Newton Day 2016:
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and Charlie Fisher for Penny, Jack, and Phoebe

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Abstract

For Sir Isaac Newton's 374th birthday, we consider the contributions of one of his greatest enemies, Robert Hooke, to classical physics. Physicists and mechanical engineers mainly know Hooke for his spring law, but he also had an important role in Newton's theory of gravity and in many other fields. Herein you will find a new telling of the Newtonian "apple" fable, a primer on Hooke's law and an application of said law where music and mechanics are entwined.

1 Introduction

This year, though it is a great year for gravity (see [1]), we will not dwell on the topic of gravity for too long. Instead, we will concentrate on Robert Hooke's (1635-1703) "true theory of Elasticity or Springiness," which he announced as an anagram in 1676 [8] and published in 1678 [9]. After projectile motion and rigid-body dynamics (tops), Hooke's Law is one of the staples of early physics education, and its consequences and utility carry deeply into physics and engineering. But before we think about springs, we should get to know a bit about this contemporary adversary of Isaac Newton's, who has (possibly) the lowest fame to contribution ratio of any scientist.

1.1 Legends

1.1.1 Traditional version

In the classic telling of the Legend of Gravity, the Great Plague of 1665 sent Newton away from Cambridge back home to Woolsthorpe. The following year, while sitting beneath an apple tree in the garden, cogitating on optics and basking in his discovery of the generalized binomial theorem, an apple falls from the tree and strikes him on the head. Newton's inverse-square Law of universal gravitation crystallizes into being. Newton then waits for nearly two decades to publish his results in Latin in order to limit its dissemination to only the very, very highly educated.

1.1.2 Fractured Fairy-tale version

In 1679, we find Newton, sitting under an apple tree. It is autumn, so the apples are ripe. Why he is sitting there, we do not know. What he is thinking about, also a mystery. Theology? Alchemy? There is a rustling in the branches above and an apple strikes him on the head. Then another. The third misses him by a few microns, but only because of the motion of the Earth. Newton looks up into the tree and sees his Optics Nemesis, Robert Hooke, up in the tree ready to drop another apple on his head. The episode inspires Newton to return to the study of celestial motions, resulting in the publications of *De motu corporum in gyrum* ("On the motion of bodies in an orbit") and the famous *Philosophiæ Naturalis Principia Mathematica* over the following decade. Rejoice.

1.2 Robert Hooke

Today, Robert Hooke is known for only two things: his spring law and his illustrations of observations made with microscopes and telescopes in his book, *Micrographia* (1665). Furthermore, people are often surprised to find out that those two contributions came from the very same person! To put him in the context of Isaac Newton's life, Hooke was appointed as the Gresham College Professor of Geometry and published the book that made him internationally famous while Newton was a student at Trinity College, Cambridge.

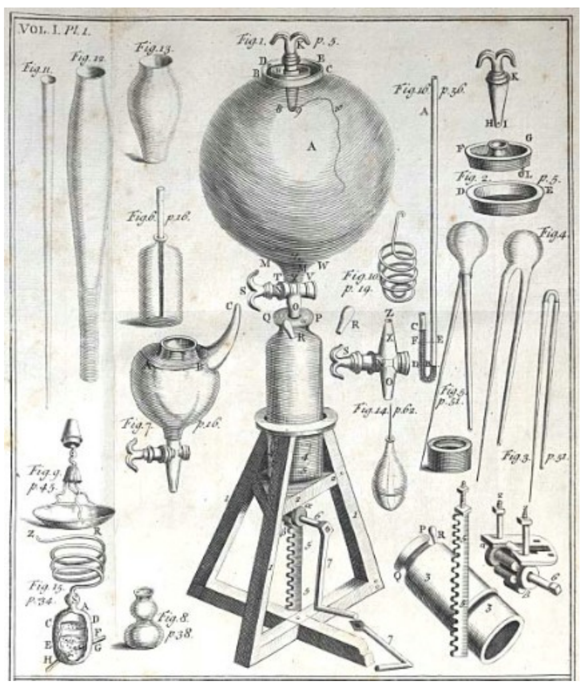

Figure 1: Hooke's pump and vacuum chamber built for Robert Boyle's experiments on air [3].

