Math 175: Combinatorics
Homework 2: Solutions

1. **Exercise 2.6.2.5 of HHM**: Suppose that an unlimited number of jelly beans is available in each of five different colors: red, green, yellow, white, and black.

   (a) How many ways are there to select twenty jelly beans?

   Solution: We write the number of red jelly beans as x_1, the number of green as x_2, the number of yellow as x_3, the number of white as x_4, and the number of black as x_5. The ways of choosing 20 jellybeans are in bijection with non-negative integer solutions

   $$x_1 + x_2 + x_3 + x_4 + x_5 = 20,$$

   or equivalently, weak compositions of 20 into five parts. By adding one to each part, these are in bijection with compositions of 25 into 5 parts. By the 'Stars and Bars' theorem we proved in class, this is $\binom{20+5-1}{5-1}$.

   (b) How many ways are there to select twenty jelly beans if we must select at least two jelly beans of each color?

   Solution: The ways of choosing 20 jellybeans where we must choose at least 2 in each color are in bijection with solutions to

   $$x_1 + x_2 + x_3 + x_4 + x_5 = 20,$$

   where each $x_i \geq 2$. Subtracting 1 from each x_i shows that the set of such ways is in bijection with the set of compositions of 15 into 5 parts. We recall from our 'Stars and Bars' result that this is $\binom{15-1}{5-1}$.

2. **Exercise 2.6.2.6 of HHM**: A catering company brings fifty identical hamburgers to a party with twenty guests.

   (a) How many ways can the hamburgers be divided among the guests, if none is left over?

   Solution: The number of ways of giving out the hamburgers is in bijection with non-negative integer solutions to

   $$\sum_{i=1}^{20} x_i = 50,$$

   that is, weak compositions of 50 into exactly 20 parts. These are in bijection with compositions of 70 into 20 parts. There are $\binom{70-1}{20-1}$ of these.

   (b) How many ways can the hamburgers be divided among the guests, if every guest receives at least one hamburger, and none is left over?

   Solution: The only difference between this part and the previous one is that now each $x_i \geq 1$. So we just need to count compositions of 50 into 20 parts. There are $\binom{50-1}{20-1}$ of these.

   (c) Repeat these problems if there may be burgers left over.

   Solution: There is a nice interpretation here if we now think of there being 21 people who receive hamburgers now instead of 20, where Person 21 is 'Garbage', the hamburgers that get left over. We are looking for weak compositions of 50 into 21 parts. There are $\binom{50+21-1}{21-1} = \binom{70}{20}$ of these.

So the ways of handing out 20 hamburgers, where some can be left over, to 20 people are in bijection with compositions of $50 - 20$ into 21 parts, but where the 21^{th} person is allowed to get 0 hamburgers (there which is given by $\binom{50-20-1}{21-1}$.

In the second part, the ways of handing out 50 hamburgers, where some can be left over, to 20 people, where each person must receive at least one are in bijection with solutions to

$$x_1 + x_2 + \cdots + x_{20} + x_{21} = 50,$$

where each $x_i \geq 1$ except that x_{21} can be equal to 0. These solutions are in bijection with compositions of 51 into 21 parts. There are $\binom{51-1}{21-1} = \binom{50}{20}$ of these.

3. (a) How many compositions of n use only parts that are even?

Solution: If n is odd, there are zero. This is because the sum of any number of even numbers is even. If n is even, by considering small numbers we may guess that the answer is $2^{n/2-1}$.

We will show that this is correct by giving a bijection between compositions of an even number n into even parts, and compositions of $n/2$. The bijection just takes each part and divides by two. Clearly, two compositions into even parts cannot be sent to the same composition of $n/2$. Also, given a composition of $n/2$ we can see that the composition where we double each part is a composition of n into even parts. This shows that we have found a bijection.

(b) How many compositions of n only use parts that are odd?

Solution: We see that for $n = 1$ we get one way, for $n = 2$ we get one way, for $n = 3$ we get two ways, for $n = 4$ we get three ways, and for $n = 5$ we get five ways. At this point we might guess that the answer is the nth Fibonacci number F_n.

