

# Machine Learning with Kernel Methods

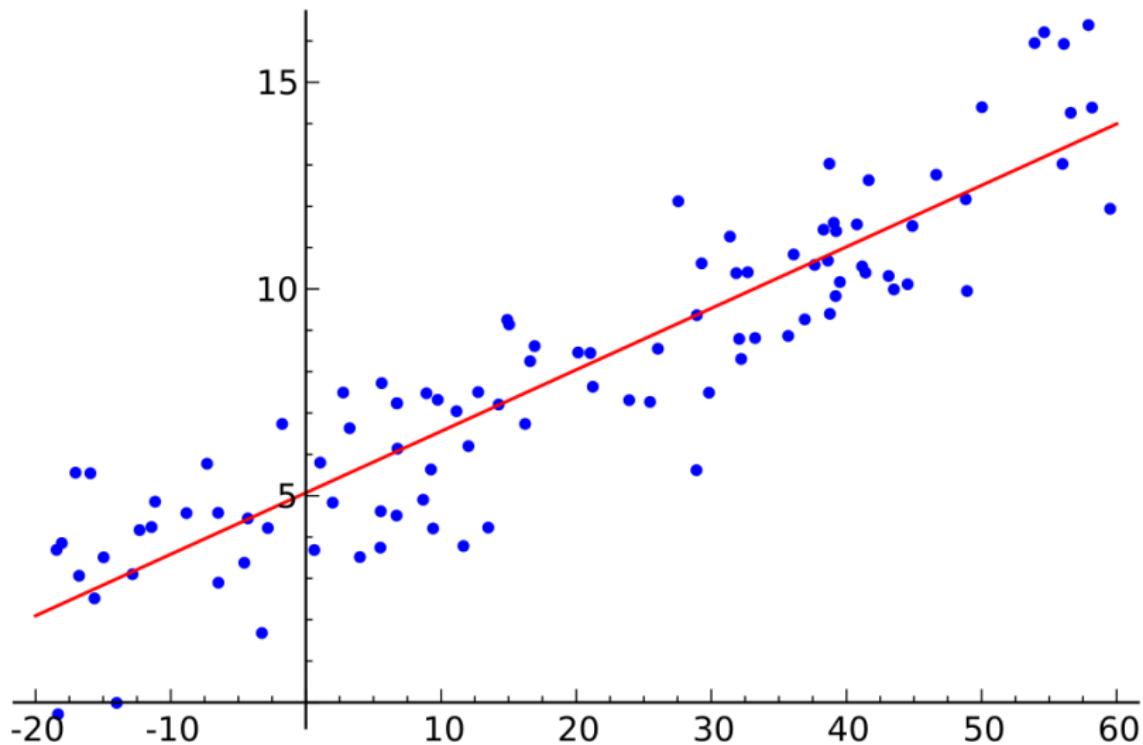


Julien Mairal & Jean-Philippe Vert

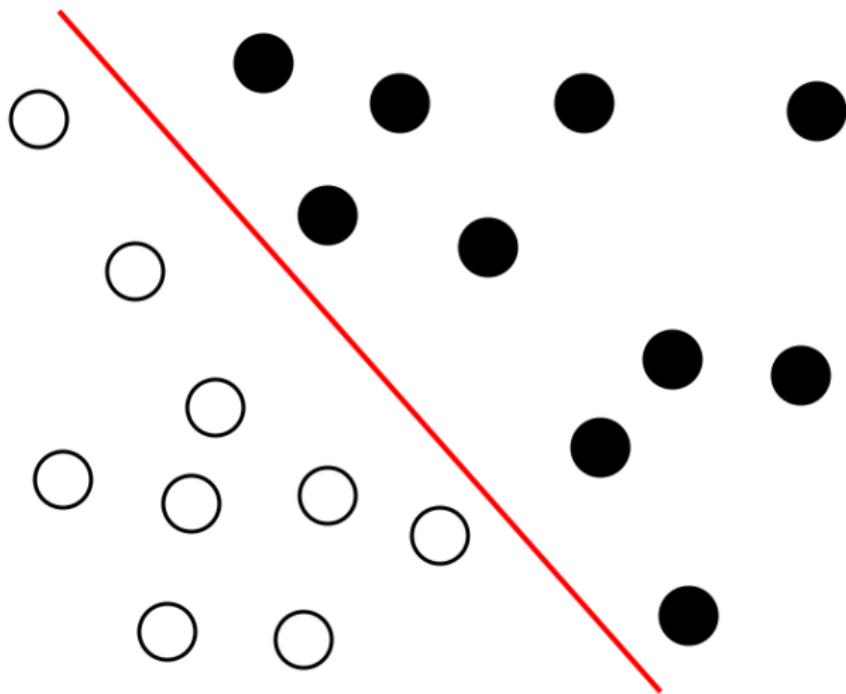
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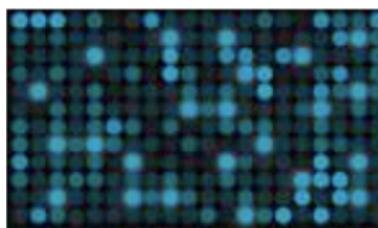
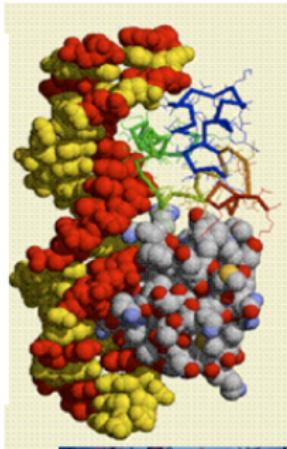
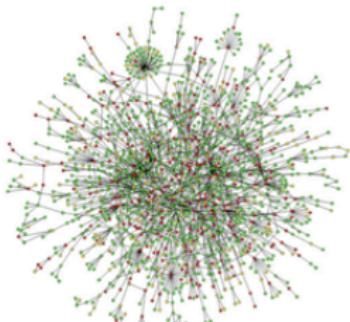
## Starting point: what we know is how to solve



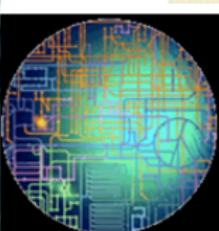
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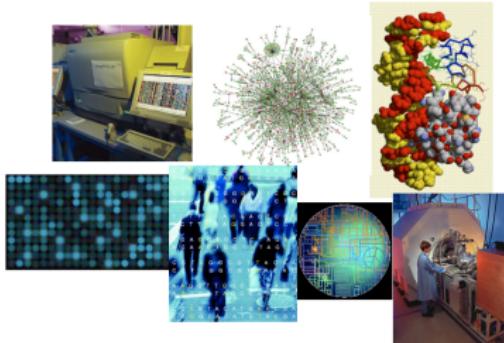
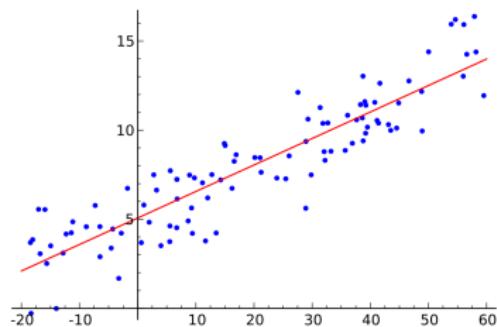
# But real data are often more complicated...



AAGCTT GCGGAGA  
ATGGTT ATGGGAGA  
CGCTT CCGGAATTCCT  
TGTAT GCGGATCCCGC  
GCTT GCGGATCCCT  
GAGCT TGGTCTAGAA  
GAACG TGGTAGAGGT  
CCTTG GCGGTAAACAC  
AAGCTT GCGGAGA  
AAGCTT TACCCGAAAC  
CCAGTACATGAAACGA  
CCGGTACATGTAACGA



# Main goal of this course



Extend  
well-understood, linear statistical learning techniques  
to  
real-world, complicated, structured, high-dimensional data  
based on  
a rigorous mathematical framework  
leading to  
practical modelling tools and algorithms

# Organization of the course

## Contents

- ① Present the **basic mathematical theory** of kernel methods.
- ② Introduce algorithms for **supervised** and **unsupervised** machine learning with kernels.
- ③ Develop a working knowledge of **kernel engineering** for specific data and applications (graphs, biological sequences, images).
- ④ Discuss **open research topics** related to kernels such as large-scale learning with kernels and “deep kernel learning” .

## Practical

- Course homepage with slides, schedules, homework etc...:  
`http://cbio.mines-paristech.fr/~jvert/svn/kernelcourse/course/2015mva.`
- Evaluation: 60% homework (3)+ 40% data challenge.

# Outline

## 1 Kernels and RKHS

- Positive Definite Kernels
- Reproducing Kernel Hilbert Spaces (RKHS)
- Examples
- Smoothness functional

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- The kernel trick
- The representer theorem

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## 2 Kernel tricks

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## 3 Kernel Methods: Supervised Learning

- Kernel ridge regression
- Kernel logistic regression
- Large-margin classifiers
- Interlude: convex optimization and duality
- Support vector machines

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## 4 Kernel Methods: Unsupervised Learning

- Kernel K-means and spectral clustering
- Kernel PCA
- A quick note on kernel CCA

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## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- Kernels on graphs

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- Kernels on graphs

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
- “Deep” learning with kernels

# Kernels and RKHS

# Overview

## Motivations

- Develop **versatile** algorithms to process and analyze data...
- ...without making any assumptions regarding the **type of data** (vectors, strings, graphs, images, ...)

## The approach

- Develop methods based on **pairwise comparisons**.
- By imposing constraints on the pairwise comparison function (positive definite kernels), we obtain a **general framework for learning from data** (optimization in RKHS).

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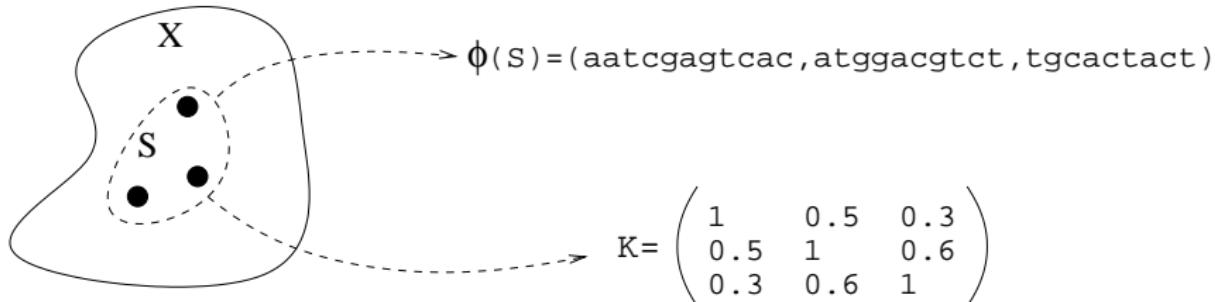
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## Representation by pairwise comparisons



### Idea

- Define a “comparison function”:  $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ .
- Represent a set of  $n$  data points  $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  by the  $n \times n$  matrix:

$$[K]_{ij} := K(\mathbf{x}_i, \mathbf{x}_j)$$

# Representation by pairwise comparisons

## Remarks

- $\mathbf{K}$  is always an  $n \times n$  matrix, whatever the nature of data: **the same algorithm will work for any type of data** (vectors, strings, ...).
- Total modularity between the **choice of  $K$**  and the **choice of the algorithm**.
- Poor scalability w.r.t. the dataset size ( $n^2$  to compute and store  $\mathbf{K}$ )... but wait until the end of the course to see how to deal with large-scale problems
- We will restrict ourselves to a **particular class** of pairwise comparison functions.

# Positive Definite (p.d.) Kernels

## Definition

A **positive definite (p.d.) kernel** on a set  $\mathcal{X}$  is a function  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  that is **symmetric**:

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = K(\mathbf{x}', \mathbf{x}),$$

and which satisfies, for all  $N \in \mathbb{N}$ ,  $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \in \mathcal{X}^N$  and  $(a_1, a_2, \dots, a_N) \in \mathbb{R}^N$ :

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j K(\mathbf{x}_i, \mathbf{x}_j) \geq 0.$$

# Similarity matrices of p.d. kernels

## Remarks

- Equivalently, a kernel  $K$  is p.d. if and only if, for any  $N \in \mathbb{N}$  and any set of points  $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \in \mathcal{X}^N$ , the **similarity matrix**  $[\mathbf{K}]_{ij} := K(\mathbf{x}_i, \mathbf{x}_j)$  is positive semidefinite.
- **Kernel methods** are algorithms that take such matrices as input.

# The simplest p.d. kernel, for real numbers

## Lemma

Let  $\mathcal{X} = \mathbb{R}$ . The function  $K : \mathbb{R}^2 \mapsto \mathbb{R}$  defined by:

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathbb{R}^2, \quad K(\mathbf{x}, \mathbf{x}') = \mathbf{x}\mathbf{x}'$$

is p.d.

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is p.d.

Proof:

- $\mathbf{x}\mathbf{x}' = \mathbf{x}'\mathbf{x}$
- $\sum_{i=1}^N \sum_{j=1}^N a_i a_j \mathbf{x}_i \mathbf{x}_j = \left( \sum_{i=1}^N a_i \mathbf{x}_i \right)^2 \geq 0$

□

## The simplest p.d. kernel, for vectors

### Lemma

Let  $\mathcal{X} = \mathbb{R}^d$ . The function  $K : \mathcal{X}^2 \mapsto \mathbb{R}$  defined by:

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x}, \mathbf{x}' \rangle_{\mathbb{R}^d}$$

is p.d. (it is often called the linear kernel).

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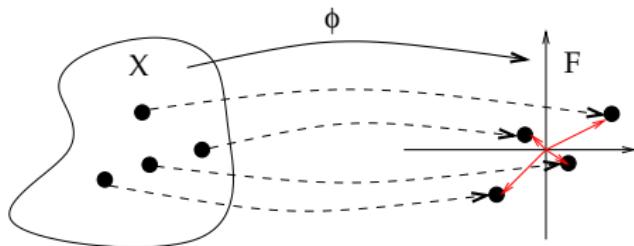
is p.d. (it is often called the **linear kernel**).

Proof:

- $\langle \mathbf{x}, \mathbf{x}' \rangle_{\mathbb{R}^d} = \langle \mathbf{x}', \mathbf{x} \rangle_{\mathbb{R}^d}$
- $\sum_{i=1}^N \sum_{j=1}^N a_i a_j \langle \mathbf{x}_i, \mathbf{x}_j \rangle_{\mathbb{R}^d} = \| \sum_{i=1}^N a_i \mathbf{x}_i \|_{\mathbb{R}^d}^2 \geq 0$

□

## A more ambitious p.d. kernel

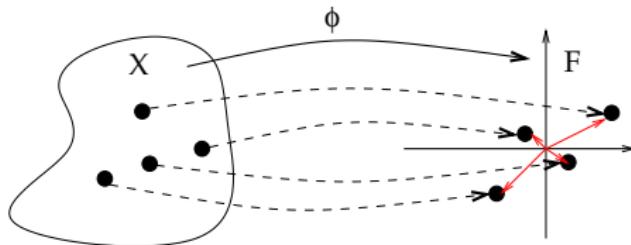


### Lemma

Let  $\mathcal{X}$  be any set, and  $\Phi : \mathcal{X} \mapsto \mathbb{R}^d$ . Then, the function  $K : \mathcal{X}^2 \mapsto \mathbb{R}$  defined as follows is p.d.:

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = \langle \Phi(\mathbf{x}), \Phi(\mathbf{x}') \rangle_{\mathbb{R}^d}.$$

## A more ambitious p.d. kernel



### Lemma

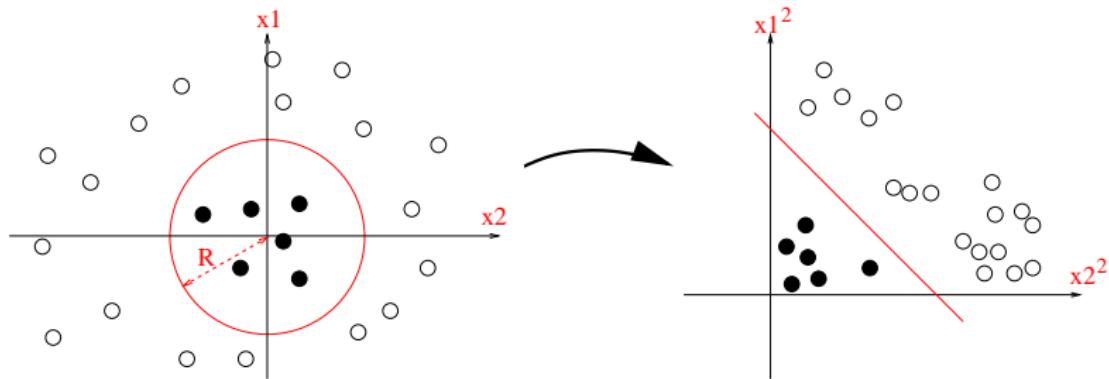
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Proof:

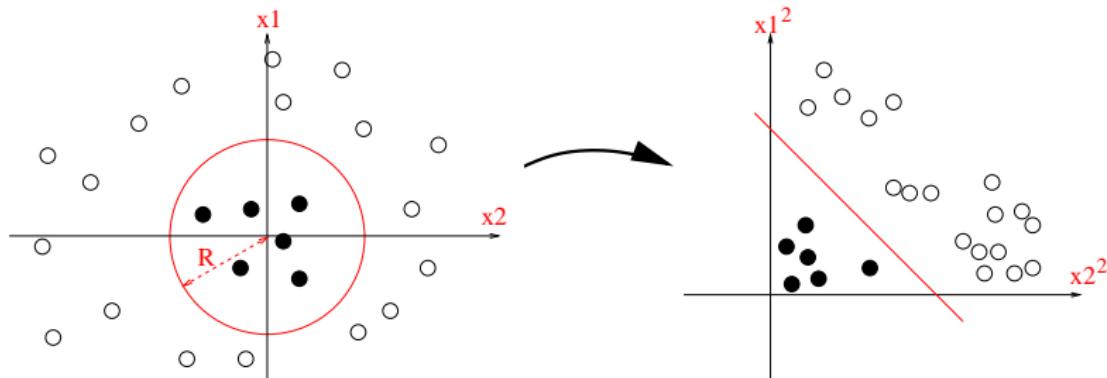
- $\langle \Phi(\mathbf{x}), \Phi(\mathbf{x}') \rangle_{\mathbb{R}^d} = \langle \Phi(\mathbf{x}'), \Phi(\mathbf{x}) \rangle_{\mathbb{R}^d}$
- $\sum_{i=1}^N \sum_{j=1}^N a_i a_j \langle \Phi(\mathbf{x}_i), \Phi(\mathbf{x}_j) \rangle_{\mathbb{R}^d} = \| \sum_{i=1}^N a_i \Phi(\mathbf{x}_i) \|_{\mathbb{R}^d}^2 \geq 0$  □

## Example: polynomial kernel



For  $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$ , let  $\Phi(\mathbf{x}) = (x_1^2, \sqrt{2}x_1x_2, x_2^2) \in \mathbb{R}^3$ :

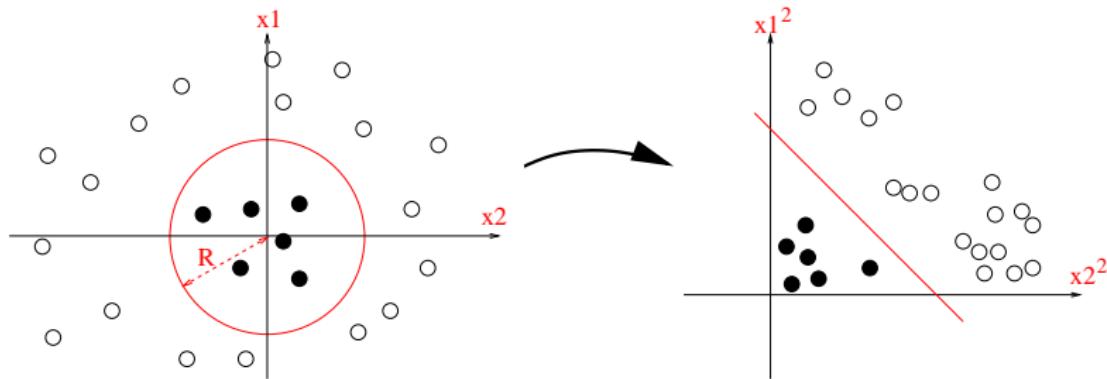
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$$\begin{aligned} K(\mathbf{x}, \mathbf{x}') &= x_1^2 x_1'^2 + 2x_1 x_2 x_1' x_2' + x_2^2 x_2'^2 \\ &= (x_1 x_1' + x_2 x_2')^2 \\ &= \langle \mathbf{x}, \mathbf{x}' \rangle_{\mathbb{R}^2}^2 . \end{aligned}$$

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Exercise: show that  $\langle \mathbf{x}, \mathbf{x}' \rangle_{\mathbb{R}^p}^d$  is p.d. on  $\mathcal{X} = \mathbb{R}^p$  for any  $d \in \mathbb{N}$ .



## Conversely: Kernels as inner products

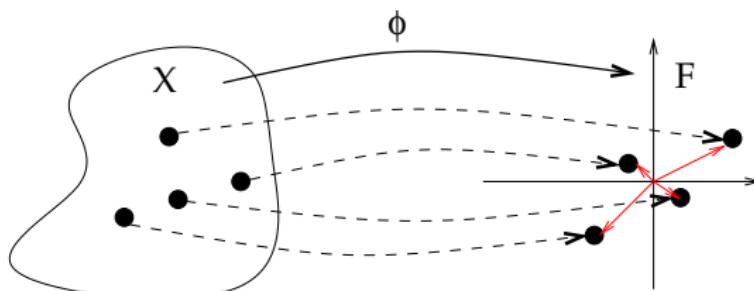
Theorem [Aronszajn, 1950]

$K$  is a p.d. kernel on the set  $\mathcal{X}$  if and only if there exists a Hilbert space  $\mathcal{H}$  and a mapping

$$\Phi : \mathcal{X} \mapsto \mathcal{H}$$

such that, for any  $x, x'$  in  $\mathcal{X}$ :

$$K(x, x') = \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}} .$$



## In case of ...

### Definitions

- An **inner product** on an  $\mathbb{R}$ -vector space  $\mathcal{H}$  is a mapping  $(f, g) \mapsto \langle f, g \rangle_{\mathcal{H}}$  from  $\mathcal{H}^2$  to  $\mathbb{R}$  that is **bilinear, symmetric** and such that  $\langle f, f \rangle_{\mathcal{H}} > 0$  for all  $f \in \mathcal{H} \setminus \{0\}$ .
- A vector space endowed with an inner product is called **pre-Hilbert**. It is endowed with a **norm** defined as  $\|f\|_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}^{\frac{1}{2}}$ .
- A **Cauchy sequence**  $(f_n)_{n \geq 0}$  is a sequence whose elements become progressively arbitrarily close to each other:

$$\lim_{N \rightarrow +\infty} \sup_{n, m \geq N} \|f_n - f_m\|_{\mathcal{H}} = 0.$$

- A **Hilbert space** is a pre-Hilbert space **complete** for the norm  $\|\cdot\|_{\mathcal{H}}$ . That is, any Cauchy sequence in  $\mathcal{H}$  converges in  $\mathcal{H}$ .

Completeness is necessary to keep “good” convergence properties of Euclidean spaces in an infinite-dimensional context.

## Proof: finite case

- Assume  $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$  is finite of size  $N$ .
- Any p.d. kernel  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  is entirely defined by the  $N \times N$  symmetric positive semidefinite matrix  $[\mathbf{K}]_{ij} := K(\mathbf{x}_i, \mathbf{x}_j)$ .
- It can therefore be diagonalized on an orthonormal basis of eigenvectors  $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N)$ , with non-negative eigenvalues  $0 \leq \lambda_1 \leq \dots \leq \lambda_N$ , i.e.,

$$K(\mathbf{x}_i, \mathbf{x}_j) = \left[ \sum_{l=1}^N \lambda_l \mathbf{u}_l \mathbf{u}_l^\top \right]_{ij} = \sum_{l=1}^N \lambda_l \mathbf{u}_l(i) \mathbf{u}_l(j) = \langle \Phi(\mathbf{x}_i), \Phi(\mathbf{x}_j) \rangle_{\mathbb{R}^N},$$

with

$$\Phi(\mathbf{x}_i) = \begin{pmatrix} \sqrt{\lambda_1} \mathbf{u}_1(i) \\ \vdots \\ \sqrt{\lambda_N} \mathbf{u}_N(i) \end{pmatrix}. \quad \square$$

## Proof: general case

- Mercer (1909) for  $\mathcal{X} = [a, b] \subset \mathbb{R}$  (more generally  $\mathcal{X}$  compact) and  $K$  continuous.
- Kolmogorov (1941) for  $\mathcal{X}$  countable.
- Aronszajn (1944, 1950) for the general case.

We will go through the proof of the general case by introducing the concept of Reproducing Kernel Hilbert Spaces (RKHS).

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# RKHS Definition

## Definition

Let  $\mathcal{X}$  be a set and  $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$  be a class of functions forming a (real) Hilbert space with inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ . The function  $K : \mathcal{X}^2 \mapsto \mathbb{R}$  is called a reproducing kernel (r.k.) of  $\mathcal{H}$  if

- ①  $\mathcal{H}$  contains all functions of the form

$$\forall \mathbf{x} \in \mathcal{X}, \quad K_{\mathbf{x}} : \mathbf{t} \mapsto K(\mathbf{x}, \mathbf{t}) .$$

- ② For every  $\mathbf{x} \in \mathcal{X}$  and  $f \in \mathcal{H}$  the reproducing property holds:

$$f(\mathbf{x}) = \langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}} .$$

If a r.k. exists, then  $\mathcal{H}$  is called a reproducing kernel Hilbert space (RKHS).

# An equivalent definition of RKHS

## Theorem

The Hilbert space  $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$  is a RKHS if and only if for any  $\mathbf{x} \in \mathcal{X}$ , the mapping:

$$\begin{aligned} F : \quad & \mathcal{H} \rightarrow \mathbb{R} \\ f & \mapsto f(\mathbf{x}) \end{aligned}$$

is **continuous**.

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is continuous.

## Corollary

Convergence in a RKHS implies pointwise convergence, i.e., if  $(f_n)_{n \in \mathbb{N}}$  converges to  $f$  in  $\mathcal{H}$ , then  $(f_n(\mathbf{x}))_{n \in \mathbb{N}}$  converges to  $f(\mathbf{x})$  for any  $\mathbf{x} \in \mathcal{X}$ .

## Proof

If  $\mathcal{H}$  is a RKHS then  $f \mapsto f(\mathbf{x})$  is continuous

If a r.k.  $K$  exists, then for any  $(\mathbf{x}, f) \in \mathcal{X} \times \mathcal{H}$ :

$$\begin{aligned}|f(\mathbf{x})| &= |\langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}}| \\&\leq \|f\|_{\mathcal{H}} \cdot \|K_{\mathbf{x}}\|_{\mathcal{H}} \quad (\text{Cauchy-Schwarz}) \\&\leq \|f\|_{\mathcal{H}} \cdot K(\mathbf{x}, \mathbf{x})^{\frac{1}{2}},\end{aligned}$$

because  $\|K_{\mathbf{x}}\|_{\mathcal{H}}^2 = \langle K_{\mathbf{x}}, K_{\mathbf{x}} \rangle_{\mathcal{H}} = K(\mathbf{x}, \mathbf{x})$ . Therefore  $f \in \mathcal{H} \mapsto f(\mathbf{x}) \in \mathbb{R}$  is a continuous linear mapping.  $\square$

Since  $F$  is linear, it is indeed sufficient to show that  $f \rightarrow 0 \Rightarrow f(x) \rightarrow 0$ .



## Proof (Converse)

If  $f \mapsto f(\mathbf{x})$  is continuous then  $\mathcal{H}$  is a RKHS

Conversely, let us assume that for any  $\mathbf{x} \in \mathcal{X}$  the linear form  $f \in \mathcal{H} \mapsto f(\mathbf{x})$  is continuous.

Then by Riesz representation theorem (general property of Hilbert spaces) there exists a unique  $g_{\mathbf{x}} \in \mathcal{H}$  such that:

$$f(\mathbf{x}) = \langle f, g_{\mathbf{x}} \rangle_{\mathcal{H}} .$$

The function  $K(\mathbf{x}, \mathbf{y}) = g_{\mathbf{x}}(\mathbf{y})$  is then a r.k. for  $\mathcal{H}$ . □

# Unicity of r.k. and RKHS

## Theorem

- If  $\mathcal{H}$  is a RKHS, then it has a unique r.k.
- Conversely, a function  $K$  can be the r.k. of at most one RKHS.

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- Conversely, a function  $K$  can be the r.k. of at most one RKHS.

## Consequence

This shows that we can talk of "the" kernel of a RKHS, or "the" RKHS of a kernel.

## Proof

If a r.k. exists then it is unique

Let  $K$  and  $K'$  be two r.k. of a RKHS  $\mathcal{H}$ . Then for any  $\mathbf{x} \in \mathcal{X}$ :

$$\begin{aligned}\|K_{\mathbf{x}} - K'_{\mathbf{x}}\|_{\mathcal{H}}^2 &= \langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K_{\mathbf{x}} - K'_{\mathbf{x}} \rangle_{\mathcal{H}} \\&= \langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K_{\mathbf{x}} \rangle_{\mathcal{H}} - \langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K'_{\mathbf{x}} \rangle_{\mathcal{H}} \\&= K_{\mathbf{x}}(\mathbf{x}) - K'_{\mathbf{x}}(\mathbf{x}) - K_{\mathbf{x}}(\mathbf{x}) + K'_{\mathbf{x}}(\mathbf{x}) \\&= 0.\end{aligned}$$

This shows that  $K_{\mathbf{x}} = K'_{\mathbf{x}}$  as functions, i.e.,  $K_{\mathbf{x}}(\mathbf{y}) = K'_{\mathbf{x}}(\mathbf{y})$  for any  $\mathbf{y} \in \mathcal{X}$ . In other words,  $\mathbf{K} = \mathbf{K}'$ . □



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The RKHS of a r.k.  $K$  is unique

Left as exercise.



# An important result

## Theorem

A function  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  is **p.d.** if and only if it is a **r.k.**

## Proof

A r.k. is p.d.

- ① A r.k. is **symmetric** because, for any  $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^2$ :

$$K(\mathbf{x}, \mathbf{y}) = \langle K_{\mathbf{x}}, K_{\mathbf{y}} \rangle_{\mathcal{H}} = \langle K_{\mathbf{y}}, K_{\mathbf{x}} \rangle_{\mathcal{H}} = K(\mathbf{y}, \mathbf{x}).$$

- ② It is **p.d.** because for any  $N \in \mathbb{N}, (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \in \mathcal{X}^N$ , and  $(a_1, a_2, \dots, a_N) \in \mathbb{R}^N$ :

$$\begin{aligned} \sum_{i,j=1}^N a_i a_j K(\mathbf{x}_i, \mathbf{x}_j) &= \sum_{i,j=1}^N a_i a_j \langle K_{\mathbf{x}_i}, K_{\mathbf{x}_j} \rangle_{\mathcal{H}} \\ &= \| \sum_{i=1}^N a_i K_{\mathbf{x}_i} \|_{\mathcal{H}}^2 \\ &\geq 0. \quad \square \end{aligned}$$

## Proof

A p.d. kernel is a r.k. (1/4)

- Let  $\mathcal{H}_0$  be the vector subspace of  $\mathbb{R}^{\mathcal{X}}$  spanned by the functions  $\{K_{\mathbf{x}}\}_{\mathbf{x} \in \mathcal{X}}$ .
- For any  $f, g \in \mathcal{H}_0$ , given by:

$$f = \sum_{i=1}^m a_i K_{\mathbf{x}_i}, \quad g = \sum_{j=1}^n b_j K_{\mathbf{y}_j},$$

let:

$$\langle f, g \rangle_{\mathcal{H}_0} := \sum_{i,j} a_i b_j K(\mathbf{x}_i, \mathbf{y}_j).$$

## Proof

A p.d. kernel is a r.k. (2/4)

- $\langle f, g \rangle_{\mathcal{H}_0}$  does not depend on the expansion of  $f$  and  $g$  because:

$$\langle f, g \rangle_{\mathcal{H}_0} = \sum_{i=1}^m a_i g(\mathbf{x}_i) = \sum_{j=1}^n b_j f(\mathbf{y}_j).$$

- This also shows that  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$  is a **symmetric bilinear form**.
- This also shows that for any  $\mathbf{x} \in \mathcal{X}$  and  $f \in \mathcal{H}_0$ :

$$\langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}_0} = f(\mathbf{x}).$$

## Proof

A p.d. kernel is a r.k. (3/4)

- $K$  is assumed to be p.d., therefore:

$$\|f\|_{\mathcal{H}_0}^2 = \sum_{i,j=1}^m a_i a_j K(\mathbf{x}_i, \mathbf{x}_j) \geq 0.$$

In particular Cauchy-Schwarz is valid with  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ .

- By Cauchy-Schwarz we deduce that  $\forall \mathbf{x} \in \mathcal{X}$ :

$$|f(\mathbf{x})| = |\langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}_0}| \leq \|f\|_{\mathcal{H}_0} K(\mathbf{x}, \mathbf{x})^{\frac{1}{2}},$$

therefore  $\|f\|_{\mathcal{H}_0} = 0 \implies f = 0$ .

- $\mathcal{H}_0$  is therefore a **pre-Hilbert space** endowed with the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ .

## Proof

### A p.d. kernel is a r.k. (4/4)

- For any Cauchy sequence  $(f_n)_{n \geq 0}$  in  $(\mathcal{H}_0, \langle \cdot, \cdot \rangle_{\mathcal{H}_0})$ , we note that:

$$\forall (\mathbf{x}, m, n) \in \mathcal{X} \times \mathbb{N}^2, \quad |f_m(\mathbf{x}) - f_n(\mathbf{x})| \leq \|f_m - f_n\|_{\mathcal{H}_0} K(\mathbf{x}, \mathbf{x})^{\frac{1}{2}}.$$

Therefore for any  $\mathbf{x}$  the sequence  $(f_n(\mathbf{x}))_{n \geq 0}$  is Cauchy in  $\mathbb{R}$  and has therefore a limit.

- If we add to  $\mathcal{H}_0$  the functions defined as the pointwise limits of Cauchy sequences, then the space becomes complete and is therefore a **Hilbert space**, with  $K$  as r.k. (up to a few technicalities, left as exercise).  $\square$

## Application: back to Aronszajn's theorem

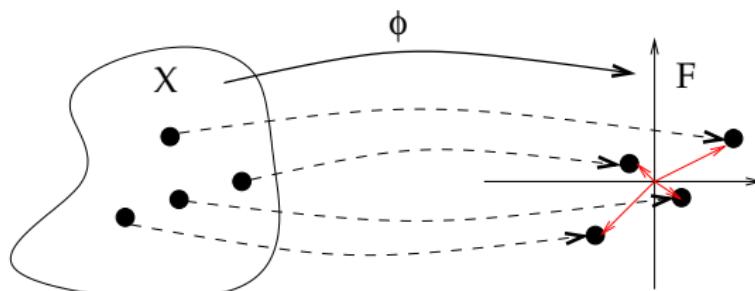
Theorem (Aronszajn, 1950)

$K$  is a p.d. kernel on the set  $\mathcal{X}$  if and only if there exists a Hilbert space  $\mathcal{H}$  and a mapping

$$\Phi : \mathcal{X} \mapsto \mathcal{H},$$

such that, for any  $\mathbf{x}, \mathbf{x}'$  in  $\mathcal{X}$ :

$$K(\mathbf{x}, \mathbf{x}') = \langle \Phi(\mathbf{x}), \Phi(\mathbf{x}') \rangle_{\mathcal{H}}.$$



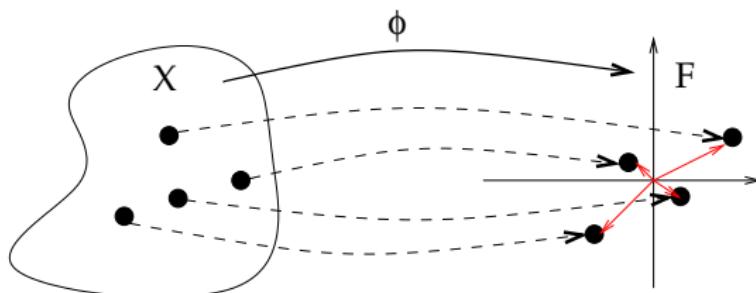
## Proof of Aronzsajn's theorem

- If  $K$  is p.d. over a set  $\mathcal{X}$  then it is the r.k. of a Hilbert space  $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ .
- Let the mapping  $\Phi : \mathcal{X} \rightarrow \mathcal{H}$  defined by:

$$\forall \mathbf{x} \in \mathcal{X}, \quad \Phi(\mathbf{x}) = K_{\mathbf{x}}.$$

- By the reproducing property we have:

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathcal{X}^2, \quad \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}} = \langle K_{\mathbf{x}}, K_{\mathbf{y}} \rangle_{\mathcal{H}} = K(\mathbf{x}, \mathbf{y}). \quad \square$$



# Outline

## 1 Kernels and RKHS

- Positive Definite Kernels
- Reproducing Kernel Hilbert Spaces (RKHS)
- Examples
- Smoothness functional

## 2 Kernel tricks

## 3 Kernel Methods: Supervised Learning

## 4 Kernel Methods: Unsupervised Learning

## 5 The Kernel Jungle

## 6 Open Problems and Research Topics

## The linear kernel

Take  $\mathcal{X} = \mathbb{R}^d$  and the **linear kernel**:

$$K(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^d}.$$

### Theorem

The RKHS of the linear kernel is the set of linear functions of the form

$$f_{\mathbf{w}}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle_{\mathbb{R}^d} \quad \text{for } \mathbf{w} \in \mathbb{R}^d,$$

endowed with the inner product

$$\forall \mathbf{w}, \mathbf{v} \in \mathbb{R}^d, \quad \langle f_{\mathbf{w}}, f_{\mathbf{v}} \rangle_{\mathcal{H}} = \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbb{R}^d}$$

and corresponding norm

$$\forall \mathbf{w} \in \mathbb{R}^d, \quad \| f_{\mathbf{w}} \|_{\mathcal{H}} = \| \mathbf{w} \|_2.$$

## Proof

- The RKHS of the linear kernel consists of functions:

$$\mathbf{x} \in \mathbb{R}^d \mapsto f(\mathbf{x}) = \sum_i a_i \langle \mathbf{x}_i, \mathbf{x} \rangle_{\mathbb{R}^d} = \langle \mathbf{w}, \mathbf{x} \rangle_{\mathbb{R}^d},$$

with  $\mathbf{w} = \sum_i a_i \mathbf{x}_i$ .

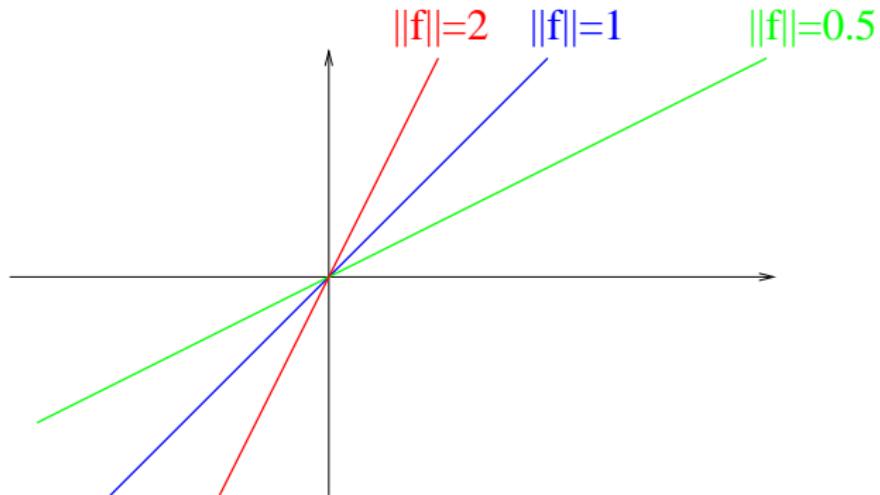
- The RKHS is therefore the set of linear forms endowed with the following inner product:

$$\langle f, g \rangle_{\mathcal{H}_K} = \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbb{R}^d},$$

when  $f(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$  and  $g(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ .

## RKHS of the linear kernel (cont.)

$$\begin{cases} K_{lin}(\mathbf{x}, \mathbf{x}') &= \mathbf{x}^\top \mathbf{x}' . \\ f(\mathbf{x}) &= \mathbf{w}^\top \mathbf{x} , \\ \|f\|_{\mathcal{H}} &= \|\mathbf{w}\|_2 . \end{cases}$$



## The polynomial kernel

Let's find the RKHS of the **polynomial kernel**:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \quad K(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^d}^2 = (\mathbf{x}^\top \mathbf{y})^2$$

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**First step: Look for an inner-product.**

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) &= \text{trace} \left( \mathbf{x}^\top \mathbf{y} \mathbf{x}^\top \mathbf{y} \right) \\ &= \text{trace} \left( \mathbf{y}^\top \mathbf{x} \mathbf{x}^\top \mathbf{y} \right) \\ &= \text{trace} \left( \mathbf{x} \mathbf{x}^\top \mathbf{y} \mathbf{y}^\top \right) \\ &= \left\langle \mathbf{x} \mathbf{x}^\top, \mathbf{y} \mathbf{y}^\top \right\rangle_F, \end{aligned}$$

where  $F$  is the Frobenius norm for matrices in  $\mathbb{R}^{d \times d}$ .

# The polynomial kernel

**Second step: propose a candidate RKHS.**

We know that  $\mathcal{H}$  contains all the functions

$$f(\mathbf{x}) = \sum_i a_i K(\mathbf{x}_i, \mathbf{x}) = \sum_i a_i \left\langle \mathbf{x}_i \mathbf{x}_i^\top, \mathbf{x} \mathbf{x}^\top \right\rangle_F = \left\langle \sum_i a_i \mathbf{x}_i \mathbf{x}_i^\top, \mathbf{x} \mathbf{x}^\top \right\rangle.$$

Any symmetric matrix in  $\mathbb{R}^{d \times d}$  may be decomposed as  $\sum_i a_i \mathbf{x}_i \mathbf{x}_i^\top$ . Our candidate RKHS  $\mathcal{H}$  will be the set of quadratic functions

$$f_{\mathbf{S}}(\mathbf{x}) = \left\langle \mathbf{S}, \mathbf{x} \mathbf{x}^\top \right\rangle_F = \mathbf{x}^\top \mathbf{S} \mathbf{x} \quad \text{for } \mathbf{S} \in \mathcal{S}^{d \times d},$$

where  $\mathcal{S}^{d \times d}$  is the set of **symmetric**<sup>1</sup> matrices in  $\mathbb{R}^{d \times d}$ , endowed with the inner-product  $\langle f_{\mathbf{S}_1}, f_{\mathbf{S}_2} \rangle_{\mathcal{H}} = \langle \mathbf{S}_1, \mathbf{S}_2 \rangle_F$ .

---

<sup>1</sup>Why is it important?



# The polynomial kernel

## Third step: check that the candidate is a Hilbert space.

This step is trivial in the present case since it is easy to see that  $\mathcal{H}$  a Euclidean space, isomorphic to  $\mathcal{S}^{d \times d}$  by  $\Phi : \mathbf{S} \mapsto f_{\mathbf{S}}$ . Sometimes, things are not so simple and we need to prove the completeness explicitly.

## Fourth step: check that $\mathcal{H}$ is the RKHS.

- ①  $\mathcal{H}$  contains all the functions  $K_{\mathbf{x}} : \mathbf{t} \mapsto K(\mathbf{x}, \mathbf{t}) = \langle \mathbf{x}\mathbf{x}^T, \mathbf{t}\mathbf{t}^T \rangle_F$ .
- ② For all  $f_{\mathbf{S}}$  in  $\mathcal{H}$  and  $\mathbf{x}$  in  $\mathcal{X}$ ,

$$f_{\mathbf{S}}(\mathbf{x}) = \left\langle \mathbf{S}, \mathbf{x}\mathbf{x}^T \right\rangle_F = \langle f_{\mathbf{S}}, f_{\mathbf{x}\mathbf{x}^T} \rangle_{\mathcal{H}} = \langle f_{\mathbf{S}}, K_{\mathbf{x}} \rangle_{\mathcal{H}} .$$

□

## Remark

All points  $\mathbf{x}$  in  $\mathcal{X}$  are mapped to a rank-one matrix  $\mathbf{x}\mathbf{x}^T$ , hence to a function  $K_{\mathbf{x}} = f_{\mathbf{x}^T \mathbf{x}}$  in  $\mathcal{H}$ . However, most of points in  $\mathcal{H}$  do not admit a pre-image (why?).

*Exercise: what is the RKHS of the general polynomial kernel?*

# Combining kernels

## Theorem

- If  $K_1$  and  $K_2$  are p.d. kernels, then:

$$K_1 + K_2,$$

$$K_1 K_2, \text{ and}$$

$$cK_1, \text{ for } c \geq 0,$$

are also p.d. kernels

- If  $(K_i)_{i \geq 1}$  is a sequence of p.d. kernels that converges pointwisely to a function  $K$ :

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = \lim_{n \rightarrow \infty} K_i(\mathbf{x}, \mathbf{x}'),$$

then  $K$  is also a p.d. kernel.



*Proof: left as exercise*

# Examples

## Theorem

If  $K$  is a kernel, then  $e^K$  is a kernel too.

Proof:

$$e^{K(\mathbf{x}, \mathbf{x}')} = \lim_{n \rightarrow +\infty} \sum_{i=0}^n \frac{K(\mathbf{x}, \mathbf{x}')^i}{i!}$$

Quizz : which of the following are p.d. kernels?

- $\mathcal{X} = (-1, 1), \quad K(\mathbf{x}, \mathbf{x}') = \frac{1}{1 - \mathbf{x}\mathbf{x}'}$

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- $\mathcal{X} = \mathbb{N}, \quad K(\mathbf{x}, \mathbf{x}') = 2^{\mathbf{x}\mathbf{x}'}$
- $\mathcal{X} = \mathbb{R}_+, \quad K(\mathbf{x}, \mathbf{x}') = \log(1 + \mathbf{x}\mathbf{x}')$



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- $\mathcal{X} = \mathbb{R}_+, \quad K(\mathbf{x}, \mathbf{x}') = \min(\mathbf{x}, \mathbf{x}') / \max(\mathbf{x}, \mathbf{x}')$
- $\mathcal{X} = \mathbb{N}, \quad K(\mathbf{x}, \mathbf{x}') = GCD(\mathbf{x}, \mathbf{x}')$
- $\mathcal{X} = \mathbb{N}, \quad K(\mathbf{x}, \mathbf{x}') = LCM(\mathbf{x}, \mathbf{x}')$
- $\mathcal{X} = \mathbb{N}, \quad K(\mathbf{x}, \mathbf{x}') = GCD(\mathbf{x}, \mathbf{x}') / LCM(\mathbf{x}, \mathbf{x}')$

# Outline

## 1 Kernels and RKHS

- Positive Definite Kernels
- Reproducing Kernel Hilbert Spaces (RKHS)
- Examples
- Smoothness functional

## 2 Kernel tricks

## 3 Kernel Methods: Supervised Learning

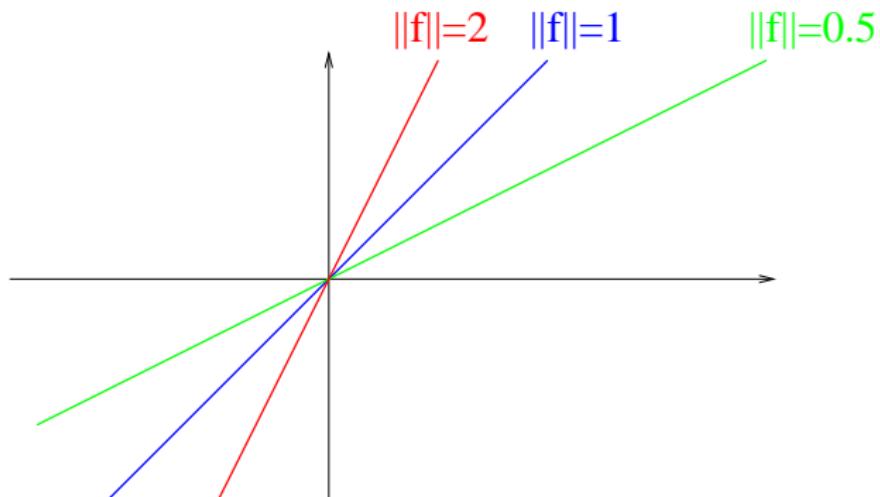
## 4 Kernel Methods: Unsupervised Learning

## 5 The Kernel Jungle

## 6 Open Problems and Research Topics

## Remember the RKHS of the linear kernel

$$\begin{cases} K_{lin}(\mathbf{x}, \mathbf{x}') &= \mathbf{x}^\top \mathbf{x}' . \\ f(\mathbf{x}) &= \mathbf{w}^\top \mathbf{x} , \\ \|f\|_{\mathcal{H}} &= \|\mathbf{w}\|_2 . \end{cases}$$



# Smoothness functional

## A simple inequality

- By Cauchy-Schwarz we have, for any function  $f \in \mathcal{H}$  and any two points  $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ :

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{x}')| &= |\langle f, K_{\mathbf{x}} - K_{\mathbf{x}'} \rangle_{\mathcal{H}}| \\ &\leq \|f\|_{\mathcal{H}} \times \|K_{\mathbf{x}} - K_{\mathbf{x}'}\|_{\mathcal{H}} \\ &= \|f\|_{\mathcal{H}} \times d_K(\mathbf{x}, \mathbf{x}'). \end{aligned}$$

- The norm of a function in the RKHS controls **how fast** the function varies over  $\mathcal{X}$  with respect to the **geometry defined by the kernel** (Lipschitz with constant  $\|f\|_{\mathcal{H}}$ ).

## Important message

Small norm  $\implies$  slow variations.

## Kernels and RKHS : Summary

- P.d. kernels can be thought of as **inner product** after embedding the data space  $\mathcal{X}$  in some Hilbert space. As such a p.d. kernel defines a **metric** on  $\mathcal{X}$ .
- A realization of this embedding is the **RKHS**, valid without restriction on the space  $\mathcal{X}$  nor on the kernel.
- The RKHS is a space of functions over  $\mathcal{X}$ . The **norm** of a function in the RKHS is related to its degree of **smoothness** w.r.t. the metric defined by the kernel on  $\mathcal{X}$ .
- We will now see some applications of kernels and RKHS in statistics, before coming back to the problem of **choosing (and eventually designing) the kernel**.

# Kernel tricks

## Motivations

Two theoretical results underpin a family of powerful algorithms for data analysis using positive definite kernels, collectively known as **kernel methods**:

- The **kernel trick**, based on the representation of p.d. kernels as inner products,
- The **representer theorem**, based on some properties of the regularization functional defined by the RKHS norm.

# Outline

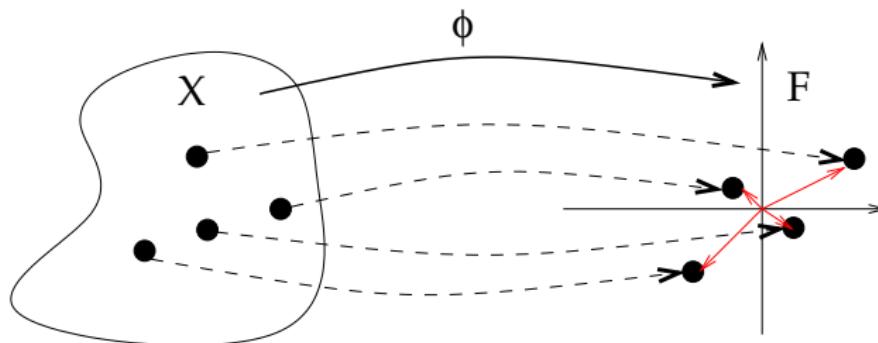
- 1 Kernels and RKHS
- 2 Kernel tricks
  - The kernel trick
  - The representer theorem
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
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## Motivations

- Choosing a p.d. kernel  $K$  on a set  $\mathcal{X}$  amounts to **embedding the data in a Hilbert space**: there exists a Hilbert space  $\mathcal{H}$  and a mapping  $\Phi : \mathcal{X} \mapsto \mathcal{H}$  such that, for all  $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ ,

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = \langle \Phi(\mathbf{x}), \Phi(\mathbf{x}') \rangle_{\mathcal{H}}.$$

- However this mapping might **not be explicitly given**, nor convenient to work with in practice (e.g., large or even infinite dimensions).
- A solution is to work implicitly in the feature space!



# The kernel trick

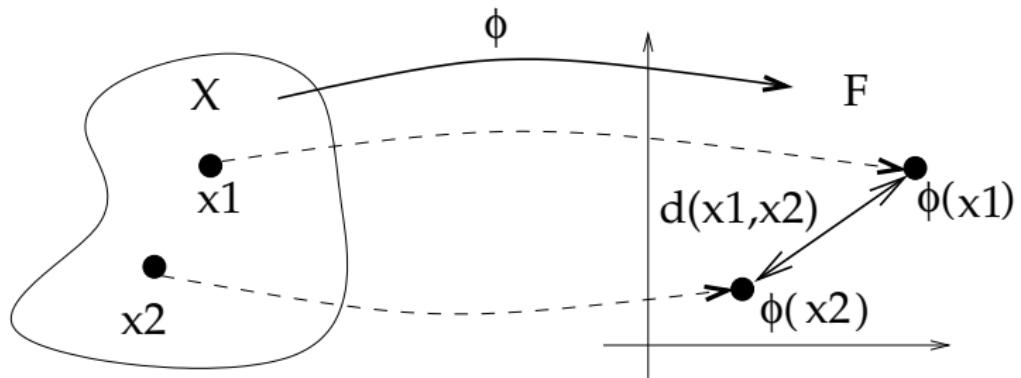
## Proposition

Any algorithm to process finite-dimensional vectors that can be expressed only in terms of pairwise inner products can be applied to potentially infinite-dimensional vectors in the feature space of a p.d. kernel by replacing each inner product evaluation by a kernel evaluation.

## Remarks:

- The proof of this proposition is trivial, because the kernel is exactly the inner product in the feature space.
- This trick has huge practical applications.
- Vectors in the feature space are only manipulated implicitly, through pairwise inner products.

## Example 1: computing distances in the feature space



$$\begin{aligned} d_K(x_1, x_2)^2 &= \| \Phi(x_1) - \Phi(x_2) \|_{\mathcal{H}}^2 \\ &= \langle \Phi(x_1) - \Phi(x_2), \Phi(x_1) - \Phi(x_2) \rangle_{\mathcal{H}} \\ &= \langle \Phi(x_1), \Phi(x_1) \rangle_{\mathcal{H}} + \langle \Phi(x_2), \Phi(x_2) \rangle_{\mathcal{H}} - 2 \langle \Phi(x_1), \Phi(x_2) \rangle_{\mathcal{H}} \end{aligned}$$

$$d_K(x_1, x_2)^2 = K(x_1, x_1) + K(x_2, x_2) - 2K(x_1, x_2)$$

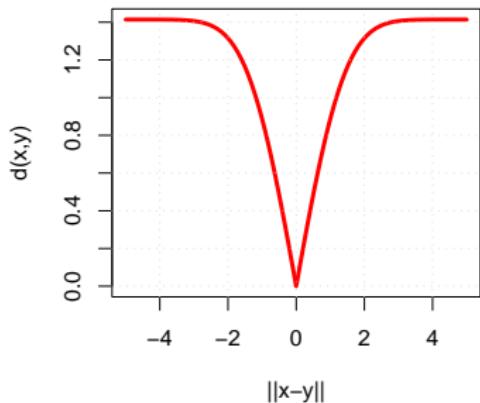
## Distance for the Gaussian kernel

- The Gaussian kernel with bandwidth  $\sigma$  on  $\mathbb{R}^d$  is:

$$K(\mathbf{x}, \mathbf{y}) = e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{2\sigma^2}},$$

- $K(\mathbf{x}, \mathbf{x}) = 1 = \|\Phi(\mathbf{x})\|_{\mathcal{H}}^2$ , so all points are on the unit sphere in the feature space.
- The distance between the images of two points  $\mathbf{x}$  and  $\mathbf{y}$  in the feature space is given by:

$$d_K(\mathbf{x}, \mathbf{y}) = \sqrt{2 \left[ 1 - e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{2\sigma^2}} \right]}$$



## Example 2: distance between a point and a set

### Problem

- Let  $\mathcal{S} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  be a finite set of points in  $\mathcal{X}$ .
- How to define and compute the **similarity** between any point  $\mathbf{x}$  in  $\mathcal{X}$  and the set  $\mathcal{S}$ ?

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A solution:

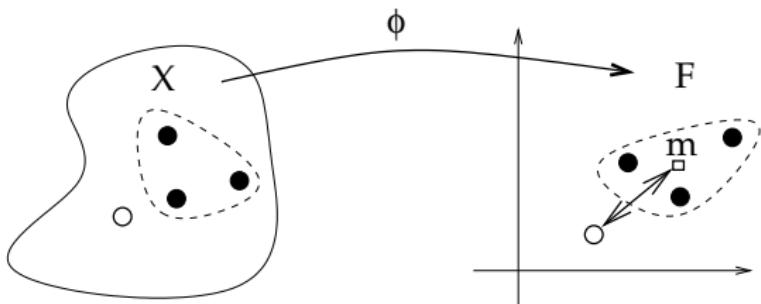
- Map all points to the feature space.
- Summarize  $\mathcal{S}$  by the **barycenter** of the points:

$$\boldsymbol{\mu} := \frac{1}{n} \sum_{i=1}^n \Phi(\mathbf{x}_i) .$$

- Define the distance between  $\mathbf{x}$  and  $\mathcal{S}$  by:

$$d_K(\mathbf{x}, \mathcal{S}) := \| \Phi(\mathbf{x}) - \boldsymbol{\mu} \|_{\mathcal{H}} .$$

# Computation



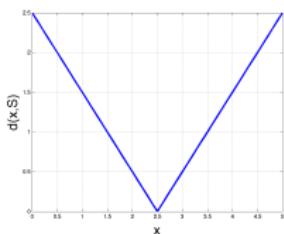
$$\begin{aligned} d_K(\mathbf{x}, \mathcal{S}) &= \| \Phi(\mathbf{x}) - \frac{1}{n} \sum_{i=1}^n \Phi(\mathbf{x}_i) \|_{\mathcal{H}} \\ &= \sqrt{K(\mathbf{x}, \mathbf{x}) - \frac{2}{n} \sum_{i=1}^n K(\mathbf{x}, \mathbf{x}_i) + \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n K(\mathbf{x}_i, \mathbf{x}_j)}. \end{aligned}$$

## Remark

The barycentre  $\mu$  only exists in the feature space in general: it does not necessarily have a pre-image  $\mathbf{x}_\mu$  such that  $\Phi(\mathbf{x}_\mu) = \mu$ .

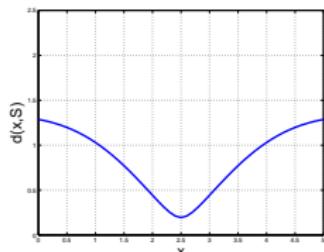
# 1D illustration

- $\mathcal{S} = \{2, 3\}$
- Plot  $f(x) = d(x, \mathcal{S})$



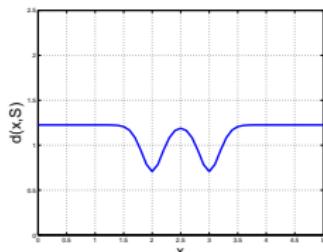
$$k(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}.$$

(linear)



$$k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2\sigma^2}}.$$

with  $\sigma = 1$ .

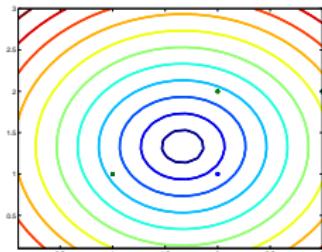


$$k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2\sigma^2}}.$$

with  $\sigma = 0.2$ .

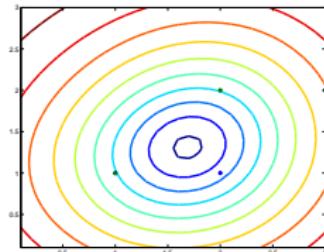
## 2D illustration

- $\mathcal{S} = \{(1, 1)', (1, 2)', (2, 2)'\}$
- Plot  $f(x) = d(x, \mathcal{S})$



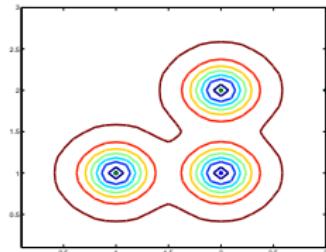
$$k(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}.$$

(linear)



$$k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2\sigma^2}}.$$

with  $\sigma = 1$ .

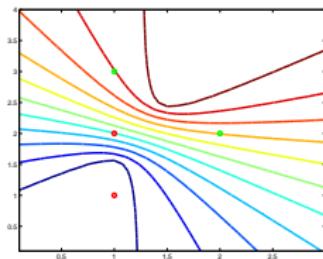


$$k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2\sigma^2}}.$$

with  $\sigma = 0.2$ .

# Application in discrimination

- $\mathcal{S}_1 = \{(1, 1)', (1, 2)'\}$  and  $\mathcal{S}_2 = \{(1, 3)', (2, 2)'\}$
- Plot  $f(x) = d(\mathbf{x}, \mathcal{S}_1)^2 - d(\mathbf{x}, \mathcal{S}_2)^2$

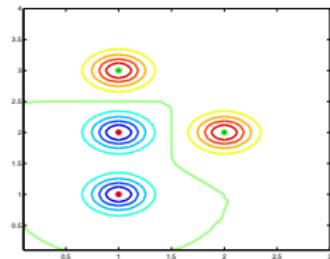


$$k(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}.$$

(linear)

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with  $\sigma = 1$ .



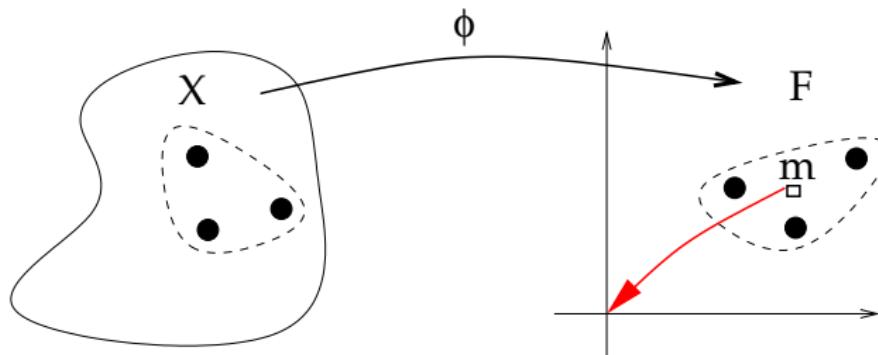
$$k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2\sigma^2}}.$$

with  $\sigma = 0.2$ .

## Example 3: Centering data in the feature space

### Problem

- Let  $\mathcal{S} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  be a finite set of points in  $\mathcal{X}$  endowed with a p.d. kernel  $K$ . Let  $\mathbf{K}$  be their  $n \times n$  Gram matrix:  $[\mathbf{K}]_{ij} = K(\mathbf{x}_i, \mathbf{x}_j)$ .
- Let  $\mu = 1/n \sum_{i=1}^n \Phi(\mathbf{x}_i)$  their barycenter, and  $\mathbf{u}_i = \Phi(\mathbf{x}_i) - \mu$  for  $i = 1, \dots, n$  be centered data in  $\mathcal{H}$ .
- How to compute the centered Gram matrix  $[\mathbf{K}^c]_{i,j} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle_{\mathcal{H}}$ ?



## Computation

- A direct computation gives, for  $0 \leq i, j \leq n$ :

$$\begin{aligned}\mathbf{K}_{i,j}^c &= \langle \Phi(\mathbf{x}_i) - \boldsymbol{\mu}, \Phi(\mathbf{x}_j) - \boldsymbol{\mu} \rangle_{\mathcal{H}} \\ &= \langle \Phi(\mathbf{x}_i), \Phi(\mathbf{x}_j) \rangle_{\mathcal{H}} - \langle \boldsymbol{\mu}, \Phi(\mathbf{x}_i) + \Phi(\mathbf{x}_j) \rangle_{\mathcal{H}} + \langle \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\mathcal{H}} \\ &= \mathbf{K}_{i,j} - \frac{1}{n} \sum_{k=1}^n (\mathbf{K}_{i,k} + \mathbf{K}_{j,k}) + \frac{1}{n^2} \sum_{k,l=1}^n \mathbf{K}_{k,l}.\end{aligned}$$

- This can be rewritten in matricial form:

$$\mathbf{K}^c = \mathbf{K} - \mathbf{U}\mathbf{K} - \mathbf{K}\mathbf{U} + \mathbf{U}\mathbf{K}\mathbf{U} = (\mathbf{I} - \mathbf{U})\mathbf{K}(\mathbf{I} - \mathbf{U}),$$

where  $\mathbf{U}_{i,j} = 1/n$  for  $1 \leq i, j \leq n$ .

## Kernel trick Summary

- The kernel trick is a trivial statement with **important applications**.
- It can be used to obtain **nonlinear** versions of well-known linear algorithms, e.g., by replacing the classical inner product by a Gaussian kernel.
- It can be used to apply classical algorithms to **non vectorial** data (e.g., strings, graphs) by again replacing the classical inner product by a valid kernel for the data.
- It allows in some cases to embed the initial space to a **larger feature space** and involve points in the feature space with no pre-image (e.g., barycenter).

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
  - The kernel trick
  - The representer theorem
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics

## Motivation

- An RKHS is a space of (potentially nonlinear) functions, and  $\|f\|_{\mathcal{H}}$  measures the smoothness of  $f$
- Given a set of data  $(\mathbf{x}_i \in \mathcal{X}, y_i \in \mathbb{R})_{i=1,\dots,n}$ , a natural to estimate a regression function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is to solve something like:

$$\min_{f \in \mathcal{H}} \underbrace{\frac{1}{n} \sum_{i=1}^n \ell(y_i, f(\mathbf{x}_i))}_{\text{empirical risk, data fit}} + \underbrace{\lambda \|f\|_{\mathcal{H}}^2}_{\text{regularization}}. \quad (1)$$

for a loss function  $\ell$  such as  $\ell(y, t) = (y - t)^2$

- How to solve in practice this problem, potentially in infinite dimension?

# The Theorem

## Representer Theorem

- Let  $\mathcal{X}$  be a set endowed with a p.d. kernel  $K$ ,  $\mathcal{H}_K$  the corresponding RKHS, and  $\mathcal{S} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathcal{X}$  a finite set of points in  $\mathcal{X}$ .
- Let  $\Psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a function of  $n + 1$  variables, strictly increasing with respect to the last variable.
- Then, any solution to the optimization problem:

$$\min_{f \in \mathcal{H}_K} \Psi(f(\mathbf{x}_1), \dots, f(\mathbf{x}_n), \|f\|_{\mathcal{H}_K}),$$

admits a representation of the form:

$$\forall \mathbf{x} \in \mathcal{X}, \quad f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}).$$

## Proof (1/2)

- Let  $\xi(f, \mathcal{S})$  be the functional that is minimized in the statement of the representer theorem, and  $\mathcal{H}_K^{\mathcal{S}}$  the linear span in  $\mathcal{H}_K$  of the vectors  $K_{x_i}$ , i.e.,

$$\mathcal{H}_K^{\mathcal{S}} = \left\{ f \in \mathcal{H}_K : f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}), (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \right\}.$$

- $\mathcal{H}_K^{\mathcal{S}}$  finite-dimensional subspace, therefore any function  $f \in \mathcal{H}_K$  can be uniquely decomposed as:

$$f = f_{\mathcal{S}} + f_{\perp},$$

with  $f_{\mathcal{S}} \in \mathcal{H}_K^{\mathcal{S}}$  and  $f_{\perp} \perp \mathcal{H}_K^{\mathcal{S}}$  (by orthogonal projection).

## Proof (2/2)

- $\mathcal{H}_K$  being a RKHS it holds that:

$$\forall i = 1, \dots, n, \quad f_{\perp}(\mathbf{x}_i) = \langle f_{\perp}, K(\mathbf{x}_i, \cdot) \rangle_{\mathcal{H}_K} = 0,$$

because  $K(\mathbf{x}_i, \cdot) \in \mathcal{H}_K$ , therefore:

$$\forall i = 1, \dots, n, \quad f(\mathbf{x}_i) = f_S(\mathbf{x}_i).$$

- Pythagoras' theorem in  $\mathcal{H}_K$  then shows that:

$$\|f\|_{\mathcal{H}_K}^2 = \|f_S\|_{\mathcal{H}_K}^2 + \|f_{\perp}\|_{\mathcal{H}_K}^2.$$

- As a consequence,  $\xi(f, \mathcal{S}) \geq \xi(f_S, \mathcal{S})$ , with equality if and only if  $\|f_{\perp}\|_{\mathcal{H}_K} = 0$ . **The minimum of  $\Psi$  is therefore necessarily in  $\mathcal{H}_K^S$ .**

□

## Remarks

Often the function  $\Psi$  has the form:

$$\Psi(f(\mathbf{x}_1), \dots, f(\mathbf{x}_n), \|f\|_{\mathcal{H}_K}) = c(f(\mathbf{x}_1), \dots, f(\mathbf{x}_n)) + \lambda \Omega(\|f\|_{\mathcal{H}_K})$$

where  $c(\cdot)$  measures the “fit” of  $f$  to a given problem (regression, classification, dimension reduction, ...) and  $\Omega$  is strictly increasing. This formulation has two important consequences:

- **Theoretically**, the minimization will enforce the norm  $\|f\|_{\mathcal{H}_K}$  to be “small”, which can be beneficial by ensuring a sufficient level of smoothness for the solution (regularization effect).
- **Practically**, we know by the representer theorem that the solution lives in a **subspace of dimension  $n$** , which can lead to efficient algorithms although the RKHS itself can be of infinite dimension.

## Practical use of the representer theorem (1/2)

- When the representer theorem holds, we know that we can look for a solution of the form

$$f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}), \quad \text{for some } \boldsymbol{\alpha} \in \mathbb{R}^n$$

- For any  $j = 1, \dots, n$ , we have

$$f(\mathbf{x}_j) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}_j) = [\mathbf{K}\boldsymbol{\alpha}]_j$$

- Furthermore,

$$\|f\|_{\mathcal{H}_K}^2 = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j K(\mathbf{x}_i, \mathbf{x}_j) = \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha}$$

## Practical use of the representer theorem (2/2)

- Therefore, a problem of the form

$$\min_{f \in \mathcal{H}_K} \Psi(f(x_1), \dots, f(x_n), \|f\|_{\mathcal{H}_K}^2)$$

is equivalent to the following  $n$ -dimensional optimization problem:

$$\min_{\alpha \in \mathbb{R}^n} \Psi([K\alpha]_1, \dots, [K\alpha]_n, \alpha^\top K \alpha)$$

- This problem can usually be solved analytically or by numerical methods; we will see many examples in the next sections.