Hooke made important contributions in so many fields that it is difficult to fathom or convey the breadth and depth of his interests, At a minimum, he made contributions to physics, astronomy, engineering, biology, geology, paleontology, architecture, and navigation (horology, time keeping). Hooke started his scientific career as an assistant to Robert Boyle, of the gas law[1]. In fact, Hooke is credited with designing the vacuum chamber and pumps that Boyle used in his seminal experiments[2] [4, 2].

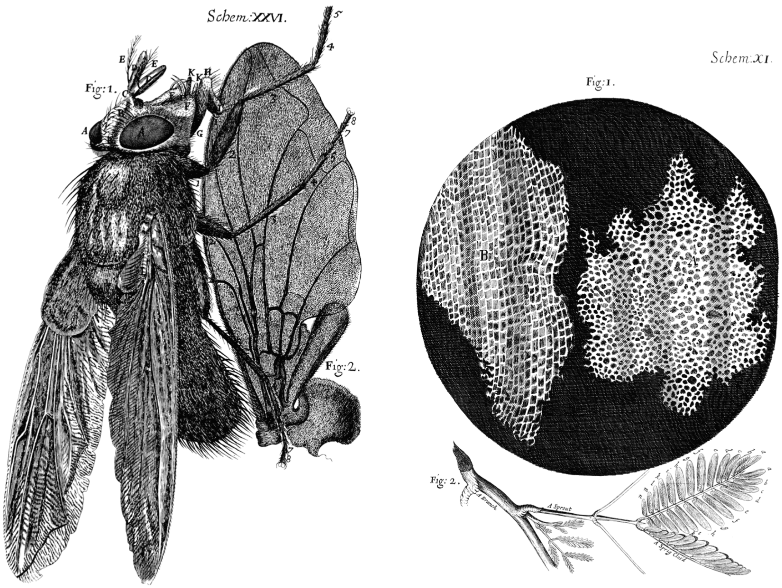

Figure 2: Fly's wing and cork cells from *Micrographia*.

In *Micrographia*, we find the first descriptions of insect anatomy: eyes, wings, feet (see Figure2. He coined the word "cell" in the biological context. He observed petrified wood, setting forth hypotheses on how the wood came to be replaced by minerals while keeping the physical form of wood down to microscopic details. There are claims that he made the first observations of foraminifera (I don't have time to hunt this down), which are a key piece of evidence in reconstruction of the KT event 65 million years ago.

Among his seminal inventions is the iris aperture, a fundamental component of any imaging device, including modern cameras. In the records of the Royal Society of London from July 27, 1681,

> *Mr. Hooke shewed his new-contrived aperture for long telescopes, which would open and close just like the pupil of a man's eye, leaving a round hole in the middle of the glass of any size desired ; which was well approved of.*

Exemplary of his diverse interests, the record continues:

[1] Under isothermal conditions, pressure and volume are inversely proportional, ie, PV = constant

[2] "I put both Mr.G[reatorex] and R. Hook . . . to contrive some Air Pump . . . [that] might be more easily manag'd: And after an unsuccessful tryal or two of ways propos'd by others, the last nam'd Person [Hooke] fitted me with a Pump, anon to be describ'd." R. Boyle in [2]

He shewed an experiment of making musical and other sounds by the help of teeth of brass-wheels ; which teeth were made of equal bigness for musical sounds, but of unequal for vocal sounds.

In the biographical sketch in *The Posthumous Works of Robert Hooke*, it is reported that seventeen years earlier (1664), Hooke had used these brass wheels to determine the frequency of the musical note G to be 272 vibrations per second. Today, we would call that frequency a sharp C, but it was with this knowledge that he estimated how fast the fly's wings move.