We first ask a more specific question. How many compositions of n use exactly k odd parts? If n is even, then the answer is zero when k is odd, and when n is odd, the answer is zero when k is even. We note that by adding 1 to each part we get a composition of $n + k$ into exactly k even parts. It is easy to verify that this is a bijection. Using the reasoning in part (a), dividing each of these parts by two gives a bijection with compositions of $\frac{n+k}{2}$ into exactly k parts. There are $\binom{\frac{n+k}{2}-1}{k-1}$.

We suppose that n is odd. The argument for n even is almost exactly the same. What are the possible values of k? It has to be of the same parity (even or odd) as n, and can be anything less than or equal to n. This gives a sum of binomial coefficients:

$$\binom{n-1}{n-1} + \binom{n-2}{n-3} + \cdots + \binom{\frac{n+1}{2}}{1}.$$

Noting that $\binom{n}{k} = \binom{n}{n-k}$ this is equal to

$$\binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{\frac{n+1}{2}}{\frac{n+1}{2}-1}.$$

Applying a Fibonacci identity, this is equal to F_n.

Solution 2: Let $f(n)$ count the number of compositions of n into odd parts. We note that $f(1) = f(2) = 1$. We will show that this sequence satisfies the same recurrence as the Fibonacci sequence. This will show that for all n, the answer is F_n.

Suppose $n \geq 2$. We give a bijection from the set of compositions of n into odd parts and the set of compositions of $n-1$ or $n-2$ into odd parts. Given a composition of n into odd parts that ends in 1, delete the last 1. This gives a composition of $n-1$ into odd parts and it is clear that this is a one-to-one correspondence. Given a composition of n into odd parts that ends in something larger than 1, subtract two from the last part. This gives a composition of $n-2$ into odd parts. This is also a one-to-one correspondence. Taking these two maps together gives our bijection and shows that $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$. This completes the proof.

(c) How many weak compositions of n into exactly k parts use only parts that are even?

Solution: When n is odd, the answer is zero, so we suppose n is even.

We give a bijection between the set of weak compositions of n into exactly k parts using only even parts, and compositions of $n+2k$ using only even parts. Using the reasoning above the set of such compositions is in bijection with the set of compositions of $\frac{n}{2} + k$ into exactly k parts. There are $\binom{\frac{n}{2}+k-1}{k-1}$ of these.

Given a weak composition of n into exactly k parts using only even parts, we first add two to every part, then we divide every part by 2. The first step gives a bijection with compositions of $n + 2k$ into exactly k even parts, and the second step gives a bijection with compositions of $\frac{n+2k}{2}$ into exactly k parts.

(d) How many compositions of n into an even number of parts only use parts that are even?

Solution: If n is odd the answer is zero, so we suppose n is even. We know that there is a bijection between compositions of n using exactly k even parts and compositions of $\frac{n}{2}$ using exactly k parts. By adding up the relevant binomial coefficients, we know that half of all compositions of $\frac{n}{2}$ have an even number of parts. There are $2^{\frac{n}{2}-1}$ total compositions of $\frac{n}{2}$, so $2^{\frac{n}{2}-2}$ have an even number of parts. Our bijection shows that this also counts the number of compositions of n into an even number of even parts.

4. A standard die has six faces labeled $\{1, 2, 3, 4, 5, 6\}$. When you roll the die, each face comes up with probability exactly $\frac{1}{6}$.

(a) Suppose you roll a die k times. What is the probability that you get k distinct numbers? (Your answer should be a function of k.)

Solution: Rolling a die k times gives a sequence of length k from the alphabet $\{1, 2, 3, 4, 5, 6\}$. We want to count the number of such sequences with no repeated letter, divided by the total number of sequences. The probability is therefore

$$\frac{\frac{6!}{(6-k)!}}{6^k} = \frac{6!}{(6-k)!6^k}.$$

(b) Write down a table that shows these values for all $1 \leq k \leq 6$. What is the smallest value of k for which this is less than $1/2$?

Note: You will probably need to use a computer or calculator to compute these values.

Solution: A little computation with a calculator or computer gives the following table:

	1	2	3	4	5	6	$k \geq 7$
Probability	1	$\frac{5}{6}$	$\frac{5}{9}$	$\frac{5}{18}$	$\frac{5}{54}$	$\frac{5}{324}$	0

We note that $k = 4$ is the first time this probability is less than 50%.