## Remarks

### Dual interpretations of kernel methods

Most kernel methods have two complementary interpretations:

- A **geometric interpretation** in the feature space, thanks to the kernel trick. Even when the feature space is “large”, most kernel methods work in the linear span of the embeddings of the points available.
- A **functional interpretation**, often as an optimization problem over (subsets of) the RKHS associated to the kernel.

The representer theorem has important consequences, but it is in fact rather trivial. We are looking for a function  $f$  in  $\mathcal{H}$  such that for all  $\mathbf{x}$  in  $\mathcal{X}$ ,  $f(\mathbf{x}) = \langle K_{\mathbf{x}}, f \rangle_{\mathcal{H}}$ . The part  $f^\perp$  that is orthogonal to the  $K_{\mathbf{x}_i}$ 's is thus “useless” to explain the training data.

# Kernel Methods

## Supervised Learning

# Supervised learning

## Definition

Given:

- $\mathcal{X}$ , a space of **inputs**,
- $\mathcal{Y}$ , a space of **outputs**,
- $\mathcal{S}_n = (\mathbf{x}_i, \mathbf{y}_i)_{i=1, \dots, n}$ , a **training set** of (input,output) pairs,

the **supervised learning problem** is to estimate a function  $h : \mathcal{X} \rightarrow \mathcal{Y}$  to **predict** the output for any future input.

# Supervised learning

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Depending on the nature of the output, this covers:

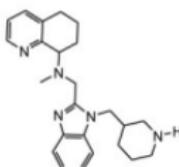
- **Regression** when  $\mathcal{Y} = \mathbb{R}$
- **Classification** when  $\mathcal{Y} = \{-1, 1\}$  or any set of two labels
- **Structured output** regression or classification when  $\mathcal{Y}$  is more general.

## Example: regression

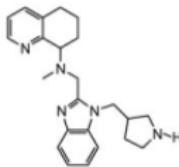
Task: predict the capacity of a small molecule to inhibit a drug target

$\mathcal{X}$  = set of molecular structures (graphs?)

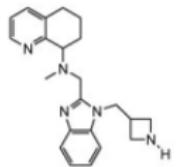
$\mathcal{Y} = \mathbb{R}$



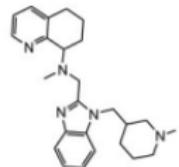
78  
 $IC_{50}=51\pm2\text{nM}$



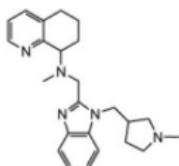
79  
 $IC_{50}=24\pm1\text{nM}$



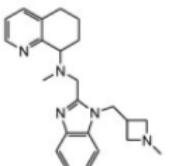
80  
 $IC_{50}=84\pm7\text{nM}$



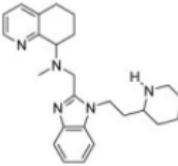
81  
 $IC_{50}=19\pm2\text{nM}$



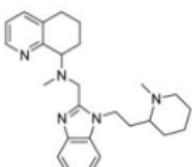
82  
 $IC_{50}=57\pm2\text{nM}$



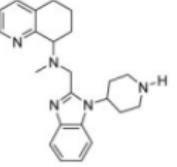
83  
 $IC_{50}=66\pm5\text{nM}$



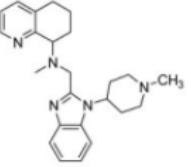
84  
 $IC_{50}=64\pm4\text{nM}$



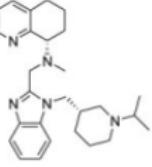
85  
 $IC_{50}=33\pm1\text{nM}$



86  
 $IC_{50}=99\pm6\text{nM}$



87  
 $IC_{50}=95\pm8\text{nM}$



88  
 $IC_{50}=2.0\pm1\text{nM}$

## Example: classification

Task: recognize if an image is a dog or a cat

$\mathcal{X}$  = set of images ( $\mathbb{R}^d$ )

$\mathcal{Y} = \{\text{cat}, \text{dog}\}$



## Example: classification

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## Example: structured output

Task: translate from Japanese to French

$\mathcal{X}$  = finite-length strings of japanese characters

$\mathcal{Y}$  = finite-length strings of french characters

The screenshot shows a web browser window for translate.google.fr. The interface includes a navigation bar with back, forward, and search icons, and a user profile icon. The main area has a "Google" logo and a "Traduction" section. On the left, there's a dropdown menu for selecting languages: "Anglais", "Français", "Arabe", and "Japonais - détecté". On the right, another dropdown menu shows "Français", "Anglais", "Arabe", and a "Traduire" button. Below these, the Japanese input "猿も木から落ちる" is displayed in a text box, with its English translation "Même les singes tombent des arbres" shown in a larger text box to its right. Both text boxes have edit icons (A, B, C) and a character count indicator (8/5000). At the bottom, there are links for "À propos de Google Traduction", "Communauté", "Mobile", social media links for G+ and B, and footer links for "À propos de Google", "Confidentialité et conditions d'utilisation", "Aide", and "Envoyer des commentaires".

# Supervised learning with kernels: general principles

- ① Express  $h : \mathcal{X} \rightarrow \mathcal{Y}$  using a real-valued function  $f : \mathcal{Z} \rightarrow \mathbb{R}$ :

- regression  $\mathcal{Y} = \mathbb{R}$  :

$$h(\mathbf{x}) = f(\mathbf{x}) \quad \text{with} \quad f : \mathcal{X} \rightarrow \mathbb{R} \quad (\mathcal{Z} = \mathcal{X})$$

- classification  $\mathcal{Y} = \{-1, 1\}$  :

$$h(\mathbf{x}) = \text{sign}(f(\mathbf{x})) \quad \text{with} \quad f : \mathcal{X} \rightarrow \mathbb{R} \quad (\mathcal{Z} = \mathcal{X})$$

- structured output:

$$h(\mathbf{x}) = \arg \max_{\mathbf{y} \in \mathcal{Y}} f(\mathbf{x}, \mathbf{y}) \quad \text{with} \quad f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \quad (\mathcal{Z} = \mathcal{X} \times \mathcal{Y})$$

- ② Define an empirical risk function  $R_n(f)$  to assess how "good" a candidate function  $f$  is on the training set  $\mathcal{S}_n$ , typically the average of a loss:

$$R_n(f) := \frac{1}{n} \sum_{i=1}^n \ell(f(\mathbf{x}_i), \mathbf{y}_i)$$

- ③ Define a p.d. kernel on  $\mathcal{Z}$  and solve

$$\min_{f \in \mathcal{H}_K, \|f\|_{\mathcal{H}_K} \leq B} R_n(f) \quad \text{or} \quad \min_{f \in \mathcal{H}_K} R_n(f) + \lambda \|f\|_{\mathcal{H}_K}^2$$

## Remarks

$$\min_{f \in \mathcal{H}_K} \underbrace{\frac{1}{n} \sum_{i=1}^n \ell(f(\mathbf{x}_i), \mathbf{y}_i)}_{\text{empirical risk, data fit}} + \underbrace{\lambda \|f\|_{\mathcal{H}_K}^2}_{\text{regularization}}.$$

- Regularization is important, particularly in high dimension, to prevent **overfitting**
- When  $\mathcal{Z} = \mathbb{R}^d$  and  $K$  is the linear kernel,  $f$  is a linear model and the regularization is  $\|w\|^2$
- Using more general spaces  $\mathcal{Z}$  and kernels  $K$  allows to
  - learn **non-linear functions** over a functional space endowed with a natural regularization (remember, small norm in RKHS = "smooth")
  - learn functions over **non-vectorial data**, such as strings and graphs

We will now see a few methods in more details

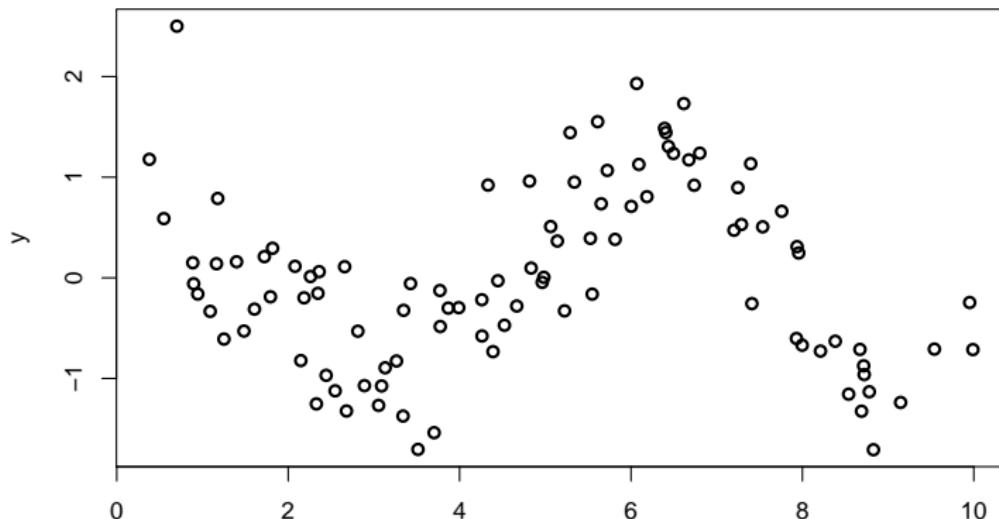
# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
  - Kernel ridge regression
  - Kernel logistic regression
  - Large-margin classifiers
  - Interlude: convex optimization and duality
  - Support vector machines
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics

# Regression

## Setup

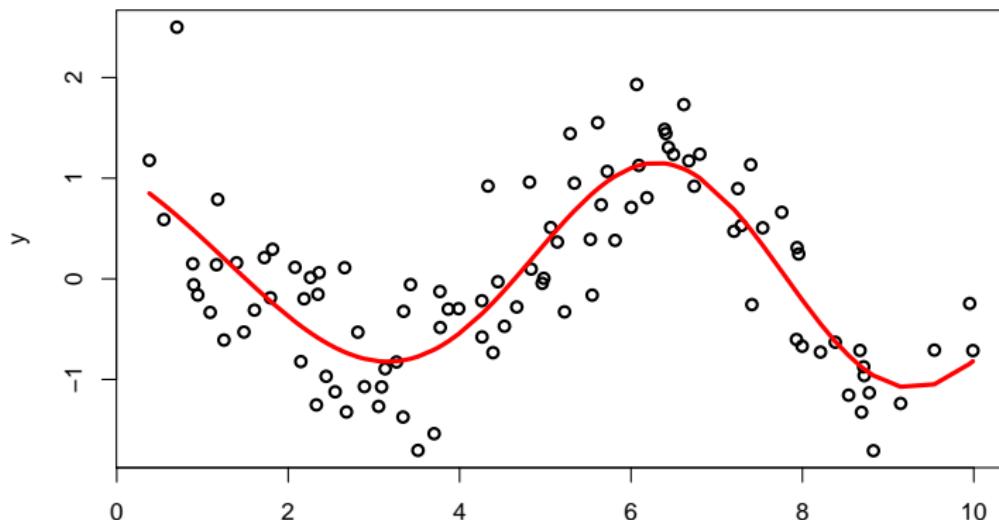
- $\mathcal{X}$  set of inputs
- $\mathcal{Y} = \mathbb{R}$  real-valued outputs
- $\mathcal{S}_n = (\mathbf{x}_i, \mathbf{y}_i)_{i=1, \dots, n} \in (\mathcal{X} \times \mathbb{R})^n$  a training set of  $n$  pairs
- Goal = find a function  $f : \mathcal{X} \rightarrow \mathbb{R}$  to predict  $\mathbf{y}$  by  $f(\mathbf{x})$



# Regression

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## Least-square regression over a general functional space

- Let us quantify the error if  $f$  predicts  $f(\mathbf{x})$  instead of  $y$  by the squared error:

$$\ell(f(\mathbf{x}), \mathbf{y}) = (\mathbf{y} - f(\mathbf{x}))^2$$

- Fix a set of functions  $\mathcal{H}$ .
- Least-square regression** amounts to finding the function in  $\mathcal{H}$  with the smallest empirical risk, called in this case the mean squared error (MSE):

$$\hat{f} \in \arg \min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n (\mathbf{y}_i - f(\mathbf{x}_i))^2$$

- Issues: unstable (especially in large dimensions), overfitting if  $\mathcal{H}$  is too “large”.

## Kernel ridge regression (KRR)

- Let us consider a RKHS  $\mathcal{H}_K$ , associated to a p.d. kernel  $K$  on  $\mathcal{X}$ .
- KRR is obtained by taking  $\mathcal{H} = \mathcal{H}_K$ , and **regularizing** the MSE criterion by the RKHS norm:

$$\hat{f} = \arg \min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n (\mathbf{y}_i - f(\mathbf{x}_i))^2 + \lambda \|f\|_{\mathcal{H}_K}^2 \quad (2)$$

- 1st effect = prevent overfitting by penalizing non-smooth functions.*

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- 1st effect = prevent overfitting by penalizing non-smooth functions.*
- By the representer theorem, any solution of (2) can be expanded as

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}).$$

- 2nd effect = simplifying the solution.*

## Solving KRR

- Let  $\mathbf{Y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)^\top \in \mathbb{R}^n$
- Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$
- Let  $\mathbf{K}$  be the  $n \times n$  Gram matrix:  $\mathbf{K}_{ij} = K(\mathbf{x}_i, \mathbf{x}_j)$
- We can then write:

$$\left( \hat{f}(\mathbf{x}_1), \dots, \hat{f}(\mathbf{x}_n) \right)^\top = \mathbf{K}\boldsymbol{\alpha}$$

- The following holds as usual:

$$\| \hat{f} \|_{\mathcal{H}_K}^2 = \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha}$$

- The KRR problem (2) is therefore equivalent to:

$$\arg \min_{\boldsymbol{\alpha} \in \mathbb{R}^n} \frac{1}{n} (\mathbf{K}\boldsymbol{\alpha} - \mathbf{Y})^\top (\mathbf{K}\boldsymbol{\alpha} - \mathbf{Y}) + \lambda \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha}$$

## Solving KRR

$$\arg \min_{\alpha \in \mathbb{R}^n} \frac{1}{n} (\mathbf{K}\alpha - \mathbf{Y})^\top (\mathbf{K}\alpha - \mathbf{Y}) + \lambda \alpha^\top \mathbf{K}\alpha$$

- This is a convex and differentiable function of  $\alpha$ . Its minimum can therefore be found by setting the gradient in  $\alpha$  to zero:

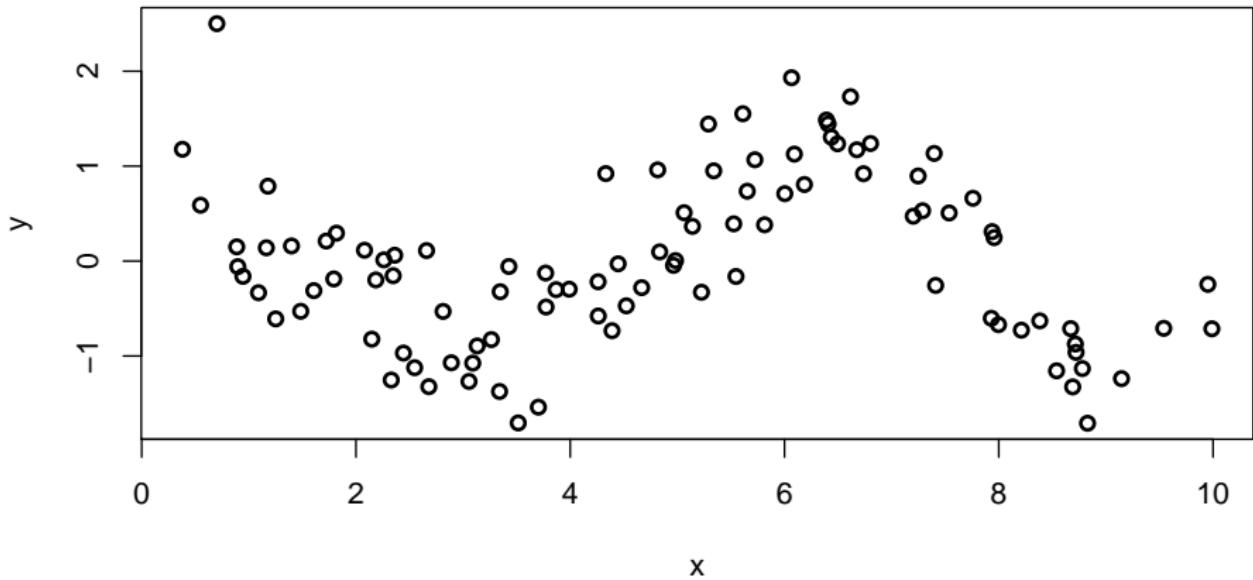
$$\begin{aligned} 0 &= \frac{2}{n} \mathbf{K} (\mathbf{K}\alpha - \mathbf{Y}) + 2\lambda \mathbf{K}\alpha \\ &= \mathbf{K} [(\mathbf{K} + \lambda n \mathbf{I}) \alpha - \mathbf{Y}] \end{aligned}$$

- For  $\lambda > 0$ ,  $\mathbf{K} + \lambda n \mathbf{I}$  is invertible (because  $\mathbf{K}$  is positive semidefinite) so one solution is to take:

$$\alpha = (\mathbf{K} + \lambda n \mathbf{I})^{-1} \mathbf{Y}.$$

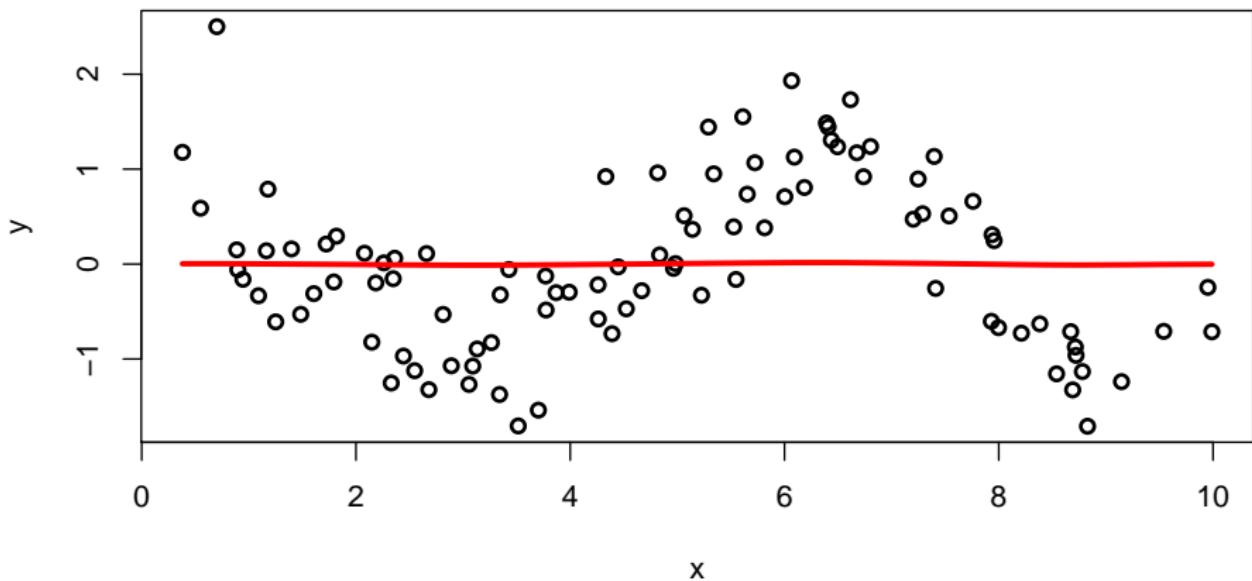


## Example (KRR with Gaussian RBF kernel)



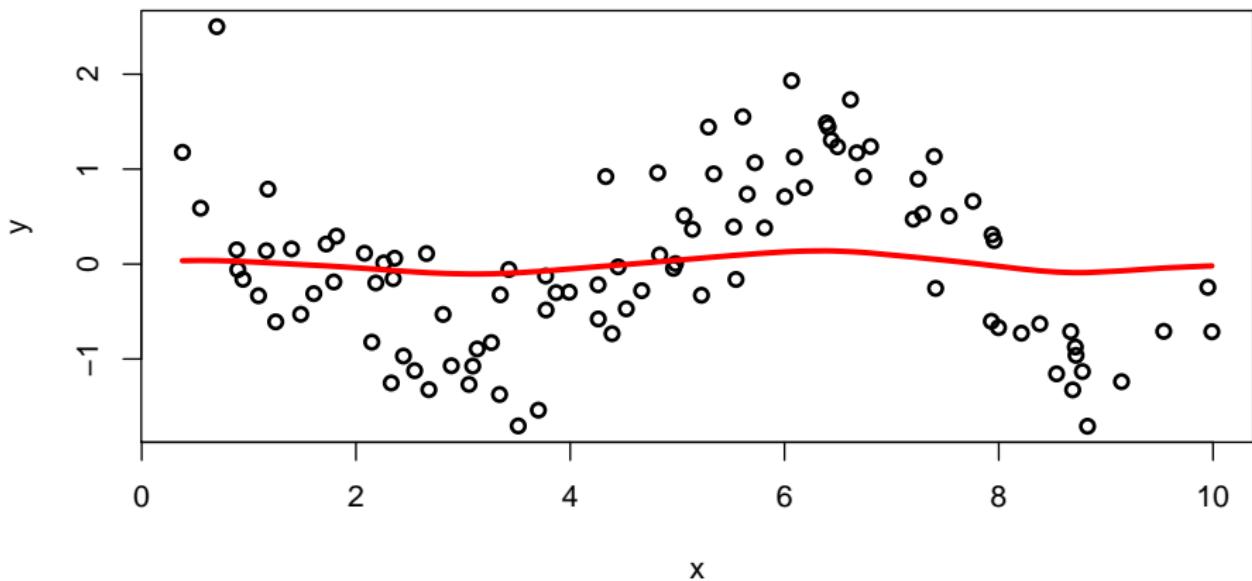
## Example (KRR with Gaussian RBF kernel)

**lambda = 1000**



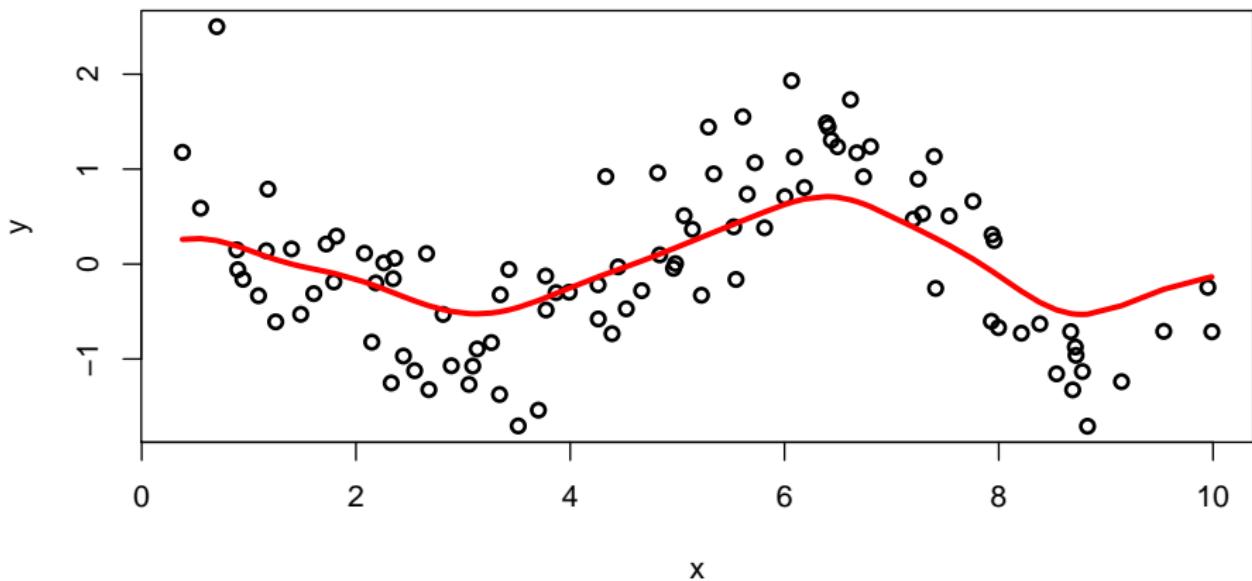
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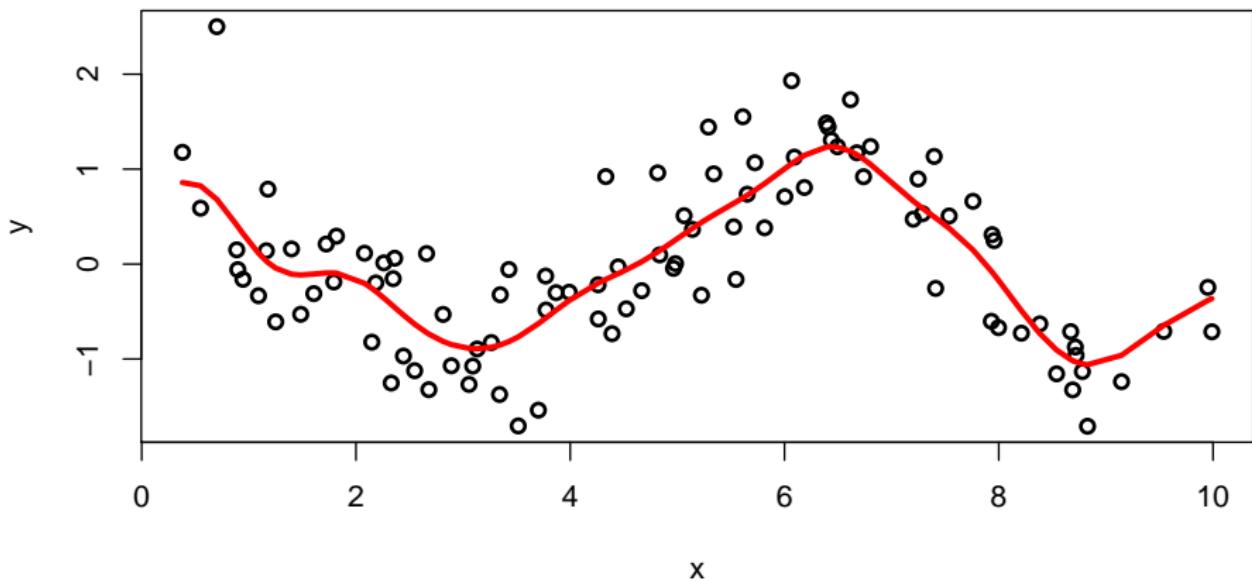
## Example (KRR with Gaussian RBF kernel)

**lambda = 10**



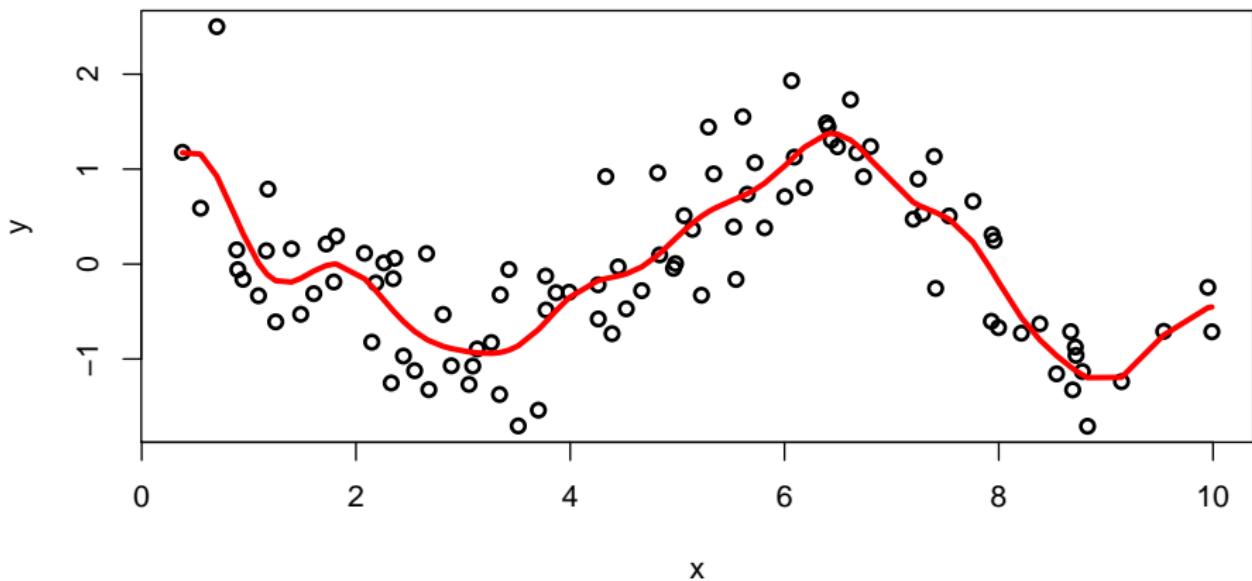
## Example (KRR with Gaussian RBF kernel)

**lambda = 1**



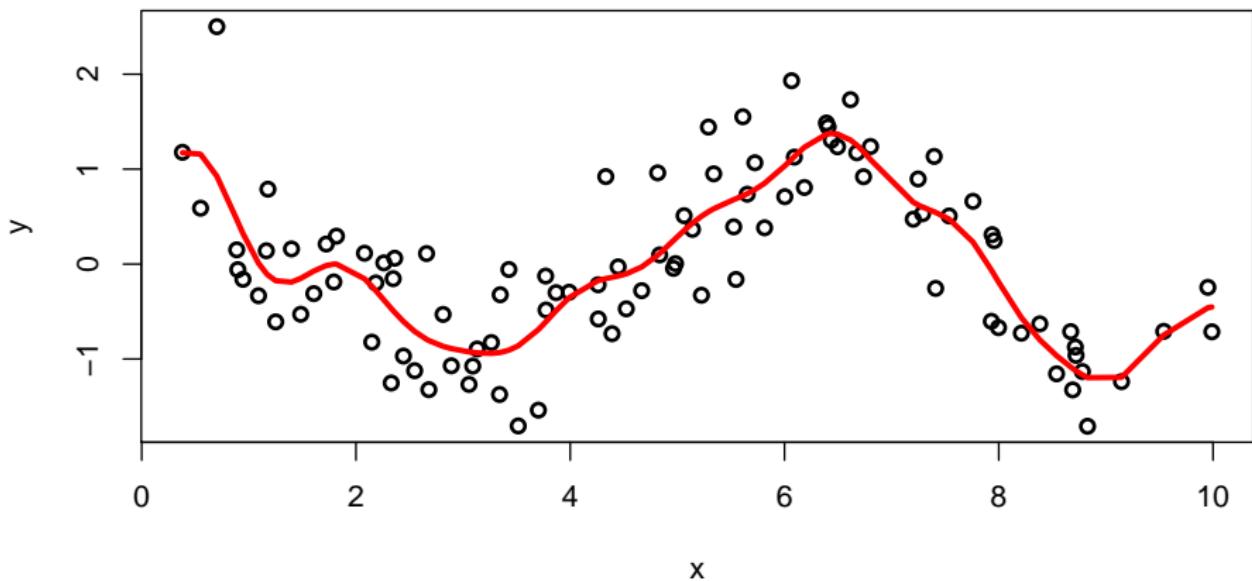
## Example (KRR with Gaussian RBF kernel)

**lambda = 0.1**



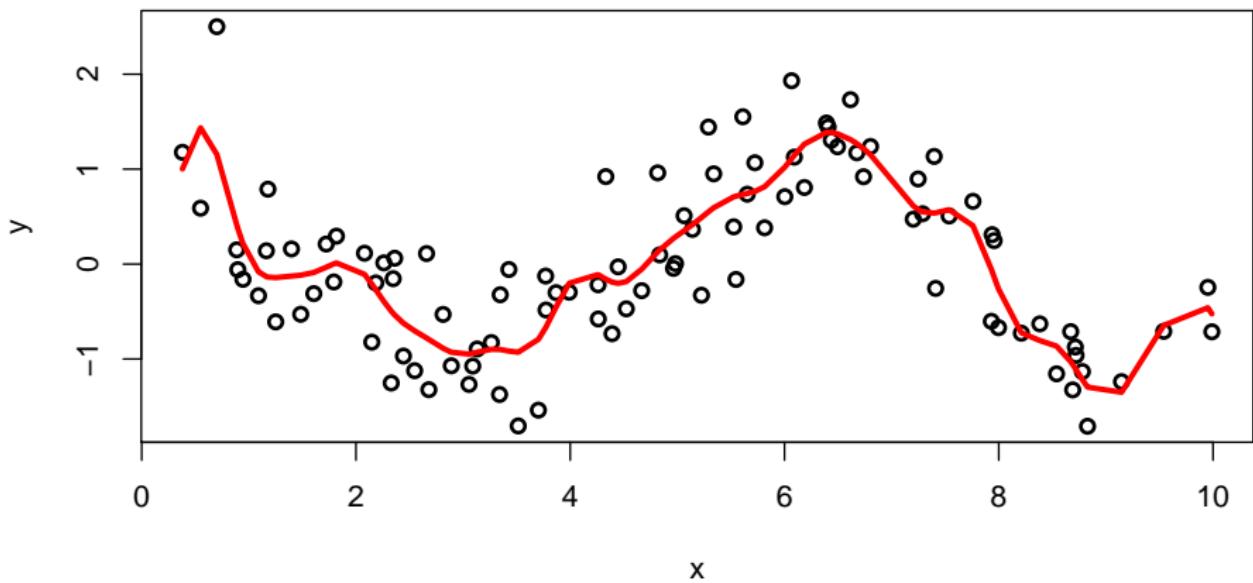
## Example (KRR with Gaussian RBF kernel)

**lambda = 0.01**



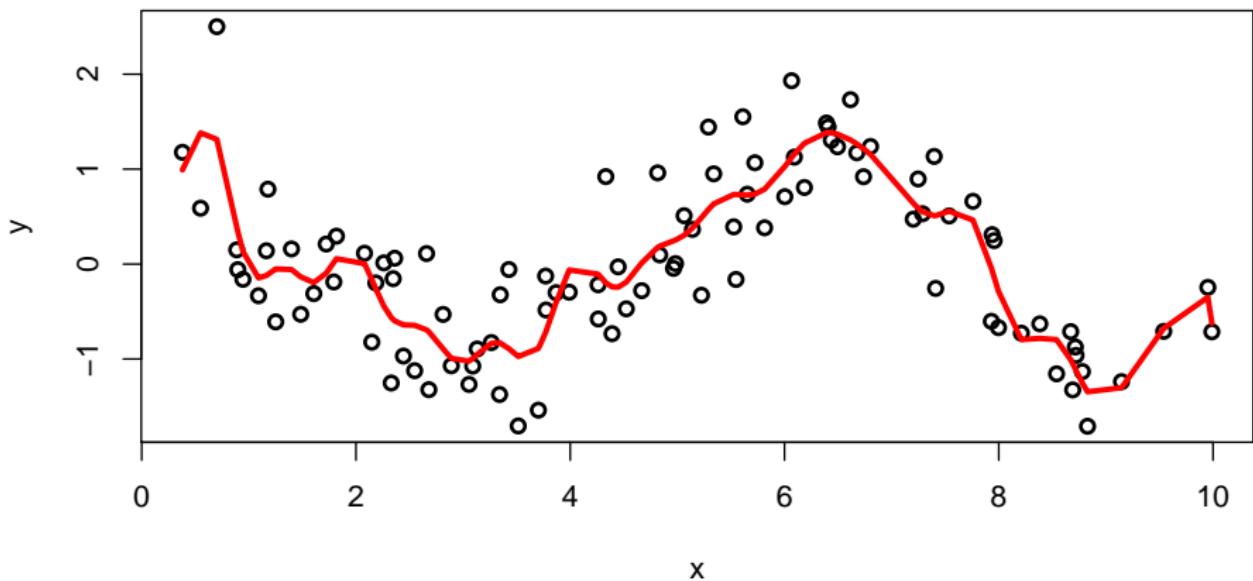
## Example (KRR with Gaussian RBF kernel)

**lambda = 0.001**



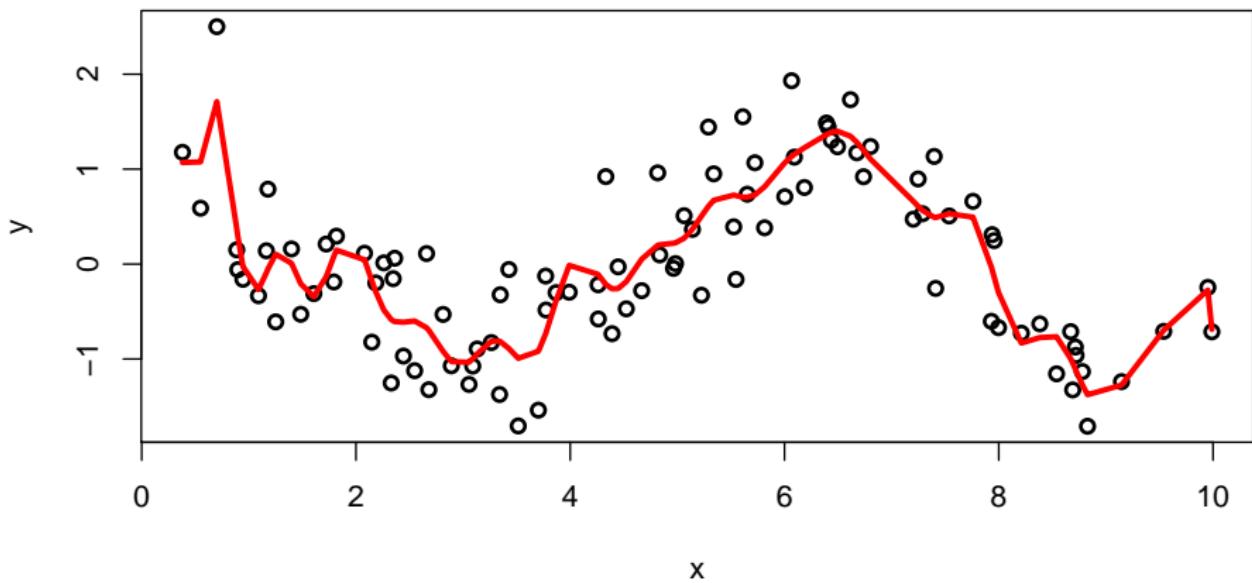
## Example (KRR with Gaussian RBF kernel)

**lambda = 0.0001**



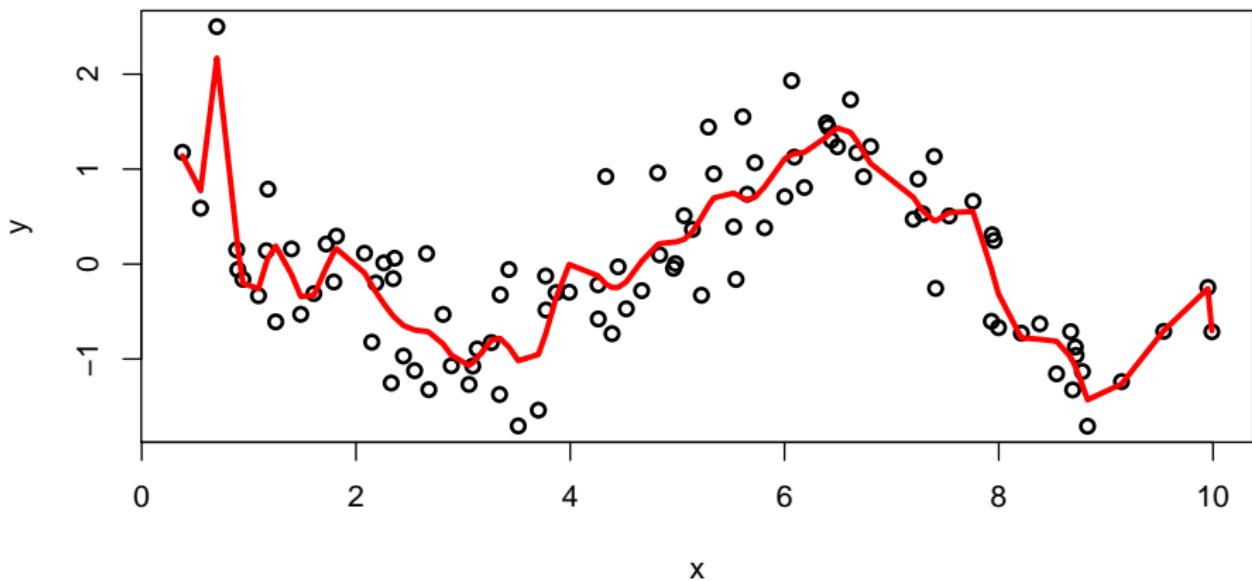
## Example (KRR with Gaussian RBF kernel)

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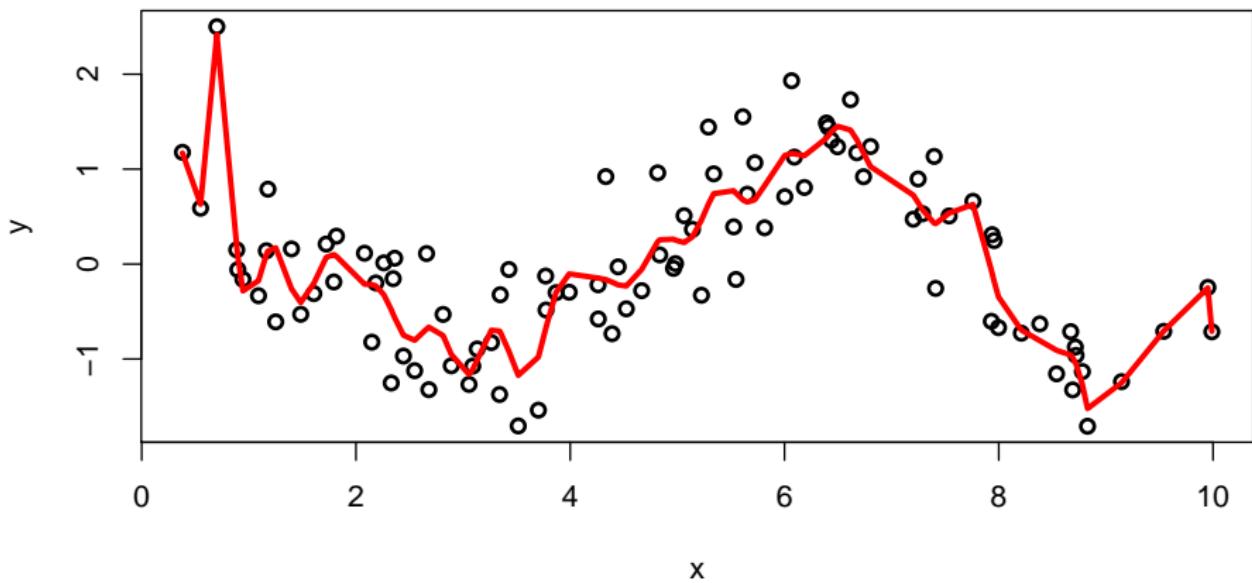
## Example (KRR with Gaussian RBF kernel)

**lambda = 0.000001**



## Example (KRR with Gaussian RBF kernel)

**lambda = 0.0000001**



## Remark: unicity of the solution

Let us find *all*  $\alpha$ 's that solve

$$\mathbf{K} [(\mathbf{K} + \lambda n \mathbf{I}) \alpha - \mathbf{Y}] = 0$$

- $\mathbf{K}$  being a symmetric matrix, it can be diagonalized in an orthonormal basis and  $\text{Ker}(\mathbf{K}) \perp \text{Im}(\mathbf{K})$ .
- In this basis we see that  $(\mathbf{K} + \lambda n \mathbf{I})^{-1}$  leaves  $\text{Im}(\mathbf{K})$  and  $\text{Ker}(\mathbf{K})$  invariant.
- The problem is therefore equivalent to:

$$\begin{aligned} & (\mathbf{K} + \lambda n \mathbf{I}) \alpha - \mathbf{Y} \in \text{Ker}(\mathbf{K}) \\ \Leftrightarrow & \alpha - (\mathbf{K} + \lambda n \mathbf{I})^{-1} \mathbf{Y} \in \text{Ker}(\mathbf{K}) \\ \Leftrightarrow & \alpha = (\mathbf{K} + \lambda n \mathbf{I})^{-1} \mathbf{Y} + \epsilon, \text{ with } \mathbf{K}\epsilon = 0. \end{aligned}$$

- However, if  $\alpha' = \alpha + \epsilon$  with  $\mathbf{K}\epsilon = 0$ , then:

$$\| f - f' \|_{\mathcal{H}}^2 = (\alpha - \alpha')^\top \mathbf{K} (\alpha - \alpha') = 0,$$

therefore  $f = f'$ . KRR has a unique solution  $f \in \mathcal{H}_K$ , which can possibly be expressed by several  $\alpha$ 's if  $K$  is singular.

## Remark: link with "standard" ridge regression

- Take  $\mathcal{X} = \mathbb{R}^d$  and the linear kernel  $K(\mathbf{x}, \mathbf{x}') = \mathbf{x}^\top \mathbf{x}'$
- Let  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^\top$  the  $n \times d$  data matrix
- The kernel matrix is then  $\mathbf{K} = \mathbf{XX}^\top$
- The function learned by KRR in that case is linear:

$$f_{KRR}(\mathbf{x}) = \mathbf{w}_{KRR}^\top \mathbf{x}$$

with

$$\mathbf{w}_{KRR} = \sum_{i=1}^n \alpha_i \mathbf{x}_i = \mathbf{X}^\top \boldsymbol{\alpha} = \mathbf{X}^\top (\mathbf{XX}^\top + \lambda n \mathbf{I})^{-1} \mathbf{Y}$$

## Remark: link with "standard" ridge regression

- On the other hand, the RKHS is the set of linear functions  $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$  and the RKHS norm is  $\|f\|_{\mathcal{H}_K} = \|\mathbf{w}\|$
- We can therefore directly rewrite the original KRR problem (2) as

$$\begin{aligned} \arg \min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (\mathbf{y}_i - \mathbf{w}^\top \mathbf{x}_i)^2 + \lambda \|\mathbf{w}\|^2 \\ = \arg \min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{n} (\mathbf{Y} - \mathbf{X}\mathbf{w})^\top (\mathbf{Y} - \mathbf{X}\mathbf{w}) + \lambda \mathbf{w}^\top \mathbf{w} \end{aligned}$$

- Setting the gradient to 0 gives the solution:

$$\mathbf{w}_{RR} = (\mathbf{X}^\top \mathbf{X} + \lambda n \mathbf{I})^{-1} \mathbf{X}^\top \mathbf{Y}$$

- Oups, looks different from  $\mathbf{w}_{KRR} = \mathbf{X}^\top (\mathbf{X}\mathbf{X}^\top + \lambda n \mathbf{I})^{-1} \mathbf{Y} \dots ?$

## Remark: link with "standard" ridge regression

### Matrix inversion lemma

For any matrices  $B$  and  $C$ , and  $\gamma > 0$  the following holds (when it makes sense):

$$B(CB + \gamma \mathbf{I})^{-1} = (BC + \gamma \mathbf{I})^{-1} B$$

We deduce that (of course...):

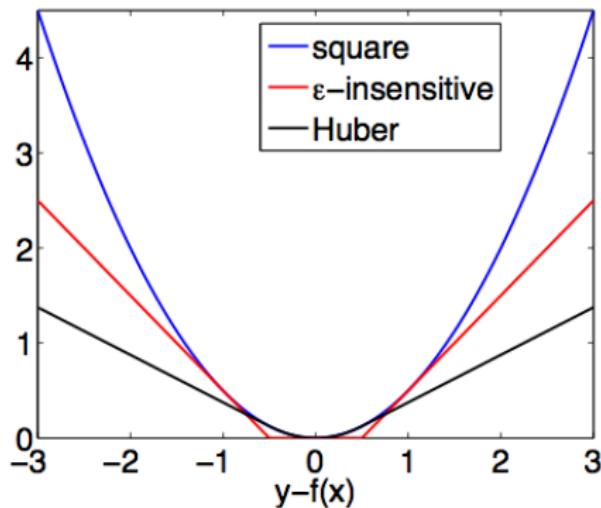
$$\mathbf{w}_{RR} = \underbrace{\left( \mathbf{X}^\top \mathbf{X} + \lambda n \mathbf{I} \right)^{-1}}_{d \times d} \mathbf{X}^\top \mathbf{Y} = \mathbf{X}^\top \underbrace{\left( \mathbf{X} \mathbf{X}^\top + \lambda n \mathbf{I} \right)^{-1}}_{n \times n} \mathbf{Y} = \mathbf{w}_{KRR}$$

Computationally, inverting the matrix is the expensive part, which suggest to implement:

- KRR when  $d > n$  (high dimension)
- RR when  $d < n$  (many points)

## Robust regression

- The squared error  $\ell(t, y) = (t - y)^2$  is arbitrary and sensitive to outliers
- Many other loss functions exist for regression, e.g.:



- Any loss function leads to a valid kernel method, which is usually solved by numerical optimization as there is usually no analytical solution beyond the squared error.

## Weighted regression

- Given weights  $W_1, \dots, W_n \in \mathbb{R}$ , a variant of ridge regression is to weight differently the error at different points:

$$\arg \min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n W_i (\mathbf{y}_i - f(\mathbf{x}_i))^2 + \lambda \|f\|_{\mathcal{H}_K}^2$$

- By the representer theorem the solution is  $f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x})$  where  $\boldsymbol{\alpha}$  solves, with  $\mathbf{W} = \text{diag}(W_1, \dots, W_n)$ :

$$\arg \min_{\boldsymbol{\alpha} \in \mathbb{R}^n} \frac{1}{n} (\mathbf{K}\boldsymbol{\alpha} - \mathbf{Y})^\top \mathbf{W} (\mathbf{K}\boldsymbol{\alpha} - \mathbf{Y}) + \lambda \boldsymbol{\alpha}^\top \mathbf{K}\boldsymbol{\alpha}$$

## Weighted regression

- Setting the gradient to zero gives

$$\begin{aligned} 0 &= \frac{2}{n} (\mathbf{K} \mathbf{W} \mathbf{K} \boldsymbol{\alpha} - \mathbf{K} \mathbf{W} \mathbf{Y}) + 2\lambda \mathbf{K} \boldsymbol{\alpha} \\ &= \frac{2}{n} \mathbf{K} \mathbf{W}^{\frac{1}{2}} \left[ \left( \mathbf{W}^{\frac{1}{2}} \mathbf{K} \mathbf{W}^{\frac{1}{2}} + n\lambda \mathbf{I} \right) \mathbf{W}^{-\frac{1}{2}} \boldsymbol{\alpha} - \mathbf{W}^{\frac{1}{2}} \mathbf{Y} \right] \end{aligned}$$

- A solution is therefore given by

$$\left( \mathbf{W}^{\frac{1}{2}} \mathbf{K} \mathbf{W}^{\frac{1}{2}} + n\lambda \mathbf{I} \right) \mathbf{W}^{-\frac{1}{2}} \boldsymbol{\alpha} - \mathbf{W}^{\frac{1}{2}} \mathbf{Y} = 0$$

therefore

$$\boldsymbol{\alpha} = \mathbf{W}^{\frac{1}{2}} \left( \mathbf{W}^{\frac{1}{2}} \mathbf{K} \mathbf{W}^{\frac{1}{2}} + n\lambda \mathbf{I} \right)^{-1} \mathbf{W}^{\frac{1}{2}} \mathbf{Y}$$

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  - Kernel ridge regression
  - **Kernel logistic regression**
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# Binary classification

## Setup

- $\mathcal{X}$  set of inputs
- $\mathcal{Y} = \{-1, 1\}$  binary outputs
- $\mathcal{S}_n = (\mathbf{x}_i, \mathbf{y}_i)_{i=1, \dots, n} \in (\mathcal{X} \times \mathcal{Y})^n$  a training set of  $n$  pairs
- Goal = find a function  $f : \mathcal{X} \rightarrow \mathbb{R}$  to predict  $y$  by  $\text{sign}(f(\mathbf{x}))$



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## The 0/1 loss

- The 0/1 loss measures if a prediction is correct or not:

$$\ell_{0/1}(f(\mathbf{x}), \mathbf{y}) = \mathbf{1}(\mathbf{y}f(\mathbf{x}) < 0) = \begin{cases} 0 & \text{if } \mathbf{y} = \text{sign}(f(\mathbf{x})) \\ 1 & \text{otherwise.} \end{cases}$$

- It is then tempting to learn  $f$  by solving:

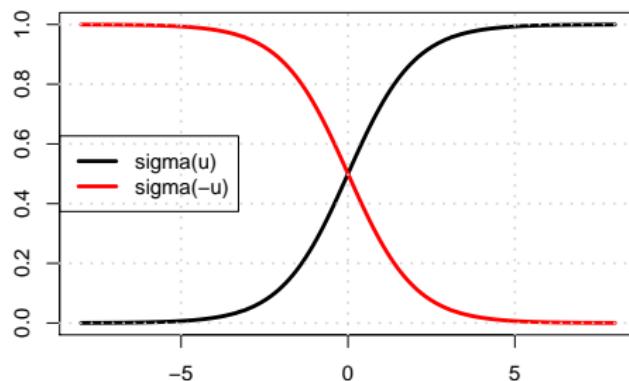
$$\min_{f \in \mathcal{H}_K} \underbrace{\frac{1}{n} \sum_{i=1}^n \ell_{0/1}(f(\mathbf{x}_i), \mathbf{y}_i)}_{\text{misclassification rate}} + \underbrace{\lambda \|f\|_{\mathcal{H}_K}^2}_{\text{regularization}}$$

- However:
  - The problem is non-smooth, and typically NP-hard to solve
  - The regularization has **no effect** since the 0/1 loss is invariant by scaling of  $f$
  - In fact, no function achieves the minimum when  $\lambda > 0$  (*why?*)

## The logistic loss

- An alternative is to define a probabilistic model of  $y$  parametrized by  $f(x)$ , e.g.:

$$\forall y \in \{-1, 1\}, \quad p(y | f(x)) = \frac{1}{1 + e^{-yf(x)}} = \sigma(yf(x))$$



- The **logistic loss** is the negative conditional likelihood:

$$\ell_{logistic}(f(x), y) = -\ln p(y | f(x)) = \ln(1 + e^{-yf(x)})$$

## Kernel logistic regression (KLR)

$$\begin{aligned}\hat{f} &= \arg \min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n \ell_{\text{logistic}}(f(\mathbf{x}_i), \mathbf{y}_i) + \frac{\lambda}{2} \|f\|_{\mathcal{H}_K}^2 \\ &= \arg \min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n \ln \left( 1 + e^{-\mathbf{y}_i f(\mathbf{x}_i)} \right) + \frac{\lambda}{2} \|f\|_{\mathcal{H}_K}^2\end{aligned}$$

- Can be interpreted as a regularized conditional maximum likelihood estimator
- No explicit solution, but smooth convex optimization problem that can be solved numerically

## Solving KLR

- By the representer theorem, any solution of KLR can be expanded as

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x})$$

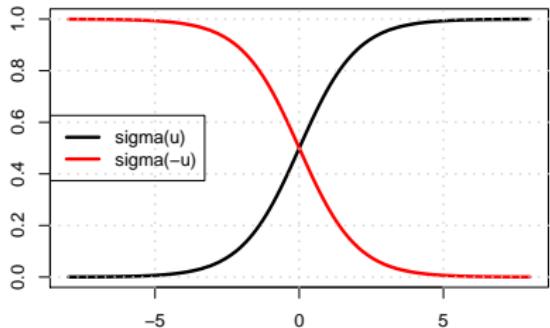
and as always we have:

$$\left( \hat{f}(\mathbf{x}_1), \dots, \hat{f}(\mathbf{x}_n) \right)^\top = \mathbf{K}\boldsymbol{\alpha} \quad \text{and} \quad \| \hat{f} \|^2_{\mathcal{H}_K} = \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha}$$

- To find  $\boldsymbol{\alpha}$  we therefore need to solve:

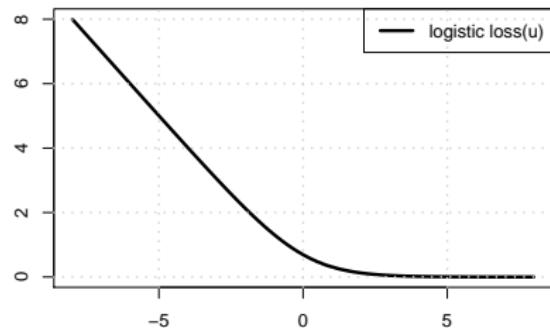
$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n \ln \left( 1 + e^{-y_i [\mathbf{K}\boldsymbol{\alpha}]_i} \right) + \frac{\lambda}{2} \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha}$$

# Technical facts



Sigmoid:

- $\sigma(u) = \frac{1}{1+e^{-u}}$
- $\sigma(-u) = 1 - \sigma(u)$
- $\sigma'(u) = \sigma(u)\sigma(-u) \geq 0$



Logistic loss:

- $\ell_{logistic}(u) = \ln(1 + e^{-u})$
- $\ell'_{logistic}(u) = -\sigma(-u)$
- $\ell''_{logistic}(u) = \sigma(u)\sigma(-u) \geq 0$

## Back to KLR

$$\min_{\alpha \in \mathbb{R}^n} J(\alpha) = \frac{1}{n} \sum_{i=1}^n \ell_{logistic} (\mathbf{y}_i [\mathbf{K}\alpha]_i) + \frac{\lambda}{2} \alpha^\top \mathbf{K} \alpha$$

This is a smooth convex optimization problem, that can be solved by many numerical methods. Let us explicit one of them, **Newton's method**, which iteratively approximates  $J$  by a quadratic function and solves the quadratic problem.

The quadratic approximation near a point  $\alpha_0$  is the function:

$$J_q(\alpha) = J_q(\alpha_0) + (\alpha - \alpha_0)^\top \nabla J(\alpha_0) + \frac{1}{2} (\alpha - \alpha_0)^\top \nabla^2 J(\alpha_0) (\alpha - \alpha_0)^\top$$

Let us compute the different terms...

## Computing the quadratic approximation

$$\frac{\partial J}{\partial \alpha_j} = \frac{1}{n} \sum_{i=1}^n \underbrace{\ell'_{logistic}(\mathbf{y}_i[\mathbf{K}\boldsymbol{\alpha}]_i)}_{P_i(\boldsymbol{\alpha})} \mathbf{y}_i \mathbf{K}_{ij} + \lambda [\mathbf{K}\boldsymbol{\alpha}]_j$$

therefore

$$\nabla J(\boldsymbol{\alpha}) = \frac{1}{n} \mathbf{K} \mathbf{P}(\boldsymbol{\alpha}) \mathbf{Y} + \lambda \mathbf{K} \boldsymbol{\alpha}$$

where  $\mathbf{P}(\boldsymbol{\alpha}) = \text{diag}(P_1(\boldsymbol{\alpha}), \dots, P_n(\boldsymbol{\alpha}))$ .

$$\frac{\partial^2 J}{\partial \alpha_j \partial \alpha_l} = \frac{1}{n} \sum_{i=1}^n \underbrace{\ell''_{logistic}(\mathbf{y}_i[\mathbf{K}\boldsymbol{\alpha}]_i) \mathbf{y}_i \mathbf{K}_{ij} \mathbf{y}_i \mathbf{K}_{il}}_{W_i(\boldsymbol{\alpha})} + \lambda \mathbf{K}_{jl}$$

therefore

$$\nabla^2 J(\boldsymbol{\alpha}) = \frac{1}{n} \mathbf{K} \mathbf{W}(\boldsymbol{\alpha}) \mathbf{K} + \lambda \mathbf{K}$$

where  $\mathbf{W}(\boldsymbol{\alpha}) = \text{diag}(W_1(\boldsymbol{\alpha}), \dots, W_n(\boldsymbol{\alpha}))$ .

## Computing the quadratic approximation

$$J_q(\alpha) = J_q(\alpha_0) + (\alpha - \alpha_0)^\top \nabla J(\alpha_0) + \frac{1}{2} (\alpha - \alpha_0)^\top \nabla^2 J(\alpha_0) (\alpha - \alpha_0)^\top$$

Terms that depend on  $\alpha$ , with  $\mathbf{P} = \mathbf{P}(\alpha_0)$  and  $\mathbf{W} = \mathbf{W}(\alpha_0)$ :

- $\alpha^\top \nabla J(\alpha_0) = \frac{1}{n} \alpha^\top \mathbf{K} \mathbf{P} \mathbf{Y} + \lambda \alpha^\top \mathbf{K} \alpha_0$
- $\frac{1}{2} \alpha^\top \nabla^2 J(\alpha_0) \alpha = \frac{1}{2n} \alpha^\top \mathbf{K} \mathbf{W} \mathbf{K} \alpha + \frac{\lambda}{2} \alpha^\top \mathbf{K} \alpha$
- $-\alpha^\top \nabla^2 J(\alpha_0) \alpha_0 = -\frac{1}{n} \alpha^\top \mathbf{K} \mathbf{W} \mathbf{K} \alpha_0 - \lambda \alpha^\top \mathbf{K} \alpha_0$

Putting it all together:

$$\begin{aligned} 2J_q(\alpha) &= -\frac{2}{n} \alpha^\top \mathbf{K} \mathbf{W} \underbrace{(\mathbf{K} \alpha_0 - \mathbf{W}^{-1} \mathbf{P} \mathbf{Y})}_{:= \mathbf{Z}} + \frac{1}{n} \alpha^\top \mathbf{K} \mathbf{W} \mathbf{K} \alpha + \lambda \alpha^\top \mathbf{K} \alpha + C \\ &= \frac{1}{n} (\mathbf{K} \alpha - \mathbf{Z})^\top \mathbf{W} (\mathbf{K} \alpha - \mathbf{Z}) + \lambda \alpha^\top \mathbf{K} \alpha + C \end{aligned}$$

This is a standard weighted kernel ridge regression (WKRR) problem!

## Solving KLR by IRLS

In summary, one way to solve KLR is to iteratively solve a WKRR problem until convergence:

$$\alpha^{t+1} \leftarrow \text{solveWKRR}(\mathbf{K}, \mathbf{W}^t, \mathbf{Z}^t)$$

where we update  $\mathbf{W}^t$  and  $\mathbf{Z}^t$  from  $\alpha^t$  as follows ( for  $i = 1, \dots, n$ ):

- $m_i \leftarrow [\mathbf{K}\alpha^t]_i$
- $P_i^t \leftarrow \ell'_{logistic}(\mathbf{y}_i m_i) = -\sigma(-\mathbf{y}_i m_i)$
- $W_i^t \leftarrow \ell''_{logistic}(\mathbf{y}_i m_i) = \sigma(m_i)\sigma(-m_i)$
- $Z_i^t \leftarrow m_i - P_i^t \mathbf{y}_i / W_i^t = m_i + \mathbf{y}_i / \sigma(-\mathbf{y}_i m_i)$

This is the kernelized version of the famous *iteratively reweighted least-square* (IRLS) method to solve the standard linear logistic regression.

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
  - Kernel ridge regression
  - Kernel logistic regression
  - **Large-margin classifiers**
  - Interlude: convex optimization and duality
  - Support vector machines
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics

## Loss functions for classifications

We already saw two loss functions for binary classification problems

- The 0/1 loss  $\ell_{0/1}(f(\mathbf{x}), \mathbf{y}) = \mathbf{1}(\mathbf{y}f(\mathbf{x}) < 0)$
- The logistic loss  $\ell_{logistic}(f(\mathbf{x}), \mathbf{y}) = \ln(1 + e^{-\mathbf{y}f(\mathbf{x})})$

In both cases, the loss is a function of the margin defined as follows

### Definition

In binary classification ( $\mathcal{Y} = \{-1, 1\}$ ), the **margin** of the function  $f$  for a pair  $(\mathbf{x}, \mathbf{y})$  is:

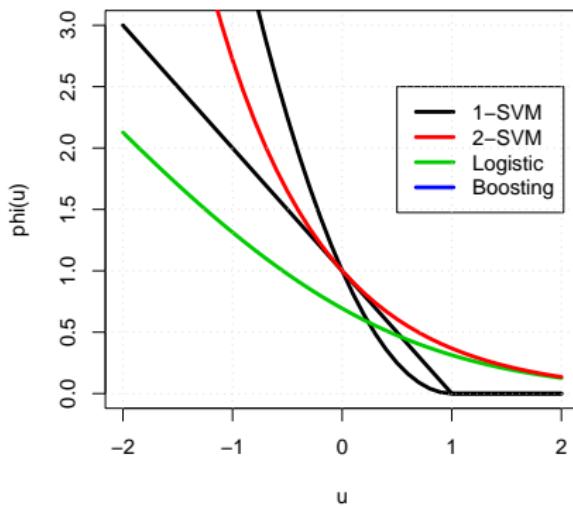
$$\mathbf{y}f(\mathbf{x}).$$

In both cases the loss is a decreasing function of the margin, i.e.,

$$\ell(f(\mathbf{x}), \mathbf{y}) = \varphi(\mathbf{y}f(\mathbf{x})) , \quad \text{with } \varphi \text{ non-increasing}$$

What about other similar loss functions?

## Loss function examples



Method	$\varphi(u)$
Kernel logistic regression	$\log(1 + e^{- u })$
Support vector machine (1-SVM)	$\max(1 - u, 0)$
Support vector machine (2-SVM)	$\max(1 - u, 0)^2$
Boosting	$e^{- u }$

# Large-margin classifiers

## Definition

Given a non-increasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ , a (kernel) large-margin classifier is an algorithm that estimates a function  $f : \mathcal{X} \rightarrow \mathbb{R}$  by solving

$$\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}_K}^2$$

Hence, KLR is a large-margin classifier, corresponding to  $\varphi(u) = \ln(1 + e^{-u})$ . Many more are possible.

## Questions:

- ① Can we solve the optimization problem for other  $\varphi$ 's?
- ② Is it a good idea to optimize this objective function, if at the end of the day we are interested in the  $\ell_{0/1}$  loss, i.e., learning models that make few errors?

## Solving large-margin classifiers

$$\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}_K}^2$$

- By the representer theorem, the solution of the unconstrained problem can be expanded as:

$$f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}).$$

- Plugging into the original problem we obtain the following unconstrained and convex optimization problem in  $\mathbb{R}^n$ :

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i [\mathbf{K}\boldsymbol{\alpha}]_i) + \lambda \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha} \right\},.$$

- When  $\varphi$  is convex, this can be solved using general tools for **convex optimization**, or specific algorithms (e.g., for SVM, see later).

# A tiny bit of learning theory

## Assumptions and notations

- Let  $\mathbb{P}$  be an (unknown) distribution on  $\mathcal{X} \times \mathcal{Y}$ , and  $\eta(\mathbf{x}) = \mathbb{P}(Y = 1 | X = \mathbf{x})$  a measurable version of the conditional distribution of  $Y$  given  $X$
- Assume the training set  $\mathcal{S}_n = (X_i, Y_i)_{i=1,\dots,n}$  are i.i.d. random variables according to  $\mathbb{P}$ .
- The **risk** of a classifier  $f : \mathcal{X} \rightarrow \mathbb{R}$  is  $R(f) = \mathbb{P}(\text{sign}(f(X)) \neq Y)$
- The **Bayes risk** is

$$R^* = \inf_{f \text{ measurable}} R(f)$$

which is attained for  $f^*(\mathbf{x}) = \eta(\mathbf{x}) - 1/2$

- The **empirical risk** of a classifier  $f : \mathcal{X} \rightarrow \mathbb{R}$  is

$$R^n(f) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(\text{sign}(f(X_i)) \neq Y_i)$$

## $\varphi$ -risk

- Let the **empirical  $\varphi$ -risk** be the empirical risk optimized by a large-margin classifier:

$$R_\varphi^n(f) = \frac{1}{n} \sum_{i=1}^n \varphi(Y_i f(X_i))$$

- It is the empirical version of the  **$\varphi$ -risk**

$$R_\varphi(f) = \mathbb{E}[\varphi(Y f(X))]$$

- Can we hope to have a small risk  $R(f)$  if we focus instead on the  $\varphi$ -risk  $R_\varphi(f)$ ?

## A small $\varphi$ -risk ensures a small 0/1 risk

### Theorem [Bartlett et al., 2003]

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$  be convex, non-increasing, differentiable at 0 with  $\varphi'(0) < 0$ . Let  $f : \mathcal{X} \rightarrow \mathbb{R}$  measurable such that

$$R_\varphi(f) = \min_{g \text{ measurable}} R_\varphi(g) = R_\varphi^*.$$

Then

$$R(f) = \min_{g \text{ measurable}} R(g) = R^*.$$

Remarks:

- This tells us that, if we know  $\mathbb{P}$ , then minimizing the  $\varphi$ -risk is a good idea even if our focus is on the classification error.
- The assumptions on  $\varphi$  can be relaxed; it works for the broader class of *classification-calibrated* loss functions [Bartlett et al., 2003].
- More generally, we can show that **if  $R_\varphi(f) - R_\varphi^*$  is small, then  $R(f) - R^*$  is small too** [Bartlett et al., 2003].

## A small $\varphi$ -risk ensures a small 0/1 risk

Proof sketch:

Condition on  $X = \mathbf{x}$ :

$$R_\varphi(f | X = \mathbf{x}) = \mathbb{E} [\varphi(Yf(X)) | X = \mathbf{x}] = \eta(\mathbf{x})\varphi(f(\mathbf{x})) + (1 - \eta(\mathbf{x}))\varphi(-f(\mathbf{x}))$$

$$R_\varphi(-f | X = \mathbf{x}) = \mathbb{E} [\varphi(-Yf(X)) | X = \mathbf{x}] = \eta(\mathbf{x})\varphi(-f(\mathbf{x})) + (1 - \eta(\mathbf{x}))\varphi(f(\mathbf{x}))$$

Therefore:

$$R_\varphi(f | X = \mathbf{x}) - R_\varphi(-f | X = \mathbf{x}) = [2\eta(\mathbf{x}) - 1] \times [\varphi(f(\mathbf{x})) - \varphi(-f(\mathbf{x}))]$$

This must be a.s.  $\leq 0$  because  $R_\varphi(f) \leq R_\varphi(-f)$ , which implies:

- if  $\eta(\mathbf{x}) > \frac{1}{2}$ ,  $\varphi(f(\mathbf{x})) \leq \varphi(-f(\mathbf{x})) \implies f(\mathbf{x}) \geq 0$
- if  $\eta(\mathbf{x}) < \frac{1}{2}$ ,  $\varphi(f(\mathbf{x})) \geq \varphi(-f(\mathbf{x})) \implies f(\mathbf{x}) \leq 0$

These inequalities are in fact strict thanks to the assumptions we made on  $\varphi$  (*left as exercise*). □

## Empirical risk minimization (ERM)

To find a function with a small  $\varphi$ -risk, the following is a good candidate:

### Definition

The **ERM estimator** on a functional class  $\mathcal{F}$  is the solution (when it exists) of:

$$\hat{f}_n = \arg \min_{f \in \mathcal{F}} R_\varphi^n(f).$$

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$$\hat{f}_n = \arg \min_{f \in \mathcal{F}} R_\varphi^n(f).$$

## Questions

- Is  $R_\varphi^n(f)$  a good estimate of the true risk  $R_\varphi(f)$ ?
- Is  $R_\varphi(\hat{f}_n)$  small?