Hooke is credited with invention of the universal joint, which he describes in the Cutlerian Lectures, *Animadversions* and *Helioscopes*. Historical priority, however, appears to go to Gaspar Schott, a German Jesuit, who describes the mechanical contrivance in *Technica curiosa sive mirabilia artis* (1664). Like Newton's *Principia*, but probably for different reasons, *Technica curiosa* is written in Latin, and I am not aware of any English translation. Whether Hooke invented the U-joint independently or not is a different project from this one. As an interesting aside, the U-joint is also known as the Cardan joint for Gerolamo Cardano (1501-1576) of Italy because he wrote extensively about gimbal mounts, which are a precursor to the U-joint. I only mention Cardano here because he is also the author of *Ars Magna* (1545) which contains the solution to the general cubic polynomial (due to Scipione del Ferro (1465-1526) and Niccolò Fontana Tartaglia (1499-1557) independently) and the first use of square roots of negative numbers.

Hooke was also a surveyor and architect. After the Great Fire of London in 1666, Hooke submitted a plan, through the Royal Society, for rebuilding London. The Common Council of London were so impressed with his proposal (though it was not implemented) that he was invited to join the committee responsible for the reconstruction of the city [5]. During the period in which London was rebuilt, Hooke recognized the connection between the catenary (the shape of a chain or rope suspended at its ends) and structural arches. Though arches have been used in architecture at least since the Romans, Hooke was one to optimize the design.[3]

The contributions to society recounted above are but a tiny fraction of Hooke's works. Despite considerable scholarship into his place in the history of science, Hooke is still only well known for his law of springs and for *Micrographia*.

2 Hooke and Newton

Today, Newton and Hooke are pedagogically linked in introductory classical mechanics. During their lifetimes, they were linked by their famous conflicts on optics and celestial mechanics. Ultimately, and probably unjustly, the interaction led to the obscuration of Hooke's posthumous reputation, largely by Newton himself.

In the 1671, Newton published his "Theory about Light and Colors" [12] in which he reports on the nature of white light, being compounded of all colors, and on the immutability of the color of light by reflection or refraction. The next year, Hooke criticized Newton's paper point by point and claimed that Newton did not add anything new to the discourse on light: "I agree with the observations of the ninth, tenth, and eleventh, though not with

[3] The anagram for catenaries, like the spring anagram, was published in *Helioscopes*; however, the solution was published posthumously: *Ut pendet continuum flexile, sic stabit contiguum rigidum inversum* meaning "As hangs a flexible cable so, inverted, stand the touching pieces of an arch." (from Wikipedia's entry on catenary)

his theory, as finding it not absolutely necessary, being as easily and naturally explained and solved by my hypothesis." [7]

The optics controversy, for which fuller accounts are readily available [5, 11], escalates until January 20, 1675, when Hooke sends Newton a conciliatory letter, the reply to which contains Newton's reference to "standing on the shoulders of giants"[4] [13].

Far worse was their dispute over gravitation after the publication of *Principia*. The general consensus today is that, based on their correspondence in 1679 and 1680 [11, 15, 16], Hooke certainly influenced Newton on the nature of gravitation, but did not have either the mathematics nor the single-minded focus to solve the problem as Newton did. The lack of any attribution to Hooke's contribution soured their relationship irreparably. Newton waited until 1704, a year after Hooke passed away, to publish *Opticks*, and in moving the Royal Society from Gresham College to Crane Court (1710), managed to "lose" many of Hooke's papers, experimental apparatuses and even his portrait.

3 Latitude, longitude, and springs

Put away the GPS. Turn off the phone. Where on Earth are you? If you are on land, there are mountains, hills, rivers, outcroppings, buildings, roads, and other landmarks to guide you. What if you are in the middle of the ocean with no land in sight? Here looks a lot like there. If the sky is clear, you can find the sun or identify the stars. Check your compass and your calendar. If it's daytime, you can figure out your latitude (N-S position) by measuring the sun's maximum elevation (which will be your local noon) and knowing the day of year. If it's night-time, measure the elevation of a known star on the N-S meridian to determine your latitude.

