Newton Day 2018:
Measurement of the Doppler effect in radar

by Peter H. Mao for Margaret and Owen*

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Abstract

For Sir Isaac Newton's 376th birthday I will discuss the Doppler effect, as it is measured by a continuous-wave radar. The tenuous connection with Newton, in this case, is that we are discussing light in its most wave-like presentation.

1 Introduction

Sirens and train whistles are the classic examples of the Doppler effect. When the fire engine, train or musician races past us, we notice that the pitch of the siren, whistle or musical note transitions from high during approach to low during departure. The change in frequency due to a moving source or observer is a general wave phenomenon, so it applies also to surface, seismic and electromagnetic waves (light). When the speed of the object, v, is much smaller than the speed of wave, c, the fractional change in frequency is (to a very good approximation) the ratio of the two speeds:

$$\Delta f/f \approx v/c. \tag{1.1}$$

For example, a minor second (as in Beethoven's Fur Elise) corresponds to a fractional change of approximately 0.06 in the frequency of the note. The speed of sound on Earth is approximately 350 m/s; therefore, an E turns into an E$^\sharp$ when a musician approaches you at 20 m/s (approximately 45 mph) and it turns into an E$^\flat$ when the musician flees from you at 20 m/s. Detailed derivations of the Doppler effect abound, so I will omit them here. Equation 1.1 suffices for the discussion here, since radar applications are typically nonrelativistic.

With electromagnetic radiation, line-of-sight velocity is determined by the Doppler effect by knowing the energy (or wavelength or frequency) of the radiation in the rest frame of the emitter and measuring the energy of the detected radiation. For astronomical sources, the rest frame energy is determined by identifying the atomic/molecular transitions associated with the radiation. Bulk motion, such as receding galaxies, moves the centroid of emission lines to lower energies, whereas local motion, like the rapid motion at the inner edges of accretion disks around neutron stars and black holes, can be a significant factor in the broadening of emission lines. In a radar system, the energy of the emission is well known, because it is produced by the radar itself.

*See Figure 5.

Methods for measuring the energy of electromagnetic radiation vary across the spectrum due to the nature of the interaction of light with detectors. At high energies, in the γ and x-ray bands, photons are sparse and highly energetic. Most of our detectors determine the energy of a photon by measuring the quantity of electrons it liberated in the detecting medium. At lower energies ($<$ a few keV), spectral resolution suffers because each photon liberates fewer electrons in the detecting medium. Wavelength dispersion with gratings is used to obtain high resolution spectra of low energy x-rays (e.g., Chandra's transmission gratings and XMM-Newton's reflection grating). At optical wavelengths, diffraction gratings are the most common tool for high-resolution spectroscopy. At radio frequencies (below 100 GHz), wavelength dispersion gratings would be onerously large; regardless, the frequency of the radiation is almost low enough to detect directly. The fastest analog-to-digital converters (ADCs) today can sample a waveform at just over 5 GHz. The ease with which radio frequency (RF) signals can be mixed, filtered and modulated allows us to measure the frequency or change in frequency of a signal directly.

The simplest radar is the unmodulated continuous wave (CW) radar. It uses the Doppler effect to determine the line-of-sight velocity of the target but is incapable of determining the distance to the target. This is the type of radar that we encounter most in our daily lives: the street signs that tell us how fast we are driving, the "radar guns" used by police to catch speeding vehicles, pitch-speed radars used in baseball, etc. CW radars are inexpensive – about a decade ago, Mattel was making one as a children's toy.

In the next section, I'll give a quick overview of the mathematical tools needed to understand Doppler radar, and then I'll present a time-domain picture of how the system makes the measurement of velocity. The frequency-domain picture is the typical presentation, but it assumes that the Fourier transform is deeply embedded in your way of thinking, which I can't expect of 8 and 12 year old children.

2 Tools

First, you will need some tools. If you already have these, then skip ahead. There is only basic trigonometry and algebra in this section.

2.1 $x - vt$

When you see this, you will think to yourself that the thing in question is moving in the $+x$ direction at velocity v. Of course, you would rather have it that $x + vt$ means the thing is moving in the $+x$ direction, but that expression sends your stuff in the $-x$ direction.

Pick your favorite number. Is it 4? When I start my clock, $t = 0$, so the x that returns my favorite number is 4. Now we wait until the clock says $t = 1$. Where did my 4 go? If $v = 1$, then I will find my 4 at $x = 5$. If $v = 10$, then my 4 is all the way out at $x = 14$ (it's moving faster).