# Class capacity

## Motivations

- The ERM principle gives a good solution if  $R_\varphi(\hat{f}_n)$  is similar to the minimum achievable risk  $\inf_{f \in \mathcal{F}} R_\varphi(f)$ .
- This can be ensured if  $\mathcal{F}$  is **not “too large”**.
- We need a measure of the “**capacity**” of  $\mathcal{F}$ .

## Definition: Rademacher complexity

The **Rademacher complexity** of a class of functions  $\mathcal{F}$  is:

$$\text{Rad}_n(\mathcal{F}) = \mathbf{E}_{X,\sigma} \left[ \sup_{f \in \mathcal{F}} \left| \frac{2}{n} \sum_{i=1}^n \sigma_i f(X_i) \right| \right],$$

where the expectation is over  $(X_i)_{i=1,\dots,n}$  and the independent uniform  $\{\pm 1\}$ -valued (Rademacher) random variables  $(\sigma_i)_{i=1,\dots,n}$ .

# Basic learning bounds

## Theorem

Suppose  $\varphi$  is Lipschitz with constant  $L_\varphi$ :

$$\forall u, u' \in \mathbb{R}, \quad |\varphi(u) - \varphi(u')| \leq L_\varphi |u - u'|.$$

Then the  $\varphi$ -risk of the ERM estimator satisfies (on average over the sampling of training set)

$$\underbrace{\mathbb{E}_{\mathcal{S}_n} R_\varphi(\hat{f}_n) - R_\varphi^*}_{\text{Excess } \varphi\text{-risk}} \leq \underbrace{4L_\varphi \text{Rad}_n(\mathcal{F})}_{\text{Estimation error}} + \underbrace{\inf_{f \in \mathcal{F}} R_\varphi(f) - R_\varphi^*}_{\text{Approximation error}}$$

This quantifies a trade-off between:

- $\mathcal{F}$  "large" = overfitting (approximation error small, estimation error large)
- $\mathcal{F}$  "small" = underfitting (estimation error small, approximation error large)

# ERM in RKHS balls

## Principle

- Assume  $\mathcal{X}$  is endowed with a p.d. kernel.
- We consider the ball of radius  $B$  in the RKHS as function class for the ERM:

$$\mathcal{F}_B = \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq B\} .$$

## Theorem (capacity control of RKHS balls)

$$\text{Rad}_n(\mathcal{F}_B) \leq \frac{2B\sqrt{\mathbf{E}K(X, X)}}{\sqrt{n}} .$$

## Proof (1/2)

$$\begin{aligned}
 \text{Rad}_n(\mathcal{F}_B) &= \mathbf{E}_{X,\sigma} \left[ \sup_{f \in \mathcal{F}_B} \left| \frac{2}{n} \sum_{i=1}^n \sigma_i f(X_i) \right| \right] \\
 &= \mathbf{E}_{X,\sigma} \left[ \sup_{f \in \mathcal{F}_B} \left| \left\langle f, \frac{2}{n} \sum_{i=1}^n \sigma_i K_{X_i} \right\rangle \right| \right] \quad (\text{RKHS}) \\
 &= \mathbf{E}_{X,\sigma} \left[ B \left\| \frac{2}{n} \sum_{i=1}^n \sigma_i K_{X_i} \right\|_{\mathcal{H}} \right] \quad (\text{Cauchy-Schwarz}) \\
 &= \frac{2B}{n} \mathbf{E}_{X,\sigma} \left[ \sqrt{\left\| \sum_{i=1}^n \sigma_i K_{X_i} \right\|_{\mathcal{H}}^2} \right] \\
 &\leq \frac{2B}{n} \sqrt{\mathbf{E}_{X,\sigma} \left[ \sum_{i,j=1}^n \sigma_i \sigma_j K(X_i, X_j) \right]} \quad (\text{Jensen})
 \end{aligned}$$

## Proof (2/2)

But  $\mathbf{E}_\sigma [\sigma_i \sigma_j]$  is 1 if  $i = j$ , 0 otherwise. Therefore:

$$\begin{aligned}\text{Rad}_n(\mathcal{F}_B) &\leq \frac{2B}{n} \sqrt{\mathbf{E}_X \left[ \sum_{i,j=1}^n \mathbf{E}_\sigma [\sigma_i \sigma_j] K(X_i, X_j) \right]} \\ &\leq \frac{2B}{n} \sqrt{\mathbf{E}_X \sum_{i=1}^n K(X_i, X_i)} \\ &= \frac{2B \sqrt{\mathbf{E}_X K(X, X)}}{\sqrt{n}}.\end{aligned}$$

□

## Basic learning bounds in RKHS balls

### Corollary

Suppose  $K(X, X) \leq \kappa^2$  a.s. (e.g., Gaussian kernel and  $\kappa = 1$ ). Then the ERM estimator in  $\mathcal{F}_B$  satisfies

$$\mathbf{E}R_\varphi(\hat{f}_n) - R_\varphi^* \leq \frac{8L_\varphi\kappa B}{\sqrt{n}} + \left[ \inf_{f \in \mathcal{F}_B} R_\varphi(f) - R_\varphi^* \right].$$

### Remarks

- $B$  controls the trade-off between approximation and estimation error
- The bound on expression error is independent of  $\mathcal{P}$  and decreases with  $n$
- The approximation error is harder to analyze in general
- In practice,  $B$  (or  $\lambda$ , next slide) is tuned by cross-validation

## ERM as penalized risk minimization

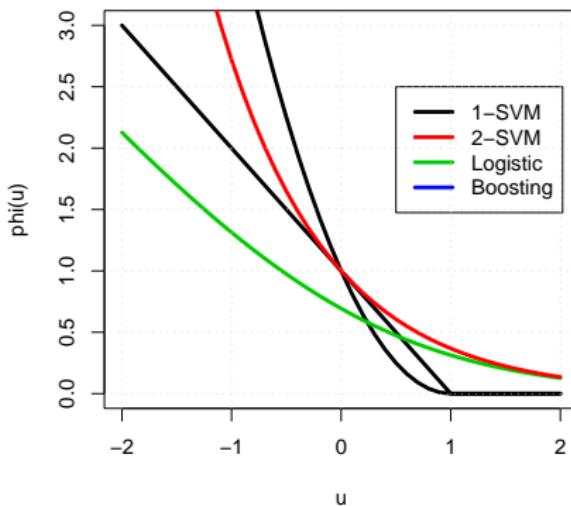
- ERM over  $\mathcal{F}_B$  solves the **constrained minimization problem**:

$$\begin{cases} \min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i f(\mathbf{x}_i)) \\ \text{subject to } \|f\|_{\mathcal{H}} \leq B. \end{cases}$$

- To make this practical we assume that  $\varphi$  is convex.
- The problem is then a **convex problem** in  $f$  for which **strong duality holds**. In particular  $f$  solves the problem if and only if it solves for some dual parameter  $\lambda$  the **unconstrained problem**:

$$\min_{f \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}}^2 \right\}.$$

## Summary: large margin classifiers



$$\min_{f \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{y}_i f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}}^2 \right\}$$

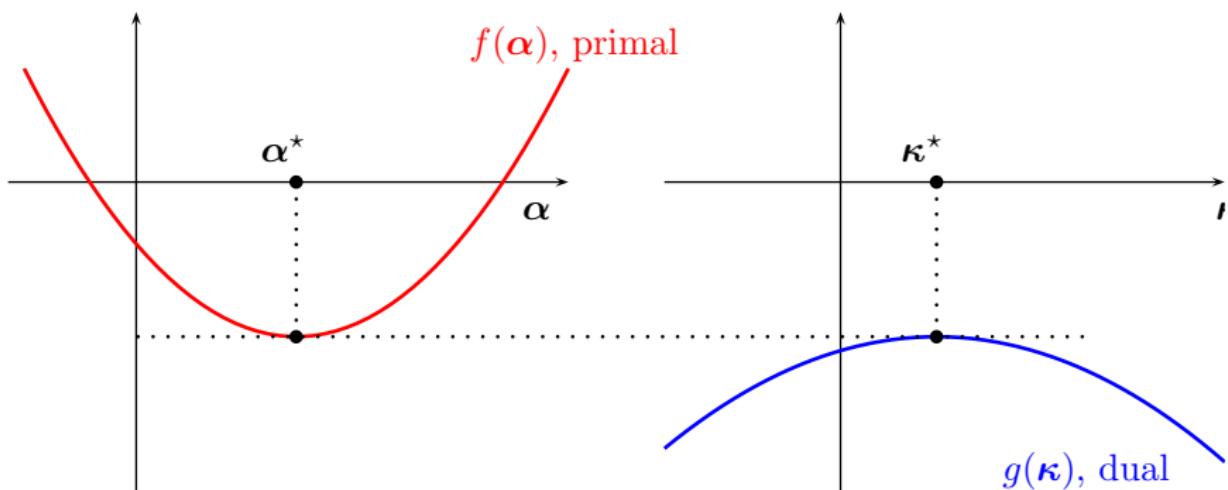
- $\varphi$  calibrated (e.g., decreasing,  $\varphi'(0) < 0$ )  $\implies$  good proxy for classification error
- $\varphi$  convex + representer theorem  $\implies$  efficient algorithms

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
  - Kernel ridge regression
  - Kernel logistic regression
  - Large-margin classifiers
  - **Interlude: convex optimization and duality**
  - Support vector machines
- 4 Kernel Methods: Unsupervised Learning
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# A few slides on convex duality

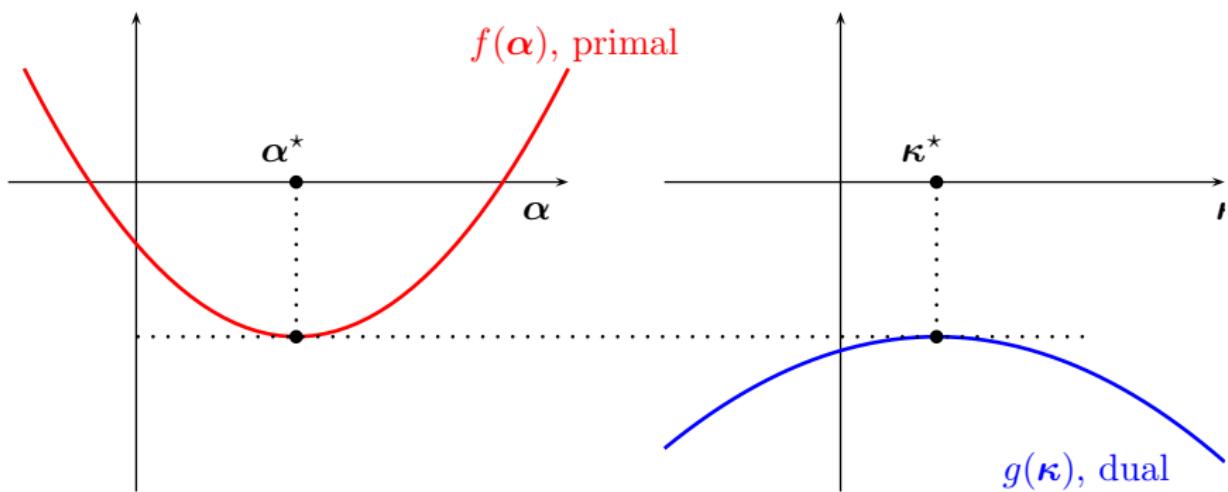
## Strong Duality



- Strong duality means that  $\max_{\kappa} g(\kappa) = \min_{\alpha} f(\alpha)$
- Strong duality holds in most “reasonable cases” for convex optimization (to be detailed soon).

# A few slides on convex duality

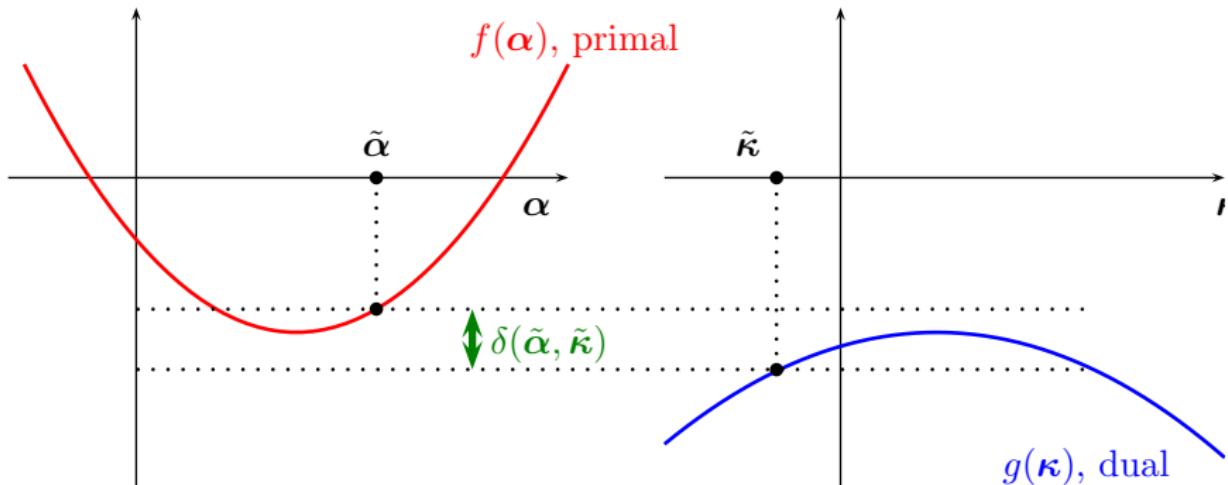
## Strong Duality



- The relation between  $\kappa^*$  and  $\alpha^*$  is not always known a priori.

# A few slides on convex duality

## Parenthesis on duality gaps



- The duality gap guarantees us that  $0 \leq f(\tilde{\alpha}) - f(\alpha^*) \leq \delta(\tilde{\alpha}, \tilde{\kappa})$ .
- Dual problems are often obtained by Lagrangian or Fenchel duality.

# A few slides on Lagrangian duality

## Setting

- We consider an equality and inequality constrained optimization problem over a variable  $x \in \mathcal{X}$ :

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && h_i(x) = 0, \quad i = 1, \dots, m, \\ & && g_j(x) \leq 0, \quad j = 1, \dots, r, \end{aligned}$$

making **no assumption** of  $f$ ,  $g$  and  $h$ .

- Let us denote by  $f^*$  the optimal value of the decision function under the constraints, i.e.,  $f^* = f(x^*)$  if the minimum is reached at a global minimum  $x^*$ .

# A few slides on Lagrangian duality

## Lagrangian

The **Lagrangian** of this problem is the function  $L : \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R}$  defined by:

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x).$$

## Lagrangian dual function

The **Lagrange dual function**  $g : \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R}$  is:

$$\begin{aligned} q(\lambda, \mu) &= \inf_{x \in \mathcal{X}} L(x, \lambda, \mu) \\ &= \inf_{x \in \mathcal{X}} \left( f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x) \right). \end{aligned}$$

# A few slides on convex Lagrangian duality

For the (primal) problem:

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && h(x) = 0, \quad g(x) \leq 0, \end{aligned}$$

the Lagrange dual problem is:

$$\begin{aligned} & \text{maximize} && q(\lambda, \mu) \\ & \text{subject to} && \mu \geq 0, \end{aligned}$$

## Proposition

- $q$  is concave in  $(\lambda, \mu)$ , even if the original problem is not convex.
- The dual function yields lower bounds on the optimal value  $f^*$  of the original problem when  $\mu$  is nonnegative:

$$q(\lambda, \mu) \leq f^*, \quad \forall \lambda \in \mathbb{R}^m, \forall \mu \in \mathbb{R}^r, \mu \geq 0.$$

## Proofs

- For each  $x$ , the function  $(\lambda, \mu) \mapsto L(x, \lambda, \mu)$  is linear, and therefore both convex and concave in  $(\lambda, \mu)$ . The pointwise minimum of concave functions is concave, therefore  $q$  is concave.
- Let  $\bar{x}$  be any feasible point, i.e.,  $h(\bar{x}) = 0$  and  $g(\bar{x}) \leq 0$ . Then we have, for any  $\lambda$  and  $\mu \geq 0$ :

$$\sum_{i=1}^m \lambda_i h_i(\bar{x}) + \sum_{i=1}^r \mu_i g_i(\bar{x}) \leq 0 ,$$

$$\implies L(\bar{x}, \lambda, \mu) = f(\bar{x}) + \sum_{i=1}^m \lambda_i h_i(\bar{x}) + \sum_{i=1}^r \mu_i g_i(\bar{x}) \leq f(\bar{x}) ,$$

$$\implies q(\lambda, \mu) = \inf_x L(x, \lambda, \mu) \leq L(\bar{x}, \lambda, \mu) \leq f(\bar{x}) , \quad \forall \bar{x} . \quad \square$$

## Weak duality

- Let  $d^*$  the optimal value of the Lagrange dual problem. Each  $q(\lambda, \mu)$  is an lower bound for  $f^*$  and by definition  $d^*$  is the best lower bound that is obtained. The following **weak duality inequality** therefore **always hold**:

$$d^* \leq f^*.$$

- This inequality holds when  $d^*$  or  $f^*$  are infinite. The difference  $d^* - f^*$  is called the **optimal duality gap** of the original problem.

## Strong duality

- We say that **strong duality** holds if the optimal duality gap is zero, i.e.:

$$d^* = f^* .$$

- If strong duality holds, then the best lower bound that can be obtained from the Lagrange dual function is **tight**
- Strong duality does **not hold** for general nonlinear problems.
- It usually holds for **convex problems**.
- Conditions that ensure strong duality for convex problems are called **constraint qualification**.
- in that case, we have for all feasible primal and dual points  $x, \lambda, \mu$ ,

$$q(\lambda, \mu) \leq q(\lambda^*, \mu^*) = L(x^*, \lambda^*, \mu^*) = f(x^*) \leq f(x).$$

## Slater's constraint qualification

Strong duality holds for a **convex** problem:

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && g_j(x) \leq 0, \quad j = 1, \dots, r, \\ & && Ax = b, \end{aligned}$$

if it is **strictly feasible**, i.e., there exists at least one **feasible point** that satisfies:

$$g_j(x) < 0, \quad j = 1, \dots, r, \quad Ax = b.$$

## Remarks

- Slater's conditions also ensure that the maximum  $d^*$  (if  $> -\infty$ ) is attained, i.e., there exists a point  $(\lambda^*, \mu^*)$  with

$$q(\lambda^*, \mu^*) = d^* = f^*$$

- They can be sharpened. For example, strict feasibility is not required for affine constraints.
- There exist many other types of constraint qualifications

## Dual optimal pairs

Suppose that strong duality holds,  $x^*$  is primal optimal,  $(\lambda^*, \mu^*)$  is dual optimal. Then we have:

$$\begin{aligned} f(x^*) &= q(\lambda^*, \mu^*) \\ &= \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i^* h_i(x) + \sum_{j=1}^r \mu_j^* g_j(x) \right\} \\ &\leq f(x^*) + \sum_{i=1}^m \lambda_i^* h_i(x^*) + \sum_{j=1}^r \mu_j^* g_j(x^*) \\ &\leq f(x^*) \end{aligned}$$

Hence both inequalities are in fact **equalities**.

## Complementary slackness

The first equality shows that:

$$L(x^*, \lambda^*, \mu^*) = \inf_{x \in \mathbb{R}^n} L(x, \lambda^*, \mu^*) ,$$

showing that  $x^*$  minimizes the Lagrangian at  $(\lambda^*, \mu^*)$ . The second equality shows the following important property:

### Complementary slackness

Each optimal Lagrange multiplier is zero unless the corresponding constraint is active at the optimum:

$$\mu_j g_j(x^*) = 0 , \quad j = 1, \dots, r .$$

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
  - Kernel ridge regression
  - Kernel logistic regression
  - Large-margin classifiers
  - Interlude: convex optimization and duality
  - Support vector machines
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics

## Support vector machines (SVM)

- Historically the first “kernel method” for pattern recognition, still the **most popular**.
- Often **state-of-the-art** in performance.
- One particular choice of loss function (**hinge loss**).
- Leads to a **sparse solution**, i.e., not all points are involved in the decomposition (**compression**).
- Particular algorithm for fast optimization (**decomposition by chunking methods**).

# Support vector machines (SVM)

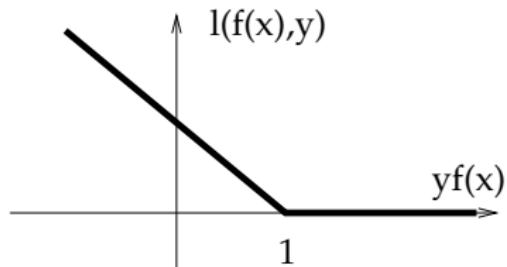
## Definition

- The **hinge loss** is the function  $\mathbb{R} \rightarrow \mathbb{R}_+$ :

$$\varphi_{\text{hinge}}(u) = \max(1 - u, 0) = \begin{cases} 0 & \text{if } u \geq 1, \\ 1 - u & \text{otherwise.} \end{cases}$$

- SVM is the corresponding large-margin classifier, which solves:

$$\min_{f \in \mathcal{H}_K} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi_{\text{hinge}}(y_i f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}_K}^2 \right\}.$$



## Problem reformulation (1/3)

- By the representer theorem, the solution satisfies

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}),$$

where  $\boldsymbol{\alpha}$  solves

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi_{\text{hinge}}(\mathbf{y}_i [\mathbf{K}\boldsymbol{\alpha}]_i) + \lambda \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha} \right\}$$

- This is a **convex** optimization problem
- But the objective function is not smooth (because of the hinge loss)

## Problem reformulation (2/3)

- Let us introduce additional **slack variables**  $\xi_1, \dots, \xi_n \in \mathbb{R}$ . The problem is equivalent to:

$$\min_{\alpha \in \mathbb{R}^n, \xi \in \mathbb{R}^n} \left\{ \frac{1}{n} \sum_{i=1}^n \xi_i + \lambda \alpha^\top \mathbf{K} \alpha \right\},$$

subject to:

$$\xi_i \geq \varphi_{\text{hinge}}(\mathbf{y}_i[\mathbf{K}\alpha]_i).$$

- The objective function is now smooth, but not the constraints
- However it is easy to replace the non-smooth constraint by a conjunction of two smooth constraints, because:

$$u \geq \varphi_{\text{hinge}}(v) \Leftrightarrow \begin{cases} u & \geq 1 - v \\ u & \geq 0 \end{cases}$$

## Problem reformulation (3/3)

In summary, the SVM solution is

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}),$$

where  $\boldsymbol{\alpha}$  solves:

### SVM (primal formulation)

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n, \boldsymbol{\xi} \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n \xi_i + \lambda \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha},$$

subject to:

$$\begin{cases} \mathbf{y}_i [\mathbf{K} \boldsymbol{\alpha}]_i + \xi_i - 1 \geq 0, & \text{for } i = 1, \dots, n, \\ \xi_i \geq 0, & \text{for } i = 1, \dots, n. \end{cases}$$

## Solving the SVM problem

- This is a classical **quadratic program** (minimization of a convex quadratic function with linear constraints) for which any out-of-the-box optimization package can be used.
- The **dimension** of the problem and the **number of constraints**, however, are  $2n$  where  $n$  is the number of points. General-purpose QP solvers will have difficulties when  $n$  exceeds a few thousands.
- Solving the **dual** of this problem (also a QP) will be more convenient and lead to faster algorithms (due to the sparsity of the final solution).

## Lagrangian

- Let us introduce the **Lagrange multipliers**  $\mu \in \mathbb{R}^n$  and  $\nu \in \mathbb{R}^n$ .
- The Lagrangian of the problem is:

$$\begin{aligned} L(\alpha, \xi, \mu, \nu) = & \frac{1}{n} \sum_{i=1}^n \xi_i + \lambda \alpha^\top \mathbf{K} \alpha \\ & - \sum_{i=1}^n \mu_i [\mathbf{y}_i [\mathbf{K} \alpha]_i + \xi_i - 1] - \sum_{i=1}^n \nu_i \xi_i \end{aligned}$$

or, in matrix notations:

$$\begin{aligned} L(\alpha, \xi, \mu, \nu) = & \xi^\top \frac{\mathbf{1}}{n} + \lambda \alpha^\top \mathbf{K} \alpha \\ & - (\text{diag}(\mathbf{y})\mu)^\top \mathbf{K} \alpha - (\mu + \nu)^\top \xi + \mu^\top \mathbf{1} \end{aligned}$$

## Minimizing $L(\alpha, \xi, \mu, \nu)$ w.r.t. $\alpha$

- $L(\alpha, \xi, \mu, \nu)$  is a convex quadratic function in  $\alpha$ . It is minimized whenever its gradient is null:

$$\nabla_{\alpha} L = 2\lambda \mathbf{K}\alpha - \mathbf{K} \operatorname{diag}(\mathbf{y})\mu = \mathbf{K} (2\lambda\alpha - \operatorname{diag}(\mathbf{y})\mu)$$

- The following solves  $\nabla_{\alpha} L = 0$ :

$$\alpha^* = \frac{\operatorname{diag}(\mathbf{y})\mu}{2\lambda}$$

## Minimizing $L(\alpha, \xi, \mu, \nu)$ w.r.t. $\xi$

- $L(\alpha, \xi, \mu, \nu)$  is a linear function in  $\xi$ .
- Its minimum is  $-\infty$  except when it is constant, i.e., when:

$$\nabla_{\xi} L = \frac{1}{n} - \mu - \nu = 0$$

or equivalently

$$\mu + \nu = \frac{1}{n}$$

## Dual function

- We therefore obtain the **Lagrange dual function**:

$$\begin{aligned} q(\mu, \nu) &= \inf_{\alpha \in \mathbb{R}^n, \xi \in \mathbb{R}^n} L(\alpha, \xi, \mu, \nu) \\ &= \begin{cases} \mu^\top \mathbf{1} - \frac{1}{4\lambda} \mu^\top \text{diag}(\mathbf{y}) \mathbf{K} \text{diag}(\mathbf{y}) \mu & \text{if } \mu + \nu = \frac{1}{n}, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

- The dual problem is:

$$\begin{aligned} &\text{maximize} && q(\mu, \nu) \\ &\text{subject to} && \mu \geq 0, \nu \geq 0. \end{aligned}$$

## Dual problem

- If  $\mu_i > 1/n$  for some  $i$ , then there is no  $\nu_i \geq 0$  such that  $\mu_i + \nu_i = 1/n$ , hence  $q(\boldsymbol{\mu}, \boldsymbol{\nu}) = -\infty$ .
- If  $0 \leq \mu_i \leq 1/n$  for all  $i$ , then the dual function takes finite values that depend only on  $\boldsymbol{\mu}$  by taking  $\nu_i = 1/n - \mu_i$ .
- The dual problem is therefore equivalent to:

$$\max_{0 \leq \boldsymbol{\mu} \leq \mathbf{1}/n} \quad \boldsymbol{\mu}^\top \mathbf{1} - \frac{1}{4\lambda} \boldsymbol{\mu}^\top \text{diag}(\mathbf{y}) \mathbf{K} \text{diag}(\mathbf{y}) \boldsymbol{\mu}$$

or with indices:

$$\max_{0 \leq \boldsymbol{\mu} \leq \mathbf{1}/n} \quad \sum_{i=1}^n \mu_i - \frac{1}{4\lambda} \sum_{i,j=1}^n \mathbf{y}_i \mathbf{y}_j \mu_i \mu_j K(\mathbf{x}_i, \mathbf{x}_j).$$

## Back to the primal

- Once the dual problem is solved in  $\mu$  we get a solution of the primal problem by  $\alpha = \text{diag}(\mathbf{y})\mu/2\lambda$ .
- Because the link is so simple, we can therefore directly plug this into the dual problem to obtain the QP that  $\alpha$  must solve:

### SVM (dual formulation)

$$\max_{\alpha \in \mathbb{R}^n} 2 \sum_{i=1}^n \alpha_i \mathbf{y}_i - \sum_{i,j=1}^n \alpha_i \alpha_j K(\mathbf{x}_i, \mathbf{x}_j) = 2\boldsymbol{\alpha}^\top \mathbf{y} - \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha},$$

subject to:

$$0 \leq \mathbf{y}_i \alpha_i \leq \frac{1}{2\lambda n}, \quad \text{for } i = 1, \dots, n.$$

## Complementary slackness conditions

- The complementary slackness conditions are, for  $i = 1, \dots, n$ :

$$\begin{cases} \mu_i [\mathbf{y}_i f(\mathbf{x}_i) + \xi_i - 1] = 0, \\ \nu_i \xi_i = 0, \end{cases}$$

- In terms of  $\alpha$  this can be rewritten as:

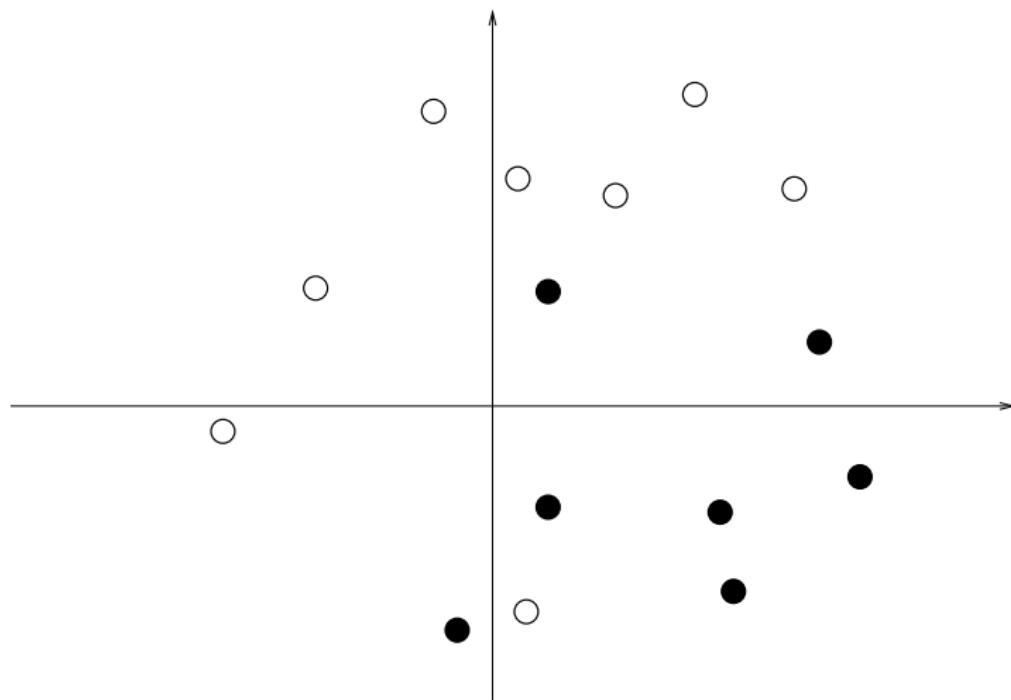
$$\begin{cases} \alpha_i [y_i f(\mathbf{x}_i) + \xi_i - 1] = 0, \\ (\alpha_i - \frac{y_i}{2\lambda n}) \xi_i = 0. \end{cases}$$

## Analysis of KKT conditions

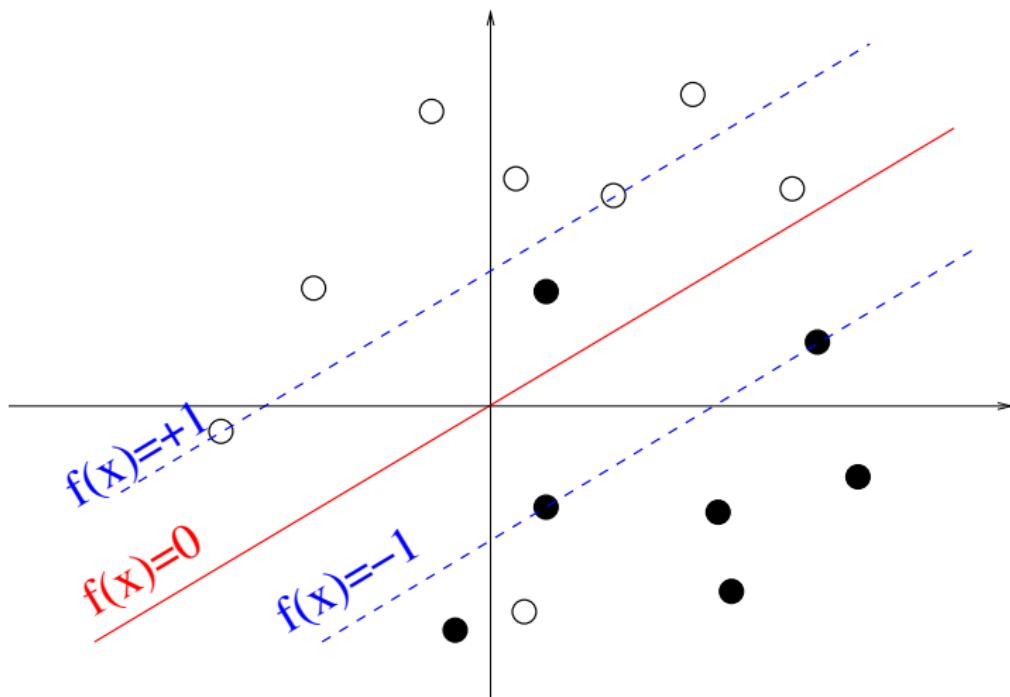
$$\begin{cases} \alpha_i [y_i f(\mathbf{x}_i) + \xi_i - 1] = 0, \\ (\alpha_i - \frac{y_i}{2\lambda n}) \xi_i = 0. \end{cases}$$

- If  $\alpha_i = 0$ , then the second constraint is active:  $\xi_i = 0$ . This implies  $y_i f(\mathbf{x}_i) \geq 1$ .
- If  $0 < y_i \alpha_i < \frac{1}{2\lambda n}$ , then both constraints are active:  $\xi_i = 0$  et  $y_i f(\mathbf{x}_i) + \xi_i - 1 = 0$ . This implies  $y_i f(\mathbf{x}_i) = 1$ .
- If  $\alpha_i = \frac{y_i}{2\lambda n}$ , then the second constraint is not active ( $\xi_i \geq 0$ ) while the first one is active:  $y_i f(\mathbf{x}_i) + \xi_i = 1$ . This implies  $y_i f(\mathbf{x}_i) \leq 1$

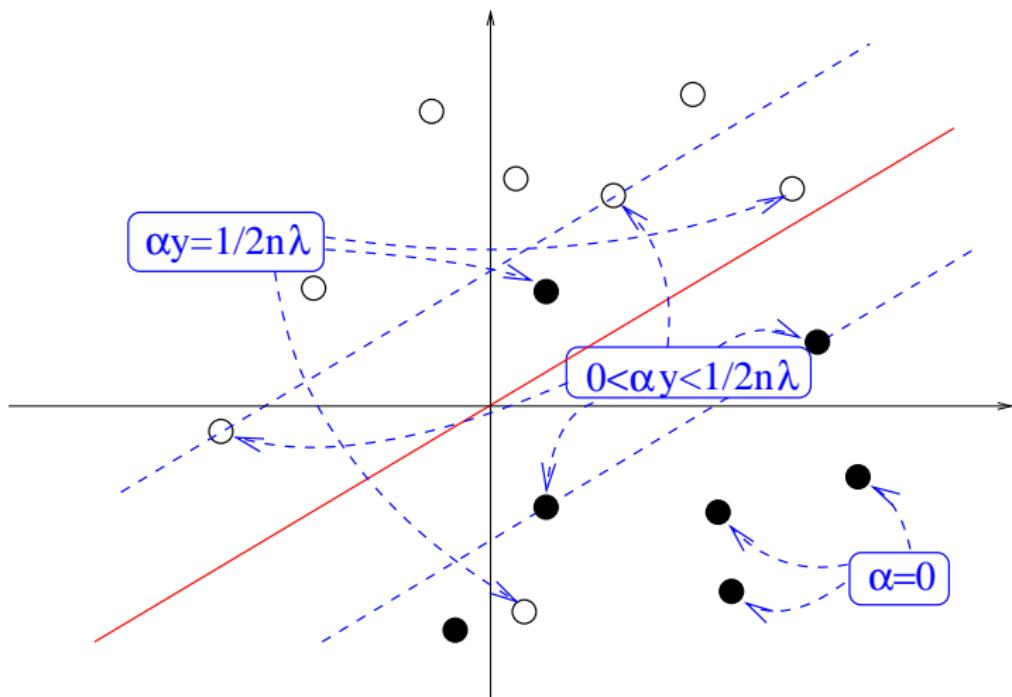
## Geometric interpretation



## Geometric interpretation



## Geometric interpretation



# Support vectors

## Consequence of KKT conditions

- The training points with  $\alpha_i \neq 0$  are called **support vectors**.
- Only support vectors are important for the classification of new points:

$$\forall \mathbf{x} \in \mathcal{X}, \quad f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}) = \sum_{i \in SV} \alpha_i K(\mathbf{x}_i, \mathbf{x}),$$

where  $SV$  is the set of support vectors.

## Consequences

- The solution is **sparse** in  $\alpha$ , leading to **fast algorithms** for training (use of decomposition methods).
- The **classification** of a new point only involves kernel evaluations with support vectors (fast).

## Remark: C-SVM

- Often the SVM optimization problem is written in terms of a regularization parameter  $C$  instead of  $\lambda$  as follows:

$$\arg \min_{f \in \mathcal{H}} \frac{1}{2} \|f\|_{\mathcal{H}}^2 + C \sum_{i=1}^n L_{\text{hinge}}(f(\mathbf{x}_i), y_i).$$

- This is equivalent to our formulation with  $C = \frac{1}{2n\lambda}$ .
- The SVM optimization problem is then:

$$\max_{\alpha \in \mathbb{R}^d} 2 \sum_{i=1}^n \alpha_i y_i - \sum_{i,j=1}^n \alpha_i \alpha_j K(\mathbf{x}_i, \mathbf{x}_j),$$

subject to:

$$0 \leq y_i \alpha_i \leq C, \quad \text{for } i = 1, \dots, n.$$

- This formulation is often called **C-SVM**.

## Remark: 2-SVM

- A variant of the SVM, sometimes called 2-SVM, is obtained by replacing the hinge loss by the square hinge loss:

$$\min_{f \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^n \varphi_{\text{hinge}}(\mathbf{y}_i f(\mathbf{x}_i))^2 + \lambda \|f\|_{\mathcal{H}}^2 \right\}.$$

- After some computation (left as exercice) we find that the dual problem of the 2-SVM is:

$$\max_{\alpha \in \mathbb{R}^d} 2\alpha^\top \mathbf{y} - \alpha^\top (\mathbf{K} + n\lambda I) \alpha,$$

subject to:

$$0 \leq y_i \alpha_i, \quad \text{for } i = 1, \dots, n.$$

- This is therefore **equivalent** to the previous SVM with the kernel  $\mathbf{K} + n\lambda I$  and  $C = +\infty$

# Kernel Methods

## Unsupervised Learning

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
  - Kernel K-means and spectral clustering
  - Kernel PCA
  - A quick note on kernel CCA
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics

# The K-means algorithm

K-means is probably the most popular algorithm for **clustering**.

## Optimization point of view

Given data points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathbb{R}^p$ , it consists of performing alternate minimization steps for optimizing the following cost function

$$\min_{\substack{\mu_j \in \mathbb{R}^p \text{ for } j=1, \dots, k \\ s_i \in \{1, \dots, k\}, \text{ for } i=1, \dots, n}} \sum_{i=1}^n \|\mathbf{x}_i - \mu_{s_i}\|_2^2.$$

K-means alternates between two steps:

### 1 cluster assignment:

Given fixed  $\mu_1, \dots, \mu_k$ , assign each  $\mathbf{x}_i$  to its closest centroid

$$\forall i, \quad s_i \in \operatorname{argmin}_{s \in \{1, \dots, k\}} \|\mathbf{x}_i - \mu_s\|_2^2.$$

# The K-means algorithm

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K-means alternates between two steps:

### 2 centroids update:

Given the previous assignments  $s_1, \dots, s_n$ , update the centroids

$$\forall j, \quad \boldsymbol{\mu}_j = \operatorname{argmin}_{\boldsymbol{\mu} \in \mathbb{R}^p} \sum_{i:s_i=j} \|\mathbf{x}_i - \boldsymbol{\mu}\|_2^2.$$

# The K-means algorithm

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K-means alternates between two steps:

### 2 centroids update:

Given the previous assignments  $s_1, \dots, s_n$ , update the centroids

$$\Leftrightarrow \forall j, \quad \boldsymbol{\mu}_j = \frac{1}{n_j} \sum_{i:s_i=j} \mathbf{x}_i.$$

## Kernel K-means and spectral clustering

We may now modify the objective to **operate in a RKHS**. Given data points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathcal{X}$  and a p.d. kernel  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  with  $\mathcal{H}$  its RKHS, the new objective becomes

$$\min_{\substack{\mu_j \in \mathcal{H} \\ s_i \in \{1, \dots, k\}}} \sum_{j=1, \dots, k} \sum_{i=1}^n \|\varphi(\mathbf{x}_i) - \mu_{s_i}\|_{\mathcal{H}}^2.$$

To optimize the cost function, we will first use the following Proposition

### Proposition

The center of mass  $\varphi_n = \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_i)$  solves the following optimization problem

$$\min_{\mu \in \mathcal{H}} \sum_{i=1}^n \|\varphi(\mathbf{x}_i) - \mu\|_{\mathcal{H}}^2.$$

# Kernel K-means and spectral clustering

## Proof

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n \|\varphi(\mathbf{x}_i) - \boldsymbol{\mu}\|_{\mathcal{H}}^2 &= \frac{1}{n} \sum_{i=1}^n \|\varphi(\mathbf{x}_i)\|_{\mathcal{H}}^2 - \left\langle \frac{2}{n} \sum_{i=1}^n \varphi(\mathbf{x}_i), \boldsymbol{\mu} \right\rangle_{\mathcal{H}} + \|\boldsymbol{\mu}\|_{\mathcal{H}}^2 \\ &= \frac{1}{n} \sum_{i=1}^n \|\varphi(\mathbf{x}_i)\|_{\mathcal{H}}^2 - 2 \langle \varphi_n, \boldsymbol{\mu} \rangle_{\mathcal{H}} + \|\boldsymbol{\mu}\|_{\mathcal{H}}^2 \\ &= \frac{1}{n} \sum_{i=1}^n \|\varphi(\mathbf{x}_i)\|_{\mathcal{H}}^2 - \|\varphi_n\|_{\mathcal{H}}^2 + \|\varphi_n - \boldsymbol{\mu}\|_{\mathcal{H}}^2,\end{aligned}$$

which is minimum for  $\boldsymbol{\mu} = \varphi_n$ .

## Kernel K-means and spectral clustering

Back with the objective,

$$\min_{\substack{\mu_j \in \mathcal{H} \text{ for } j=1, \dots, k \\ s_i \in \{1, \dots, k\} \text{ for } i=1, \dots, n}} \sum_{i=1}^n \|\varphi(\mathbf{x}_i) - \mu_{s_i}\|_{\mathcal{H}}^2,$$

we know that given assignments  $s_i$ , the optimal  $\mu_j$  are the centers of mass of the respective clusters and we obtain the equivalent objective:

$$\min_{\substack{s_i \in \{1, \dots, k\} \text{ for } i=1, \dots, n}} \sum_{i=1}^n \left\| \varphi(\mathbf{x}_i) - \frac{1}{|C_{s_i}|} \sum_{j \in C_{s_i}} \varphi(\mathbf{x}_j) \right\|_{\mathcal{H}}^2,$$

or, after short calculations,

$$\min_{\substack{s_i \in \{1, \dots, k\} \text{ for } i=1, \dots, n}} \sum_{i=1}^n K(\mathbf{x}_i, \mathbf{x}_i) - \frac{2}{|C_{s_i}|} \sum_{j \in C_{s_i}} K(\mathbf{x}_i, \mathbf{x}_j) + \frac{1}{|C_{s_i}|^2} \sum_{j \in C_{s_i}} \sum_{l \in C_{s_i}} K(\mathbf{x}_j, \mathbf{x}_l).$$

## Kernel K-means and spectral clustering

and, after removing the constant terms, we obtain the objective

$$\min_{\substack{s_i \in \{1, \dots, k\} \\ \text{for } i=1, \dots, n}} \sum_{i=1}^n -\frac{1}{|C_{s_i}|} \sum_{j \in C_{s_i}} K(\mathbf{x}_i, \mathbf{x}_j), \quad (*)$$

The objective can be expressed with **pairwise kernel comparisons**.

Unfortunately, the problem is hard and we need an appropriate strategy to obtain an **approximate solution**.

### Greedy approach: kernel K-means

At every iteration,

- Update the sets  $C_l$ ,  $l = 1, \dots, k$  given current assignments  $s_i$ 's.
- Update the assignments by minimizing  $(*)$  keeping the sets  $C_l$  fixed.

The algorithm is similar to the traditional K-means algorithm.

## Kernel K-means and spectral clustering

Another approach consists of **relaxing the non-convex problem with a feasible one**, which yields a class of algorithms called **spectral clustering**. First, we rewrite the objective function as

$$\min_{\substack{s_i \in \{1, \dots, k\} \\ \text{for } i=1, \dots, n}} \sum_{l=1}^k \sum_{i,j \in C_l} -\frac{1}{|C_l|} K(\mathbf{x}_i, \mathbf{x}_j).$$

and we introduce

- the binary matrix  $\mathbf{A}$  in  $\{0, 1\}^{n \times k}$  such that  $[\mathbf{A}]_{ij} = 1$  if  $s_i = j$  and 0 otherwise.
- a diagonal matrix  $\mathbf{D}$  in  $\mathbb{R}^{l \times l}$  with diagonal entries  $[\mathbf{D}]_{jj}$  equal to the inverse of the number of elements in cluster  $j$ .

and the objective can be rewritten (proof is easy and left as an exercise)

$$\min_{\mathbf{A}, \mathbf{D}} \left[ -\text{trace}(\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{K} \mathbf{A} \mathbf{D}^{1/2}) \right].$$

## Kernel K-means and spectral clustering

$$\min_{\mathbf{A}, \mathbf{D}} \left[ -\text{trace}(\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{K} \mathbf{A} \mathbf{D}^{1/2}) \right].$$

The constraints on  $\mathbf{A}, \mathbf{D}$  are such that  $\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{A} \mathbf{D}^{1/2} = \mathbf{I}$  (exercise). A natural relaxation consists of dropping the constraints on  $\mathbf{A}$  and instead optimize over  $\mathbf{Z} = \mathbf{A} \mathbf{D}^{1/2}$ :

$$\max_{\mathbf{Z} \in \mathbb{R}^{n \times k}} \text{trace}(\mathbf{Z}^\top \mathbf{K} \mathbf{Z}) \text{ s.t. } \mathbf{Z}^\top \mathbf{Z} = \mathbf{I}.$$

A solution  $\mathbf{Z}^*$  to this problem may be obtained by computing the eigenvectors of  $\mathbf{K}$  associated to the  $k$ -largest eigenvalues. As we will see in a few slides, this procedure is related to the **kernel PCA** algorithm.

## Kernel K-means and spectral clustering

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### Question

How do we obtain an approximate solution  $(\mathbf{A}, \mathbf{D})$  of the original problem from  $\mathbf{Z}^*$ ?

## Kernel K-means and spectral clustering

$$\min_{\mathbf{A}, \mathbf{D}} \left[ -\text{trace}(\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{K} \mathbf{A} \mathbf{D}^{1/2}) \right].$$

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### Answer 1

With the original constraints on  $\mathbf{A}$ , every row of  $\mathbf{A}$  has a single non-zero entry  $\Rightarrow$  compute the maximum entry of every row of  $\mathbf{Z}^*$ .

## Kernel K-means and spectral clustering

$$\min_{\mathbf{A}, \mathbf{D}} \left[ -\text{trace}(\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{K} \mathbf{A} \mathbf{D}^{1/2}) \right].$$

The constraints on  $\mathbf{A}, \mathbf{D}$  are such that  $\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{A} \mathbf{D}^{1/2} = \mathbf{I}$  (exercise). A natural relaxation consists of dropping the constraints on  $\mathbf{A}$  and instead optimize over  $\mathbf{Z} = \mathbf{A} \mathbf{D}^{1/2}$ :

$$\max_{\mathbf{Z} \in \mathbb{R}^{n \times k}} \text{trace}(\mathbf{Z}^\top \mathbf{K} \mathbf{Z}) \text{ s.t. } \mathbf{Z}^\top \mathbf{Z} = \mathbf{I}.$$

A solution  $\mathbf{Z}^*$  to this problem may be obtained by computing the eigenvectors of  $\mathbf{K}$  associated to the  $k$ -largest eigenvalues. As we will see in a few slides, this procedure is related to the **kernel PCA** algorithm.

### Answer 2

Normalize the rows of  $\mathbf{Z}^*$  to have unit  $\ell_2$ -norm, and apply the traditional K-means algorithm on the rows. This is called **spectral clustering**.

## Kernel K-means and spectral clustering

$$\min_{\mathbf{A}, \mathbf{D}} \left[ -\text{trace}(\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{K} \mathbf{A} \mathbf{D}^{1/2}) \right].$$

The constraints on  $\mathbf{A}, \mathbf{D}$  are such that  $\mathbf{D}^{1/2} \mathbf{A}^\top \mathbf{A} \mathbf{D}^{1/2} = \mathbf{I}$  (exercise). A natural relaxation consists of dropping the constraints on  $\mathbf{A}$  and instead optimize over  $\mathbf{Z} = \mathbf{A} \mathbf{D}^{1/2}$ :

$$\max_{\mathbf{Z} \in \mathbb{R}^{n \times k}} \text{trace}(\mathbf{Z}^\top \mathbf{K} \mathbf{Z}) \text{ s.t. } \mathbf{Z}^\top \mathbf{Z} = \mathbf{I}.$$

A solution  $\mathbf{Z}^*$  to this problem may be obtained by computing the eigenvectors of  $\mathbf{K}$  associated to the  $k$ -largest eigenvalues. As we will see in a few slides, this procedure is related to the **kernel PCA** algorithm.

### Answer 3

Choose another variant of the previous procedures.

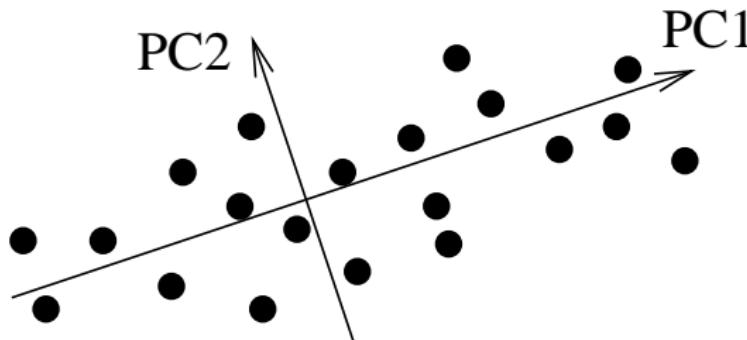
# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
  - Kernel K-means and spectral clustering
  - Kernel PCA
  - A quick note on kernel CCA
- 5 The Kernel Jungle
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# Principal Component Analysis (PCA)

## Classical setting

- Let  $\mathcal{S} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  be a set of vectors ( $\mathbf{x}_i \in \mathbb{R}^d$ )
- PCA is a classical algorithm in multivariate statistics to define a set of orthogonal directions that capture the maximum variance
- Applications: low-dimensional representation of high-dimensional points, visualization



# Principal Component Analysis (PCA)

## Formalization

- Assume that the data are **centered** (otherwise center them as preprocessing), i.e.:

$$\sum_{i=1}^n \mathbf{x}_i = 0.$$

- The **orthogonal projection** onto a direction  $\mathbf{w} \in \mathbb{R}^d$  is the function  $h_{\mathbf{w}} : \mathcal{X} \rightarrow \mathbb{R}$  defined by:

$$h_{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^\top \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

# Principal Component Analysis (PCA)

## Formalization

- The empirical variance captured by  $h_{\mathbf{w}}$  is:

$$\hat{\text{var}}(h_{\mathbf{w}}) := \frac{1}{n} \sum_{i=1}^n h_{\mathbf{w}}(\mathbf{x}_i)^2 = \frac{1}{n} \sum_{i=1}^n \frac{(\mathbf{x}_i^\top \mathbf{w})^2}{\|\mathbf{w}\|^2}.$$

- The  $i$ -th principal direction  $\mathbf{w}_i$  ( $i = 1, \dots, d$ ) is defined by:

$$\mathbf{w}_i = \underset{\mathbf{w} \perp \{\mathbf{w}_1, \dots, \mathbf{w}_{i-1}\}}{\arg \max} \hat{\text{var}}(h_{\mathbf{w}}).$$

# Principal Component Analysis (PCA)

## Solution

- Let  $\mathbf{X}$  be the  $n \times d$  data matrix whose rows are the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . We can then write:

$$\hat{\text{var}}(h_{\mathbf{w}}) = \frac{1}{n} \sum_{i=1}^n \frac{(\mathbf{x}_i^\top \mathbf{w})^2}{\|\mathbf{w}\|^2} = \frac{1}{n} \frac{\mathbf{w}^\top \mathbf{X}^\top \mathbf{X} \mathbf{w}}{\mathbf{w}^\top \mathbf{w}}.$$

- The solutions of:

$$\mathbf{w}_i = \arg \max_{\mathbf{w} \perp \{\mathbf{w}_1, \dots, \mathbf{w}_{i-1}\}} \frac{1}{n} \frac{\mathbf{w}^\top \mathbf{X}^\top \mathbf{X} \mathbf{w}}{\mathbf{w}^\top \mathbf{w}}$$

are the **successive eigenvectors of  $\mathbf{K} = \mathbf{X}^\top \mathbf{X}$** , ranked by decreasing eigenvalues.

## Functional point of view

- Let  $K(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y}$  be the linear kernel.
- The associated RKHS  $\mathcal{H}$  is the set of linear functions:

$$f_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^\top \mathbf{x},$$

endowed with the norm  $\| f_{\mathbf{w}} \|_{\mathcal{H}} = \| \mathbf{w} \|_{\mathbb{R}^d}$ .

- Therefore we can write:

$$\hat{var}(h_{\mathbf{w}}) = \frac{1}{n} \sum_{i=1}^n \frac{(\mathbf{x}_i^\top \mathbf{w})^2}{\| \mathbf{w} \|^2} = \frac{1}{n \| f_{\mathbf{w}} \|^2} \sum_{i=1}^n f_{\mathbf{w}}(\mathbf{x}_i)^2.$$

- Moreover,  $\mathbf{w} \perp \mathbf{w}' \Leftrightarrow f_{\mathbf{w}} \perp f_{\mathbf{w}'}$ .

## Functional point of view

- In other words, PCA solves, for  $i = 1, \dots, d$ :

$$f_i = \arg \max_{f \perp \{f_1, \dots, f_{i-1}\}} \frac{1}{n \|f\|^2} \sum_{i=1}^n f(\mathbf{x}_i)^2.$$

- We can apply the representer theorem (*exercise: check that this is also valid in a linear subspace*): for  $i = 1, \dots, d$ , we have:

$$\forall \mathbf{x} \in \mathcal{X}, \quad f_i(\mathbf{x}) = \sum_{j=1}^n \alpha_{i,j} K(\mathbf{x}_j, \mathbf{x}),$$

with  $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,n})^\top \in \mathbb{R}^n$ .

## Functional point of view

- Therefore we have:

$$\|f_i\|_{\mathcal{H}}^2 = \sum_{k,l=1}^d \alpha_{i,k} \alpha_{i,l} K(\mathbf{x}_k, \mathbf{x}_l) = \boldsymbol{\alpha}_i^\top \mathbf{K} \boldsymbol{\alpha}_i,$$

- Similarly:

$$\sum_{k=1}^n f_i(\mathbf{x}_k)^2 = \boldsymbol{\alpha}_i^\top \mathbf{K}^2 \boldsymbol{\alpha}_i.$$

## Functional point of view

PCA maximizes in  $\alpha$  the function:

$$\alpha_i = \arg \max_{\alpha} \frac{\alpha^\top K^2 \alpha}{n \alpha^\top K \alpha},$$

under the constraints:

$$\alpha_i^\top K \alpha_j = 0 \quad \text{for } j = 1, \dots, i-1.$$

## Solution

- Let  $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$  be an orthonormal basis of eigenvectors of  $\mathbf{K}$  with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ .
- Let  $\alpha_i = \sum_{j=1}^n \beta_{ij} \mathbf{u}_j$ , then

$$\frac{\alpha_i^\top \mathbf{K}^2 \alpha_i}{n \alpha_i^\top \mathbf{K} \alpha_i} = \frac{\sum_{j=1}^n \beta_{ij}^2 \lambda_j^2}{n \sum_{j=1}^n \beta_{ij}^2 \lambda_j},$$

which is maximized at  $\alpha_1 = \beta_{11} \mathbf{u}_1$ ,  $\alpha_2 = \beta_{22} \mathbf{u}_2$ , etc...

## Normalization

- For  $\alpha_i = \beta_{ii}\mathbf{u}_i$ , we want:

$$1 = \|f_i\|_{\mathcal{H}}^2 = \alpha_i^\top \mathbf{K} \alpha_i = \beta_{ii}^2 \lambda_i.$$

- Therefore:

$$\alpha_i = \frac{1}{\sqrt{\lambda_i}} \mathbf{u}_i.$$

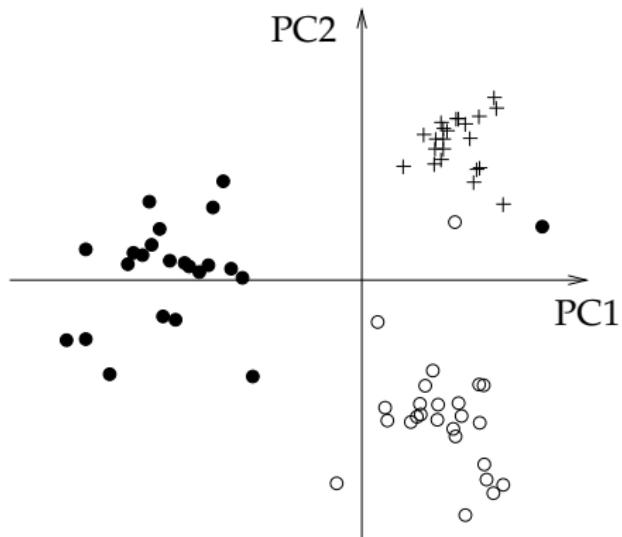
## Kernel PCA: summary

- ① Center the Gram matrix
- ② Compute the first eigenvectors  $(\mathbf{u}_i, \lambda_i)$
- ③ Normalize the eigenvectors  $\alpha_i = \mathbf{u}_i / \sqrt{\lambda_i}$
- ④ The projections of the points onto the  $i$ -th eigenvector is given by  $\mathbf{K}\alpha_i$

## Kernel PCA: remarks

- In this formulation, we must **diagonalize the centered kernel Gram matrix**, instead of the covariance matrix in the classical setting
- *Exercise: check that  $\mathbf{X}^\top \mathbf{X}$  and  $\mathbf{X}\mathbf{X}^\top$  have the same spectrum (up to 0 eigenvalues) and that the eigenvectors are related by a simple relationship.*
- This formulation remains valid for any p.d. kernel: this is **kernel PCA**
- **Applications:** nonlinear PCA with nonlinear kernels for vectors, PCA of non-vector objects (strings, graphs..) with specific kernels...

## Example



A set of 74 human tRNA sequences is analyzed using a kernel for sequences (the second-order marginalized kernel based on SCFG). This set of tRNAs contains three classes, called Ala-AGC (*white circles*), Asn-GTT (*black circles*) and Cys-GCA (*plus symbols*) (from Tsuda et al., 2003).

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# Canonical Correlation Analysis (CCA)

Given two views  $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_n]$  in  $\mathbb{R}^{p \times n}$  and  $\mathbf{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_n]$  in  $\mathbb{R}^{d \times n}$  of the same dataset, the goal of canonical correlation analysis (CCA) is to find **pairs of directions** in the two views that are **maximally correlated**.

## Formulation

Assuming that the datasets are centered, we want to maximize

$$\max_{\mathbf{w}_a \in \mathbb{R}^p, \mathbf{w}_b \in \mathbb{R}^d} \frac{\frac{1}{n} \sum_{i=1}^n \mathbf{w}_a^\top \mathbf{x}_i \mathbf{y}_i^\top \mathbf{w}_b}{\left( \frac{1}{n} \sum_{i=1}^n \mathbf{w}_a^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{w}_a \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^n \mathbf{w}_b^\top \mathbf{y}_i \mathbf{y}_i^\top \mathbf{w}_b \right)^{1/2}}.$$

Assuming that the pairs  $(\mathbf{x}_i, \mathbf{y}_i)$  are i.i.d. samples from an unknown distribution, CCA seeks to maximize

$$\max_{\mathbf{w}_a \in \mathbb{R}^p, \mathbf{w}_b \in \mathbb{R}^d} \frac{\text{cov}(\mathbf{w}_a^\top \mathbf{X}, \mathbf{w}_b^\top \mathbf{Y})}{\sqrt{\text{var}(\mathbf{w}_a^\top \mathbf{X})} \sqrt{\text{var}(\mathbf{w}_b^\top \mathbf{Y})}}.$$

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It is possible to show that this is an **generalized eigenvalue problem** (see next slide or see Section 6.5 of Shawe-Taylor and Cristianini 2004b).

The above problem provides the **first pair of canonical directions**. Next directions can be obtained by solving the same problem under the constraint that they are **orthogonal to the previous canonical directions**.

# Canonical Correlation Analysis (CCA)

## Formulation

Assuming that the datasets are centered,

$$\max_{\mathbf{w}_a \in \mathbb{R}^p, \mathbf{w}_b \in \mathbb{R}^d} \frac{\mathbf{w}_a^\top \mathbf{X}^\top \mathbf{Y} \mathbf{w}_b}{(\mathbf{w}_a^\top \mathbf{X}^\top \mathbf{X} \mathbf{w}_a)^{1/2} (\mathbf{w}_b^\top \mathbf{Y}^\top \mathbf{Y} \mathbf{w}_b)^{1/2}}.$$

can be formulated, after removing the scaling ambiguity, as

$$\max_{\mathbf{w}_a \in \mathbb{R}^p, \mathbf{w}_b \in \mathbb{R}^d} \mathbf{w}_a^\top \mathbf{X}^\top \mathbf{Y} \mathbf{w}_b \text{ s.t. } \mathbf{w}_a^\top \mathbf{X}^\top \mathbf{X} \mathbf{w}_a = 1 \text{ and } \mathbf{w}_b^\top \mathbf{Y}^\top \mathbf{Y} \mathbf{w}_b = 1.$$

Then, there exists  $\lambda_a$  and  $\lambda_b$  such that the problem is equivalent to

$$\min_{\mathbf{w}_a \in \mathbb{R}^p, \mathbf{w}_b \in \mathbb{R}^d} -\mathbf{w}_a^\top \mathbf{X}^\top \mathbf{Y} \mathbf{w}_b + \frac{\lambda_a}{2} (\mathbf{w}_a^\top \mathbf{X}^\top \mathbf{X} \mathbf{w}_a - 1) + \frac{\lambda_b}{2} (\mathbf{w}_b^\top \mathbf{Y}^\top \mathbf{Y} \mathbf{w}_b - 1).$$

## Canonical Correlation Analysis (CCA)

Taking the derivatives and setting the gradient to zero, we obtain

$$-\mathbf{X}^T \mathbf{Y} \mathbf{w}_b + \lambda_a \mathbf{X}^T \mathbf{X} \mathbf{w}_a = 0$$

$$-\mathbf{Y}^T \mathbf{X} \mathbf{w}_a + \lambda_b \mathbf{Y}^T \mathbf{Y} \mathbf{w}_b = 0$$

Multiply first equality by  $\mathbf{w}_a^T$  and second equality by  $\mathbf{w}_b^T$ ; subtract the two resulting equalities and we get

$$\lambda_a \mathbf{w}_a^T \mathbf{X}^T \mathbf{X} \mathbf{w}_a = \lambda_b \mathbf{w}_b^T \mathbf{Y}^T \mathbf{Y} \mathbf{w}_b = \lambda_a = \lambda_b = \lambda,$$

and then, we obtain the **generalized eigenvalue problem**:

$$\begin{bmatrix} 0 & \mathbf{X}^T \mathbf{Y} \\ \mathbf{Y}^T \mathbf{X} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_a \\ \mathbf{w}_b \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{X}^T \mathbf{X} & 0 \\ 0 & \mathbf{Y}^T \mathbf{Y} \end{bmatrix} \begin{bmatrix} \mathbf{w}_a \\ \mathbf{w}_b \end{bmatrix}$$

## Canonical Correlation Analysis (CCA)

Let us define

$$\boldsymbol{\Sigma}_A = \begin{bmatrix} 0 & \mathbf{X}^\top \mathbf{Y} \\ \mathbf{Y}^\top \mathbf{X} & 0 \end{bmatrix}, \quad \boldsymbol{\Sigma}_B = \begin{bmatrix} \mathbf{X}^\top \mathbf{X} & 0 \\ 0 & \mathbf{Y}^\top \mathbf{Y} \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_a \\ \mathbf{w}_b \end{bmatrix}$$

Assuming the covariances are invertible, the generalized eigenvalue problem is equivalent to

$$\boldsymbol{\Sigma}_B^{-1/2} \boldsymbol{\Sigma}_A \mathbf{w} = \lambda \boldsymbol{\Sigma}_B^{1/2} \mathbf{w}$$

which is also equivalent to the eigenvalue problem

$$\boldsymbol{\Sigma}_B^{-1/2} \boldsymbol{\Sigma}_A \boldsymbol{\Sigma}_B^{-1/2} (\boldsymbol{\Sigma}_B^{-1/2} \mathbf{w}) = \lambda (\boldsymbol{\Sigma}_B^{-1/2} \mathbf{w}).$$

## Kernel Canonical Correlation Analysis

Similar to kernel PCA, it is possible to operate in a RKHS. Given two p.d. kernels  $K_a, K_b : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , we can obtain two “views” of a dataset  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathcal{X}^n$ :

$$(\varphi_a(\mathbf{x}_1), \dots, \varphi_a(\mathbf{x}_n)) \quad \text{and} \quad (\varphi_b(\mathbf{x}_1), \dots, \varphi_b(\mathbf{x}_n)),$$

where  $\varphi_a : \mathcal{X} \rightarrow \mathcal{H}_a$  and  $\varphi_b : \mathcal{X} \rightarrow \mathcal{H}_b$  are the embeddings in the RKHSs  $\mathcal{H}_a$  of  $K_a$  and  $\mathcal{H}_b$  of  $K_b$ , respectively. Then, we may formulate **kernel CCA** as the following optimization problem

$$\max_{f_a \in \mathcal{H}_a, f_b \in \mathcal{H}_b} \frac{\frac{1}{n} \sum_{i=1}^n \langle f_a, \varphi_a(\mathbf{x}_i) \rangle_{\mathcal{H}_a} \langle \varphi_b(\mathbf{x}_i), f_b \rangle_{\mathcal{H}_b}}{\left( \frac{1}{n} \sum_{i=1}^n \langle f_a, \varphi_a(\mathbf{x}_i) \rangle_{\mathcal{H}_a}^2 \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^n \langle f_b, \varphi_b(\mathbf{x}_i) \rangle_{\mathcal{H}_b}^2 \right)^{1/2}}.$$

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$$\max_{f_a \in \mathcal{H}_a, f_b \in \mathcal{H}_b} \frac{\frac{1}{n} \sum_{i=1}^n f_a(\mathbf{x}_i) f_b(\mathbf{x}_i)}{\left( \frac{1}{n} \sum_{i=1}^n f_a(\mathbf{x}_i)^2 \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^n f_b(\mathbf{x}_i)^2 \right)^{1/2}}.$$

# Kernel Canonical Correlation Analysis

Up to a few technical details (exercise), we can apply the **representer theorem** and look for solutions  $f_a(.) = \sum_{i=1}^n \alpha_i K_a(\mathbf{x}_i, .)$  and  $f_b(.) = \sum_{i=1}^n \beta_i K_b(\mathbf{x}_i, .)$ . We finally obtain the formulation

$$\max_{\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^n} \frac{\frac{1}{n} \sum_{i=1}^n [\mathbf{K}_a \alpha]_i [\mathbf{K}_b \beta]_i}{\left( \frac{1}{n} \sum_{i=1}^n [\mathbf{K}_a \alpha]_i^2 \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^n [\mathbf{K}_b \beta]_i^2 \right)^{1/2}},$$

which is equivalent to

$$\max_{\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^n} \frac{\alpha^\top \mathbf{K}_a \mathbf{K}_b \beta}{(\alpha^\top \mathbf{K}_a^2 \alpha)^{1/2} (\beta^\top \mathbf{K}_b^2 \beta)^{1/2}},$$

or, after removing the scaling ambiguity for  $\alpha$  and  $\beta$ ,

$$\max_{\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^n} \color{red}{\alpha^\top \mathbf{K}_a \mathbf{K}_b \beta} \text{ s.t. } \color{red}{\alpha^\top \mathbf{K}_a^2 \alpha = 1} \text{ and } \color{red}{\beta^\top \mathbf{K}_b^2 \beta = 1}.$$

# Kernel Canonical Correlation Analysis

## Remarks

- kernel CCA also yields a generalized eigenvalue problem.
- the subsequent canonical directions are obtained by solving the same problem with additional orthogonality constraints.
- in practice, kernel CCA is numerically unstable; it requires regularization to replace the constraints  $\alpha^\top \mathbf{K}_a^2 \alpha$  by  $\alpha^\top (\mathbf{K}_a^2 + \mu_a \mathbf{I}) \alpha = 1$  (same for  $\mathbf{K}_b$ ), which improves the condition number of the matrix  $\mathbf{K}_a^2$ .

# The Kernel Jungle

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
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- 5 The Kernel Jungle
  - Kernels for probabilistic models
  - Kernels for biological sequences
  - Mercer kernels and shift-invariant kernels
  - Kernels for graphs
  - Kernels on graphs
- 6 Open Problems and Research Topics

## Motivation

Kernel methods are sometimes criticized for their **lack of flexibility**: a large effort is spent in designing by hand the kernel.