What about longitude (E-W position)? Assuming you haven't changed the time on your watch since you left port, check the time at local noon and read off your longitude directly. If your watch says 13:00 at local noon, then you're 15 degrees West of where you started and if it says 11:00 AM, then you're 15 degrees East of where you started. One small problem I forgot to tell you. It's the 17th century and the only clocks are pendulum clocks that can't keep time on ships that pitch and roll with the waves. Good luck.

Until John Harrison (1693-1776) built his H-1 marine chronometer in 1737, sailors had no idea of their longitudinal position on the open ocean. Columbus, the Pilgrims, Magellan – none of them had any idea of where they were (longitudinally) during their travels. Before Harrison, others, notably Hooke and Christian Huygens (1629-1693), attempted to build a timepiece that would maintain accuracy in the harsh environment of sea-faring vessels. Hooke's interest in springs, his cryptic announcements prior to publication, and many of his disputes with people not named Isaac Newton stem from his attempts to build and commercialize a marine chronometer.

Although Hooke did not make a successful marine chronometer, certainly his interest in the problem, his collaboration with the watch-maker Thomas Tompion (1639-1713) and the publication of his findings in *Potentia Restitutiva* fostered the intellectual climate in England that paved the way for Harrison's success in solving the longitude problem only 60 years later.

[4] Personally, I was surprised to find out that the context of that letter is optics and not celestial mechanics, and that the "giants" are Descartes and (possibly) Hooke, not Galileo, Copernicus, Brahe and Kepler.

4 Hooke's Law and its consequences

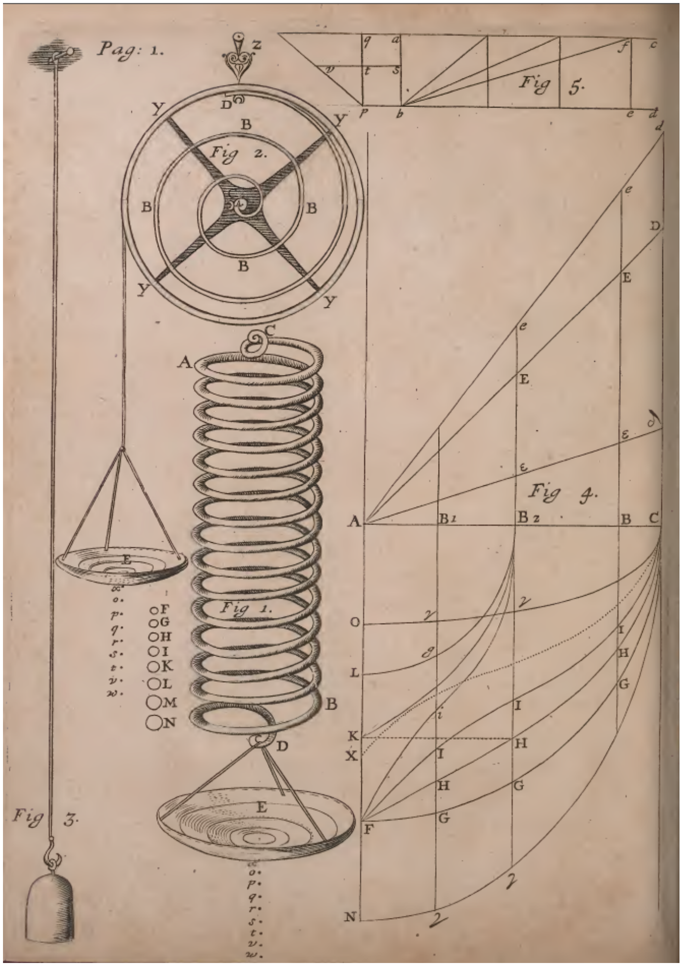

Figure 3: Hooke's illustration of springs from *Potentia Restitutiva*

Let us return to the title of this year's paper: ceiiinosssttuu. In 1678, Hooke published *Potentia Restitutiva, or on Spring* where he reveals the solution to the *Helioscopes* anagram:

> Ut tensio sic vis; *That is, The Power of any Spring is in the same proportion with the Tension thereof : That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward. Now as the Theory is very short, so the way of trying it is very easie.*

Note that this publication precedes *Principia* by almost a decade, so the term force and mass are not defined in the modern sense. Consequently, Hooke's power and tension are not the same as our power and tension. Furthermore, knowledge of the conservation of momentum and energy was not established at this time, though some see hints of the conservation of energy in Hooke's work [14].

In modern notation, Hooke's Law is given (in one dimension) as

$$F = -kx \tag{4.1}$$

where x is the displacement from the resting length of the spring, k is the spring constant, and the negative sign indicates that the force is a restoring force, pointing in the direction opposite the displacement. This is the mathematical embodiment of Hooke's description above, which in modern terms, translates to: If a force of one stretches or bends the spring by a distance of one, then a force of two bends it a distance of two, etc.

In the standard freshman-physics setup of the mass and spring system, the spring is fixed at one end with a mass at the other end (usually hanging, under gravity). The mass at the free end greatly exceeds that of the spring, so the spring itself is considered massless. With the standard assumptions along with Newton's $F = ma = m\ddot{x}$ (and a tiny bit of rearranging), Hooke's Law becomes the differential equation (in time)

$$\ddot{x} + \frac{k}{m}x = 0. \tag{4.2}$$

The dynamics of the mass-spring system are described by the general solution to our new spring equation:

$$x = Ae^{i\omega t} + \bar{A}e^{-i\omega t} \tag{4.3}$$

where $\omega = \sqrt{k/m}$ is the radial frequency of oscillation and A is an arbitrary complex-valued constant (incorporating both amplitude and phase information) that depends on the spatial and temporal Randbedingung[5], and $\bar{A}$ is its complex conjugate.[6] Have you seen this before? In form, it is identical to the solution to the pendulum, except that no small angle approximation is needed to arrive at this solution. As Hooke noted, the frequency of oscillation of a spring is independent of the initial conditions, hence its application to mechanical time-keeping.

What about the energetics of the system? How much energy is required to pull the free end of the spring to a displacement x_0? Given the relationship between force and scalar potential (detailed discussion of which I have to leave for a later Newton Day paper), $F = -dV/dx$, we have a differential equation (and its solution) in space:

$$\frac{dV}{dx} = kx \tag{4.4}$$

$$V = \frac{1}{2}kx^2 \tag{4.5}$$

where V is the potential energy of the mass-spring system, which we define to be zero at equilibrium ($x = 0$), thus removing the constant of integration. Note that the potential energy is independent of the mass on the free end of the spring and only depends on the spring constant and the displacement. By the conservation of energy, the kinetic energy of the mass,

[5] Boundary conditions. This is the only word in German that I learned this year. "Ansatz" and "gedanken" were getting lonely as imported scientific German.

[6] Mathematically, the second coefficient does not have to be the complex conjugate of the first, but physically it does so that the displacement is a real number.

$\frac{1}{2}mv^2$, at $x = 0$ is equal to the initial potential energy of the mass-spring system, $\frac{1}{2}kx_0^2$. Thus we see that the exit velocity of a projectile launched by a spring mechanism is directly proportional to the initial displacement of the spring from equilibrium. This is *very* important for games like Mr. Mouth.

5 Flies and cantilever sag - Callahan's method

Using his experimentally determined vibrational frequency for the music note G at 272 Hz, Hooke estimated the wing-beat frequency of a blue fly (from *Micrographia*):

> *And these vibrations or motions to and fro between the two limits seem so swift, that 'tis very probable (from the sound it affords, if it be compar'd with the vibration of a musical string, tun'd unison to it) it makes many hundreds, if not some thousands of vibrations in a second minute of time.*

He goes on to propose that insect wing motions "may be one of the quickest vibrating spontaneous motions of any in the world."

