Newton Day 2020:
Eigenvalues & Eigenvectors

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and Chaz Shapiro for Sylvie and Willow

May you find the hidden beauty in all things

December 25, 2020

Abstract

For Sir Isaac Newton's 378th birthday, we will pay homage to the great geometer by bringing some geometry back into linear algebra. This year, we give you a short geometric exploration on the eigenvalues and eigenvectors of real 2×2 matrices and an application of eigenvectors to the calculation of Fibonacci numbers.

Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.' . . . Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

Sir Michael Atiyah [1]

Eigenvalues and eigenvectors, properties of linear transformations, sit at the interface of geometry and algebra. As algebraic results, eigenvalues and eigenvectors are the answers to many of the problems we come across in physics, engineering and many other areas of study. In school, most of us are taught how to extract eigenvalues and eigenvectors from a strictly algebraic approach. The geometric meaning of these answers, in the context of both mathematics and physics, is also taught; nonetheless, the computations are tedious (much better for computers than humans to complete), so it is easy to lose the intuitive connection between the transformation and the results we seek.

1 Introduction

I still vaguely recall my first introduction to the terms "eigenvector" and "eigenvalue." They must have appeared in the syllabus of an early math or physics class in college, and I recall being highly amused by the peculiarity of the words. Bennet Brown happened to be in the room[1] when I expressed my amusement, so in a calm and reassuring way he explained the

[1] I think we were in the Runkle-Atkinson kitchen of Senior House.

concept to me – that the eigenvectors of a matrix are those vectors that, when multiplied by the matrix, only change by a scaling factor, the eigenvalue. Of course, his explanation was not enough for me to understand eigenvalues and eigenvectors deeply, but he did leave me with the distinct impression that this was a central concept, which indeed it is.

Mathematically, eigenvectors and eigenvalues can be traced back to Euler's and Lagrange's work on the rotation of rigid bodies in free space (tops, frisbees, footballs, both spherical and prolate, spacecraft, etc.). Linguistically, the terms came into English from German in the context of quantum mechanics, via Dirac's[3] and Eddington's[4] publications on Schrodinger's wave mechanics in the mid 1920's.[5]

I learned the algebraic method for finding the eigenvalues of a linear transform way back in college, but it occurred to me, early in the pandemic lock-down, that I had no sense for reliably figuring out the eigenvalues and eigenvectors of the simplest class of matrices without resorting to computations. Here, I will show you a (mostly) visual way to find the eigenvalues and eigenvectors of a 2-dimensional real (as opposed to complex) linear transformation. I will skip over most of the formal mathematical development – you can find that in either Wikipedia or any decent linear algebra text[2] If you're already familiar with linear transformations, eigenvalues, and eigenvectors, skip ahead; otherwise, the next section is (I hope) a gentle introduction to those concepts.

2 Primers

2.1 Linear transformations

Eigenvalues and eigenvectors are properties of linear transformations, so it makes no sense to talk about eigen-things without at least a cursory introduction to linear transformations. Rather than carry on with a precise definition of linear transformations that you can find elsewhere, let's just look at a few linear transformations of a photograph of the Dog Town mural in my neighborhood (Figure 1).

I superimposed a square grid pattern on the image so that you can precisely see how the geometry is changed with each transformation. If you imagine the mural as a mosaic of square tiles, then each transformation consists of choosing a new tile shape and rotation angle for the entire mosaic. In the transformed mosaic, the only additional constraint is that if two tiles shared a pair of vertices in the original square-tiled mosaic, the corresponding tiles in the new mosaic must share the same vertices.

Surprisingly simple, isn't it? Just replace squares with some parallelogram and specify an angle. One of the important features of linear transformations is that they transform lines into lines. Circles and curves get distorted and most point-to-point distances change, but every line segment from the original image maps to a line segment in the transformed image.

[2]My current favorite is Sheldon Axler's *Linear Algebra Done Right*[2].