## Question

How do we design a kernel **adapted** to the data?

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Kernel methods are sometimes criticized for their **lack of flexibility**: a large effort is spent in designing by hand the kernel.

## Question

How do we design a kernel **adapted** to the data?

## Answer

A successful strategy is given by kernels for generative models, which are/have been the state of the art in many fields, including image and sequence representations.

### Parametric model

A **model** is a family of distributions

$$\{P_\theta, \theta \in \Theta \subset \mathbb{R}^m\} \subseteq \mathcal{M}_1^+(\mathcal{X}).$$

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
  - Fisher kernel
  - Mutual information kernels
  - Marginalized kernels
- Kernels for biological sequences
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- Kernels on graphs

# Fisher kernel

## Definition

- Fix a parameter  $\theta_0 \in \Theta$  (e.g., by maximum likelihood over a training set of sequences)
- For each sequence  $\mathbf{x}$ , compute the Fisher score vector:

$$\Phi_{\theta_0}(\mathbf{x}) = \nabla_{\theta} \log P_{\theta}(\mathbf{x})|_{\theta=\theta_0} .$$

- Form the kernel (Jaakkola et al., 2000):

$$K(\mathbf{x}, \mathbf{x}') = \Phi_{\theta_0}(\mathbf{x})^{\top} \mathbf{I}(\theta_0)^{-1} \Phi_{\theta_0}(\mathbf{x}') ,$$

where  $\mathbf{I}(\theta_0) = E[\Phi_{\theta_0}(\mathbf{x}) \Phi_{\theta_0}(\mathbf{x})^{\top}]$  is the Fisher information matrix.

## Fisher kernel properties (1/2)

- The Fisher score describes how **each parameter contributes** to the process of generating a particular example
- A kernel classifier employing the Fisher kernel derived from a model that contains the label as a latent variable is, asymptotically, **at least as good a classifier as the MAP labelling** based on the model (Jaakkola and Haussler, 1999).
- A variant of the Fisher kernel (called the Tangent of Posterior kernel) can also improve over the direct posterior classification by helping to **correct the effect of estimation errors** in the parameter (Tsuda et al., 2002).

## Fisher kernel properties (2/2)

### Lemma

The Fisher kernel is **invariant** under change of parametrization.

- Consider indeed different parametrization given by some diffeomorphism  $\lambda = f(\theta)$ . The **Jacobian** matrix relating the parametrization is denoted by  $[J]_{ij} = \frac{\partial \theta_j}{\partial \lambda_i}$ .
- The gradient of log-likelihood w.r.t. to the new parameters is

$$\Phi_{\lambda_0}(\mathbf{x}) = \nabla_{\lambda} \log P_{\lambda_0}(\mathbf{x}) = \mathbf{J} \nabla_{\theta} \log P_{\theta_0}(\mathbf{x}) = \mathbf{J} \Phi_{\theta_0}(\mathbf{x}).$$

- the Fisher information matrix is

$$\mathbf{I}(\lambda_0) = E \left[ \Phi_{\theta_0}(\mathbf{x}) \Phi_{\theta_0}(\mathbf{x})^\top \right] = \mathbf{J} \mathbf{I}(\theta_0) \mathbf{J}^\top.$$

- we conclude by noticing that  $\mathbf{I}(\lambda_0)^{-1} = \mathbf{J}^{-1} \mathbf{I}(\theta_0)^{-1} \mathbf{J}^{\top -1}$ .

## Fisher kernel in practice

- $\Phi_{\theta_0}(\mathbf{x})$  can be computed explicitly for many models (e.g., HMMs), where the model is **first estimated from data**.
- $\mathbf{I}(\theta_0)$  is often replaced by the identity matrix for simplicity.
- Several different models (i.e., different  $\theta_0$ ) can be trained and combined.
- The **Fisher vectors** are defined as  $\varphi_{\theta_0}(\mathbf{x}) = \mathbf{I}(\theta_0)^{-1/2}\Phi_{\theta_0}(\mathbf{x})$ . They are explicitly computed and correspond to an explicit embedding:  
 $K(\mathbf{x}, \mathbf{x}') = \varphi_{\theta_0}(\mathbf{x})^\top \varphi_{\theta_0}(\mathbf{x}')$ .

## Fisher kernels: example with Gaussian data model (1/2)

Consider a normal distribution  $\mathcal{N}(\mu, \sigma^2)$  and denote by  $\alpha = 1/\sigma^2$  the inverse variance, i.e., precision parameter. With  $\theta = (\mu, \alpha)$ , we have

$$\log P_\theta(x) = \frac{1}{2} \log \alpha - \frac{1}{2} \log(2\pi) - \frac{1}{2} \alpha(x - \mu)^2,$$

and thus

$$\frac{\partial \log P_\theta(x)}{\partial \mu} = \alpha(x - \mu), \quad \frac{\partial \log P_\theta(x)}{\partial \alpha} = \frac{1}{2} \left[ \frac{1}{\alpha} - (x - \mu)^2 \right],$$

and (exercise)

$$\mathbf{I}(\theta) = \begin{pmatrix} \alpha & 0 \\ 0 & (1/2)\alpha^{-2} \end{pmatrix}.$$

The **Fisher vector** is then

$$\varphi_\theta(\mathbf{x}) = \begin{pmatrix} (x - \mu)/\sigma \\ (1/\sqrt{2})(1 - (x - \mu)^2/\sigma^2) \end{pmatrix}.$$

## Fisher kernels: example with Gaussian data model (2/2)

Now consider an i.i.d. data model over a set of data points  $x_1, \dots, x_n$  all distributed according to  $\mathcal{N}(\mu, \sigma^2)$ :

$$P_\theta(x_1, \dots, x_n) = \prod_{i=1}^n P_\theta(x_i).$$

Then, the Fisher vector is given by the sum of Fisher vectors of the points.

- Encodes the **discrepancy in the first and second order moment** of the data w.r.t. those of the model.

$$\varphi(x_1, \dots, x_n) = \sum_{i=1}^n \varphi(x_i) = n \begin{pmatrix} (\hat{\mu} - \mu)/\sigma \\ (\sigma^2 - \hat{\sigma}^2)/(\sqrt{2}\sigma^2) \end{pmatrix},$$

- where

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{and} \quad \hat{\sigma} = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2.$$

## Application: Aggregation of visual words (1/4)

- **Patch extraction and description stage:**

In various contexts, images may be described as a set of patches  $\mathbf{x}_1, \dots, \mathbf{x}_n$  computed at interest points. For example, SIFT, HOG, LBP, color histograms, convolutional features...

- **Coding stage:** The set of patches is then encoded into a single representation  $\varphi(\mathbf{x}_i)$ , typically in a high-dimensional space.
- **Pooling stage:** For example, sum pooling

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{i=1}^n \varphi(\mathbf{x}_i).$$

Fisher vectors with a Gaussian Mixture Model (GMM) is considered to be a state-of-the-art aggregation technique [Perronnin and Dance, 2007].

## Application: Aggregation of visual words (2/4)

Let  $\theta = (\pi_j, \mu_j, \Sigma_j)_{j=1 \dots k}$  be the parameters of a GMM with  $k$  Gaussian components. Then, the probabilistic model is given by

$$P_\theta(\mathbf{x}) = \sum_{j=1}^k \pi_j \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j).$$

### Remarks

- Each mixture component corresponds to a **visual word**, with a mean, variance, and mixing weight.
- Diagonal covariances  $\boldsymbol{\Sigma}_j = \text{diag}(\sigma_{j1}, \dots, \sigma_{jp}) = \text{diag}(\boldsymbol{\sigma}_j)$  are often used for simplicity.
- This is a richer model than the traditional “bag of words” approach.
- The probabilistic model is learned offline beforehand.

## Application: Aggregation of visual words (3/4)

After a few calculations (exercise), we obtain  $\varphi_{\theta}(\mathbf{x}_1, \dots, \mathbf{x}_n) =$

$$[\varphi_{\pi_1}(\mathbf{X}), \dots, \varphi_{\pi_p}(\mathbf{X}), \varphi_{\boldsymbol{\mu}_1}(\mathbf{X})^\top, \dots, \varphi_{\boldsymbol{\mu}_p}(\mathbf{X})^\top, \varphi_{\sigma_1}(\mathbf{X})^\top, \dots, \varphi_{\sigma_p}(\mathbf{X})^\top]^\top,$$

with

$$\varphi_{\boldsymbol{\mu}_j}(\mathbf{X}) = \frac{1}{n\sqrt{\pi_j}} \sum_{i=1}^n \gamma_{ij} (\mathbf{x}_i - \boldsymbol{\mu}_j) / \sigma_j$$

$$\varphi_{\sigma_j}(\mathbf{X}) = \frac{1}{n\sqrt{2\pi_j}} \sum_{i=1}^n \gamma_{ij} [(x_i - \boldsymbol{\mu}_j)^2 / \sigma_j^2 - 1],$$

where with an abuse of notation, the division between two vectors is meant elementwise and the scalars  $\gamma_{ij}$  can be interpreted as the **soft-assignment** of word  $i$  to component  $j$ :

$$\gamma_{ij} = \frac{\pi_j \mathcal{N}(\mathbf{x}_i; \boldsymbol{\mu}_j, \boldsymbol{\sigma}_j)}{\sum_{l=1}^k \pi_l \mathcal{N}(\mathbf{x}_i; \boldsymbol{\mu}_l, \boldsymbol{\sigma}_l)}.$$

## Application: Aggregation of visual words (4/4)

Finally, we also have the following interpretation of encoding first and second-order statistics:

$$\varphi_{\mu_j}(\mathbf{X}) = \frac{\gamma_j}{\sqrt{\pi_j}} (\hat{\mu}_j - \mu_j) / \sigma_j$$

$$\varphi_{\sigma_j}(\mathbf{X}) = \frac{\gamma_j}{\sqrt{2\pi_j}} (\hat{\sigma}_j^2 - \sigma_j^2) / \sigma_j^2,$$

with

$$\gamma_j = \sum_{i=1}^n \gamma_{ij} \quad \text{and} \quad \hat{\mu}_j = \frac{1}{\gamma_j} \sum_{i=1}^n \gamma_{ij} \mathbf{x}_i \quad \text{and} \quad \hat{\sigma}_j = \frac{1}{\gamma_j} \sum_{i=1}^n \gamma_{ij} (\mathbf{x}_i - \mu_j)^2.$$

The component  $\varphi_{\pi}(\mathcal{X})$  is often dropped due to its negligible contribution in practice, and the resulting representation is of dimension  $2kp$  where  $p$  is the dimension of the  $\mathbf{x}_i$ 's.

## Relation to classification with generative models (1/3)

Assume that we have a **generative probabilistic model**  $P_\theta$  to model random variables  $(X, Y)$  where  $Y$  is a label in  $\{1, \dots, p\}$ .

Assume that the marginals  $P_\theta(Y = k) = \pi_k$  are among the model parameters  $\theta$ , which we can also parametrize as

$$P_\theta(Y = k) = \pi_k = \frac{e^{\alpha_k}}{\sum_{k'=1}^p e^{\alpha_{k'}}}.$$

The classification of a new point  $x$  can be obtained via **Bayes' rule**:

$$\hat{y}(x) = \operatorname{argmax}_{k=1,\dots,p} P_\theta(Y = k|x),$$

where  $P_\theta(Y = k|x)$  is short for  $P_\theta(Y = k|X = x)$  and

$$\begin{aligned} P_\theta(Y = k|x) &= P_\theta(x|Y = k)P_\theta(Y = k)/P_\theta(x) \\ &= P_\theta(x|Y = k)\pi_k / \sum_{k'=1}^p P_\theta(x|Y = k')\pi_{k'} \end{aligned}$$

## Relation to classification with generative models (2/3)

Then, consider the Fisher score

$$\begin{aligned}\nabla_{\theta} \log P_{\theta}(x) &= \frac{1}{P_{\theta}(x)} \nabla_{\theta} P_{\theta}(x) \\&= \frac{1}{P_{\theta}(x)} \nabla_{\theta} \sum_{k=1}^p P_{\theta}(x, Y = k) \\&= \frac{1}{P_{\theta}(x)} \sum_{k=1}^p P_{\theta}(x, Y = k) \nabla_{\theta} \log P_{\theta}(x, Y = k) \\&= \sum_{k=1}^p P_{\theta}(Y = k|x) [\nabla_{\theta} \log \pi_k + \nabla_{\theta} \log P_{\theta}(x|Y = k)].\end{aligned}$$

In particular (exercise)

$$\frac{\partial \log P_{\theta}(x)}{\partial \alpha_k} = P_{\theta}(Y = k|x) - \pi_k.$$

## Relation to classification with generative models (3/3)

The first  $p$  elements in the Fisher score are given by class posteriors minus a constant

$$\varphi_{\theta}(x) = [P_{\theta}(Y = 1|x) - \pi_1, \dots, P_{\theta}(Y = p|x) - \pi_p, \dots].$$

Consider a multi-class linear classifier on  $\varphi_{\theta}(x)$  such that for class  $k$

- The weights are zero except one for the  $k$ -th position;
- The intercept  $b_k$  be  $-\pi_k$ ;

Then,

$$\hat{y}(x) = \operatorname{argmax}_{k=1,\dots,p} \varphi_{\theta}(x)^{\top} \mathbf{w}_k + b_k$$

$$\hat{y}(x) = \operatorname{argmax}_{k=1,\dots,p} P_{\theta}(Y = k|x).$$

Bayes' rule is implemented via this simple classifier using Fisher kernel.

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
  - Fisher kernel
  - Mutual information kernels
  - Marginalized kernels
- Kernels for biological sequences
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- Kernels on graphs

# Mutual information kernels

## Definition

- Choose a prior  $w(d\theta)$  on the measurable set  $\Theta$ .
- Form the kernel (Seeger, 2002):

$$K(\mathbf{x}, \mathbf{x}') = \int_{\theta \in \Theta} P_\theta(\mathbf{x})P_\theta(\mathbf{x}')w(d\theta).$$

- No explicit computation of a finite-dimensional feature vector.
- $K(\mathbf{x}, \mathbf{x}') = \langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle_{L_2(w)}$  with

$$\varphi(\mathbf{x}) = (P_\theta(\mathbf{x}))_{\theta \in \Theta}.$$

## Example: coin toss

- Let  $P_\theta(X = 1) = \theta$  and  $P_\theta(X = 0) = 1 - \theta$  a model for random coin toss, with  $\theta \in [0, 1]$ .
- Let  $d\theta$  be the Lebesgue measure on  $[0, 1]$
- The mutual information kernel between  $\mathbf{x} = 001$  and  $\mathbf{x}' = 1010$  is:

$$\begin{cases} P_\theta(\mathbf{x}) &= \theta(1-\theta)^2, \\ P_\theta(\mathbf{x}') &= \theta^2(1-\theta)^2, \end{cases}$$

$$K(\mathbf{x}, \mathbf{x}') = \int_0^1 \theta^3(1-\theta)^4 d\theta = \frac{3!4!}{8!} = \frac{1}{280}.$$

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# Marginalized kernels

## Definition

- For any observed data  $\mathbf{x} \in \mathcal{X}$ , let a latent variable  $\mathbf{y} \in \mathcal{Y}$  be associated probabilistically through a conditional probability  $P_{\mathbf{x}}(d\mathbf{y})$ .
- Let  $K_{\mathcal{Z}}$  be a kernel for the complete data  $\mathbf{z} = (\mathbf{x}, \mathbf{y})$
- Then the following kernel is a valid kernel on  $\mathcal{X}$ , called a marginalized kernel (Tsuda et al., 2002):

$$\begin{aligned} K_{\mathcal{X}}(\mathbf{x}, \mathbf{x}') &:= E_{P_{\mathbf{x}}(d\mathbf{y}) \times P_{\mathbf{x}'}(d\mathbf{y}')} K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') \\ &= \int \int K_{\mathcal{Z}}((\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}')) P_{\mathbf{x}}(d\mathbf{y}) P_{\mathbf{x}'}(d\mathbf{y}') . \end{aligned}$$

## Marginalized kernels: proof of positive definiteness

- $K_{\mathcal{Z}}$  is p.d. on  $\mathcal{Z}$ . Therefore there exists a Hilbert space  $\mathcal{H}$  and  $\Phi_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{H}$  such that:

$$K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') = \langle \Phi_{\mathcal{Z}}(\mathbf{z}), \Phi_{\mathcal{Z}}(\mathbf{z}') \rangle_{\mathcal{H}}.$$

- Marginalizing therefore gives:

$$\begin{aligned} K_{\mathcal{X}}(\mathbf{x}, \mathbf{x}') &= E_{P_{\mathbf{x}}(d\mathbf{y}) \times P_{\mathbf{x}'}(d\mathbf{y}')} K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') \\ &= E_{P_{\mathbf{x}}(d\mathbf{y}) \times P_{\mathbf{x}'}(d\mathbf{y}')} \langle \Phi_{\mathcal{Z}}(\mathbf{z}), \Phi_{\mathcal{Z}}(\mathbf{z}') \rangle_{\mathcal{H}} \\ &= \langle E_{P_{\mathbf{x}}(d\mathbf{y})} \Phi_{\mathcal{Z}}(\mathbf{z}), E_{P_{\mathbf{x}'}(d\mathbf{y}')} \Phi_{\mathcal{Z}}(\mathbf{z}') \rangle_{\mathcal{H}}, \end{aligned}$$

therefore  $K_{\mathcal{X}}$  is p.d. on  $\mathcal{X}$ .  $\square$

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
  - Kernels for probabilistic models
  - **Kernels for biological sequences**
  - Mercer kernels and shift-invariant kernels
  - Kernels for graphs
  - Kernels on graphs
- 6 Open Problems and Research Topics

# Outline

## 5 The Kernel Jungle

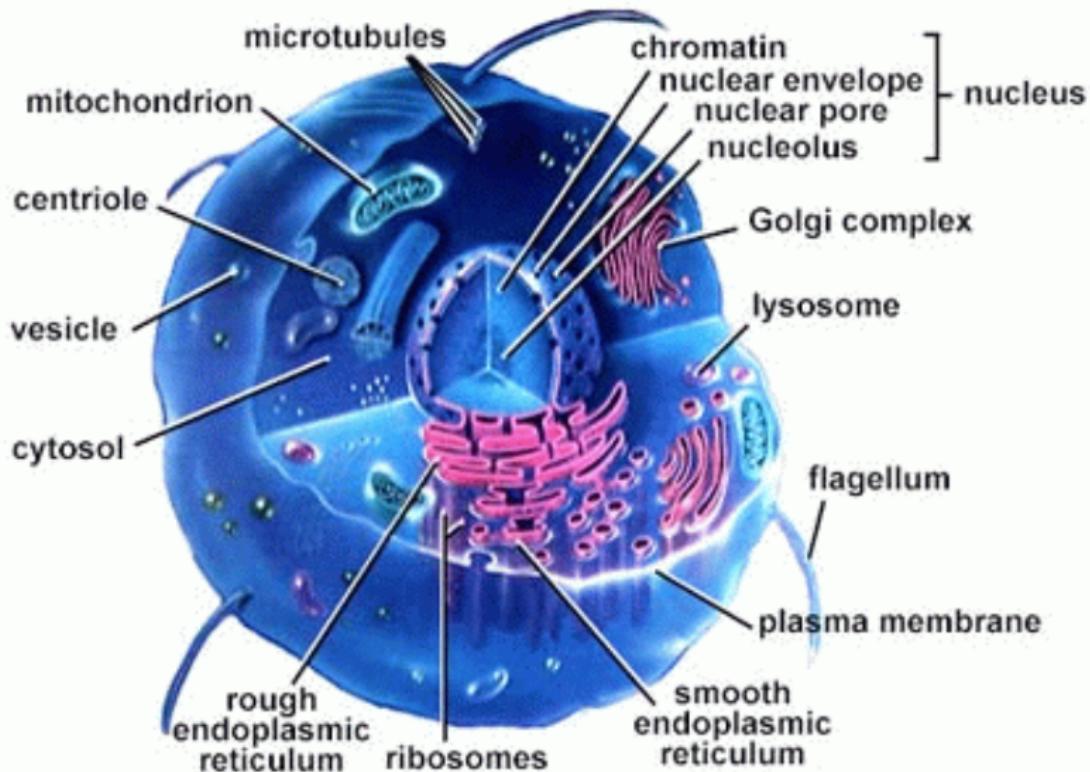
- Kernels for probabilistic models
- **Kernels for biological sequences**
  - Motivations and history of genomics
  - Kernels derived from large feature spaces
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# Short history of genomics

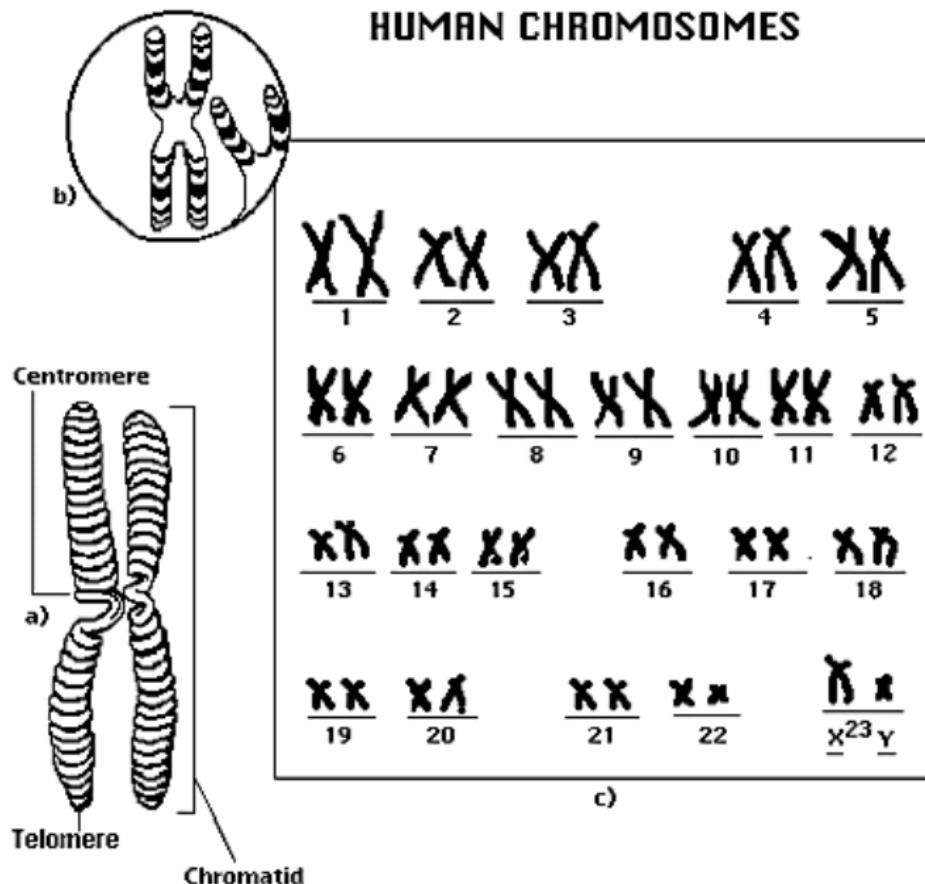


- 1866 : Laws of heredity (Mendel)
- 1909 : Morgan and the drosophilists
- 1944 : DNA supports heredity (Avery)
- 1953 : Structure of DNA (Crick and Watson)**
- 1966 : Genetic code (Nirenberg)
- 1960-70 : Genetic engineering
- 1977 : Method for sequencing (Sanger)
- 1982 : Creation of Genbank
- 1990 : Human genome project launched
- 2003 : Human genome project completed**

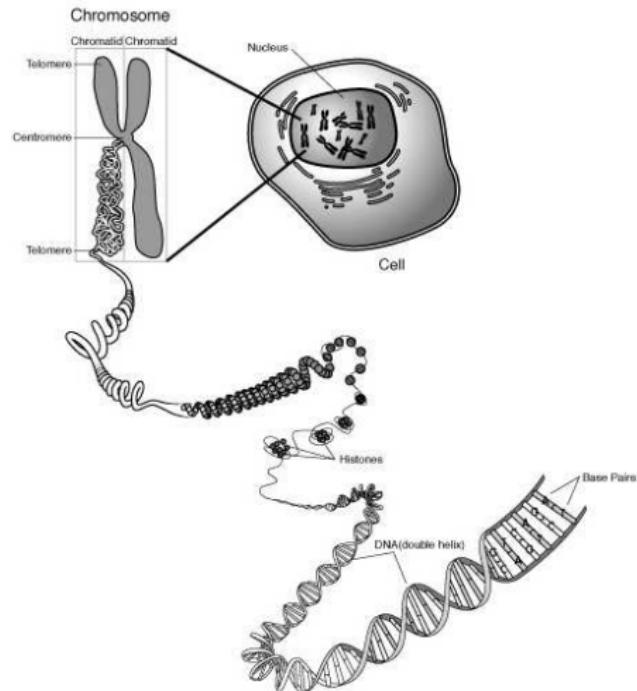
# A cell



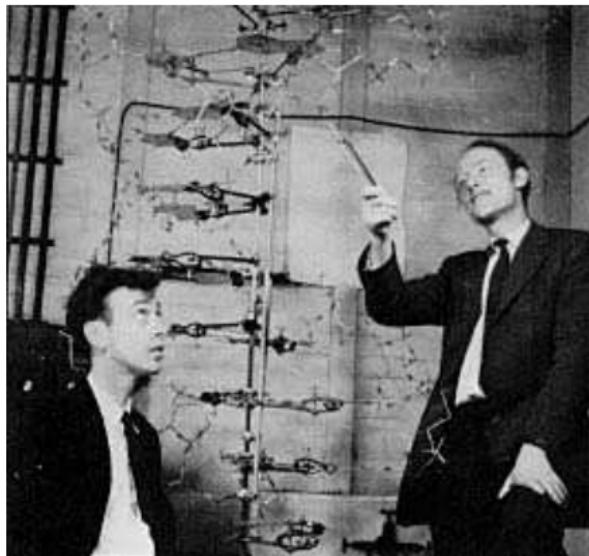
# Chromosomes



# Chromosomes and DNA

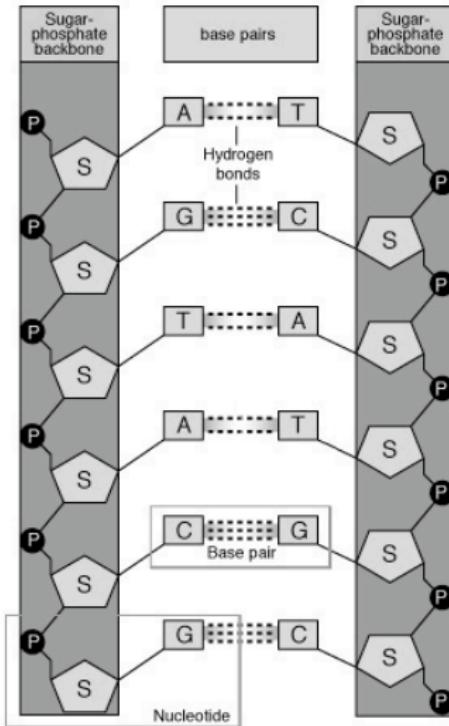
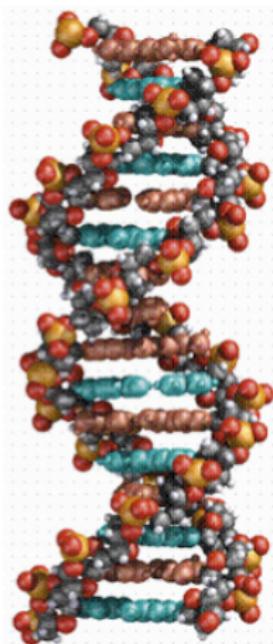


# Structure of DNA

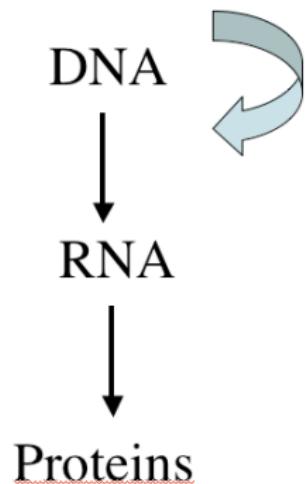
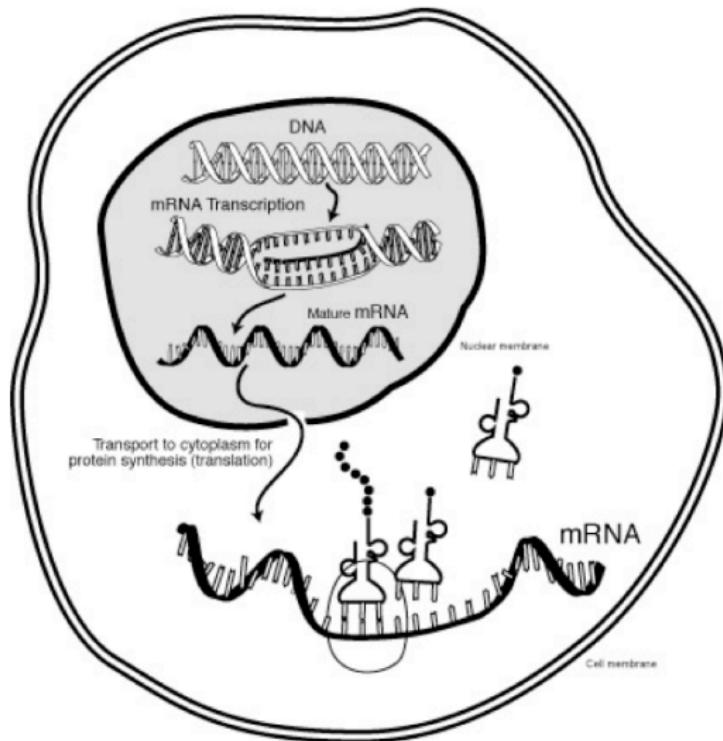


“We wish to suggest a structure for the salt of deoxyribose nucleic acid (D.N.A.). This structure have novel features which are of considerable biological interest” (Watson and Crick, 1953)

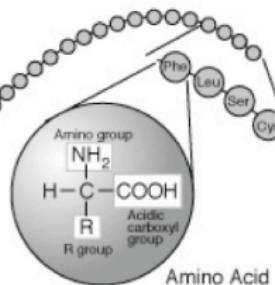
# The double helix



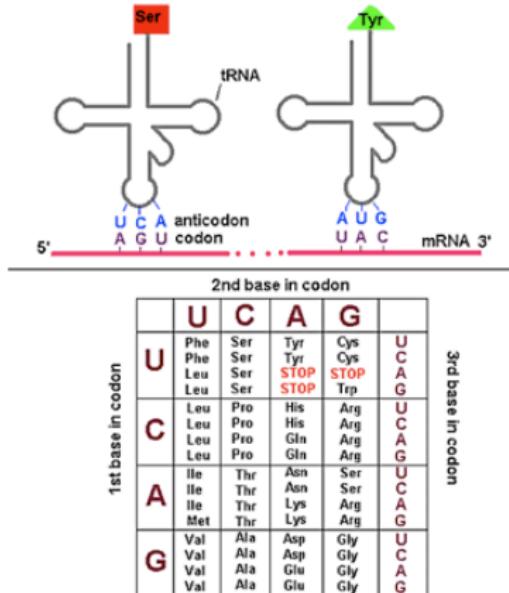
# Central dogma



# Proteins



# Genetic code



DNA = 4 letters (ATCG)

↓

RNA = 4 letters (AUCG)

↓

Protein = 20 letters (amino acids)

1 amino acid

=

3 nucleotides

The Genetic Code

## Human genome project

- Goal : sequence the 3,000,000,000 bases of the human genome
- Consortium with 20 labs, 6 countries
- Cost : about 3,000,000,000 USD



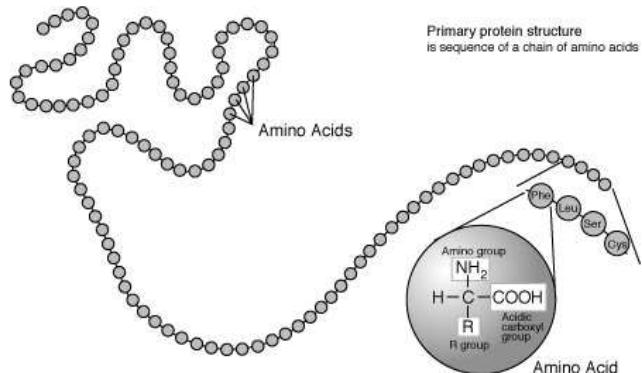
## 2003: End of genomics era



### Findings

- About 25,000 genes only (representing 1.2% of the genome).
- Automatic gene finding with graphical models.
- 97% of the genome is considered “junk DNA”.
- Superposition of a variety of signals (many to be discovered).

# Protein sequence



A : Alanine	V : Valine	L : Leucine
F : Phenylalanine	P : Proline	M : Methionine
E : Glutamic acid	K : Lysine	R : Arginine
T : Threonine	C : Cysteine	N : Asparagine
H : Histidine	Y : Tyrosine	W : Tryptophane
I : Isoleucine	S : Serine	Q : Glutamine
D : Aspartic acid	G : Glycine	

## Challenges with protein sequences

- A protein sequences can be seen as a **variable-length sequence** over the **20-letter alphabet** of amino-acids, e.g., insulin:  
FVNQHLCGSHLVEALYLVCGERGFFYTPKA
- These sequences are produced at a fast rate (result of the **sequencing programs**)
- Need for algorithms to **compare, classify, analyze** these sequences
- Applications: classification into **functional or structural** classes, prediction of **cellular localization** and **interactions**, ...

## Example: supervised sequence classification

### Data (training)

- Secreted proteins:

MASKATLLLAF~~TLLFATCIARHQQRQQQNQCQLQNIEA~~...

MARSSLFTFLCLAVFINGCLSQIEQQSPWEFQGSEVW...

MALHTVLIMLSLLPMLEAQNPEHANITIGEPI~~TNETLGWL~~...

...

- Non-secreted proteins:

MAPPSVFAEV~~PQAQPVLVF~~KLIADFREDPDRKVNLGVG...

MAHTLGLTQP~~NSTEPHKISFTA~~EIDVIEWKGDI~~L~~VVG...

MSI~~SESYAKEIKTA~~RQFTDFPIEGEQFEDFLPIIGNP..

...

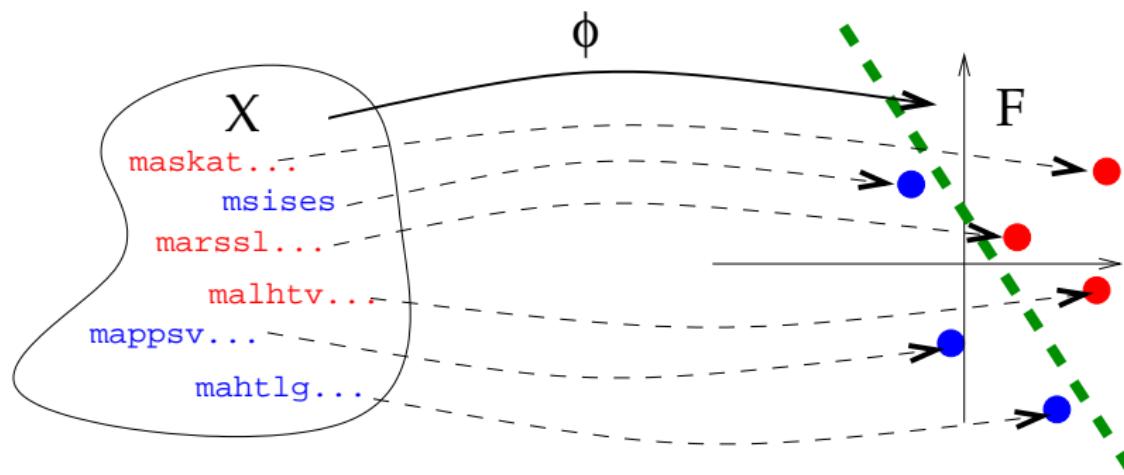
### Goal

- Build a **classifier** to **predict** whether new proteins are secreted or not.

# Supervised classification with vector embedding

## The idea

- Map each string  $x \in \mathcal{X}$  to a vector  $\Phi(x) \in \mathcal{F}$ .
- Train a **classifier for vectors** on the images  $\Phi(x_1), \dots, \Phi(x_n)$  of the training set (nearest neighbor, linear perceptron, logistic regression, support vector machine...)



## Kernels for protein sequences

- Kernel methods have been widely investigated since Jaakkola et al.'s seminal paper (1998).
- What is a good kernel?
  - it should be mathematically valid (symmetric, p.d. or c.p.d.)
  - fast to compute
  - adapted to the problem (give good performances)

# Kernel engineering for protein sequences

- Define a (possibly high-dimensional) **feature space** of interest
  - Physico-chemical kernels
  - Spectrum, mismatch, substring kernels
  - Pairwise, motif kernels

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# Kernel engineering for protein sequences

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## 5 The Kernel Jungle

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# Vector embedding for strings

## The idea

Represent each sequence  $x$  by a **fixed-length numerical vector**  
 $\Phi(x) \in \mathbb{R}^n$ . How to perform this embedding?

# Vector embedding for strings

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Represent each sequence  $x$  by a **fixed-length numerical vector**  $\Phi(x) \in \mathbb{R}^n$ . How to perform this embedding?

## Physico-chemical kernel

Extract **relevant features**, such as:

- length of the sequence
- time series analysis of numerical physico-chemical properties of amino-acids along the sequence (e.g., polarity, hydrophobicity), using for example:
  - Fourier transforms (Wang et al., 2004)
  - Autocorrelation functions (Zhang et al., 2003)

$$r_j = \frac{1}{n-j} \sum_{i=1}^{n-j} h_i h_{i+j}$$

# Substring indexation

## The approach

Alternatively, index the feature space by fixed-length strings, i.e.,

$$\Phi(\mathbf{x}) = (\Phi_u(\mathbf{x}))_{u \in \mathcal{A}^k}$$

where  $\Phi_u(\mathbf{x})$  can be:

- the number of occurrences of  $u$  in  $\mathbf{x}$  (without gaps) : **spectrum kernel** (Leslie et al., 2002)
- the number of occurrences of  $u$  in  $\mathbf{x}$  up to  $m$  mismatches (without gaps) : **mismatch kernel** (Leslie et al., 2004)
- the number of occurrences of  $u$  in  $\mathbf{x}$  allowing gaps, with a weight decaying exponentially with the number of gaps : **substring kernel** (Lohdi et al., 2002)

## Example: spectrum kernel (1/2)

### Kernel definition

- The 3-spectrum of

$$\mathbf{x} = \text{CGGSLIAMMWFGV}$$

is:

(CGG, GGS, GSL, SLI, LIA, IAM, AMM, MMW, MWF, WFG, FGV) .

- Let  $\Phi_u(\mathbf{x})$  denote the number of occurrences of  $u$  in  $\mathbf{x}$ . The  $k$ -spectrum kernel is:

$$K(\mathbf{x}, \mathbf{x}') := \sum_{u \in \mathcal{A}^k} \Phi_u(\mathbf{x}) \Phi_u(\mathbf{x}') .$$

## Example: spectrum kernel (2/2)

### Implementation

- The computation of the kernel is formally a sum over  $|\mathcal{A}|^k$  terms, but at most  $|\mathbf{x}| - k + 1$  terms are non-zero in  $\Phi(\mathbf{x}) \implies$  Computation in  $O(|\mathbf{x}| + |\mathbf{x}'|)$  with pre-indexation of the strings.
- Fast classification of a sequence  $\mathbf{x}$  in  $O(|\mathbf{x}|)$ :

$$f(\mathbf{x}) = \mathbf{w} \cdot \Phi(\mathbf{x}) = \sum_u w_u \Phi_u(\mathbf{x}) = \sum_{i=1}^{|\mathbf{x}|-k+1} w_{x_i \dots x_{i+k-1}}.$$

### Remarks

- Work with any string (natural language, time series...)
- Fast and scalable, a good default method for string classification.
- Variants allow matching of  $k$ -mers up to  $m$  mismatches.

## Example 2: Substring kernel (1/11)

### Definition

- For  $1 \leq k \leq n \in \mathbb{N}$ , we denote by  $\mathcal{I}(k, n)$  the set of sequences of indices  $\mathbf{i} = (i_1, \dots, i_k)$ , with  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ .
- For a string  $\mathbf{x} = x_1 \dots x_n \in \mathcal{X}$  of length  $n$ , for a sequence of indices  $\mathbf{i} \in \mathcal{I}(k, n)$ , we define a **substring** as:

$$\mathbf{x}(\mathbf{i}) := x_{i_1} x_{i_2} \dots x_{i_k}.$$

- The **length** of the substring is:

$$l(\mathbf{i}) = i_k - i_1 + 1.$$

## Example 2: Substring kernel (2/11)

### Example

ABRACADABRA

- $\mathbf{i} = (3, 4, 7, 8, 10)$
- $x(\mathbf{i}) = \text{RADAR}$
- $l(\mathbf{i}) = 10 - 3 + 1 = 8$

## Example 2: Substring kernel (3/11)

### The kernel

- Let  $k \in \mathbb{N}$  and  $\lambda \in \mathbb{R}^+$  fixed. For all  $\mathbf{u} \in \mathcal{A}^k$ , let  $\Phi_{\mathbf{u}} : \mathcal{X} \rightarrow \mathbb{R}$  be defined by:

$$\forall \mathbf{x} \in \mathcal{X}, \quad \Phi_{\mathbf{u}}(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{I}(k, |\mathbf{x}|) : \mathbf{x}(\mathbf{i}) = \mathbf{u}} \lambda^{l(\mathbf{i})}.$$

- The **substring kernel** is the p.d. kernel defined by:

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K_{k,\lambda}(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{u} \in \mathcal{A}^k} \Phi_{\mathbf{u}}(\mathbf{x}) \Phi_{\mathbf{u}}(\mathbf{x}').$$

## Example 2: Substring kernel (4/11)

### Example

u	ca	ct	at	ba	bt	cr	ar	br
$\Phi_u(\text{cat})$	$\lambda^2$	$\lambda^3$	$\lambda^2$	0	0	0	0	0
$\Phi_u(\text{car})$	$\lambda^2$	0	0	0	0	$\lambda^3$	$\lambda^2$	0
$\Phi_u(\text{bat})$	0	0	$\lambda^2$	$\lambda^2$	$\lambda^3$	0	0	0
$\Phi_u(\text{bar})$	0	0	0	$\lambda^2$	0	0	$\lambda^2$	$\lambda^3$

$$\begin{cases} K(\text{cat}, \text{cat}) = K(\text{car}, \text{car}) = 2\lambda^4 + \lambda^6 \\ K(\text{cat}, \text{car}) = \lambda^4 \\ K(\text{cat}, \text{bar}) = 0 \end{cases}$$

## Example 2: Substring kernel (5/11)

### Kernel computation

- We need to compute, for any pair  $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ , the kernel:

$$\begin{aligned} K_{n,\lambda}(\mathbf{x}, \mathbf{x}') &= \sum_{\mathbf{u} \in \mathcal{A}^k} \Phi_{\mathbf{u}}(\mathbf{x}) \Phi_{\mathbf{u}}(\mathbf{x}') \\ &= \sum_{\mathbf{u} \in \mathcal{A}^k} \sum_{\mathbf{i}: \mathbf{x}(\mathbf{i}) = \mathbf{u}} \sum_{\mathbf{i}': \mathbf{x}'(\mathbf{i}') = \mathbf{u}} \lambda^{l(\mathbf{i}) + l(\mathbf{i}')} . \end{aligned}$$

- Enumerating the substrings is **too slow** (of order  $|\mathbf{x}|^k$ ).

## Example 2: Substring kernel (6/11)

### Kernel computation (cont.)

- For  $\mathbf{u} \in \mathcal{A}^k$  remember that:

$$\Phi_{\mathbf{u}}(\mathbf{x}) = \sum_{\mathbf{i}: \mathbf{x}(\mathbf{i})=\mathbf{u}} \lambda^{i_n - i_1 + 1}.$$

- Let now:

$$\Psi_{\mathbf{u}}(\mathbf{x}) = \sum_{\mathbf{i}: \mathbf{x}(\mathbf{i})=\mathbf{u}} \lambda^{|\mathbf{x}| - i_1 + 1}.$$

## Example 2: Substring kernel (7/11)

### Kernel computation (cont.)

Let us note  $\mathbf{x}(1, j) = x_1 \dots x_j$ . A simple rewriting shows that, if we note  $a \in \mathcal{A}$  the last letter of  $\mathbf{u}$  ( $\mathbf{u} = \mathbf{v}a$ ):

$$\Phi_{\mathbf{v}a}(\mathbf{x}) = \sum_{j \in [1, |\mathbf{x}|]: x_j = a} \Psi_{\mathbf{v}}(\mathbf{x}(1, j - 1)) \lambda,$$

and

$$\Psi_{\mathbf{v}a}(\mathbf{x}) = \sum_{j \in [1, |\mathbf{x}|]: x_j = a} \Psi_{\mathbf{v}}(\mathbf{x}(1, j - 1)) \lambda^{|\mathbf{x}| - j + 1}.$$

## Example 2: Substring kernel (8/11)

### Kernel computation (cont.)

Moreover we observe that if the string is of the form  $xa$  (i.e., the last letter is  $a \in \mathcal{A}$ ), then:

- If the last letter of  $\mathbf{u}$  is not  $a$ :

$$\begin{cases} \Phi_{\mathbf{u}}(xa) = \Phi_{\mathbf{u}}(x) , \\ \Psi_{\mathbf{u}}(xa) = \lambda \Psi_{\mathbf{u}}(x) . \end{cases}$$

- If the last letter of  $\mathbf{u}$  is  $a$  (i.e.,  $\mathbf{u} = \mathbf{v}a$  with  $\mathbf{v} \in \mathcal{A}^{n-1}$ ):

$$\begin{cases} \Phi_{\mathbf{v}a}(xa) = \Phi_{\mathbf{v}a}(x) + \lambda \Psi_{\mathbf{v}}(x) , \\ \Psi_{\mathbf{v}a}(xa) = \lambda \Psi_{\mathbf{v}a}(x) + \lambda \Psi_{\mathbf{v}}(x) . \end{cases}$$

## Example 2: Substring kernel (9/11)

### Kernel computation (cont.)

Let us now show how the function:

$$B_n(\mathbf{x}, \mathbf{x}') := \sum_{\mathbf{u} \in \mathcal{A}^n} \Psi_{\mathbf{u}}(\mathbf{x}) \Psi_{\mathbf{u}}(\mathbf{x}')$$

and the kernel:

$$K_n(\mathbf{x}, \mathbf{x}') := \sum_{\mathbf{u} \in \mathcal{A}^n} \Phi_{\mathbf{u}}(\mathbf{x}) \Phi_{\mathbf{u}}(\mathbf{x}')$$

can be computed recursively. We note that:

$$\begin{cases} B_0(\mathbf{x}, \mathbf{x}') = K_0(\mathbf{x}, \mathbf{x}') = 0 & \text{for all } \mathbf{x}, \mathbf{x}' \\ B_k(\mathbf{x}, \mathbf{x}') = K_k(\mathbf{x}, \mathbf{x}') = 0 & \text{if } \min(|\mathbf{x}|, |\mathbf{x}'|) < k \end{cases}$$

## Example 2: Substring kernel (10/11)

Recursive computation of  $B_n$

$$\begin{aligned} & B_n(\mathbf{x}a, \mathbf{x}') \\ &= \sum_{\mathbf{u} \in \mathcal{A}^n} \Psi_{\mathbf{u}}(\mathbf{x}a) \Psi_{\mathbf{u}}(\mathbf{x}') \\ &= \lambda \sum_{\mathbf{u} \in \mathcal{A}^n} \Psi_{\mathbf{u}}(\mathbf{x}) \Psi_{\mathbf{u}}(\mathbf{x}') + \lambda \sum_{\mathbf{v} \in \mathcal{A}^{n-1}} \Psi_{\mathbf{v}}(\mathbf{x}) \Psi_{\mathbf{v}a}(\mathbf{x}') \\ &= \lambda B_n(\mathbf{x}, \mathbf{x}') + \\ &\quad \lambda \sum_{\mathbf{v} \in \mathcal{A}^{n-1}} \Psi_{\mathbf{v}}(\mathbf{x}) \left( \sum_{j \in [1, |\mathbf{x}'|] : x'_j = a} \Psi_{\mathbf{v}}(\mathbf{x}'(1, j-1)) \lambda^{|\mathbf{x}'|-j+1} \right) \\ &= \lambda B_n(\mathbf{x}, \mathbf{x}') + \sum_{j \in [1, |\mathbf{x}'|] : x'_j = a} B_{n-1}(\mathbf{x}, \mathbf{x}'(1, j-1)) \lambda^{|\mathbf{x}'|-j+2} \end{aligned}$$

## Example 2: Substring kernel (10/11)

Recursive computation of  $B_n$

$$B_n(xa, x'b)$$

$$\begin{aligned} &= \lambda B_n(x, x'b) + \lambda \sum_{j \in [1, |x'|] : x'_j = a} B_{n-1}(x, x'(1, j-1)) \lambda^{|x'|-j+2} \\ &\quad + \delta_{a=b} B_{n-1}(x, x') \lambda^2 \end{aligned}$$

$$\begin{aligned} &= \lambda B_n(x, x'b) + \lambda(B_n(xa, x') - \lambda B_n(x, x')) + \delta_{a=b} B_{n-1}(x, x') \lambda^2 \\ &= \lambda B_n(x, x'b) + \lambda B_n(xa, x') - \lambda^2 B_n(x, x') + \delta_{a=b} B_{n-1}(x, x') \lambda^2. \end{aligned}$$

The dynamic programming table can be filled in  $O(n|x||x'|)$  operations.

## Example 2: Substring kernel (10/11)

Recursive computation of  $K_n$

$$\begin{aligned} K_n(xa, x') &= \sum_{\mathbf{u} \in \mathcal{A}^n} \Phi_{\mathbf{u}}(xa) \Phi_{\mathbf{u}}(x') \\ &= \sum_{\mathbf{u} \in \mathcal{A}^n} \Phi_{\mathbf{u}}(x) \Phi_{\mathbf{u}}(x') + \lambda \sum_{\mathbf{v} \in \mathcal{A}^{n-1}} \Psi_{\mathbf{v}}(x) \Phi_{\mathbf{v}a}(x') \\ &= K_n(x, x') + \\ &\quad \lambda \sum_{\mathbf{v} \in \mathcal{A}^{n-1}} \Psi_{\mathbf{v}}(x) \left( \sum_{j \in [1, |x'|]: x'_j = a} \Psi_{\mathbf{v}}(x'(1, j-1)) \right) \lambda \\ &= \lambda K_n(x, x') + \lambda^2 \sum_{j \in [1, |x'|]: x'_j = a} B_{n-1}(x, x'(1, j-1)) \end{aligned}$$

## Summary: Substring indexation

- Implementation in  $O(|\mathbf{x}| + |\mathbf{x}'|)$  in memory and time for the spectrum and mismatch kernels (with suffix trees)
- Implementation in  $O(k(|\mathbf{x}| + |\mathbf{x}'|))$  in memory and time for the spectrum and mismatch kernels (with tries)
- Implementation in  $O(k|\mathbf{x}| \times |\mathbf{x}'|)$  in memory and time for the substring kernels
- The feature space has high dimension ( $|\mathcal{A}|^k$ ), so learning requires **regularized methods** (such as SVM)

# Dictionary-based indexation

## The approach

- Choose a **dictionary** of sequences  $\mathcal{D} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$
- Choose a **measure of similarity**  $s(\mathbf{x}, \mathbf{x}')$
- Define the mapping  $\Phi_{\mathcal{D}}(\mathbf{x}) = (s(\mathbf{x}, \mathbf{x}_i))_{\mathbf{x}_i \in \mathcal{D}}$

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## Examples

This includes:

- **Motif kernels** (Logan et al., 2001): the dictionary is a library of motifs, the similarity function is a matching function
- **Pairwise kernel** (Liao & Noble, 2003): the dictionary is the training set, the similarity is a classical measure of similarity between sequences.

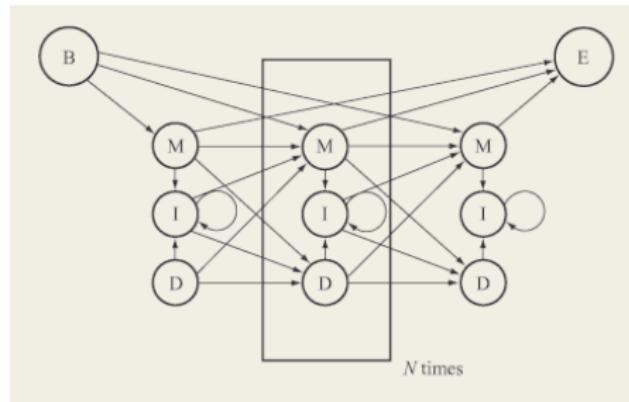
# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- **Kernels for biological sequences**
  - Motivations and history of genomics
  - Kernels derived from large feature spaces
  - **Kernels derived from generative models**
    - Kernels derived from a similarity measure
    - Application to remote homology detection
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- Kernels on graphs

# Probabilistic models for sequences

Probabilistic modeling of biological sequences is older than kernel designs. Important models include HMM for protein sequences, SCFG for RNA sequences.



Recall: parametric model

A **model** is a family of distributions

$$\{P_\theta, \theta \in \Theta \subset \mathbb{R}^m\} \subset \mathcal{M}_1^+(\mathcal{X})$$

# Context-tree model

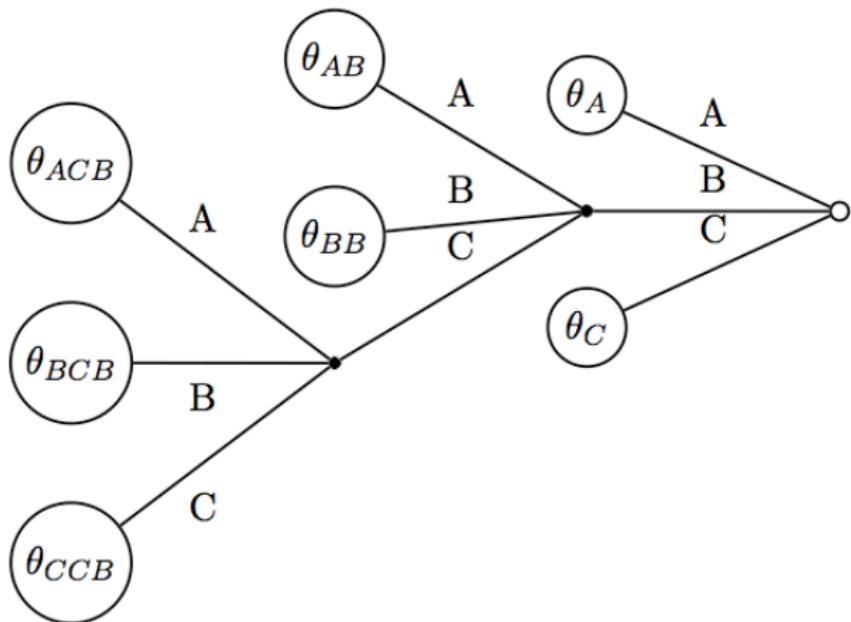
## Definition

A context-tree model is a **variable-memory Markov chain**:

$$P_{\mathcal{D}, \theta}(\mathbf{x}) = P_{\mathcal{D}, \theta}(x_1 \dots x_D) \prod_{i=D+1}^n P_{\mathcal{D}, \theta}(x_i | x_{i-D} \dots x_{i-1})$$

- $\mathcal{D}$  is a suffix tree
- $\theta \in \Sigma^{\mathcal{D}}$  is a set of conditional probabilities (multinomials)

## Context-tree model: example



$$P(AABACBACC) = P(AAB)\theta_{AB}(A)\theta_A(C)\theta_C(B)\theta_{ACB}(A)\theta_A(C)\theta_C(A).$$

# The context-tree kernel

Theorem (Cuturi et al., 2005)

- For particular choices of priors, the context-tree kernel:

$$K(\mathbf{x}, \mathbf{x}') = \sum_{\mathcal{D}} \int_{\theta \in \Sigma^{\mathcal{D}}} P_{\mathcal{D}, \theta}(\mathbf{x}) P_{\mathcal{D}, \theta}(\mathbf{x}') w(d\theta | \mathcal{D}) \pi(\mathcal{D})$$

can be computed in  $O(|\mathbf{x}| + |\mathbf{x}'|)$  with a variant of the Context-Tree Weighting algorithm.

- This is a valid mutual information kernel.
- The similarity is related to information-theoretical measure of mutual information between strings.

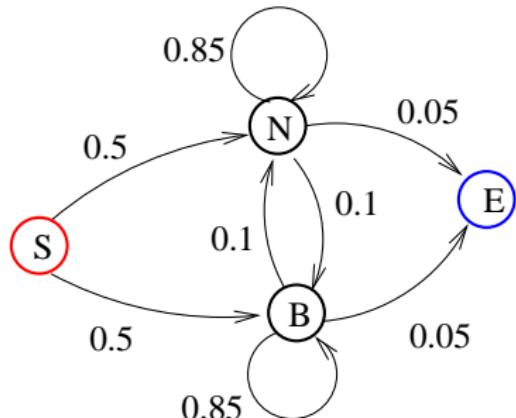
# Marginalized kernels

## Recall: Definition

- For any observed data  $\mathbf{x} \in \mathcal{X}$ , let a latent variable  $\mathbf{y} \in \mathcal{Y}$  be associated probabilistically through a conditional probability  $P_{\mathbf{x}}(d\mathbf{y})$ .
- Let  $K_{\mathcal{Z}}$  be a kernel for the complete data  $\mathbf{z} = (\mathbf{x}, \mathbf{y})$
- Then the following kernel is a valid kernel on  $\mathcal{X}$ , called a marginalized kernel (Tsuda et al., 2002):

$$\begin{aligned} K_{\mathcal{X}}(\mathbf{x}, \mathbf{x}') &:= E_{P_{\mathbf{x}}(d\mathbf{y}) \times P_{\mathbf{x}'}(d\mathbf{y}')} K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') \\ &= \int \int K_{\mathcal{Z}}((\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}')) P_{\mathbf{x}}(d\mathbf{y}) P_{\mathbf{x}'}(d\mathbf{y}') . \end{aligned}$$

## Example: HMM for normal/biased coin toss



- Normal ( $N$ ) and biased ( $B$ ) coins (not observed)
- Observed output are 0/1 with probabilities:

$$\begin{cases} \pi(0|N) = 1 - \pi(1|N) = 0.5, \\ \pi(0|B) = 1 - \pi(1|B) = 0.8. \end{cases}$$

- Example of realization (complete data):

NNNNNNBBBBBBBBBBNNNNNNNNNNNNBBBBBB  
1001011101111010010111001111011

## 1-spectrum kernel on complete data

- If both  $\mathbf{x} \in \mathcal{A}^*$  and  $\mathbf{y} \in \mathcal{S}^*$  were observed, we might rather use the 1-spectrum kernel on the complete data  $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ :

$$K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') = \sum_{(a,s) \in \mathcal{A} \times \mathcal{S}} n_{a,s}(\mathbf{z}) n_{a,s}(\mathbf{z}'),$$

where  $n_{a,s}(\mathbf{x}, \mathbf{y})$  for  $a = 0, 1$  and  $s = N, B$  is the number of occurrences of  $s$  in  $\mathbf{y}$  which emit  $a$  in  $\mathbf{x}$ .

- Example:

$$\begin{aligned}\mathbf{z} &= 1001011101111010010111001111011, \\ \mathbf{z}' &= 0011010110011111011010111101100101,\end{aligned}$$

$$\begin{aligned}K_{\mathcal{Z}}(\mathbf{z}, \mathbf{z}') &= n_0(\mathbf{z}) n_0(\mathbf{z}') + n_0(\mathbf{z}) n_0(\mathbf{z}') + n_1(\mathbf{z}) n_1(\mathbf{z}') + n_1(\mathbf{z}) n_1(\mathbf{z}') \\ &= 7 \times 15 + 9 \times 12 + 13 \times 6 + 2 \times 1 = 293.\end{aligned}$$

## 1-spectrum marginalized kernel on observed data

- The marginalized kernel for observed data is:

$$\begin{aligned} K_{\mathcal{X}}(\mathbf{x}, \mathbf{x}') &= \sum_{\mathbf{y}, \mathbf{y}' \in \mathcal{S}^*} K_{\mathcal{Z}}((\mathbf{x}, \mathbf{y}), (\mathbf{x}, \mathbf{y})) P(\mathbf{y}|\mathbf{x}) P(\mathbf{y}'|\mathbf{x}') \\ &= \sum_{(a,s) \in \mathcal{A} \times \mathcal{S}} \Phi_{a,s}(\mathbf{x}) \Phi_{a,s}(\mathbf{x}'), \end{aligned}$$

with

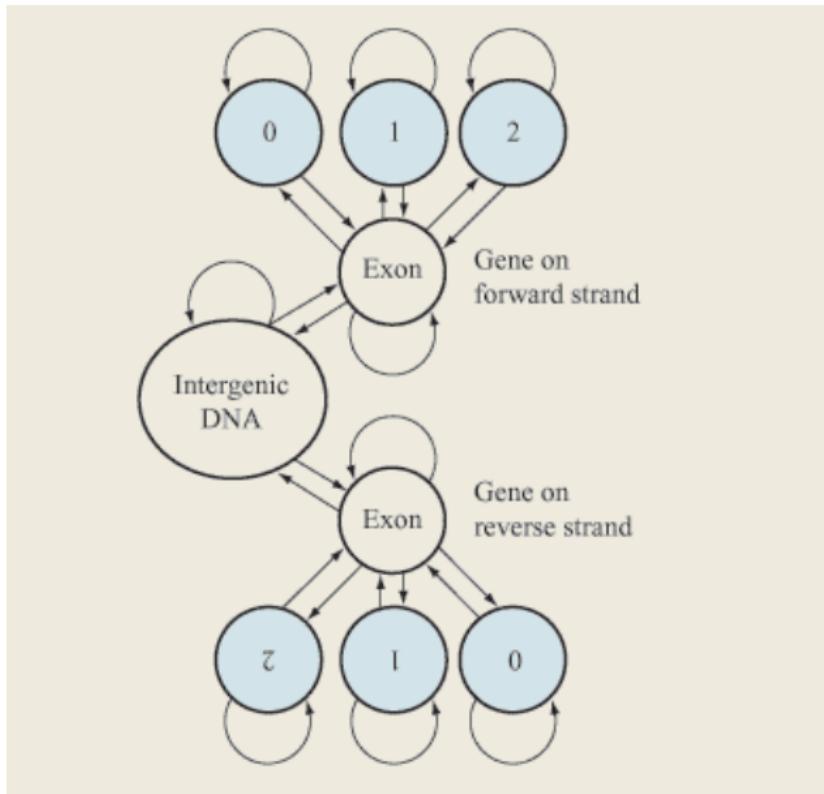
$$\Phi_{a,s}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{S}^*} P(\mathbf{y}|\mathbf{x}) n_{a,s}(\mathbf{x}, \mathbf{y})$$

## Computation of the 1-spectrum marginalized kernel

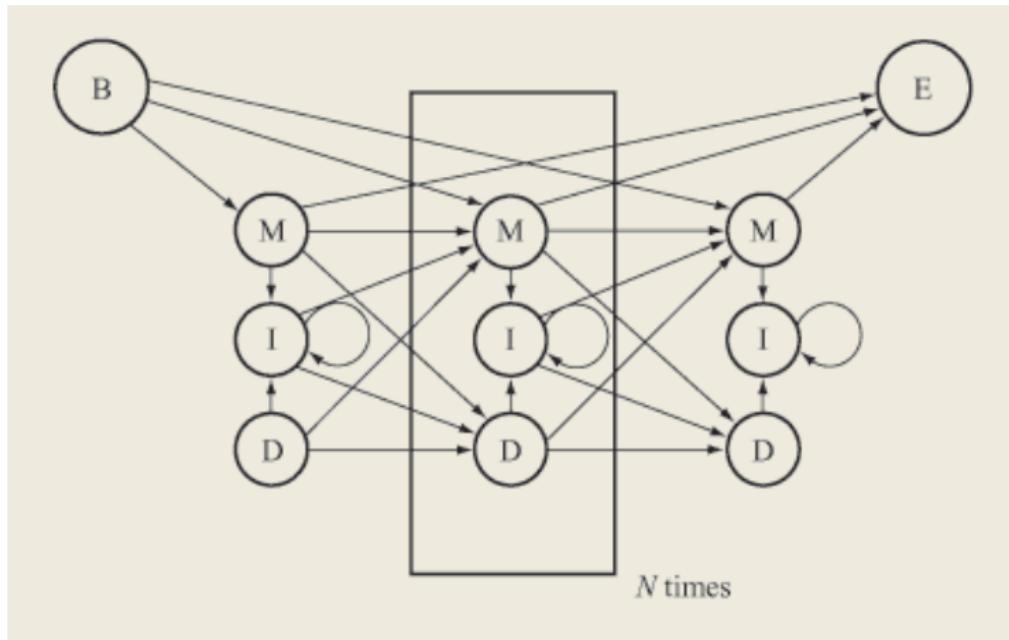
$$\begin{aligned}\Phi_{a,s}(\mathbf{x}) &= \sum_{\mathbf{y} \in \mathcal{S}^*} P(\mathbf{y}|\mathbf{x}) n_{a,s}(\mathbf{x}, \mathbf{y}) \\&= \sum_{\mathbf{y} \in \mathcal{S}^*} P(\mathbf{y}|\mathbf{x}) \left\{ \sum_{i=1}^n \delta(x_i, a) \delta(y_i, s) \right\} \\&= \sum_{i=1}^n \delta(x_i, a) \left\{ \sum_{\mathbf{y} \in \mathcal{S}^*} P(\mathbf{y}|\mathbf{x}) \delta(y_i, s) \right\} \\&= \sum_{i=1}^n \delta(x_i, a) P(y_i = s|\mathbf{x}).\end{aligned}$$

and  $P(y_i = s|\mathbf{x})$  can be computed efficiently by forward-backward algorithm!

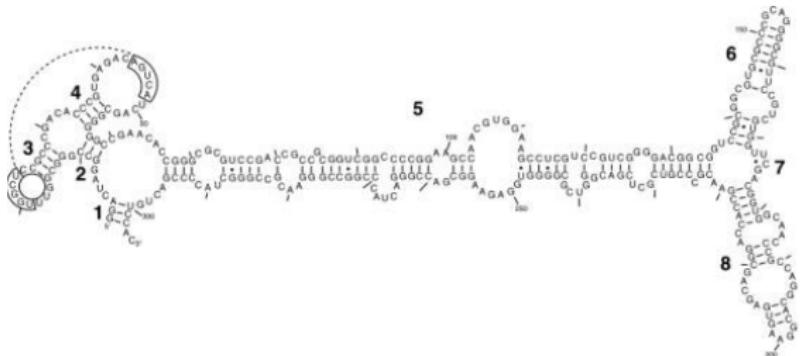
## HMM example (DNA)



## HMM example (protein)



# SCFG for RNA sequences



## SCFG rules

- $S \rightarrow SS$
- $S \rightarrow aSa$
- $S \rightarrow aS$
- $S \rightarrow a$

## Marginalized kernel (Kin et al., 2002)

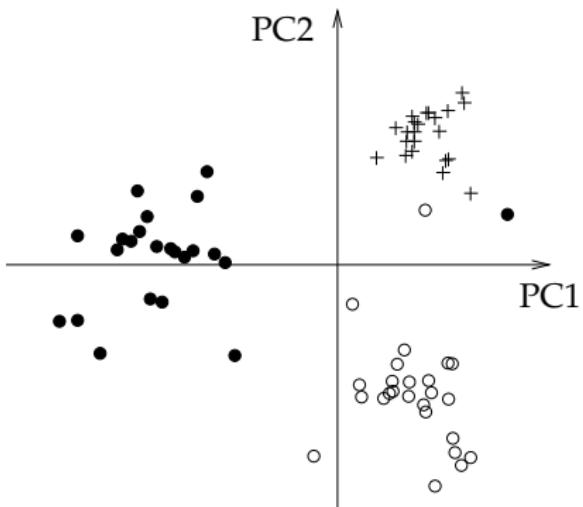
- Feature: number of occurrences of each (base,state) combination
- Marginalization using classical inside/outside algorithm

# Marginalized kernels in practice

## Examples

- Spectrum kernel on the hidden states of a HMM for **protein sequences** (Tsuda et al., 2002)
- Kernels for **RNA sequences** based on SCFG (Kin et al., 2002)
- Kernels for **graphs** based on random walks on graphs (Kashima et al., 2004)
- Kernels for **multiple alignments** based on phylogenetic models (Vert et al., 2006)

## Marginalized kernels: example



A set of 74 human tRNA sequences is analyzed using a kernel for sequences (the second-order marginalized kernel based on SCFG). This set of tRNAs contains three classes, called Ala-AGC (*white circles*), Asn-GTT (*black circles*) and Cys-GCA (*plus symbols*) (from Tsuda et al., 2002).

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# Sequence alignment

## Motivation

How to compare 2 sequences?

$$x_1 = \text{CGGSLIAMMWFGV}$$

$$x_2 = \text{CLIVMMNRLMWFGV}$$

Find a good **alignment**:

CGGSLIAMM	-----	WFGV
. .         . . .		
C-----LIVMMNRLMWFGV		

## Alignment score

In order to quantify the relevance of an alignment  $\pi$ , define:

- a substitution matrix  $S \in \mathbb{R}^{\mathcal{A} \times \mathcal{A}}$
- a gap penalty function  $g : \mathbb{N} \rightarrow \mathbb{R}$

Any alignment is then scored as follows

CGGSLIAMM-----WFGV
...         . . . .
C----LIVMMNRLMWFGV

$$\begin{aligned}s_{S,g}(\pi) = & S(C, C) + S(L, L) + S(I, I) + S(A, V) + 2S(M, M) \\ & + S(W, W) + S(F, F) + S(G, G) + S(V, V) - g(3) - g(4)\end{aligned}$$

## Local alignment kernel

### Smith-Waterman score (Smith and Waterman, 1981)

- The widely-used Smith-Waterman local alignment score is defined by:

$$SW_{S,g}(x, y) := \max_{\pi \in \Pi(x, y)} s_{S,g}(\pi).$$

- It is symmetric, but not positive definite...

## Local alignment kernel

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- It is symmetric, but not positive definite...

### LA kernel (Saigo et al., 2004)

The local alignment kernel:

$$K_{LA}^{(\beta)}(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \Pi(\mathbf{x}, \mathbf{y})} \exp(\beta s_{S,g}(\mathbf{x}, \mathbf{y}, \pi)),$$

is symmetric positive definite.