In physics, we like to keep terms unitless, so we can put them up into exponents, so v is usually taken apart into k and ω, and the $x - vt$ terms will usually look like $kx - \omega t$. You can work out that $v = \omega/k$.

2.2 $\cos(\phi)$ and $\sin(\phi)$

In school, you will learn that $\cos(\phi)$ and $\sin(\phi)$ are the leg-length to hypotenuse-length of a right triangle. From the vertex with angle ϕ, $\cos(\phi)$ uses the close leg and $\sin(\phi)$ uses the far leg in the ratio. You will become great friends with Pythagoras.

Here, I will give you a definition that is only different in words because I want you to think about waves and circles rather than triangles. You know that in a 2-dimensional space, like a piece of graph paper, after defining your x and y axes, you can name any location in the space by a Cartesian pair, such as (3,4).

Let's now restrict our attention to the unit circle – a circle of radius 1 centered at the origin (Figure 1). I can describe any point on the circle by a single number, ϕ, which represents the *counterclockwise* angular distance along the circle from the point (1,0). Given this convention (from Euler?), the Cartesian coordinates of the point on the unit circle at ϕ is $(\cos(\phi), \sin(\phi))$.

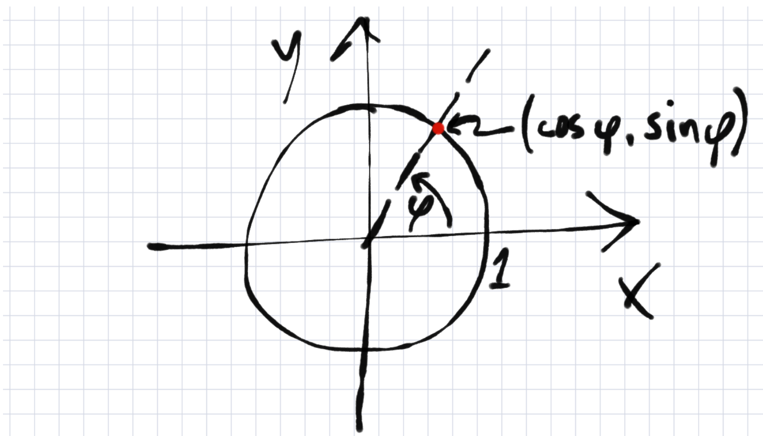

Figure 1: Definition of $\cos(\phi)$ and $\sin(\phi)$ as the x and y coordinates of a point on the unit circle, parameterized by the counterclockwise angle from the x-axis.

If we think of the y-axis of the 2-dimensional space as the complex number i, then the location of the point on the unit circle can be written as $\cos(\phi) + i\sin(\phi)$, or equivalently as $e^{i\phi}$, the gift to humanity from Euler.[1]

Why are we going through so much trouble to name points on a unit circle? It's because sometimes, it's better to think of something as moving on a unit circle with constant speed and other times, it's better to think of the same thing as a wave. In fact, if we let $\phi = kx - \omega t$, then $cos(\phi)$ is the canonical wave traveling in the $+x$ direction. Physically, it will represent the temporal and spatial variation of an electric field, and hence a voltage measured by the radar receiver.

2.3 k and ω

We've already established that the ratio ω/k represents a velocity, for example, of a particular crest of a wave described by $\cos(kx - \omega t)$. What do k and ω represent, then?

[1]See Chapter I-22 of the Feynman Lectures for a beautiful exposition on this relationship.

Suppose we fix our location at $x = 0$. Then the phase (angular location on the conceptual circle) will constantly decrease with increasing time, i.e., it will move clockwise with increasing time. Furthermore, each time the quantity ωt reaches an integer multiple of 2π, we return to the same location on the circle, so $\frac{\omega}{2\pi}$ tells us the number of waves we see in each unit of time (the frequency).

Now, lets fix time instead of location. Take a snapshot of the wave at some time (let's say $t = 0$ to make our lives easy) and survey the wave in the spatial dimension. At $x = 0$, we see a peak in the wave. We also see peaks at each value where kx is a multiple of 2π, so $\frac{k}{2\pi}$ tells us the number of waves we see in each unit of length. If you would rather think in terms of wavelength, then you get the wavelength of the wave from the expression $\lambda = \frac{2\pi}{k}$.