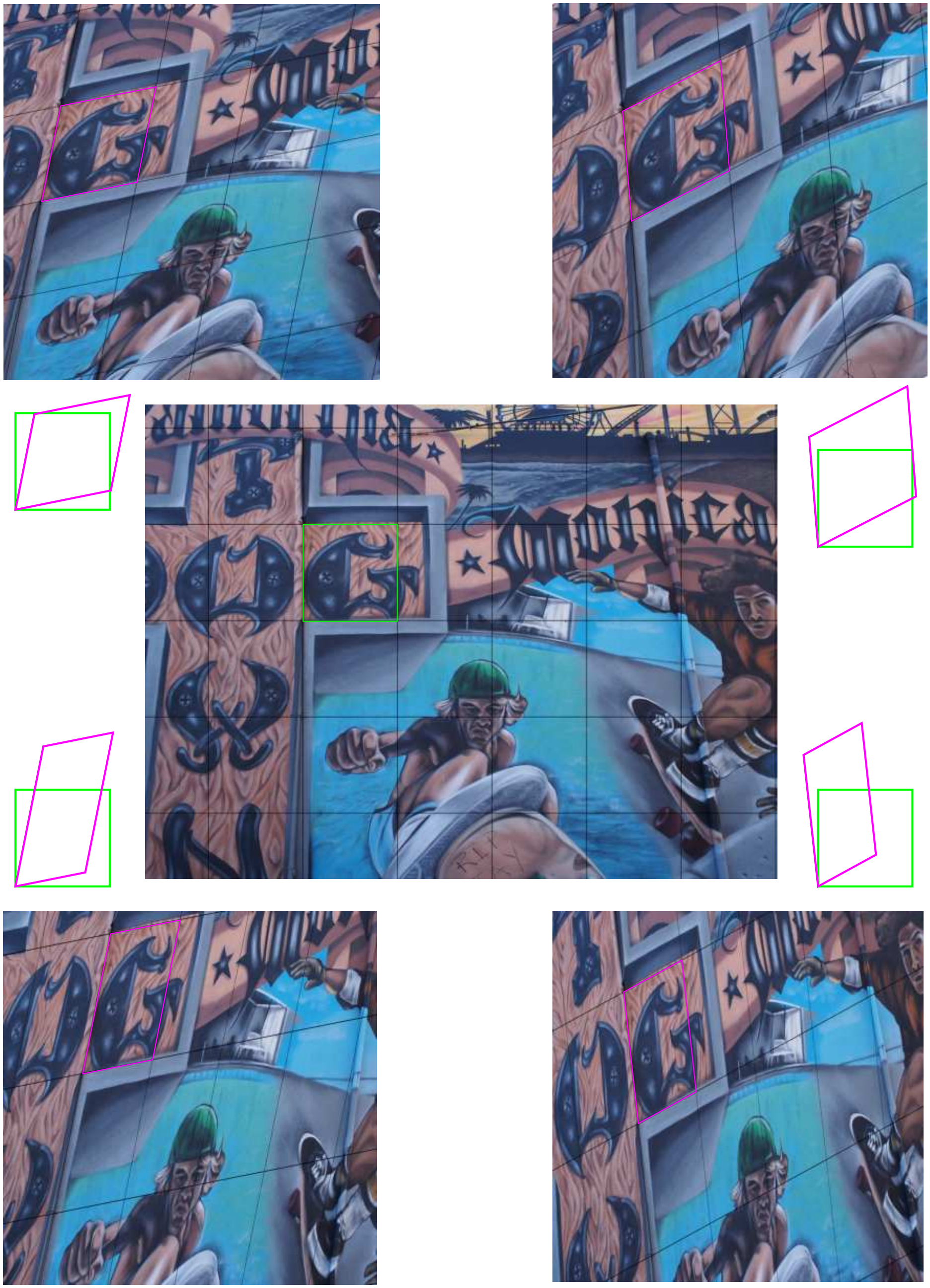


Figure 1: Linear transformations of the Dog Town mural near El Cholo (restaurant) on Wilshire Blvd. in Santa Monica (by LUKEK2S and QUIK_K2S). The grid lines are not part of the mural – I added them to guide your eyes. In the top row, the two transformations use geometrically similar rhombi, but at different angles. Likewise, the transformations in the bottom row use geometrically similar parallelograms set at different rotation angles.

2.2 Eigenvalues and eigenvectors

Linear transformations map line segments to line segments, but most of the line segments are rotated by the transformation. The eigenvectors of a linear transformation are special because they define the directions along which line segments are **not** rotated by the transformation. All line segments parallel to an eigenvector are scaled (stretched or compressed) by the same value; this scaling ratio is called an eigenvalue.

The eigenvector(s) of a linear transformation tell us the directions along which the image is stretched or compressed. If we were to apply the same transformation to an image over and over again, we would find that the resulting images would be repeatedly scaled along the eigenvectors by powers of the eigenvalues. In effect, this gives us a different understanding of the linear transformation that may not be immediately evident from the "tile-shape and angle" representation of the transformation. If you look at the Dog Town images again, can you pick out the eigenvector directions and estimate the eigenvalues? It's not always easy to identify them!