## LA kernel is p.d.: proof (1/11)

### Lemma

- If  $K_1$  and  $K_2$  are p.d. kernels, then:

$$K_1 + K_2,$$

$$K_1 K_2, \text{ and}$$

$$cK_1, \text{ for } c \geq 0,$$

are also p.d. kernels

- If  $(K_i)_{i \geq 1}$  is a sequence of p.d. kernels that converges pointwisely to a function  $K$ :

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathcal{X}^2, \quad K(\mathbf{x}, \mathbf{x}') = \lim_{n \rightarrow \infty} K_i(\mathbf{x}, \mathbf{x}'),$$

then  $K$  is also a p.d. kernel.

## LA kernel is p.d.: proof (2/11)

### Proof of lemma

Let  $A$  and  $B$  be  $n \times n$  positive semidefinite matrices. By diagonalization of  $A$ :

$$A_{i,j} = \sum_{p=1}^n f_p(i) f_p(j)$$

for some vectors  $f_1, \dots, f_n$ . Then, for any  $\alpha \in \mathbb{R}^n$ :

$$\sum_{i,j=1}^n \alpha_i \alpha_j A_{i,j} B_{i,j} = \sum_{p=1}^n \sum_{i,j=1}^n \alpha_i f_p(i) \alpha_j f_p(j) B_{i,j} \geq 0.$$

The matrix  $C_{i,j} = A_{i,j} B_{i,j}$  is therefore p.d. Other properties are obvious from definition.  $\square$

## LA kernel is p.d.: proof (3/11)

### Lemma (direct sum and product of kernels)

Let  $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ . Let  $K_1$  be a p.d. kernel on  $\mathcal{X}_1$ , and  $K_2$  be a p.d. kernel on  $\mathcal{X}_2$ . Then the following functions are p.d. kernels on  $\mathcal{X}$ :

- the **direct sum**,

$$K((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) = K_1(\mathbf{x}_1, \mathbf{y}_1) + K_2(\mathbf{x}_2, \mathbf{y}_2),$$

- The **direct product**:

$$K((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) = K_1(\mathbf{x}_1, \mathbf{y}_1) K_2(\mathbf{x}_2, \mathbf{y}_2).$$

## LA kernel is p.d.: proof (4/11)

### Proof of lemma

If  $K_1$  is a p.d. kernel, let  $\Phi_1 : \mathcal{X}_1 \mapsto \mathcal{H}$  be such that:

$$K_1(\mathbf{x}_1, \mathbf{y}_1) = \langle \Phi_1(\mathbf{x}_1), \Phi_1(\mathbf{y}_1) \rangle_{\mathcal{H}}.$$

Let  $\Phi : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{H}$  be defined by:

$$\Phi((\mathbf{x}_1, \mathbf{x}_2)) = \Phi_1(\mathbf{x}_1).$$

Then for  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$  and  $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \mathcal{X}$ , we get

$$\langle \Phi((\mathbf{x}_1, \mathbf{x}_2)), \Phi((\mathbf{y}_1, \mathbf{y}_2)) \rangle_{\mathcal{H}} = K_1(\mathbf{x}_1, \mathbf{x}_2),$$

which shows that  $K(\mathbf{x}, \mathbf{y}) := K_1(\mathbf{x}_1, \mathbf{y}_1)$  is p.d. on  $\mathcal{X}_1 \times \mathcal{X}_2$ . The lemma follows from the properties of sums and products of p.d. kernels.  $\square$

## LA kernel is p.d.: proof (5/11)

Lemma: kernel for sets

Let  $K$  be a p.d. kernel on  $\mathcal{X}$ , and let  $\mathcal{P}(\mathcal{X})$  be the set of **finite subsets** of  $\mathcal{X}$ . Then the function  $K_P$  on  $\mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$  defined by:

$$\forall A, B \in \mathcal{P}(\mathcal{X}), \quad K_P(A, B) := \sum_{x \in A} \sum_{y \in B} K(x, y)$$

is a p.d. kernel on  $\mathcal{P}(\mathcal{X})$ .

## LA kernel is p.d.: proof (6/11)

### Proof of lemma

Let  $\Phi : \mathcal{X} \mapsto \mathcal{H}$  be such that

$$K(\mathbf{x}, \mathbf{y}) = \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}}.$$

Then, for  $A, B \in \mathcal{P}(\mathcal{X})$ , we get:

$$\begin{aligned} K_P(A, B) &= \sum_{\mathbf{x} \in A} \sum_{\mathbf{y} \in B} \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}} \\ &= \left\langle \sum_{\mathbf{x} \in A} \Phi(\mathbf{x}), \sum_{\mathbf{y} \in B} \Phi(\mathbf{y}) \right\rangle_{\mathcal{H}} \\ &= \langle \Phi_P(A), \Phi_P(B) \rangle_{\mathcal{H}}, \end{aligned}$$

with  $\Phi_P(A) := \sum_{\mathbf{x} \in A} \Phi(\mathbf{x})$ .  $\square$

## LA kernel is p.d.: proof (7/11)

### Definition: Convolution kernel (Haussler, 1999)

Let  $K_1$  and  $K_2$  be two p.d. kernels for strings. The **convolution** of  $K_1$  and  $K_2$ , denoted  $K_1 \star K_2$ , is defined for any  $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$  by:

$$K_1 \star K_2(\mathbf{x}, \mathbf{y}) := \sum_{\mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}, \mathbf{y}_1 \mathbf{y}_2 = \mathbf{y}} K_1(\mathbf{x}_1, \mathbf{y}_1) K_2(\mathbf{x}_2, \mathbf{y}_2).$$

### Lemma

If  $K_1$  and  $K_2$  are p.d. then  $K_1 \star K_2$  is p.d..

## LA kernel is p.d.: proof (8/11)

### Proof of lemma

Let  $\mathcal{X}$  be the set of finite-length strings. For  $\mathbf{x} \in \mathcal{X}$ , let

$$R(\mathbf{x}) = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathcal{X} \times \mathcal{X} : \mathbf{x} = \mathbf{x}_1 \mathbf{x}_2\} \subset \mathcal{X} \times \mathcal{X}.$$

We can then write

$$K_1 * K_2(\mathbf{x}, \mathbf{y}) = \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in R(\mathbf{x})} \sum_{(\mathbf{y}_1, \mathbf{y}_2) \in R(\mathbf{y})} K_1(\mathbf{x}_1, \mathbf{y}_1) K_2(\mathbf{x}_2, \mathbf{y}_2)$$

which is a p.d. kernel by the previous lemmas.  $\square$

## LA kernel is p.d.: proof (9/11)

### 3 basic string kernels

- The constant kernel:

$$K_0(\mathbf{x}, \mathbf{y}) := 1.$$

- A kernel for letters:

$$K_a^{(\beta)}(\mathbf{x}, \mathbf{y}) := \begin{cases} 0 & \text{if } |\mathbf{x}| \neq 1 \text{ where } |\mathbf{y}| \neq 1, \\ \exp(\beta S(\mathbf{x}, \mathbf{y})) & \text{otherwise.} \end{cases}$$

- A kernel for gaps:

$$K_g^{(\beta)}(\mathbf{x}, \mathbf{y}) = \exp[\beta(g(|\mathbf{x}|) + g(|\mathbf{y}|))].$$

## LA kernel is p.d.: proof (10/11)

### Remark

- $S : \mathcal{A}^2 \rightarrow \mathbb{R}$  is the similarity function between letters used in the alignment score.  $K_a^{(\beta)}$  is only p.d. when the matrix:

$$(\exp(\beta s(a, b)))_{(a,b) \in \mathcal{A}^2}$$

is positive semidefinite (this is true for all  $\beta$  when  $s$  is **conditionally p.d.**)

- $g$  is the gap penalty function used in alignment score. **The gap kernel is always p.d.** (with no restriction on  $g$ ) because it can be written as:

$$K_g^{(\beta)}(\mathbf{x}, \mathbf{y}) = \exp(\beta g(|\mathbf{x}|)) \times \exp(\beta g(|\mathbf{y}|)) .$$

## LA kernel is p.d.: proof (11/11)

### Lemma

The local alignment kernel is a (limit) of convolution kernel:

$$K_{LA}^{(\beta)} = \sum_{n=0}^{\infty} K_0 \star \left( K_a^{(\beta)} \star K_g^{(\beta)} \right)^{(n-1)} \star K_a^{(\beta)} \star K_0.$$

As such it is p.d..

### Proof (sketch)

- By induction on  $n$  (simple but long to write).
- See details in Vert et al. (2004).

## LA kernel computation

- We assume an **affine gap penalty**:

$$\begin{cases} g(0) &= 0, \\ g(n) &= d + e(n - 1) \text{ si } n \geq 1, \end{cases}$$

- The LA kernel can then be computed by **dynamic programming** by:

$$K_{LA}^{(\beta)}(\mathbf{x}, \mathbf{y}) = 1 + X_2(|\mathbf{x}|, |\mathbf{y}|) + Y_2(|\mathbf{x}|, |\mathbf{y}|) + M(|\mathbf{x}|, |\mathbf{y}|),$$

where  $M(i, j)$ ,  $X(i, j)$ ,  $Y(i, j)$ ,  $X_2(i, j)$ , and  $Y_2(i, j)$  for  $0 \leq i \leq |\mathbf{x}|$ , and  $0 \leq j \leq |\mathbf{y}|$  are defined recursively.

## LA kernel is p.d.: proof (/)

### Initialization

$$\begin{cases} M(i, 0) = M(0, j) = 0, \\ X(i, 0) = X(0, j) = 0, \\ Y(i, 0) = Y(0, j) = 0, \\ X_2(i, 0) = X_2(0, j) = 0, \\ Y_2(i, 0) = Y_2(0, j) = 0, \end{cases}$$

## LA kernel is p.d.: proof (/)

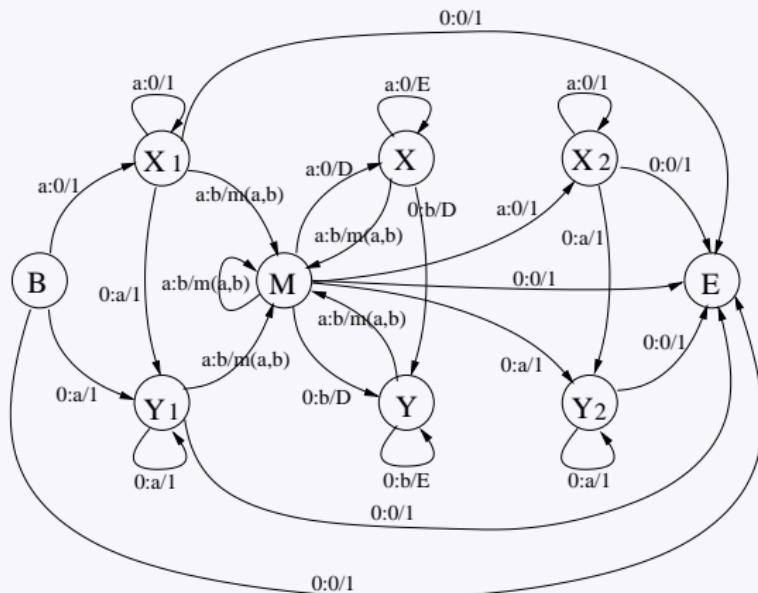
### Recursion

For  $i = 1, \dots, |\mathbf{x}|$  and  $j = 1, \dots, |\mathbf{y}|$ :

$$\begin{cases} M(i,j) &= \exp(\beta S(x_i, y_j)) \left[ 1 + X(i-1, j-1) \right. \\ &\quad \left. + Y(i-1, j-1) + M(i-1, j-1) \right], \\ X(i,j) &= \exp(\beta d) M(i-1, j) + \exp(\beta e) X(i-1, j), \\ Y(i,j) &= \exp(\beta d) [M(i, j-1) + X(i, j-1)] \\ &\quad + \exp(\beta e) Y(i, j-1), \\ X_2(i,j) &= M(i-1, j) + X_2(i-1, j), \\ Y_2(i,j) &= M(i, j-1) + X_2(i, j-1) + Y_2(i, j-1). \end{cases}$$

## LA kernel in practice

- Implementation by a finite-state transducer in  $O(|\mathbf{x}| \times |\mathbf{x}'|)$



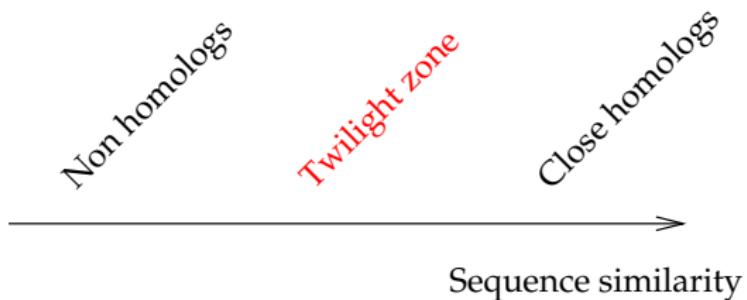
- In practice, **values are too large** (exponential scale) so taking its logarithm is a safer choice (but not p.d. anymore!)

# Outline

## 5 The Kernel Jungle

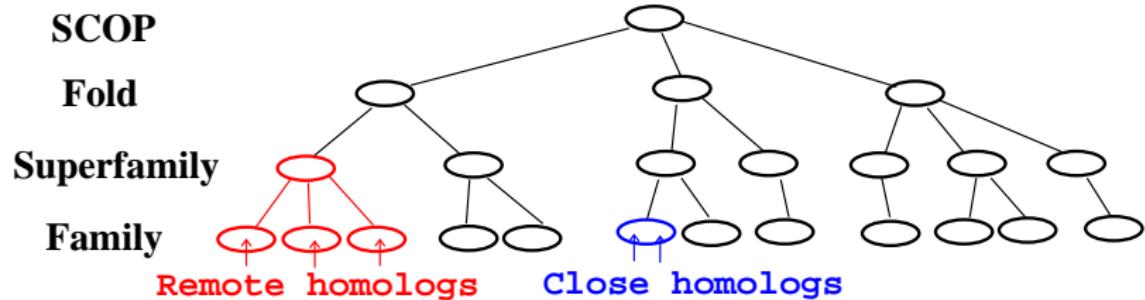
- Kernels for probabilistic models
- **Kernels for biological sequences**
  - Motivations and history of genomics
  - Kernels derived from large feature spaces
  - Kernels derived from generative models
  - Kernels derived from a similarity measure
  - **Application to remote homology detection**
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- Kernels on graphs

## Remote homology



- Homologs have **common ancestors**
- Structures and functions are more conserved than sequences
- **Remote homologs** can not be detected by direct sequence comparison

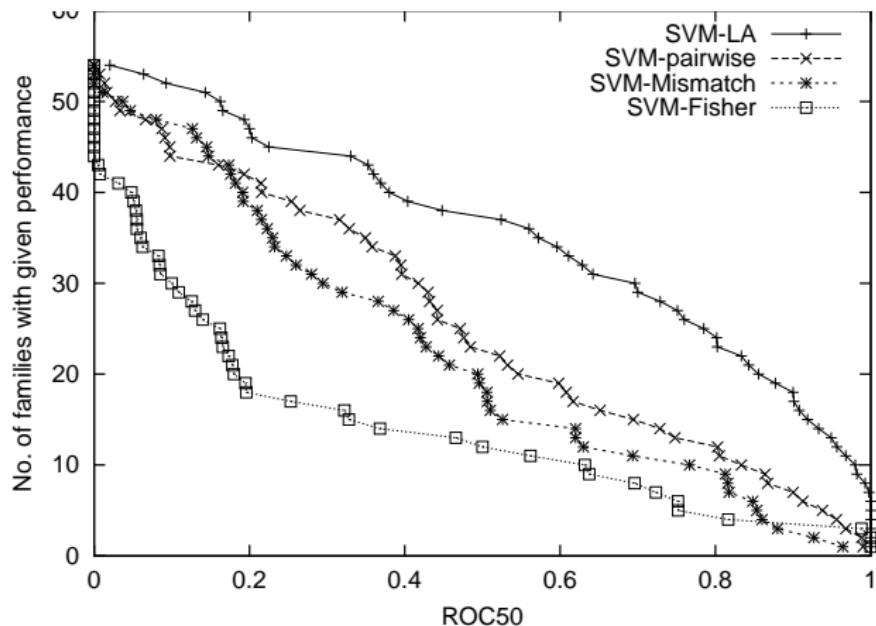
# SCOP database



## A benchmark experiment

- **Goal:** recognize directly the superfamily
- **Training:** for a sequence of interest, positive examples come from the same superfamily, but different families. Negative from other superfamilies.
- **Test:** predict the superfamily.

## Difference in performance



Performance on the SCOP superfamily recognition benchmark (from Saigo et al., 2004).

## String kernels: Summary

- A variety of principles for string kernel design have been proposed.
- Good **kernel design** is **important** for each data and each task.  
Performance is not the only criterion.
- Still an **art**, although principled ways have started to emerge.
- **Fast implementation** with string algorithms is often possible.
- Their application goes well beyond computational biology.

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
  - Kernels for probabilistic models
  - Kernels for biological sequences
  - **Mercer kernels and shift-invariant kernels**
  - Kernels for graphs
  - Kernels on graphs
- 6 Open Problems and Research Topics

## Motivations

- The RKHS norm is related to the **smoothness** of functions.
- Smoothness of a function is naturally quantified by **Sobolev norms** (in particular  $L_2$  norms of derivatives), or by the decay of the Fourier transform.
- In this section, we introduce several kernels where this link is explicit, and we make a general link between RKHS and **Green functions defined by differential operators**.

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- **Mercer kernels and shift-invariant kernels**
  - Shift-invariant kernels
    - Generalization to semigroups
    - Mercer kernels
    - RKHS and Green functions
  - Kernels for graphs
  - Kernels on graphs

# Translation invariant kernels

## Definition

A kernel  $K : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$  is called **translation invariant** (t.i.) if it only depends on the difference between its argument, i.e.:

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}, \quad K(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x} - \mathbf{y}).$$

## Examples

- Gaussian kernel (or RBF kernel)

$$K(\mathbf{x}, \mathbf{y}) = e^{-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{y}\|_2^2}.$$

- Laplace kernel

$$K(\mathbf{x}, \mathbf{y}) = e^{-\alpha \|\mathbf{x} - \mathbf{y}\|_1}.$$

## In case of...

### Definition

Let  $f \in L^1(\mathbb{R}^d)$ . The **Fourier transform** of  $f$ , denoted  $\hat{f}$  or  $\mathcal{F}[f]$ , is the function defined for all  $\omega \in \mathbb{R}^d$  by:

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\omega} f(\mathbf{x}) d\mathbf{x}.$$

## In case of...

### Properties

- $\hat{f}$  is complex-valued, continuous, tends to 0 at infinity and  $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ .
- If  $\hat{f} \in L^1(\mathbb{R}^d)$ , then the **inverse Fourier formula** holds:

$$\forall x \in \mathbb{R}^d, \quad f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \omega} \hat{f}(\omega) d\omega.$$

- If  $f \in L^1(\mathbb{R}^d)$  is square integrable, then **Parseval's formula** holds:

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 d\omega.$$

# Translation invariant kernels

## Definition

A kernel  $K : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$  is called **translation invariant** (t.i.) if it only depends on the difference between its argument, i.e.:

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}, \quad K(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x} - \mathbf{y}).$$

## Intuition

If  $K$  is t.i. and  $\kappa \in L^1(\mathbb{R}^d)$ , then

$$\begin{aligned}\kappa(\mathbf{x} - \mathbf{y}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y}) \cdot \omega} \hat{\kappa}(\omega) d\omega \\ &= \int_{\mathbb{R}^d} \frac{\hat{\kappa}(\omega)}{(2\pi)^d} e^{i\omega \cdot \mathbf{x}} e^{-i\omega \cdot \mathbf{y}} d\omega.\end{aligned}$$

# Characterization of p.d. t.i. kernels

## Theorem (Bochner)

A real-valued continuous function  $\kappa(\mathbf{x} - \mathbf{y})$  on  $\mathbb{R}^d$  is positive definite if and only if it is the Fourier-Stieltjes transform of a **symmetric, positive, and finite** Borel measure  $\mu$ :

$$\kappa(\mathbf{z}) = \int_{\mathbb{R}^d} e^{i\mathbf{z}\cdot\omega} \mu(d\omega).$$

## Remarks

- If  $\kappa(0) = 1$ ,  $\kappa$  is a characteristic function—that is,  $\kappa(\mathbf{z}) = \mathbb{E}_{\omega}[e^{i\mathbf{z}\cdot\omega}]$ .
- $\Leftarrow$  is easy:

$$\sum_{k,l} \alpha_k \alpha_l \kappa(\mathbf{x}_k - \mathbf{x}_l) = \int_{\mathbb{R}^d} \left| \sum_k \alpha_k e^{i\mathbf{x}_k \cdot \omega} \right|^2 \mu(d\omega) \geq 0.$$

# RKHS of translation invariant kernels

## Theorem

Let  $K$  be a translation invariant p.d. kernel, such that  $\kappa$  is integrable on  $\mathbb{R}^d$  as well as its Fourier transform  $\hat{\kappa}$ . The subset  $\mathcal{H}_K$  of  $L_2(\mathbb{R}^d)$  that consists of integrable and continuous functions  $f$  such that:

$$\|f\|_K^2 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2}{\hat{\kappa}(\omega)} d\omega < +\infty,$$

endowed with the inner product:

$$\langle f, g \rangle := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\omega) \hat{g}(\omega)^*}{\hat{\kappa}(\omega)} d\omega$$

is a RKHS with  $K$  as r.k.

## Proof

$\mathcal{H}_K$  is a Hilbert space: exercise.

For  $\mathbf{x} \in \mathbb{R}^d$ ,  $K_{\mathbf{x}}(\mathbf{y}) = K(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x} - \mathbf{y})$  therefore:

$$\hat{K}_{\mathbf{x}}(\omega) = \int e^{-i\omega \cdot \mathbf{u}} \kappa(\mathbf{u} - \mathbf{x}) d\mathbf{u} = e^{-i\omega \cdot \mathbf{x}} \hat{\kappa}(\omega).$$

This leads to  $K_{\mathbf{x}} \in \mathcal{H}$ , because:

$$\int_{\mathbb{R}^d} \frac{|\hat{K}_{\mathbf{x}}(\omega)|^2}{\hat{\kappa}(\omega)} \leq \int_{\mathbb{R}^d} |\hat{\kappa}(\omega)| < \infty,$$

Moreover, if  $f \in \mathcal{H}$  and  $\mathbf{x} \in \mathbb{R}^d$ , we have:

$$\langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{K}_{\mathbf{x}}(\omega) \hat{f}(\omega)^*}{\hat{\kappa}(\omega)} d\omega = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\omega)^* e^{-i\omega \cdot \mathbf{x}} = f(\mathbf{x})$$

## Example

Gaussian kernel

$$K(x, y) = e^{-\frac{(x-y)^2}{2\sigma^2}}$$

corresponds to:

$$\hat{\kappa}(\omega) = e^{-\frac{\sigma^2 \omega^2}{2}}$$

and

$$\mathcal{H} = \left\{ f : \int \left| \hat{f}(\omega) \right|^2 e^{\frac{\sigma^2 \omega^2}{2}} d\omega < \infty \right\}.$$

In particular, all functions in  $\mathcal{H}$  are **infinitely differentiable** with all derivatives in  $L^2$ .

## Example

### Laplace kernel

$$K(x, y) = \frac{1}{2} e^{-\gamma|x-y|}$$

corresponds to:

$$\hat{\kappa}(\omega) = \frac{\gamma}{\gamma^2 + \omega^2}$$

and

$$\mathcal{H} = \left\{ f : \int \left| \hat{f}(\omega) \right|^2 \frac{(\gamma^2 + \omega^2)}{\gamma} d\omega < \infty \right\},$$

the set of functions  $L^2$  differentiable with derivatives in  $L^2$  (Sobolev norm).

## Example

Low-frequency filter

$$K(x, y) = \frac{\sin(\Omega(x - y))}{\pi(x - y)}$$

corresponds to:

$$\hat{\kappa}(\omega) = U(\omega + \Omega) - U(\omega - \Omega)$$

and

$$\mathcal{H} = \left\{ f : \int_{|\omega| > \Omega} |\hat{f}(\omega)|^2 d\omega = 0 \right\},$$

the set of functions whose spectrum is included in  $[-\Omega, \Omega]$ .

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- **Mercer kernels and shift-invariant kernels**
  - Shift-invariant kernels
  - **Generalization to semigroups**
  - Mercer kernels
  - RKHS and Green functions
- Kernels for graphs
- Kernels on graphs

# Generalization to semigroups (cf Berg et al., 1983)

## Definition

- A **semigroup**  $(S, \circ)$  is a nonempty set  $S$  equipped with an associative composition  $\circ$  and a neutral element  $e$ .
- A **semigroup with involution**  $(S, \circ, *)$  is a semigroup  $(S, \circ)$  together with a mapping  $* : S \rightarrow S$  called **involution** satisfying:
  - ①  $(s \circ t)^* = t^* \circ s^*$ , for  $s, t \in S$ .
  - ②  $(s^*)^* = s$  for  $s \in S$ .

## Examples

- Any **group**  $(G, \circ)$  is a semigroup with involution when we define  $s^* = s^{-1}$ .
- Any **abelian semigroup**  $(S, +)$  is a semigroup with involution when we define  $s^* = s$ , the **identical involution**.

# Positive definite functions on semigroups

## Definition

Let  $(S, \circ, *)$  be a semigroup with involution. A function  $\varphi : S \rightarrow \mathbb{R}$  is called **positive definite** if the function:

$$\forall s, t \in S, \quad K(s, t) = \varphi(s^* \circ t)$$

is a p.d. kernel on  $S$ .

## Example: translation invariant kernels

$(\mathbb{R}^d, +, -)$  is an abelian group with involution. A function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  is p.d. if the function

$$K(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x} - \mathbf{y})$$

is p.d. on  $\mathbb{R}^d$  (translation invariant kernels).

# Semicharacters

## Definition

A function  $\rho : S \rightarrow \mathbb{C}$  on an **abelian** semigroup with involution  $(S, +, *)$  is called a **semicharacter** if

- ①  $\rho(0) = 1$ ,
- ②  $\rho(s + t) = \rho(s)\rho(t)$  for  $s, t \in S$ ,
- ③  $\rho(s^*) = \overline{\rho(s)}$  for  $s \in S$ .

The set of semicharacters on  $S$  is denoted by  $S^*$ .

## Remarks

- If  $*$  is the **identity**, a semicharacter is automatically **real-valued**.
- If  $(S, +)$  is an **abelian group** and  $s^* = -s$ , a semicharacter has its values in the circle group  $\{z \in \mathbb{C} \mid |z| = 1\}$  and is a **group character**.

# Semicharacters are p.d.

## Lemma

Every semicharacter is p.d., in the sense that:

- $K(s, t) = \overline{K(t, s)}$ ,
- $\sum_{i,j=1}^n a_i \overline{a_j} K(x_i, x_j) \geq 0$ .

## Proof

Direct from definition, e.g.,

$$\sum_{i,j=1}^n a_i \overline{a_j} \rho(x_i + x_j^*) = \sum_{i,j=1}^n a_i \overline{a_j} \rho(x_i) \overline{\rho(x_j)} \geq 0.$$

## Examples

- $\varphi(t) = e^{\beta t}$  on  $(\mathbb{R}, +, Id)$ .
- $\varphi(t) = e^{i\omega t}$  on  $(\mathbb{R}, +, -)$ .

# Integral representation of p.d. functions

## Definition

- An function  $\alpha : S \rightarrow \mathbb{R}$  on a semigroup with involution is called an **absolute value** if (i)  $\alpha(e) = 1$ , (ii)  $\alpha(s \circ t) \leq \alpha(s)\alpha(t)$ , and (iii)  $\alpha(s^*) = \alpha(s)$ .
- A function  $f : S \rightarrow \mathbb{R}$  is called **exponentially bounded** if there exists an absolute value  $\alpha$  and a constant  $C > 0$  s.t.  $|f(s)| \leq C\alpha(s)$  for  $s \in S$ .

## Theorem

Let  $(S, +, *)$  an abelian semigroup with involution. A function  $\varphi : S \rightarrow \mathbb{R}$  is p.d. and exponentially bounded (resp. bounded) if and only if it has a representation of the form:

$$\varphi(s) = \int_{S^*} \rho(s)d\mu(\rho).$$

where  $\mu$  is a Radon measure with compact support on  $S^*$  (resp. on  $\hat{S}$ , the set of bounded semicharacters).

## Proof

Sketch (details in Berg et al., 1983, Theorem 4.2.5)

- For an absolute value  $\alpha$ , the set  $P_1^\alpha$  of  $\alpha$ -bounded p.d. functions that satisfy  $\varphi(0) = 1$  is a **compact convex set** whose **extreme points are precisely the  $\alpha$ -bounded semicharacters**.
- If  $\varphi$  is p.d. and exponentially bounded then there exists an absolute value  $\alpha$  such that  $\varphi(0)^{-1}\varphi \in P_1^\alpha$ .
- By the **Krein-Milman theorem** there exists a Radon probability measure on  $P_1^\alpha$  having  $\varphi(0)^{-1}\varphi$  as **barycentre**.

## Remarks

- The result is **not true** without the assumption of **exponentially bounded** semicharacters.
- In the case of abelian groups with  $s^* = -s$  this reduces to **Bochner's theorem** for discrete abelian groups, cf. Rudin (1962).

## Example 1: $(R_+, +, Id)$

### Semicharacters

- $S = (\mathbb{R}_+, +, Id)$  is an **abelian semigroup**.
- P.d. functions are **nonnegative**, because  $\varphi(x) = \varphi(\sqrt{x})^2$ .
- The set of **bounded semicharacters** is exactly the set of functions:

$$s \in \mathbb{R}_+ \mapsto \rho_a(s) = e^{-as},$$

for  $a \in [0, +\infty]$  (left as exercise).

- **Non-bounded** semicharacters are more difficult to characterize; in fact there exist nonmeasurable solutions of the equation  $h(x+y) = h(x)h(y)$ .

## Example 1: $(R_+, +, Id)$ (cont.)

### P.d. functions

- By the integral representation theorem for bounded semi-characters we obtain that a function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  is p.d. and bounded if and only if it has the form:

$$\varphi(s) = \int_0^\infty e^{-as} d\mu(a) + b\rho_\infty(s)$$

where  $\mu \in \mathcal{M}_+^b(\mathbb{R}_+)$  and  $b \geq 0$ .

- The first term is the Laplace transform of  $\mu$ .  $\varphi$  is p.d., bounded and continuous iff it is the Laplace transform of a measure in  $\mathcal{M}_+^b(\mathbb{R})$ .

## Example 2: Semigroup kernels for finite measures (1/6)

### Setting

- We assume that data to be processed are “**bags-of-points**”, i.e., sets of points (with repeats) of a space  $\mathcal{U}$ .
- **Example** : a finite-length string as a set of  $k$ -mers.
- How to define a p.d. kernel between any two bags that only depends on the **union of the bags**?
- See details and proofs in Cuturi et al. (2005).

## Example 2: Semigroup kernels for finite measures (2/6)

### Semigroup of bounded measures

- We can represent any bag-of-point  $\mathbf{x}$  as a finite measure on  $\mathcal{U}$ :

$$\mathbf{x} = \sum_i a_i \delta_{x_i},$$

where  $a_i$  is the number of occurrences on  $x_i$  in the bag.

- The measure that represents the union of two bags is the **sum of the measures** that represent each individual bag.
- This suggests to look at the semigroup  $(\mathcal{M}_+^b(\mathcal{U}), +, \text{Id})$  of bounded Radon measures on  $\mathcal{U}$  and to search for p.d. functions  $\varphi$  on this semigroup.

## Example 2: Semigroup kernels for finite measures (3/6)

### Semicharacters

- For any Borel measurable function  $f : \mathcal{U} \rightarrow \mathbb{R}$  the function  $\rho_f : \mathcal{M}_+^b(\mathcal{U}) \rightarrow \mathbb{R}$  defined by:

$$\rho_f(\mu) = e^{\mu[f]}$$

is a semicharacter on  $(\mathcal{M}_+^b(\mathcal{U}), +)$ .

- Conversely,  $\rho$  is **continuous** semicharacter (for the topology of weak convergence) if and only if there exists a continuous function  $f : \mathcal{U} \rightarrow \mathbb{R}$  such that  $\rho = \rho_f$ .
- No such characterization for non-continuous characters, even bounded.

## Example 2: Semigroup kernels for finite measures (4/6)

### Corollary

Let  $\mathcal{U}$  be a Hausdorff space. For any Radon measure  $\mu \in \mathcal{M}_+^c(C(\mathcal{U}))$  with compact support on the Hausdorff space of continuous real-valued functions on  $\mathcal{U}$  endowed with the topology of pointwise convergence, the following function  $K$  is a continuous p.d. kernel on  $\mathcal{M}_+^b(\mathcal{U})$  (endowed with the topology of weak convergence):

$$K(\mu, \nu) = \int_{C(X)} e^{\mu[f] + \nu[f]} d\mu(f).$$

### Remarks

The converse is not true: there exist continuous p.d. kernels that do not have this integral representation (it might include non-continuous semicharacters)

## Example 2: Semigroup kernels for finite measures (5/6)

### Example : entropy kernel

- Let  $\mathcal{X}$  be the set of probability densities (w.r.t. some reference measure) on  $\mathcal{U}$  with finite entropy:

$$h(\mathbf{x}) = - \int_{\mathcal{U}} \mathbf{x} \ln \mathbf{x}.$$

- Then the following **entropy kernel** is a p.d. kernel on  $\mathcal{X}$  for all  $\beta > 0$ :

$$K(\mathbf{x}, \mathbf{x}') = e^{-\beta h\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)}.$$

- Remark: only valid for **densities** (e.g., for a kernel density estimator from a bag-of-parts)

## Example 2: Semigroup kernels for finite measures (6/6)

Examples : inverse generalized variance kernel

- Let  $\mathcal{U} = \mathbb{R}^d$  and  $\mathcal{M}_+^V(\mathcal{U})$  be the set of finite measure  $\mu$  with second order moment and non-singular variance

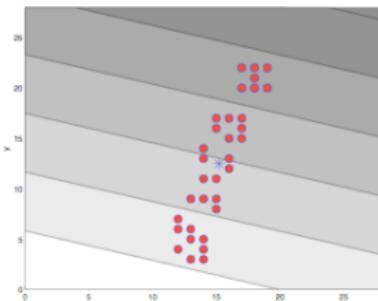
$$\Sigma(\mu) = \mu [xx^\top] - \mu [x]\mu [x]^\top.$$

- Then the following function is a p.d. kernel on  $\mathcal{M}_+^V(\mathcal{U})$ , called the **inverse generalized variance kernel**:

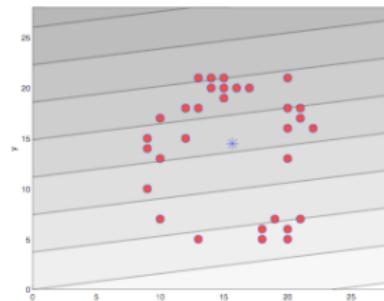
$$K(\mu, \mu') = \frac{1}{\det \Sigma\left(\frac{\mu+\mu'}{2}\right)}.$$

- Generalization possible with **regularization** and **kernel trick**.

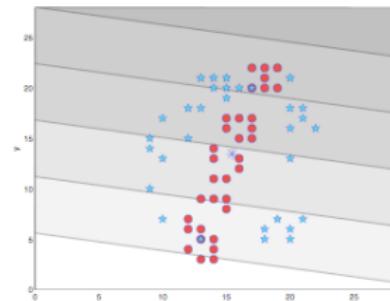
# Application of semigroup kernel



$$\Sigma_{1,1} = 0.0552$$
$$\Sigma_{2,2} = 0.0013$$



$$\Sigma'_{1,1} = 0.0441$$
$$\Sigma'_{2,2} = 0.0237$$



$$\Sigma''_{1,1} = 0.0497$$
$$\Sigma''_{2,2} = 0.0139$$

Weighted linear PCA of two different measures, with the first PC shown. Variances captured by the first and second PC are shown. The generalized variance kernel is the inverse of the product of the two values.

# Kernelization of the IGV kernel

## Motivations

- Gaussian distributions may be poor models.
- The method fails in large dimension

## Solution

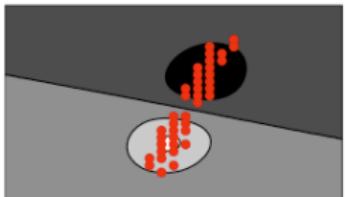
- ① Regularization:

$$K_\lambda(\mu, \mu') = \frac{1}{\det\left(\Sigma\left(\frac{\mu+\mu'}{2}\right) + \lambda I_d\right)}.$$

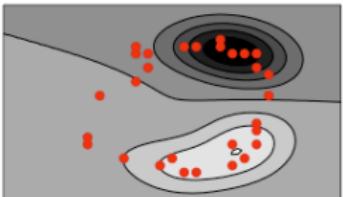
- ② Kernel trick: the non-zero eigenvalues of  $UU^\top$  and  $U^\top U$  are the same  $\implies$  replace the covariance matrix by the centered Gram matrix (technical details in Cuturi et al., 2005).

# Illustration of kernel IGV kernel

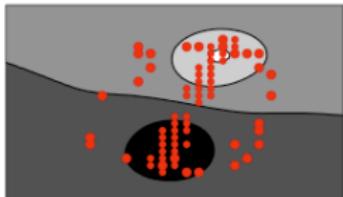
0.276



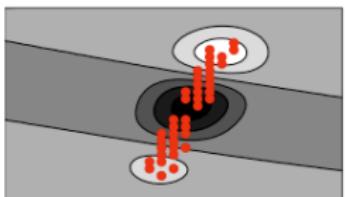
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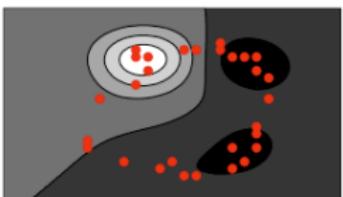
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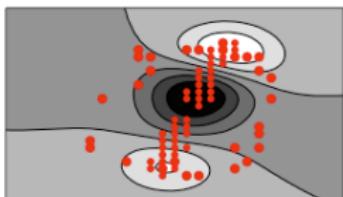
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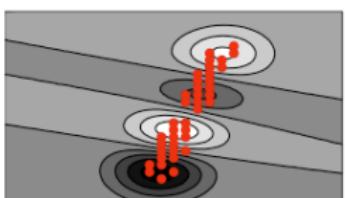
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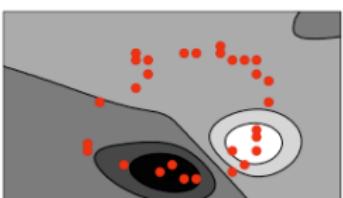
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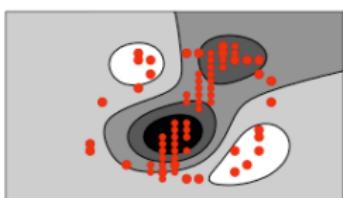
0.124



0.119



0.0934



# Semigroup kernel remarks

## Motivations

- A **very general formalism** to exploit an **algebraic structure** of the data.
- Kernel IVG kernel has given good results for character recognition from a subsampled image.
- The main motivation is more generally to develop kernels for **complex objects** from which **simple “patches”** can be extracted.
- The extension to **nonabelian groups** (e.g., permutation in the symmetric group) might find natural applications.

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- **Mercer kernels and shift-invariant kernels**
  - Shift-invariant kernels
  - Generalization to semigroups
  - **Mercer kernels**
  - RKHS and Green functions
- Kernels for graphs
- Kernels on graphs

# Mercer kernels

## Definition

A kernel  $K$  on a set  $\mathcal{X}$  is called a **Mercer kernel** if:

- ①  $\mathcal{X}$  is a **compact metric space** (typically, a closed bounded subset of  $\mathbb{R}^d$ ).
- ②  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  is a **continuous** p.d. kernel (w.r.t. the Borel topology)

## Motivations

- We can exhibit an explicit and intuitive feature space for a large class of p.d. kernels
- Historically, provided the first proof that a p.d. kernel is an inner product for non-finite sets  $\mathcal{X}$  (Mercer, 1905).
- Can be thought of as the natural generalization of the factorization of positive semidefinite matrices over infinite spaces.

## Sketch of the proof

- ① The kernel matrix when  $\mathcal{X}$  is finite becomes a **linear operator** when  $\mathcal{X}$  is a metric space.
- ② The matrix was positive semidefinite in the finite case, the linear operator is **self-adjoint** and **positive** in the metric case.
- ③ The **spectral theorem** states that any **compact** linear operator admits a complete orthonormal basis of eigenfunctions, with non-negative eigenvalues (just like positive semidefinite matrices can be diagonalized with nonnegative eigenvalues).
- ④ The kernel function can then be expanded over basis of eigenfunctions as:

$$K(\mathbf{x}, \mathbf{t}) = \sum_{k=1}^{\infty} \lambda_k \psi_k(\mathbf{x}) \psi_k(\mathbf{t}),$$

where  $\lambda_i \geq 0$  are the non-negative eigenvalues.

## In case of...

### Definition

Let  $\mathcal{H}$  be a Hilbert space

- A **linear operator** is a continuous linear mapping from  $\mathcal{H}$  to itself.
- A linear operator  $L$  is called **compact** if, for any bounded sequence  $\{f_n\}_{n=1}^{\infty}$ , the sequence  $\{Lf_n\}_{n=1}^{\infty}$  has a subsequence that converges.
- $L$  is called **self-adjoint** if, for any  $f, g \in \mathcal{H}$ :

$$\langle f, Lg \rangle = \langle Lf, g \rangle .$$

- $L$  is called **positive** if it is self-adjoint and, for any  $f \in \mathcal{H}$ :

$$\langle f, Lf \rangle \geq 0 .$$

# An important lemma

## The linear operator

- Let  $\nu$  be any Borel measure on  $\mathcal{X}$ , and  $L_2^\nu(\mathcal{X})$  the Hilbert space of square integrable functions on  $\mathcal{X}$ .
- For any function  $K : \mathcal{X}^2 \mapsto \mathbb{R}$ , let the transform:

$$\forall f \in L_2^\nu(\mathcal{X}), \quad (L_K f)(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{t}) f(\mathbf{t}) d\nu(\mathbf{t}).$$

## Lemma

If  $K$  is a Mercer kernel, then  $L_K$  is a compact and bounded linear operator over  $L_2^\nu(\mathcal{X})$ , self-adjoint and positive.

## Proof (1/6)

$L_K$  is a mapping from  $L_2^\nu(\mathcal{X})$  to  $L_2^\nu(\mathcal{X})$

For any  $f \in L_2^\nu(\mathcal{X})$  and  $(\mathbf{x}_1, \mathbf{x}_2) \in \mathcal{X}^2$ :

$$\begin{aligned}|L_K f(\mathbf{x}_1) - L_K f(\mathbf{x}_2)| &= \left| \int (K(\mathbf{x}_1, \mathbf{t}) - K(\mathbf{x}_2, \mathbf{t})) f(\mathbf{t}) d\nu(\mathbf{t}) \right| \\&\leq \|K(\mathbf{x}_1, \cdot) - K(\mathbf{x}_2, \cdot)\| \|f\| \quad (\text{Cauchy-Schwarz}) \\&\leq \sqrt{\nu(\mathcal{X})} \max_{\mathbf{t} \in \mathcal{X}} |K(\mathbf{x}_1, \mathbf{t}) - K(\mathbf{x}_2, \mathbf{t})| \|f\|.\end{aligned}$$

$K$  being continuous and  $\mathcal{X}$  compact,  $K$  is uniformly continuous, therefore  $L_K f$  is continuous. In particular,  $L_K f \in L_2^\nu(\mathcal{X})$  (with the slight abuse of notation  $\mathcal{C}(\mathcal{X}) \subset L_2^\nu(\mathcal{X})$ ).  $\square$

## Proof (2/6)

$L_K$  is linear and continuous

- Linearity is obvious (by definition of  $L_K$  and linearity of the integral).
- For continuity, we observe that for all  $f \in L_2^\nu(\mathcal{X})$  and  $\mathbf{x} \in \mathcal{X}$ :

$$\begin{aligned}|(L_K f)(\mathbf{x})| &= \left| \int K(\mathbf{x}, \mathbf{t}) f(\mathbf{t}) d\nu(\mathbf{t}) \right| \\&\leq \sqrt{\nu(\mathcal{X})} \max_{\mathbf{t} \in \mathcal{X}} |K(\mathbf{x}, \mathbf{t})| \|f\| \\&\leq \sqrt{\nu(\mathcal{X})} C_K \|f\|.\end{aligned}$$

with  $C_K = \max_{\mathbf{x}, \mathbf{t} \in \mathcal{X}} |K(\mathbf{x}, \mathbf{t})|$ . Therefore:

$$\|L_K f\| = \left( \int L_K f(\mathbf{t})^2 d\nu(\mathbf{t}) \right)^{\frac{1}{2}} \leq \sqrt{\nu(\mathcal{X})} C_K \|f\|. \quad \square$$

## Proof (3/6)

### Criterion for compactness

In order to prove the compactness of  $L_K$  we need the following criterion.  
Let  $C(\mathcal{X})$  denote the set of continuous functions on  $\mathcal{X}$  endowed with infinite norm  $\|f\|_\infty = \max_{x \in \mathcal{X}} |f(x)|$ .

A set of functions  $G \subset C(\mathcal{X})$  is called **equicontinuous** if:

$$\forall \epsilon > 0, \exists \delta > 0, \forall (\mathbf{x}, \mathbf{y}) \in \mathcal{X}^2,$$

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \forall g \in G, |g(\mathbf{x}) - g(\mathbf{y})| < \epsilon.$$

### Ascoli Theorem

A part  $H \subset C(\mathcal{X})$  is **relatively compact** (i.e., its closure is compact) if and only if it is **uniformly bounded** and **equicontinuous**.

## Proof (4/6)

$L_K$  is compact

Let  $(f_n)_{n \geq 0}$  be a bounded sequence of  $L_2^\nu(\mathcal{X})$  ( $\|f_n\| < M$  for all  $n$ ).  
The sequence  $(L_K f_n)_{n \geq 0}$  is a sequence of continuous functions, uniformly bounded because:

$$\|L_K f_n\|_\infty \leq \sqrt{\nu(\mathcal{X})} C_K \|f_n\| \leq \sqrt{\nu(\mathcal{X})} C_K M.$$

It is equicontinuous because:

$$|L_K f_n(\mathbf{x}_1) - L_K f_n(\mathbf{x}_2)| \leq \sqrt{\nu(\mathcal{X})} \max_{\mathbf{t} \in \mathcal{X}} |K(\mathbf{x}_1, \mathbf{t}) - K(\mathbf{x}_2, \mathbf{t})| M.$$

By Ascoli theorem, we can extract a sequence uniformly convergent in  $C(\mathcal{X})$ , and therefore in  $L_2^\nu(\mathcal{X})$ .  $\square$

## Proof (5/6)

$L_K$  is self-adjoint

$K$  being symmetric, we have for all  $f, g \in \mathcal{H}$ :

$$\begin{aligned}\langle f, Lg \rangle &= \int f(\mathbf{x}) (Lg)(\mathbf{x}) \nu(d\mathbf{x}) \\ &= \int \int f(\mathbf{x}) g(\mathbf{t}) K(\mathbf{x}, \mathbf{t}) \nu(d\mathbf{x}) \nu(d\mathbf{t}) \quad (\text{Fubini}) \\ &= \langle Lf, g \rangle.\end{aligned}$$

## Proof (6/6)

$L_K$  is positive

We can approximate the integral by finite sums:

$$\begin{aligned}\langle f, Lf \rangle &= \int \int f(\mathbf{x}) f(\mathbf{t}) K(\mathbf{x}, \mathbf{t}) \nu(d\mathbf{x}) \nu(d\mathbf{t}) \\ &= \lim_{k \rightarrow \infty} \frac{\nu(\mathcal{X})}{k^2} \sum_{i,j=1}^k K(\mathbf{x}_i, \mathbf{x}_j) f(\mathbf{x}_i) f(\mathbf{x}_j) \\ &\geq 0,\end{aligned}$$

because  $K$  is positive definite.  $\square$

# Diagonalization of the operator

We need the following general result:

## Spectral theorem

Let  $L$  be a **compact** linear operator on a Hilbert space  $\mathcal{H}$ . Then there exists in  $\mathcal{H}$  a **complete orthonormal system**  $(\psi_1, \psi_2, \dots)$  of eigenvectors of  $L$ . The eigenvalues  $(\lambda_1, \lambda_2, \dots)$  are **real** if  $L$  is self-adjoint, and **non-negative** if  $L$  is positive.

## Remark

This theorem can be applied to  $L_K$ . In that case the eigenfunctions  $\psi_k$  associated to the eigenfunctions  $\lambda_k \neq 0$  can be considered as **continuous functions**, because:

$$\psi_k = \frac{1}{\lambda_k} L \psi_K .$$

## Main result

### Mercer Theorem

Let  $\mathcal{X}$  be a compact metric space,  $\nu$  a Borel measure on  $\mathcal{X}$ , and  $K$  a continuous p.d. kernel. Let  $(\lambda_1, \lambda_2, \dots)$  denote the nonnegative eigenvalues of  $L_K$  and  $(\psi_1, \psi_2, \dots)$  the corresponding eigenfunctions. Then all functions  $\psi_k$  are continuous, and for any  $\mathbf{x}, \mathbf{t} \in \mathcal{X}$ :

$$K(\mathbf{x}, \mathbf{t}) = \sum_{k=1}^{\infty} \lambda_k \psi_k(\mathbf{x}) \psi_k(\mathbf{t}),$$

where the convergence is absolute for each  $\mathbf{x}, \mathbf{t} \in \mathcal{X}$ , and uniform on  $\mathcal{X} \times \mathcal{X}$ .

# Mercer kernels as inner products

## Corollary

The mapping

$$\begin{aligned}\Phi : \mathcal{X} &\mapsto l^2 \\ \mathbf{x} &\mapsto \left( \sqrt{\lambda_k} \psi_k(\mathbf{x}) \right)_{k \in \mathbb{N}}\end{aligned}$$

is well defined, continuous, and satisfies

$$K(\mathbf{x}, \mathbf{t}) = \langle \Phi(\mathbf{x}), \Phi(\mathbf{t}) \rangle_{l^2}.$$

## Proof of the corollary

### Proof

By Mercer theorem we see that for all  $\mathbf{x} \in \mathcal{X}$ ,  $\sum \lambda_k \psi_k^2(\mathbf{x})$  converges to  $K(\mathbf{x}, \mathbf{x}) < \infty$ , therefore  $\Phi(\mathbf{x}) \in l^2$ .

The continuity of  $\Phi$  results from:

$$\begin{aligned}\|\Phi(\mathbf{x}) - \Phi(\mathbf{t})\|_{l^2}^2 &= \sum_{k=1}^{\infty} \lambda_k (\psi_k(\mathbf{x}) - \psi_k(\mathbf{t}))^2 \\ &= K(\mathbf{x}, \mathbf{x}) + K(\mathbf{t}, \mathbf{t}) - 2K(\mathbf{x}, \mathbf{t})\end{aligned}$$

## Summary

- This proof extends the proof valid when  $\mathcal{X}$  is finite.
- This is a **constructive** proof, developed by Mercer (1905).
- **Compactness** and **continuity** are required. For instance, for  $\mathcal{X} = \mathbb{R}^d$ , the eigenvalues of:

$$\int_{\mathcal{X}} K(\mathbf{x}, \mathbf{t}) \psi(\mathbf{t}) = \lambda \psi(\mathbf{t})$$

are not necessarily countable, Mercer theorem does not hold. Other tools are thus required such as the Fourier transform for shift-invariant kernels.

## RKHS of Mercer kernels

- Let  $\mathcal{X}$  be a compact metric space, and  $K$  a **Mercer kernel** on  $\mathcal{X}$  (symmetric, continuous and positive definite).
- We have expressed a decomposition of the kernel in terms of the **eigenfunctions** of the linear **convolution operator**.
- In some cases this provides an **intuitive** feature space.
- The kernel also has a **RKHS**, like any p.d. kernel.
- Can we get an **intuition of the RKHS norm** in terms of the **eigenfunctions and eigenvalues** of the convolution operator?

## Reminder: expansion of Mercer kernel

### Theorem

Denote by  $L_K$  the linear operator of  $L_2^\nu(\mathcal{X})$  defined by:

$$\forall f \in L_2^\nu(\mathcal{X}), (L_K f)(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{t}) f(\mathbf{t}) d\nu(\mathbf{t}).$$

Let  $(\lambda_1, \lambda_2, \dots)$  denote the eigenvalues of  $L_K$  in decreasing order, and  $(\psi_1, \psi_2, \dots)$  the corresponding eigenfunctions. Then it holds that for any  $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ :

$$K(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \lambda_k \psi_k(\mathbf{x}) \psi_k(\mathbf{y}) = \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{l^2},$$

with  $\Phi : \mathcal{X} \mapsto l^2$  defined par  $\Phi(\mathbf{x}) = (\sqrt{\lambda_k} \psi_k(\mathbf{x}))_{k \in \mathbb{N}}$ .

# RKHS construction

## Theorem

Assuming that all eigenvalues are positive, the RKHS is the Hilbert space:

$$H_K = \left\{ f \in L_2^\nu(\mathcal{X}) : f = \sum_{i=1}^{\infty} a_i \psi_i, \quad \text{with} \quad \sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k} < \infty \right\}$$

endowed with the inner product:

$$\langle f, g \rangle_K = \sum_{k=1}^{\infty} \frac{a_k b_k}{\lambda_k}, \quad \text{for } f = \sum_k a_k \psi_k, g = \sum_k b_k \psi_k.$$

## Remark

If some eigenvalues are equal to zero, then the result and the proof remain valid on the subspace spanned by the eigenfunctions with positive eigenvalues.

## Proof (1/6)

### Sketch

In order to show that  $H_K$  is the RKHS of the kernel  $K$  we need to show that:

- ① it is a Hilbert space of functions from  $\mathcal{X}$  to  $\mathbb{R}$ ,
- ② for any  $\mathbf{x} \in \mathcal{X}$ ,  $K_{\mathbf{x}} \in H_K$  ,
- ③ for any  $\mathbf{x} \in \mathcal{X}$  and  $f \in H_K$ ,  $f(\mathbf{x}) = \langle f, K_{\mathbf{x}} \rangle_{H_K}$  .

## Proof (2/6)

$H_K$  is a Hilbert space

Indeed the function:

$$L_K^{\frac{1}{2}} : L_2^\nu(\mathcal{X}) \rightarrow H_K$$
$$\sum_{i=1}^{\infty} a_i \psi_i \mapsto \sum_{i=1}^{\infty} a_i \sqrt{\lambda_i} \psi_i$$

is an isomorphism, therefore  $H_K$  is a Hilbert space, like  $L_2^\nu(\mathcal{X})$ . □

## Proof (3/6)

$H_K$  is a space of continuous functions

For any  $f = \sum_{i=1}^{\infty} a_i \psi_i \in H_K$ , and  $\mathbf{x} \in \mathcal{X}$ , we have (if  $f(\mathbf{x})$  makes sense):

$$\begin{aligned}|f(\mathbf{x})| &= \left| \sum_{i=1}^{\infty} a_i \psi_i(\mathbf{x}) \right| = \left| \sum_{i=1}^{\infty} \frac{a_i}{\sqrt{\lambda_i}} \sqrt{\lambda_i} \psi_i(\mathbf{x}) \right| \\&\leq \left( \sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i} \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^{\infty} \lambda_i \psi_i(\mathbf{x})^2 \right)^{\frac{1}{2}} \\&= \|f\|_{H_K} K(\mathbf{x}, \mathbf{x})^{\frac{1}{2}} \\&= \|f\|_{H_K} \sqrt{C_K}.\end{aligned}$$

Therefore convergence in  $\|\cdot\|_{H_K}$  implies uniform convergence for functions.

## Proof (4/6)

$H_K$  is a space of continuous functions (cont.)

Let now  $f_n = \sum_{i=1}^n a_i \psi_i \in H_K$ . The functions  $\psi_i$  are continuous functions, therefore  $f_n$  is also continuous, for all  $n$ . The  $f_n$ 's are convergent in  $H_K$ , therefore also in the (complete) space of continuous functions endowed with the uniform norm.

Let  $f_c$  the continuous limit function. Then  $f_c \in L_2^\nu(\mathcal{X})$  and

$$\|f_n - f_c\|_{L_2^\nu(\mathcal{X})} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$\|f - f_n\|_{L_2^\nu(\mathcal{X})} \leq \lambda_1 \|f - f_n\|_{H_K} \xrightarrow{n \rightarrow \infty} 0,$$

therefore  $f = f_c$ . □

## Proof (5/6)

$$K_x \in H_K$$

For any  $\mathbf{x} \in \mathcal{X}$  let, for all  $i$ ,  $a_i = \lambda_i \psi_i(\mathbf{x})$ . We have:

$$\sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i} = \sum_{i=1}^{\infty} \lambda_i \psi_i(\mathbf{x})^2 = K(\mathbf{x}, \mathbf{x}) < \infty,$$

therefore  $\varphi_x := \sum_{i=1}^{\infty} a_i \psi_i \in H_K$ . As seen earlier the convergence in  $H_K$  implies pointwise convergence, therefore for any  $\mathbf{t} \in \mathcal{X}$ :

$$\varphi_x(\mathbf{t}) = \sum_{i=1}^{\infty} a_i \psi_i(\mathbf{t}) = \sum_{i=1}^{\infty} \lambda_i \psi_i(\mathbf{x}) \psi_i(\mathbf{t}) = K(\mathbf{x}, \mathbf{t}),$$

therefore  $\varphi_x = K_x \in H_K$ .  $\square$

## Proof (6/6)

$$f(\mathbf{x}) = \langle f, K_x \rangle_{H_K}$$

Let  $f = \sum_{i=1}^{\infty} a_i \psi_i \in H_K$ , et  $\mathbf{x} \in \mathcal{X}$ . We have seen that:

$$K_x = \sum_{i=1}^{\infty} \lambda_i \psi_i(\mathbf{x}) \psi_i,$$

therefore:

$$\langle f, K_x \rangle_{H_K} = \sum_{i=1}^{\infty} \frac{\lambda_i \psi_i(\mathbf{x}) a_i}{\lambda_i} = \sum_{i=1}^{\infty} a_i \psi_i(\mathbf{x}) = f(\mathbf{x}),$$

which concludes the proof.  $\square$

## Remarks

- Although  $H_K$  was built from the eigenfunctions of  $L_K$ , which depend on the choice of the measure  $\nu(\mathbf{x})$ , we know by uniqueness of the RKHS that  $H_K$  is independant of  $\nu$  and  $L_K$ .
- Mercer theorem provides a **concrete way** to build the RKHS, by taking linear combinations of the eigenfunctions of  $L_K$  (with adequately chosen weights).
- The eigenfunctions  $(\psi_i)_{i \in \mathbb{N}}$  form an **orthogonal basis** of the RKHS:

$$\langle \psi_i, \psi_j \rangle_{H_K} = 0 \quad \text{si } i \neq j, \quad \|\psi_i\|_{H_K} = \frac{1}{\sqrt{\lambda_i}}.$$

The RKHS is a well-defined **ellipsoid** with axes given by the eigenfunctions.

# Outline

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- Kernels on graphs

## Motivations

- The RKHS norm is related to the **smoothness** of functions.
- Smoothness of a function is naturally quantified by **Sobolev norms** (in particular  $L_2$  norms of derivatives).
- In this section we make a general link between RKHS and **Green functions defined by differential operators**.

## A simple example

### Explicit choice of smoothness

Let

$$\mathcal{H} = \{f : [0, 1] \mapsto \mathbb{R}, \text{absolutely continuous, } f' \in L^2([0, 1]), f(0) = 0\}.$$

endowed with the bilinear form:

$$\forall (f, g) \in \mathcal{F}^2 \langle f, g \rangle_{\mathcal{H}} = \int_0^1 f'(u) g'(u) du.$$

Note that  $\langle f, f \rangle_{\mathcal{H}}$  measures the smoothness of  $f$ :

$$\langle f, f \rangle_{\mathcal{H}} = \int_0^1 f'(u)^2 du = \|f'\|_{L^2([0,1])}^2.$$

# The RKHs point of view

## Theorem

$\mathcal{H}$  is a RKHS with r.k. given by:

$$\forall (x, y) \in [0, 1]^2, \quad K(x, y) = \min(x, y).$$

## Remark

Therefore,  $\|f\|_{\mathcal{H}} = \|f'\|_{L^2}$ : the RKHS norm is precisely the smoothness functional defined in the simple example.

## Proof (1/3)

### Sketch

We need to show that

- $\mathcal{H}$  is a Hilbert space
- $\forall x \in [0, 1], K_x \in \mathcal{H},$
- $\forall (x, f) \in [0, 1] \times \mathcal{H}, \langle f, K_x \rangle_{\mathcal{H}} = f(x).$

## Proof (1/3)

### Sketch

We need to show that

- $\mathcal{H}$  is a Hilbert space
- $\forall x \in [0, 1], K_x \in \mathcal{H},$
- $\forall (x, f) \in [0, 1] \times \mathcal{H}, \langle f, K_x \rangle_{\mathcal{H}} = f(x).$

## Proof (2/3)

$\mathcal{H}$  is a pre-Hilbert space

- $f$  absolutely continuous implies differentiable almost everywhere, and

$$\forall x \in [0, 1], \quad f(x) = f(0) + \int_0^x f'(u)du.$$

- For any  $f \in \mathcal{H}$ ,  $f(0) = 0$  implies by Cauchy-Schwarz:

$$|f(x)| = \left| \int_0^x f'(u)du \right| \leq \sqrt{x} \left( \int_0^1 f'(u)^2 du \right)^{\frac{1}{2}} = \sqrt{x} \|f\|_{\mathcal{H}}.$$

Therefore,  $\|f\|_{\mathcal{H}} = 0 \implies f = 0$ , showing that  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is an inner product.  $\mathcal{H}$  is thus a pre-Hilbert space.

## Proof (2/3)

$\mathcal{H}$  is a Hilbert space

- To show that  $\mathcal{H}$  is complete, let  $(f_n)_{n \in \mathbb{N}}$  a Cauchy sequence in  $\mathcal{H}$
- $(f'_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2[0, 1]$ , thus converges to  $g \in L^2[0, 1]$
- By the previous inequality,  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence and thus converges to a real number  $f(x)$ , for any  $x \in [0, 1]$ . Moreover:

$$f(x) = \lim_n f_n(x) = \lim_n \int_0^x f'_n(u) du = \int_0^x g(u) du,$$

showing that  $f$  is absolutely continuous and  $f' = g$  almost everywhere; in particular,  $f' \in L^2[0, 1]$ .

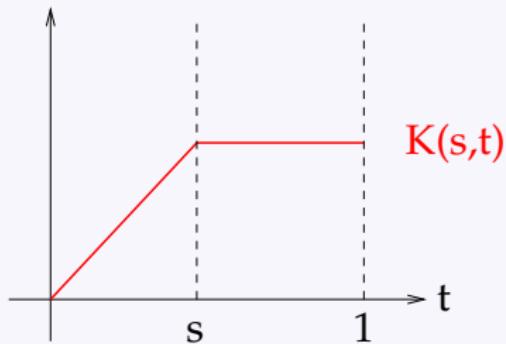
- Finally,  $f(0) = \lim_n f_n(0) = 0$ , therefore  $f \in \mathcal{H}$  and

$$\lim_n \|f_n - f\|_{\mathcal{H}} = \|f' - g\|_{L^2[0,1]} = 0.$$

## Proof (2/3)

$\forall x \in [0, 1], K_x \in \mathcal{H}$

Let  $K_x(y) = K(x, y) = \min(x, y)$  sur  $[0, 1]^2$ :



$K_x$  is differentiable except at  $s$ , has a square integrable derivative, and  $K_x(0) = 0$ , therefore  $K_x \in \mathcal{H}$  for all  $x \in [0, 1]$ .  $\square$

## Proof (3/3)

For all  $x, f$ ,  $\langle f, K_x \rangle_{\mathcal{H}} = f(x)$

For any  $x \in [0, 1]$  and  $f \in \mathcal{H}$  we have:

$$\langle f, K_x \rangle_{\mathcal{H}} = \int_0^1 f'(u) K'_x(u) du = \int_0^x f'(u) du = f(x),$$

which shows that  $K$  is the r.k. associated to  $\mathcal{H}$ .  $\square$

# Generalization

## Theorem

Let  $\mathcal{X} = \mathbb{R}^d$  and  $D$  a differential operator on a class of functions  $\mathcal{H}$  such that, endowed with the inner product:

$$\forall (f, g) \in \mathcal{H}^2, \quad \langle f, g \rangle_{\mathcal{H}} = \langle Df, Dg \rangle_{L^2(\mathcal{X})},$$

it is a Hilbert space.

Then  $\mathcal{H}$  is a RKHS that admits as r.k. the Green function of the operator  $D^*D$ , where  $D^*$  denotes the adjoint operator of  $D$ .

In case of...

## Green functions

Let the differential equation on  $\mathcal{H}$ :

$$f = Dg ,$$

where  $g$  is unknown. In order to solve it we can look for  $g$  of the form:

$$g(x) = \int_{\mathcal{X}} k(x, y) f(y) dy$$

for some function  $k : \mathcal{X}^2 \mapsto \mathbb{R}$ .  $k$  must then satisfy, for all  $x \in \mathcal{X}$ ,

$$f(x) = Dg(x) = \langle Dk_x, f \rangle_{L^2(\mathcal{X})} .$$

$k$  is called the **Green function** of the operator  $D$ .

## Proof

Let  $\mathcal{H}$  be a Hilbert space endowed with the inner product:

$$\langle f, g \rangle_{\mathcal{X}} = \langle Df, Dg \rangle_{L^2(\mathcal{X})},$$

and  $K$  be the Green function of the operator  $D^*D$ . For all  $x \in \mathcal{X}$ ,  $K_x \in \mathcal{H}$  because:

$$\langle DK_x, DK_x \rangle_{L^2(\mathcal{X})} = \langle D^*DK_x, K_x \rangle_{L^2(\mathcal{X})} = K_x(x) < \infty.$$

Moreover, for all  $f \in \mathcal{H}$  and  $x \in \mathcal{X}$ , we have:

$$f(x) = \langle D^*DK_x, f \rangle_{L^2(\mathcal{X})} = \langle DK_x, Df \rangle_{L^2(\mathcal{X})} = \langle K_x, f \rangle_{\mathcal{H}},$$

which shows that  $\mathcal{H}$  is a RKHS with  $K$  as r.k.  $\square$

## Kernel examples: Summary

- Many notions of **smoothness** can be translated as **RKHS norms** for particular kernels (eigenvalues convolution operator, Sobolev norms and Green operators, Fourier transforms...).
- There is **no “uniformly best kernel”**, but rather a large **toolbox** of methods and tricks to **encode prior knowledge** and exploit the **nature or structure** of the data.
- In the following sections we focus on **particular data and applications** to illustrate the process of **kernel design**.

# Outline

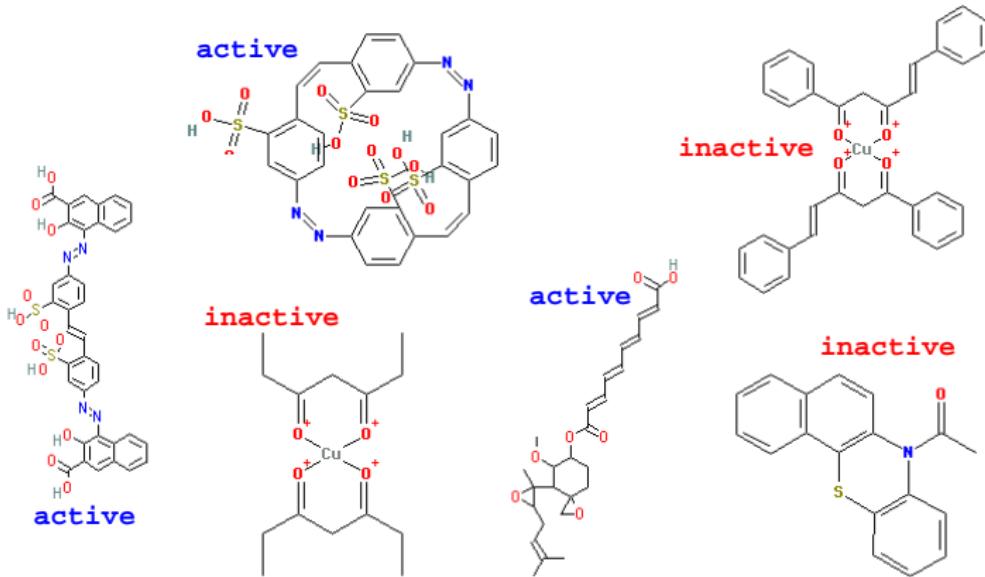
- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
  - Kernels for probabilistic models
  - Kernels for biological sequences
  - Mercer kernels and shift-invariant kernels
  - **Kernels for graphs**
  - Kernels on graphs
- 6 Open Problems and Research Topics

# Outline

## 5 The Kernel Jungle

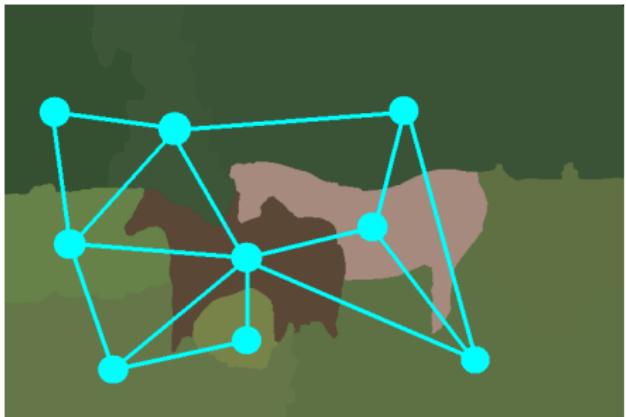
- Kernels for probabilistic models
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- **Kernels for graphs**
  - Motivation
  - Explicit enumeration of features
  - Challenges
  - Walk-based kernels
  - Applications
- Kernels on graphs

# Virtual screening for drug discovery



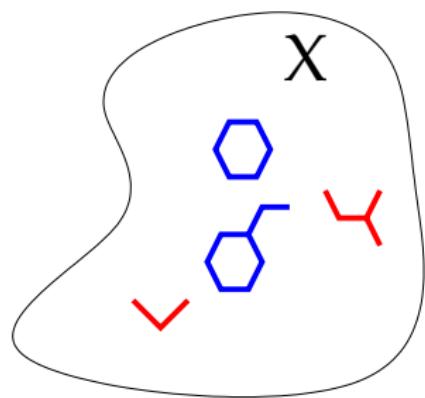
NCI AIDS screen results (from <http://cactus.nci.nih.gov>).

