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L13

Aside: HW3 Q7c.

We flip a fair coin a number of times.

Show that getting $n-1$ or n heads in $2n$ flips is as likely as getting n or $n+1$ heads in $2n+1$ flips.

A.

# flips	# H
$2n$	$n-1$ or n
$2n+1$	n or $n+1$

$P(nH \text{ or } (n+1)H \text{ in } 2n+1 \text{ flips})$

$$= P((n-1)H \text{ in } 2n \text{ flips}) \cdot \underbrace{P(H \text{ in next flip})}_{1/2}$$
$$+ P(nH \text{ in } 2n \text{ flips}) \cdot \underbrace{P(H \text{ or } T \text{ in next flip})}_{1}$$
$$+ P((n+1)H \text{ in } 2n \text{ flips}) \cdot \underbrace{P(T \text{ in next flip})}_{1/2} \qquad (*)$$

Now:

$$P((n-1)H \text{ in } 2n \text{ flips}) = \frac{\binom{2n}{n-1}}{2^{2n}}$$

$$P((n+1)H \text{ in } 2n \text{ flips}) = \frac{\binom{2n}{n+1}}{2^{2n}}$$

$$\binom{2n}{n-1} = \binom{2n}{n+1} \quad (\text{symmetry of Pascals } \Delta)$$

Thus:

$$P((n-1)H \text{ in } 2n \text{ flips}) = P((n+1)H \text{ in } 2n \text{ flips})$$

Thus

$$(*) = P((n-1)H \text{ in } 2n \text{ flips}) + P(nH \text{ in } 2n \text{ flips})$$
$$= P((n-1)H \text{ or } nH \text{ in } 2n \text{ flips})$$

The Subtraction Principle / The Power of Negative Thinking

Q Flip a coin 5 times. How many ways can we get at least one H and at least one T?

A1 Enumerate:

	# H	# T	# ways
	0 ×	5 ✓	1
favorable outcomes	1 ✓	4 ✓	$\binom{5}{1}$
	2 ✓	3 ✓	$\binom{5}{2}$
	3 ✓	2 ✓	$\binom{5}{3}$
	4 ✓	1 ✓	$\binom{5}{4}$
	5 ✓	0 ×	1

Thus answer is $\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}$.

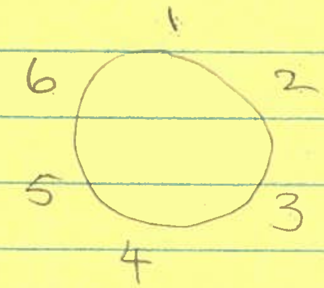
A2 Table highlights that it is easier to count the unfavorable outcomes, of which there are only two: all heads and all tails.

Since we know the # possible outcomes, 2^5, the answer can also be written:

$$2^5-2$$

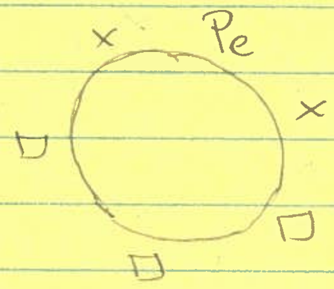
Q There are 6 people sitting around a table on labelled chairs. Two of them, Peter and Paul, are bitter enemies who refuse to sit together. How many seating arrangements are there?

A1



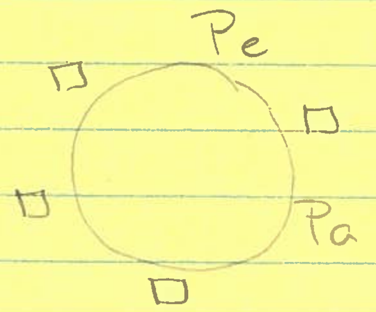
Note: if everyone just rotates one seat to the left, say, then this counts as a different arrangement.

1. Pick a seat for Peter. (6 possibilities)
2. Paul can sit in neither the seat to the left nor right of Peter:



$\Rightarrow$ Paul can sit in one of the 3 □'s.

3. Seat the remaining people:



$\Rightarrow$ 4 choices for 3^{rd} person
3 " " 4^{th} "
2 " " 5^{th} "
1 " " 6^{th} "

Thus, there are $6 \cdot 3 \cdot 4!$ ways to seat the people

A2 The "negative" approach to this question is to subtract # unfavorable arrangements from # possible arrangements.

An unfavorable arrangement is one where Peter and Paul are seated next to one another. In how many ways can this occur?

1. # pairs of adjacent seats $= 6$
2. # ways to seat Peter and Paul in chosen seats $= 2$
3. # ways to seat remaining people $= 4!$

$\Rightarrow$ # unfavorable arrangements $= 6 \cdot 2 \cdot 4!$

possible arrangements $= 6!$

$\Rightarrow$ # favorable arrangements $= 6! - 6 \cdot 2 \cdot 4!$

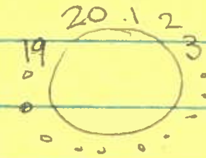
$$= 6(5-2)4!$$

$$= 6 \cdot 3 \cdot 4!$$

which is the same answer we got before.

Q There are 20 people seated at a circular table (with labeled chairs). How many ways can we pick a subset of 3 people containing no neighbors?

A1. Negative Approach.



Count #ways we can pick the people s.t. at least two of them are neighbors.

1. ways to pick two neighbors:

(1,2) , (2,3) , ... (18,19), (19, 20) , (20,1)

$\Rightarrow$ 20 ways.

