A NOVEL COMPUTATIONAL MODEL OF THE AXONEME FOR MICROSWIMMERS IN STOKES FLOW

Peter M N Hull

MATHS FOR SPERM CELLS

A New Approach to Modelling Microswimmers in Stokes Flow

Peter M N Hull

Overview

Overview

1. What is the problem?

Overview

1. What is the problem?
2. What is my solution?

Overview

1. What is the problem?
2. What is my solution?
3. How can I test it?

Overview

1. What is the problem?
2. What is my solution?
3. How can I test it?
4. Did it work?

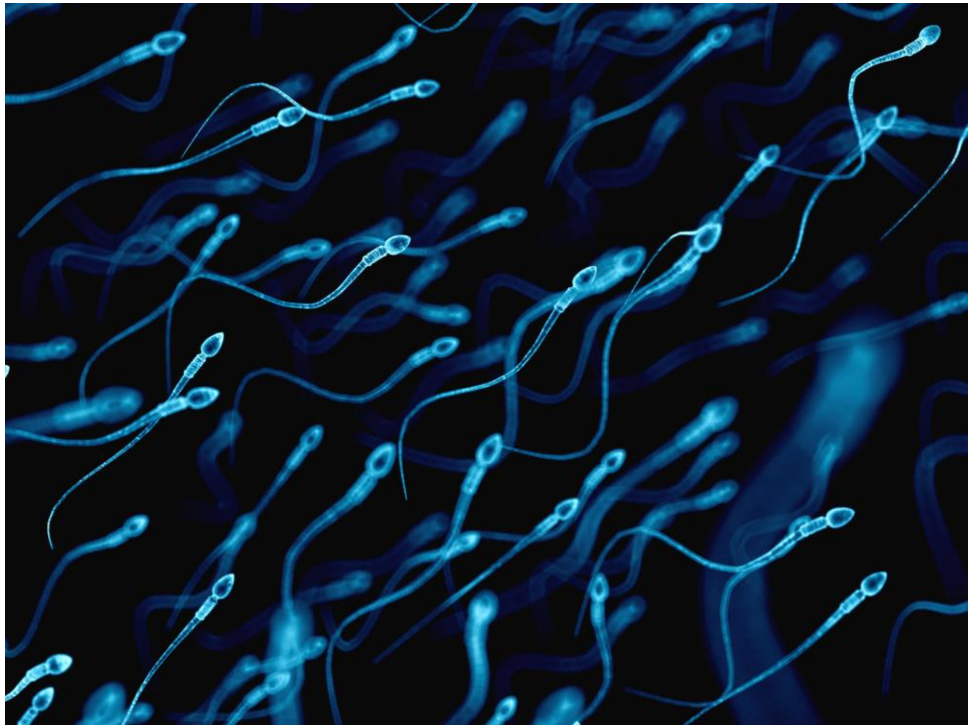
Overview

1. What is the problem?
2. What is my solution?
3. How can I test it?
4. Did it work?
5. What's next?

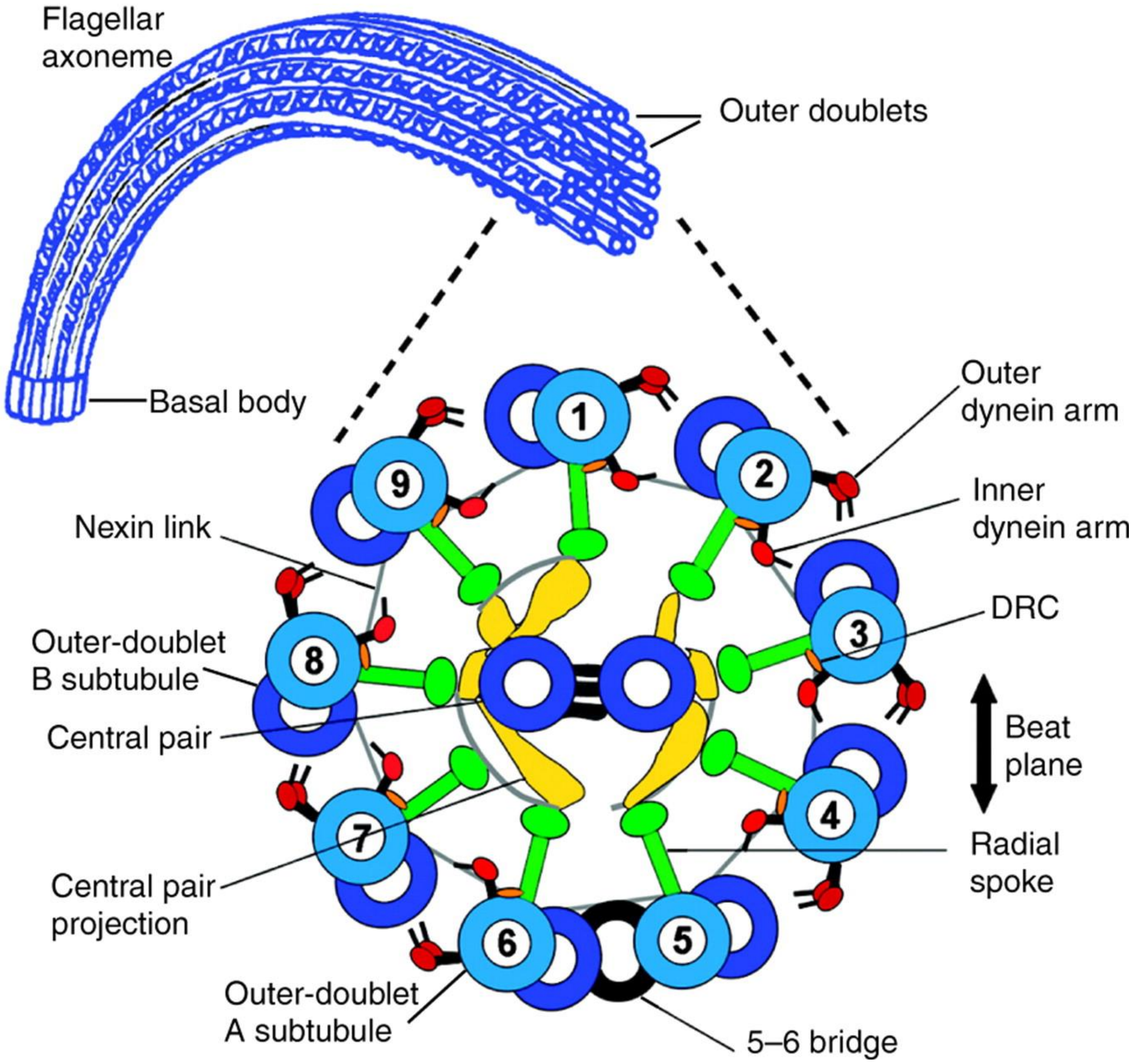
WHAT IS THE PROBLEM?

PART I

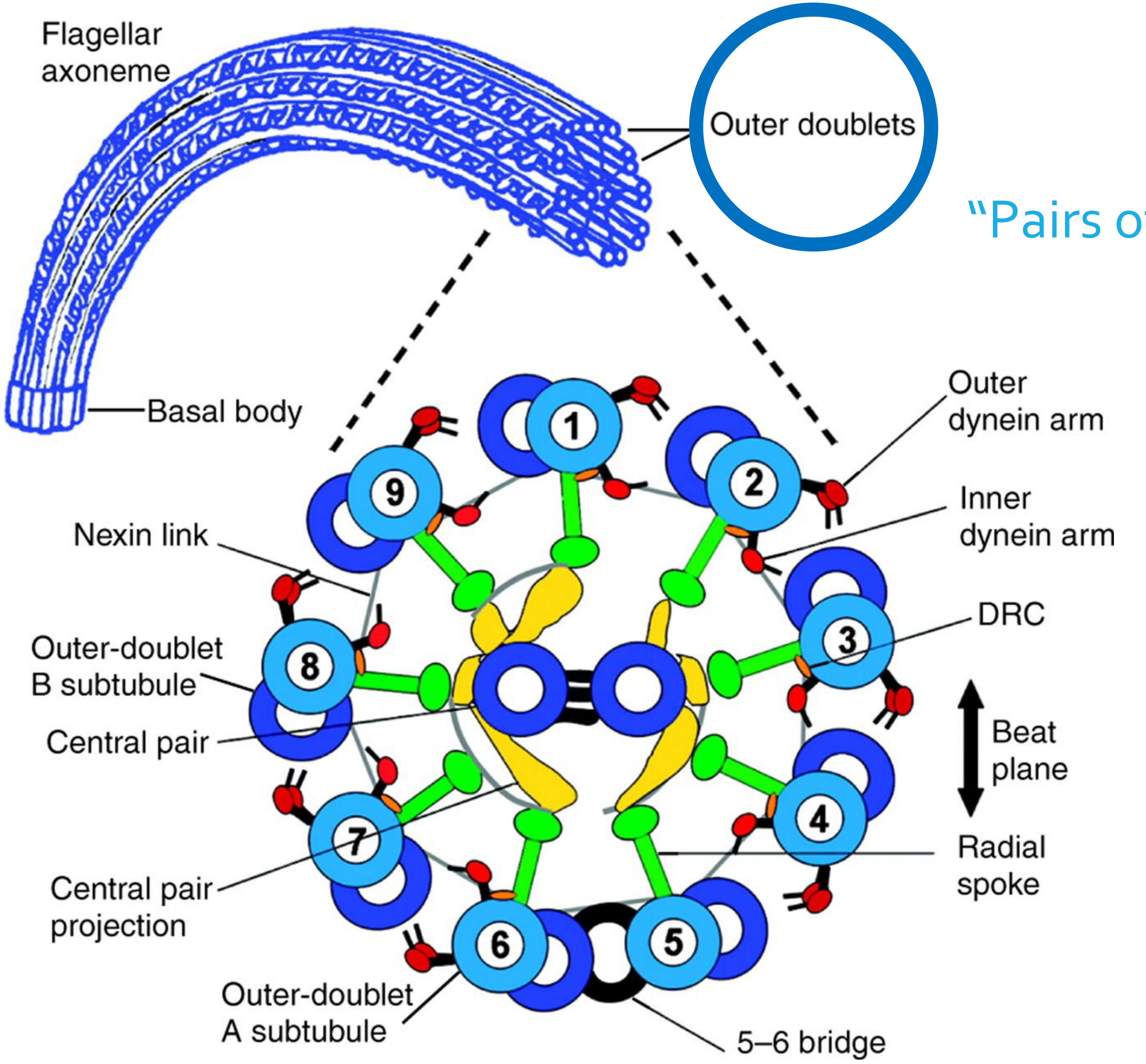
Problem: Sperm cells are hard to understand



Lindemann & Lesich (2010)

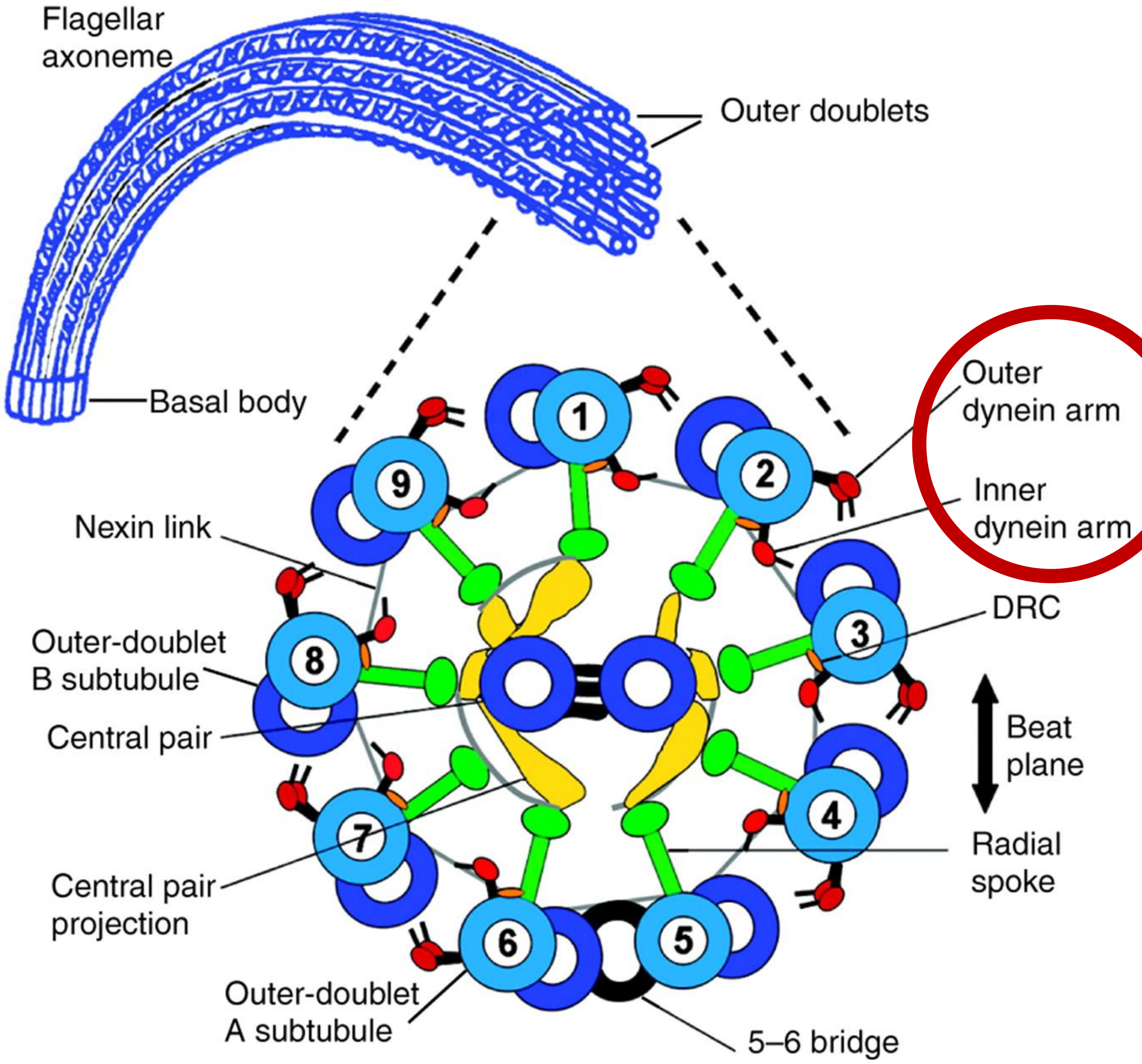


Lindemann & Lesich (2010)