# Image retrieval and classification



*From Harchaoui and Bach (2007).*

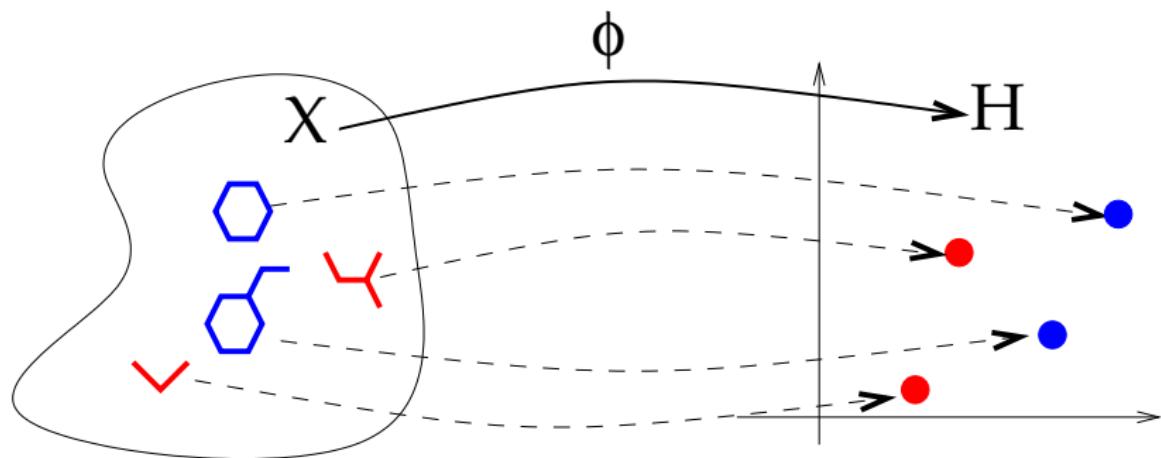
## Our approach



## Our approach

- ① Represent each graph  $\mathbf{x}$  in  $\mathcal{X}$  by a vector  $\Phi(\mathbf{x}) \in \mathcal{H}$ , either **explicitly** or **implicitly** through the kernel

$$K(\mathbf{x}, \mathbf{x}') = \Phi(\mathbf{x})^\top \Phi(\mathbf{x}').$$

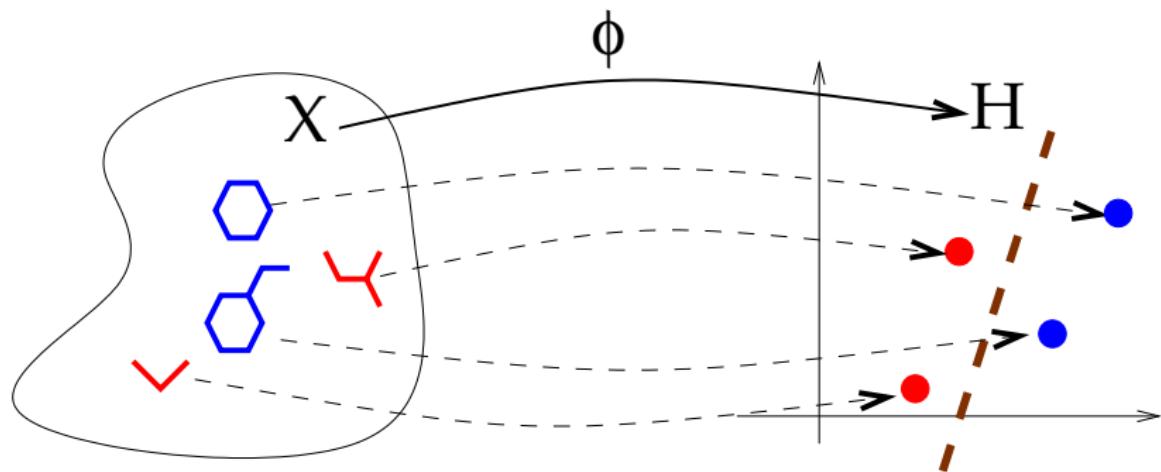


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- ② Use a linear method for classification in  $\mathcal{H}$ .



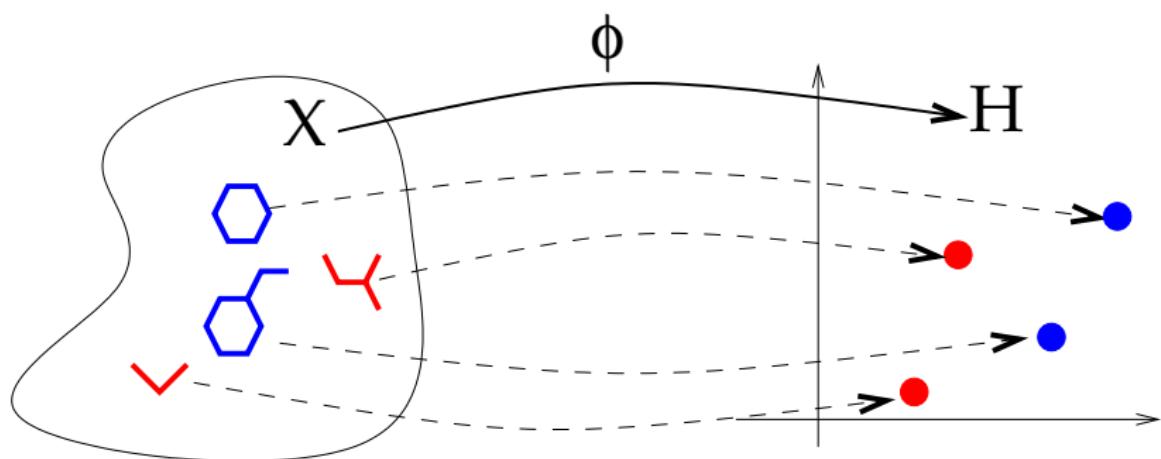
# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
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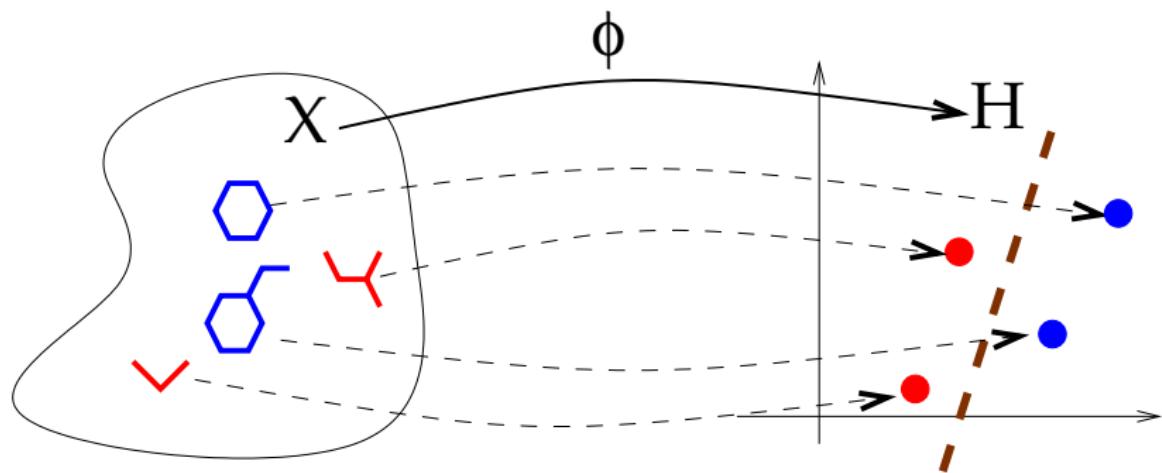
## The approach

- ① Represent explicitly each graph  $x$  by a **vector of fixed dimension**  $\Phi(x) \in \mathbb{R}^p$ .



## The approach

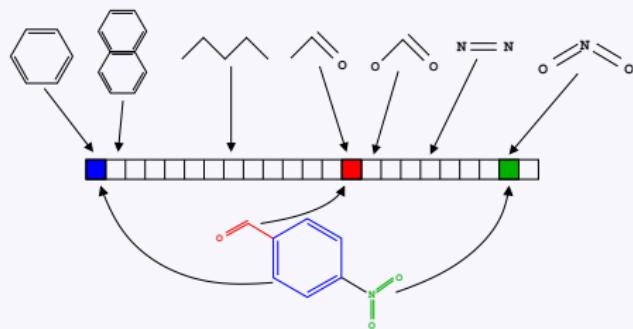
- ① Represent explicitly each graph  $x$  by a vector of fixed dimension  $\Phi(x) \in \mathbb{R}^p$ .
- ② Use an algorithm for regression or pattern recognition in  $\mathbb{R}^p$ .



# Example

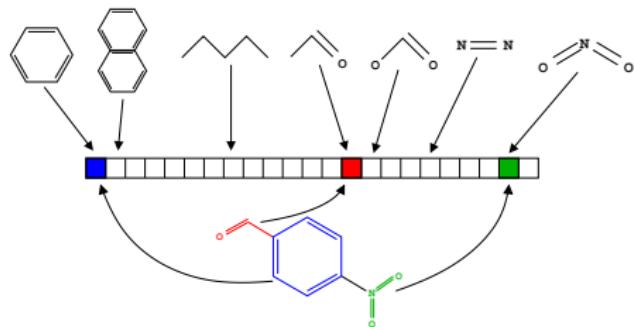
## 2D structural keys in chemoinformatics

- Index a molecule by a binary fingerprint defined by a limited set of **predefined** structures



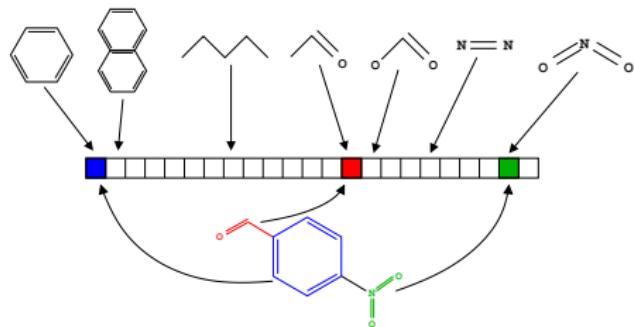
- Use a machine learning algorithm such as SVM,  $k$ NN, PLS, decision tree, etc.

## Challenge: which descriptors (patterns)?



- **Expressiveness:** they should retain as much information as possible from the graph
- **Computation:** they should be fast to compute
- **Large dimension** of the vector representation: memory storage, speed, statistical issues

## Indexing by substructures

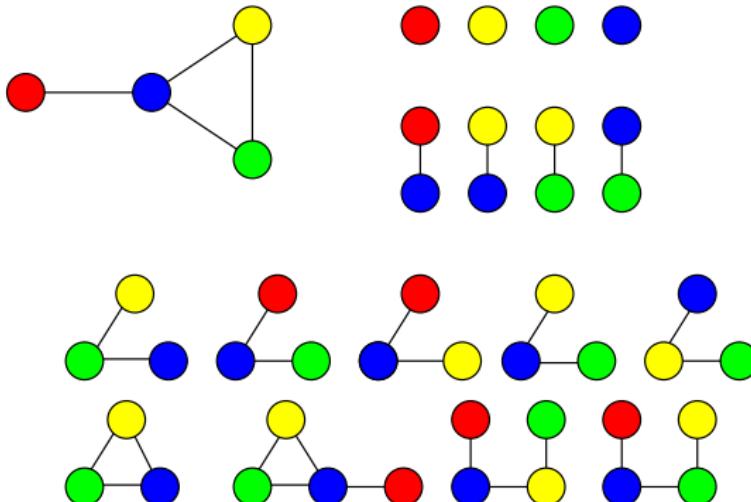


- Often we believe that **the presence or absence of particular substructures** may be important predictive patterns
- Hence it makes sense to represent a graph by **features** that indicate the presence (or the number of occurrences) of these substructures
- However, detecting the presence of particular substructures may be **computationally challenging**...

# Subgraphs

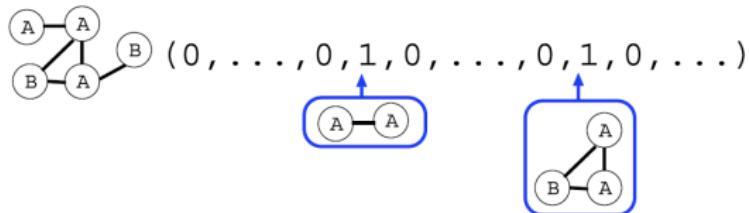
## Definition

A **subgraph** of a graph  $(V, E)$  is a graph  $(V', E')$  with  $V' \subset V$  and  $E' \subset E$ .

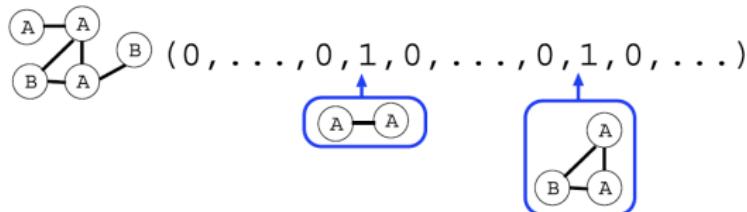


A graph and all its connected subgraphs.

## Indexing by all subgraphs?



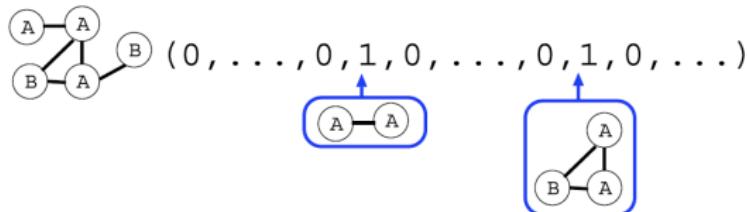
# Indexing by all subgraphs?



## Theorem

*Computing all subgraph occurrences is NP-hard.*

# Indexing by all subgraphs?



## Theorem

Computing all subgraph occurrences is **NP-hard**.

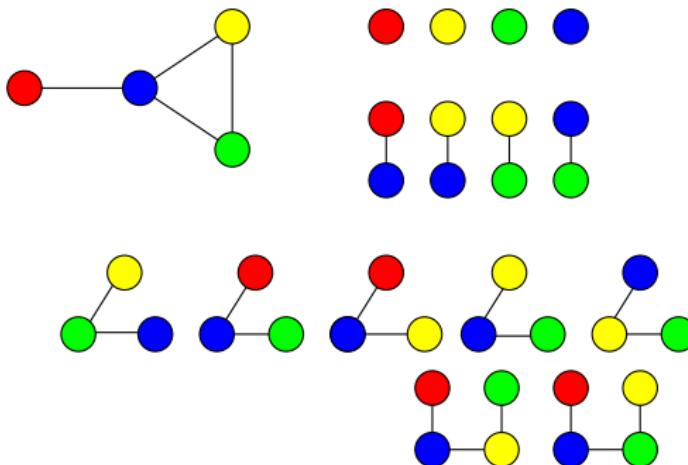
## Proof

- The linear graph of size  $n$  is a subgraph of a graph  $X$  with  $n$  vertices iff  $X$  has a Hamiltonian path;
- The decision problem whether a graph has a Hamiltonian path is NP-complete.

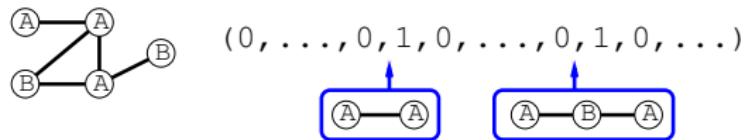
# Paths

## Definition

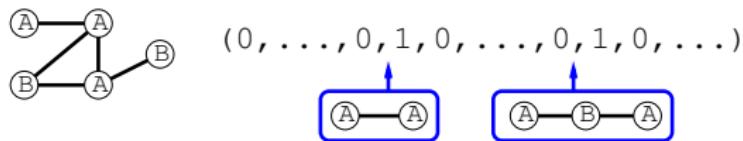
- A **path** of a graph  $(V, E)$  is a sequence of **distinct vertices**  $v_1, \dots, v_n \in V$  ( $i \neq j \implies v_i \neq v_j$ ) such that  $(v_i, v_{i+1}) \in E$  for  $i = 1, \dots, n - 1$ .
- Equivalently the paths are the **linear subgraphs**.



## Indexing by all paths?



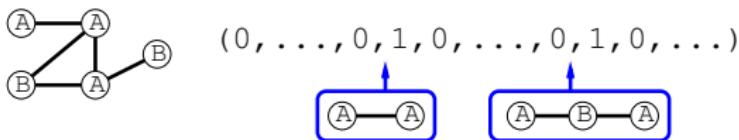
# Indexing by all paths?



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## Theorem

Computing all path occurrences is **NP-hard**.

## Proof

Same as for subgraphs.

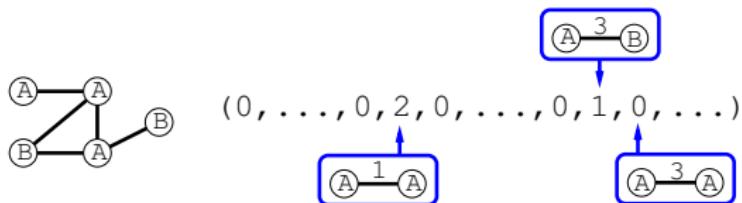
# Indexing by what?

## Substructure selection

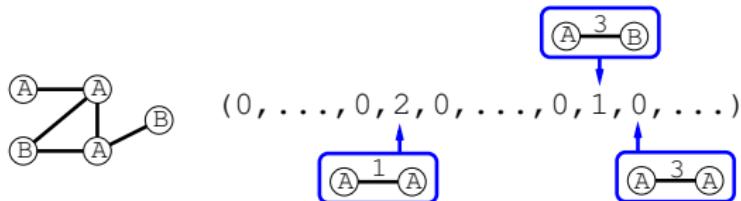
We can imagine more limited sets of substructures that lead to more computationnally efficient indexing (non-exhaustive list)

- substructures selected by **domain knowledge** (MDL fingerprint)
- all paths **up to length  $k$**  (Openeye fingerprint, Nicholls 2005)
- all **shortest path lengths** (Borgwardt and Kriegel, 2005)
- all subgraphs **up to  $k$  vertices** (graphlet kernel, Shervashidze et al., 2009)
- all **frequent** subgraphs in the database (Helma et al., 2004)

Example: Indexing by all shortest path lengths and their endpoint labels



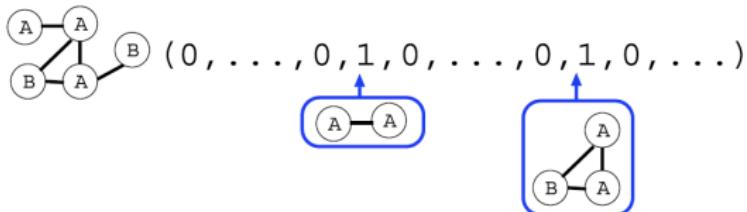
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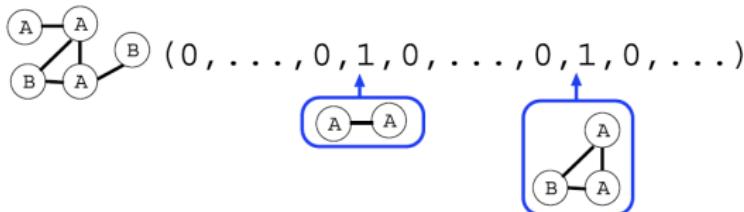
### Properties (Borgwardt and Kriegel, 2005)

- There are  $O(n^2)$  shortest paths.
- The vector of counts can be computed in  $O(n^3)$  with the Floyd-Warshall algorithm.

## Example: Indexing by all subgraphs up to $k$ vertices



## Example: Indexing by all subgraphs up to $k$ vertices



### Properties (Shervashidze et al., 2009)

- Naive enumeration scales as  $O(n^k)$ .
- Enumeration of connected graphlets in  $O(nd^{k-1})$  for graphs with degree  $\leq d$  and  $k \leq 5$ .
- Randomly sample subgraphs if enumeration is infeasible.

## Summary

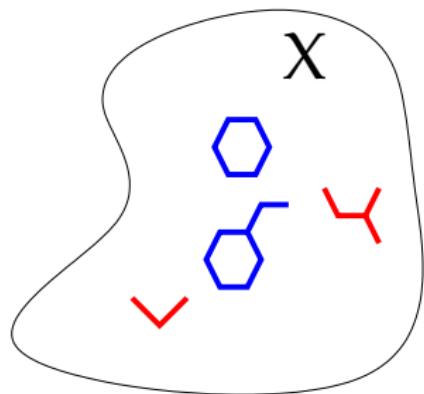
- Explicit computation of substructure occurrences can be **computationnally prohibitive** (subgraphs, paths);
- Several ideas to **reduce** the set of substructures considered;
- In practice, NP-hardness may not be so prohibitive (e.g., graphs with small degrees), the strategy followed should depend on the data considered.

# Outline

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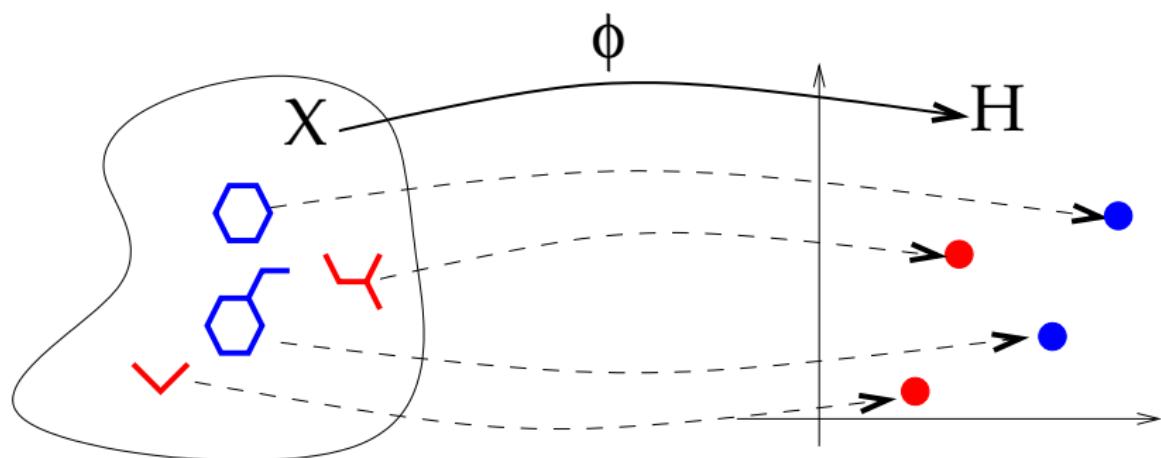
# The idea



## The idea

- ① Represent **implicitly** each graph  $\mathbf{x}$  in  $\mathcal{X}$  by a vector  $\Phi(\mathbf{x}) \in \mathcal{H}$  through the kernel

$$K(\mathbf{x}, \mathbf{x}') = \Phi(\mathbf{x})^\top \Phi(\mathbf{x}').$$

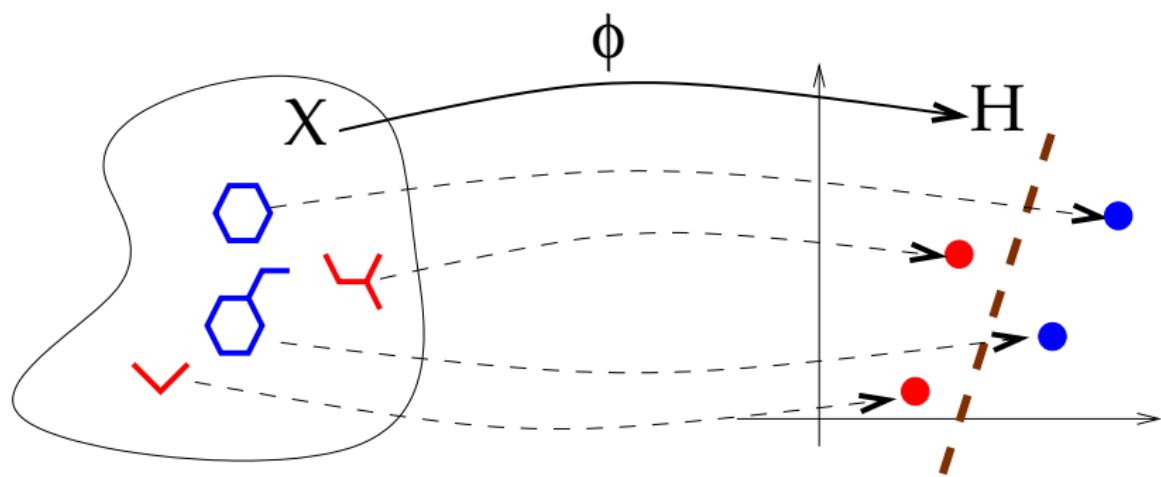


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- ② Use a kernel method for classification in  $\mathcal{H}$ .



## Expressiveness vs Complexity

Definition: Complete graph kernels

A graph kernel is **complete** if it distinguishes non-isomorphic graphs, i.e.:

$$\forall G_1, G_2 \in \mathcal{X}, \quad d_K(G_1, G_2) = 0 \implies G_1 \simeq G_2.$$

Equivalently,  $\Phi(G_1) \neq \Phi(G_2)$  if  $G_1$  and  $G_2$  are not isomorphic.

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## Expressiveness vs Complexity trade-off

- If a graph kernel is not complete, then there is **no hope** to learn all possible functions over  $\mathcal{X}$ : the kernel is not **expressive** enough.
- On the other hand, kernel **computation** must be **tractable**, i.e., no more than polynomial (with small degree) for practical applications.
- Can we define **tractable** and **expressive** graph kernels?

## Complexity of complete kernels

Proposition (Gärtner et al., 2003)

Computing **any complete graph kernel** is at least as hard as the graph isomorphism problem.

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## Proof

- For any kernel  $K$  the complexity of computing  $d_K$  is the same as the complexity of computing  $K$ , because:

$$d_K(G_1, G_2)^2 = K(G_1, G_1) + K(G_2, G_2) - 2K(G_1, G_2).$$

- If  $K$  is a complete graph kernel, then computing  $d_K$  solves the graph isomorphism problem ( $d_K(G_1, G_2) = 0$  iff  $G_1 \simeq G_2$ ).  $\square$

# Subgraph kernel

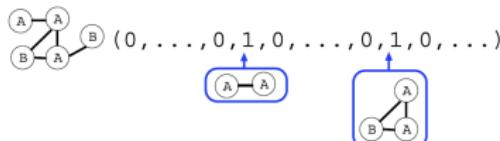
## Definition

- Let  $(\lambda_G)_{G \in \mathcal{X}}$  be a set or **nonnegative** real-valued weights
- For any graph  $G \in \mathcal{X}$  and any connected graph  $H \in \mathcal{X}$ , let

$$\Phi_H(G) = |\{G' \text{ is a subgraph of } G : G' \simeq H\}|.$$

- The **subgraph kernel** between any two graphs  $G_1$  and  $G_2 \in \mathcal{X}$  is defined by:

$$K_{\text{subgraph}}(G_1, G_2) = \sum_{\substack{H \in \mathcal{X} \\ H \text{ connected}}} \lambda_H \Phi_H(G_1) \Phi_H(G_2).$$



## Subgraph kernel complexity

Proposition (Gärtner et al., 2003)

Computing the subgraph kernel is **NP-hard**.

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## Proof (1/2)

- Let  $P_n$  be the path graph with  $n$  vertices.
- Subgraphs of  $P_n$  are path graphs:

$$\Phi(P_n) = ne_{P_1} + (n - 1)e_{P_2} + \dots + e_{P_n}.$$

- The vectors  $\Phi(P_1), \dots, \Phi(P_n)$  are linearly independent, therefore:

$$e_{P_n} = \sum_{i=1}^n \alpha_i \Phi(P_i),$$

where the coefficients  $\alpha_i$  can be found in polynomial time (solving an  $n \times n$  triangular system).

# Subgraph kernel complexity

Proposition (Gärtner et al., 2003)

Computing the subgraph kernel is **NP-hard**.

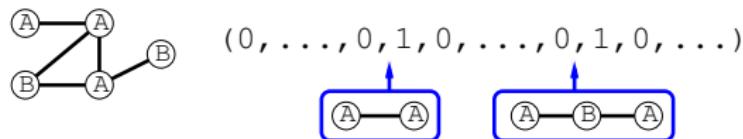
Proof (2/2)

- If  $G$  is a graph with  $n$  vertices, then it has a path that visits each node exactly once (Hamiltonian path) if and only if  $\Phi(G)^\top e_{P_n} > 0$ , i.e.,

$$\Phi(G)^\top \left( \sum_{i=1}^n \alpha_i \Phi(P_i) \right) = \sum_{i=1}^n \alpha_i K_{\text{subgraph}}(G, P_i) > 0.$$

- The decision problem whether a graph has a Hamiltonian path is NP-complete.  $\square$

# Path kernel



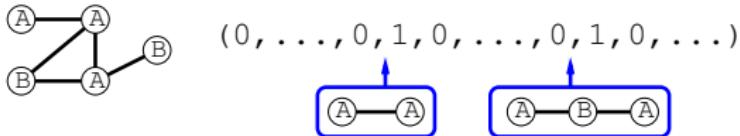
## Definition

The **path kernel** is the subgraph kernel restricted to paths, i.e.,

$$K_{\text{path}}(G_1, G_2) = \sum_{H \in \mathcal{P}} \lambda_H \Phi_H(G_1) \Phi_H(G_2),$$

where  $\mathcal{P} \subset \mathcal{X}$  is the set of path graphs.

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## Proposition (Gärtner et al., 2003)

Computing the path kernel is **NP-hard**.

# Summary

## Expressiveness vs Complexity trade-off

- It is **intractable** to compute **complete graph kernels**.
- It is **intractable** to compute the **subgraph kernels**.
- Restricting subgraphs to be linear does not help: it is also **intractable** to compute the **path kernel**.
- One approach to define polynomial time computable graph kernels is to have the feature space be made up of graphs **homomorphic** to subgraphs, e.g., to consider **walks** instead of paths.

# Outline

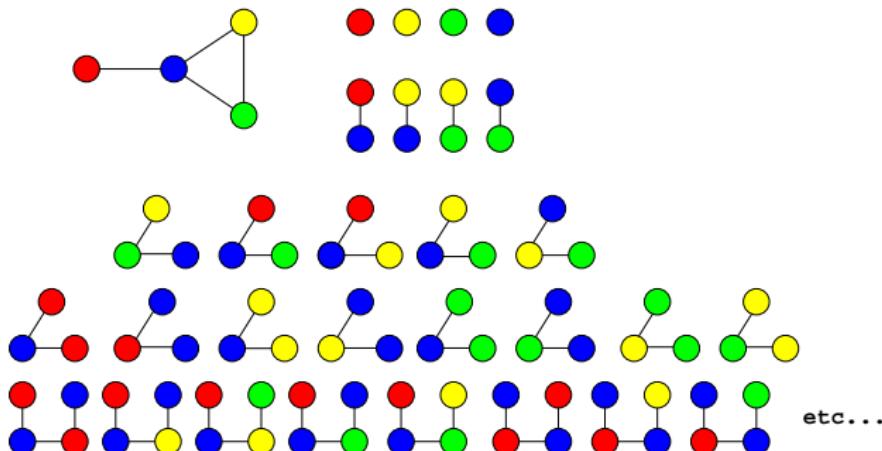
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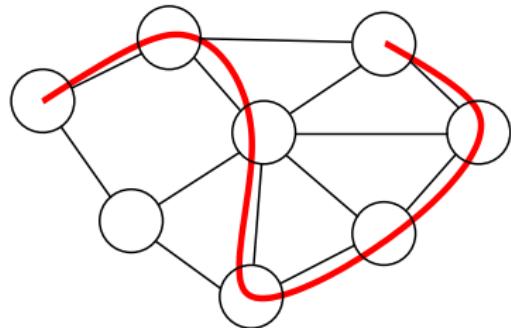
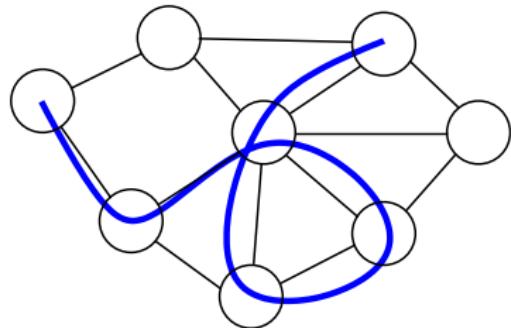
# Walks

## Definition

- A **walk** of a graph  $(V, E)$  is sequence of  $v_1, \dots, v_n \in V$  such that  $(v_i, v_{i+1}) \in E$  for  $i = 1, \dots, n - 1$ .
- We note  $\mathcal{W}_n(G)$  the set of walks with  $n$  vertices of the graph  $G$ , and  $\mathcal{W}(G)$  the set of all walks.



Walks  $\neq$  paths



# Walk kernel

## Definition

- Let  $\mathcal{S}_n$  denote the set of all possible **label sequences** of walks of length  $n$  (including vertex and edge labels), and  $\mathcal{S} = \cup_{n \geq 1} \mathcal{S}_n$ .
- For any graph  $\mathcal{X}$  let a **weight**  $\lambda_G(w)$  be associated to each walk  $w \in \mathcal{W}(G)$ .
- Let the feature vector  $\Phi(G) = (\Phi_s(G))_{s \in \mathcal{S}}$  be defined by:

$$\Phi_s(G) = \sum_{w \in \mathcal{W}(G)} \lambda_G(w) \mathbf{1} (s \text{ is the label sequence of } w).$$

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- A walk kernel is a graph kernel defined by:

$$K_{\text{walk}}(G_1, G_2) = \sum_{s \in \mathcal{S}} \Phi_s(G_1) \Phi_s(G_2).$$

## Walk kernel examples

### Examples

- The  $n$ th-order walk kernel is the walk kernel with  $\lambda_G(w) = 1$  if the length of  $w$  is  $n$ , 0 otherwise. It compares two graphs through their common walks of length  $n$ .

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- The random walk kernel is obtained with  $\lambda_G(w) = P_G(w)$ , where  $P_G$  is a Markov random walk on  $G$ . In that case we have:

$$K(G_1, G_2) = P(\text{label}(W_1) = \text{label}(W_2)),$$

where  $W_1$  and  $W_2$  are two independent random walks on  $G_1$  and  $G_2$ , respectively (Kashima et al., 2003).

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- The random walk kernel is obtained with  $\lambda_G(w) = P_G(w)$ , where  $P_G$  is a Markov random walk on  $G$ . In that case we have:

$$K(G_1, G_2) = P(\text{label}(W_1) = \text{label}(W_2)),$$

where  $W_1$  and  $W_2$  are two independent random walks on  $G_1$  and  $G_2$ , respectively (Kashima et al., 2003).

- The geometric walk kernel is obtained (when it converges) with  $\lambda_G(w) = \beta^{\text{length}(w)}$ , for  $\beta > 0$ . In that case the feature space is of infinite dimension (Gärtner et al., 2003).

# Computation of walk kernels

## Proposition

These three kernels ( $n$ th-order, random and geometric walk kernels) can be computed efficiently in **polynomial time**.

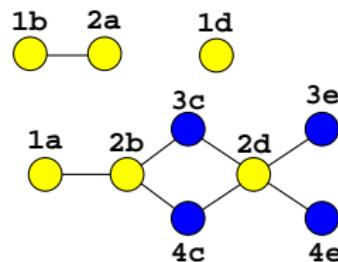
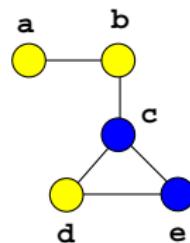
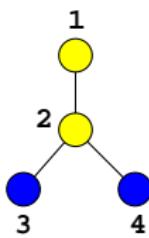
# Product graph

## Definition

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs with labeled vertices.

The **product graph**  $G = G_1 \times G_2$  is the graph  $G = (V, E)$  with:

- ①  $V = \{(v_1, v_2) \in V_1 \times V_2 : v_1 \text{ and } v_2 \text{ have the same label}\}$  ,
- ②  $E = \{((v_1, v_2), (v'_1, v'_2)) \in V \times V : (v_1, v'_1) \in E_1 \text{ and } (v_2, v'_2) \in E_2\}$ .



$G_1$

$G_2$

$G_1 \times G_2$

# Walk kernel and product graph

## Lemma

There is a **bijection** between:

- ① The **pairs of walks**  $w_1 \in \mathcal{W}_n(G_1)$  and  $w_2 \in \mathcal{W}_n(G_2)$  with the **same label sequences**,
- ② The **walks on the product graph**  $w \in \mathcal{W}_n(G_1 \times G_2)$ .

# Walk kernel and product graph

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- ② The **walks on the product graph**  $w \in \mathcal{W}_n(G_1 \times G_2)$ .

## Corollary

$$\begin{aligned} K_{\text{walk}}(G_1, G_2) &= \sum_{s \in \mathcal{S}} \Phi_s(G_1) \Phi_s(G_2) \\ &= \sum_{(w_1, w_2) \in \mathcal{W}(G_1) \times \mathcal{W}(G_2)} \lambda_{G_1}(w_1) \lambda_{G_2}(w_2) \mathbf{1}(l(w_1) = l(w_2)) \\ &= \sum_{w \in \mathcal{W}(G_1 \times G_2)} \lambda_{G_1 \times G_2}(w). \end{aligned}$$

## Computation of the $n$ th-order walk kernel

- For the  $n$ th-order walk kernel we have  $\lambda_{G_1 \times G_2}(w) = 1$  if the length of  $w$  is  $n$ , 0 otherwise.
- Therefore:

$$K_{n\text{th-order}}(G_1, G_2) = \sum_{w \in \mathcal{W}_n(G_1 \times G_2)} 1.$$

- Let  $A$  be the adjacency matrix of  $G_1 \times G_2$ . Then we get:

$$K_{n\text{th-order}}(G_1, G_2) = \sum_{i,j} [A^n]_{i,j} = \mathbf{1}^\top A^n \mathbf{1}.$$

- Computation in  $O(n|V_1||V_2|d_1 d_2)$ , where  $d_i$  is the maximum degree of  $G_i$ .

## Computation of random and geometric walk kernels

- In both cases  $\lambda_G(w)$  for a walk  $w = v_1 \dots v_n$  can be decomposed as:

$$\lambda_G(v_1 \dots v_n) = \lambda^i(v_1) \prod_{i=2}^n \lambda^t(v_{i-1}, v_i).$$

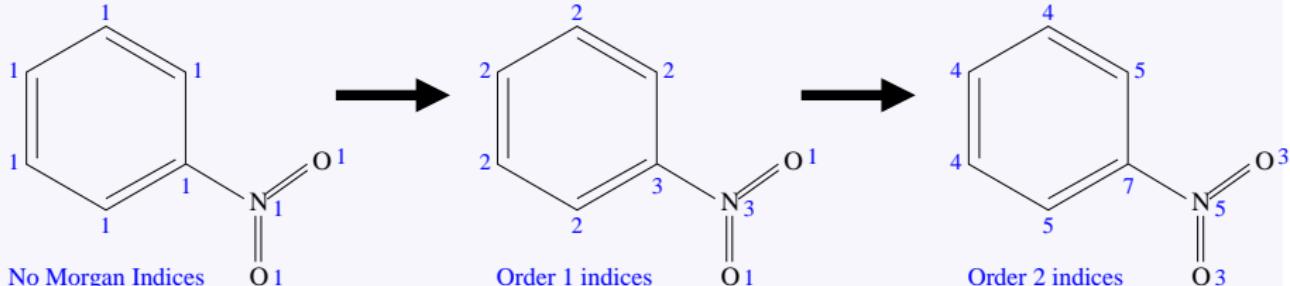
- Let  $\Lambda_i$  be the vector of  $\lambda^i(v)$  and  $\Lambda_t$  be the matrix of  $\lambda^t(v, v')$ :

$$\begin{aligned} K_{\text{walk}}(G_1, G_2) &= \sum_{n=1}^{\infty} \sum_{w \in \mathcal{W}_n(G_1 \times G_2)} \lambda^i(v_1) \prod_{i=2}^n \lambda^t(v_{i-1}, v_i) \\ &= \sum_{n=0}^{\infty} \Lambda_i \Lambda_t^n \mathbf{1} \\ &= \color{red} \Lambda_i (I - \Lambda_t)^{-1} \mathbf{1} \end{aligned}$$

- Computation in  $O(|V_1|^3 |V_2|^3)$ .

## Extensions 1: Label enrichment

Atom relabeling with the Morgan index (Mahé et al., 2004)

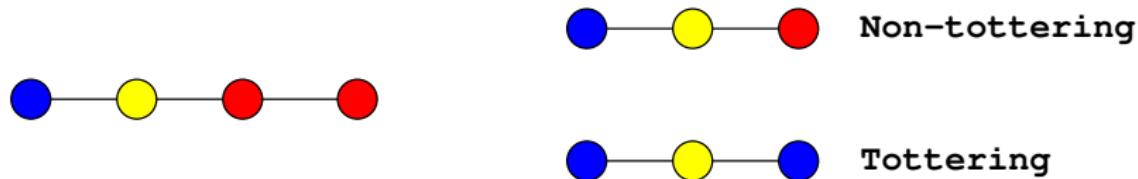


- Compromise between **fingerprints** and **structural keys**.
- Other **relabeling** schemes are possible.
- Faster computation with more labels (less matches implies a smaller product graph).

## Extension 2: Non-tottering walk kernel

### Tottering walks

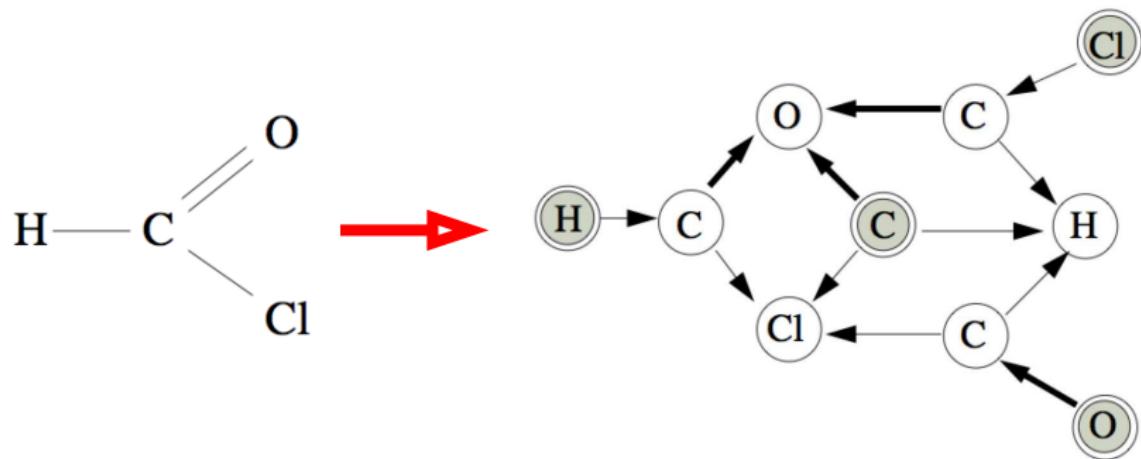
A **tottering walk** is a walk  $w = v_1 \dots v_n$  with  $v_i = v_{i+2}$  for some  $i$ .



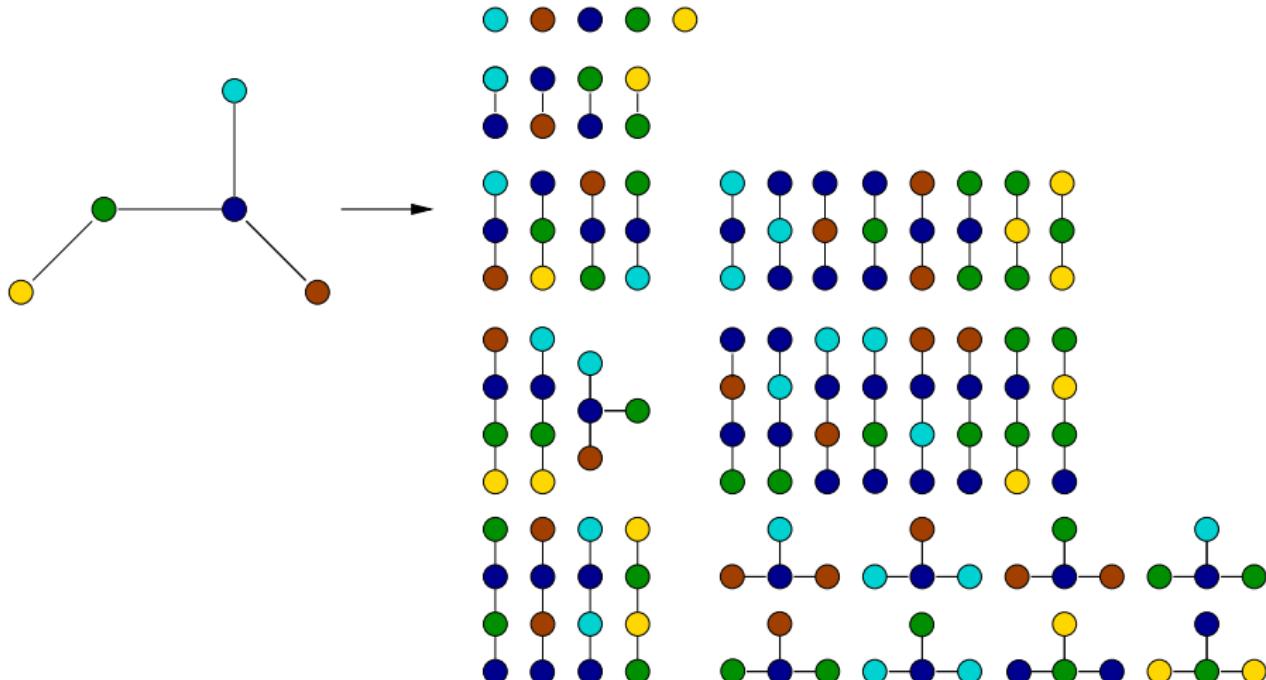
- Tottering walks seem **irrelevant** for many applications.
- Focusing on non-tottering walks is a way to get closer to the **path kernel** (e.g., equivalent on trees).

## Computation of the non-tottering walk kernel (Mahé et al., 2005)

- Second-order Markov random walk to prevent tottering walks
- Written as a first-order Markov random walk on an augmented graph
- Normal walk kernel on the augmented graph (which is always a directed graph).

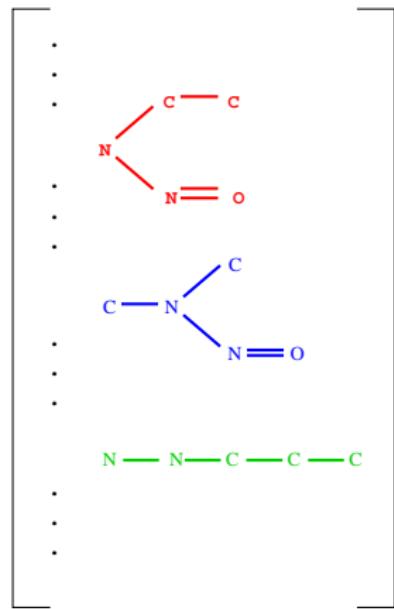
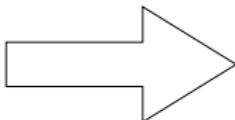
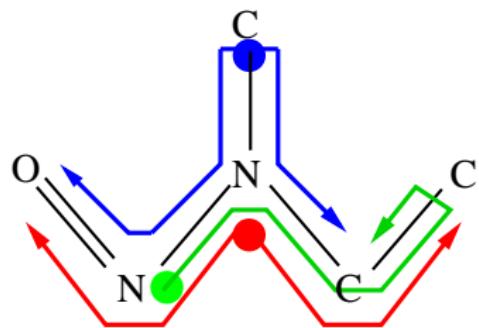


## Extension 3: Subtree kernels



Remark: Here and in subsequent slides by *subtree* we mean a tree-like pattern with potentially repeated nodes and edges.

## Example: Tree-like fragments of molecules



## Computation of the subtree kernel (Ramon and Gärtner, 2003; Mahé and Vert, 2009)

- Like the walk kernel, amounts to computing the (weighted) number of subtrees in the **product graph**.
- Recursion: if  $\mathcal{T}(v, n)$  denotes the weighted number of subtrees of depth  $n$  rooted at the vertex  $v$ , then:

$$\mathcal{T}(v, n+1) = \sum_{R \subset \mathcal{N}(v)} \prod_{v' \in R} \lambda_t(v, v') \mathcal{T}(v', n),$$

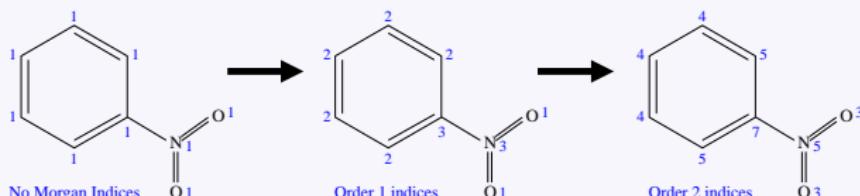
where  $\mathcal{N}(v)$  is the set of neighbors of  $v$ .

- Can be combined with the non-tottering graph transformation as preprocessing to obtain the **non-tottering subtree kernel**.

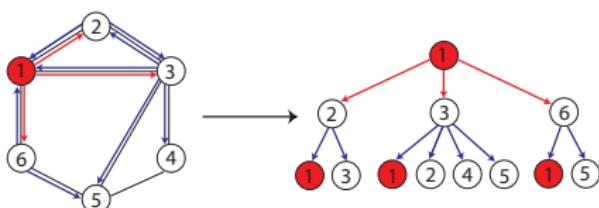
# Back to label enrichment

## Link between the Morgan index and subtrees

Recall the Morgan index:



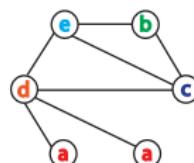
The Morgan index of order  $k$  at a node  $v$  in fact corresponds to the number of leaves in the  $k$ -th order full subtree pattern rooted at  $v$ .



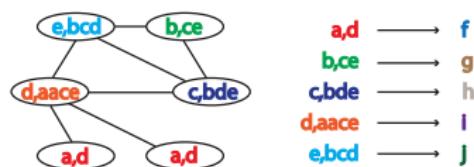
# Label enrichment via the Weisfeiler-Lehman algorithm

A slightly more involved label enrichment strategy (Weisfeiler and Lehman, 1968) is exploited in the definition and computation of the Weisfeiler-Lehman subtree kernel (Shervashidze and Borgwardt, 2009).

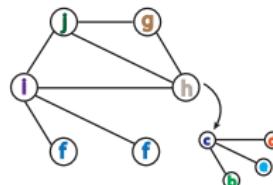
- ➊ Multiset-label determination and sorting



- ➋ Label compression



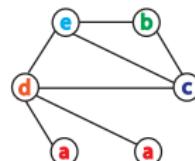
- ➌ Relabeling



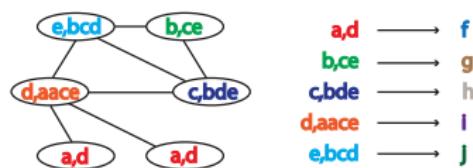
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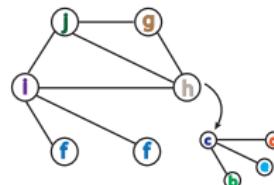
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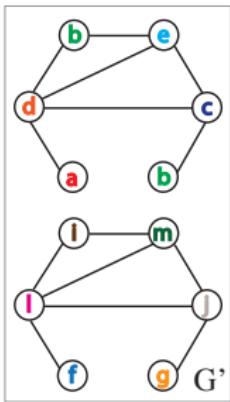
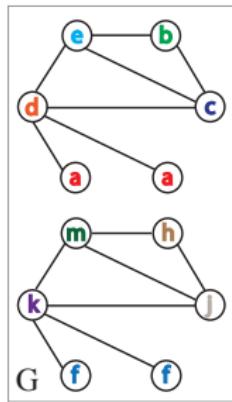


- ➌ Relabeling



Compressed labels represent full subtree patterns.

# Weisfeiler-Lehman (WL) subtree kernel



$$\phi_{WLsubtree}^{(1)}(G) = (2, 1, 1, 1, 1, 2, 0, 1, 0, 1, 1, 0, 1)$$

a b c d e f g h i j k l m

$$\phi_{WLsubtree}^{(1)}(G') = (1, 2, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1)$$

a b c d e f g h i j k l m

Counts of  
original  
node labels

Counts of  
compressed  
node labels

## Properties

- The WL features up to the  $k$ -th order are computed in  $O(|E|k)$ .
- Similarly to the Morgan index, the WL relabeling can be exploited in combination with any graph kernel (that takes into account categorical node labels) to make it more expressive (Shervashidze et al., 2011).

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- Mercer kernels and shift-invariant kernels
- **Kernels for graphs**
  - Motivation
  - Explicit enumeration of features
  - Challenges
  - Walk-based kernels
  - **Applications**
- Kernels on graphs

# Application in chemoinformatics (Mahé et al., 2005)

## MUTAG dataset

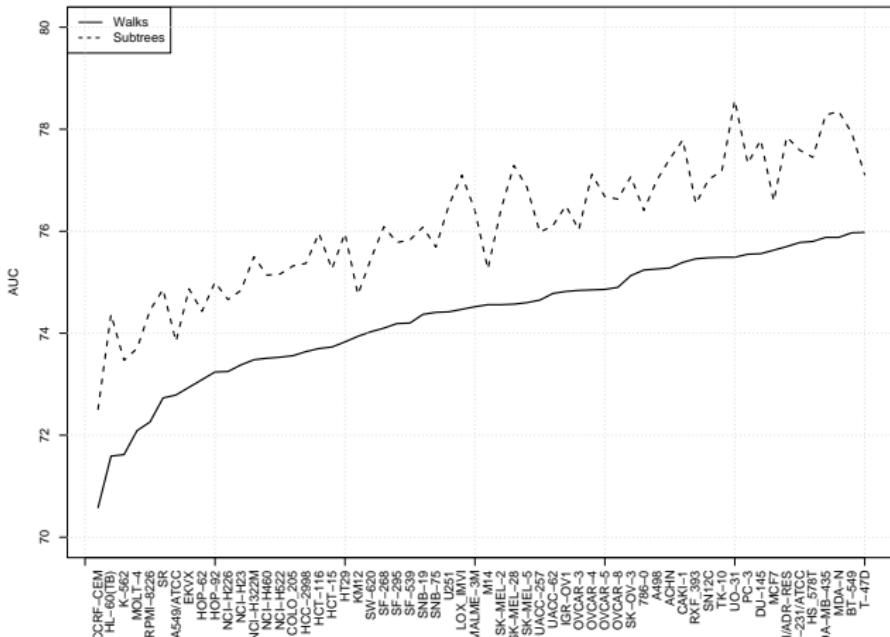
- aromatic/hetero-aromatic compounds
- high mutagenic activity / no mutagenic activity, assayed in *Salmonella typhimurium*.
- 188 compounds: 125 + / 63 -

## Results

10-fold cross-validation accuracy

Method	Accuracy
Progol1	81.4%
2D kernel	91.2%

# 2D subtree vs walk kernels



Screening of inhibitors for 60 cancer cell lines.

## Comparison of several graph feature extraction methods/kernels (Shervashidze et al., 2011)

10-fold cross-validation accuracy on graph classification problems in chemo- and bioinformatics:

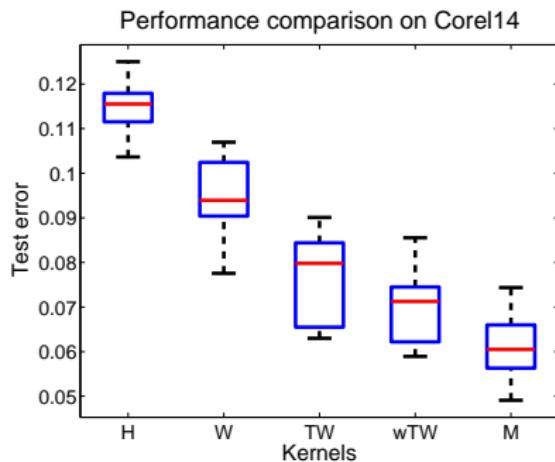
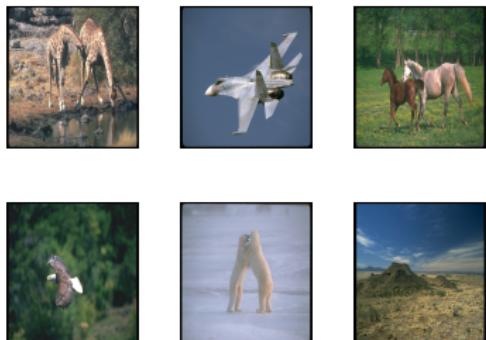
- NCI1 and NCI109 - active/inactive compounds in an anti-cancer screen
- ENZYMES - 6 types of enzymes from the BRENDA database

Method/Data Set	NCI1	NCI109	ENZYMES
WL subtree	82.19 ( $\pm 0.18$ )	82.46 ( $\pm 0.24$ )	52.22 ( $\pm 1.26$ )
WL shortest path	84.55 ( $\pm 0.36$ )	83.53 ( $\pm 0.30$ )	59.05 ( $\pm 1.05$ )
Ramon & Gärtner	61.86 ( $\pm 0.27$ )	61.67 ( $\pm 0.21$ )	13.35 ( $\pm 0.87$ )
Geometric $p$ -walk	58.66 ( $\pm 0.28$ )	58.36 ( $\pm 0.94$ )	27.67 ( $\pm 0.95$ )
Geometric walk	64.34 ( $\pm 0.27$ )	63.51 ( $\pm 0.18$ )	21.68 ( $\pm 0.94$ )
Graphlet count	66.00 ( $\pm 0.07$ )	66.59 ( $\pm 0.08$ )	32.70 ( $\pm 1.20$ )
Shortest path	73.47 ( $\pm 0.11$ )	73.07 ( $\pm 0.11$ )	41.68 ( $\pm 1.79$ )

# Image classification (Harchaoui and Bach, 2007)

## COREL14 dataset

- 1400 natural images in 14 classes
- Compare kernel between histograms (H), walk kernel (W), subtree kernel (TW), weighted subtree kernel (wTW), and a combination (M).



# Summary: graph kernels

## What we saw

- Kernels do **not allow** to overcome the NP-hardness of subgraph patterns.
- They allow to work with approximate subgraphs (walks, subtrees) in infinite dimension, thanks to the **kernel trick**.
- However: using kernels makes it difficult to **come back to patterns** after the learning stage.

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
  - Kernels for probabilistic models
  - Kernels for biological sequences
  - Mercer kernels and shift-invariant kernels
  - Kernels for graphs
  - Kernels on graphs
- 6 Open Problems and Research Topics

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
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- **Kernels on graphs**
  - Motivation
  - Graph distance and p.d. kernels
  - Construction by regularization
  - The diffusion kernel
  - Harmonic analysis on graphs
  - Applications

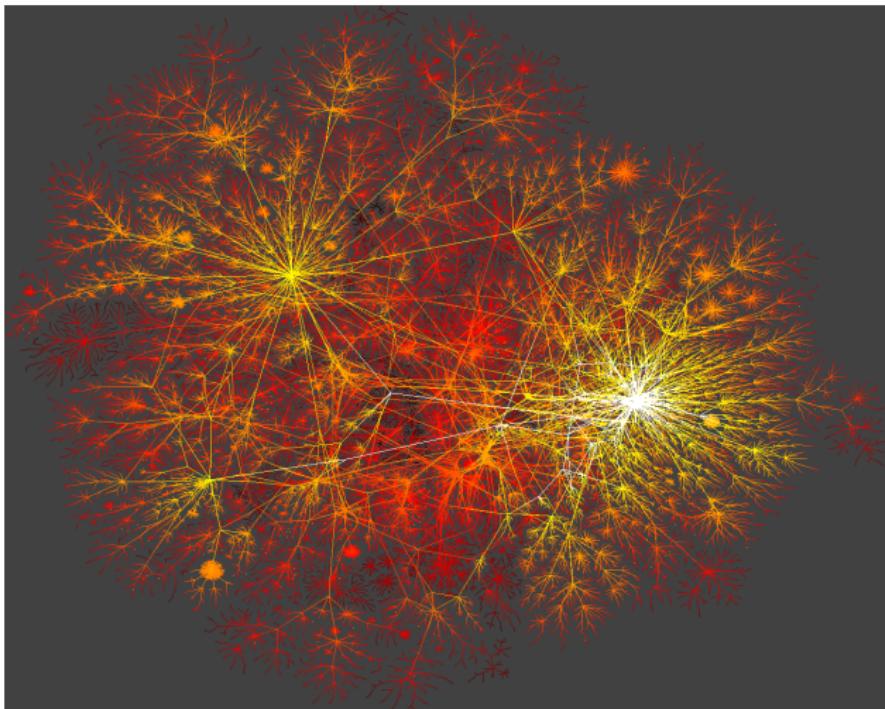
# Graphs

## Motivation

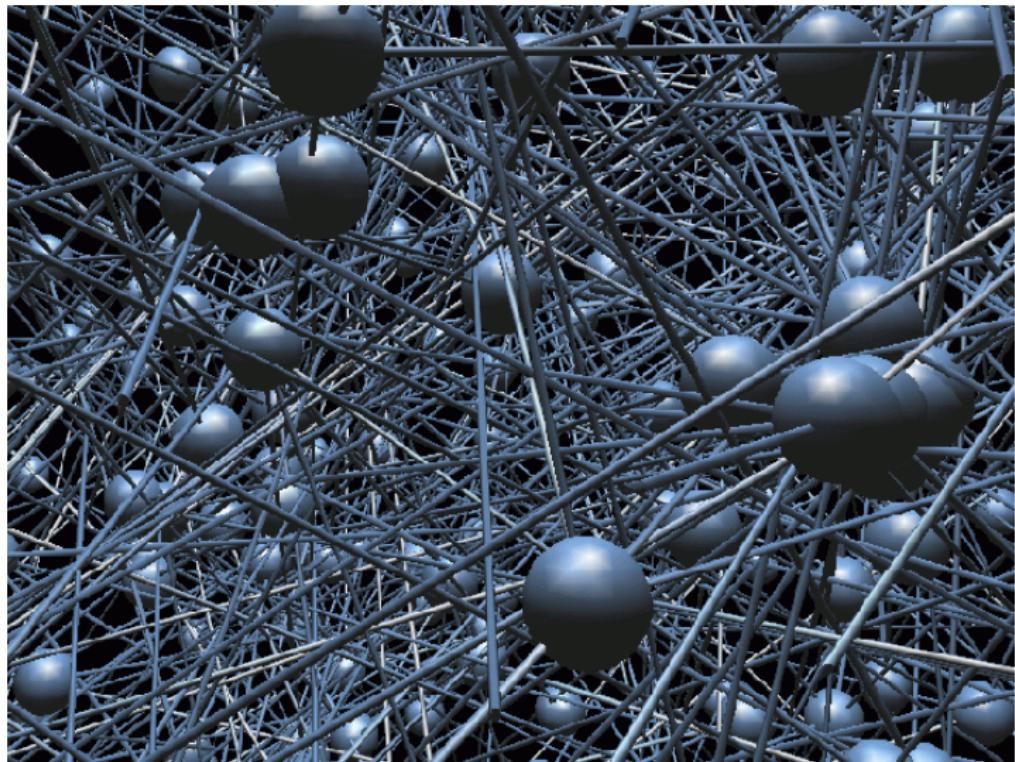
Data often come in the form of **nodes in a graph** for different reasons:

- by **definition** (interaction network, internet...)
- by **discretization**/sampling of a continuous domain
- by **convenience** (e.g., if only a similarity function is available)

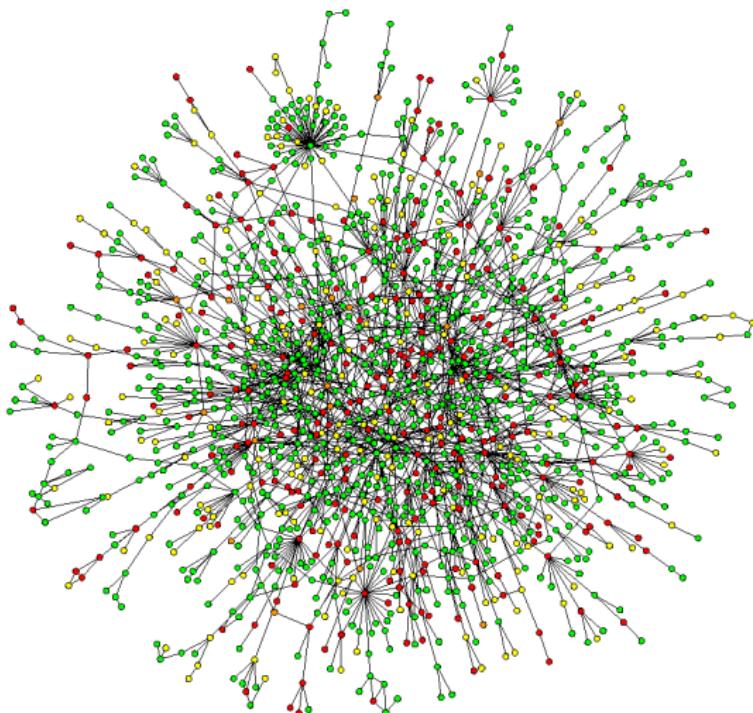
## Example: web



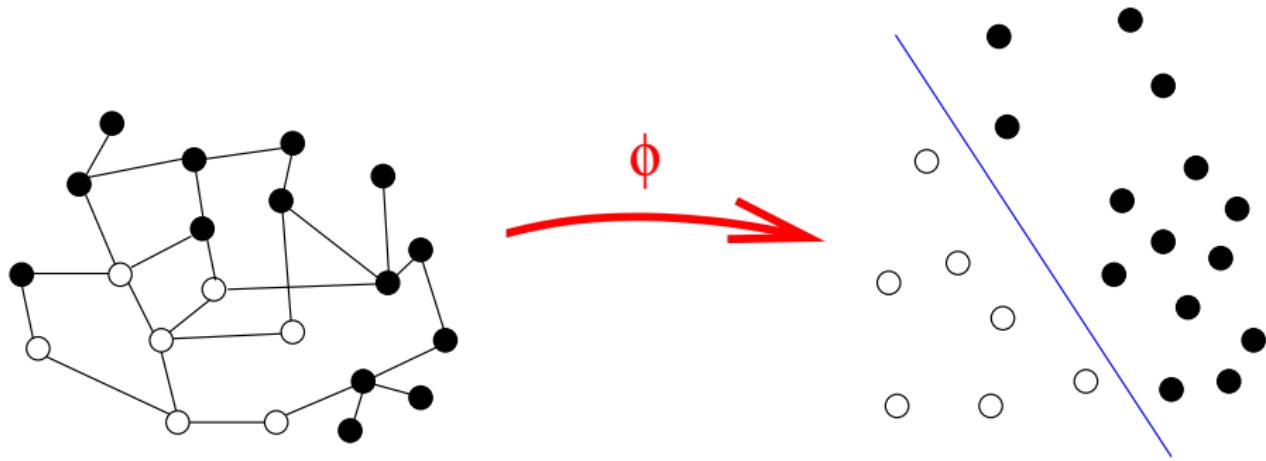
## Example: social network



## Example: protein-protein interaction



## Kernel on a graph



- We need a **kernel  $K(x, x')$  between nodes of the graph.**
- Example: predict protein functions from high-throughput protein-protein interaction data.

## General remarks

### Strategies to design a kernel on a graph

- $\mathcal{X}$  being finite, any symmetric semi-definite matrix  $K$  defines a valid p.d. kernel on  $\mathcal{X}$ .

## General remarks

### Strategies to design a kernel on a graph

- $\mathcal{X}$  being finite, any symmetric semi-definite matrix  $K$  defines a valid p.d. kernel on  $\mathcal{X}$ .
- How to “translate” the graph topology into the kernel?
  - Direct geometric approach:  $K_{i,j}$  should be “large” when  $\mathbf{x}_i$  and  $\mathbf{x}_j$  are “close” to each other on the graph?
  - Functional approach:  $\|f\|_K$  should be “small” when  $f$  is “smooth” on the graph?
  - Link discrete/continuous: is there an equivalent to the continuous Gaussian kernel on the graph (e.g., limit by fine discretization)?

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# Conditionally p.d. kernels

## Hilbert distance

- Any p.d. kernel is an inner product in a Hilbert space

$$K(\mathbf{x}, \mathbf{x}') = \langle \Phi(\mathbf{x}), \Phi(\mathbf{x}') \rangle_{\mathcal{H}}.$$

- It defines a Hilbert distance:

$$d_K(\mathbf{x}, \mathbf{x}')^2 = K(\mathbf{x}, \mathbf{x}) + K(\mathbf{x}', \mathbf{x}') - 2K(\mathbf{x}, \mathbf{x}').$$

- $-d_K^2$  is **conditionally positive definite (c.p.d.)**, i.e.:

$$\forall t > 0, \quad \exp(-td_K(\mathbf{x}, \mathbf{x}')^2) \text{ is p.d.}$$

## Example

### A direct approach

- For  $\mathcal{X} = \mathbb{R}^n$ , the inner product is p.d.:

$$K(\mathbf{x}, \mathbf{x}') = \mathbf{x}^\top \mathbf{x}'.$$

- The corresponding Hilbert distance is the Euclidean distance:

$$d_K(\mathbf{x}, \mathbf{x}')^2 = \mathbf{x}^\top \mathbf{x} + \mathbf{x}'^\top \mathbf{x}' - 2\mathbf{x}^\top \mathbf{x}' = \|\mathbf{x} - \mathbf{x}'\|^2.$$

- $-d_K^2$  is **conditionally positive definite (c.p.d.)**, i.e.:

$$\forall t > 0, \quad \exp(-t\|\mathbf{x} - \mathbf{x}'\|^2) \text{ is p.d.}$$

## Graph distance

### Graph embedding in a Hilbert space

- Given a graph  $G = (V, E)$ , the **graph distance**  $d_G(x, x')$  between any two vertices is the **length of the shortest path** between  $x$  and  $x'$ .
- We say that the graph  $G = (V, E)$  can be **embedded** (exactly) in a Hilbert space if  $-d_G$  is **c.p.d.**, which implies in particular that  $\exp(-td_G(x, x'))$  is p.d. for all  $t > 0$ .

# Graph distance

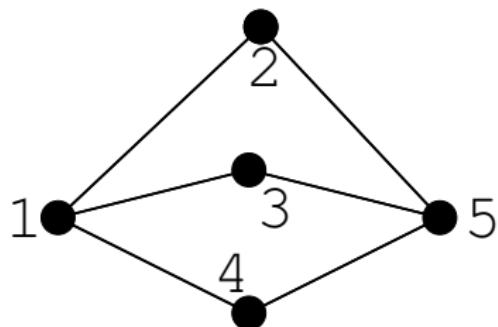
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## Lemma

- In general graphs cannot** be embedded exactly in Hilbert spaces.
- In some cases exact embeddings exist**, e.g.:
  - trees** can be embedded exactly,
  - closed chains** can be embedded exactly.

