Newton Day 2011

v^2/r

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Abstract

For Sir Isaac Newton's 369th birthday, we consider derivations of centripetal acceleration, $a_c = v^2/r$. This relationship does not have the stature of $F = ma$, but it is one of the more frequently used relationships in physics.

1 Introduction

It has been very difficult this year to formulate a theme for Newton Day. I usually start my reading after Halloween; this year, I remembered a book by Subrahmanyan Chandrasekhar that was published in 1995, the year I started graduate school. As one of his late-in-life projects, Chandrasekhar read the Motte translation of Newton's *Principia* with an eye toward translating Newton's "Jamesian prose" (as he puts it) into modern notation and terminology. This year, I turned to Chandrasekhar's *Newton's Principia for the Common Reader* for inspiration. I didn't have to look too far – on page 5, he points out that $a_c = v^2/r$ "could also be derived by considering a sequence of 'reflections from the circle at the several angular point' [of an inscribed polygon] . . ."[1].

Upon reading this passage, I realized that I have been taking $a_c = v^2/r$ for granted for many years now. The relationship is so ubiquitous, that it almost rivals $F = ma$. It applies not only to the motion of planets, but also to cyclotron and synchrotrons, to the isotopes separating in the magnetic field of a mass spectrometer, to cells being separated in a centrifuge, to race cars at the limit of adhesion, to pushing one of those old Trader Joe's shopping carts around. Quite possibly, it applies to anything not traveling in a straight line.

I recalled that when I first learned of that relationship, in high school, I was somewhat suspicious of it, having only recently been introduced to calculus and infinitesimals. Although my unease with the standard derivation of centripetal acceleration did not hinder my ability to progress in physics, I would have appreciated the geometric derivation as a confirmation of the result.

Before we get into the inscribed-polygon solution, let's first go through. . .

2 The standard derivation

Note: if you already know this, skip ahead to the next section. If you don't, then all you need to take away from this discussion is that (I believe) the next section is easier to follow.

In the standard derivation of centripetal acceleration, one considers a test mass moving on a circle at a constant speed (that is, a constant magnitude of velocity). Acceleration is (by definition) the time rate of change of the velocity vector. In one dimension, say, on a dragstrip, acceleration works out as the change in speed per unit time, but on a circle (with only centripetal force acting), the acceleration only changes the direction of velocity, not its magnitude. Graphically, the change in velocity, dv, is a vector from the head of the initial velocity, v_i, to the head of the final velocity, v_f, when the tails of the velocity vectors are translated to a common origin.

Consider the velocity of an object moving in a circle under centripetal force, at velocity, $v(t)$ (Figure 1). In a given time interval, dt, the object moves a distance $|v|dt$. The angle that the object moves through, $d\theta$ is related to the time interval by the arc length, $rd\theta = |v|dt$. The useful form of this relationship (for later use) is:

$$dt = \frac{rd\theta}{|v|}.$$

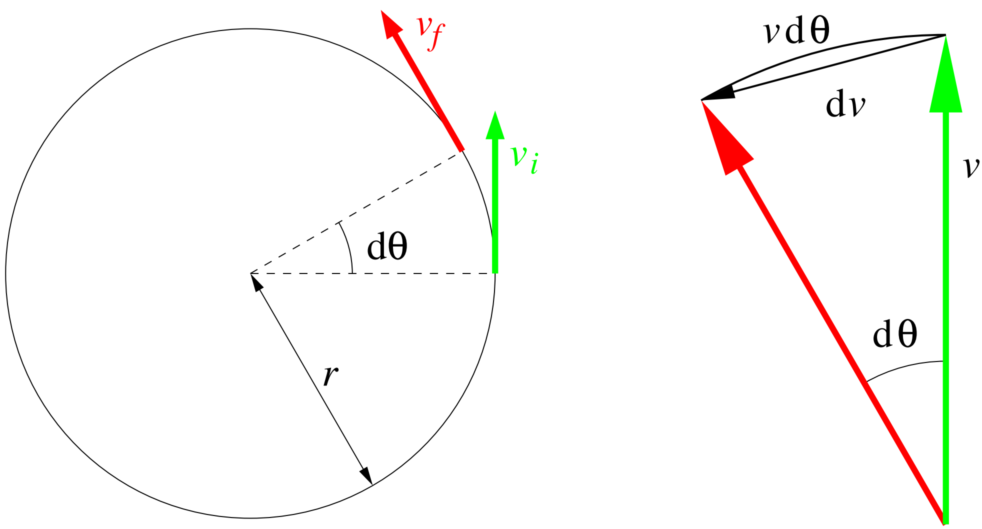

Figure 1: The standard derivation of centripetal acceleration. An object moves in a circular path at velocity v. The green arrow represents the initial velocity, while the red arrow represents the velocity after time interval dt. The angle that the object moves through and the angle swept by the velocity vector are identically $d\theta$.