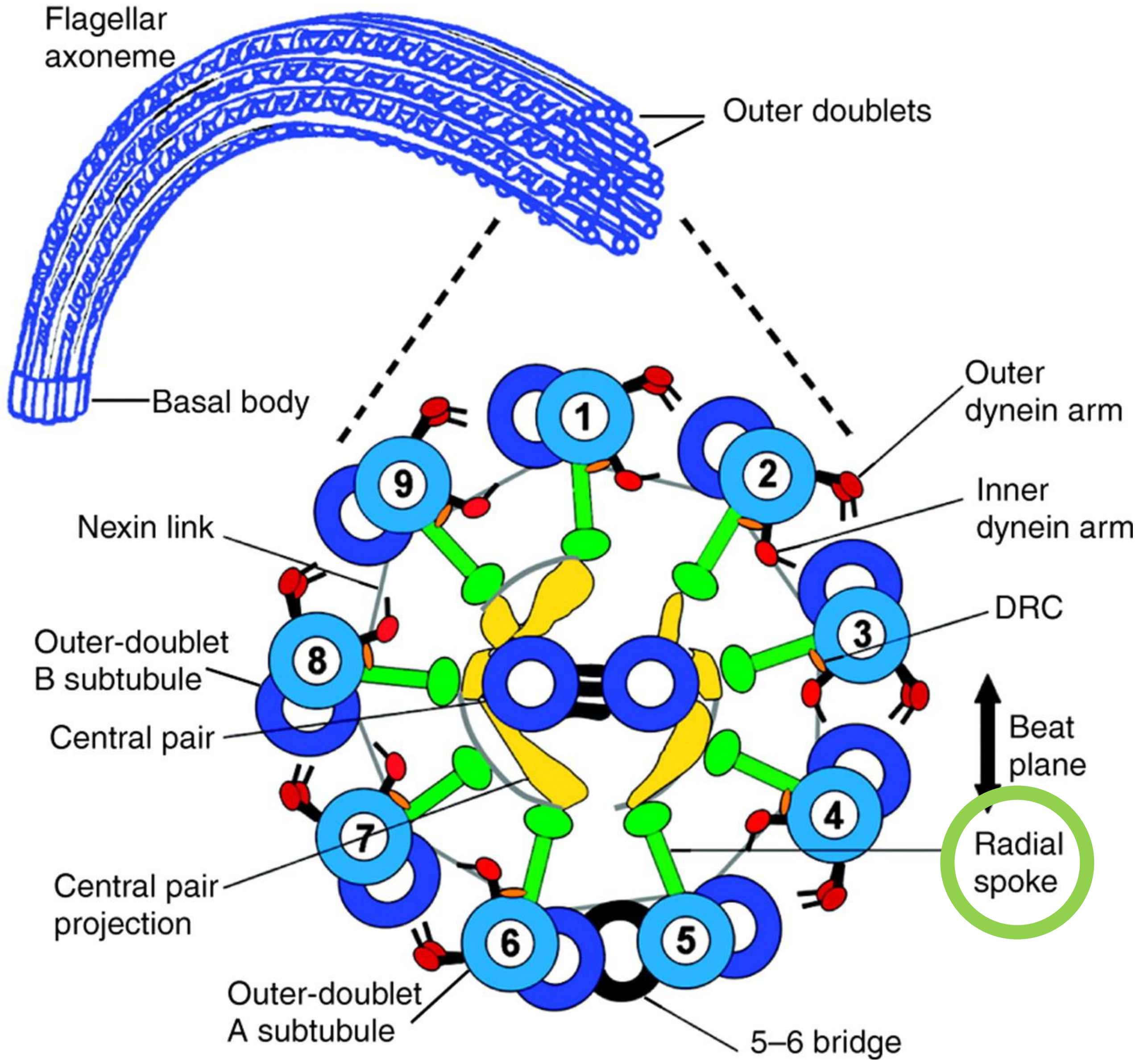


"Pairs of tubes"

Lindemann & Lesich (2010)



Lindemann & Lesich (2010)



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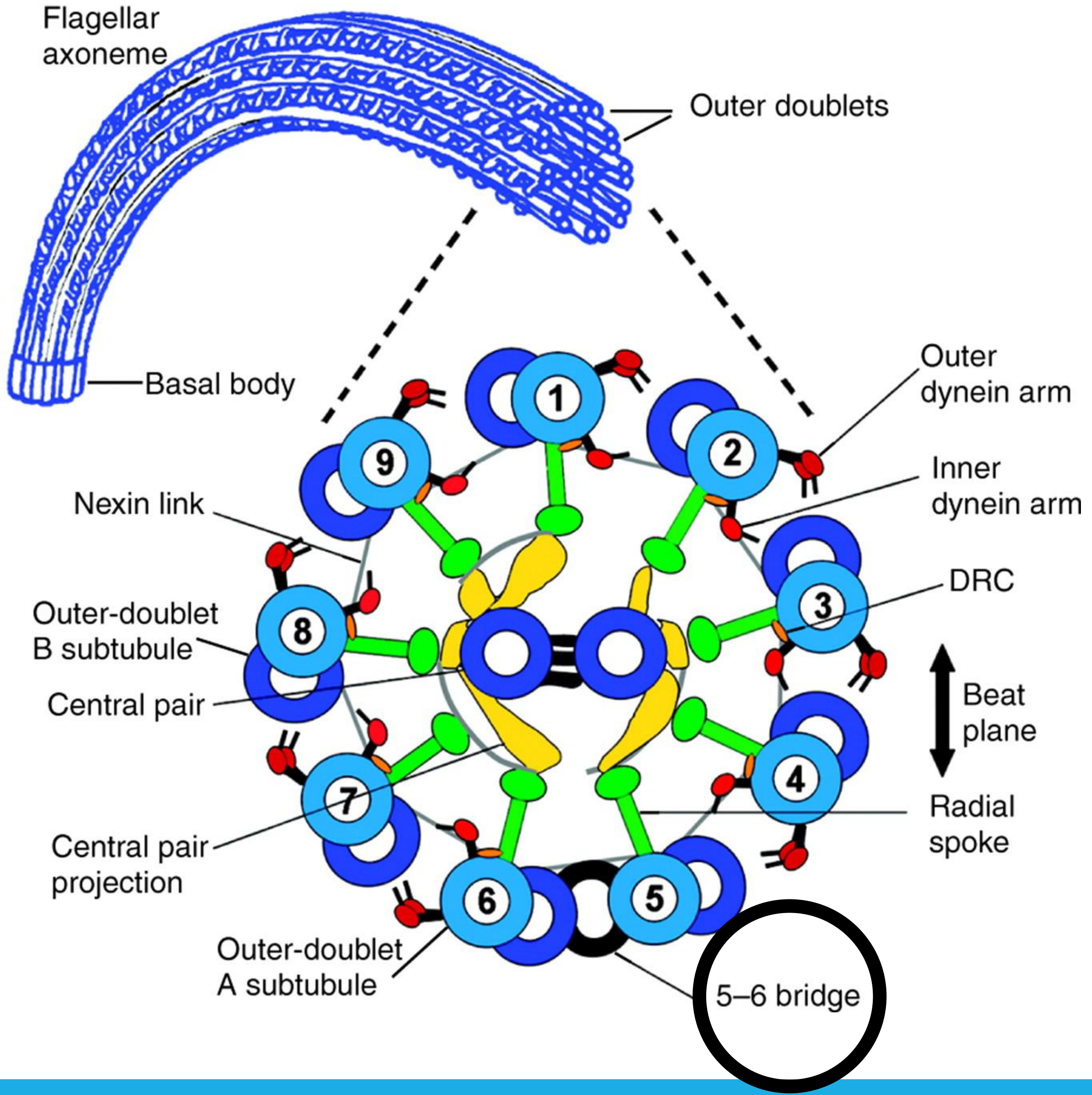
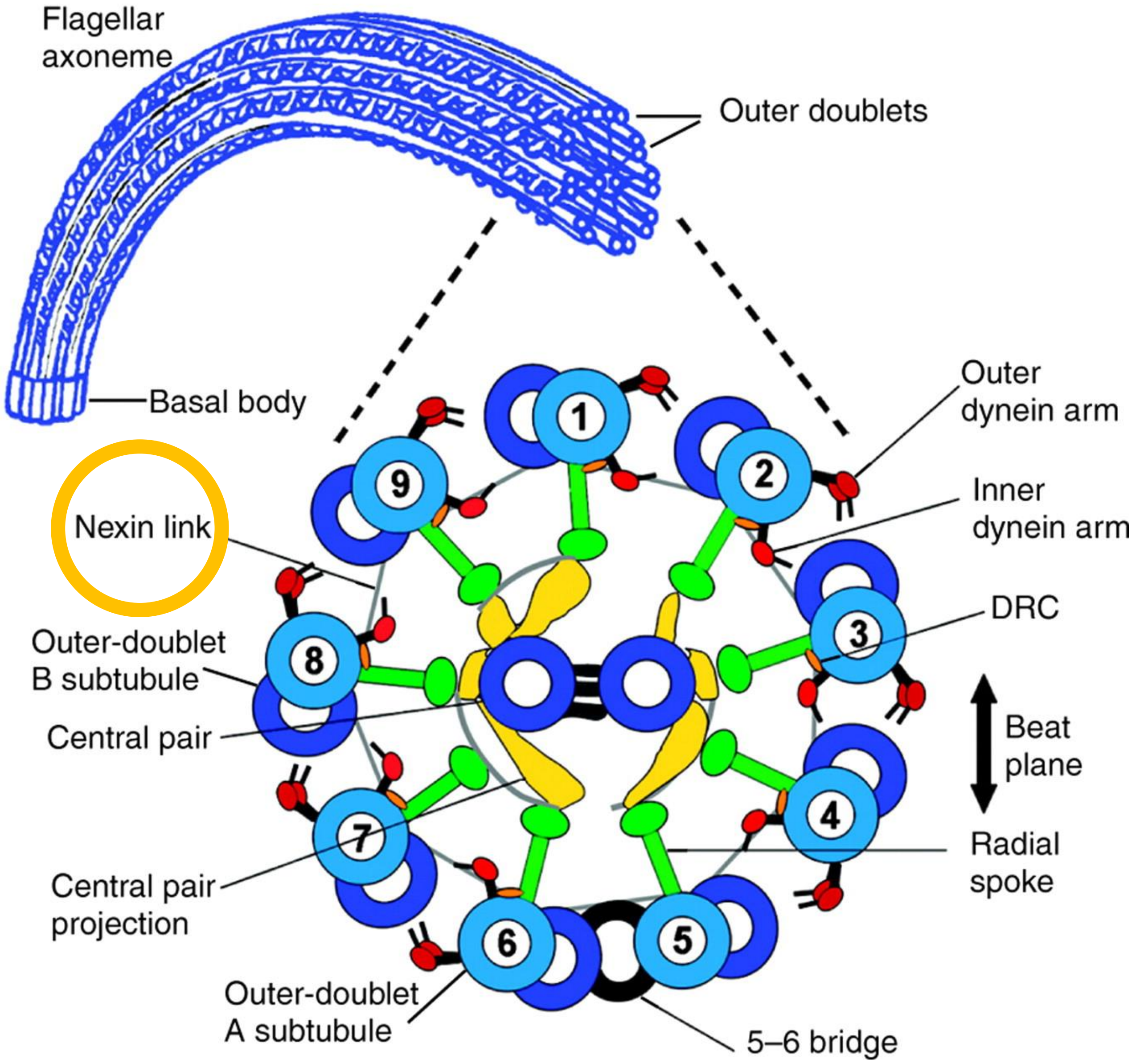
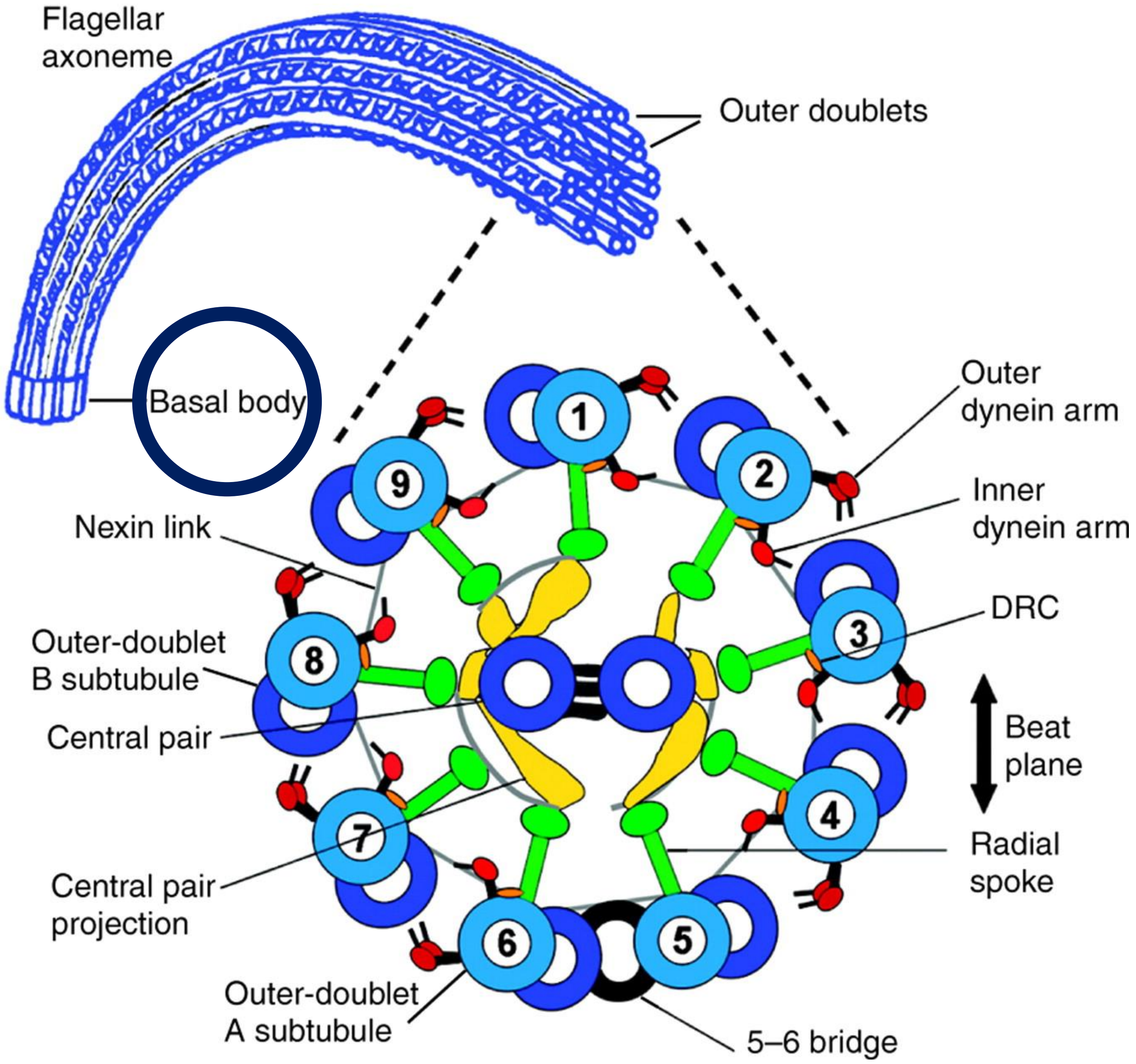


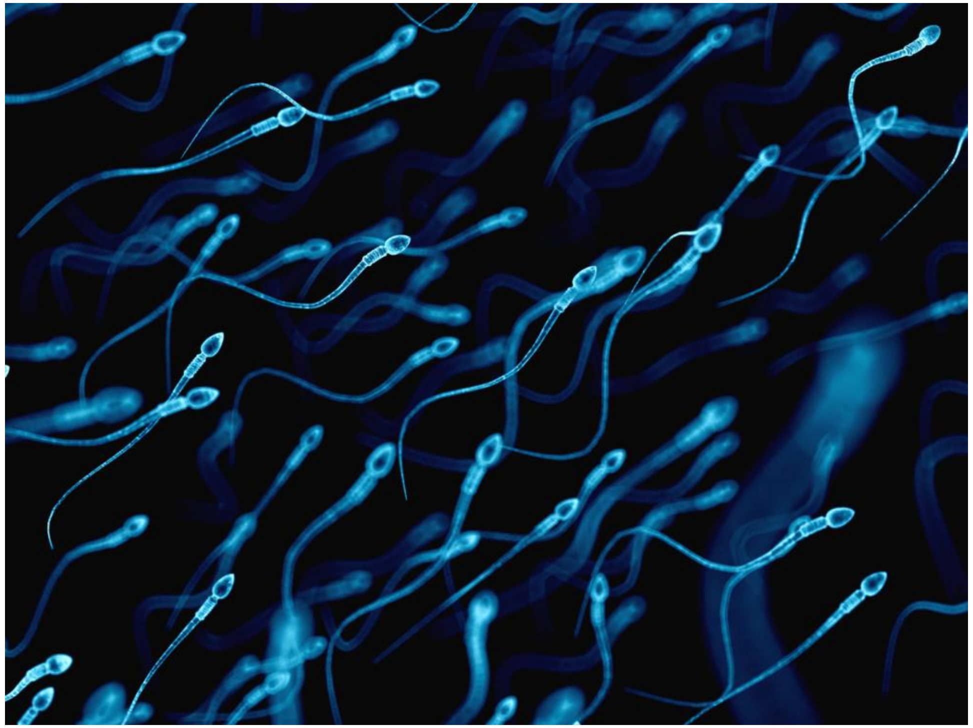
Lindemann & Lesich (2010)