## Example: non-c.p.d. graph distance



$$d_G = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\lambda_{\min} \left( \left[ e^{(-0.2d_G(i,j))} \right] \right) = -0.028 < 0.$$

## Graph distances on trees are c.p.d.

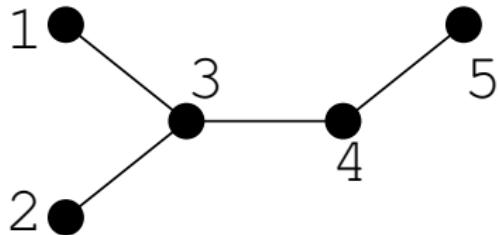
### Proof

- Let  $G = (V, E)$  be a tree;
- Fix a root  $x_0 \in V$ ;
- Represent any vertex  $x \in V$  by a vector  $\Phi(x) \in \mathbb{R}^{|E|}$ , where  $\Phi(x)_i = 1$  if the  $i$ -th edge is part of the (unique) path between  $x$  and  $x_0$ , 0 otherwise.
- Then

$$d_G(x, x') = \| \Phi(x) - \Phi(x') \|^2,$$

and therefore  $-d_G$  is c.p.d., in particular  $\exp(-td_G(x, x'))$  is p.d. for all  $t > 0$ .

## Example

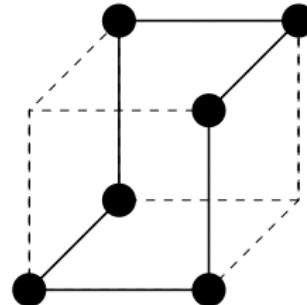
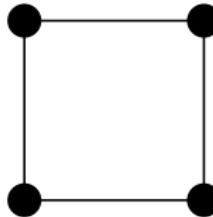


$$\left[ e^{-d_G(i,j)} \right] = \begin{pmatrix} 1 & 0.14 & 0.37 & 0.14 & 0.05 \\ 0.14 & 1 & 0.37 & 0.14 & 0.05 \\ 0.37 & 0.37 & 1 & 0.37 & 0.14 \\ 0.14 & 0.14 & 0.37 & 1 & 0.37 \\ 0.05 & 0.05 & 0.14 & 0.37 & 1 \end{pmatrix}$$

## Graph distances on closed chains are c.p.d.

Proof: case  $|V| = 2p$

- Let  $G = (V, E)$  be a directed cycle with an even number of vertices  $|V| = 2p$ .
- Fix a root  $x_0 \in V$ , number the  $2p$  edges from  $x_0$  to  $x_0$ ;
- Label the  $2p$  edges with  $e_1, \dots, e_p, -e_1, \dots, -e_p$  (vectors in  $\mathbb{R}^p$ );
- For a vertex  $v$ , take  $\Phi(v)$  to be the sum of the labels of the edges in the shortest directed path between  $x_0$  and  $v$ .



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# Functional approach

## Motivation

- How to design a p.d. kernel on **general graphs**?
- Designing a kernel is equivalent to defining an **RKHS**.
- There are intuitive notions of **smoothness** on a graph.

## Idea

- Define a priori a **smoothness functional** on the functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ ;
- Show that **it defines an RKHS** and identify the corresponding kernel.

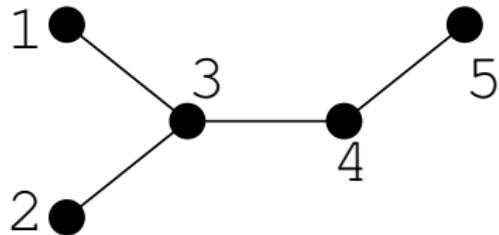
## Notations

- $\mathcal{X} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$  is finite.
- For  $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ , we note  $\mathbf{x} \sim \mathbf{x}'$  to indicate the existence of an edge between  $\mathbf{x}$  and  $\mathbf{x}'$ .
- We assume that there is no self-loop  $\mathbf{x} \sim \mathbf{x}$ , and that there is a single connected component.
- The adjacency matrix is  $A \in \mathbb{R}^{m \times m}$ :

$$A_{i,j} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

- $D$  is the diagonal matrix where  $D_{i,i}$  is the number of neighbors of  $\mathbf{x}_i$  ( $D_{i,i} = \sum_{j=1}^m A_{i,j}$ ).

## Example

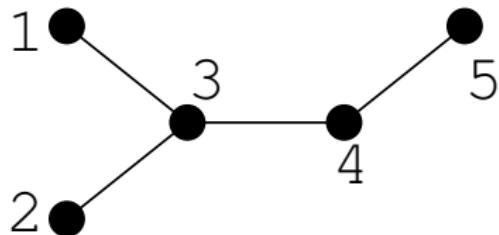


$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

# Graph Laplacian

## Definition

The Laplacian of the graph is the matrix  $L = D - A$ .



$$L = D - A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

# Properties of the Laplacian

## Lemma

Let  $L = D - A$  be the Laplacian of a **connected** graph:

- For any  $f : \mathcal{X} \rightarrow \mathbb{R}$ ,

$$\Omega(f) := \sum_{i \sim j} (f(\mathbf{x}_i) - f(\mathbf{x}_j))^2 = f^\top L f$$

- $L$  is a **symmetric positive semi-definite** matrix
- 0 is an **eigenvalue** with multiplicity 1 associated to the constant eigenvector  $\mathbf{1} = (1, \dots, 1)$
- The **image** of  $L$  is

$$Im(L) = \left\{ f \in \mathbb{R}^m : \sum_{i=1}^m f_i = 0 \right\}$$

## Proof: link between $\Omega(f)$ and $L$

$$\begin{aligned}\Omega(f) &= \sum_{i \sim j} (f(\mathbf{x}_i) - f(\mathbf{x}_j))^2 \\&= \sum_{i \sim j} \left( f(\mathbf{x}_i)^2 + f(\mathbf{x}_j)^2 - 2f(\mathbf{x}_i)f(\mathbf{x}_j) \right) \\&= \sum_{i=1}^m D_{i,i} f(\mathbf{x}_i)^2 - 2 \sum_{i \sim j} f(\mathbf{x}_i)f(\mathbf{x}_j) \\&= \mathbf{f}^\top D \mathbf{f} - \mathbf{f}^\top A \mathbf{f} \\&= \mathbf{f}^\top L \mathbf{f}\end{aligned}$$

## Proof: eigenstructure of $L$

- $L$  is symmetric because  $A$  and  $D$  are symmetric.
- For any  $f \in \mathbb{R}^m$ ,  $f^\top L f = \Omega(f) \geq 0$ , therefore the (real-valued) eigenvalues of  $L$  are  $\geq 0$  :  $L$  is therefore positive semi-definite.
- $f$  is an eigenvector associated to eigenvalue 0
  - iff  $f^\top L f = 0$
  - iff  $\sum_{i \sim j} (f(\mathbf{x}_i) - f(\mathbf{x}_j))^2 = 0$ ,
  - iff  $f(\mathbf{x}_i) = f(\mathbf{x}_j)$  when  $i \sim j$ ,
  - iff  $f$  is constant (because the graph is connected).
- $L$  being symmetric,  $Im(L)$  is the orthogonal supplement of  $Ker(L)$ , that is, the set of functions orthogonal to  $\mathbf{1}$ .  $\square$

# Our first graph kernel

## Theorem

The set  $\mathcal{H} = \{f \in \mathbb{R}^m : \sum_{i=1}^m f_i = 0\}$  endowed with the norm

$$\Omega(f) = \sum_{i \sim j} (f(\mathbf{x}_i) - f(\mathbf{x}_j))^2$$

is a RKHS whose reproducing kernel is  $L^*$ , the pseudo-inverse of the graph Laplacian.

## In case of...

### Pseudo-inverse of $L$

Remember the pseudo-inverse  $L^*$  of  $L$  is the linear application that is equal to:

- 0 on  $\text{Ker}(L)$
- $L^{-1}$  on  $\text{Im}(L)$ , that is, if we write:

$$L = \sum_{i=1}^m \lambda_i u_i u_i^\top$$

the eigendecomposition of  $L$ :

$$L^* = \sum_{\lambda_i \neq 0} (\lambda_i)^{-1} u_i u_i^\top.$$

- In particular it holds that  $L^* L = L L^* = \Pi_{\mathcal{H}}$ , the projection onto  $\text{Im}(L) = \mathcal{H}$ .

## Proof (1/2)

- Restricted to  $\mathcal{H}$ , the symmetric bilinear form:

$$\langle f, g \rangle = f^\top L g$$

is positive definite (because  $L$  is positive semi-definite, and  $\mathcal{H} = \text{Im}(L)$ ). It is therefore a scalar product, making of  $\mathcal{H}$  a **Hilbert space** (in fact Euclidean).

- The norm in this Hilbert space  $\mathcal{H}$  is:

$$\|f\|^2 = \langle f, f \rangle = f^\top L f = \Omega(f) .$$

## Proof (2/2)

To check that  $\mathcal{H}$  is a RKHS with reproducing kernel  $K = L^*$ , it suffices to show that:

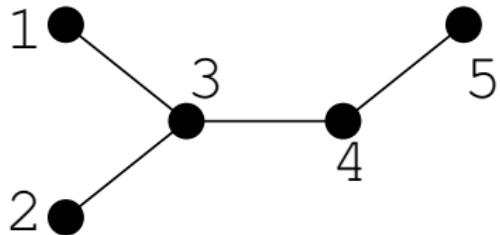
$$\begin{cases} \forall \mathbf{x} \in \mathcal{X}, & K_{\mathbf{x}} \in \mathcal{H}, \\ \forall (\mathbf{x}, f) \in \mathcal{X} \times \mathcal{H}, & \langle f, K_{\mathbf{x}} \rangle = f(\mathbf{x}) . \end{cases}$$

- $\text{Ker}(K) = \text{Ker}(L^*) = \text{Ker}(L)$ , implying  $K\mathbf{1} = 0$ . Therefore, each row/column of  $K$  is in  $\mathcal{H}$ .
- For any  $f \in \mathcal{H}$ , if we note  $g_i = \langle K(i, \cdot), f \rangle$  we get:

$$g = KLf = L^*Lf = \Pi_{\mathcal{H}}(f) = f .$$

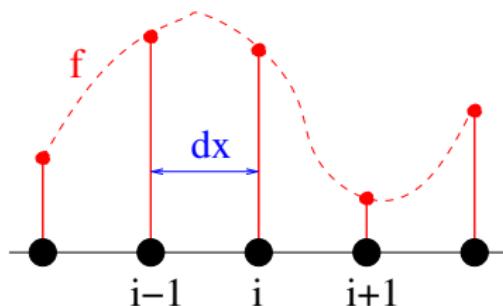
As a conclusion  $K = L^*$  is the reproducing kernel of  $\mathcal{H}$ .  $\square$

## Example



$$L^* = \begin{pmatrix} 0.88 & -0.12 & 0.08 & -0.32 & -0.52 \\ -0.12 & 0.88 & 0.08 & -0.32 & -0.52 \\ 0.08 & 0.08 & 0.28 & -0.12 & -0.32 \\ -0.32 & -0.32 & -0.12 & 0.48 & 0.28 \\ -0.52 & -0.52 & -0.32 & 0.28 & 1.08 \end{pmatrix}$$

## Interpretation of the Laplacian



$$\begin{aligned}\Delta f(x) &= f''(x) \\ &\sim \frac{f'(x + dx/2) - f'(x - dx/2)}{dx} \\ &\sim \frac{f(x + dx) - f(x) - f(x) + f(x - dx)}{dx^2} \\ &= \frac{f_{i-1} + f_{i+1} - 2f(i)}{dx^2} \\ &= -\frac{Lf(i)}{dx^2}.\end{aligned}$$

## Interpretation of regularization

For  $f = [0, 1] \rightarrow \mathbb{R}$  and  $x_i = i/m$ , we have:

$$\begin{aligned}\Omega(f) &= \sum_{i=1}^m \left( f\left(\frac{i+1}{m}\right) - f\left(\frac{i}{m}\right) \right)^2 \\ &\sim \sum_{i=1}^m \left( \frac{1}{m} \times f'\left(\frac{i}{m}\right) \right)^2 \\ &= \frac{1}{m} \times \frac{1}{m} \sum_{i=1}^m f'\left(\frac{i}{m}\right)^2 \\ &\sim \frac{1}{m} \int_0^1 f'(t)^2 dt.\end{aligned}$$

# Outline

## 5 The Kernel Jungle

- Kernels for probabilistic models
- Kernels for biological sequences
- Mercer kernels and shift-invariant kernels
- Kernels for graphs
- **Kernels on graphs**
  - Motivation
  - Graph distance and p.d. kernels
  - Construction by regularization
  - **The diffusion kernel**
  - Harmonic analysis on graphs
  - Applications

## Motivation

- Consider the normalized Gaussian kernel on  $\mathbb{R}^d$ :

$$K_t(\mathbf{x}, \mathbf{x}') = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{4t}\right).$$

- In order to transpose it to the graph, replacing the Euclidean distance by the shortest-path distance does not work.
- In this section we provide a characterization of the Gaussian kernel as the **solution of a partial differential equation** involving the Laplacian, which we can transpose to the graph: the **diffusion equation**.
- The solution of the discrete diffusion equation will be called the **diffusion kernel** or **heat kernel**.

# The diffusion equation

## Lemma

For any  $\mathbf{x}_0 \in \mathbb{R}^d$ , the function:

$$K_{\mathbf{x}_0}(\mathbf{x}, t) = K_t(\mathbf{x}_0, \mathbf{x}) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}_0\|^2}{4t}\right)$$

is solution of the *diffusion equation*:

$$\frac{\partial}{\partial t} K_{\mathbf{x}_0}(\mathbf{x}, t) = \Delta K_{\mathbf{x}_0}(\mathbf{x}, t)$$

with initial condition  $K_{\mathbf{x}_0}(\mathbf{x}, 0) = \delta_{\mathbf{x}_0}(\mathbf{x})$

(proof by direct computation).

## Discrete diffusion equation

For finite-dimensional  $f_t \in \mathbb{R}^m$ , the diffusion equation becomes:

$$\frac{\partial}{\partial t} f_t = -L f_t$$

which admits the following solution:

$$f_t = f_0 e^{-tL}$$

with

$$e^{tL} = I - tL + \frac{t^2}{2!} L^2 - \frac{t^3}{3!} L^3 + \dots$$

## Diffusion kernel (Kondor and Lafferty, 2002)

This suggest to consider:

$$K = e^{-tL}$$

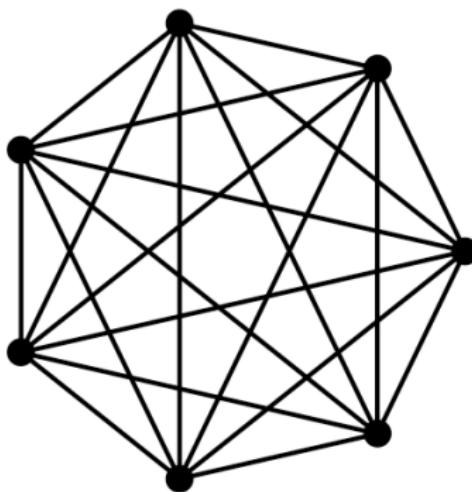
which is indeed symmetric positive semi-definite because if we write:

$$L = \sum_{i=1}^m \lambda_i u_i u_i^\top \quad (\lambda_i \geq 0)$$

we obtain:

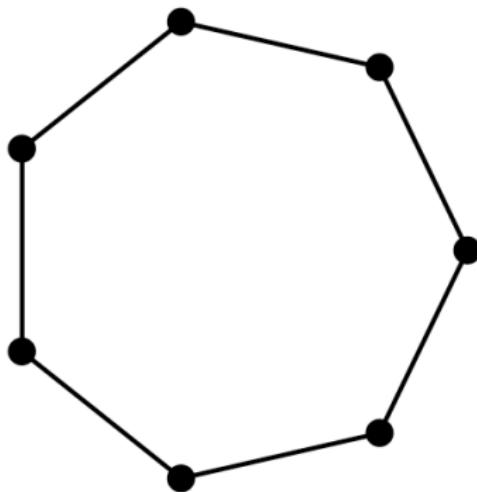
$$K = e^{-tL} = \sum_{i=1}^m e^{-t\lambda_i} u_i u_i^\top$$

## Example: complete graph



$$K_{i,j} = \begin{cases} \frac{1+(m-1)e^{-tm}}{m} & \text{for } i = j, \\ \frac{1-e^{-tm}}{m} & \text{for } i \neq j. \end{cases}$$

## Example: closed chain



$$K_{i,j} = \frac{1}{m} \sum_{\nu=0}^{m-1} \exp \left[ -2t \left( 1 - \cos \frac{2\pi\nu}{m} \right) \right] \cos \frac{2\pi\nu(i-j)}{m}.$$

# Outline

## 5 The Kernel Jungle

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  - The diffusion kernel
- **Harmonic analysis on graphs**
- Applications

## Motivation

- In this section we show that the diffusion and Laplace kernels can be interpreted in the **frequency domain** of functions
- This shows that our strategy to design kernels on graphs was based on **(discrete) harmonic analysis** on the graph
- This follows the approach we developed for semigroup kernels!

## Spectrum of the diffusion kernel

- Let  $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_m$  be the eigenvalues of the Laplacian:

$$L = \sum_{i=1}^m \lambda_i u_i u_i^\top \quad (\lambda_i \geq 0)$$

- The diffusion kernel  $K_t$  is an **invertible** matrix because its eigenvalues are strictly positive:

$$K_t = \sum_{i=1}^m e^{-t\lambda_i} u_i u_i^\top$$

## Norm in the diffusion RKHS

- Any function  $f \in \mathbb{R}^m$  can be written as  $f = K(K^{-1}f)$ , therefore its norm in the diffusion RKHS is:

$$\|f\|_{K_t}^2 = (f^\top K^{-1}) K (K^{-1}f) = f^\top K^{-1} f.$$

- For  $i = 1, \dots, m$ , let:

$$\hat{f}_i = u_i^\top f$$

be the projection of  $f$  onto the eigenbasis of  $K$ .

- We then have:

$$\|f\|_{K_t}^2 = f^\top K^{-1} f = \sum_{i=1}^m e^{t\lambda_i} \hat{f}_i^2.$$

- This looks similar to  $\int |\hat{f}(\omega)|^2 e^{\sigma^2 \omega^2} d\omega \dots$

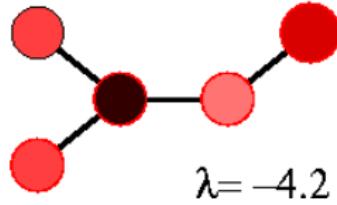
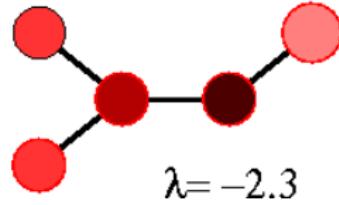
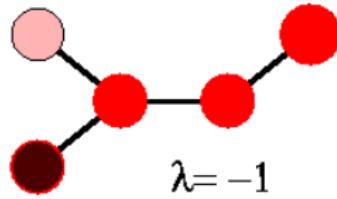
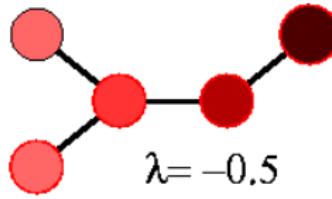
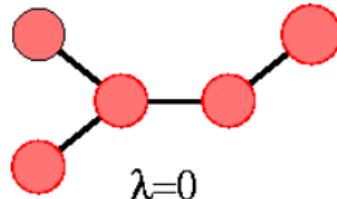
# Discrete Fourier transform

## Definition

The vector  $\hat{f} = (\hat{f}_1, \dots, \hat{f}_m)^\top$  is called the **discrete Fourier transform** of  $f \in \mathbb{R}^n$

- The eigenvectors of the Laplacian are the discrete equivalent to the sine/cosine Fourier basis on  $\mathbb{R}^n$ .
- The eigenvalues  $\lambda_i$  are the equivalent to the frequencies  $\omega^2$
- Successive eigenvectors “oscillate” increasingly as eigenvalues get more and more negative.

## Example: eigenvectors of the Laplacian



## Generalization

This observation suggests to define a whole family of kernels:

$$K_r = \sum_{i=1}^m r(\lambda_i) u_i u_i^\top$$

associated with the following RKHS norms:

$$\|f\|_{K_r}^2 = \sum_{i=1}^m \frac{\hat{f}_i^2}{r(\lambda_i)}$$

where  $r : \mathbb{R}^+ \rightarrow \mathbb{R}_*^+$  is a **non-increasing** function.

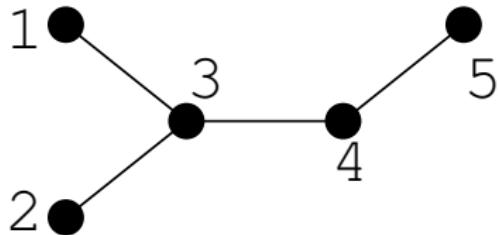
## Example : regularized Laplacian

$$r(\lambda) = \frac{1}{\lambda + \epsilon}, \quad \epsilon > 0$$

$$K = \sum_{i=1}^m \frac{1}{\lambda_i + \epsilon} u_i u_i^\top = (L + \epsilon I)^{-1}$$

$$\| f \|_K^2 = f^\top K^{-1} f = \sum_{i \sim j} (f(\mathbf{x}_i) - f(\mathbf{x}_j))^2 + \epsilon \sum_{i=1}^m f(\mathbf{x}_i)^2.$$

## Example



$$(L + I)^{-1} = \begin{pmatrix} 0.60 & 0.10 & 0.19 & 0.08 & 0.04 \\ 0.10 & 0.60 & 0.19 & 0.08 & 0.04 \\ 0.19 & 0.19 & 0.38 & 0.15 & 0.08 \\ 0.08 & 0.08 & 0.15 & 0.46 & 0.23 \\ 0.04 & 0.04 & 0.08 & 0.23 & 0.62 \end{pmatrix}$$

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- **Applications**

## Applications 1: graph partitioning

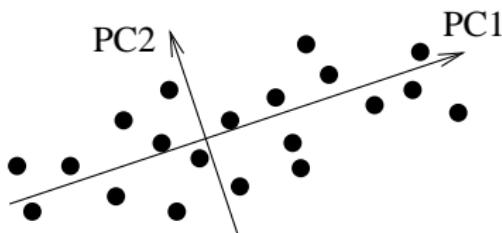
- A classical relaxation of graph partitioning is:

$$\min_{f \in \mathbb{R}^X} \sum_{i \sim j} (f_i - f_j)^2 \quad \text{s.t. } \sum_i f_i^2 = 1$$

- This can be rewritten

$$\max_f \sum_i f_i^2 \text{ s.t. } \|f\|_{\mathcal{H}} \leq 1$$

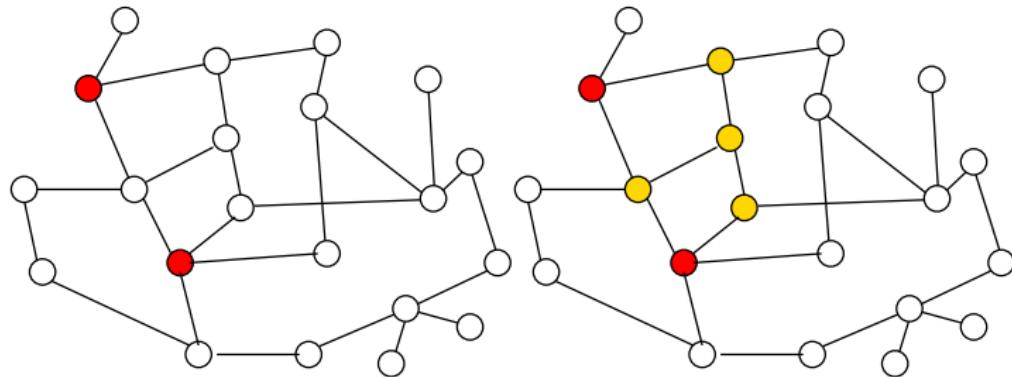
- This is **principal component analysis** in the RKHS (“kernel PCA”)



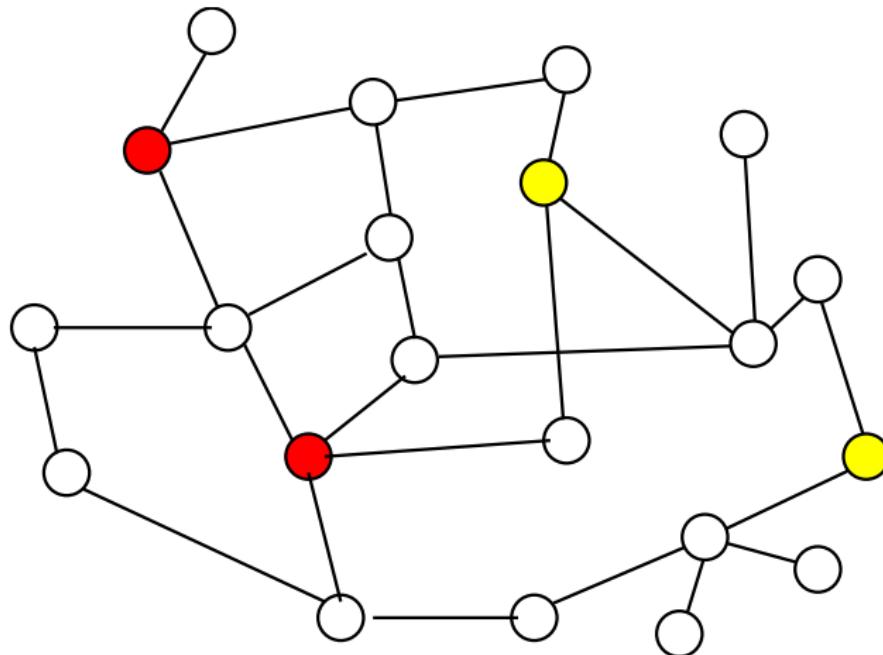
## Applications 2: search on a graph

- Let  $x_1, \dots, x_q$  be a set of  $q$  nodes (the **query**). How to find “similar” nodes (and rank them)?
- One solution:

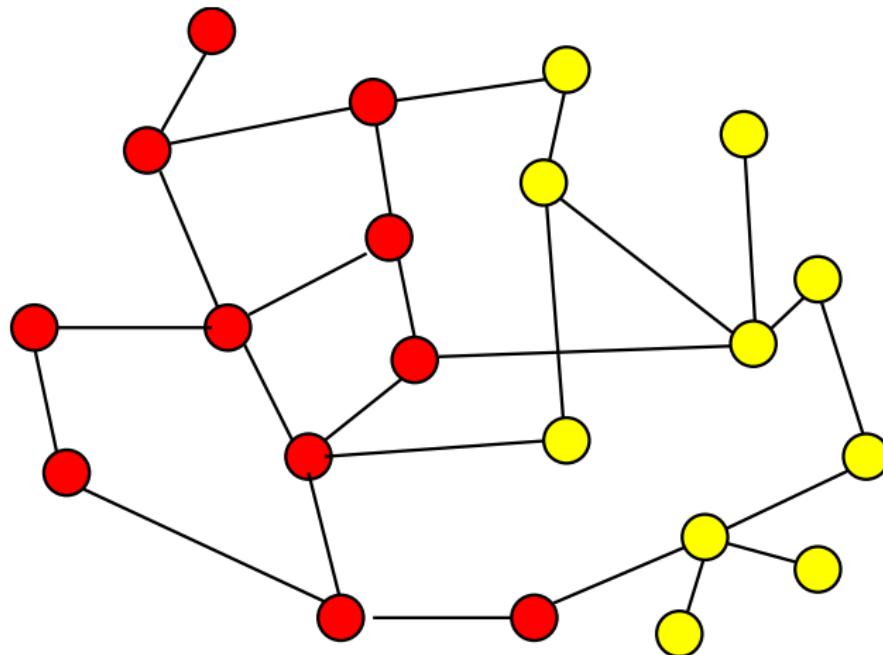
$$\min_f \|f\|_{\mathcal{H}} \quad \text{s.t.} \quad f(x_i) \geq 1 \text{ for } i = 1, \dots, q.$$



## Application 3: Semi-supervised learning



## Application 3: Semi-supervised learning



## Application 4: Tumor classification from microarray data (Rapaport et al., 2006)

### Data available

- Gene expression measures for **more than 10k genes**
- Measured on **less than 100 samples** of two (or more) different classes (e.g., different tumors)

## Application 4: Tumor classification from microarray data (Rapaport et al., 2006)

### Data available

- Gene expression measures for **more than 10k genes**
- Measured on **less than 100 samples** of two (or more) different classes (e.g., different tumors)

### Goal

- Design a **classifier** to automatically assign a class to future samples from their expression profile
- **Interpret** biologically the differences between the classes

# Linear classifiers

## The approach

- Each sample is represented by a vector  $x = (x_1, \dots, x_p)$  where  $p > 10^5$  is the number of probes
- **Classification:** given the set of labeled sample, learn a linear decision function:

$$f(x) = \sum_{i=1}^p \beta_i x_i + \beta_0 ,$$

that is positive for one class, negative for the other

- **Interpretation:** the weight  $\beta_i$  quantifies the influence of gene  $i$  for the classification

# Linear classifiers

## Pitfalls

- No robust estimation procedure exist for 100 samples in  $10^5$  dimensions!
- It is necessary to reduce the complexity of the problem with prior knowledge.

# Example : Norm Constraints

## The approach

A common method in statistics to learn with few samples in high dimension is to **constrain the norm of  $\beta$** , e.g.:

- Euclidean norm (support vector machines, ridge regression):  
$$\|\beta\|_2 = \sum_{i=1}^p \beta_i^2$$
- $L_1$ -norm (lasso regression) : 
$$\|\beta\|_1 = \sum_{i=1}^p |\beta_i|$$

## Pros

- Good performance in classification

## Cons

- Limited interpretation (small weights)
- No prior biological knowledge

## Example 2: Feature Selection

### The approach

Constrain most weights to be 0, i.e., **select a few genes** (< 20) whose expression are enough for classification. Interpretation is then about the selected genes.

#### Pros

- Good performance in classification
- Useful for **biomarker** selection
- Apparently easy interpretation

#### Cons

- The gene selection process is usually **not robust**
- Wrong interpretation is the rule (too much correlation between genes)

# Pathway interpretation

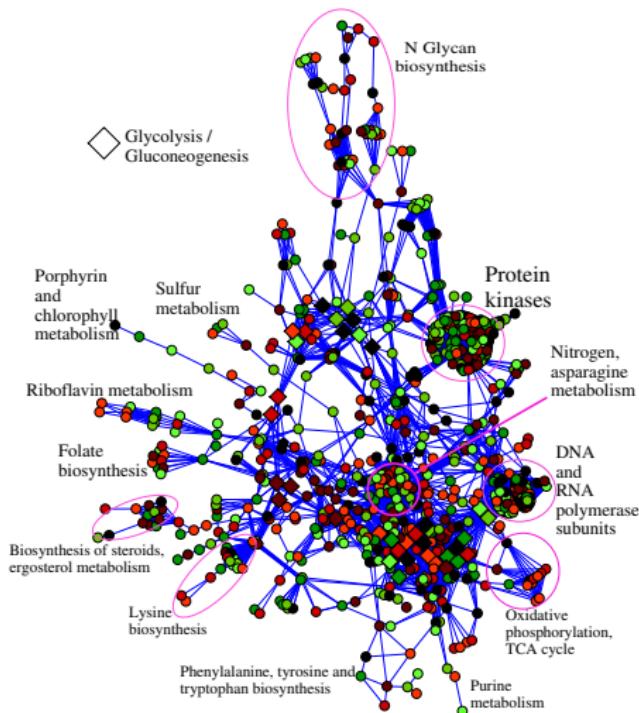
## Motivation

- Basic biological functions are usually expressed in terms of **pathways** and not of single genes (metabolic, signaling, regulatory)
- Many pathways are already known
- How to use this prior knowledge to **constrain the weights to have an interpretation at the level of pathways?**

## Solution (Rapaport et al., 2006)

- **Constrain the diffusion RKHS norm of  $\beta$**
- Relevant if the true decision function is indeed smooth w.r.t. the biological network

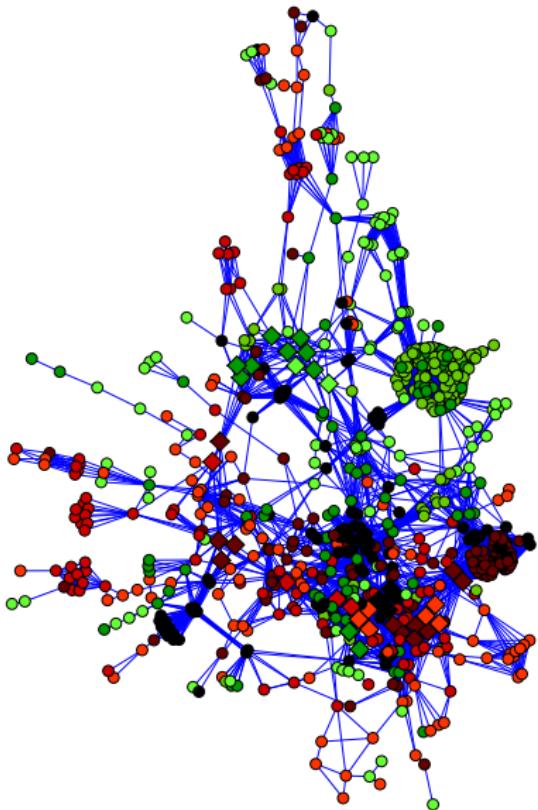
# Pathway interpretation



## Bad example

- The graph is the complete known **metabolic network** of the budding yeast (from KEGG database)
- We project the **classifier weight** learned by a SVM
- Good classification accuracy, but **no possible interpretation!**

# Pathway interpretation



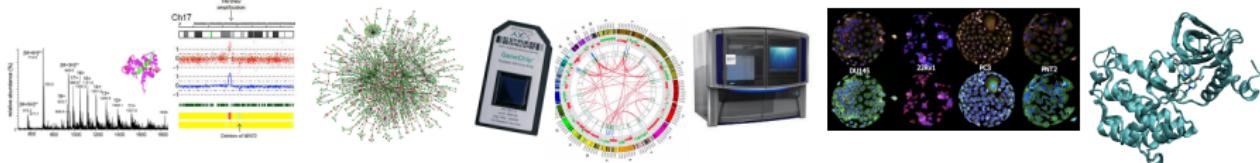
## Good example

- The graph is the complete known **metabolic network** of the budding yeast (from KEGG database)
- We project the **classifier weight** learned by a spectral SVM
- Good classification accuracy, **and good interpretation!**

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics
  - Multiple Kernel Learning (MKL)
  - Large-scale learning with kernels
  - “Deep” learning with kernels

# Motivation



- We have seen how to make learning algorithms given a kernel  $K$  on some data space  $\mathcal{X}$
- Often we may have **several possible kernels**:
  - by **varying the kernel type or parameters** on a given description of the data (eg, linear, polynomial, Gaussian kernels with different bandwidths...)
  - because we have **different views of the same data**, eg, a protein can be characterized by its sequence, its structure, its mass spectrometry profile...
- How to **choose or integrate** different kernels in a learning task?

## Setting: learning with one kernel

- For any  $f : \mathcal{X} \rightarrow \mathbb{R}$ , let  $f^n = (f(\mathbf{x}_1), \dots, f(\mathbf{x}_n)) \in \mathbb{R}^n$
- Given a p.d. kernel  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , we learn with  $K$  by solving:

$$\min_{f \in \mathcal{H}_K} R(f^n) + \lambda \|f\|_{\mathcal{H}_K}^2, \quad (3)$$

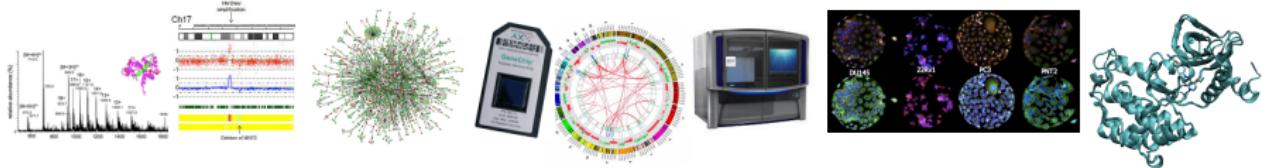
where  $\lambda > 0$  and  $R : \mathbb{R}^n \rightarrow \mathbb{R}$  is an **closed**<sup>2</sup> and **convex** empirical risk:

- $R(u) = \frac{1}{n} \sum_{i=1}^n (u_i - y_i)^2$  for kernel ridge regression
- $R(u) = \frac{1}{n} \sum_{i=1}^n \max(1 - y_i u_i, 0)$  for SVM
- $R(u) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i u_i))$  for kernel logistic regression

---

<sup>2</sup> $R$  is closed if, for each  $A \in \mathbb{R}$ , the sublevel set  $\{u \in \mathbb{R}^n : R(u) \leq A\}$  is closed. For example, if  $R$  is continuous then it is closed.

# Sum kernel



## Definition

Let  $K_1, \dots, K_M$  be  $M$  kernels on  $\mathcal{X}$ . The sum kernel  $K_S$  is the kernel on  $\mathcal{X}$  defined as

$$\forall \mathbf{x}, \mathbf{x}' \in \mathcal{X}, \quad K_S(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^M K_i(\mathbf{x}, \mathbf{x}').$$

## Sum kernel and vector concatenation

### Theorem

For  $i = 1, \dots, M$ , let  $\Phi_i : \mathcal{X} \rightarrow \mathcal{H}_i$  be a feature map such that

$$K_i(\mathbf{x}, \mathbf{x}') = \langle \Phi_i(\mathbf{x}), \Phi_i(\mathbf{x}') \rangle_{\mathcal{H}_i}.$$

Then  $K_S = \sum_{i=1}^M K_i$  can be written as:

$$K_S(\mathbf{x}, \mathbf{x}') = \langle \Phi_S(\mathbf{x}), \Phi_S(\mathbf{x}') \rangle_{\mathcal{H}_S},$$

where  $\Phi_S : \mathcal{X} \rightarrow \mathcal{H}_S = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_M$  is the **concatenation** of the feature maps  $\Phi_i$ :

$$\Phi_S(\mathbf{x}) = (\Phi_1(\mathbf{x}), \dots, \Phi_M(\mathbf{x}))^\top.$$

Therefore, summing kernels amounts to concatenating their feature space representations, which is a quite natural way to integrate different features.

## Proof

For  $\Phi_S(\mathbf{x}) = (\Phi_1(\mathbf{x}), \dots, \Phi_M(\mathbf{x}))^\top$ , we easily compute:

$$\begin{aligned}\langle \Phi_S(\mathbf{x}), \Phi_S(\mathbf{x}') \rangle_{\mathcal{H}_S} &= \sum_{i=1}^M \langle \Phi_i(\mathbf{x}), \Phi_i(\mathbf{x}') \rangle_{\mathcal{H}_i} \\ &= \sum_{i=1}^M K_i(\mathbf{x}, \mathbf{x}') \\ &= K_S(\mathbf{x}, \mathbf{x}').\end{aligned}$$

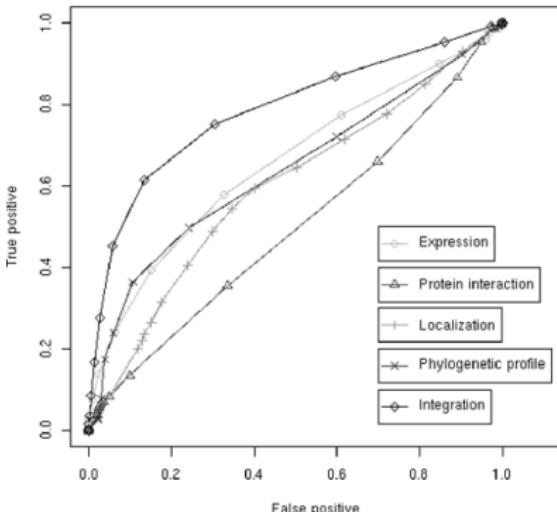
# Example: data integration with the sum kernel



## Protein network inference from multiple genomic data: a supervised approach

Y. Yamanishi<sup>1,\*</sup>, J.-P. Vert<sup>2</sup> and M. Kanehisa<sup>1</sup>

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$K_{\text{exp}}$  (Expression)

$K_{\text{ppi}}$  (Protein interaction)

$K_{\text{loc}}$  (Localization)

$K_{\text{phy}}$  (Phylogenetic profile)

$K_{\text{exp}} + K_{\text{ppi}} + K_{\text{loc}} + K_{\text{phy}}$   
(Integration)

# The sum kernel: functional point of view

## Theorem

The solution  $f^* \in \mathcal{H}_{K_S}$  when we learn with  $K_S = \sum_{i=1}^M K_i$  is equal to:

$$f^* = \sum_{i=1}^M f_i^*,$$

where  $(f_1^*, \dots, f_M^*) \in \mathcal{H}_{K_1} \times \dots \times \mathcal{H}_{K_M}$  is the solution of:

$$\min_{f_1, \dots, f_M} R \left( \sum_{i=1}^M f_i^n \right) + \lambda \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}}^2.$$

## Generalization: The weighted sum kernel

### Theorem

The solution  $f^*$  when we learn with  $K_{\eta} = \sum_{i=1}^M \eta_i K_i$ , with  $\eta_1, \dots, \eta_M \geq 0$ , is equal to:

$$f^* = \sum_{i=1}^M f_i^*,$$

where  $(f_1^*, \dots, f_M^*) \in \mathcal{H}_{K_1} \times \dots \times \mathcal{H}_{K_M}$  is the solution of:

$$\min_{f_1, \dots, f_M} R \left( \sum_{i=1}^M f_i^n \right) + \lambda \sum_{i=1}^M \frac{\| f_i \|_{\mathcal{H}_{K_i}}^2}{\eta_i}.$$

## Proof (1/4)

$$\min_{f_1, \dots, f_M} R \left( \sum_{i=1}^M f_i^n \right) + \lambda \sum_{i=1}^M \frac{\| f_i \|_{\mathcal{H}_{K_i}}^2}{\eta_i}.$$

- $R$  being convex, the problem is strictly convex and has a **unique solution**  $(f_1^*, \dots, f_M^*) \in \mathcal{H}_{K_1} \times \dots \times \mathcal{H}_{K_M}$ .
- By the representer theorem, there exists  $\alpha_1^*, \dots, \alpha_M^* \in \mathbb{R}^n$  such that

$$f_i^*(\mathbf{x}) = \sum_{j=1}^n \alpha_{ij}^* K_i(\mathbf{x}_j, \mathbf{x}).$$

- $(\alpha_1^*, \dots, \alpha_M^*)$  is the solution of

$$\min_{\alpha_1, \dots, \alpha_M \in \mathbb{R}^n} R \left( \sum_{i=1}^M \mathbf{K}_i \alpha_i \right) + \lambda \sum_{i=1}^M \frac{\alpha_i^\top \mathbf{K}_i \alpha_i}{\eta_i}.$$

## Proof (2/4)

- This is equivalent to

$$\min_{\mathbf{u}, \alpha_1, \dots, \alpha_M \in \mathbb{R}^n} R(\mathbf{u}) + \lambda \sum_{i=1}^M \frac{\alpha_i^\top \mathbf{K}_i \alpha_i}{\eta_i} \quad \text{s.t.} \quad \mathbf{u} = \sum_{i=1}^M \mathbf{K}_i \alpha_i.$$

- This is equivalent to the saddle point problem:

$$\min_{\mathbf{u}, \alpha_1, \dots, \alpha_M \in \mathbb{R}^n} \max_{\gamma \in \mathbb{R}^n} R(\mathbf{u}) + \lambda \sum_{i=1}^M \frac{\alpha_i^\top \mathbf{K}_i \alpha_i}{\eta_i} + 2\lambda \gamma^\top (\mathbf{u} - \sum_{i=1}^M \mathbf{K}_i \alpha_i).$$

- By Slater's condition, strong duality holds, meaning we can invert min and max:

$$\max_{\gamma \in \mathbb{R}^n} \min_{\mathbf{u}, \alpha_1, \dots, \alpha_M \in \mathbb{R}^n} R(\mathbf{u}) + \lambda \sum_{i=1}^M \frac{\alpha_i^\top \mathbf{K}_i \alpha_i}{\eta_i} + 2\lambda \gamma^\top (\mathbf{u} - \sum_{i=1}^M \mathbf{K}_i \alpha_i).$$

## Proof (3/4)

- Minimization in  $\mathbf{u}$ :

$$\min_{\mathbf{u}} R(\mathbf{u}) + 2\lambda\gamma^\top \mathbf{u} = -\max_{\mathbf{u}} \left\{ -2\lambda\gamma^\top \mathbf{u} - R(\mathbf{u}) \right\} = -R^*(-2\lambda\gamma),$$

where  $R^*$  is the Fenchel dual of  $R$ :

$$\forall \mathbf{v} \in \mathbb{R}^n \quad R^*(\mathbf{v}) = \sup_{\mathbf{u} \in \mathbb{R}^n} \mathbf{u}^\top \mathbf{v} - R(\mathbf{u}).$$

- Minimization in  $\alpha_i$  for  $i = 1, \dots, M$ :

$$\min_{\alpha_i} \left\{ \lambda \frac{\alpha_i^\top \mathbf{K}_i \alpha_i}{\eta_i} - 2\lambda\gamma^\top \mathbf{K}_i \alpha_i \right\} = -\lambda\eta_i \gamma^\top \mathbf{K}_i \gamma,$$

where the minimum in  $\alpha_i$  is reached for  $\alpha_i^* = \eta_i \gamma$ .

## Proof (4/4)

- The dual problem is therefore

$$\max_{\gamma \in \mathbb{R}^n} \left\{ -R^*(-2\lambda\gamma) - \lambda\gamma^\top \left( \sum_{i=1}^M \eta_i \mathbf{K}_i \right) \gamma \right\}.$$

- Note that if learn from a single kernel  $\mathbf{K}_\eta$ , we get the same dual problem

$$\max_{\gamma \in \mathbb{R}^n} \left\{ -R^*(-2\lambda\gamma) - \lambda\gamma^\top \mathbf{K}_\eta \gamma \right\}.$$

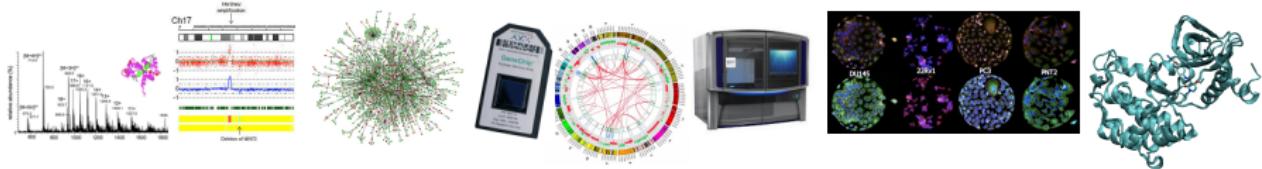
- If  $\gamma^*$  is a solution of the dual problem, then  $\alpha_i^* = \eta_i \gamma^*$  leading to:

$$\forall \mathbf{x} \in \mathcal{X}, \quad f_i^*(\mathbf{x}) = \sum_{j=1}^n \alpha_{ij}^* \mathbf{K}_i(\mathbf{x}_j, \mathbf{x}) = \sum_{j=1}^n \eta_i \gamma_j^* \mathbf{K}_i(\mathbf{x}_j, \mathbf{x})$$

- Therefore,  $f^* = \sum_{i=1}^M f_i^*$  satisfies

$$f^*(\mathbf{x}) = \sum_{i=1}^M \sum_{j=1}^n \eta_i \gamma_j^* \mathbf{K}_i(\mathbf{x}_j, \mathbf{x}) = \sum_{j=1}^n \gamma_j^* \mathbf{K}_\eta(\mathbf{x}_j, \mathbf{x}). \quad \square$$

# Learning the kernel



## Motivation

- If we know how to weight each kernel, then we can learn with the weighted kernel

$$\mathbf{K}_\eta = \sum_{i=1}^M \eta_i \mathbf{K}_i$$

- However, usually we don't know...
- Perhaps we can optimize the weights  $\eta_i$  during learning?

# An objective function for $K$

## Theorem

For any p.d. kernel  $K$  on  $\mathcal{X}$ , let

$$J(K) = \min_{f \in \mathcal{H}_K} \{R(f^n) + \lambda \|f\|_{\mathcal{H}_K}^2\}.$$

The function  $K \mapsto J(K)$  is **convex**.

This suggests a principled way to "learn" a kernel: define a convex set of candidate kernels, and minimize  $J(K)$  by convex optimization.

## Proof

- We have shown by strong duality that

$$J(K) = \max_{\gamma \in \mathbb{R}^n} \left\{ -R^*(-2\lambda\gamma) - \lambda\gamma^\top \mathbf{K}\gamma \right\}.$$

- For each  $\gamma$  fixed, this is an affine function of  $K$ , hence convex
- A supremum of convex functions is convex. □



## MKL (Lanckriet et al., 2004)

- We consider the set of **convex combinations**

$$K_{\boldsymbol{\eta}} = \sum_{i=1}^M \eta_i K_i \quad \text{with} \quad \boldsymbol{\eta} \in \Sigma_M = \left\{ \boldsymbol{\eta}_i \geq 0, \sum_{i=1}^M \eta_i = 1 \right\}$$

- We optimize both  $\boldsymbol{\eta}$  and  $f^*$  by solving:

$$\min_{\boldsymbol{\eta} \in \Sigma_M} J(K_{\boldsymbol{\eta}}) = \min_{\boldsymbol{\eta} \in \Sigma_M} \min_{f \in \mathcal{H}_{K_{\boldsymbol{\eta}}}} \left\{ R(f^n) + \lambda \|f\|_{\mathcal{H}_{K_{\boldsymbol{\eta}}}}^2 \right\}$$

- The problem is **jointly convex** in  $(\boldsymbol{\eta}, \boldsymbol{\alpha})$  and can be solved efficiently.
- The output is both a set of weights  $\boldsymbol{\eta}$ , and a predictor corresponding to the kernel method trained with kernel  $K_{\boldsymbol{\eta}}$ .
- This method is usually called **Multiple Kernel Learning (MKL)**.

# Example: protein annotation



## A statistical framework for genomic data fusion

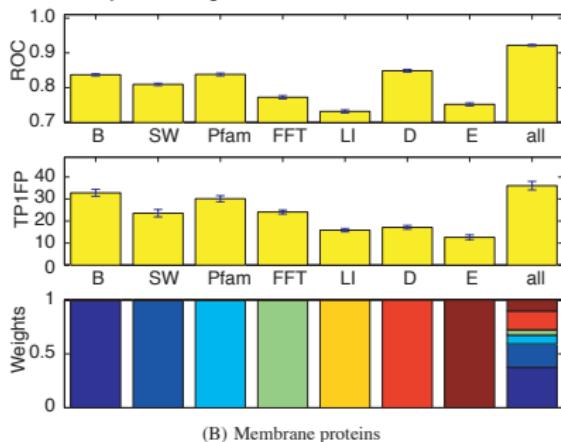
Gert R. G. Lanckriet<sup>1</sup>, Tijl De Bie<sup>3</sup>, Nello Cristianini<sup>4</sup>,  
Michael I. Jordan<sup>2</sup> and William Stafford Noble<sup>5,\*</sup>

<sup>1</sup>Department of Electrical Engineering and Computer Science, <sup>2</sup>Division of Computer Science, Department of Statistics, University of California, Berkeley 94720, USA,

<sup>3</sup>Department of Electrical Engineering, ESAT-SCD, Katholieke Universiteit Leuven 3001, Belgium,

<sup>4</sup>Department of Statistics, University of California, Davis 95618, USA and

<sup>5</sup>Department of Genome Sciences, University of Washington, Seattle 98195, USA

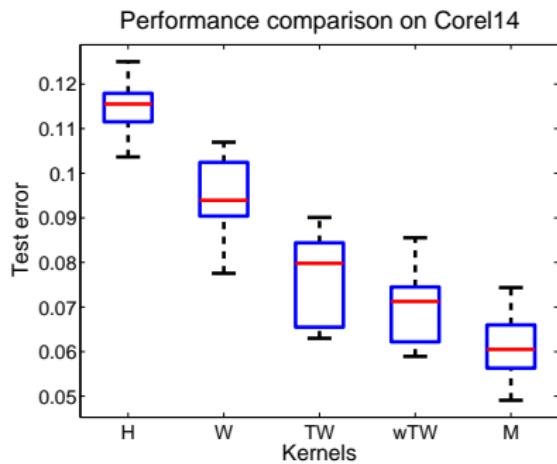


Kernel	Data	Similarity measure
$K_{SW}$	protein sequences	Smith-Waterman
$K_B$	protein sequences	BLAST
$K_{Pfam}$	protein sequences	Pfam HMM
$K_{FFT}$	hydrophathy profile	FFT
$K_{LI}$	protein interactions	linear kernel
$K_D$	protein interactions	diffusion kernel
$K_E$	gene expression	radial basis kernel
$K_{RND}$	random numbers	linear kernel

# Example: Image classification (Harchaoui and Bach, 2007)

## COREL14 dataset

- 1400 natural images in 14 classes
- Compare kernel between histograms (H), walk kernel (W), subtree kernel (TW), weighted subtree kernel (wTW), and a combination by MKL (M).



## MKL revisited (Bach et al., 2004)

$$K_{\boldsymbol{\eta}} = \sum_{i=1}^M \eta_i K_i \quad \text{with} \quad \boldsymbol{\eta} \in \Sigma_M = \left\{ \boldsymbol{\eta} \geq 0, \sum_{i=1}^M \eta_i = 1 \right\}$$

### Theorem

The solution  $f^*$  of

$$\min_{\boldsymbol{\eta} \in \Sigma_M} \min_{f \in \mathcal{H}_{K_{\boldsymbol{\eta}}}} \left\{ R(f^n) + \lambda \|f\|_{\mathcal{H}_{K_{\boldsymbol{\eta}}}}^2 \right\}$$

is  $f^* = \sum_{i=1}^M f_i^*$ , where  $(f_1^*, \dots, f_M^*) \in \mathcal{H}_{K_1} \times \dots \times \mathcal{H}_{K_M}$  is the solution of:

$$\min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \left( \sum_{i=1}^M \|f_i\|_{\mathcal{H}_{K_i}} \right)^2 \right\}.$$

## Proof (1/2)

$$\begin{aligned} & \min_{\eta \in \Sigma_M} \min_{f \in \mathcal{H}_{K_\eta}} \left\{ R(f^n) + \lambda \| f \|_{\mathcal{H}_{K_\eta}}^2 \right\} \\ &= \min_{\eta \in \Sigma_M} \min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \sum_{i=1}^M \frac{\| f_i \|_{\mathcal{H}_{K_i}}^2}{\eta_i} \right\} \\ &= \min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \min_{\eta \in \Sigma_M} \left\{ \sum_{i=1}^M \frac{\| f_i \|_{\mathcal{H}_{K_i}}^2}{\eta_i} \right\} \right\} \\ &= \min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \left( \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}} \right)^2 \right\}, \end{aligned}$$

## Proof (2/2)

where the last equality results from:

$$\forall \mathbf{a} \in \mathbb{R}_+^M, \quad \left( \sum_{i=1}^M a_i \right)^2 = \inf_{\boldsymbol{\eta} \in \Sigma_M} \sum_{i=1}^M \frac{a_i^2}{\eta_i},$$

which is a direct consequence of the Cauchy-Schwarz inequality:

$$\sum_{i=1}^M a_i = \sum_{i=1}^M \frac{a_i}{\sqrt{\eta_i}} \times \sqrt{\eta_i} \leq \left( \sum_{i=1}^M \frac{a_i^2}{\eta_i} \right)^{\frac{1}{2}} \left( \sum_{i=1}^M \eta_i \right)^{\frac{1}{2}}.$$

## Algorithm: simpleMKL (Rakotomamonjy et al., 2008)

- We want to minimize in  $\eta \in \Sigma_M$ :

$$\min_{\eta \in \Sigma_M} J(K_\eta) = \min_{\eta \in \Sigma_M} \max_{\gamma \in \mathbb{R}^n} \left\{ -R^*(-2\lambda\gamma) - \lambda\gamma^\top K_\eta \gamma \right\}.$$

- For a fixed  $\eta \in \Sigma_M$ , we can compute  $f(\eta) = J(K_\eta)$  by using a standard solver for a single kernel to find  $\gamma^*$ :

$$J(K_\eta) = -R^*(-2\lambda\gamma^*) - \lambda\gamma^{*\top} K_\eta \gamma^*.$$

- From  $\gamma^*$  we can also compute the gradient of  $J(K_\eta)$  with respect to  $\eta$ :

$$\frac{\partial J(K_\eta)}{\partial \eta_i} = -\lambda\gamma^{*\top} K_i \gamma^*.$$

- $J(K_\eta)$  can then be minimized on  $\Sigma_M$  by a projected gradient or reduced gradient algorithm.

## Sum kernel vs MKL

- Learning with the sum kernel (uniform combination) solves

$$\min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}}^2 \right\}.$$

- Learning with MKL (best convex combination) solves

$$\min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \left( \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}} \right)^2 \right\}.$$

- Although MKL can be thought of as optimizing a convex combination of kernels, it is more correct to think of it as a penalized risk minimization estimator with the **group lasso** penalty:

$$\Omega(f) = \min_{f_1 + \dots + f_M = f} \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}}.$$

## Example: ridge vs LASSO regression

- Take  $\mathcal{X} = \mathbb{R}^d$ , and for  $\mathbf{x} = (x_1, \dots, x_d)^\top$  consider the **rank-1 kernels**:

$$\forall i = 1, \dots, d, \quad K_i(\mathbf{x}, \mathbf{x}') = x_i x'_i.$$

- A function  $f_i \in \mathcal{H}_{K_i}$  has the form  $f_i(\mathbf{x}) = \beta_i x_i$ , with  $\|f_i\|_{\mathcal{H}_{K_i}} = |\beta_i|$
- The sum kernel is  $K_S(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^d x_i x'_i = \mathbf{x}^\top \mathbf{x}$ , a function  $\mathcal{H}_{K_S}$  is of the form  $f(\mathbf{x}) = \boldsymbol{\beta}^\top \mathbf{x}$ , with norm  $\|f\|_{\mathcal{H}_{K_S}} = \|\boldsymbol{\beta}\|_{\mathbb{R}^d}$ .
- Learning with the **sum kernel** solves a **ridge regression** problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^d} \left\{ R(\mathbf{X}\boldsymbol{\beta}) + \lambda \sum_{i=1}^d \beta_i^2 \right\}.$$

- Learning with **MKL** solves a **LASSO regression** problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^d} \left\{ R(\mathbf{X}\boldsymbol{\beta}) + \lambda \left( \sum_{i=1}^d |\beta_i| \right)^2 \right\}.$$

## Extensions (Micchelli et al., 2005)

For  $r > 0$ ,  $K_\eta = \sum_{i=1}^M \eta_i K_i$  with  $\eta \in \Sigma_M^r = \left\{ \eta_i \geq 0, \sum_{i=1}^M \eta_i^r = 1 \right\}$

### Theorem

The solution  $f^*$  of

$$\min_{\eta \in \Sigma_M^r} \min_{f \in \mathcal{H}_{K_\eta}} \left\{ R(f^n) + \lambda \| f \|_{\mathcal{H}_{K_\eta}}^2 \right\}$$

is  $f^* = \sum_{i=1}^M f_i^*$ , where  $(f_1^*, \dots, f_M^*) \in \mathcal{H}_{K_1} \times \dots \times \mathcal{H}_{K_M}$  is the solution of:

$$\min_{f_1, \dots, f_M} \left\{ R \left( \sum_{i=1}^M f_i^n \right) + \lambda \left( \sum_{i=1}^M \| f_i \|_{\mathcal{H}_{K_i}}^{\frac{2r}{r+1}} \right)^{\frac{r+1}{r}} \right\}.$$

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics
  - Multiple Kernel Learning (MKL)
  - Large-scale learning with kernels
  - “Deep” learning with kernels

# Outline

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
  - Motivation
  - Large-scale learning with linear models
  - Nyström approximations
  - Random Fourier features
  - New challenges
- “Deep” learning with kernels

# Motivation

## Main problem

All methods we have seen require computing the  $n \times n$  Gram matrix, which is infeasible when  $n$  is significantly greater than 100 000 both in terms of memory and computation.

## Solutions

- low-rank approximation of the kernel;
- random Fourier features.

The goal is to find an approximate embedding  $\psi : \mathcal{X} \rightarrow \mathbb{R}^d$  such that

$$K(\mathbf{x}, \mathbf{x}') \approx \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle_{\mathbb{R}^d}.$$

## Motivation

Then, functions  $f$  in  $\mathcal{H}$  may be approximated by linear ones in  $\mathbb{R}^d$ , e.g.,.

$$f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}) \approx \left\langle \sum_{i=1}^n \alpha_i \psi(\mathbf{x}_i), \psi(\mathbf{x}) \right\rangle_{\mathbb{R}^d} = \langle \mathbf{w}, \psi(\mathbf{x}) \rangle_{\mathbb{R}^d}.$$

Then, the ERM problem

$$\min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n L(y_i, f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{H}}^2,$$

becomes, approximately,

$$\min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n L(y_i, \mathbf{w}^\top \mathbf{x}_i) + \lambda \|\mathbf{w}\|_2^2,$$

which we know how to solve when  $n$  is large.

# Outline

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
  - Motivation
  - Large-scale learning with linear models
  - Nyström approximations
  - Random Fourier features
  - New challenges
- “Deep” learning with kernels

## Large-scale learning with linear models

Let us study for a while optimization techniques for minimizing large sums of functions

$$\min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n f_i(\mathbf{w}).$$

Good candidates are

- **stochastic** optimization techniques;
- **randomized incremental** optimization techniques;

We will see a couple of such algorithms with their convergence rates and start with the (batch) gradient descent method.

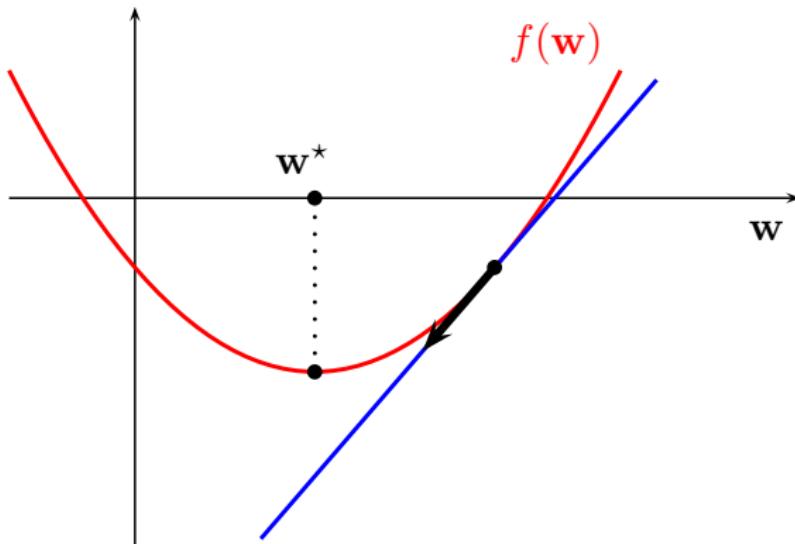
# Introduction of a few optimization principles

Why do we care about convexity?

# Introduction of a few optimization principles

Why do we care about convexity?

Local observations give information about the global optimum

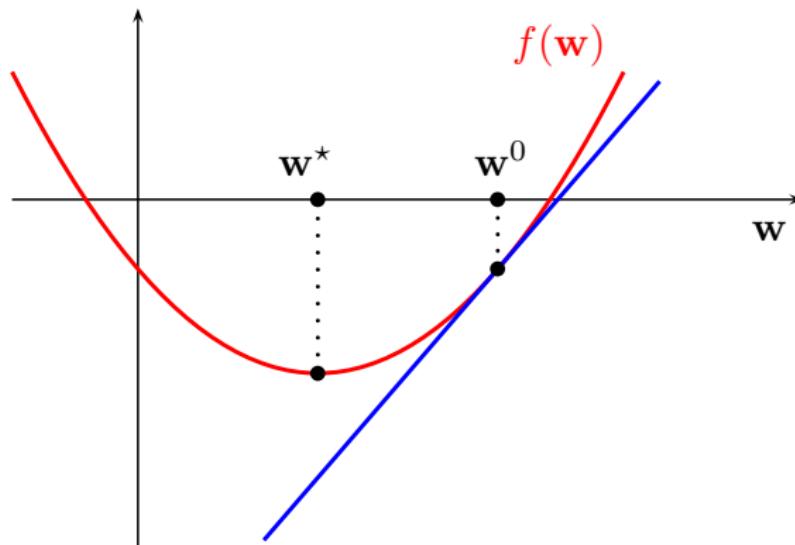


- $\nabla f(\mathbf{w}) = \mathbf{0}$  is a necessary and sufficient optimality condition for differentiable convex functions;
- it is often easy to upper-bound  $f(\mathbf{w}) - f^*$ .

# Introduction of a few optimization principles

An important inequality for smooth convex functions

If  $f$  is convex

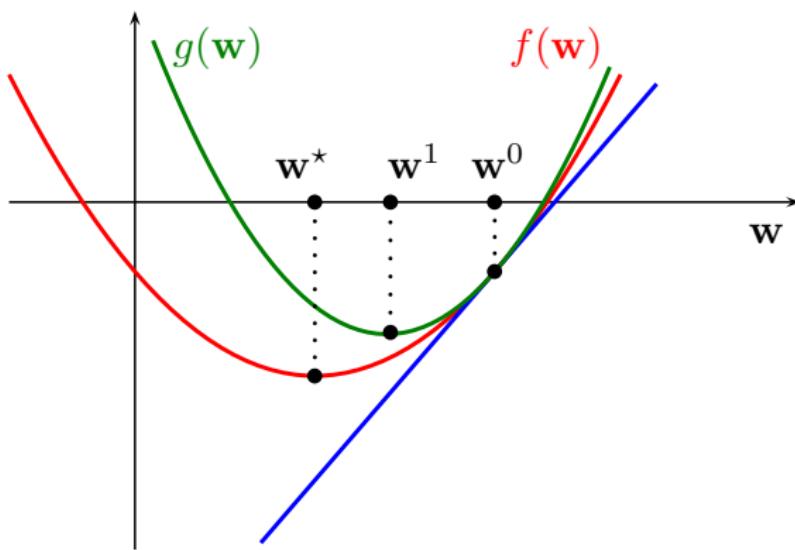


- $f(\mathbf{w}) \geq \underbrace{f(\mathbf{w}^0) + \nabla f(\mathbf{w}^0)^\top (\mathbf{w} - \mathbf{w}^0)}_{\text{linear approximation}};$
- this is an equivalent definition of convexity for smooth functions.

# Introduction of a few optimization principles

An important inequality for smooth functions

If  $\nabla f$  is  $L$ -Lipschitz continuous ( $f$  does not need to be convex)

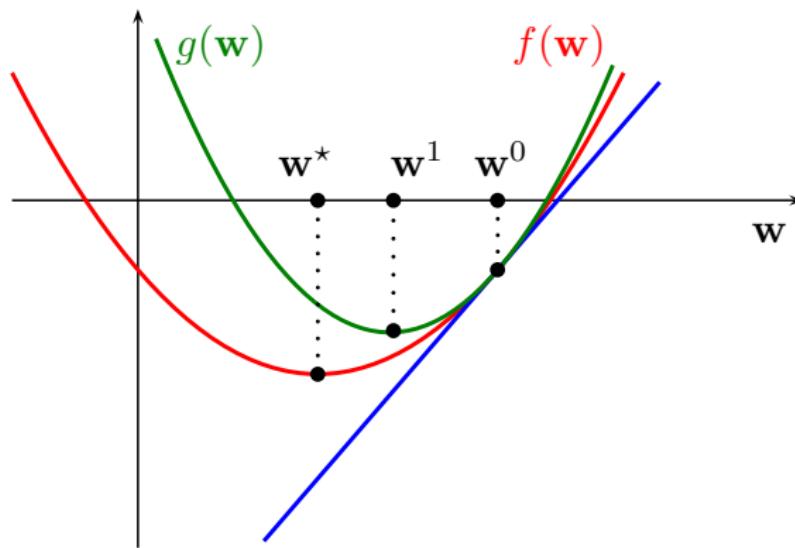


- $f(\mathbf{w}) \leq g(\mathbf{w}) = f(\mathbf{w}^0) + \nabla f(\mathbf{w}^0)^\top (\mathbf{w} - \mathbf{w}^0) + \frac{L}{2} \|\mathbf{w} - \mathbf{w}^0\|_2^2$ ;
- $g(\mathbf{w}) = C_{\mathbf{w}^0} + \frac{L}{2} \|\mathbf{w}^0 - (1/L) \nabla f(\mathbf{w}^0) - \mathbf{w}\|_2^2$ .

# Introduction of a few optimization principles

An important inequality for smooth functions

If  $\nabla f$  is  $L$ -Lipschitz continuous ( $f$  does not need to be convex)



- $f(\mathbf{w}) \leq g(\mathbf{w}) = f(\mathbf{w}^0) + \nabla f(\mathbf{w}^0)^\top (\mathbf{w} - \mathbf{w}^0) + \frac{L}{2} \|\mathbf{w} - \mathbf{w}^0\|_2^2;$
- $\boxed{\mathbf{w}^1 = \mathbf{w}^0 - \frac{1}{L} \nabla f(\mathbf{w}^0)}$  (gradient descent step).

# Introduction of a few optimization principles

## Gradient Descent Algorithm

Assume that  $f$  is convex and differentiable, and that  $\nabla f$  is  $L$ -Lipschitz.

### Theorem

Consider the algorithm

$$\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \frac{1}{L} \nabla f(\mathbf{w}^{t-1}).$$

Then,

$$f(\mathbf{w}^t) - f^* \leq \frac{L \|\mathbf{w}^0 - \mathbf{w}^*\|_2^2}{2t}.$$

### Remarks

- the convergence rate improves under additional assumptions on  $f$  (strong convexity);
- some variants have a  $O(1/t^2)$  convergence rate [Nesterov, 2004].