Figure 2: Original (untransformed or identity-transformed) image of my analog stopwatch. (A gift from my Dad when I was in high school.)

To get some practice finding eigenvalues and eigenvectors visually, let's look at some transformations, geometrically similar to those used in the Dog Town image, applied to the face of an analog stopwatch (Figure 2). The graph-paper background of the image shows us the tile shape and orientation of the transformation, and the circular clock face makes it easier to identify the eigenvector directions. Figure 3 has rhombus-tile transformations of the stopwatch and Figure 4 uses a parallelogram with 1:2 ratio in the side lengths. In both sets, the acute angle of the tile is about 1.176005207095135 radians. The numbers and dial markings from the original image are superimposed on the transformed images to help you find the eigenvector directions. The corresponding eigenvalues can be measured by taking the ratio of the diameter of the transformed image to the diameter of the original image along

Figure 3: Linear transformations using a rhombus tile at various rotation angles. Can you find the eigenvectors? Photo editing in this figure and the next one was done by Maggie.

the eigenvectors.

If you succeeded in finding the eigenvector directions, you will notice that the eigenvectors do not necessarily (in fact, rarely) line up with the edges of the tiles. You will also find that there are transformations with 0, 1, or 2 eigenvectors. Thus, we find that both the geometric shape *and* the rotation angle affect the number of independent eigenvectors, and, correspondingly, the number of distinct, real eigenvalues. To be complete, if our transformation is a square with no rotation, the transformation is called scalar and it has only one eigenvalue to which all vectors belong. Since scalar transformations are fairly trivial, they

Figure 4: Linear transformations using a parallelogram tile with 1:2 ratio in side lengths at various rotation angles. Can you find the eigenvectors? They don't show up at the same angles as with the rhombus transformations.

garner little attention.

Now that you have found the eigenvectors of the transformations of the stopwatch face, you will find it easier to identify those directions in the transformations of the Dog Town mural.

3 On 2-dimensional real linear transforms

> *'Should you just be an algebraist or a geometer?' is like saying 'Would you rather be deaf or blind?' If you are blind, you do not see space: if you are deaf, you do not hear, and hearing takes place in time. On the whole, we prefer to have both faculties.*
>
> Sir Michael Atiyah [1]

The questions that instigated this early pandemic diversion are:

1. Can we figure out (or at least make a decent guess at) the eigenvalues and eigenvectors of a matrix without computation?
2. Do most (real) linear transformations have the maximum number of real eigenvalues?
3. What does the space of transformations with real eigenvalues look like?

For dimensions greater that 2, question #1 gets increasing difficult, question #2 is probably "no," and for question #3, I have no idea. For 2-dimensional linear transformations, the answers are "yes," "yes," and "you will see."

Thus far, I have completely avoided putting any equations before your eyes, but as Atiyah alludes, a little bit of algebra can give us a much deeper geometric understanding. I will be brief on the algebra, only using it to motivate the results of the graphical method. Please consult a real text on linear algebra for a proper exposition of the subject.

3.1 Some algebra

In two dimensions, we need four numbers to specify the transformation: the side lengths of the new tile (u and v), the angle between the sides adjacent to the origin δ, and the angle between the bottom edge of the new tile and the bottom edge of the original tile, θ. As is usual in mathematics, positive angles are counter-clockwise. The first three numbers define the size and shape of the new tile, while the last value specifies the rotation of the tile in the new image. This geometric representation of a linear transformation is shown on the left side of Figure 5.

Equivalently, we could specify the Cartesian (x, y) coordinates of the two origin-adjacent vertices of the new tile, as shown on the right side of Figure 5. The Cartesian coordinates are related to the geometric description of the tile by

$$x_1 = u\cos\theta \tag{3.1}$$
$$y_1 = u\sin\theta \tag{3.2}$$
$$x_2 = v\cos(\theta+\delta) \tag{3.3}$$
$$y_2 = v\sin(\theta+\delta). \tag{3.4}$$

In both the geometric and the algebraic approaches, the length scale is set by the side length of the square tiles in the original image.