The change in velocity over the time interval is $dv = v_f - v_i$ (see right side of Figure 1). The angle between the initial and final velocity vectors is the same as the angle that the object moves through, $d\theta$, so the magnitude of dv is exactly $2|v|\sin(d\theta/2)$ or approximately $|v|d\theta$. The exact form is the chord from the head of v_i to the head of v_f, whereas the approximate form is the length of the arc that is swept by going from v_i to v_f. The trick, here, is to convince oneself that in the limit $d\theta \to 0$, the approximate form *becomes* exact. Perhaps this will be the topic of a future Newton Day, but not this year. Regardless, $d\theta$ *is* an infinitesimal, so we get:

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{|v|d\theta}{rd\theta/|v|} \qquad (1) \\ &= \frac{v^2}{r} \qquad (2) \end{aligned}$$

as we should.

3 A geometric derivation

This geometric derivation is interesting simply because, with the right assumptions (which to me, all seem perfectly reasonable), one obtains an expression for the centripetal acceleration that is true for ALL n-gons. It works just as well for a triangle as it does for a circle (an ∞-gon).

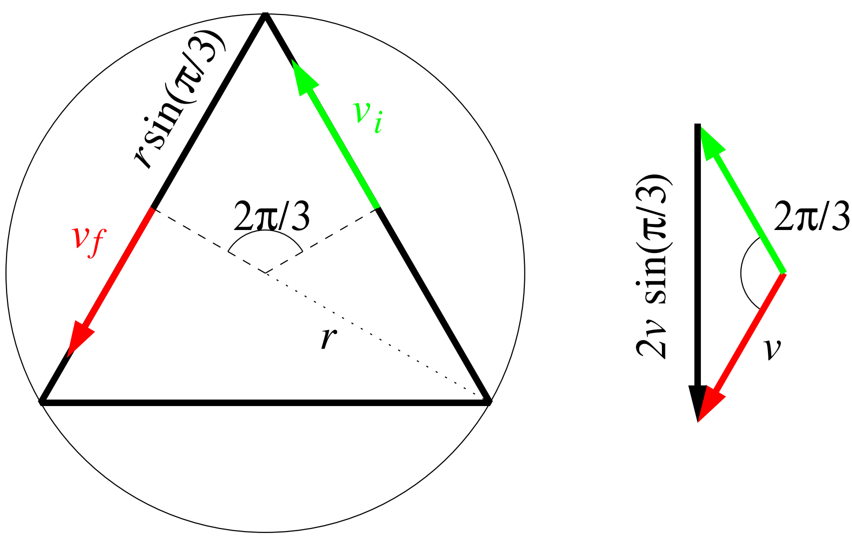

Figure 2: Inscribed polygon example – the triangle.

Let us look at the triangle first. The object in question moves at velocity v along the sides of the triangle shown in Figure 2, which is inscribed in a circle of radius r. The velocity changes direction at each vertex of the triangle by the exterior angle, $2\pi/3$, so the magnitude of the change in velocity is

$$dv = 2v\sin(\pi/3).$$

Since the interaction between the object and its attractor is assumed to be continuous, rather than discrete, we use the time to traverse two half-sides of the triangle in our calculation of the acceleration. The length of each half-side of the triangle is $r\sin(\pi/3)$, so the time to traverse one side (or two half-sides) is

$$dt = \frac{2r\sin(\pi/3)}{v}.$$

The acceleration in the case of a triangle, then, is simply

$$\begin{aligned} a &= dv/dt \\ &= \frac{2v\sin(\pi/3)}{2r\sin(\pi/3)/v} \qquad (3) \\ &= \frac{v^2}{r} \qquad (4) \end{aligned}$$

as advertised!