5. A standard deck of playing cards has 4 suits (spades $\spadesuit$, diamonds $\diamondsuit$, clubs $\clubsuit$, and hearts $\heartsuit$). In each suit there is exactly one card from each of 13 denominations $(2, 3, 4, \ldots, 10$, Jack, Queen, King, and Ace). This gives 52 total cards (for example the 2 of Diamonds and the Jack of Hearts).

   Suppose you draw 5 cards at random.

   (a) What is the probability that you do not have two cards of the same denomination?

   Solution: We first note that there are $\binom{52}{5}$ different possible collections of 5 cards. We are asking how many collections have no two cards of the same denomination. We first count sequences of five cards so that no two have the same denomination. We have 52 choices for the first card, 48 choices for the second card, 44 choices for the third card, 40 choices for the fourth card, and 36 choices for the final card. To go from sequences to sets, we divide by 5! to forget the order in which we choose the cards. This gives a probability of

$$\frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!\binom{52}{5}} = \frac{44 \cdot 40 \cdot 36}{51 \cdot 50 \cdot 49 \cdot 47} \approx 50.7\%.$$

   (b) What is the probability that you have four cards of the same denomination? (This is called a 'Four of a Kind'.)

   Solution: There are 13 denominations in which we could have 'Four of a Kind'. Once we choose which of these 13 collections of four cards we want to have, we need to choose one more card. There are 48 choices for this last card. So the probability it

$$\frac{\binom{13}{1} \cdot 48}{\binom{52}{5}} = \frac{13 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50 \cdot 49} \approx 0.00024.$$

   (c) What is the probability that you have three, but not four cards of the same denomination?

   Solution: We choose which of the 13 denominations we want to have three cards in. We then need to choose which 3 of the four cards in this denomination we would like to have. Finally, we have two remaining cards to choose, and since neither card can be in our 'Three of a Kind' denomination, we have $\binom{48}{2}$ choices for this pair of cards. The probability is then

$$\frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{48}{2}}{\binom{52}{5}} \approx 0.02257.$$

   (d) What is the probability that you have three cards in one denomination, and your remaining two cards are in the same denomination? (This is called a 'Full House'.)

Solution: This is a subcase of the previous problem. The first part is the same- we choose our 'Three of a Kind' denomination and then choose which 3 of the 4 cards in it we want to have. We have 48 remaining cards not in this denomination and want to choose a pair of two from the same denomination. We first choose which of the 12 denominations we want to have our pair in, and then choose which 2 of the 4 cards in this denomination we want to have. The probability is then

$$\frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}}{\binom{52}{5}} \approx 0.00144.$$

6. In lecture we saw that once there are 23 or more people in a room, there is more than a 50% chance of having two people with the same birthday. (For simplicity we have assumed that there are 365 possibly birthdays and that they are all equally likely.)

   For k people we can compute this probability exactly. We line up the k people in some order and ask each of them to write down their birthday. This gives a word of length k from an alphabet of 365 letters. Now we'll consider the birthday problem for triples. I will break it down into a bunch of steps to try to help guide you through it.

   (a) What is the probability that among k people the first two of them have the same birthday, nobody else has that birthday, and every other person has a distinct birthday?

   Solution: We think of the probability as being given by the following process: We ask k people in some order to say their birthdays. We count the number of possible sequences of birthday where the first two are the same, nobody else has that birthday, and every other birthday is distinct, and divide by 365^k, the total number of possible sequences.

   We first pick the birthday shared by the first two people. There are $\binom{365}{1}$ possible days. We then know that the next $k-2$ people each have distinct birthdays chosen from the 364 remaining days, giving $\frac{364!}{(364-(k-2))!}$ possible sequences. This gives the probability as

$$\frac{\binom{365}{1}\frac{364!}{(364-(k-2))!}}{365^k} = \frac{365!}{(364-(k-2))!365^k}.$$

   (b) What is the probability that among k people exactly two of them have the same birthday?