In algebra, the transformation is represented by a 2×2 matrix

$$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.5}$$

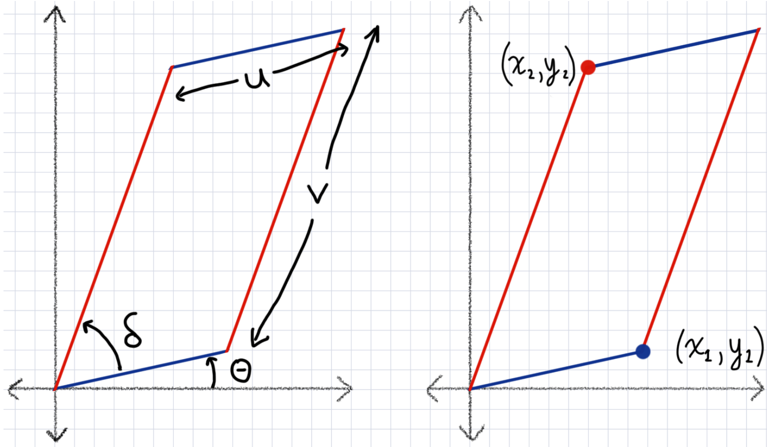


Figure 5: Geometric (left) and algebraic (right) quantifications of linear transformations in 2 dimensions.

which acts on points in the original image by matrix multiplication. Labeling the matrix elements by x's and y's shows you where the coordinates go, while the a, b, c, d labeling is common practice. I will use them interchangeably, even within the same matrix.

The change in area that a linear transformation imparts on its source image is an important quantity. Geometrically, it is simply base $\times$ height, or $uv \sin \delta$; algebraically, it is the determinant of the matrix, $ad - bc$. Non-zero linear transformations with zero area project the entire image onto a line. In the geometric view, the angle of the line is θ, in the algebraic view, the image is projected onto the line defined by either of the columns of the matrix (which are, in this case, in line with each other).

Algebraically, an eigenvector of a linear transformation (A) is a nonzero vector ($\vec{w}$) that satisfies

$$A\vec{w} = \lambda\vec{w} \tag{3.6}$$

for some number λ, which is an eigenvalue of A. This is true for any number of dimensions, not just 2. By introducing the identity transform, we can rearrange the above equation to read

$$(A - \lambda I)\vec{w} = 0. \tag{3.7}$$

Assuming A is not a scalar matrix, $A - \lambda I$ cannot be the zero matrix, so it must be a zero-determinant transformation. Restricting ourselves to 2 dimensions again, the determinant (area) of $A - \lambda I$ is

$$(a - \lambda)(d - \lambda) - bc = 0. \tag{3.8}$$

This is a quadratic function with λ as the free variable, a problem that humans have been able to solve since at least the time of the Babylonians. It is also known as the characteristic equation for A, which is a nod to Laplace's terminology for eigenvalues. There are two solutions, both, one or neither of which will be real numbers, and these are the eigenvalues of our 2-dimensional linear transformation, A. Has your geometric intuition fled you yet?

3.2 A graphical method; or, cheating the devil

In Section 2.2, you visually identified the eigenvector of some linear transformations and found that there were cases in which we find 0, 1 or 2 directions (and, correspondingly, 0, 1, or 2 eigenvalues). In Section 3.1, we algebraically came to the same conclusion. Now using that beautiful machinery just a little bit more, we will cheat the devil. We will be able to know how many real eigenvalues a 2-dimensional linear transform has, just by looking at the first column of the matrix and sketching the boundary in 2-space where we have exactly one real eigenvalue.

Going back to high-school level algebra, you know that the part of the quadratic equation inside the square root governs the number of real solutions. For the characteristic equation (Equation 3.8), the discriminant is

$$\Delta = \underbrace{(a+d)}_{\mathrm{Tr}\,A}{}^2 - 4\underbrace{(ad-bc)}_{\det A}. \tag{3.9}$$

Making the symbolic substitutions $c \to x$ and $d \to y$ and rearranging a little bit, we arrive at the highly suggestive

$$x = -\frac{(y-a)^2}{4c} \tag{3.10}$$

describing when the discriminant is zero and we have one eigenvalue. This is a sideways-facing parabola with its focus at $(-c, a)$ and directrix along the vertical line $x = c$. The negative sign out in front means that we have 2 real solutions *outside* this parabola, and 0 real solutions *inside* this parabola.

To cheat the devil, let us reframe how we think about these 2-dimensional linear transformations. In the Primer section, it was natural to think of the tile shape as the defining feature of a set of transformations, and to use the rotation angle to explore variants within that set. In other words, we fixed three parameters (u, v, δ) and allowed one (θ) to vary. Instead, as suggested by Equation 3.10, let's fix the two parameters a and c, which constitute the first column of the matrix and let the second column, $\begin{bmatrix} x \\ y \end{bmatrix}$, vary. Geometrically, we are fixing the bottom edge of the tile (u, θ) and allowing the left edge (v, δ) to run free. When we are faced with a specific linear transformations, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we only need to sketch the parabola in Equation 3.10 and identify where the point (b, d) lies in relation to that parabola.