2.4 $\cos(\phi)\cos(\theta)$

Euler's identity, $e^{i\phi} = \cos(\phi) + i\sin(\phi)$, makes it easy to calculate the trigonometric cosine product rule. The exercise is left to the reader; I will just state the result:

$$\cos(\phi)\cos(\theta) = \frac{1}{2}[\cos(\phi - \theta) + \cos(\phi + \theta)]. \tag{2.1}$$

Since cosines are canonical waves, this identity says that the product of two waves of frequencies ω_1 and ω_2 is the sum of two half-amplitude waves of frequencies $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$. Figure 2 gives an example of mixing 4 Hz ($\omega_1 = 4 \times 2\pi$) and 3 Hz ($\omega_2 = 3 \times 2\pi$) waves to produce a sum of 1 Hz and 7 Hz waves. This relationship is the heart of the Doppler measurement in radar.

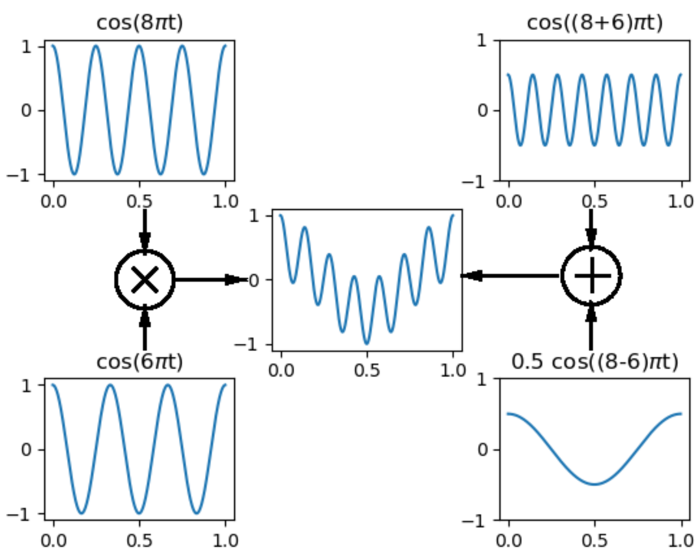

Figure 2: Graphical example of Equation 2.1. The product of the 3 and 4 Hz waves on the left side is identical to the sum of the half-amplitude 1 and 7 Hz waves.

3 Continuous Wave Radar

An unmodulated (single frequency) continuous wave (CW) radar can only measure the line-of-sight velocity of an object, and it does so by measuring the difference in frequency between the transmitted and received light, the Doppler shift.

Before getting into the inner workings of the radar, let's set up a model for the physical system of a CW radar and a moving target (ball or car) approaching the radar. You can think of the transmitter on the radar as a flashlight. When the light has been on long enough for the illumination to return to the radar, we can think of the light as having always been on. Since we are not measuring the transit time, we don't have to worry about signal delays. I like to think of the target as a mirror and unfold the model into a unidirectional one-dimensional system, as I've done in Figure 3, with the origin at the transmitter. This illustrates that when a target approaches the radar at velocity v, the receiver approaches the transmitter at velocity $2v$. This also makes life easy because we can describe both the transmitted and received signals as a wave traveling in the $+x$ direction:

$$s(t) = cos(kx - \omega t). \tag{3.1}$$

Because the light has been on long enough for us to see it illuminating the target, we can treat this equation as describing the signal at all times and locations. This unfolded model is by no means perfect – it violates relativity in many ways, but the typical velocities measured by a radar never approach the speed of light.