Lindemann & Lesich (2010)

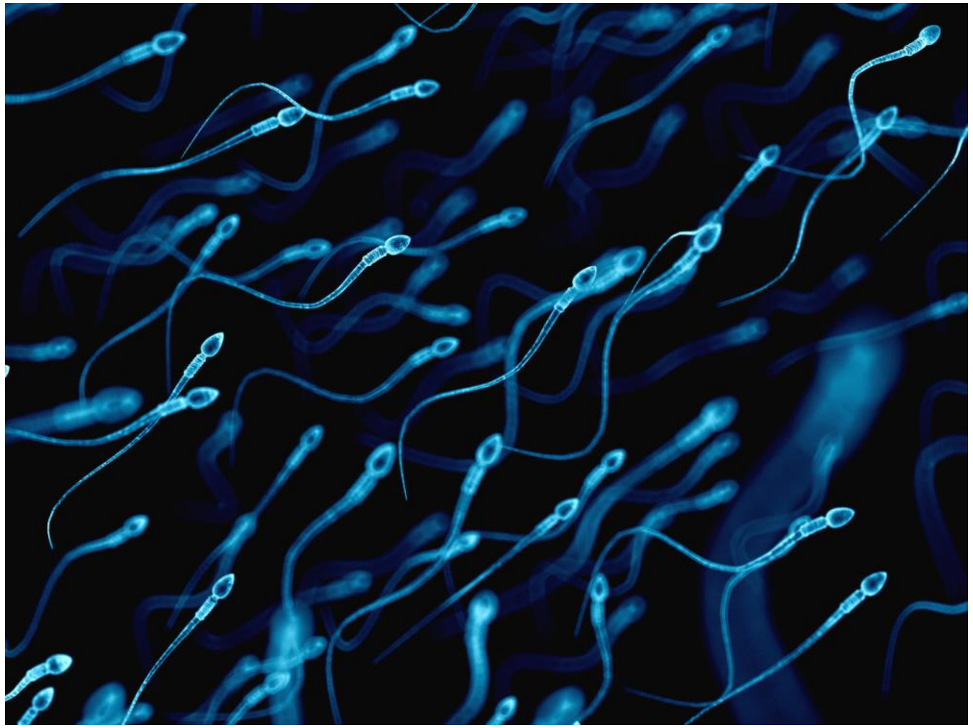


Why should we care?



Kumar & Singh (2015)

Why should we care?

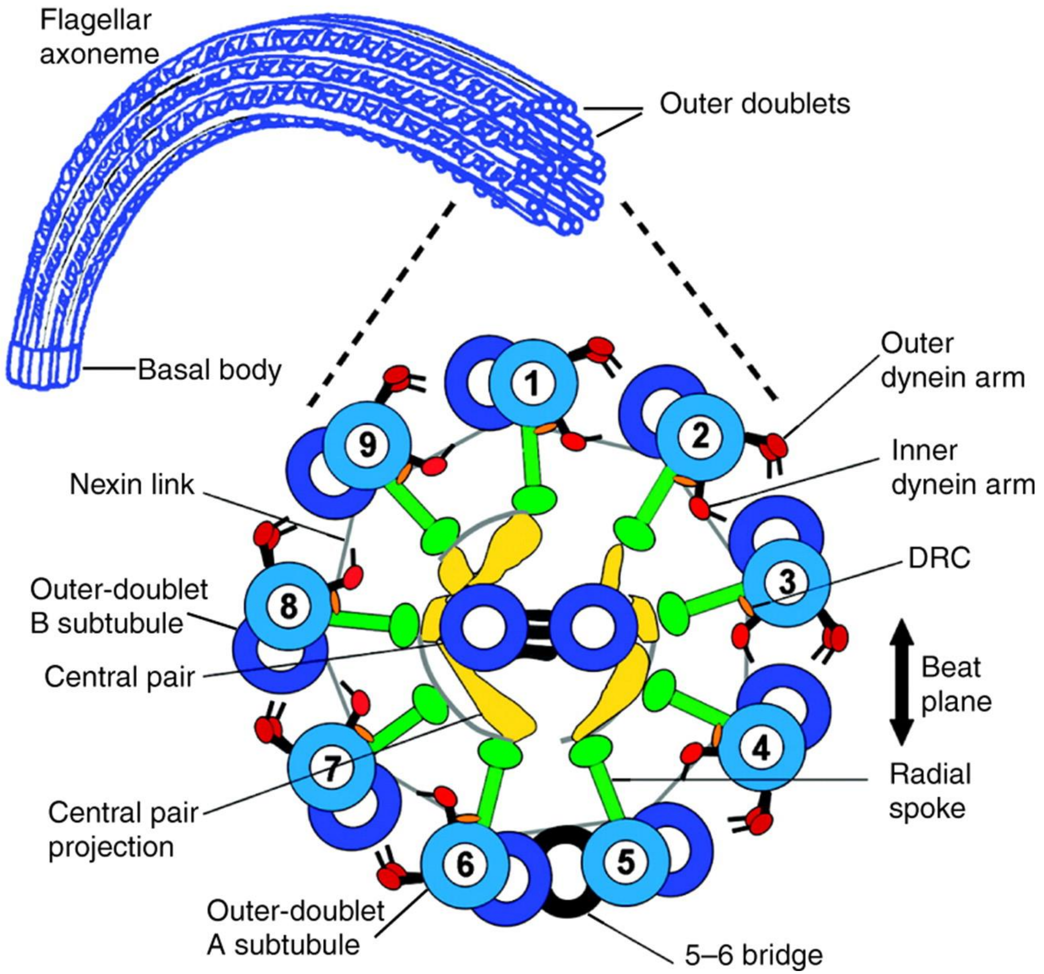


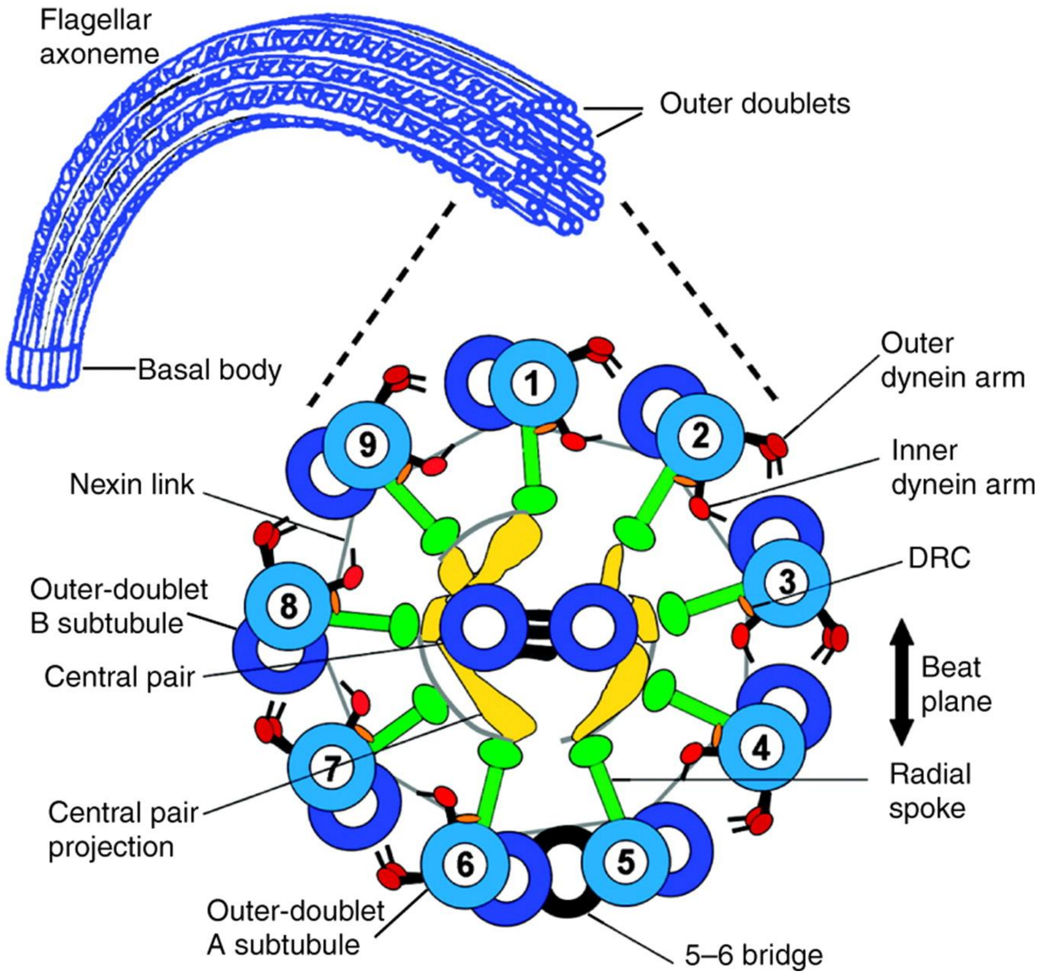
Because sperm are important!

WHAT IS MY SOLUTION?

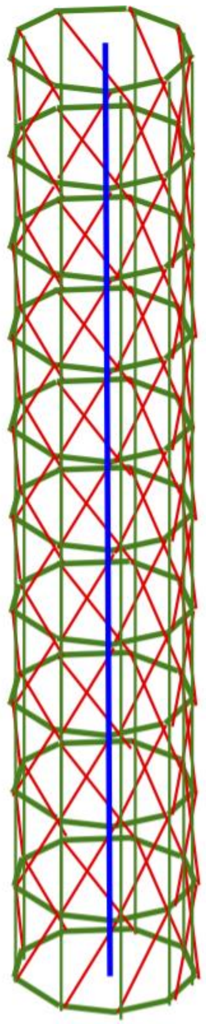
PART II

ANSWER: SIMPLIFY


Flagellar
axoneme
Outer doublets
Basal body
Outer
dynein arm
Inner
dynein arm
Nexin link
DRC
Outer-doublet
B subtubule
Central pair
Beat
plane
Radial
spoke
Central pair
projection
Outer-doublet
A subtubule
5–6 bridge
1
2
3
4
5
6
7
8
9



Han & Peskin (2018)



CAN IT BE SIMPLER?

Can it be simpler?

First Assumption:

Can it be simpler?

<u>First Assumption</u>:

Using a pair of two-dimensional rods is good enough.

Hilfinger, Jülicher & Chattopadhyay (2009)

Can it be simpler?

Second Assumption:

Can it be simpler?

Second Assumption:

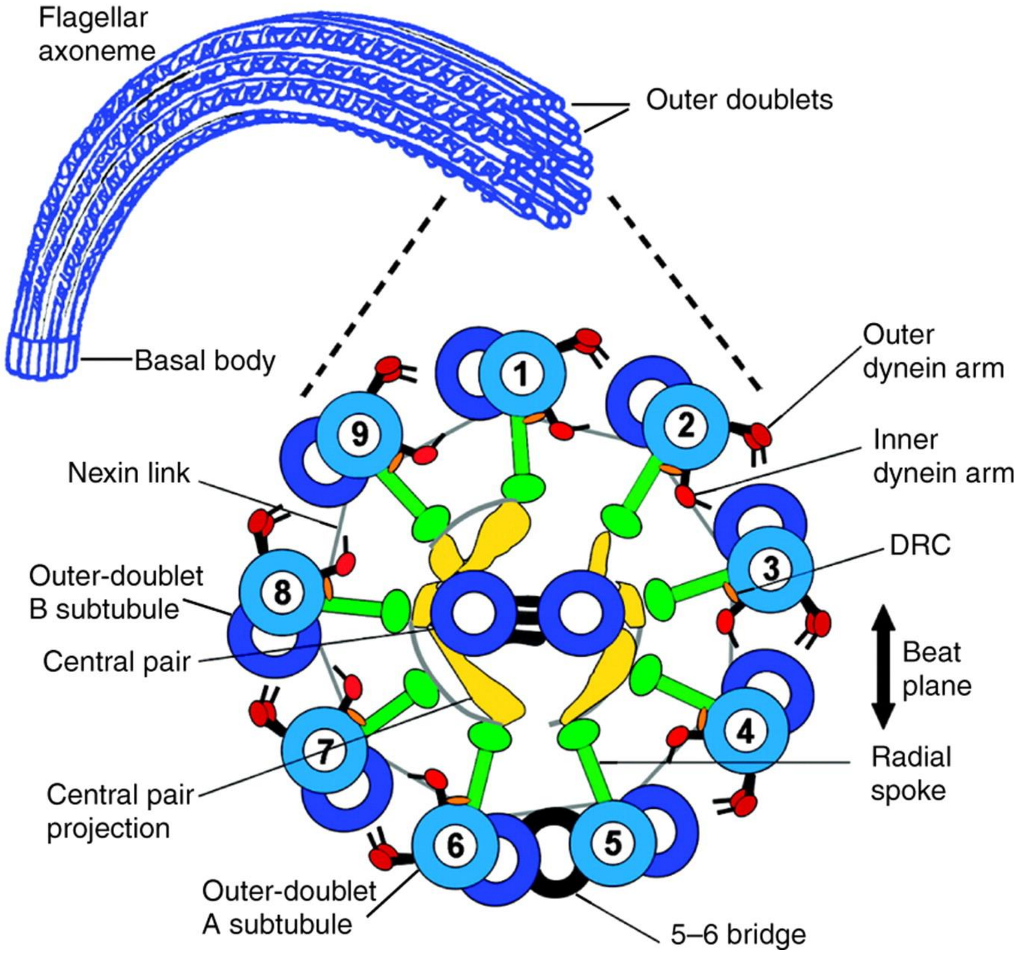
Can use internal clocks.

Can it be simpler?