The formula for the 1-eigenvalue parabola is not something I can keep in my head, but a simple geometric construction is something I can hold on to. We only need two simple geometric operations to locate the focus and directrix, which fully define the parabola. Figure 6 shows two ways to locate the defining features for transformations with $(a, c) = (5, 2)$. A classical Greek geometer would use the operations on the left side:

1. The focus is located by rotating (a, c) counter-clockwise by a quarter turn.

2. The directrix is the vertical line located by reflecting the focus across the y axis.

If you prefer Japanese origami to straight-edge and compass constructions, then you use the operations on the right side:

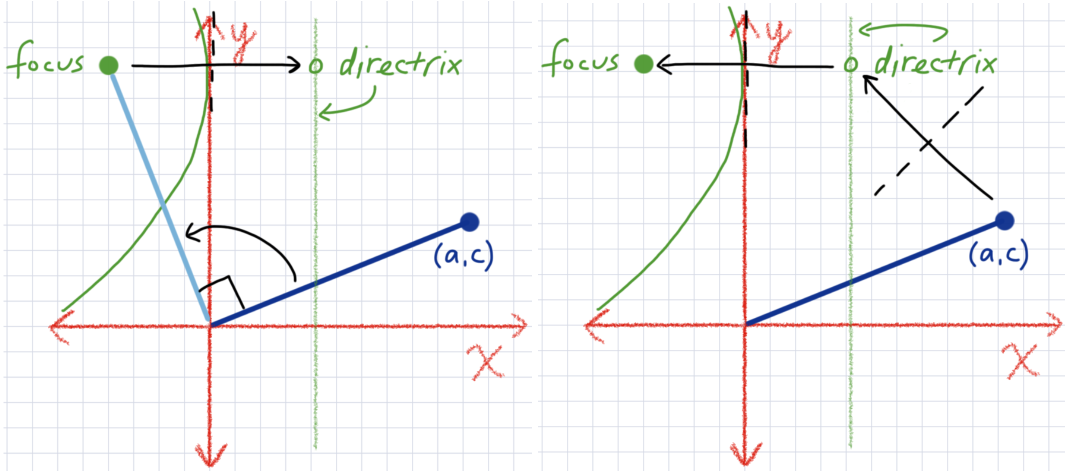


Figure 6: Two ways to find the focus and directrix of the 1-eigenvalue parabola in the space of all 2-dimensional linear transformations with bottom edge given by (a, c). On the left side, we find the focus by rotation, then the directrix by reflection; on the right side, we find the directrix and then the focus by two successive reflections.

1. The directrix is the vertical line located by reflecting (a, c) across the diagonal line $x = y$.
2. The focus is located by reflecting the directrix-defining point across the y axis.

In addition to the focus and the directrix, there is one other easily identifiable point – where the trace and the determinant of the transformation are both zero, making the discriminant trivially zero. The trace is zero when $y = -a$, and the determinant (area) is zero when the edges of the tile lie on the same line. At this point, the parabola is tangent to the line defined by the (fixed) bottom edge of the transformation. In Figure 7, we have our graphical construction zoomed out (indeed, using the previously scorned scalar transformation) so that you can see the zero trace, zero determinant tangent point.

Many general truths about 2 dimensional transformations can be gleaned from Figure 7, most of which I leave for you to verify on your own:

- The parabola is always located in the quadrant counterclockwise to the quadrant containing (a, c). This should be obvious from the rotation step. Also, if (b, d) is on the focus, the tile is square and rotated by θ.
- When (b, d) falls on the parabola, the eigenvector is parallel to the tangent to the parabola at that point. This requires just a few lines of very messy algebra.
- When (b, d) falls on the parabola, the eigenvalue is half the trace of the matrix. This is high-school algebra.
- When (b, d) falls on the directrix, the eigenvectors are orthogonal. This is beyond high-school algebra. See Chapter 7 of Axler.[2]