In a similar vein, Shawn Callahan, a mechanical engineer formerly with Caltech Optical Observatories (COO) but currently with the Large Synoptic Survey Telescope, would estimate the cantilever sag of a part by ringing it like a bell and listening for its musical note. I received this second-hand from a few other COO engineers, so be aware that I may be misrepresenting Shawn!

The simplest way to see this connection is to consider the relationship between the static and dynamic consequences of Hooke's Law. Under gravity, the equilibrium displacement of the free end of a spring due to a mass m is gm/k. If the mass is displaced from the equilibrium point, it oscillates with a frequency of $\omega = \sqrt{k/m}$ radians/sec. Therefore, with an idealized mass-spring system, if you observe the oscillation frequency, then you know the equilibrium displacement of the free end due to the mass:

$$\Delta x = g\frac{m}{k} \tag{5.1}$$

$$= \frac{g}{\omega^2} \tag{5.2}$$

$$= \frac{g}{(2\pi f)^2} \tag{5.3}$$

How does this compare with Euler-Bernoulli beam theory that mechanical engineers study? According to Euler-Bernoulli, the free-end sag of a cantilevered beam is the somewhat daunting

$$\Delta x = \frac{g\mu L^4}{8EI}, \tag{5.4}$$

where g is the acceleration due to gravity, 8 is eight (not "ate," Maggie)[7] and the rest doesn't matter for this discussion. The fundamental frequency of the cantilevered beam is the similarly daunting

$$\omega = 3.52\sqrt{\frac{EI}{\mu L^4}}. \tag{5.5}$$

[7]Numberphile, *Math Jokes Explained*

If we replace everything in the displacement expression that is not g or 8 by $(3.52/\omega)^2$, we are left with the displacement in terms of the frequency:

$$\Delta x = 1.55\frac{g}{\omega^2} \tag{5.6}$$
$$= 1.55\frac{g}{(2\pi f)^2}. \tag{5.7}$$

For an A440 tuning fork held horizontally, the ends of the tines sag by about 2 μm relative to the base of the tines. In this case, the simple application of Hooke's law gets us well within an order of magnitude of Euler-Bernoulli result. However, you should use this with caution – the fundamental frequency of a free beam is more than two and a half octaves higher, and this method is highly dependent on the Randbedingung and the loading of the part.

6 Activities

For the Newton Day activities, I obtained a set of 200 assorted springs from Harbor Freight Tools, along with some steel rulers and a plastic Vernier caliper. At a minimum, we will replicate Hooke's experiments with weights and springs. As time and interest permit, we can extend the experiments to measuring the spring constants and comparing them against the wire gauge, spring diameter, number of windings, etc. The spring constant of systems with springs arranged in parallel or series may also be explored, as it pertains to the material in the Appendix.

As an example of a non-massless spring, we have a Slinky. I was hoping that by analyzing a non-massless spring, I would have a closer approximation for the cantilever sag with easy physics, but I ran out of time to do the frequency analysis. Perhaps an enterprising reader will take it up.

7 Science

I brought Owen to the LIGO announcement on February 11 this year. As we were arriving at his childcare (Child Educational Center) about an hour behind our usual schedule he says to me, "Dad, I didn't want to go to that."

I replied, "That's ok, it wasn't your decision. I wanted to be there for that announcement, and I would have missed it if I had dropped you off first. Besides, some day you'll learn about this when you study General Relativity and you will remember that you were at Caltech for the announcement of the result."

Later that month, Owen attended his first real science talk – Alan Weinstein's talk at Caltech on the first LIGO results. We got there barely on time, so the only seats available were way down in front. Owen took in the talk like a real scientist: he paid close attention to the introduction, fell asleep when it got too technical for him, and woke up for the conclusions.