   Solution: We first pick the common birthday, which we can do in $\binom{365}{1}$ ways. We now have to pick the two spots for these people in the sequence of k people. We can do this in $\binom{k}{2}$ ways. Now there are $k-2$ distinct birthdays chosen from 364 possible birthdays and given in some order among the $k-2$ remaining spots. This gives the probability as

$$\frac{\binom{365}{1}\binom{k}{2}\frac{364!}{(364-(k-2))!}}{365^k} = \frac{365!\binom{k}{2}}{(364-(k-2))!365^k}.$$

   (c) What is the probability that among k people, the first two people have the same birthday, the second two people have the same birthday (different from the first pair), and everyone else has a distinct birthday?

   Solution: We first choose the birthday shared by the first pair. There are 365 ways to do this. We then choose the birthday shared by the second pair. There are 364 ways to do this given the choice we've already made. We then know that there are $k-4$ distinct birthdays chosen from the 363 remaining days given in some order by the $k-4$ remaining people. This gives the probability as

$$\frac{365 \cdot 364 \cdot \frac{363!}{(363-(k-4))!}}{365^k} = \frac{365!}{(363-(k-4))! \cdot 365^k}.$$

(d) What is the probability that among k people, exactly two pairs of people share a birthday (and no three people have the same birthday)?

Solution: We first pick the two shared birthdays. We then pick the spots for one of the two pairs, which we can do in $\binom{k}{2}$ ways. We then pick the spots for the next pair, which we can do in $\binom{k-2}{2}$ ways. We note that

$$\binom{k}{2}\binom{k-2}{2} = \frac{k!}{(k-4)!2^2}.$$

We then choose the remaining birthdays in some order in the remaining $k-4$ spots. This gives

$$\frac{\binom{365}{2}\frac{k!}{(k-4)!2^2}\frac{363!}{(363-(k-4))!}}{365^k} = \frac{365!k!}{2!(k-4)!2^2(363-(k-4))!365^k}.$$

(e) Show that the probability that among k people there are exactly m pairs of people with the same birthday, and no three people have the same birthday is given by

$$\left(\frac{365!k!}{m!(365-k+m)!2^m(k-2m)!}\right) \cdot \frac{1}{365^k}$$

Solution: We adapt the argument give above for the special case $m=2$. We first choose these m shared birthdays, which we can do in $\binom{365}{m}$ ways. We then choose the spots for the first pair, followed by the second pair, and so on, down to the mth pair. We can do this in $\binom{k}{2}\binom{k-2}{2}\cdots\binom{k-2(m-1)}{2}$. We note that this is equal to $\frac{k!}{(k-2m)!2^m}$. We now choose the remaining $k-2m$ distinct birthdays in some order from the remaining $365-m$ days, which we can do in $\frac{(365-m)!}{(365-m-(k-2m))!}$ ways. Note that $\binom{365}{m}(365-m)! = \frac{365!}{m!}$. This gives the probability as

$$\frac{365!k!}{(k-2m)!2^m m!(365+m-k)!365^k},$$

completing the proof.

(f) Write down a function that gives the probability that among k people, no three of them have the same birthday. This is just a sum involving the previous part of this problem.

Solution: This probability is given as a sum over the number of pairs of people with a common birthday. This number of pairs can take any value from 0, in the case when all birthdays are distinct, to $\lfloor \frac{k}{2} \rfloor$. This gives

$$\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{365!k!}{(k-2m)!2^m m!(365+m-k)!365^k}.$$

(g) **Bonus/Not Required**: Use a computer to evaluate this probability for some values of k (if you can do it without a computer, more power to you). How many people do we need to have in a room before there is more than a 50% chance that at least three people share a common birthday?

Solution: Evaluating in Mathematica with the following commands shows that for $k = 87$ the probability is approximately .500545 and for $k = 88$ the probability is around .488935. Here is the Mathematica code:

$$\begin{aligned} P[k\_, m\_] &:= 365! * k!/((k-2m)! * 2^m * m! * (365+m-k)! * 365^k) \\ PP[k\_] &:= \text{Sum}[P[k,m], \{m, 0, \text{Floor}[k/2]\}] \\ N[PP[88]], & \end{aligned}$$

which gives 0.488935.