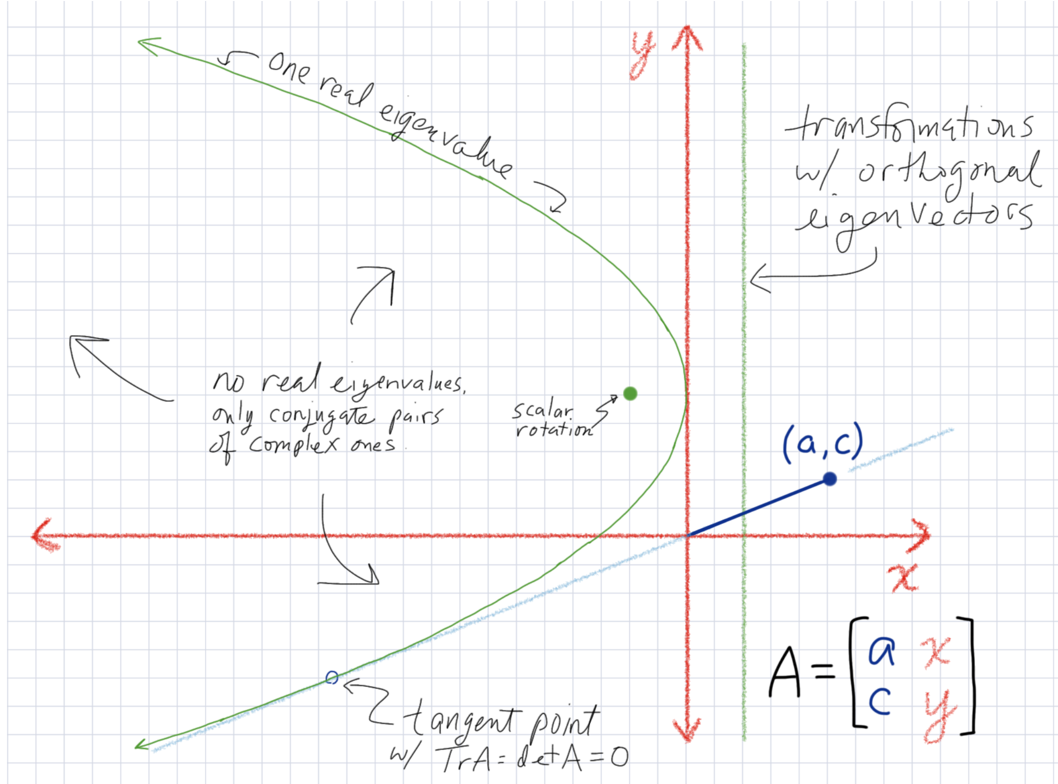


Figure 7: Map of 2-dimensional linear transformations with (a, c) fixed at (5,2). All values of x and y that fall outside the parabola have 2 distinct eigenvalues and 2 distinct eigenvectors.

- We always have two eigenvalues if (b, d) falls on the starboard side of the the bottom edge because the parabola lies entirely on the port side of the bottom edge. Having a "left edge" on the starboard side means that the image has been flipped over. It's interesting to note that a flipped image (negative area) *always* has two distinct eigenvalues.

- Most 2-dimensional linear transformations have 2 real eigenvalues.

Finally, a quick word on the 1-eigenvalue linear transforms. As you can see from the figure, these are very much in the minority of 2-dimensional linear transformations, occupying only a 1-dimensional subspace in each map. Somewhere in my reading, I came across the term "defective matrix" for these transforms, presumably because they cannot be diagonalized. Geometrically, they shear the image along the eigenvector direction. In algebra, they lead into the topics of Jordan normal form and generalized eigenvectors.

4 The Fibonacci Sequence in Closed Form (Chaz Shapiro)

As a student, I once had to derive a closed-form expression for the Fibonacci Sequence, and I've always thought that was a cool and surprising application of eigenvectors. Since the famous sequence is defined recursively, it naively seems that you need to follow the whole "trajectory" to figure out each term. But this is something like a differential equation, which describes an evolving trajectory based on the instantaneous state. Since many differential equations are solvable, perhaps we should suspect that Fibonacci is too.