Second Assumption:

Can use internal clocks. *(Spoke-Axis Hypothesis)*

Lindemann & Lesich (2010)


Flagellar axoneme
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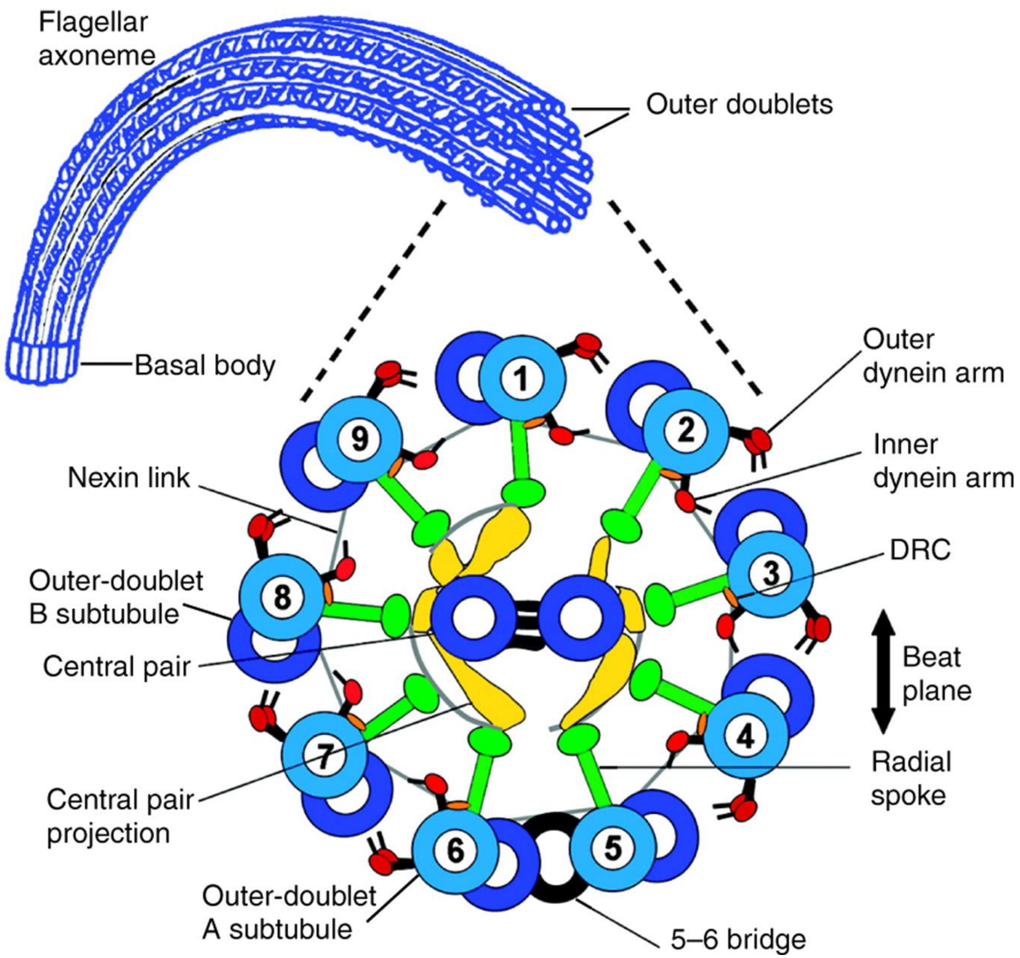
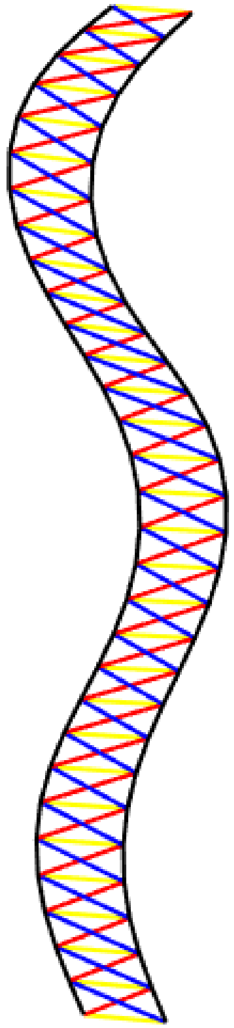


Peter Hull (2020)



AXONEME MODEL

Use Springs to Represent Axoneme Driving Forces

Two Components

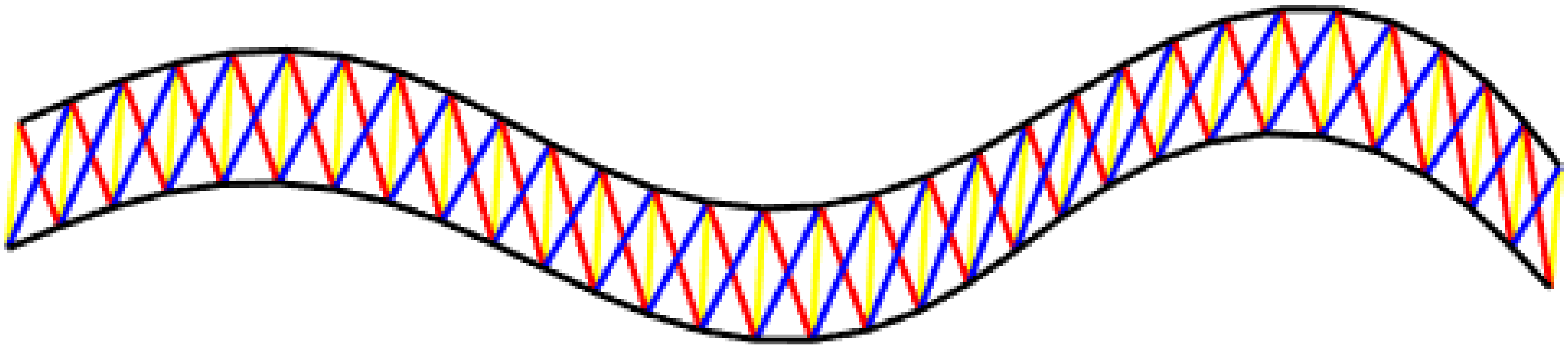
Two Components

1. Maintain body structure

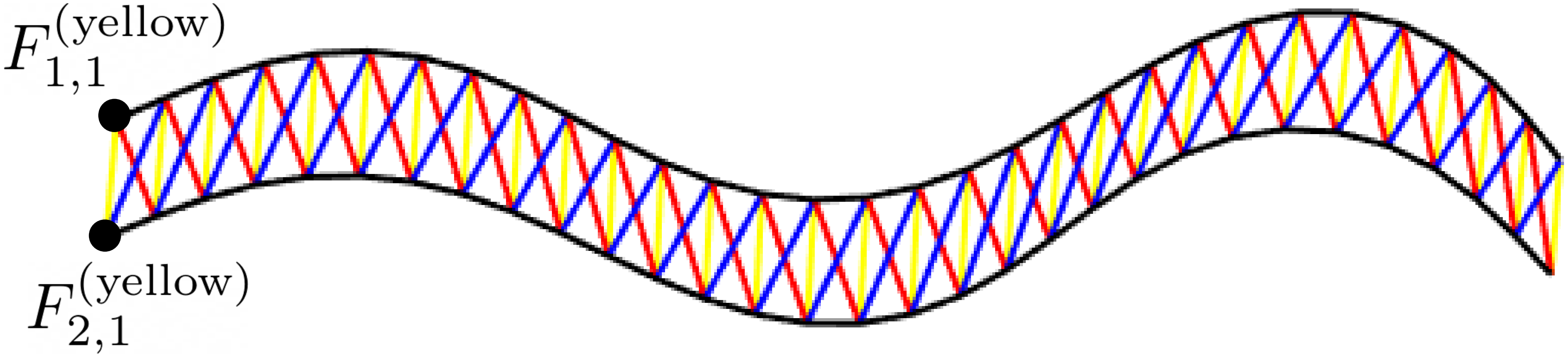
Two Components

1. Maintain body structure
2. Generate motion with active links

Maintaining Body Structure



Maintaining Body Structure



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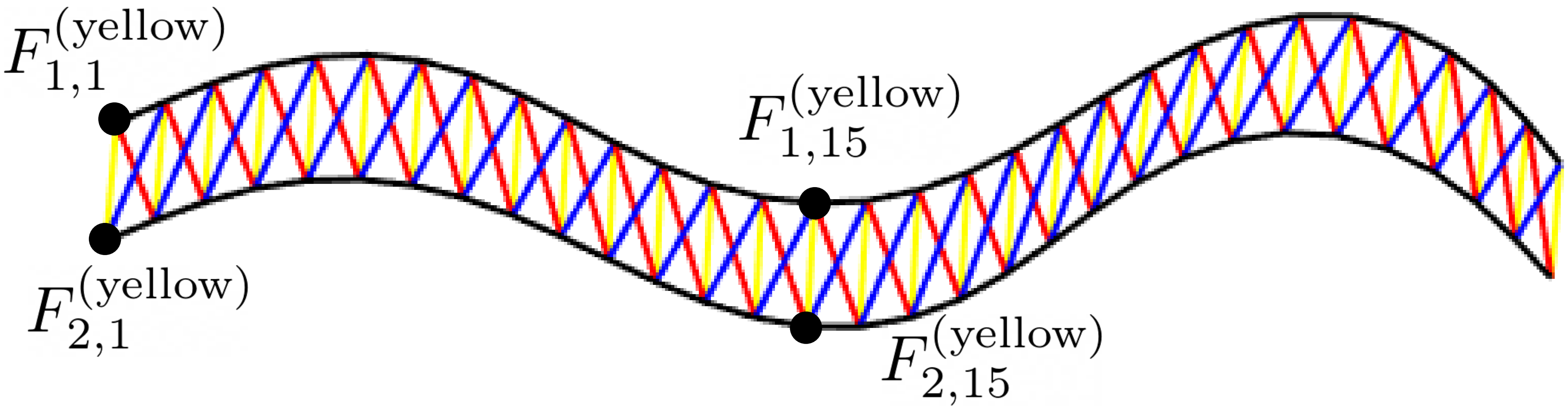

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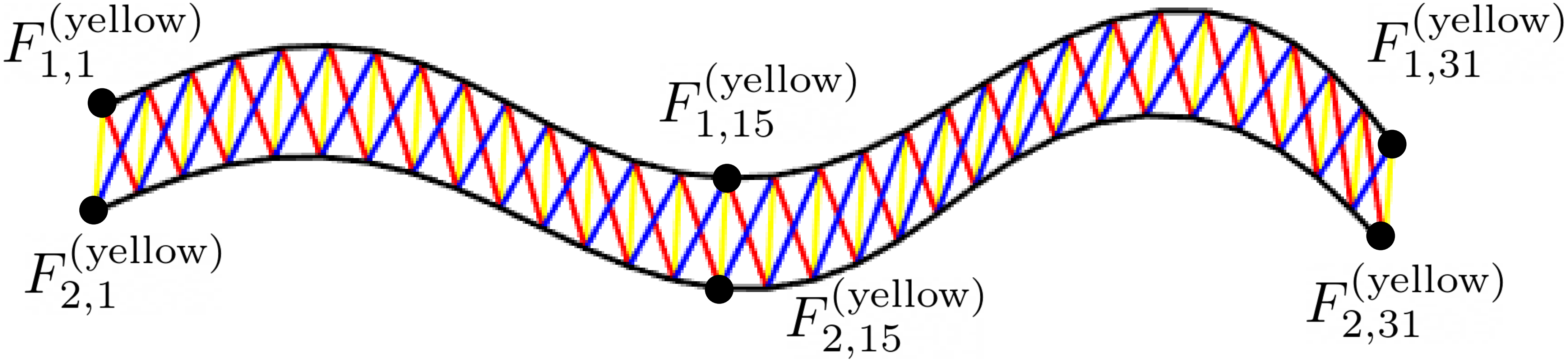
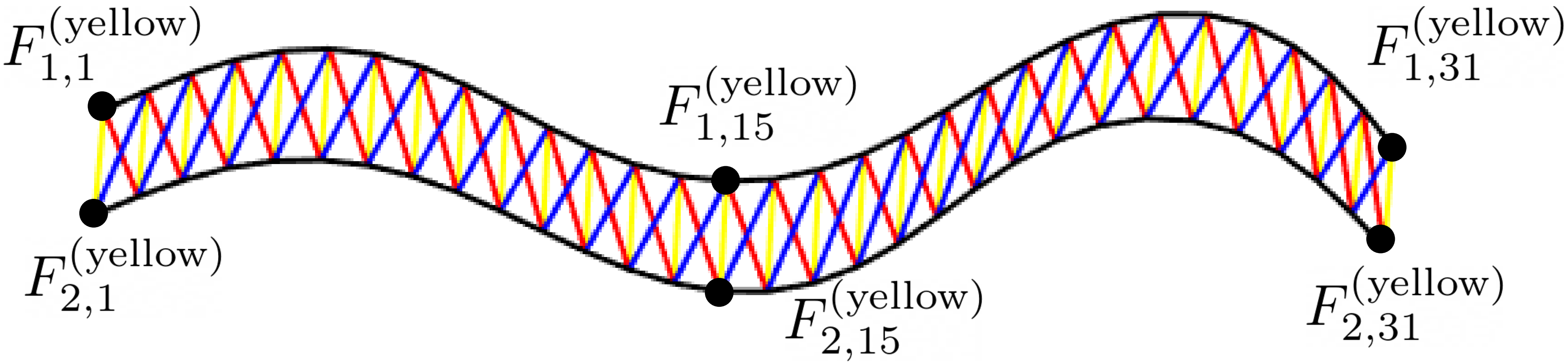
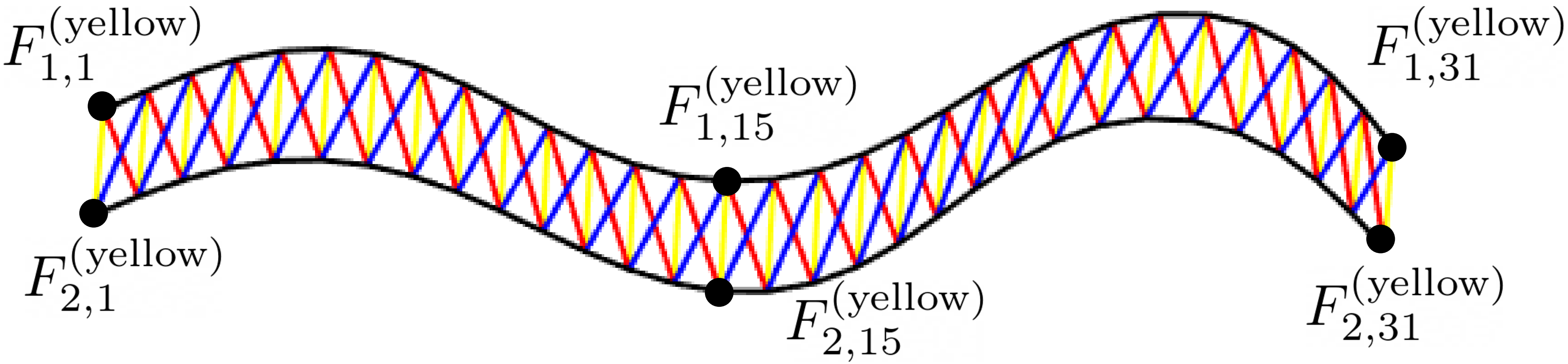


Maintaining Body Structure



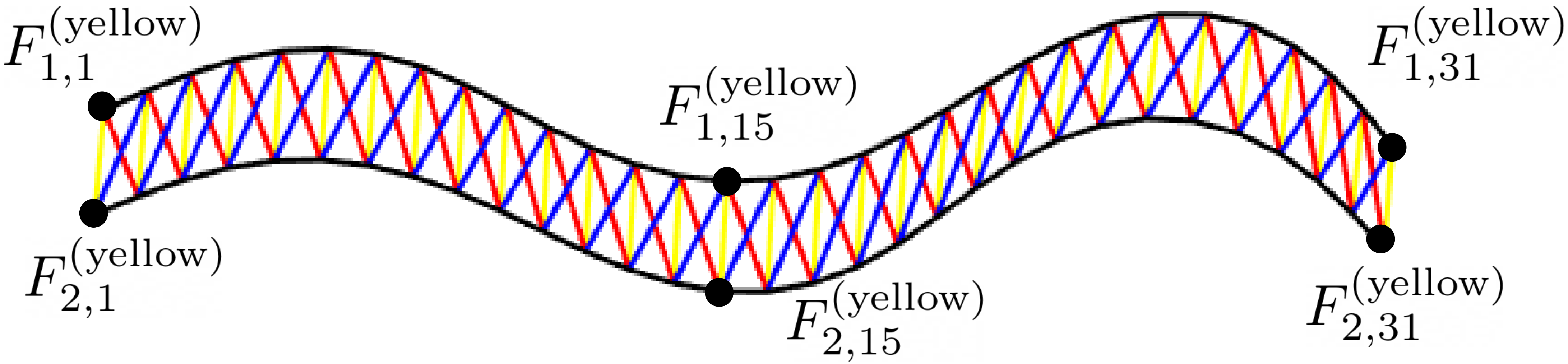
Maintaining Body Structure

$$F_{1,n}^{(\mathrm{yellow})} = -F_{2,n}^{(\mathrm{yellow})}$$



Maintaining Body Structure

$$F_{1,n}^{(\text{yellow})} = -F_{2,n}^{(\text{yellow})} \quad := \quad D - l_{n,n}(t)$$