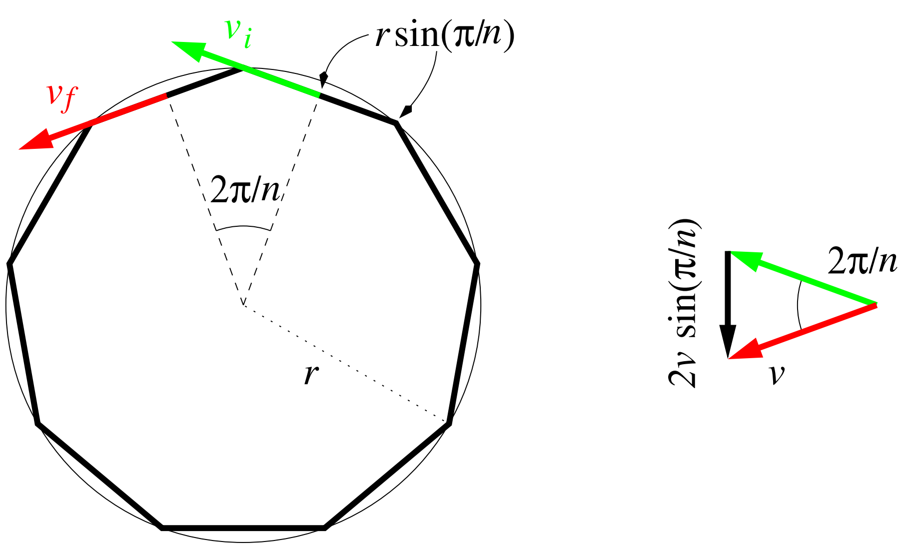

Figure 3: Inscribed polygon example - generalization to n-sided polygons.

Now consider any regular n-gon (e.g., Figure 3's nonagon). The velocity change at each vertex is $2\pi/n$, making the change in velocity

$$dv = 2v\sin(\pi/n).$$

Following the logic of the triangle example, the time to travel from the base of v_i to the base of v_f in the left side of Figure 3 is

$$dt = \frac{2r\sin(\pi/n)}{v},$$

making the acceleration, once again,

$$\begin{aligned} a &= dv/dt \\ &= \frac{2v\sin(\pi/n)}{2r\sin(\pi/n)/v} && (5) \\ &= \frac{v^2}{r}. && (6) \end{aligned}$$

4 I cheated (a little)

Actually, we still have to take limits into account. In the limit $n \to \infty$, the denominator in Equation 5 goes to zero, and dividing by zero is a no-no. Fortunately, we're saved by a guy named l'Hôpital, who popularized a certain Newtonian result about the ratio of the derivatives of terms…[1]

In any case, the offending term in Equation 5 has the form

$$\lim_{x\to 0}\frac{\sin x}{\sin x} = 1$$

and all is well.

5 For those who know, but forget

Using dimensional analysis, one can always "derive" the relationship between centripetal acceleration [L/T^2], velocity [L/T] and radius of curvature [L].

6 From the Scholium to Proposition IV, Theorem IV

The Motte translation [2]:

> The preceding Proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, is reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as that velocity and the number of reflections conjunctly; that is (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time applied to the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

[1] Wikipedia [4] cites a 1694 letter from Johann Bernoulli as the source of the rule, but it is obfuscatedly stated in Newton's *Principia*. Perhaps a topic for a future Newton Day.

The original (via Project Gutenberg [3]:

> Demonstrari etiam possunt præcedentia in hunc modum. In circulo quovis describi intelligatur Polygonum laterum quotcun Et si corpus in Polygoni lateribus data cum velocitate movendo, ad ejus angulos singulos a circulo reflectatur; vis qua singulis reflexionibus impingit in circulum erit ut ejus velocitas, adeo summa virium in dato tempore erit ut velocitas illa & numerus reflexionum conjunctim, hoc est (si Polygonum detur specie) ut longitudo dato illo tempore descripta & longitudo eadem applicata ad Radium circuli, id est ut quadratum longitudinis illius applicatum ad Radium; adeo si Polygonum lateribus infinite diminutis coincidat cum circulo, ut quadratum arcus dato tempore descripti applicatum ad radium. Hæc est vis qua corpus urget circulum, & huic æqualis est vis contraria qua circulus continuo repellit corpus centrum versus.

7 Acknowledgments

Many thanks this year to the providers of source material:

- Michael S. Hart, for creating Project Gutenberg.
- The Caltech Library Archives, for making Chandrasekhar's book available to me for such a long time!

8 Apologies

To mathematicians and physicists who may decry my loose (nonexistent?) usage of vector notation in this year's document. It wasn't *just* laziness – I thought that adding in all the $\hat{r}$'s would negatively impact readability. I hope the holiday-inspired color scheme of the velocity vectors ameliorates some of your irritation.

References

[1] Subrahmanyan Chandrasekhar, *Newton's* Principia *for the Common Reader*. Clarendon Press, Oxford, 1995.

[2] Isaac Newton (Translated by Andrew Motte), *The Principia*. Prometheus Books, New York, 1995.

[3] Isaac Newton, *Philosophiae Naturalis Principia Mathematica*. 1687.

[4] Wikipedia article on l'Hôpital