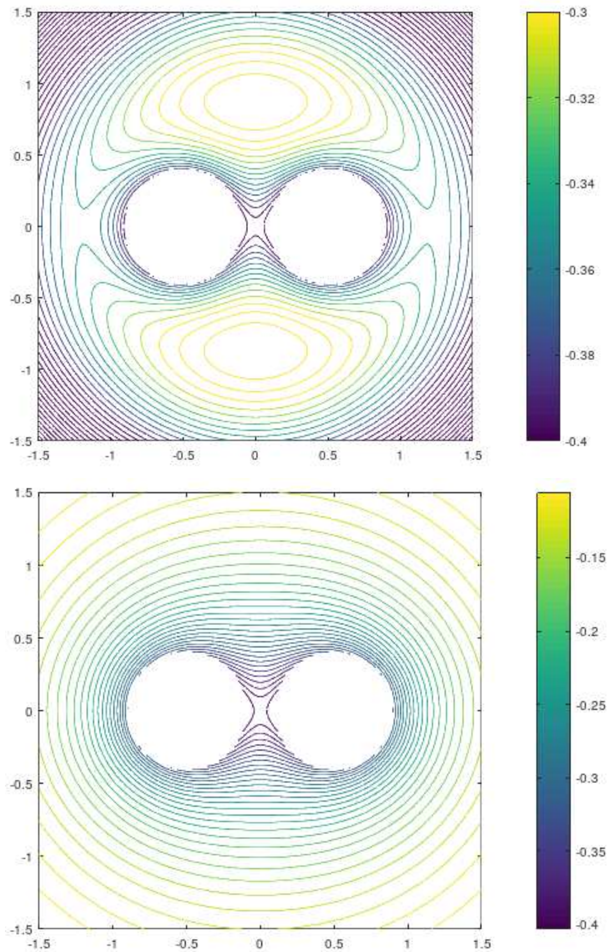


Figure 2: **Top**: Rotating frame effective potential. Masses are separated by 1 unit of length. L1-L2 are on the x-axis and L4 and L5 are on the y inside the closed yellow contours. **Bottom**: Topographic countours of the model gravitational potential. The masses are scaled to 0.1 units, so that at a distance of 1 unit, the slope of the surface is $\frac{1}{10}$.

[3]No guarantees this year – it's already 11 PM on 12/24!