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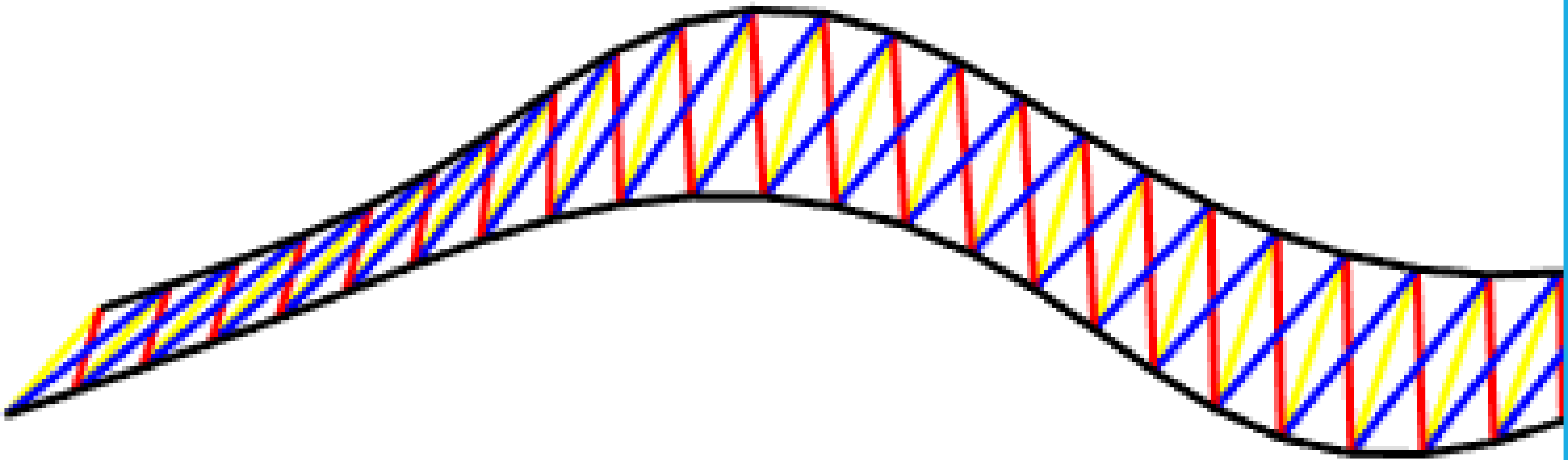

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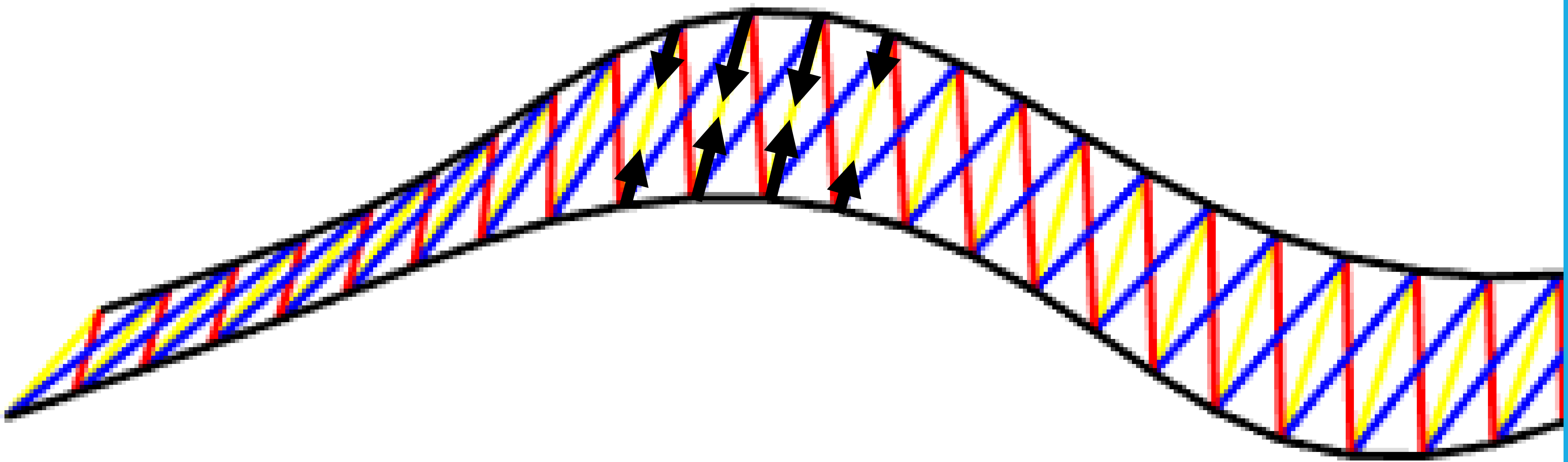

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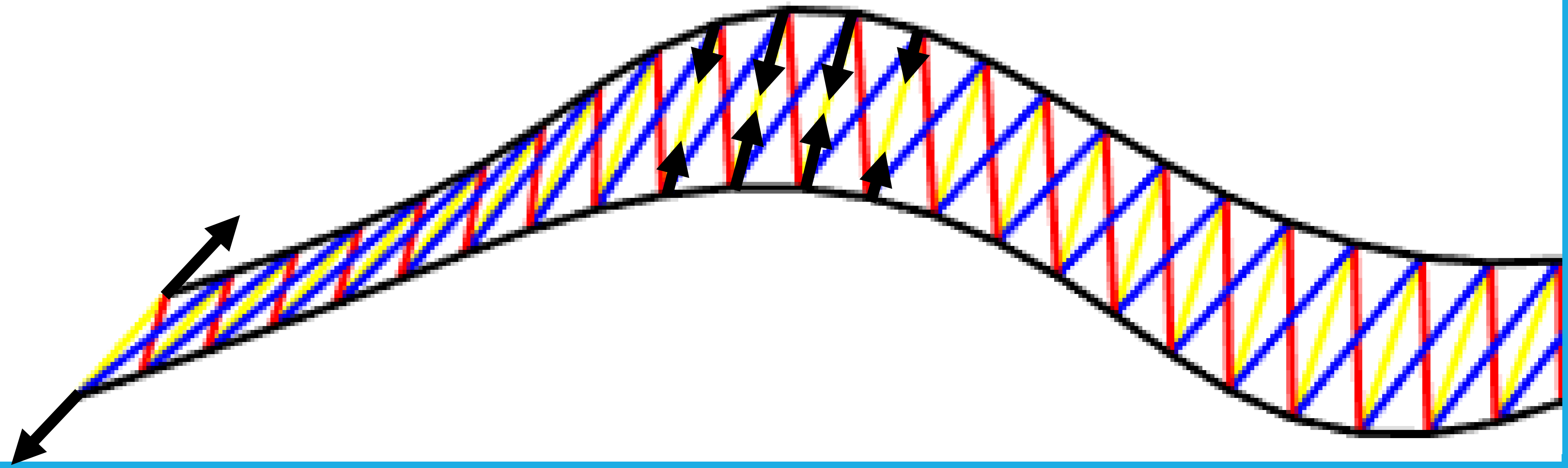
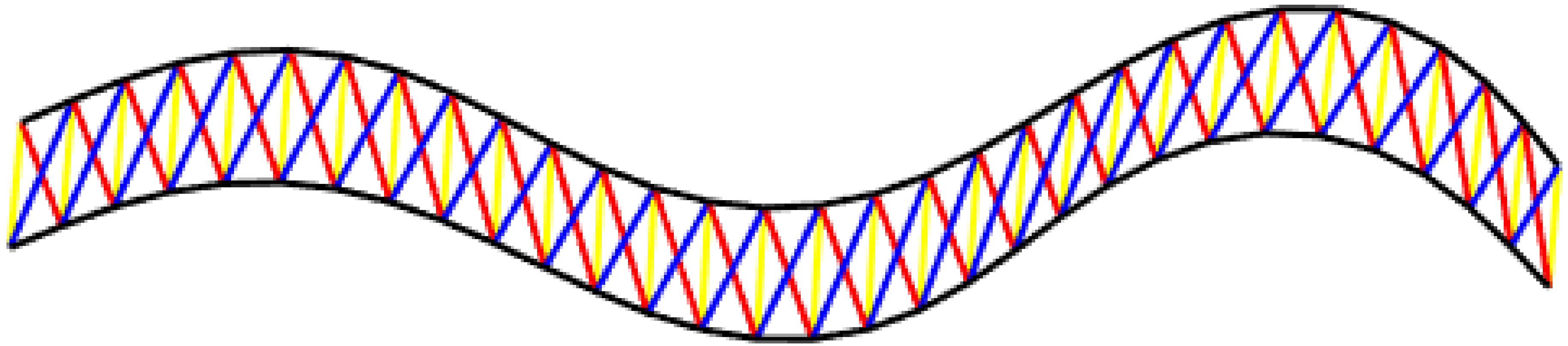


Modelling Active Dynein Motor Proteins



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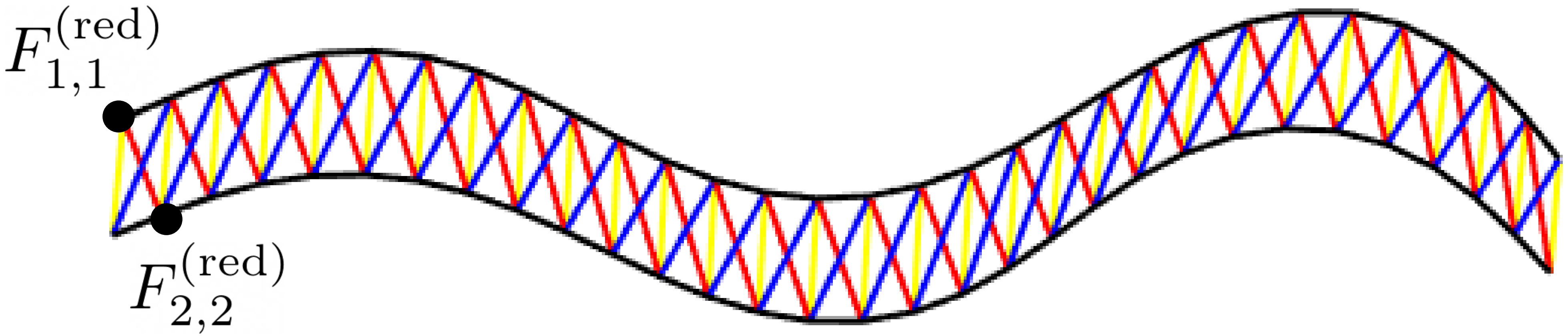

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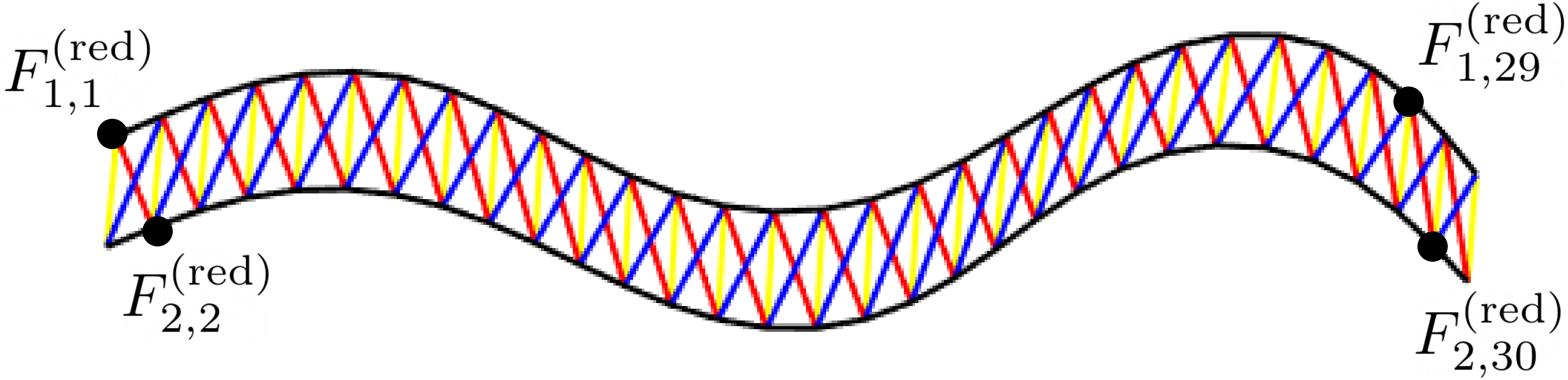

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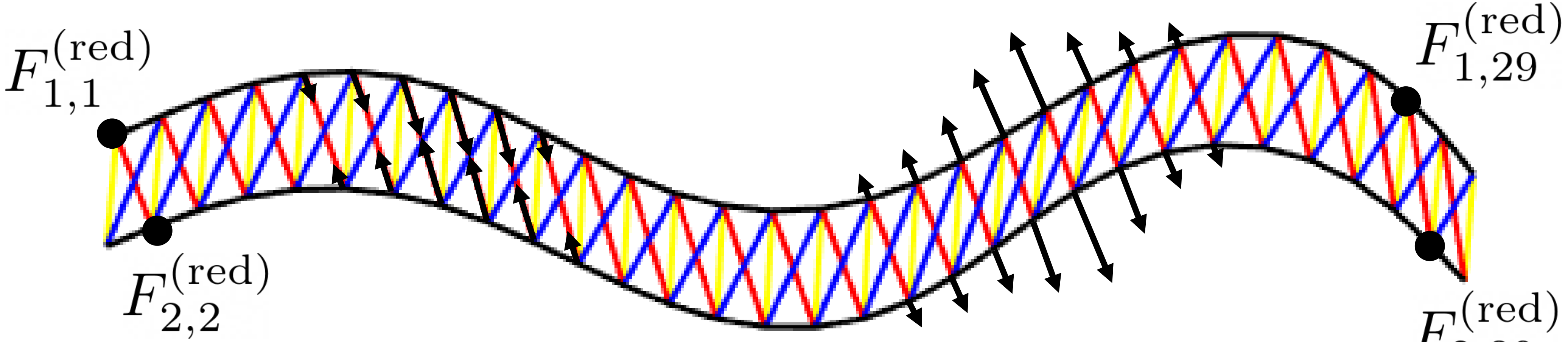

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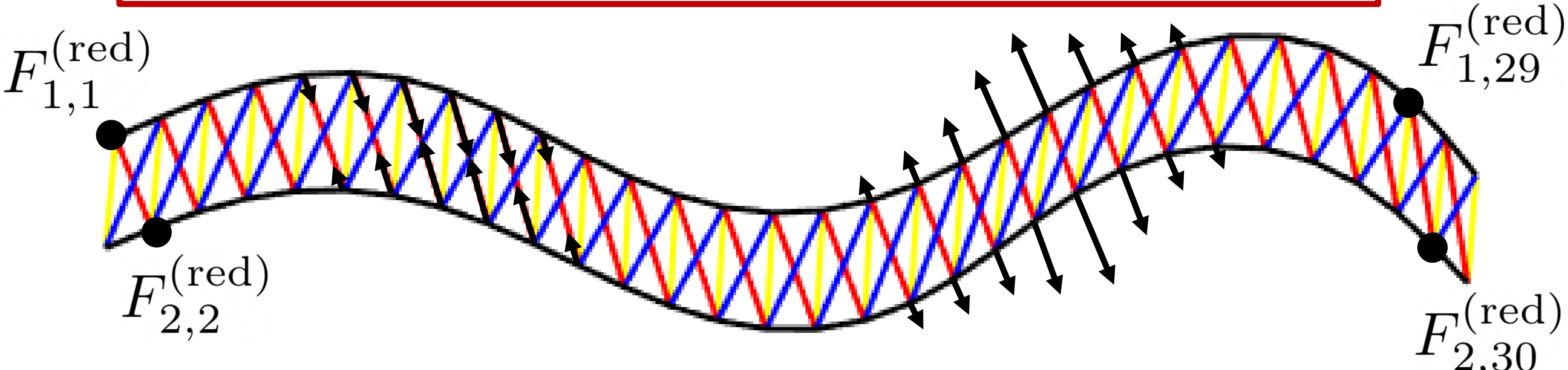

