Newton Day 2006
Celebrating 364 years of Sir Isaac Newton

Participants:
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Abstract

Sir Isaac Newton, arguably the greatest mathematical physicist of all time, was born in 1642 on the day the Brits of the time celebrated the birth of Jesus, who, by the way, was also not born on December 25. This year is baby Maggie's first Newton day and for this occasion, we will consider the natural frequency of the pendulum. Coincidentally, this problem also links us to Galileo, who is purported to have used his pulse to time a pendulum, and thereby concluded that the period of a pendulum is independent of its maximum displacement. Galileo wanted to prove that the period is strictly independent of the maximum displacement; today, we will explore the truth of that matter.

1 Statement of the problem

We consider the periodic motion of a pendulum. In physics-speak, a pendulum is a mass hanging from a massless string or bar. The mass, constrained by the string, either has some initial velocity at the equilibrium position or is pulled to some displacement angle and then released.[1]

The relevant parameters of the pendulum problem are illustrated in Figure 1, likely to appear on the next page.

2 Standard high school treatment

At any given angle θ, the force tangent to the direction of travel is

$$\vec{F} = -mg\sin(\theta)\hat{\theta}. \tag{1}$$

The negative sign is there because the force points in the opposite direction of the displacement, and $\hat{\theta}$ denotes the tangential unit vector.[2] With small angular displacements, $\sin\theta \approx \theta$. So we have

$$\vec{F} \approx -mg\theta\hat{\theta}. \tag{2}$$

Newton's fundamental contribution, and the only equation one really needs to know in mechanics is $\vec{F} = m\vec{a}$, force is the product of mass and acceleration. Acceleration is the second time derivative of

[1] . . . or some linear combination of the two.

[2] Also, g is the magnitude of the gravitational acceleration, but that's a point that only picky physicists will quibble over on a day like this.

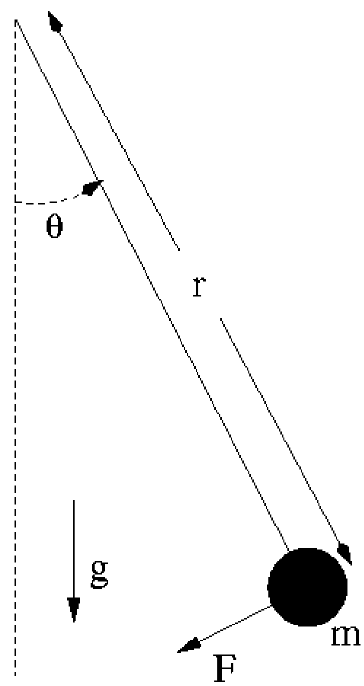

Figure 1: The classical pendulum with a massless string. Please excuse the difference in notation between the text and the drawing. Properly, the force should be labeled $\vec{F}$, the gravitational acceleration $\vec{g}$, and the displacement $\theta\hat{\theta}$.

displacement. In our case, the displacement is $r\theta\hat{\theta}$ so the acceleration is $r\ddot{\theta}\hat{\theta}$. Recall that the dots indicate a derivative with respect to time, and that r, the length of the pendulum is a constant.

Considering the force equations, we see that

$$\vec{F} = mr\ddot{\theta}\hat{\theta} = -mg\theta\hat{\theta} \tag{3}$$

which simplifies to the second order differential equation

$$r\ddot{\theta} = -g\theta. \tag{4}$$

Note that the mass in the problem cancels out. The general solution to this equation is:

$$\theta = Ae^{i(\sqrt{g/r}t+\phi)}. \tag{5}$$

The additional terms are the amplitude A and phase ϕ, which are determined by the initial conditions. If we take the case where the clock starts when we release the pendulum from some displacement angle A, then we get the very familiar

$$\theta(t) = A\cos(\sqrt{g/r}t). \tag{6}$$

The system returns to its initial state when $\sqrt{g/r}t = 2\pi$, so the period is

$$T = 2\pi\sqrt{r/g}. \tag{7}$$

That's the answer we're looking for – the period of oscillation depends only on the length of the pendulum and the local gravity. It is independent of both the amplitude of oscillation and the mass of the pendulum.

Question: Early on, we made the small angle approximation for $\sin\theta$. What happens to the period of oscillation for large amplitude swings? Is it greater or less than the value calculated in Equation 7?

Jelena's answer: $\sin\theta < \theta$ so the true force at all displacements (except $\theta = 0$) is weaker than the approximated force. The period will be longer.