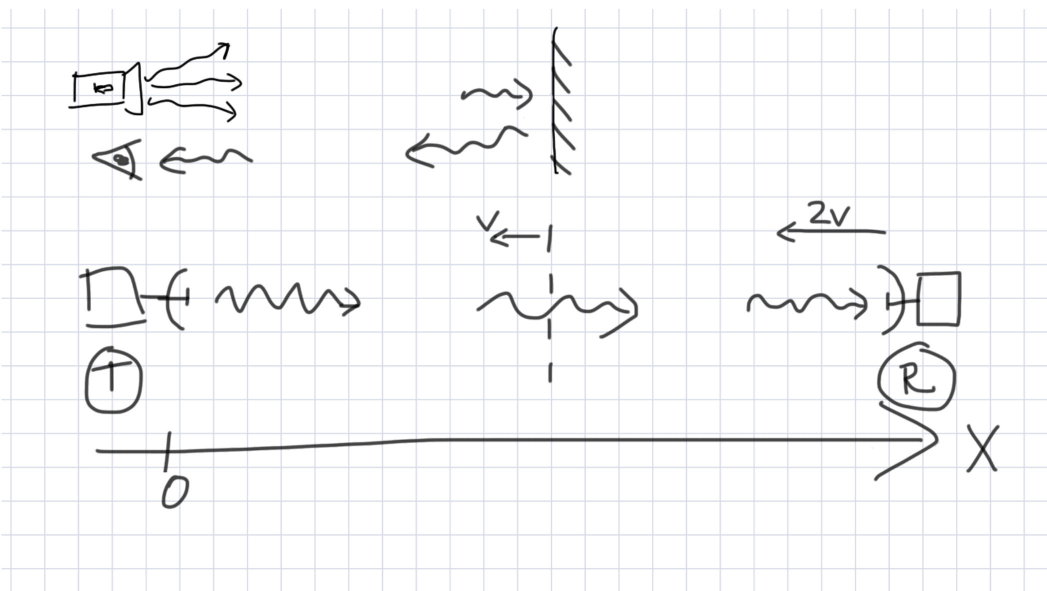

Figure 3: **Top:** Flashlight/mirror analogy for a CW radar. **Bottom:** Radar/target model with the transmitter (T) set at the origin and the receiver (R) placed at the image of the radar reflected through the target. The image of the radar moves at twice the speed of the target.

The signal at the transmitter is

$$s_t(t) = cos(-\omega t) \tag{3.2}$$

because we've set the origin of our coordinate system at the transmitter. Because the receiver has a time dependent position, $x(t)$, the received signal is

$$s_r(t) = cos(kx(t) - \omega t), \tag{3.3}$$

where $x(t) = x_0 - 2vt$ and x_0 is twice the distance from the radar to the target. Written out with time explicitly stated, the received signal is equivalently

$$s_r(t) = cos(kx_0 - k2vt - \omega t), \tag{3.4}$$

which should strongly suggest that the $k2v$ term acts like an additive modification to the signal frequency, ω. At this point, you could substitute in the relationship $k = \frac{\omega}{c}$ and have your Doppler shift formula, $\omega_r - \omega = \frac{2v}{c}$, but let's put this off for a bit to consider the measurement situation.

Most CW radars operate at frequencies in the 10-40 GHz regime, where the wavelengths are of order 1 cm. For vehicles, we could conceivably use longer wavelengths (lower frequencies), but for a baseball, which is only 7.5 cm in diameter, going much lower in frequency would make the target invisible to the radar. In order to measure these frequencies directly, we'd need to be able to digitize the voltage induced by the radiation at more than twice its frequency, and the fastest (ie, most expensive) ADCs on the market today operate at 5–10 GHz.

If we look inside the radar (Figure 4), we will find a few magical components – the mixer and the low-pass filter. The mixer multiplies its input signals and the low-pass filter eliminates high-frequency signals. The CW radar keeps some of its transmitted signal and mixes it with the received signal:

$$s_m(t) = \cos(kx_0 - k2vt - \omega t) \times \cos(-\omega t) \tag{3.5}$$

$$= \frac{1}{2}[\cos(kx_0 - k2vt - (\omega - \omega)t) + \cos(kx_0 - k2vt - (\omega + \omega)t)]. \tag{3.6}$$

The signal is then passed through the low-pass filter, which eliminates the high frequency component, leaving us with the signal

$$s_m(t) = \frac{1}{2}\cos(kx_0 - k2vt) \tag{3.7}$$

$$= \frac{1}{2}\cos(kx_0 - \omega\frac{2v}{c}t) \tag{3.8}$$

For an object moving at 15 m/s, $\frac{2v}{c} = 10^{-7}$, so our 10 GHz signal is brought down into the kHz range, where it is easy to digitize the signal. One could even route this signal to a speaker and audibly listen to the radar tone, as Dr. Ellie Arroway does in "Contact."