## Proof (1/2)

Proof of the main inequality for smooth functions

We want to show that for all  $\mathbf{w}$  and  $\mathbf{z}$ ,

$$f(\mathbf{w}) \leq f(\mathbf{z}) + \nabla f(\mathbf{z})^\top (\mathbf{w} - \mathbf{z}) + \frac{L}{2} \|\mathbf{w} - \mathbf{z}\|_2^2.$$

By using Taylor's theorem with integral form,

$$f(\mathbf{w}) - f(\mathbf{z}) = \int_0^1 \nabla f(t\mathbf{w} + (1-t)\mathbf{z})^\top (\mathbf{w} - \mathbf{z}) dt.$$

Then,

$$\begin{aligned} f(\mathbf{w}) - f(\mathbf{z}) - \nabla f(\mathbf{z})^\top (\mathbf{w} - \mathbf{z}) &\leq \int_0^1 (\nabla f(t\mathbf{w} + (1-t)\mathbf{z}) - \nabla f(\mathbf{z}))^\top (\mathbf{w} - \mathbf{z}) dt \\ &\leq \int_0^1 |(\nabla f(t\mathbf{w} + (1-t)\mathbf{z}) - \nabla f(\mathbf{z}))^\top (\mathbf{w} - \mathbf{z})| dt \\ &\leq \int_0^1 \|\nabla f(t\mathbf{w} + (1-t)\mathbf{z}) - \nabla f(\mathbf{z})\|_2 \|\mathbf{w} - \mathbf{z}\|_2 dt \quad (\text{C.-S.}) \\ &\leq \int_0^1 Lt \|\mathbf{w} - \mathbf{z}\|_2^2 dt = \frac{L}{2} \|\mathbf{w} - \mathbf{z}\|_2^2. \end{aligned}$$

## Proof (2/2)

### Proof of the theorem

We have shown that for all  $\mathbf{w}$ ,

$$f(\mathbf{w}) \leq g_t(\mathbf{w}) = f(\mathbf{w}^{t-1}) + \nabla f(\mathbf{w}^{t-1})^\top (\mathbf{w} - \mathbf{w}^{t-1}) + \frac{L}{2} \|\mathbf{w} - \mathbf{w}^{t-1}\|_2^2.$$

$g_t$  is minimized by  $\mathbf{w}^t$ ; it can be rewritten  $g_t(\mathbf{w}) = g_t(\mathbf{w}^t) + \frac{L}{2} \|\mathbf{w} - \mathbf{w}^t\|_2^2$ . Then,

$$\begin{aligned} f(\mathbf{w}^t) &\leq g_t(\mathbf{w}^t) = g_t(\mathbf{w}^*) - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2 \\ &= f(\mathbf{w}^{t-1}) + \nabla f(\mathbf{w}^{t-1})^\top (\mathbf{w}^* - \mathbf{w}^{t-1}) + \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2 - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2 \\ &\leq f^* + \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2 - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2. \end{aligned}$$

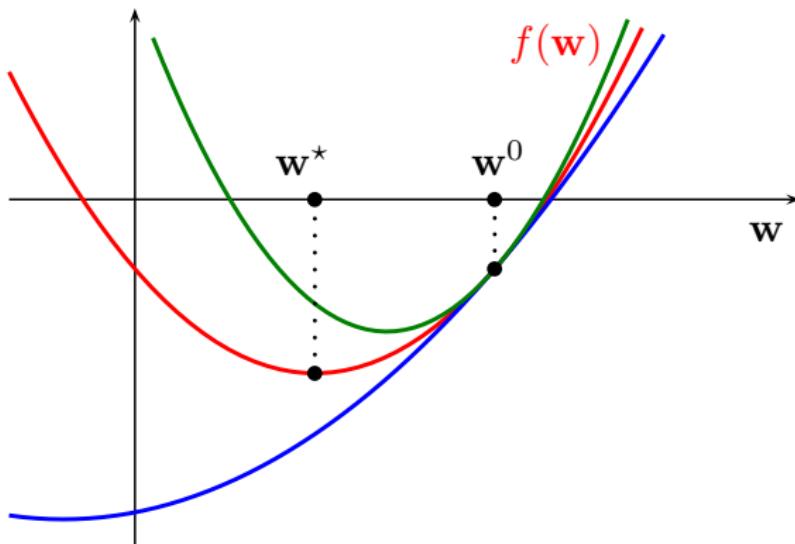
By summing from  $t = 1$  to  $T$ , we have a telescopic sum

$$T(f(\mathbf{w}^T) - f^*) \leq \sum_{t=1}^T f(\mathbf{w}^t) - f^* \leq \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^0\|_2^2 - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^T\|_2^2.$$

# Introduction of a few optimization principles

An important inequality for smooth and  $\mu$ -strongly convex functions

If  $\nabla f$  is  $L$ -Lipschitz continuous and  $f$   $\mu$ -strongly convex



- $f(\mathbf{w}) \leq f(\mathbf{w}^0) + \nabla f(\mathbf{w}^0)^\top (\mathbf{w} - \mathbf{w}^0) + \frac{L}{2} \|\mathbf{w} - \mathbf{w}^0\|_2^2;$
- $f(\mathbf{w}) \geq f(\mathbf{w}^0) + \nabla f(\mathbf{w}^0)^\top (\mathbf{w} - \mathbf{w}^0) + \frac{\mu}{2} \|\mathbf{w} - \mathbf{w}^0\|_2^2;$

# Introduction of a few optimization principles

## Proposition

When  $f$  is  $\mu$ -strongly convex, differentiable and  $\nabla f$  is  $L$ -Lipschitz, the gradient descent algorithm with step-size  $1/L$  produces iterates such that

$$f(\mathbf{w}^t) - f^* \leq \left(1 - \frac{\mu}{L}\right)^t \frac{L\|\mathbf{w}^0 - \mathbf{w}^*\|_2^2}{2}.$$

We call that a **linear** convergence rate (even though it has an exponential form).

## Proof

We start from an inequality from the previous proof

$$\begin{aligned} f(\mathbf{w}^t) &\leq f(\mathbf{w}^{t-1}) + \nabla f(\mathbf{w}^{t-1})^\top (\mathbf{w}^* - \mathbf{w}^{t-1}) + \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2 - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2 \\ &\leq f^* + \frac{L-\mu}{2} \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2 - \frac{L}{2} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2. \end{aligned}$$

In addition, we have that  $f(\mathbf{w}^t) \geq f^* + \frac{\mu}{2} \|\mathbf{w}^t - \mathbf{w}^*\|_2^2$ , and thus

$$\begin{aligned} \|\mathbf{w}^* - \mathbf{w}^t\|_2^2 &\leq \frac{L-\mu}{L+\mu} \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2 \\ &\leq \left(1 - \frac{\mu}{L}\right) \|\mathbf{w}^* - \mathbf{w}^{t-1}\|_2^2. \end{aligned}$$

Finally,

$$\begin{aligned} f(\mathbf{w}^t) - f^* &\leq \frac{L}{2} \|\mathbf{w}^t - \mathbf{w}^*\|_2^2 \\ &\leq \left(1 - \frac{\mu}{L}\right)^t \frac{L \|\mathbf{w}^* - \mathbf{w}^0\|_2^2}{2} \end{aligned}$$

# The stochastic (sub)gradient descent algorithm

Consider now the minimization of an expectation

$$\min_{\mathbf{w} \in \mathbb{R}^p} f(\mathbf{w}) = \mathbb{E}_{\mathbf{x}}[\ell(\mathbf{x}, \mathbf{w})],$$

To simplify, we assume that for all  $\mathbf{x}$ ,  $\mathbf{w} \mapsto \ell(\mathbf{x}, \mathbf{w})$  is differentiable, but everything here is true for nonsmooth functions.

## Algorithm

At iteration  $t$ ,

- Randomly draw one example  $\mathbf{x}_t$  from the training set;
- Update the current iterate

$$\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \eta_t \nabla_{\mathbf{w}} \ell(\mathbf{x}_t, \mathbf{w}_{t-1}).$$

- Perform online averaging of the iterates (optional)

$$\tilde{\mathbf{w}}^t \leftarrow (1 - \gamma_t) \tilde{\mathbf{w}}^{t-1} + \gamma_t \mathbf{w}^t.$$

## The stochastic (sub)gradient descent algorithm

There are various learning rates strategies (constant, varying step-sizes), and averaging strategies. Depending on the problem assumptions and choice of  $\eta_t$ ,  $\gamma_t$ , classical convergence rates may be obtained (see Nemirovsky et al., 2009)

- $f(\tilde{\mathbf{w}}^t) - f^* = O(1/\sqrt{t})$  for convex problems;
- $f(\tilde{\mathbf{w}}^t) - f^* = O(1/t)$  for strongly-convex ones;

### Remarks

- The convergence rates are not that great, but the complexity **per-iteration** is small (1 gradient evaluation for minimizing an empirical risk versus  $n$  for the batch algorithm).
- When the amount of data is infinite, the method **minimizes the expected risk**.
- Choosing a good learning rate automatically is an open problem.

## Randomized incremental algorithms (1/3)

Consider now the minimization of a large finite sum of smooth convex functions:

$$\min_{\mathbf{w} \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n f_i(\mathbf{w}),$$

A class of algorithms with low per-iteration complexity have been recently introduced that enjoy **exponential** (aka, linear) convergence rates for strongly-convex problems, e.g., SAG (Schmidt et al., 2013)

### SAG algorithm

$$\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \frac{\gamma}{Ln} \sum_{i=1}^n y_i^t \quad \text{with} \quad y_i^t = \begin{cases} \nabla f_i(\mathbf{w}^{t-1}) & \text{if } i = i_t \\ y_i^{t-1} & \text{otherwise} \end{cases}.$$

## Randomized incremental algorithms (2/3)

Consider now the minimization of a large finite sum of smooth convex functions:

$$\min_{\mathbf{w} \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n f_i(\mathbf{w}) + \frac{\mu}{2} \|\mathbf{w}\|_2^2,$$

A class of algorithms with low per-iteration complexity have been recently developed that enjoy **exponential** convergence rates for strongly-convex problems, e.g., MISO/Finito (Mairal, 2015; Defazio et al., 2015; Lin et al., 2015)

Basic MISO/Finito algorithm (requires  $n \geq 2L/\mu$ )

$$\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \frac{1}{\mu n} (y_{i_t}^t - y_{i_t}^{t-1}) \quad \text{with} \quad y_i^t = \begin{cases} \nabla f_i(\mathbf{w}^{t-1}) & \text{if } i = i_t \\ y_i^{t-1} & \text{otherwise} \end{cases}.$$

see also SDCA (Shalev-Shwartz and Zhang, 2012).

## Randomized incremental algorithms (3/3)

Many of these techniques are in fact performing SGD-types of steps

$$\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \eta_t g_t,$$

where  $\mathbb{E}[g_t | \mathbf{w}_{t-1}] = \nabla f(\mathbf{w}_{t-1})$ , but where the estimator of the gradient has **lower variance** than in SGD (see SVRG [Johnson and Zhang, 2013]).

Typically, these methods have the convergence rate

$$f(\mathbf{w}_t) - f^* = O\left(\left(1 - C \max\left(\frac{1}{n}, \frac{\mu}{L}\right)\right)^t\right)$$

and their complexity per-iteration is independent of  $n$ ! In addition, they are often almost parameter-free (theoretical values for their learning rates work in practice).

# Large-scale learning with linear models

## Conclusion

- we know how to deal with huge-scale problems when the models are linear;
- significant progress has been made during the last 3-4 years;
- all of this is also useful to learn with kernels!

# Outline

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
  - Motivation
  - Large-scale learning with linear models
  - Nyström approximations
  - Random Fourier features
  - New challenges
- “Deep” learning with kernels

## Nyström approximations [Williams and Seeger, 2002] (1/14)

Consider a dataset  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathcal{X}$  with a p.d. kernel  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ . Call  $\mathcal{H}$  its RKHS and  $\varphi : \mathcal{X} \rightarrow \mathcal{H}$  the mapping such that

$$K(\mathbf{x}, \mathbf{x}') = \langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle_{\mathcal{H}}.$$

A natural approximation consists of representing each data point  $\mathbf{x}_i$  as a linear combination of a few anchor points  $f_j$  in  $\mathcal{H}$ :

$$\varphi(\mathbf{x}) \approx \sum_{j=1}^d \beta_j(\mathbf{x}) f_j.$$

Then,

$$\begin{aligned} \langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle_{\mathcal{H}} &\approx \left\langle \sum_{j=1}^d \beta_j(\mathbf{x}) f_j, \sum_{j=1}^d \beta_j(\mathbf{x}') f_j \right\rangle_{\mathcal{H}} \\ &= \sum_{j,l=1}^d \beta_j(\mathbf{x}) \beta_l(\mathbf{x}') \langle f_j, f_l \rangle_{\mathcal{H}} = \boldsymbol{\beta}(\mathbf{x})^\top \mathbf{G} \boldsymbol{\beta}(\mathbf{x}'). \end{aligned}$$

## Nyström approximations (2/14)

Then, we have

$$\langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle_{\mathcal{H}} \approx \boldsymbol{\beta}(\mathbf{x})^\top \mathbf{G} \boldsymbol{\beta}(\mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle_{\mathbb{R}^d},$$

with

$$\psi(\mathbf{x}) = \mathbf{G}^{1/2} \boldsymbol{\beta}(\mathbf{x}).$$

In practice, the anchor points  $f_j$  in  $\mathcal{H}$  and the coordinates  $\boldsymbol{\beta}$  are learned by **minimizing the least square error in  $\mathcal{H}$**

$$\min_{\substack{f_1, \dots, f_d \in \mathcal{H} \\ \beta_{ij} \in \mathbb{R}}} \sum_{i=1}^n \left\| \varphi(\mathbf{x}_i) - \sum_{j=1}^d \beta_{ij} f_j \right\|_{\mathcal{H}}^2.$$

## Nyström approximations (3/14)

Note that the problem

$$\min_{\substack{f_1, \dots, f_d \in \mathcal{H} \\ \beta_{ij} \in \mathbb{R}}} \sum_{i=1}^n \left\| \varphi(\mathbf{x}_i) - \sum_{j=1}^d \beta_{ij} f_j \right\|_{\mathcal{H}}^2,$$

is equivalent, after developing the quadratic function, to

$$\min_{\substack{f_1, \dots, f_d \in \mathcal{H} \\ \beta_{ij} \in \mathbb{R}}} \sum_{i=1}^n -2 \sum_{j=1}^d \beta_{ij} \langle f_j, \varphi(\mathbf{x}_i) \rangle_{\mathcal{H}} + \sum_{j,l=1}^d \beta_{ij} \beta_{il} \langle f_j, f_l \rangle_{\mathcal{H}},$$

or also

$$\min_{\substack{f_1, \dots, f_d \in \mathcal{H} \\ \beta_{ij} \in \mathbb{R}}} \sum_{i=1}^n -2 \sum_{j=1}^d \beta_{ij} f_j(\mathbf{x}_i) + \sum_{j,l=1}^d \beta_{ij} \beta_{il} \langle f_j, f_l \rangle_{\mathcal{H}}.$$

## Nyström approximations (4/14)

Then, call  $[\mathbf{K}_f]_{jl} = \langle f_j, f_l \rangle_{\mathcal{H}}$  and  $f(\mathbf{x}_i) = [f_1(\mathbf{x}_i), \dots, f_d(\mathbf{x}_i)]$  in  $\mathbb{R}^d$ . The problem may be rewritten as

$$\min_{\substack{f_1, \dots, f_d \in \mathcal{H} \\ \beta_i \in \mathbb{R}^d}} \sum_{i=1}^n -2\beta_i^\top f(\mathbf{x}_i) + \beta_i^\top \mathbf{K}_f \beta_i,$$

and by minimizing with respect to all  $\beta_i$  with  $f$  fixed, we have that  $\beta_i = \mathbf{K}_f^{-1} f(\mathbf{x}_i)$  (assuming  $\mathbf{K}_f$  to be invertible to simplify), which leads to

$$\max_{f_1, \dots, f_d \in \mathcal{H}} \sum_{i=1}^n f(\mathbf{x}_i)^\top \mathbf{K}_f^{-1} f(\mathbf{x}_i).$$

Consider an optimal solution  $f^*$  and perform the eigenvalue decomposition of  $\mathbf{K}_{f^*} = \mathbf{U} \boldsymbol{\Delta} \mathbf{U}^\top$ . Then, define the functions  $[g_1^*(\mathbf{x}), \dots, g_d^*(\mathbf{x})] = \boldsymbol{\Delta}^{-1/2} \mathbf{U}^\top f^*(\mathbf{x})$ . **The functions  $g_j^*$  are points in the RKHS  $\mathcal{H}$**  (as linear combinations of entries of  $f^*$ ).

## Nyström approximations (5/14)

By construction

$$\begin{aligned} [\mathbf{K}_{g^*}]_{jl} &= \langle g_j^*, g_l^* \rangle_{\mathcal{H}} \\ &= \left\langle \frac{1}{\sqrt{\Delta_{jj}}} \sum_{k=1}^d [\mathbf{U}]_{kj} f_k^*, \frac{1}{\sqrt{\Delta_{ll}}} \sum_{k=1}^d [\mathbf{U}]_{kl} f_k^* \right\rangle_{\mathcal{H}} \\ &= \frac{1}{\sqrt{\Delta_{jj}}} \frac{1}{\sqrt{\Delta_{ll}}} \sum_{k,k'=1}^d [\mathbf{U}]_{kj} [\mathbf{U}]_{k'l} \langle f_k^*, f_{k'}^* \rangle_{\mathcal{H}} \\ &= \frac{1}{\sqrt{\Delta_{jj}}} \frac{1}{\sqrt{\Delta_{ll}}} \sum_{k,k'=1}^d [\mathbf{U}]_{kj} [\mathbf{U}]_{k'l} [\mathbf{K}_{f^*}]_{kk'} \\ &= \frac{1}{\sqrt{\Delta_{jj}}} \frac{1}{\sqrt{\Delta_{ll}}} \mathbf{u}_j^\top \mathbf{K}_{f^*} \mathbf{u}_l \\ &= \delta_{j=l}. \end{aligned}$$

## Nyström approximations (6/14)

Then,  $\mathbf{K}_{g^*} = \mathbf{I}$  and  $g^*$  is also a solution of the problem

$$\max_{f_1, \dots, f_d \in \mathcal{H}} \sum_{i=1}^n f(\mathbf{x}_i)^\top \mathbf{K}_f^{-1} f(\mathbf{x}_i),$$

since

$$\begin{aligned} f^*(\mathbf{x}_i)^\top \mathbf{K}_{f^*}^{-1} f^*(\mathbf{x}_i) &= f^*(\mathbf{x}_i)^\top \mathbf{U} \boldsymbol{\Delta}^{-1} \mathbf{U}^\top f^*(\mathbf{x}_i) \\ &= g^*(\mathbf{x}_i)^\top g^*(\mathbf{x}_i) = g^*(\mathbf{x}_i)^\top \mathbf{K}_{g^*}^{-1} g^*(\mathbf{x}_i), \end{aligned}$$

and also a solution of the problem

$$\max_{g_1, \dots, g_d \in \mathcal{H}} \sum_{j=1}^d \sum_{i=1}^n g_j(\mathbf{x}_i)^2 \quad \text{s.t. } g_j \perp g_k \text{ for } k \neq j.$$

## Nyström approximations (6/14)

Then,  $\mathbf{K}_{g^*} = \mathbf{I}$  and  $g^*$  is also a solution of the problem

$$\max_{f_1, \dots, f_d \in \mathcal{H}} \sum_{i=1}^n f(\mathbf{x}_i)^\top \mathbf{K}_f^{-1} f(\mathbf{x}_i),$$

since

$$\begin{aligned} f^*(\mathbf{x}_i)^\top \mathbf{K}_{f^*}^{-1} f^*(\mathbf{x}_i) &= f^*(\mathbf{x}_i)^\top \mathbf{U} \mathbf{\Delta}^{-1} \mathbf{U}^\top f^*(\mathbf{x}_i) \\ &= g^*(\mathbf{x}_i)^\top g^*(\mathbf{x}_i) = g^*(\mathbf{x}_i)^\top \mathbf{K}_{g^*}^{-1} g^*(\mathbf{x}_i), \end{aligned}$$

and also a solution of the problem

$$\max_{g_1, \dots, g_d \in \mathcal{H}} \sum_{j=1}^d \sum_{i=1}^n g_j(\mathbf{x}_i)^2 \quad \text{s.t. } g_j \perp g_k \text{ for } k \neq j.$$

This is the kernel PCA formulation!

## Nyström approximations (7/14)

### First recipe with kernel PCA

Given a dataset of  $n$  training points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathcal{X}$ ,

- randomly choose a subset  $Z = [\mathbf{x}_{z_1}, \dots, \mathbf{x}_{z_m}]$  of  $m \leq n$  training points;
- compute the  $m \times m$  kernel matrix  $\mathbf{K}_{Z,Z}$ .
- **perform kernel PCA** to find the  $d \leq m$  largest principal directions (parametrized by  $d$  vectors  $\alpha_j$  in  $\mathbb{R}^m$ );

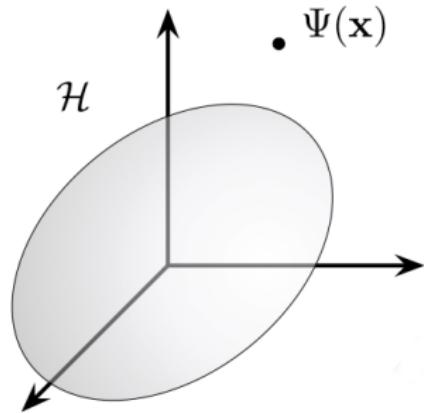
Then, every point  $\mathbf{x}$  in  $\mathcal{X}$  may be approximated by

$$\begin{aligned}\psi(\mathbf{x}) &= \beta(\mathbf{x}) = [g_1^*(\mathbf{x}), \dots, g_d^*(\mathbf{x})]^\top \\ &= \left[ \sum_{i=1}^m \alpha_{1i} K(\mathbf{x}_{z_i}, \mathbf{x}), \dots, \sum_{i=1}^m \alpha_{di} K(\mathbf{x}_{z_i}, \mathbf{x}) \right]^\top.\end{aligned}$$

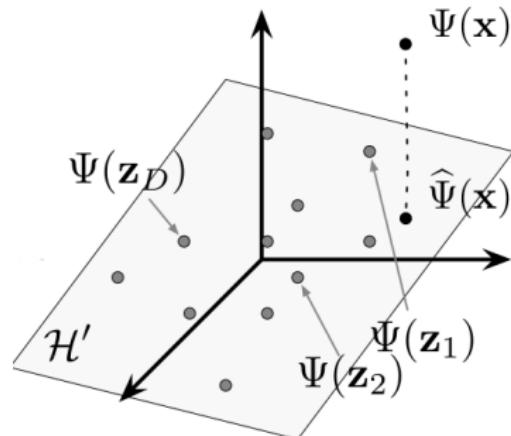
## Nyström approximations (8/14)

- The complexity of training is  $O(m^3)$  (eig decomposition) +  $O(m^2)$  kernel evaluations.
- The complexity of encoding a point  $\mathbf{x}$  is  $O(md)$  (matrix vector multiplication) +  $O(m)$  kernel evaluations.

(a) exact feature space



(b) dense approx.



Images courtesy of Vedaldi and Zisserman [2012]

## Nyström approximations (9/14)

The main issue with kernel PCA is the encoding time, which depends linearly of  $m$ . A popular alternative is instead to select the anchor points among the training data points  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . Then, choose  $f_1 = \varphi(\mathbf{x}_{z_1}), \dots, f_d = \varphi(\mathbf{x}_{z_d})$ .

### Second recipe with random point sampling

Given a dataset of  $n$  training points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  in  $\mathcal{X}$ ,

- randomly choose a subset  $Z = [\mathbf{x}_{z_1}, \dots, \mathbf{x}_{z_d}]$  of  $d$  training points;
- compute the  $d \times d$  kernel matrix  $\mathbf{K}_{Z,Z}$ .

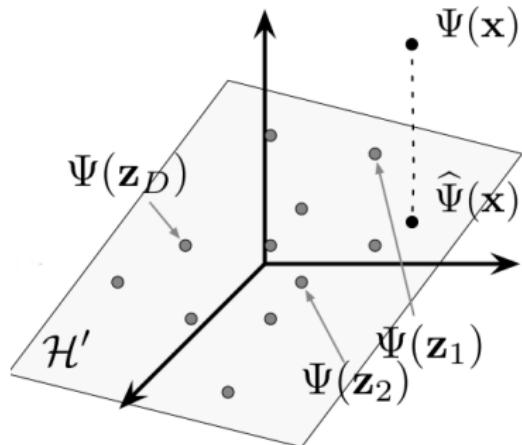
Then, a new point  $\mathbf{x}$  is encoded as

$$\begin{aligned}\psi(\mathbf{x}) &= \mathbf{K}_{Z,Z}^{1/2} \beta(\mathbf{x}) = \mathbf{K}_{Z,Z}^{1/2} \mathbf{K}_{Z,Z}^{-1} f(\mathbf{x}) \\ &= \mathbf{K}_{Z,Z}^{-1/2} [K(\mathbf{x}_{z_1}, \mathbf{x}), \dots, K(\mathbf{x}_{z_d}, \mathbf{x})]^\top \\ &= \mathbf{K}_{Z,Z}^{-1/2} \mathbf{K}_{Z,\mathbf{x}}.\end{aligned}$$

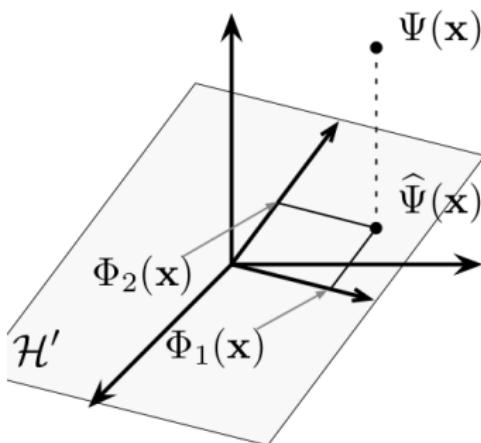
## Nyström approximations (10/14)

- The complexity of training is  $O(d^3)$  (eig decomposition) +  $O(d^2)$  kernel evaluations.
- The complexity of encoding a point  $\mathbf{x}$  is  $O(d^2)$  (matrix vector multiplication) +  $O(d)$  kernel evaluations.

(b) dense approx.



(d) Nyström's approx.



Images courtesy of Vedaldi and Zisserman [2012]

## Nyström approximations (11/14)

The encoding time is now low, but the (random) choice of anchor points is not clever. Better approximation can be obtained with a **greedy algorithm** that iteratively selects one column at a time with largest residual (Bach and Jordan, 2002; Smola and Shölkopf, 2000).

At iteration  $k$ , assume that  $Z = [z_1, \dots, z_k]$ ; then, the residual for a data point  $\mathbf{x}$  encoded with  $k$  anchor points  $f_1, \dots, f_k$  is

$$\min_{\beta \in \mathbb{R}^k} \|\varphi(\mathbf{x}) - \sum_{j=1}^k \beta_j f_j\|_{\mathcal{H}}^2,$$

which is equal to

$$\|\varphi(\mathbf{x})\|_{\mathcal{H}}^2 - f(\mathbf{x})^\top \mathbf{K}_f^{-1} f(\mathbf{x}),$$

and since  $f_j = \varphi(\mathbf{x}_{z_j})$  for all  $j$ , the data point  $\mathbf{x}_i$  with largest residual is the one that maximizes

$$K(\mathbf{x}_i, \mathbf{x}_i) - \mathbf{K}_{x_i, Z} \mathbf{K}_{Z, Z}^{-1} \mathbf{K}_{Z, x_i}.$$

## Nyström approximations (12/14)

This brings us to the following algorithm

### Third recipe with greedy anchor point selection

Initialize  $Z = \emptyset$ . For  $k = 1, \dots, d$  do

- **data point selection**

$$z_k \leftarrow \operatorname{argmax}_{i \in \{1, \dots, n\}} K(\mathbf{x}_i, \mathbf{x}_i) - \mathbf{K}_{x_i, Z} \mathbf{K}_{Z, Z}^{-1} \mathbf{K}_{Z, x_i};$$

- **update the set  $Z$**

$$Z \leftarrow [Z, z_k].$$

A naive implementation is slow ( $O(j^2n + j^3)$  at every iteration). To get a reasonable complexity, one has to use simple linear algebra tricks (see next slide).

## Nyström approximations (13/14)

$$\mathbf{K}_{[Z,z],[Z,z]}^{-1} = \begin{bmatrix} \mathbf{K}_{Z,Z} & \mathbf{K}_{Z,z} \\ \mathbf{K}_{z,Z} & \mathbf{K}_{z,z} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{K}_{Z,Z}^{-1} + \frac{1}{s} \mathbf{b} \mathbf{b}^\top & -\frac{1}{s} \mathbf{b} \\ -\frac{1}{s} \mathbf{b}^\top & \frac{1}{s} \end{bmatrix},$$

$s$  is the Schur complement  $s = \mathbf{K}_{z,z} - \mathbf{K}_{z,Z} \mathbf{K}_{Z,Z}^{-1} \mathbf{K}_{Z,z}$ , and

$$\mathbf{b} = \mathbf{K}_{Z,Z}^{-1} \mathbf{K}_{Z,z}.$$

- the matrix  $\mathbf{K}_{[Z,z],[Z,z]}^{-1}$  can be obtained from  $\mathbf{K}_{Z,Z}^{-1}$  and  $\mathbf{K}_{Z,z}$  in  $O(j^2)$  float operations; for that we need to always keep into memory the  $j \times n$  matrix  $\mathbf{K}_{Z,X}$ .
- computing the matrix  $\mathbf{K}_{[Z,z],X}$  from  $\mathbf{K}_{Z,X}$  requires  $n$  kernel evaluations;
- the quantity  $\mathbf{K}_{x_i,[Z,z]} \mathbf{K}_{[Z,z],[Z,z]}^{-1} \mathbf{K}_{[Z,z],x_i}$  can be computed from  $\mathbf{K}_{x_i,Z} \mathbf{K}_{Z,Z}^{-1} \mathbf{K}_{Z,x_i}$  in  $O(j)$  float operations.

The total training complexity is  $O(d^2n)$  float operations and  $O(dn)$  kernel evaluations

## Nyström approximations (14/14)

### Concluding remarks

- The last technique is equivalent to computing an **incomplete Cholesky factorization** of the kernel matrix (Bach and Jordan, 2002; Fine and Scheinberg, 2001);
- The techniques we have seen produce low-rank approximations of the kernel matrix  $\mathbf{K} \approx \mathbf{L}\mathbf{L}^\top$ ;
- When  $\mathcal{X} = \mathbb{R}^d$ , it is also possible to synthesize training points  $\mathbf{z}_1, \dots, \mathbf{z}_d$  and use anchor points  $\varphi(\mathbf{z}_1), \dots, \varphi(\mathbf{z}_d)$ , e.g., with a K-means algorithms.

# Outline

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
  - Motivation
  - Large-scale learning with linear models
  - Nyström approximations
  - Random Fourier features
  - New challenges
- “Deep” learning with kernels

## Random Fourier features [Rahimi and Recht, 2007] (1/5)

A large class of approximations for shift-invariant kernels are based on sampling techniques. Consider a real-valued positive-definite continuous translation-invariant kernel  $K(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x} - \mathbf{y})$  with  $\kappa : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then, if  $\kappa(0) = 1$ , Bochner theorem tells us that  $\kappa$  is a **valid characteristic function for some probability measure**

$$\kappa(\mathbf{z}) = \mathbb{E}_{\mathbf{w}}[e^{i\mathbf{w}^\top \mathbf{z}}].$$

Remember indeed that, with the right assumptions on  $\kappa$ ,

$$\kappa(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\kappa}(\mathbf{w}) e^{i\mathbf{w}^\top \mathbf{x}} e^{-i\mathbf{w}^\top \mathbf{y}} d\mathbf{w},$$

and the probability measure admits a density  $p(\mathbf{w}) = \frac{1}{(2\pi)^d} \hat{\kappa}(\mathbf{w})$  (non-negative, real-valued, sum to 1 since  $\kappa(0) = 1$ ).

## Random Fourier features (2/5)

Then,

$$\begin{aligned}\kappa(\mathbf{x} - \mathbf{y}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\kappa}(\mathbf{w}) e^{i\mathbf{w}^\top \mathbf{x}} e^{-i\mathbf{w}^\top \mathbf{y}} d\mathbf{w} \\ &= \int_{\mathbb{R}^d} p(\mathbf{w}) \cos(\mathbf{w}^\top \mathbf{x} - \mathbf{w}^\top \mathbf{y}) d\mathbf{w} \\ &= \int_{\mathbb{R}^d} p(\mathbf{w}) \left( \cos(\mathbf{w}^\top \mathbf{x}) \cos(\mathbf{w}^\top \mathbf{y}) + \sin(\mathbf{w}^\top \mathbf{x}) \sin(\mathbf{w}^\top \mathbf{y}) \right) d\mathbf{w} \\ &= \int_{\mathbb{R}^d} \int_{b=0}^{2\pi} \frac{p(\mathbf{w})}{2\pi} 2 \cos(\mathbf{w}^\top \mathbf{x} + b) \cos(\mathbf{w}^\top \mathbf{y} + b) d\mathbf{w} db \quad (\text{exercise}) \\ &= \mathbb{E}_{\mathbf{w} \sim p(\mathbf{w}), b \sim \mathcal{U}[0, 2\pi]} \left[ \sqrt{2} \cos(\mathbf{w}^\top \mathbf{x} + b) \sqrt{2} \cos(\mathbf{w}^\top \mathbf{y} + b) \right]\end{aligned}$$

## Random Fourier features (3/5)

### Random Fourier features recipe

- Compute the Fourier transform of the kernel  $\hat{\kappa}$  and define the probability density  $p(\mathbf{w}) = \hat{\kappa}(\mathbf{w})/(2\pi)^d$ ;
- Draw  $d$  i.i.d. samples  $\mathbf{w}_1, \dots, \mathbf{w}_d$  from  $p$  and  $d$  i.i.d. samples  $b_1, \dots, b_d$  from the uniform distribution on  $[0, 2\pi]$ ;
- define the mapping

$$\mathbf{x} \mapsto \psi(\mathbf{x}) = \sqrt{\frac{2}{d}} \left[ \cos(\mathbf{w}_1^\top \mathbf{x} + b_1), \dots, \cos(\mathbf{w}_d^\top \mathbf{x} + b_d) \right]^\top.$$

Then, we have that

$$\kappa(\mathbf{x} - \mathbf{y}) \approx \langle \psi(\mathbf{x}), \psi(\mathbf{y}) \rangle_{\mathbb{R}^d}.$$

The two quantities are equal in expectation.

## Random Fourier features (4/5)

Theorem, [Rahimi and Recht, 2007]

On any compact subset  $\mathcal{X}$  of  $\mathbb{R}^m$ , for all  $\varepsilon > 0$ ,

$$\mathbb{P} \left[ \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{X}} |\kappa(\mathbf{x} - \mathbf{y}) - \langle \psi(\mathbf{x}), \psi(\mathbf{y}) \rangle_{\mathbb{R}^d}| \geq \varepsilon \right] \leq 2^8 \left( \frac{\sigma_p \text{diam}(\mathcal{X})}{\varepsilon} \right)^2 e^{-\frac{d\varepsilon^2}{4(m+2)}},$$

where  $\sigma_p^2 = \mathbb{E}_{\mathbf{w} \sim p(\mathbf{w})} [\mathbf{w}^\top \mathbf{w}]$  is the second moment of the Fourier transform of  $\kappa$ .

### Remarks

- The convergence is uniform, **not data dependent**;
- Take the sequence  $\varepsilon_d = \sqrt{\frac{\log(d)}{d}} \sigma_p \text{diam}(\mathcal{X})$ ; Then the term on the right converges to zero when  $d$  grows to infinity;
- **Prediction functions with Random Fourier features are not in  $\mathcal{H}$ .**

## Random Fourier features (5/5)

### Ingredients of the proof

- For a *fixed* pair of points  $\mathbf{x}, \mathbf{y}$ , Hoeffding's inequality says that

$$\mathbb{P}\left[\underbrace{|\kappa(\mathbf{x} - \mathbf{y}) - \langle \psi(\mathbf{x}), \psi(\mathbf{y}) \rangle_{\mathbb{R}^d}|}_{f(\mathbf{x}, \mathbf{y})} \geq \varepsilon\right] \leq 2e^{-\frac{d\varepsilon^2}{4}}.$$

- Consider a net (set of balls of radius  $r$ ) that covers  $\mathcal{X}_\Delta = \{\mathbf{x} - \mathbf{y} : (\mathbf{x}, \mathbf{y}) \in \mathcal{X}\}$  with at most  $T = (4\text{diam}(\mathcal{X})/r)^m$  balls.
- Apply the Hoeffding's inequality to the centers  $\mathbf{x}_i - \mathbf{y}_i$  of the balls;
- Use a basic union bound

$$\mathbb{P}\left[\sup_i f(\mathbf{x}_i, \mathbf{y}_i) \geq \frac{\varepsilon}{2}\right] \leq \sum_i \mathbb{P}\left[f(\mathbf{x}_i, \mathbf{y}_i) \geq \frac{\varepsilon}{2}\right] \leq 2Te^{-\frac{d\varepsilon^2}{8}}.$$

- Glue things together: control the probability for points  $(\mathbf{x}, \mathbf{y})$  inside each ball, and adjust the radius  $r$  (a bit technical).

# Outline

## 6 Open Problems and Research Topics

- Multiple Kernel Learning (MKL)
- Large-scale learning with kernels
  - Motivation
  - Large-scale learning with linear models
  - Nyström approximations
  - Random Fourier features
  - New challenges
- “Deep” learning with kernels

## New challenges

We have seen two classes of kernel approximation techniques. Several challenges remain

- make random Fourier features **data dependent** (e.g., Bach, 2015);
- make these approximation techniques **data and task dependent**;
- reduce the number of dimensions;
- find more explicit approximate feature maps dedicated to useful kernels [e.g., Vedaldi and Zisserman, 2012];

# Outline

- 1 Kernels and RKHS
- 2 Kernel tricks
- 3 Kernel Methods: Supervised Learning
- 4 Kernel Methods: Unsupervised Learning
- 5 The Kernel Jungle
- 6 Open Problems and Research Topics
  - Multiple Kernel Learning (MKL)
  - Large-scale learning with kernels
  - “Deep” learning with kernels

# Outline

## 6 Open Problems and Research Topics

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# Deep learning with kernels

## Main question

- in some fields producing large amounts of labeled data (notably in computer vision), kernel methods are not performing as well as multilayer neural networks. **Why? How to improve kernel methods?**

## Possible angles of attack

- are multilayer neural networks close to a kernel machine?
- building multilayer kernels with successful principles from multilayer neural networks (successful=“convolutional” or “recurrent”).
- perform end-to-end-learning with kernels (crafting the kernel);

## Perspectives

- build multilayer architectures that are **easy to regularize** and that may work without (or with less) supervision.
- build **versatile** architectures to process structured data.

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*most critical point, important open problem;*
- kernel methods are glorified template matching algorithms?

## Classical criticisms of kernel methods

- lack of adaptivity to data?  
if necessary, use kernels for probabilistic models;
- lack of adaptivity to the task (end-to-end learning)?  
most critical point, important open problem;
- kernel methods are glorified template matching algorithms?  
irrelevant, only true for Gaussian kernel with  $\sigma$  too small;

$$f(\mathbf{x}) = \sum_{i=1}^n \alpha_i K(\mathbf{x}_i, \mathbf{x}) \quad \underset{??}{\approx} \quad \sum_{i=1}^n \frac{y_i}{\sum_{l=1}^n K(\mathbf{x}_l, \mathbf{x})} K(\mathbf{x}_i, \mathbf{x}).$$

The representer theorem simply tells us that the prediction function  $f$  lies in a subspace spanned by the data (nothing to do with the “template-matching” Nadaraya-Watson estimator on the right).

The  $\alpha_i$ 's do not have the same sign as the  $y_i$ 's in general.

The theorem also applies to the last layer of neural networks...

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## Links between kernels and neural networks

A large class of kernels on  $\mathbb{R}^p$  may be defined as an expectation

$$K(\mathbf{x}, \mathbf{y}) = \mathbb{E}_{\mathbf{w}}[s(\mathbf{w}^\top \mathbf{x})s(\mathbf{w}^\top \mathbf{y})],$$

where  $s : \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear function. Then, approximating the expectation by a finite sum yields

$$K(\mathbf{x}, \mathbf{y}) \approx \frac{1}{d} \sum_{j=1}^d s(\mathbf{w}_j^\top \mathbf{x})s(\mathbf{w}_j^\top \mathbf{y}) = \langle \psi(\mathbf{x}), \psi(\mathbf{y}) \rangle_{\mathbb{R}^d},$$

where  $\psi(\mathbf{x})$  may be interpreted as a one-layer neural network.

### Example

Any shift-invariant kernel with random Fourier features!

$$\psi(\mathbf{x}) = \sqrt{\frac{2}{d}} \left[ \cos(\mathbf{w}_1^\top \mathbf{x} + b_1), \dots, \cos(\mathbf{w}_d^\top \mathbf{x} + b_d) \right]^\top.$$

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### Example

The Gaussian kernel on the hypersphere:

$$\begin{aligned} e^{-\frac{1}{2\sigma^2}\|\mathbf{x}-\mathbf{y}\|_2^2} &= \left(\frac{2}{\pi\sigma^2}\right)^{\frac{m}{2}} \int_{\mathbf{w} \in \mathbb{R}^m} e^{-\frac{1}{\sigma^2}\|\mathbf{x}-\mathbf{w}\|_2^2} e^{-\frac{1}{\sigma^2}\|\mathbf{y}-\mathbf{w}\|_2^2} d\mathbf{w} \\ &= \int_{\mathbf{w} \in \mathbb{R}^m} p(\mathbf{w}) e^{-\frac{1}{\sigma^2} + \frac{2}{\sigma^2}\mathbf{w}^\top \mathbf{x}} e^{-\frac{1}{\sigma^2} + \frac{2}{\sigma^2}\mathbf{w}^\top \mathbf{y}} d\mathbf{w}, \end{aligned}$$

where  $p(\mathbf{w})$  is the density of the multivariate normal distribution  $\mathcal{N}(0, \sigma^2/4\mathbf{I})$ .

# Links between kernels and neural networks

## Example, arc-cosine kernels

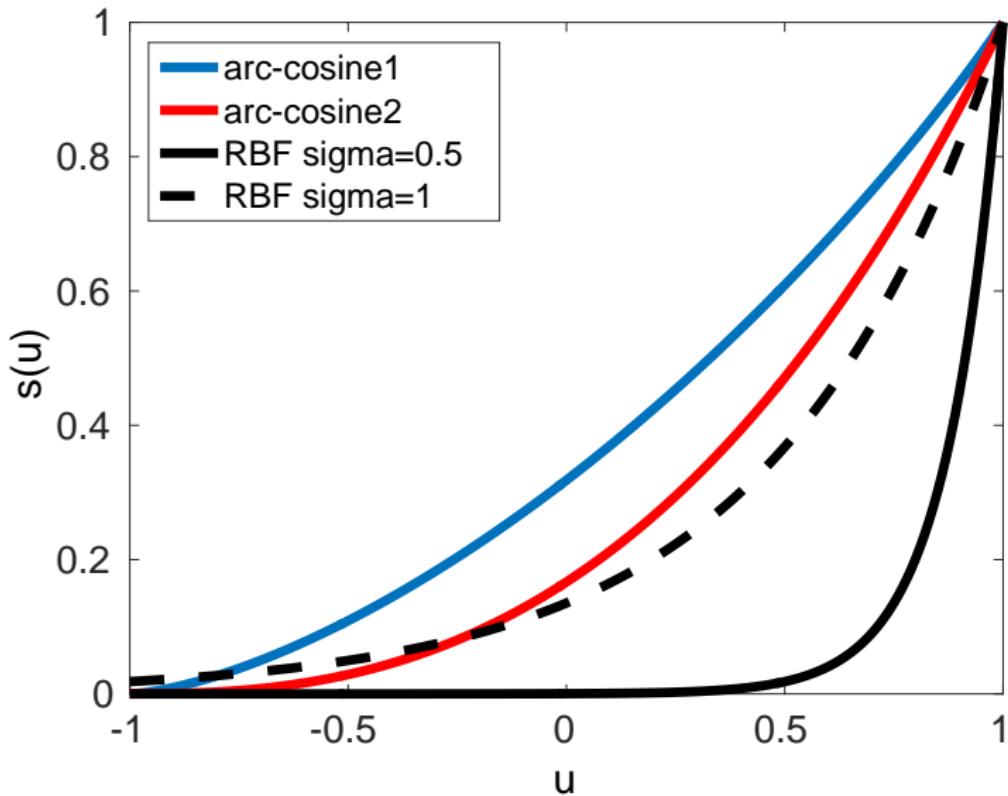
Cho and Saul, 2009 have proposed a collection of kernels defined as

$$K(\mathbf{x}, \mathbf{y}) = 2 \int_{\mathbf{w} \in \mathbb{R}^m} p(\mathbf{w}) s(\mathbf{w}^\top \mathbf{x}) s(\mathbf{w}^\top \mathbf{y}) d\mathbf{w},$$

for  $\mathbf{x}, \mathbf{y}$  on the hyper-sphere  $\mathbb{S}^{m-1}$  and  $p(\mathbf{w})$  is the density of the multivariate normal distribution  $\mathcal{N}(0, \mathbf{I})$ . Interestingly, the non-linearity  $s$  are typical ones from the neural network literature.

- $s(u) = \max(0, u)$  (rectified linear units) leads to  
 $K_1(\mathbf{x}, \mathbf{y}) = \sin(\theta) + (\pi - \theta) \cos(\theta)$  with  $\theta = \cos^{-1}(\mathbf{x}^\top \mathbf{y})$ ;
- $s(u) = \max(0, u)^2$  (squared rectified linear units) leads to  
 $K_2(\mathbf{x}, \mathbf{y}) = 3 \sin(\theta) \cos(\theta) + (\pi - \theta)(1 + 2 \cos^2(\theta))$ ;
- and also a general formula for  $s(u) = \max(0, u)^p$ , with  $d \geq 0$ .

## Links between kernels and neural networks



## Links between kernels and neural networks

We have seen that some kernels admit an interpretation as one-layer neural networks with random weights and infinite number of neurons.

Another common features between neural networks and kernel method is **the composition of feature maps** [Cho and Saul, 2009].

Consider kernels with the form

$$K_1(\mathbf{x}, \mathbf{y}) = \kappa(\|\varphi_0(\mathbf{x})\|_{\mathcal{H}_0}, \|\varphi_0(\mathbf{y})\|_{\mathcal{H}_0}, \langle \varphi_0(\mathbf{x}), \varphi_0(\mathbf{y}) \rangle_{\mathcal{H}_0}) = \langle \varphi_1(\mathbf{x}), \varphi_1(\mathbf{y}) \rangle_{\mathcal{H}_1},$$

e.g., linear, polynomial, Gaussian, arc-cosine with  $\varphi_0(\mathbf{x}) = \mathbf{x}$ .

Then, it is easy to obtain a new kernel  $K_2$  by **composition**:

$$K_2(\mathbf{x}, \mathbf{y}) = \kappa(\|\varphi_1(\mathbf{x})\|_{\mathcal{H}_1}, \|\varphi_1(\mathbf{y})\|_{\mathcal{H}_1}, \langle \varphi_1(\mathbf{x}), \varphi_1(\mathbf{y}) \rangle_{\mathcal{H}_1}) = \langle \varphi_2(\mathbf{x}), \varphi_2(\mathbf{y}) \rangle_{\mathcal{H}_2},$$

and recursively build multilayer kernels.

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## Motivation

We have made explicit some links between neural networks  
(approximation by linear operations followed by pointwise non-linearities,  
and composition of feature maps leading to multilayer kernels).

However, one important ingredient in the kernel world is still missing:  
The main deep learning success, convolutional neural networks, is able to

- learn local structures in images (**local stationarity**);
- learn how to combine these local structures into mid and high-level  
ones (**spatial composition**).

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- learn local structures in images (**local stationarity**);
- learn how to combine these local structures into mid and high-level  
ones (**spatial composition**).

From a tutorial of Y. LeCun, quoting Stuart Geman “the world is  
compositional or there is a God”.

## Motivation

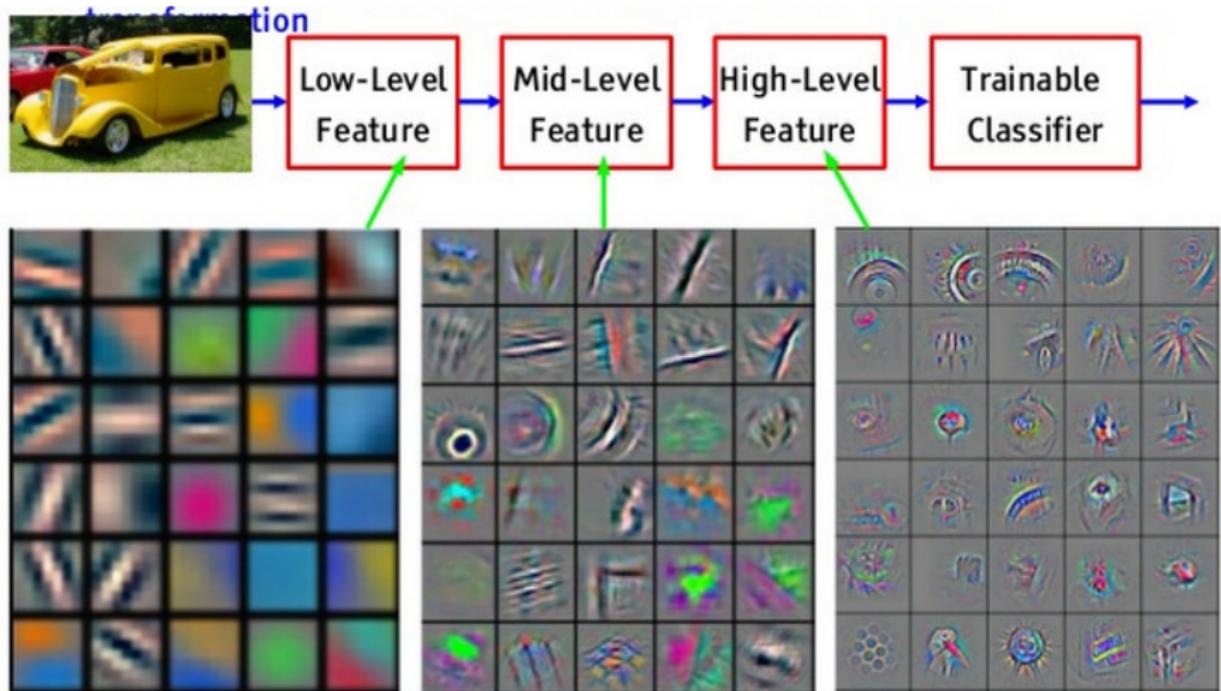


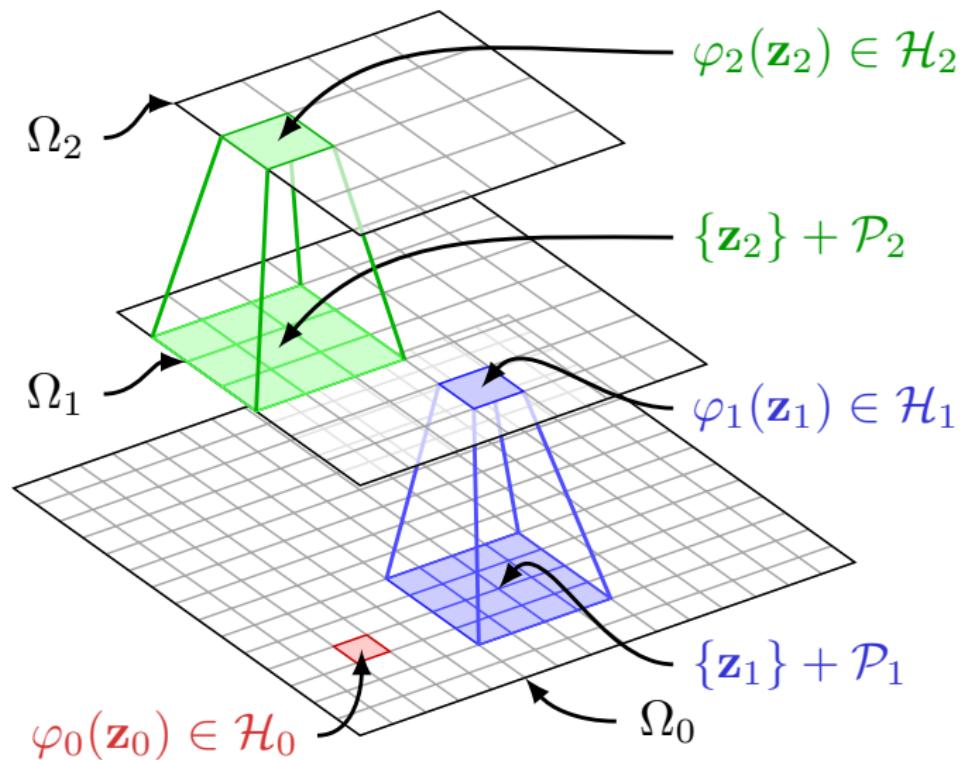
Figure: Picture from Yann Lecun's tutorial, based on [Zeiler and Fergus, 2013].

## Convolutional kernel networks

A few words about convolutional kernel networks [Mairal et al., 2014]

- Unsupervised representation of images based on a multilayer kernel, along with a finite-dimensional embedding  $\psi$ , which is a new type of convolutional neural network;
- Competitive results for image retrieval [Paulin et al., 2016];
- New principles to perform end-to-end supervised learning with multilayer kernels (unpublished yet).

## Convolutional kernel networks



## Main properties of CKNs

- CKNs are organized in a **multi-layer** fashion.
- Each layer produces an image feature map.

An **image feature map**  $\varphi$  is a function  $\varphi : \Omega \rightarrow \mathcal{H}$ , where  $\Omega \subseteq [0, 1]^2$  is a set of “coordinates” and  $\mathcal{H}$  is a Hilbert space.

Concretely, these are similar to feature maps of CNNs.

- Each layer defines a kernel between **patches** of the previous layer.
- The approximation scheme requires learning each layer sequentially, and can be interpreted as a CNN layer with a different objective.

## Image feature maps and convolutional kernels

An *image feature map*  $\varphi$  is a function  $\varphi : \Omega \rightarrow \mathcal{H}$ , where  $\Omega \subseteq [0, 1]^2$  is a set of “coordinates” in the image and  $\mathcal{H}$  is a Hilbert space.

It is possible to define a *convolutional kernel* between  $\varphi$  and  $\varphi'$

$$K(\varphi, \varphi') := \sum_{\mathbf{z} \in \Omega} \sum_{\mathbf{z}' \in \Omega} \|\varphi(\mathbf{z})\|_{\mathcal{H}} \|\varphi'(\mathbf{z}')\|_{\mathcal{H}} e^{-\frac{1}{2\beta^2} \|\mathbf{z} - \mathbf{z}'\|_2^2} e^{-\frac{1}{2\sigma^2} \|\tilde{\varphi}(\mathbf{z}) - \tilde{\varphi}'(\mathbf{z}')\|_{\mathcal{H}}^2},$$

- when  $\beta$  is large,  $K$  is **invariant to the positions**  $\mathbf{z}$  and  $\mathbf{z}'$ .
- when  $\beta$  is small, only features **placed at the same location**  $\mathbf{z} = \mathbf{z}'$  are compared to each other.

The kernel is inspired from the kernel descriptors of Bo et al., 2011.

## Image feature maps and convolutional kernels

An *image feature map*  $\varphi$  is a function  $\varphi : \Omega \rightarrow \mathcal{H}$ , where  $\Omega \subseteq [0, 1]^2$  is a set of “coordinates” in the image and  $\mathcal{H}$  is a Hilbert space.

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The kernel can be defined on *patches*

$$\sum_{\mathbf{z} \in \mathcal{P}} \sum_{\mathbf{z}' \in \mathcal{P}} \|\varphi(\mathbf{u} + \mathbf{z})\|_{\mathcal{H}} \|\varphi'(\mathbf{u}' + \mathbf{z}')\|_{\mathcal{H}} e^{-\frac{1}{2\beta^2} \|\mathbf{z}-\mathbf{z}'\|_2^2} e^{-\frac{1}{2\sigma^2} \|\tilde{\varphi}(\mathbf{u}+\mathbf{z}) - \tilde{\varphi}'(\mathbf{u}'+\mathbf{z}')\|_{\mathcal{H}}^2},$$

where  $\mathcal{P}$  is a patch shape and  $\mathbf{u}, \mathbf{u}'$  are locations in  $\Omega$ .

## Zoom on the zero-th layer

Before we build a hierarchy, we can specify two simple zero-th layer feature maps  $\varphi_0$ .

### Gradient map

$\mathcal{H}_0 = \mathbb{R}^2$  and  $\varphi_0(\mathbf{z})$  is the two-dimensional gradient of the image at pixel  $\mathbf{z}$ . Then, the quantity  $\|\varphi_0(\mathbf{z})\|_{\mathcal{H}_0}$  is the **gradient intensity**, and  $\tilde{\varphi}_0(\mathbf{z})$  is its **orientation**  $[\cos(\theta), \sin(\theta)]$ .

### Patch map

$\varphi_0$  associates to a location  $\mathbf{z}$  an image patch of size  $m \times m$  centered at  $\mathbf{z}$ . Then,  $\mathcal{H}_0 = \mathbb{R}^{m^2}$ , and  $\tilde{\varphi}_0(\mathbf{z})$  is a **contrast-normalized** version of the patch.

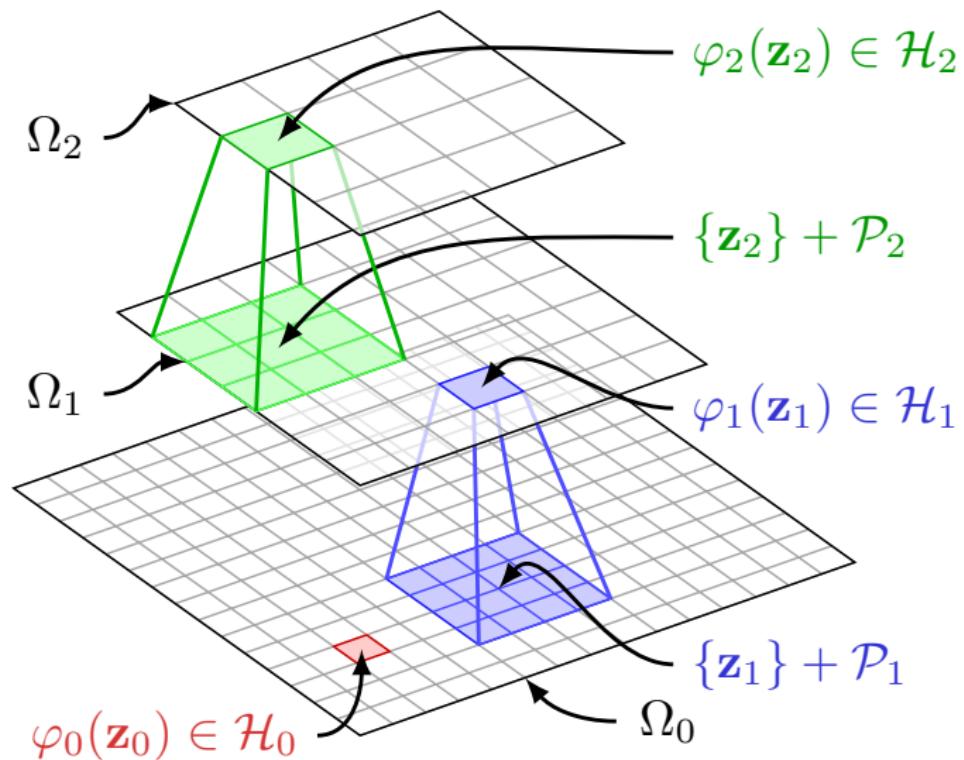
## Multilayer kernels

Let us consider a set of coordinates  $\Omega_{k-1}$  and a Hilbert space  $\mathcal{H}_{k-1}$ . We build a new set  $\Omega_k$  and a new Hilbert space  $\mathcal{H}_k$  as follows:

- choose a patch shape  $\mathcal{P}_k$  and a set of coordinates  $\Omega_k$  such that for each  $\mathbf{z}_k$  in  $\Omega_k$  corresponds to a patch in  $\Omega_{k-1}$  centered at  $\mathbf{z}_k$ .
- call  $K_k$  the kernel of the previous slide on the “patch” feature maps  $\mathcal{P}_k \rightarrow \mathcal{H}_{k-1}$  (with parameters  $\beta_k, \sigma_k$ ). We denote by  $\mathcal{H}_k$  the Hilbert space for which the p.d. kernel  $K_k$  is reproducing.

An image represented by a feature map  $\varphi_{k-1} : \Omega_{k-1} \rightarrow \mathcal{H}_{k-1}$  at layer  $k-1$  is now encoded in the  $k$ -th layer as  $\varphi_k : \Omega_k \rightarrow \mathcal{H}_k$ , where  $\varphi_k(\mathbf{z}_k)$  is the representation in  $\mathcal{H}_k$  of the patch of  $\varphi_{k-1}$  centered at  $\mathbf{z}_k$ .

## Convolutional kernel networks



# Optimization

## Key approximation

When  $\mathbf{x}$  and  $\mathbf{y}$  are on the sphere,

$$e^{-\frac{1}{2\alpha^2}\|\mathbf{x}-\mathbf{y}\|_2^2} = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})}[s(\mathbf{z}^\top \mathbf{x})s(\mathbf{z}^\top \mathbf{y})],$$

where  $s(u) \propto e^{-\frac{1}{\alpha^2} + \frac{2u}{\alpha^2}}$  and  $p(\mathbf{z})$  is the density of the multivariate normal distribution  $\mathcal{N}(0, (\alpha^2/4)\mathbf{I})$ . Then,

$$e^{-\frac{1}{2\alpha^2}\|\mathbf{x}-\mathbf{y}\|_2^2} \approx \frac{1}{p} \sum_{j=1}^p \eta_j s(\mathbf{z}_j^\top \mathbf{x})s(\mathbf{z}_j^\top \mathbf{y}).$$

Instead of random sampling,  $\mathbf{z}_j$  and  $\eta_j$  are learned on training data:

$$\min_{\mathbf{z}, \eta} \sum_{i=1}^n \left( e^{-\frac{1}{2\alpha^2}\|\mathbf{x}_i-\mathbf{y}_i\|_2^2} - \frac{1}{p} \sum_{j=1}^p \eta_j s(\mathbf{z}_j^\top \mathbf{x}_i)s(\mathbf{z}_j^\top \mathbf{y}_i) \right)^2.$$

## Approximation principles

We proceed by recursion, with the approximation holding for  $k = 0$ .

Main ingredients for approximating  $K(\varphi_{k-1}, \varphi'_{k-1})$ .

- replace  $\varphi_{k-1}$  by its finite-dimensional approximation  $\psi_{k-1}$ ;

$$\approx \sum_{\mathbf{z}, \mathbf{z}' \in \Omega_{k-1}} \|\psi_{k-1}(\mathbf{z})\|_2 \|\psi'_{k-1}(\mathbf{z}')\|_2 e^{-\frac{1}{2\beta_k^2} \|\mathbf{z}-\mathbf{z}'\|_2^2} e^{-\frac{1}{2\sigma_k^2} \|\tilde{\psi}_{k-1}(\mathbf{z}) - \tilde{\psi}'_{k-1}(\mathbf{z}')\|_2^2};$$

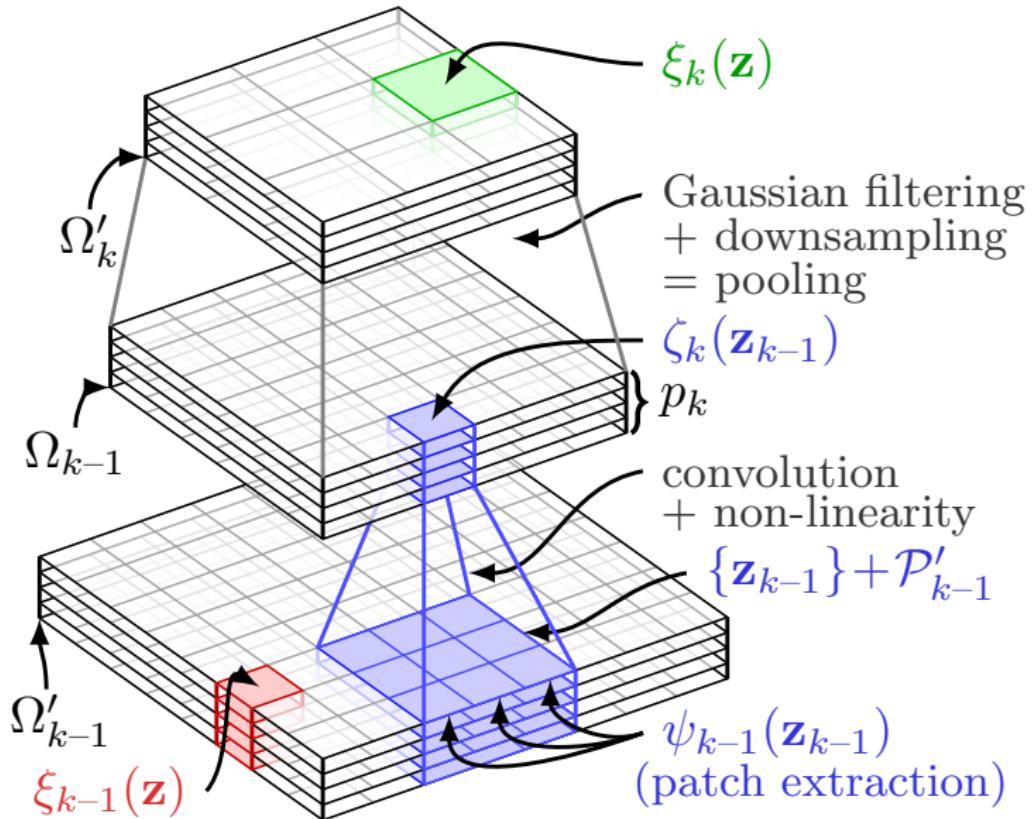
- use the finite-dimensional approximation of the Gaussian kernel

$$\approx \sum_{\mathbf{z}, \mathbf{z}' \in \Omega_{k-1}} \zeta_k(\mathbf{z})^\top \zeta'_k(\mathbf{z}') e^{-\frac{1}{2\beta_k^2} \|\mathbf{z}-\mathbf{z}'\|_2^2};$$

- approximate the remaining Gaussian kernel

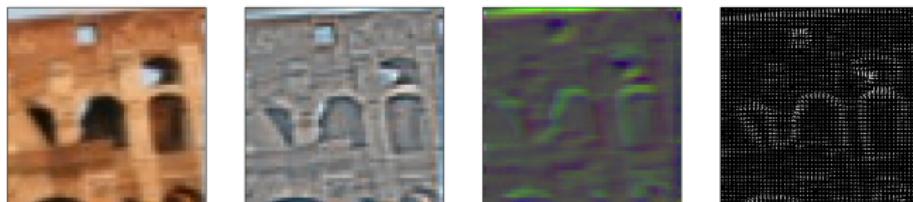
$$\approx \frac{2}{\pi} \sum_{\mathbf{u} \in \Omega'_k} \left( \sum_{\mathbf{z} \in \Omega_{k-1}} e^{-\frac{1}{\beta_k^2} \|\mathbf{z}-\mathbf{u}\|_2^2} \zeta_k(\mathbf{z}) \right)^\top \left( \sum_{\mathbf{z}' \in \Omega_{k-1}} e^{-\frac{1}{\beta_k^2} \|\mathbf{z}'-\mathbf{u}\|_2^2} \zeta'_k(\mathbf{z}') \right);$$

## Zoom between layers $k-1$ and $k$



# Application to image retrieval

- Encoding of interest points with CKN + VLAD.
- Possible inputs:



Results (mAP or true positives in top-4 for UKB)

Method \ Dataset	Holidays	UKB	Oxford
<b>VLAD+SIFT</b> [Jegou et al., 2012]	63.4	3.47	-
<b>VLAD++</b> [Arandjelovic and Zissermann, 2013]	64.6	-	<b>55.5</b>
<b>CNN</b> [Babenko et al., 2014]	79.3	3.56	54.5
<b>CNN2</b> [Gong et al., 2014]	80.2	-	-
<b>Sum-pooling VGG</b> [Babenko et al., 2015]	80.2	3.65	53.1
<b>Ours (vanilla, high-dimensional)</b>	79.3	3.76	49.8
<b>Ours + PCA 4096 + whitening</b>	<b>82.9</b>	<b>3.77</b>	47.2

## What about image classification?

First proof of concept was evaluated on classical “deep learning” datasets. **without data augmentation or data pre-processing**:

Tr. size	CNN [25]	Scat-1 [8]	Scat-2 [8]	CKN-GM1 (12/50)	CKN-GM2 (12/400)	CKN-PM1 (200)	CKN-PM2 (50/200)	[32]	[18]	[19]
300	7.18	4.7	5.6	4.39	4.24	5.98	<b>4.15</b>		NA	
1K	3.21	2.3	2.6	2.60	<b>2.05</b>	3.23	2.76		NA	
2K	2.53	<b>1.3</b>	1.8	1.85	1.51	1.97	2.28		NA	
5K	1.52	<b>1.03</b>	1.4	1.41	1.21	1.41	1.56		NA	
10K	0.85	<b>0.88</b>	1	1.17	<b>0.88</b>	1.18	1.10		NA	
20K	0.76	0.79	<b>0.58</b>	0.89	0.60	0.83	0.77		NA	
40K	0.65	0.74	0.53	0.68	<b>0.51</b>	0.64	0.58		NA	
60K	0.53	0.70	0.4	0.58	<b>0.39</b>	0.63	0.53	0.47	0.45	0.53

Table: Test error in % for various approaches on the MNIST dataset.

Method	[12]	[27]	[18]	[13]	[4]	[17]	[32]	CKN-GM	CKN-PM	CKN-CO
CIFAR-10	82.0	82.2	<b>88.32</b>	79.6	NA	83.96	84.87	74.84	78.30	82.18
STL-10	60.1	58.7	NA	51.5	<b>64.5</b>	62.3	NA	60.04	60.25	62.32

Table: Classification accuracy in % on CIFAR-10 and STL-10.

# Current Perspectives

## Engineering effort helps

- higher (huge)-dimensional models may be learned; they give about 86% on CIFAR-10 ( $\approx 88\%$  with data augmentation);

## Supervision helps

- preliminary supervised models are already close to 90% (single model, no data augmentation);

## Future (ongoing) research directions

- finish adding supervision.
- dealing with video data;
- extending the framework to structured data, sequences, graphs;
- theory and faster learning algorithms;

# Conclusion of the course

## What we saw

- Basic definitions of p.d. kernels and RKHS
- How to use RKHS in machine learning
- The importance of the choice of kernels, and how to include “prior knowledge” there.
- Several approaches for kernel design (there are many!)
- Review of kernels for strings and on graphs
- Recent research topics about kernel methods

## What we did not see

- How to **automatize** the process of kernel design (kernel selection? kernel optimization?)
- How to deal with **non p.d.** kernels
- Bayesian view of kernel methods, called **Gaussian processes**.

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