2. #ways to pick 3rd member of set? You might think the answer is 18 since we have already chosen 2 of the 20 people. However this overcounts:

1 2 (3 4) 5 6 7 .. $\Rightarrow$ ..., {2,3,4}, {3,4,5}, ...

1 2 3 (4 5) 6 7 .. $\Rightarrow$ ..., {3,4,5}, {4,5,6}, ..

1 2 3 4 (5 6) 7 .. $\Rightarrow$ ..., {4,5,6}, {5,6,7}, ..

Evidently, the overcounting arises

when the 3[rd] member is chosen to be a neighbor of the 1[st] or 2[nd] member.

Overcounting can be avoided by

(i) excluding one neighbor of the original pair from becoming the 3[rd] member
$\Rightarrow$ # ways to choose 3[rd] member = 17.
$\Rightarrow$ # 3-sets w/ at least 2 neighbors = $20 \cdot 17$.

or

(ii) subtract 1 for each time a 3-set is double-counted, which is 20 (once for each of (1,2), (2,3), ..., (19,20), (20,1)). Thus:

$$\# \text{3-sets w/ at least 2 neighbors} = 20 \cdot 18 - 20 = 20(18-1) = 20 \cdot 17.$$

There is another way to count the 3-sets w/ at least two neighbors. Partition the sets into those containing 2 and 3 neighbors.

1. 3-sets w/ 2 neighbors (only):

20 choices for the pair, and 16 choices for the remaining person:

	3-sets to exclude
1 2 (3 4) $\cancel{5}$ 6 7 ...	{2,3,4}, {3,4,5}
1 2 $\cancel{3}$ (4 5) $\cancel{6}$ 7 ...	{3,4,5}, {4,5,6}
1 2 3 $\cancel{4}$ (5 6) $\cancel{7}$ ...	{4,5,6}, {5,6,7}.
⋮	⋮

$\Rightarrow$ #ways = 20·16

2. 3-sets w/ 3 neighbors: 20 choices:
(1 2 3), (2 3 4), (3 4 5), ..., (19 20 1), (20, 1, 2)

Then

3-sets w/ at least 2 nbrs = 20·16 + 20
= 20·17

as before.

Finally: #ways to pick 3 people s.t. <u>no</u> 2 of them are neighbors is:

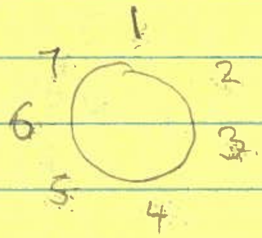
$$\underbrace{\binom{20}{3}}_{\text{\#ways to choose 3 people from 20.}} - 20\cdot 17$$

A2: "Direct Approach"

Consider simplest example:

#ways to choose A : 7

#ways to choose B : 7-3=4 (exclude A and his nbrs)



(A,B,C)	#ways	shared nbr?
(1,3,5)	2	✓
(1,3,6)	2	✓
(1,4,6)	1	×
(1,5,3)	1	×
(1,6,3)	2	✓
(1,6,4)	2	✓

(2,4,6)	2	(3,5,7)	2
(2,4,7)		(3,5,1)	
(2,5,7)	1	(3,6,1)	1
(2,6,4)	1	(3,7,5)	1
(2,7,4)	2	(3,1,5)	2
(2,7,5)		(3,1,6)	

...

Pattern:

- shared nbr: $\cdots \underbrace{XAXBX}\cdots$ not candidates for C $\Rightarrow$ #ways $= 7-5$
- no shared nbr: $\cdots \underbrace{XAXXBX}_{6}\cdots$ or $\cdots\underbrace{XAX}_{3}\cdots\underbrace{XBX}_{3}\cdots$ either way, 6 cannot be candidates for C $\Rightarrow$ #ways $= 7-6$

How many ways to choose B s.t. it shares a nbr with A?

$$\left.\begin{array}{l} \cdots XAXB \cdots \\ \cdots BXAX \cdots \end{array}\right\} \text{ 2 ways.}$$

How many ways to choose B s.t. it does not share a nbr with A?

$$\cdots \underbrace{XXAXX}_{\text{none of these can be B}} \cdots \implies \#\text{ways} = 7-5 = 2$$

Return to original problem involving 20 people. #ways to choose a tuple (A, B, C) s.t. no two elements are nbrs is:

$$\underset{\text{choose A}}{20} \cdot \Big\{ \underset{\text{choose B s.t. it shares nbr with A}}{2} \cdot \underset{\text{choose C}}{(20-5)} + \underset{\text{choose B s.t. it doesn't share nbr with A.}}{(20-5)} \underset{\text{choose C.}}{(20-6)} \Big\}.$$

$$= 20(2 \cdot 15 + 15 \cdot 14) = 20 \cdot 15 \cdot 16$$

Finally, observe that the question asked for a set of 3 people, i.e. the order in which the people are picked is irrelevant. We can "forget" this order by dividing by 3!

Thus

ways to choose 3 people s.t. no 2 are nbrs

$= \frac{20.15.16}{3!}$

This is the same as the original answer, $\binom{20}{3} - 20.17$, since

$$\frac{20.15.16}{3!} = 20\cdot 5\cdot 8 = 20\cdot 40$$

and:

$$\binom{20}{3} - 20.17 = \frac{20.19.18}{6} - 20.17$$

$$= 20\,[19\cdot 3 - 17]$$

$$= 20\,[57 - 17]$$

$$= 20\cdot 40$$

$$\begin{array}{r} 19 \\ \times\ 3 \\ \hline 57 \end{array}$$