I like to think about the measured signal in Equation 3.7 in the $\cos(kx(t))$ form. The effect of downmixing and low-pass filtering the received signal is essentially to freeze the temporal oscillations of the wave at the input to the ADC. In effect, we are freezing time. With the ω term gone, the remaining signal is only a function of the spatial location of the target relative

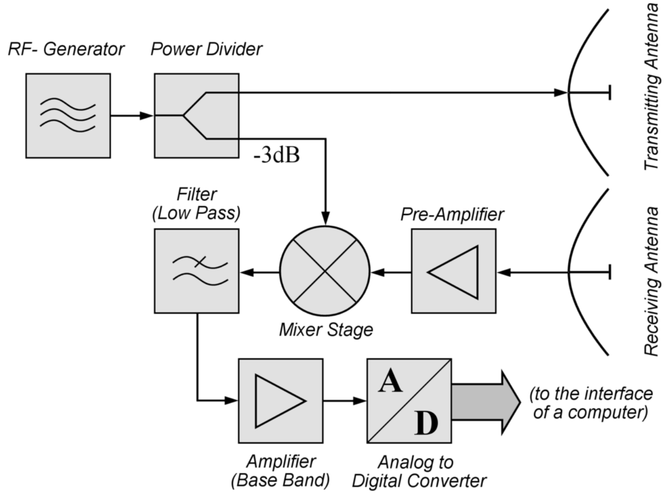

Figure 4: CW radar block diagram. For our purposes, the important components are the mixer and the low-pass filter. The output of the mixer is the product of its inputs and the filter only allows the low frequency component of the mixed signal through. After the low-pass filter, the frequency of the signal is proportional to the target velocity. Credit: Charly Whisky [GFDL or CC BY-SA 3.0], from Wikimedia Commons.

to the radar. For our 10 GHz transmit signal, the ADC will see one period of oscillation for each 1.5 cm that the target moves (3 cm of transmitter to receiver distance). We can imagine that the transmitted radio signal is a corrugated road with bumps spaced about 3 cm apart. To measure velocity, we simply count the number of bumps we drive over per unit time (and divide by 2).

4 Pulse-Doppler Radar

The next-simplest radar is the pulse-Doppler radar, the electromagnetic analogue to a bat's echolocation technique. The "pulse" refers to the burst of radio-frequency light that is transmitted. The radar measures the time between transmission and detection of the pulse. Each nanosecond of wait-time corresponds to 150 mm of range, with typical wait-times ranging from < 100 ns (< 15 m) to several milliseconds (150 km/ms). The "Doppler" part is almost the same as a CW radar, but, in order to detect stationary targets, the down-mixing of the received signal is not taken all the way to DC. Consequently, an ω term remains in the measured signal. The trick with pulse-Doppler radar is in the timing of the measurement. After round-trip time to the target is identified, the signal is measured at the same round-trip time over several pulses.[2] This nulls the ω term in the signal, and we are back to the CW picture of driving over bumps.

I was surprised to learn that there are bats that take their echolocation technique to the next level – frequency modulated pulse Doppler, which is used in radar to improve range

[2]A typical pulse repetition interval is 100 μs (or 10 kHz).

resolution.

5 Games

In keeping with this year's radar theme, I found an inexpensive CW radar make by Netplayz that does not look like a gun. I didn't think it would be a good idea for the kids to be running around Los Angeles with anything looking remotely like a gun! We will build corner cubes of various sizes out of aluminum foil and cardboard and see if we can place some constraints on the frequency of this radar.

(Quiz time!) I'm not saying that we'll do this, but suppose we used an Arduino as the ADC in a CW radar. The Arduino's ADC can sample at roughly 10 kHz. Let's say we have a 10 GHz oscillator. What's the maximum speed measurable by this system?

6 Conclusion

Across the electromagnetic spectrum, the Doppler effect is used to measure the velocities of things, from nearby nonrelativistic things like balls and cars to the recession of galaxies due to cosmic expansion and the orbital velocity of accretion disks around black holes. Radar, which is used for the nearby nonrelativistic things, measures the *change* in frequency due to the Doppler effect rather than the frequency of the received radiation. This was a surprise to me because I had only worked in fields where the absolute energy of the detected radiation was the quantity of interest.

I've presented a time-domain explanation of the Doppler measurement, even though the frequency-domain explanation is more elegant. I hope that the measurement-centric "corrugated road" picture gives you an intuitive sense for how Doppler velocities are measured by radar systems.

7 Acknowledgments

Special thanks this year go to Dr. Ninoslav Majurec, my office-mate at JPL, for bringing me up to speed on radar and leading me around some of my *Denkfehlers* in radar measurement. Not only is Nino an expert in radar and RF electronics, he is also a fount of knowledge of American popular culture, with a specialty in obscure 1980's television.

Figure 5: Owen and Maggie at 8 and 12 years.