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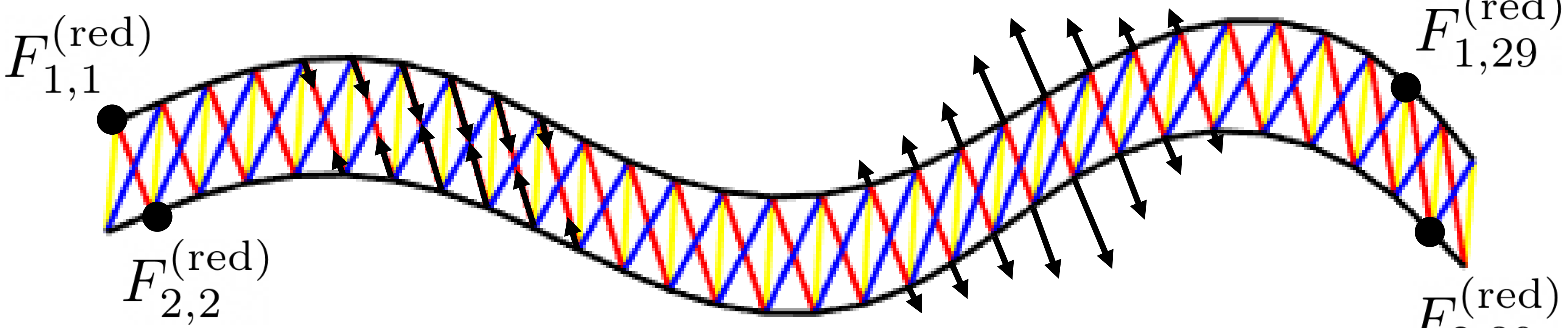
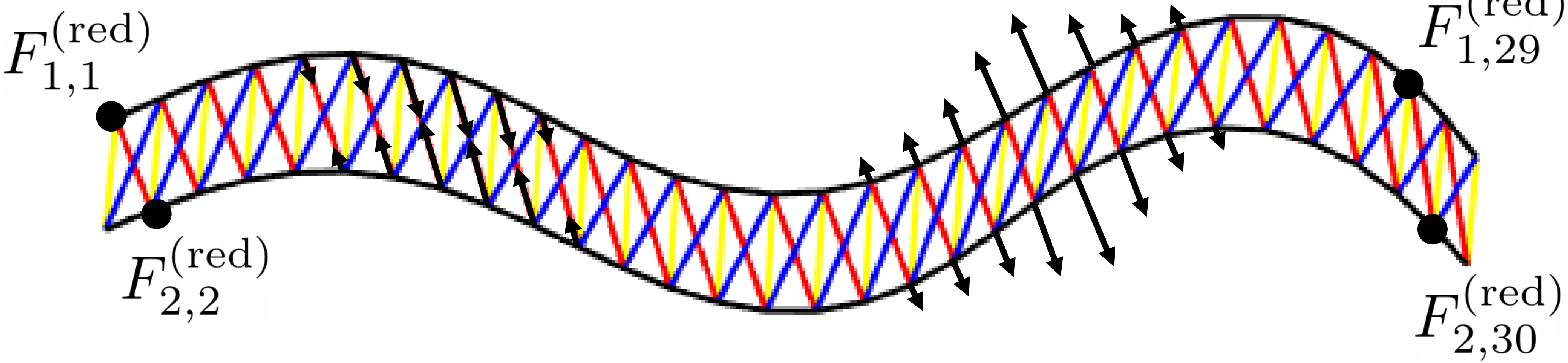
$$F_{1,n}^{(\mathrm{red})} = -F_{2,n+1}^{(\mathrm{red})}$$



Modelling Active Dynein Motor Proteins

$$F^{(\mathrm{red})}_{1,n} = -F^{(\mathrm{red})}_{2,n+1} := r^{(\mathrm{red})}_{n,n+1}(t) - l_{n,n+1}(t)$$



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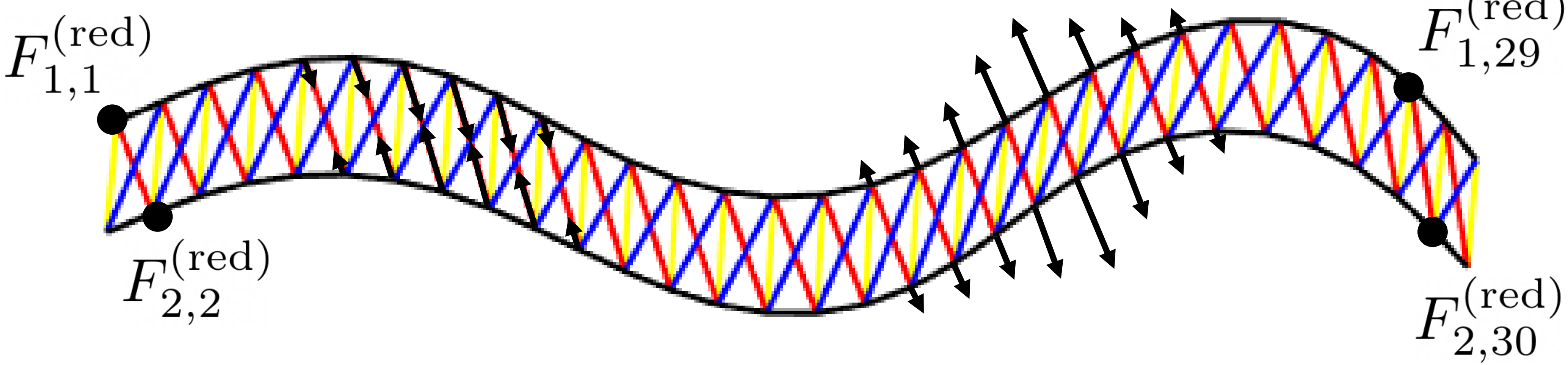
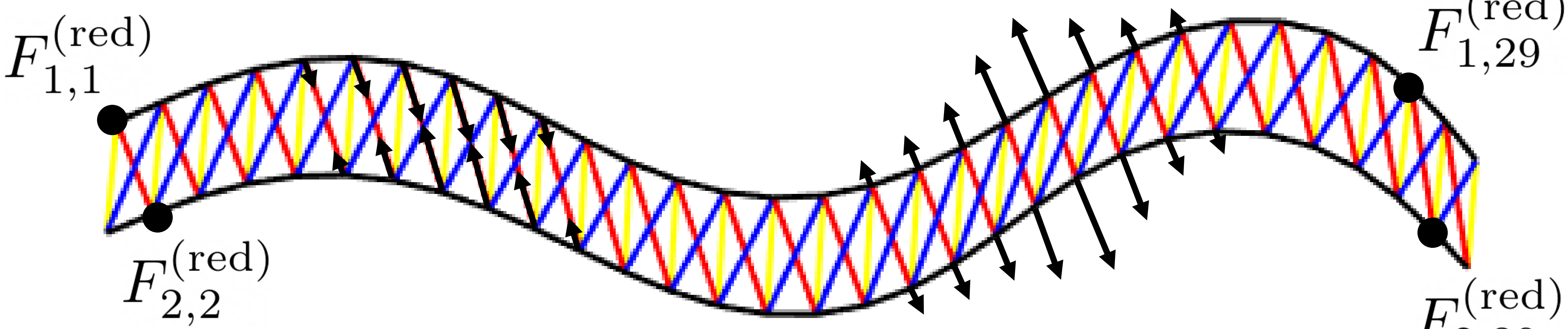


Modelling Active Dynein Motor Proteins

$$F_{1,n}^{(\mathrm{red})} = -F_{2,n+1}^{(\mathrm{red})} := r_{n,n+1}^{(\mathrm{red})}(t) - l_{n,n+1}(t)$$

$$r_{n,n+1}^{(\mathrm{red})}(t) = C + f_n^{(\mathrm{red})}(t)$$



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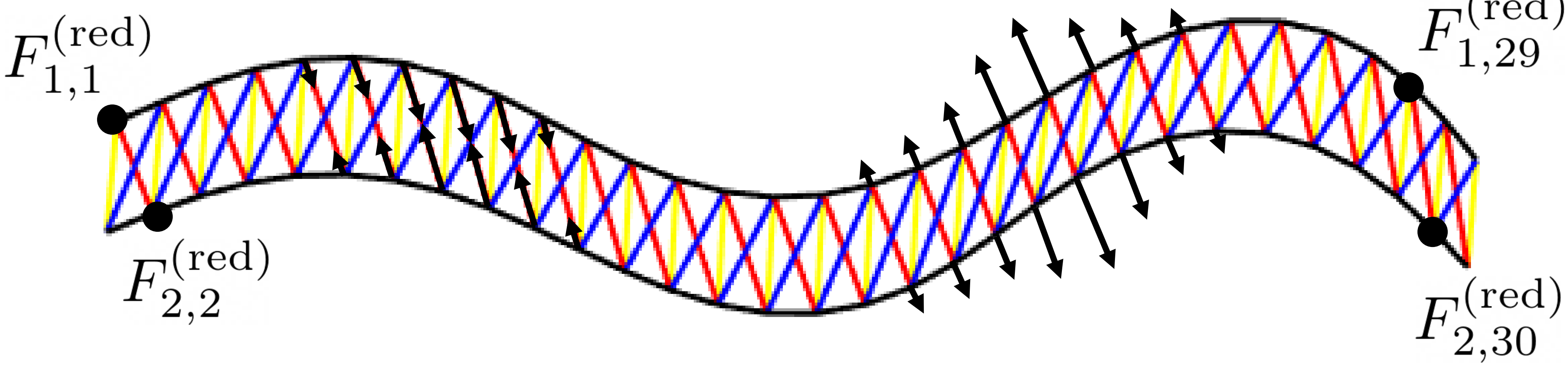

Modelling Active Dynein Motor Proteins

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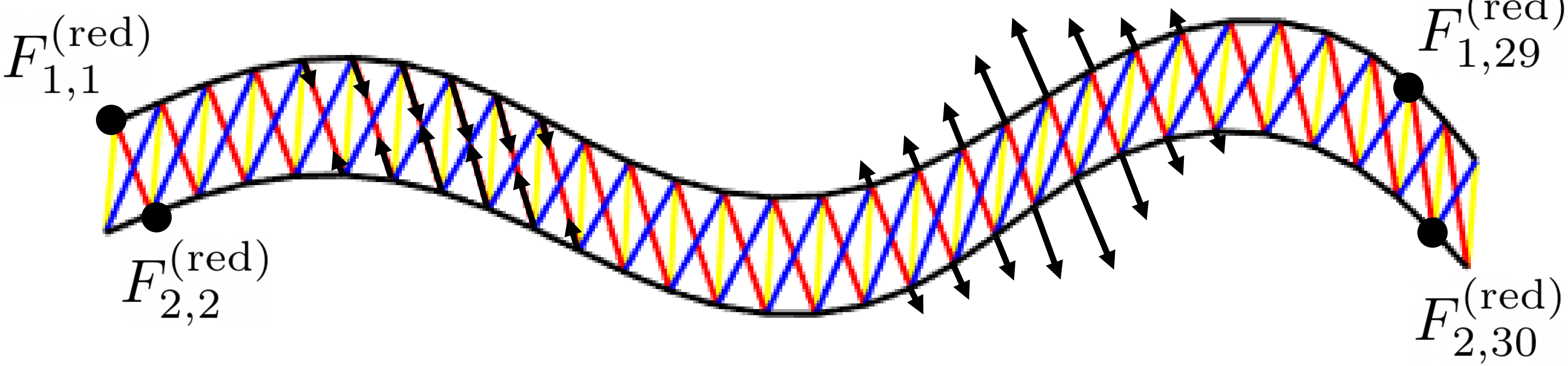
$$\boxed{f_n^{(\mathrm{red})}(t)} = \lambda \sin\left(\omega t + \frac{kn}{N}\right)$$



Modelling Active Dynein Motor Proteins

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\begin{aligned}
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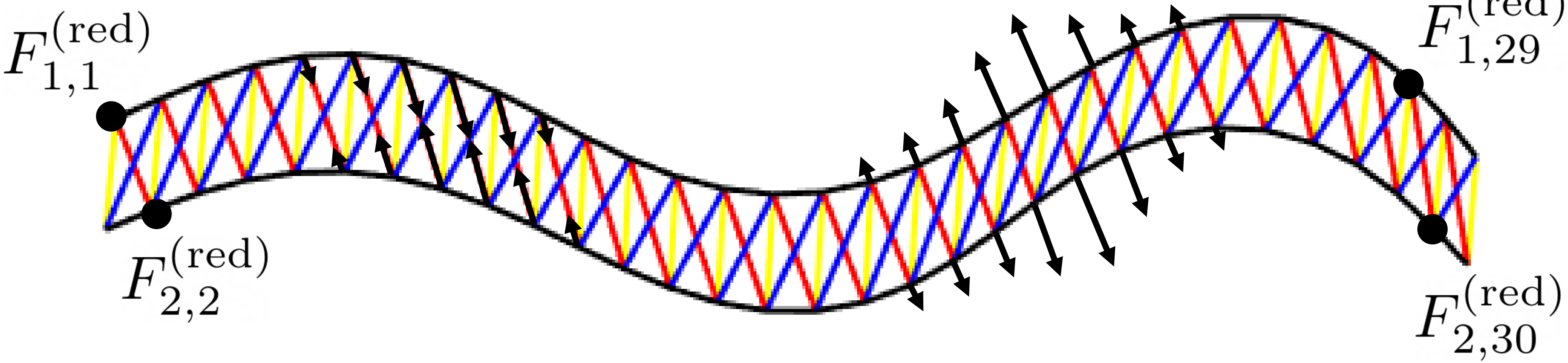
f^{(\mathrm{red})}_{n}(t) \quad &= \quad \lambda \sin\left(\omega t + \frac{kn}{N}\right)
\end{aligned}
$$



Modelling Active Dynein Motor Proteins

$$
\begin{aligned}
F^{(\text{blue})}_{1,n+1} = -F^{(\text{blue})}_{2,n} &:= r^{(\text{blue})}_{n+1,n}(t) - l_{n+1,n}(t) \\
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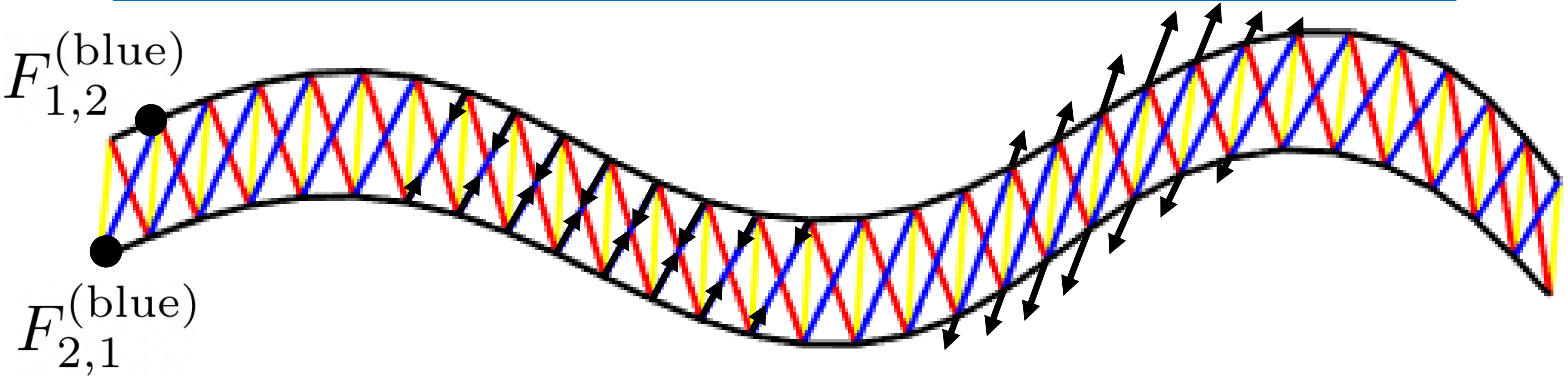