[3]

The Fibonacci Sequence is defined by:

$$x_{n+1} = x_n + x_{n-1} \tag{4.1}$$

for $n > 1$ where $x_0 \equiv 0$ and $x_1 \equiv 1$. The clever move is to rewrite this as a system of equations:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} . \tag{4.2}$$

The 1st equation in the system corresponds to (4.1) while the 2nd equation just accounts for x_n moving down a slot in the vector after one step. We rewrite the above equation after defining some notation:

$$\vec{x}_n \equiv \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} \quad ; \quad \hat{A} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

$$\vec{x}_n = \hat{A}\vec{x}_{n-1} . \tag{4.4}$$

We now have a compact way to express any term of the sequence:

$$\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; \vec{x}_n = \hat{A}^n \vec{x}_0 . \tag{4.5}$$

[3] Photo by Etsy seller LaserTrees

Finding the nth term is equivalent to applying the matrix $\hat{A}$ n times to the initial condition, $\vec{x}_0$. If we can decompose $\vec{x}_0$ into eigenvectors of $\hat{A}$, then operating with $\hat{A}^n$ will be simply achieved by multiplying the eigenvectors by their eigenvalues to the nth power. In physics, we think of eigenvectors as modes that evolve relatively simply on their own. The idea that there are such modes hidden in the familiar Fibonacci Sequence might make your heart grow three sizes.

The two eigenvalues of $\hat{A}$ are straightforward to find in the usual way using the determinant. However, I noticed a way to find them that's a bit more Fibonacci-y.[4] If we start with a generic vector defining any orientation, $\vec{v} = (v, 1)$, and assume it is an eigenvector, $\hat{A}\vec{v} = \lambda\vec{v}$, then by the definition of $\hat{A}$,

$$\begin{pmatrix} 1+v \\ v \end{pmatrix} = \lambda \begin{pmatrix} v \\ 1 \end{pmatrix} . \tag{4.6}$$

Since these two vectors are the same, the ratios of their elements are the same:

$$\frac{1+v}{v} = \frac{v}{1}, \tag{4.7}$$

which is the defining property of the Golden Ratio, φ, i.e. that the ratio of the two parts equals the ratio of one part to the sum. Like a Christmas ghost, we should have expected φ to make an appearance.

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618034... \tag{4.8}$$

The conjugate of φ is also a solution to (4.7):

$$1 - \varphi = \frac{-1}{\varphi} = \frac{1-\sqrt{5}}{2} = 0.618034... \tag{4.9}$$

By (4.6) and (4.7), we can deduce $v = \lambda = \varphi$ or $-1/\varphi$. Thus, the eigenvalues are

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2} = \varphi \text{ or } -1/\varphi , \tag{4.10}$$

and the eigenvectors satisfying $\hat{A}\vec{e}_\pm = \lambda_\pm \vec{e}_\pm$ are

$$\vec{e}_+ = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} ; \vec{e}_- = \begin{pmatrix} -1/\varphi \\ 1 \end{pmatrix} . \tag{4.11}$$

Finally, we can express the initial condition in the eigenvector basis,

$$\vec{x}_0 = \frac{\vec{e}_+ - \vec{e}_-}{2\varphi - 1} , \tag{4.12}$$

and see what happens when we apply $\hat{A}$ to it n times:

$$\vec{x}_n = \hat{A}^n \vec{x}_0 = \frac{\varphi^n \vec{e}_+ - (-1/\varphi)^n \vec{e}_-}{2\varphi - 1} . \tag{4.13}$$

[4] Alternatively: Fibonaccical, Fibonaccious

The bottom part of the vector equation gives us the equation we want:

$$x_n = \frac{\varphi^n - (-1/\varphi)^n}{2\varphi - 1} \,. \tag{4.14}$$

This is a closed form for the Fibonacci Sequence. Hallelujah. QED. Where's the Tylenol? For $n \gg 1$, the $1/\varphi$ term decays away, leaving

$$x_n \approx \frac{\varphi^n}{2\varphi - 1} \;\; (n \gg 1) \,. \tag{4.15}$$

After some light Googling, I learned that (4.14) is known as the Binet Formula and can be derived in other ways.[5] It is one of various ways to compute the Fibonacci Sequence faster than the obvious algorithm based on the definition, which is the least efficient method.[6] However, care must be taken in programming or you'll run into numerical precision problems. Here's what I get when defining an unsophisticated function that uses (4.14) with $\varphi = 1.618034$ as a single precision float in Python2: (Python converts answers to double precision)

0 0.0
1 1.0
2 1.0
3 2.000000036705515
4 3.00000007341103
5 5.000000183527576
...
31 1346269.4036351882
32 2178309.675194743

...well that escalated quickly. If we're rounding, we'll be wrong by 1 on the 32nd term. Switching φ to double precision does a little better:

70 190392490709135.44
71 308061521170129.7

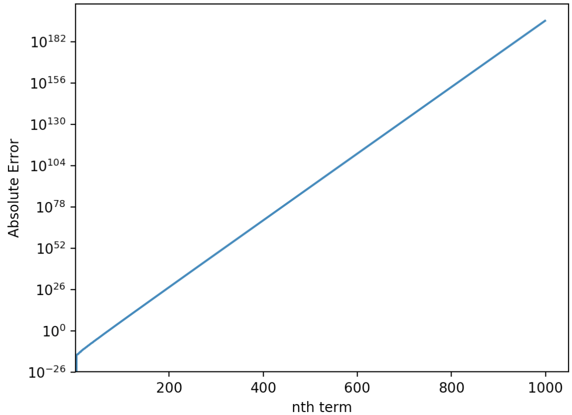
Your mileage may vary depending on how your programming language's numerical precision works. Assuming a fixed precision for φ, we'll have a fixed error $\Delta\varphi$ in that variable. Expanding the numerator of (4.15),

$$(\varphi + \Delta\varphi)^n = \varphi^n + n\frac{\Delta\varphi}{\varphi}\varphi^n + ... \tag{4.16}$$

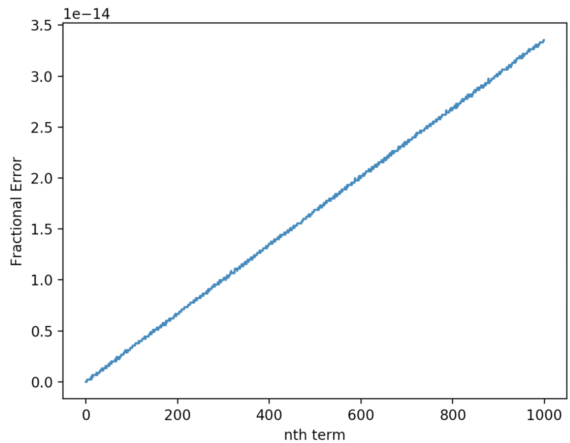
the error given by the 2nd term is growing a bit faster than exponentially, while the relative error (divide out φ^n) is growing linearly. This behavior is confirmed by computing the Sequence two ways in Python.

[5] e.g. http://mathonline.wikidot.com/a-closed-form-of-the-fibonacci-sequence.
[6] e.g. Ali Dasdan, "Twelve Simple Algorithms to Compute Fibonacci Numbers".

Absolute error (closed form minus recursive form):



Relative error (absolute error divided by exact term):



If more precision is needed, you'll need a more careful algorithm, but a quick and dirty function is good to 1% up to about the 3×10^{14}th term! Dear, dear reader... I hope this adds to your enjoyment of the Fibonacci Sequence.

5 Conclusion

Using geometry and algebra together, I've described a way to quickly sketch a graph to determine if the 2-dimensional linear transformation at hand has 0, 1 or 2 real eigenvalues. The graph can also be used to quickly estimate the eigenvector directions and the eigenvalues, but I forwent that discussion in favor of pointing out some general properties of 2-dimensional linear transformations from the graph. The discussion here does not easily extend to higher dimensions, which is why you won't see this in any linear algebra textbook.

Perhaps the only real utility of this year's Newton Day topic is that it serves as an example of dimensionality reduction as a means to understanding a problem. As visual creatures, we are very good at figuring out 2-dimensional problems (jigsaw puzzles), fairly decent at 3-dimensional ones (Rubik's cube), and pretty bad at 4+dimensional problems (special relativity). The space of all 2-dimensional linear transformations is 4-dimensional within which simple questions, such as those posed at the beginning of Section 3 are difficult to answer. By reducing it to a 2-dimensional problem (all 2-dimensional linear transformations with the first column fixed) and understanding the general features of the map, the aforementioned questions become easy to answer.

References

[1] M. Atiyah, "Mathematics in the 20th Century," *Bull. London Math. Soc.*, **34** (2002) 1-15. DOI:10.1112/S0024609301008566

[2] S. Axler, *Linear Algebra Done Right*, *3rd edition*. Springer, 2015.
This book lives up to its audacious title.

[3] P.A.M. Dirac, "On the Theory of Quantum Mechanics," *Proc. Royal Soc. A*, **112** (1926) 661-677.
Dirac uses "eigenfunction" throughout, which is the continuous analog of "eigenvector."

[4] A.S. Eddington, "Eigenvalues and Whittaker's Function," *Nature*, **120** (July 23, 1927) 117.

[5] J. Miller, "Earliest Known Uses of Some of the Words of Mathematics" https://jeff560.tripod.com/mathword.html, accessed December 2020.
This is a really cool website that looks like it will be gone in a decade.

Figure 8: Left: Sylvie & Willow. Right: Maggie & Owen.