Pete's answer: The potential goes as $1 - \cos\theta \approx \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots$. The second term in the approximation means that the true potential is always lower in energy than the harmonic oscillator potential ($\frac{x^2}{2}$). I come to the same conclusion as Jelena, but with a more convoluted thought process. Sometimes it *is* advantageous to think in terms of forces rather than energy.

3 Cheater-physicist method

Consider the physical quantities involved in the problem: mass (m), length of the pendulum (r), the gravitational constant (g), and the amplitude of oscillation (A). We are looking for a quantity with dimensions of time, so we consider the dimensions of the quantities in question: The answer has units of time, so we look

quantity	dimensions
mass	m
length	l
acceleration due to gravity	l/t^2
amplitude of motion	nondimensional!

for quantities that have units of time when multiplied together. $\sqrt{r/g}$ has units of time ($\sqrt{\frac{l}{l/t^2}} = t$), so the answer must have that form.

The inverse of $\sqrt{r/g}$ has units of $1/t$, or frequency. The frequency is never in term of cycles per second; rather, it will give a value in terms of radians per second.[3] To convert to cycles per second, we divide the frequency by 2π, or equivalently, multiply the period by 2π. The period of oscillation, therefore, is

$$T = 2\pi\sqrt{r/g}. \tag{8}$$

Question: The amplitude is dimensionless, so if it were inserted into the equation, the units would not change. Why doesn't the amplitude show up in the expression for the period? [This topic was not thoroughly discussed.]

4 Clever show-off method

The only generalized coordinate of the problem is the angular displacement, θ. The Lagrangian is defined as

$$L = K - V \tag{9}$$

where K is the kinetic energy[4] and V is the potential energy in terms of the generalized coordinate. In this case

$$K = \frac{1}{2}mr^2\dot{\theta}^2 \tag{10}$$

and

$$V = mgr(1 - \cos\theta) \tag{11}$$

so the Lagrangian is

$$L = \frac{1}{2}mr^2\dot{\theta}^2 - mgr(1 - \cos\theta). \tag{12}$$

[3] Recall that a radian is the angle subtended by a 1-radius long arc.

[4] Normally, T is used to denote the kinetic energy, but we are already using it for the oscillation period.

For the uninitiated, the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0. \tag{13}$$

The Euler-Lagrange equation comes from the calculus of variations and it gives us an extremum of the action integral. With this beautiful equation in hand, one can quickly show that the equation of motion is

$$mr^2\ddot{\theta} + mgr\sin\theta = 0. \tag{14}$$

This quickly reduces to the equation of motion (Eqn. 4) derived in Section 2.

To the untrained eye, this method looks overly complicated, since after all the trouble we go through to determine K and V and apply the Euler-Lagrange equation, we still have three steps to get to the answer. The beauty of the Lagrangian method is that we never have to consider vector quantities! This is a HUGE advantage in a limited set of more complicated problems, such as the double pendulum.

5 Experiments

Now, using a stopwatch and a ruler, we can predict and then experimentally verify the natural periods of oscillation of Maggie in her swing and other household objects, such as the Newton-day ornaments on the ceremonial apple tree. Alternatively, if we trust our calculations, then we can determine the gravitational acceleration at the surface of the Earth.

6 Results

We measured the oscillation period of Maggie in her swing and of the Santa Cruz slug ornament. Maggie's swing has very low friction in the pivot, but the slug's oscillations die out after only ~ 7 periods. Time was measured with a mechanical stopwatch with 0.2 s increments and length was measured with a tape-measure (in inches). Results tabulated below.

Table 1: Today's experimental results.

description	r [cm]	n periods	Δt [s]	T_{expt} [s]	T_{calc} [s]
Maggie	30.5	10	11.0	1.1	1.11
slug	12.7	6	4.2	0.7	0.72

7 Conclusions

Experiment agreed very well with theory, as one can see from the last two columns of Table 1. The main source of uncertainty with Maggie in the swing was determining the location of her center of mass in the swing. With the slug, the undetermined damping coefficient was a cause for concern, but the experimental result still turned out close to the theoretical ideal-case value.

Figure 2: (1) Everyone but Maggie [L→R, back→front]: Mischa, Rudy, Dan, Deirdre, Jelena, Pete. (2) The swing sans test mass. (3) The Santa Cruz slug (damped oscillator).

Figure 3: Maternal grandparents with the test mass.

Figure 4: (1) Raw data, dessert. (2) Rudy. (3) Jelena with Maggie. (4) Deirdre with Maggie. (5) Cookie dough that never got used. (6) Pete.