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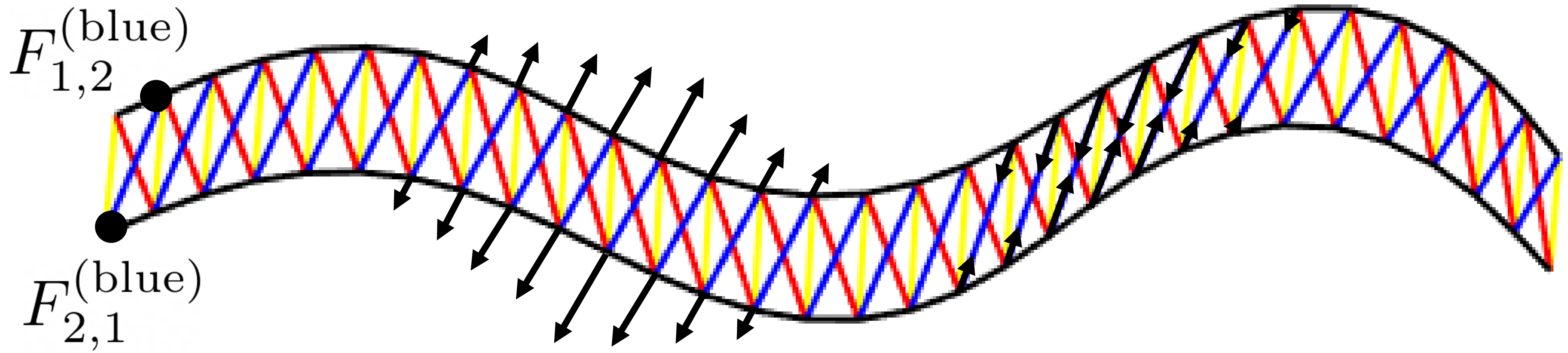
f^{(\mathrm{blue})}_{n}(t) \quad &= \quad \lambda \sin\left(\omega t + \frac{kn}{N} + \pi\right)
\end{aligned}
$$



$$\boxed{F_{1,n}^{(\text{yellow})} = -F_{2,n}^{(\text{yellow})} \quad := \quad D - l_{n,n}(t)}$$

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$$

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\end{aligned}
$$

$$
\mathbf{F}_{i,n}^{A} = \mathbf{F}_{i,n}^{(\mathrm{yellow})} + \mathbf{F}_{i,n}^{(\mathrm{red})} + \mathbf{F}_{i,n}^{(\mathrm{blue})}
$$

$$
F_{1,n}^{(\mathrm{yellow})} = -F_{2,n}^{(\mathrm{yellow})} := D - l_{n,n}(t)
$$

$$
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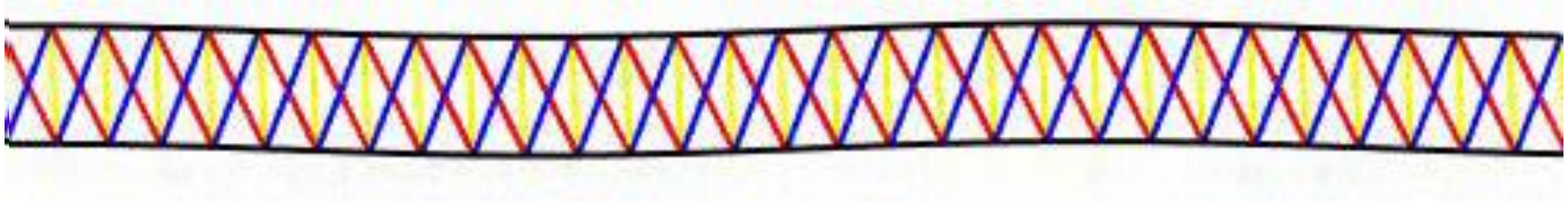


$$
F^{(\text{yellow})}_{1,n} = -F^{(\text{yellow})}_{2,n} \quad := \quad D - l_{n,n}(t)
$$

HOW CAN I TEST IT?

PART III

HOW CAN I TEST IT?

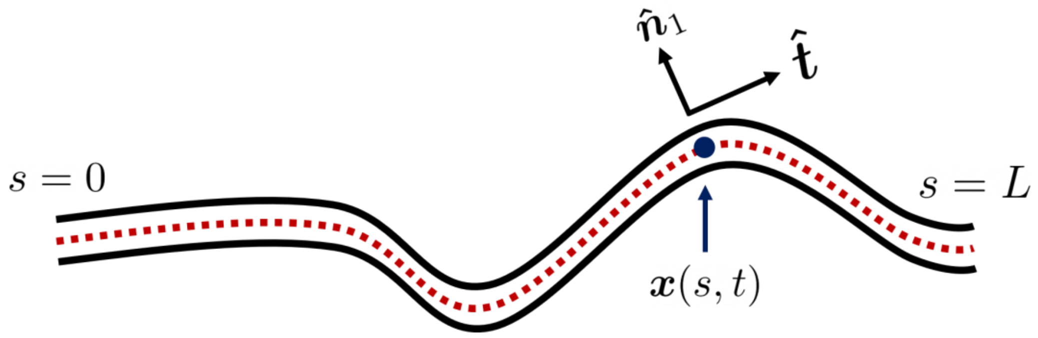
PART III

WARNING : TECHNICAL DETAIL

FOUR STEPS TO SUCCESS

Schoeller et. al (2019)

1. Model each of the two walls as a Kirchhoff Rod



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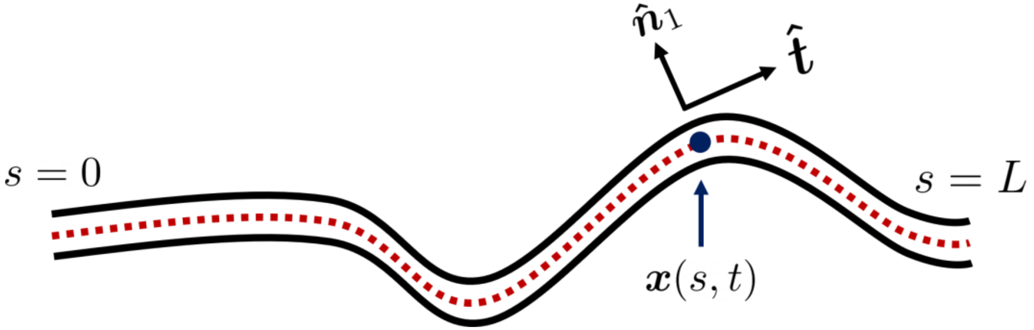
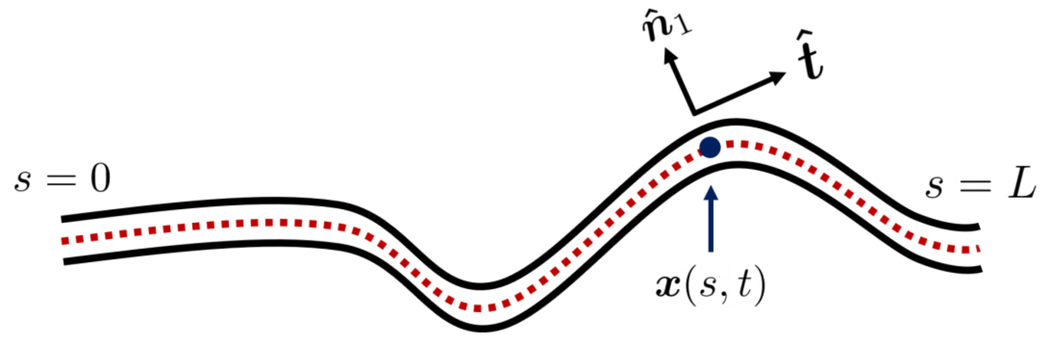


$$
\begin{aligned}
\mathbf{F}_n^C &= \mathbf{\Lambda}_{n+\frac{1}{2}} - \mathbf{\Lambda}_{n-\frac{1}{2}} \\
\mathbf{T}_n^C &= \frac{\Delta L}{2} \hat{\mathbf{t}}_n \times \left(\mathbf{\Lambda}_{n+\frac{1}{2}} + \mathbf{\Lambda}_{n-\frac{1}{2}} \right) + \left(\mathbf{M}_{n-\frac{1}{2}} - \mathbf{M}_{n+\frac{1}{2}} \right)
\end{aligned}
$$

Laundau & Lifschitz (1970)

1. Model each of the two walls as a Kirchhoff Rod



$$\mathbf{F}_n^C = \boxed{\boldsymbol{\Lambda}_{n+\frac{1}{2}} - \boldsymbol{\Lambda}_{n-\frac{1}{2}}}$$

$$\mathbf{T}_n^C = \frac{\Delta L}{2}\hat{\mathbf{t}}_n \times \boxed{\left(\boldsymbol{\Lambda}_{n+\frac{1}{2}} + \boldsymbol{\Lambda}_{n-\frac{1}{2}}\right)} + \left(\mathbf{M}_{n-\frac{1}{2}} - \mathbf{M}_{n+\frac{1}{2}}\right)$$

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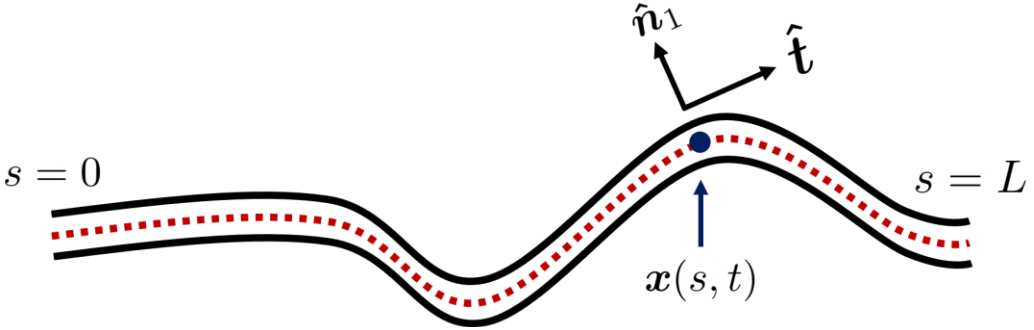

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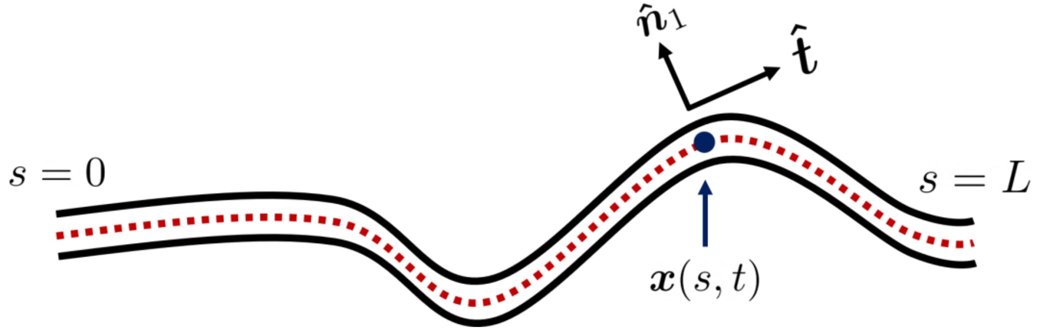

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Laundau & Lifschitz (1970)

2. Use rod forces to get hydrodynamics

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$$\mathbf{F}_n^H =$$

$$\mathbf{T}_n^H =$$

2. Use rod forces to get hydrodynamics

$$
\begin{aligned}
\mathbf{F}_n^H &= \mathbf{F}_n^C + \mathbf{F}_n^A + \mathbf{F}_n^S \\
\mathbf{T}_n^H &= \mathbf{T}_n^C
\end{aligned}
$$

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$$\mathbf{F}_n^H = \boxed{\mathbf{F}_n^C} + \mathbf{F}_n^A + \mathbf{F}_n^S$$

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2. Use rod forces to get hydrodynamics

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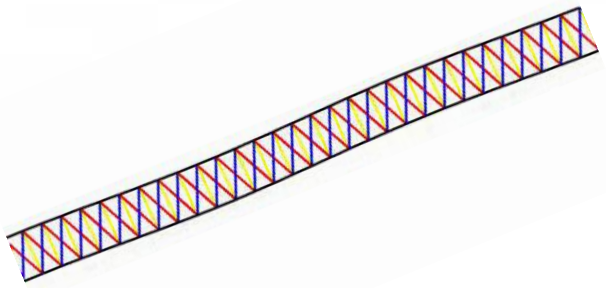

2. Use rod forces to get hydrodynamics

$$\begin{aligned}\mathbf{F}_n^H &= \mathbf{F}_n^C + \mathbf{F}_n^A + \boxed{\mathbf{F}_n^S} \\ \mathbf{T}_n^H &= \mathbf{T}_n^C\end{aligned}$$

Anything extra we want to add

3. Use hydrodynamics to get velocities

3. Use hydrodynamics to get velocities

$$\begin{pmatrix} \mathbf{u}_n \\ \mathbf{\Omega}_n \end{pmatrix} = \sum_{m=1}^{N} \mathbf{M}_{n,m} \begin{pmatrix} \mathbf{F}_m^H \\ \mathbf{T}_m^H \end{pmatrix}$$

Wajnryb et al. (2013)

3. Use hydrodynamics to get velocities

$$\begin{pmatrix} \mathbf{u}_n \\ \mathbf{\Omega}_n \end{pmatrix} = \sum_{m=1}^{N} \boxed{\mathbf{M}_{n,m}} \begin{pmatrix} \mathbf{F}_m^H \\ \mathbf{T}_m^H \end{pmatrix}$$

Generalisation of Stokes Law

Wajnryb et al. (2013)

4. Use velocities to get displacement

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$$\frac{\mathrm{d}\mathbf{x}_n}{dt} = \mathbf{u}_n$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \Omega_n$$

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$$\frac{\mathrm{d}\mathbf{x}_n}{dt} = \mathbf{u}_n$$

$$\boxed{\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \Omega_n}$$

Restrict movement to a plane

4. Use velocities to get displacement

$$\frac{\mathrm{d}\mathbf{x}_n}{dt} = \mathbf{u}_n$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \Omega_n$$

Shortcut

$$\boxed{\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{\Delta L}{2}\left(\hat{\mathbf{t}}_n + \hat{\mathbf{t}}_{n+1}\right)}$$

4. Use velocities to get displacement

$$\Delta t \left\{ \begin{aligned} \frac{\mathrm{d}\mathbf{x}_n}{dt} &= \mathbf{u}_n \\ \frac{\mathrm{d}\theta_n}{\mathrm{d}t} &= \Omega_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{\Delta L}{2}\left(\hat{\mathbf{t}}_n + \hat{\mathbf{t}}_{n+1}\right) \end{aligned} \right.$$

RECAP

Recap: Four steps to success…

1. Use two **Kirchhoff Rods** for the walls.

$$\begin{aligned}
\mathbf{F}_n^C &= \mathbf{\Lambda}_{n+\frac{1}{2}} - \mathbf{\Lambda}_{n-\frac{1}{2}} \\
\mathbf{T}_n^C &= \frac{\Delta L}{2}\hat{\mathbf{t}}_n \times \left(\mathbf{\Lambda}_{n+\frac{1}{2}} + \mathbf{\Lambda}_{n-\frac{1}{2}}\right) + \left(\mathbf{M}_{n-\frac{1}{2}} - \mathbf{M}_{n+\frac{1}{2}}\right)
\end{aligned}$$

Recap: Four steps to success…

1. Use two **Kirchhoff Rods** for the walls.
2. Use the rods to get **hydrodynamic forces.**

$$\begin{aligned} \mathbf{F}_n^H &= \mathbf{F}_n^C + \mathbf{F}_n^A + \mathbf{F}_n^S \\ \mathbf{T}_n^H &= \mathbf{T}_n^C \end{aligned}$$

Recap: Four steps to success…

1. Use two **Kirchhoff Rods** for the walls.
2. Use the rods to get **hydrodynamic forces**.
3. Use hydrodynamics to get **velocities**.