Figure 4: Owen at the big LIGO announcement, Maggie at the Getty.

Figure 5: Jack, Phoebe and Penny.

A From Hooke to Young: Stress, Strain and Failure

The spring constant is dependent on the material of the spring as well as its geometry or shape. For engineers this presents a problem in defining how a material will react when a force is applied to it. To make the spring equation more generic (remove the geometric dependencies) it needs to be rearranged in different terms. Before deriving this new form of the spring constant let us first define a few new terms. The first new term is stress (σ), which has units of pressure – force per unit area. The next is strain (ε), which is fractional change in length (change in length divided by original length). The last is called the elastic modulus or Young's modulus, E, which relates stress and strain (see Equation A.1). The elastic modulus is a property with a unique value for every material.

$$\sigma = E\varepsilon \tag{A.1}$$

In the world of engineering, the stress-strain equation does not have a negative sign because it relates an *applied* stress to the strain in the material. In the physics formulation of Hooke's Law, the force is exerted by the spring on its surroundings.

We can see the connection between Hooke's Law (Equation 4.1) and the stress-strain relationship by making use of the results from our exploration of series and parallel systems of springs carried out in Section 6. Through either experiment or pure thought, you discovered that for a system of A springs in parallel, each with spring constant E, the spring constant of the system, $k = EA$. For a system of L springs arranged in series, you found $k = E/L$. What if we had $A \times L$ springs arranged in a grid? The spring constant of this system is $k = EA/L$, so by Hooke's Law, we have

$$F_{\text{applied}} = k\Delta x \tag{A.2}$$

$$= \frac{EA}{L}\Delta x. \tag{A.3}$$

Rearranging and regrouping terms we have the equivalent relationship

$$\underbrace{\frac{F_{\text{applied}}}{A}}_{\sigma} = E\underbrace{\frac{\Delta x}{L}}_{\varepsilon}, \tag{A.4}$$

which looks a whole lot like the stress-strain equation. In this discrete formulation of the stress-strain equation, $\sigma = F/A$ is the force applied to an individual spring and $\varepsilon = \Delta x/L$ is the displacement of an individual spring. Also, A and L are integers, so dimensionally, all of the terms match Hooke's Law.

When we consider the mechanical properties of solids from an engineering standpoint, we model solids as three dimensional arrays of springs. A, which was formerly the number of parallel "strings" of springs, becomes the cross-sectional area of the object, and L, the number of springs in each string, becomes the length of the object. Giving physical dimensions to A and L changes the meaning of the terms in Equation A.4 to match those of Equation A.1. Now, σ has units of pressure or stress and ε becomes the dimensionless strain. What about the elastic modulus, E? We can turn around the relationship between E and k to get

$$E = \frac{kL}{A}, \tag{A.5}$$

which allows us to interpret the elastic modulus as the spring constant of a unit length per unit area. The elastic modulus has units of force per unit area, ie, pressure.

We can describe Equation A.1 in words to be the same as the spring equation. If you apply a pressure to an object, it will change in size in proportion with its elastic modulus.

There are a number of material properties that are expressed in terms of stress and strain. These properties are determined empirically through testing under controlled environments.

Figure 6 identifies a number of these material properties.

Definitions:

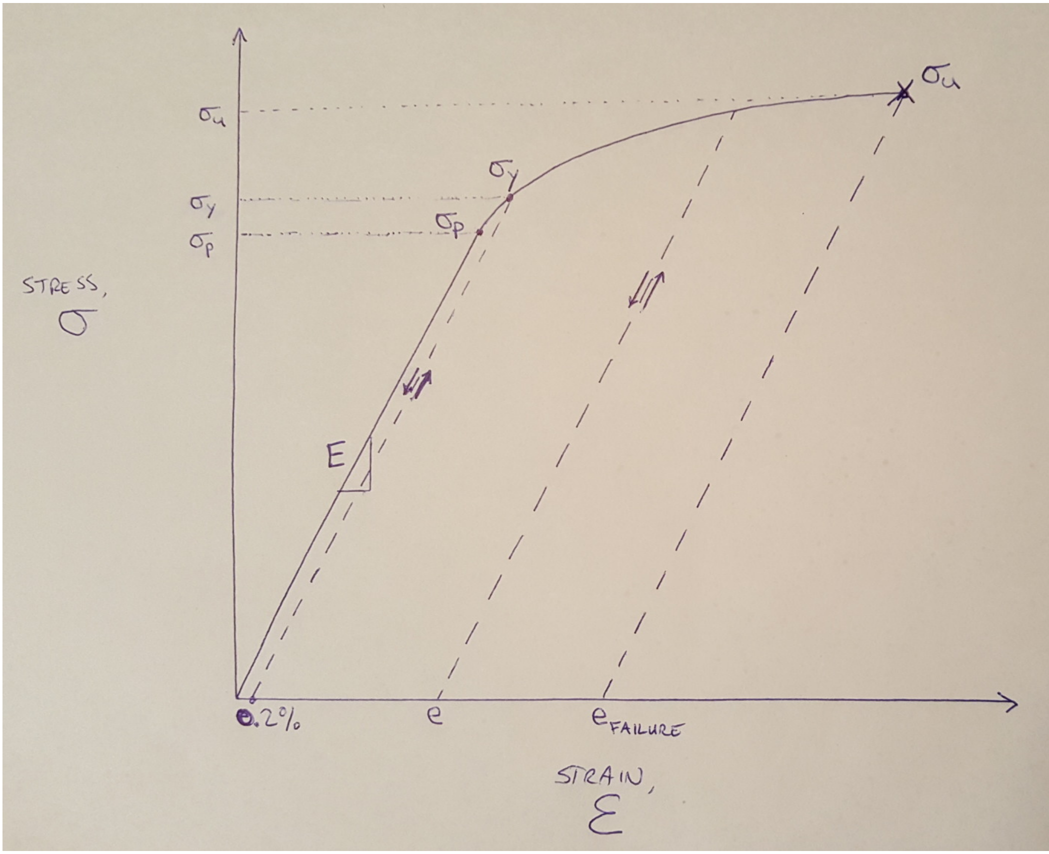

Figure 6: Stress and strain beyond the elastic regime.

- σ_p = proportional limit or proportional stress. This is the stress at which point a material no longer behaves elastically. Stresses larger than this will result in plastic deformation. This exact stress can be difficult to measure and is rarely known as an exact value.
- σ_y = yield stress. This is the stress at which 0.2% elongation (residual strain) occurs. This is a commonly reported value for most materials .
- σ_u = ultimate stress or failure stress. This is the stress at which point a material fails or breaks.
- E = Elastic modulus or Young's modulus. This is the slope of the stress-strain curve below the proportional limit.
- e = elongation or residual strain. This is how much the material will still be deformed after all load is removed.
- e_{failure} = elongation at failure. This is how much a material will stretch beyond its proportional limit before it fails. Materials that have large elongation at failure are

called ductile. Metals are the most common example of ductile materials. Materials that have small elongation at failure are called brittle. Glasses and ceramics are the common examples of brittle materials.

As long as the stress in the material does not exceed the proportional stress (σ_p) it will always return to its original shape once the load is removed from the object. This behavior is known as elastic deformation. If the stress does exceed the proportional limit then the object will not return to its original shape. It will have some amount of residual strain. The amount of strain that remains in the object after load is removed is known as elongation. Once an object that has residual strain is put under load again it will behave elastically from that point forward until it reaches the stress defined by the solid line in Figure 6. The maximum stress that a material can withstand without failing (breaking) is known as the ultimate stress or failure stress.

Every material in the world will follow the basic curve shown in Figure 6, but with different values for all of the parameters defined.

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