Math 175: Combinatorics
Homework 5

Due in class Friday, February 24.
Please email `nckaplan@math.uci.edu` with questions.

After the last homework we started by discussing some examples about counting labeled versus unlabeled teams. We explained why there is less than a 50% chance that you end up on the same team as your friend.

We then introduced the subtraction principle. The main idea is that we want to count sequences/outcomes with some property. We instead count the total number of possible sequences/outcomes and then subtract off the number of sequences/outcomes without this property. We gave examples involving people sitting around a circular table. There is no section of our textbooks that goes over the subtraction principle specifically, but these ideas can be applied to several of the problems in Chapter 1 of LVP.

We discussed lattice paths on a grid and the Catalan numbers. We showed that there are $\binom{n+m}{m}$ paths of length $n+m$ from $(0,0)$ to (m,n). When $n=m$ we showed that the number of these paths that do not cross above the diagonal line $x=y$ is given by the Catalan number C_n. We did this by using the reflection principle to show that the number of paths that do cross above $x=y$ is given by $\binom{2n}{n-1}$.

Catalan numbers are the subject of Section 2.6.6 of the HHM book, but the perspective is very different. The Wikipedia entry for Catalan numbers does a good job presenting the material that we covered on this topic.

In class I derived the probability that a point is visited by minimal length paths from $(0,0)$ to (m,n). Here is a reference in case you would like to know which points are visited most often:
`http://www.math.uci.edu/~nckaplan/research_files/Kaplan-RestaurantLocation.pdf`

We then gave some examples involving the Pigeonhole principle. This is the subject of Section 2.4 in LVP and Section 2.4 in HHM.

Finally, we started to talk about the Principle of Inclusion-Exclusion. This is the subject of Section 2.3 of LVP and Section 2.5 of HHM.

Problems

1. (a) How many ways are there to line up 4 boys and 8 girls so that no two boys are next to each other?

   Note: Each child is a unique, so $B_1B_2B_3B_4G_1G_2\cdots G_8$ is a different lineup than $B_2B_1B_3B_4G_1G_2\cdots G_8$, (although neither one counts for this question.)

   (b) How many ways are there to seat these 4 boys and 8 girls at a circular table (with 12 labeled chairs) so that no two boys are next to each other?

2. How many pairs of subsets A, B of $\{1, 2, \ldots, n\}$ are there such that A is contained in B?

3. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$. Show that for $n \geq 1$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

   Hint: Can you think of a way to break up a path from $(0,0)$ to (n,n) that does not cross above the diagonal line $x = y$ into two parts?

4. Nathan and Amir are running for President of Math 175. There are $2n+1$ students voting in the election. Each student writes their favorite candidate's name on a piece of paper and throws it in a box. The ballots are shuffled and then counted one by one. This particular election is extremely close- Nathan wins by a single vote.

   What is the probability that once the first ballot is read out, Nathan is strictly ahead of Amir for the entire ballot counting process?

   Example: When there are 5 votes, N gets 3 and A gets 2. Here is one possible way for the ballots to be read out where Nathan is always ahead: NNANA. The reading NNAAN does not count, because after four votes are read the candidates are tied.

5. Suppose $2k$ people are seated around a table with labeled chairs. How many ways are there for k pairs of people to shake hands simultaneously across the table in such a way that no arms cross?

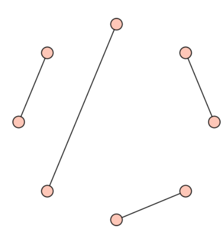

Figure 1: A way for $n = 8$ people to shake hands with no arms crossing

6. In class we saw that the number of paths from $(0,0)$ to (m,n) of the minimal length $m+n$ is $\binom{m+n}{m}$. We saw that the number of these paths passing through (a,b) is given by

$$f_{m,n}(a,b) = \binom{a+b}{a} \cdot \binom{(m+n)-(a+b)}{m-a}.$$

For any $n \geq 1$, it is clear from the definition that $f_{n,n}(1,1) = f_{n,n}(n-1,n-1)$. Show that for any a satisfying $1 < a < n-1$,

$$f_{n,n}(1,1) > f_{n,n}(a,a).$$

Hint: What can you say about the ratio $f_{n,n}(a,a)/f_{n,n}(a+1,a+1)$?

7. Suppose there are $m+n$ students in a class, $a+b$ of them are girls, and $m \geq a$. If we randomly choose m students, what is the probability that exactly a of them are girls?

8. We select 38 even positive integers all less than 1000. Prove that there will be two of them whose difference is at most 26.

9. (a) Show that any 4-element subset of $\{1,2,\ldots,6\}$ has two disjoint subsets that have the same sum.

   (b) Show that there is a 4-element subset of $\{1,2,\ldots,7\}$ such that no two disjoint subsets have the same sum.

10. Suppose we shoot five arrows at a target that is 6 inches by 6 inches. Since we are awesome at archery (no big deal) we hit the target every time. Prove that there are two arrows with distance between them at most $3\sqrt{2}$ inches.