$$\begin{pmatrix} \mathbf{u}_n \\ \mathbf{\Omega}_n \end{pmatrix} = \sum_{m=1}^{N} \mathbf{M}_{n,m} \begin{pmatrix} \mathbf{F}_m^H \\ \mathbf{T}_m^H \end{pmatrix}$$

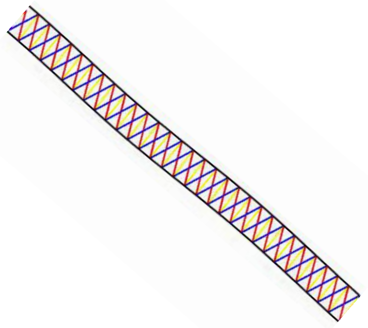
Recap: Four steps to success…

1. Use two **Kirchhoff Rods** for the walls.
2. Use the rods to get **hydrodynamic forces.**
3. Use hydrodynamics to get **velocities**.
4. Use velocities to get **displacement**.

$$\boxed{\begin{aligned} \frac{\mathrm{d}\mathbf{x}_n}{dt} &= \mathbf{u}_n \\ \frac{\mathrm{d}\theta_n}{\mathrm{d}t} &= \Omega_n \end{aligned}}$$

Recap: Five ~~four~~ steps to success…

1. Use two **Kirchhoff Rods** for the walls.
2. Use the rods to get **hydrodynamic forces**.
3. Use hydrodynamics to get **velocities**.
4. Use velocities to get **displacement**.
5. Watch it **swim**…

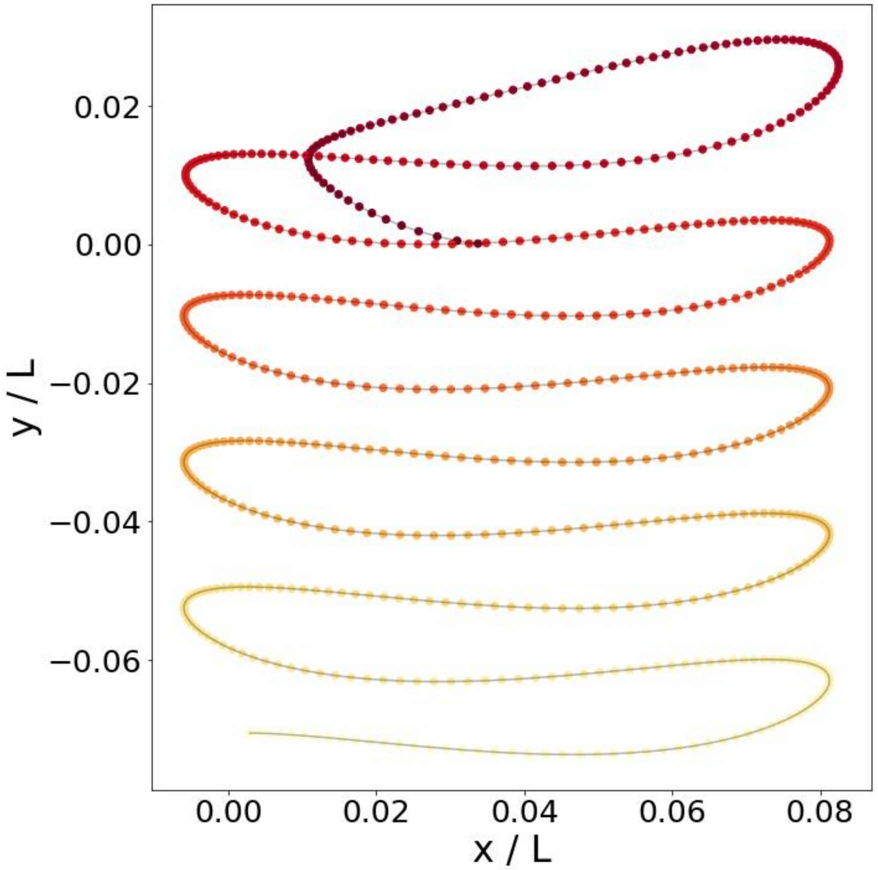


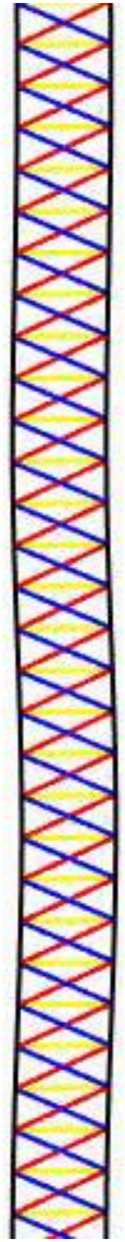
DID IT WORK?

PART IV

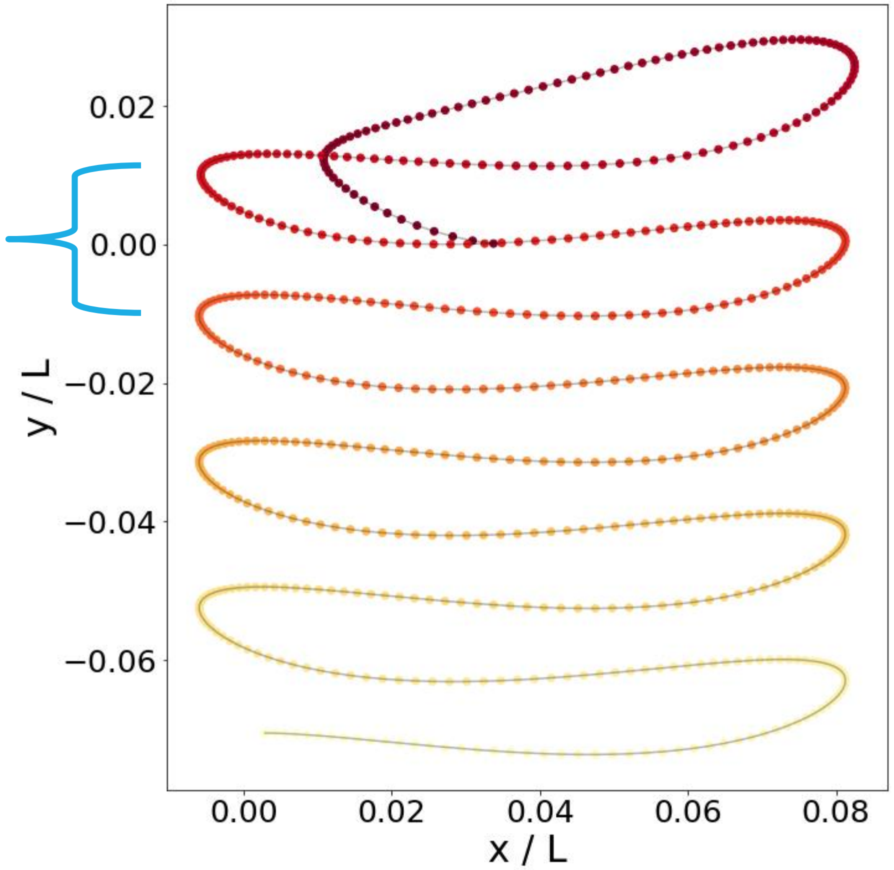
THE GOOD

They swim as expected

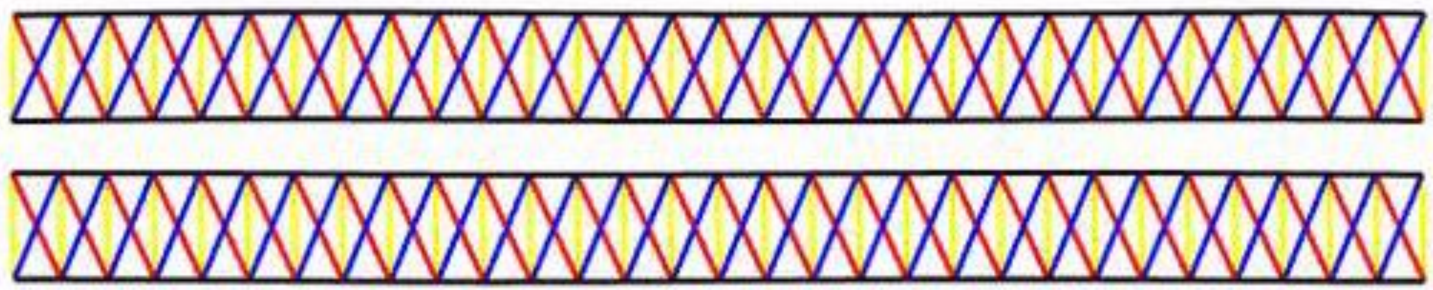




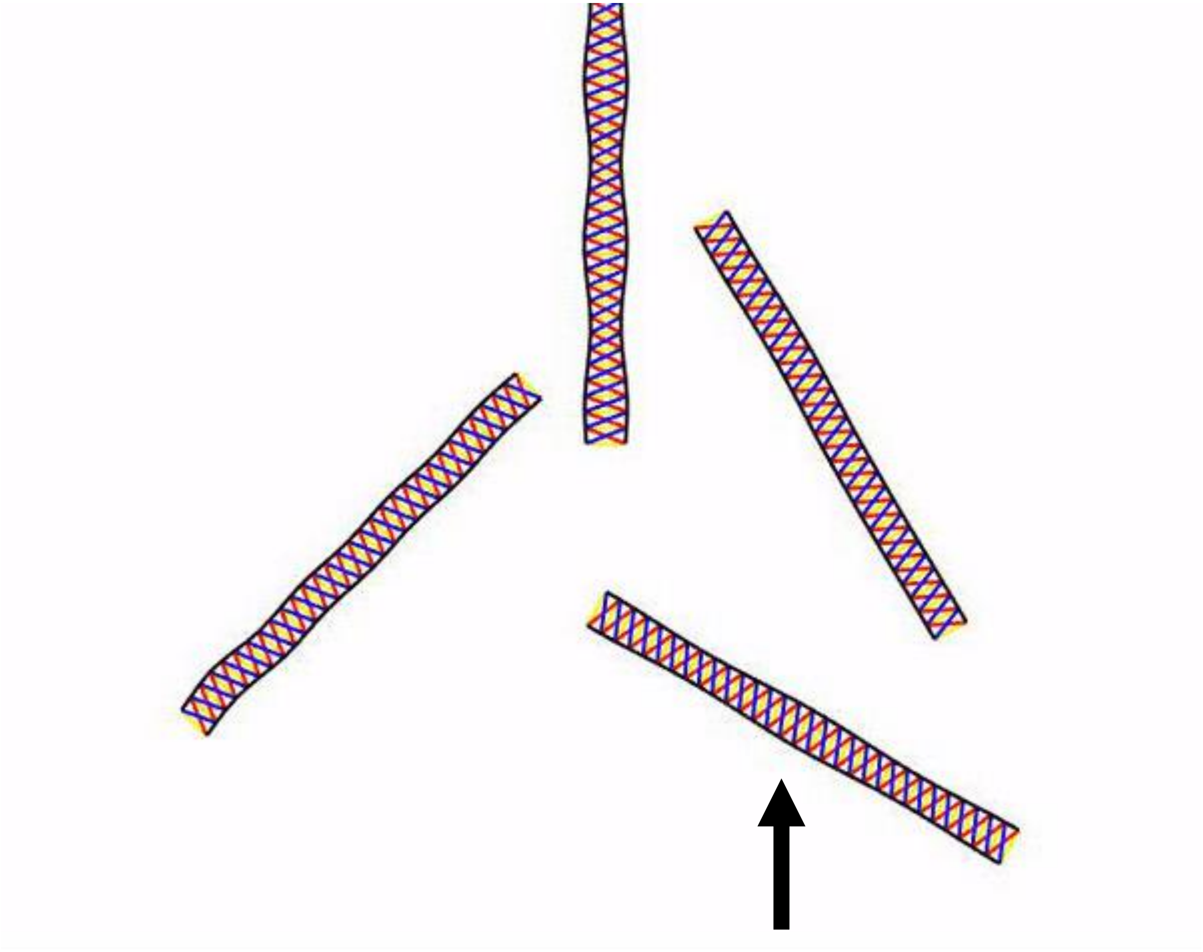
They swim as expected



They coordinate



Variety of swimming styles is possible

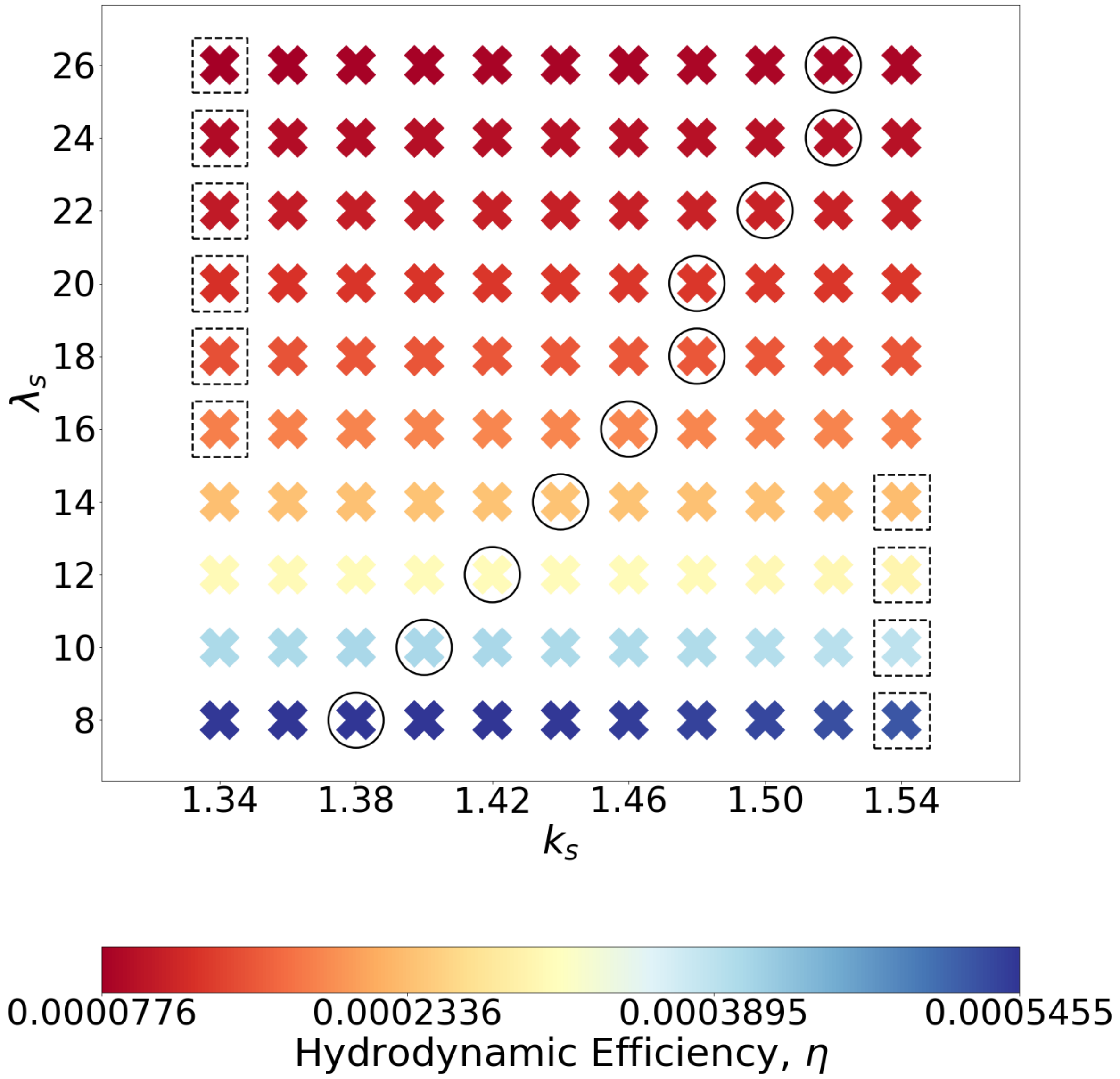


This one not possible in a single filament model.

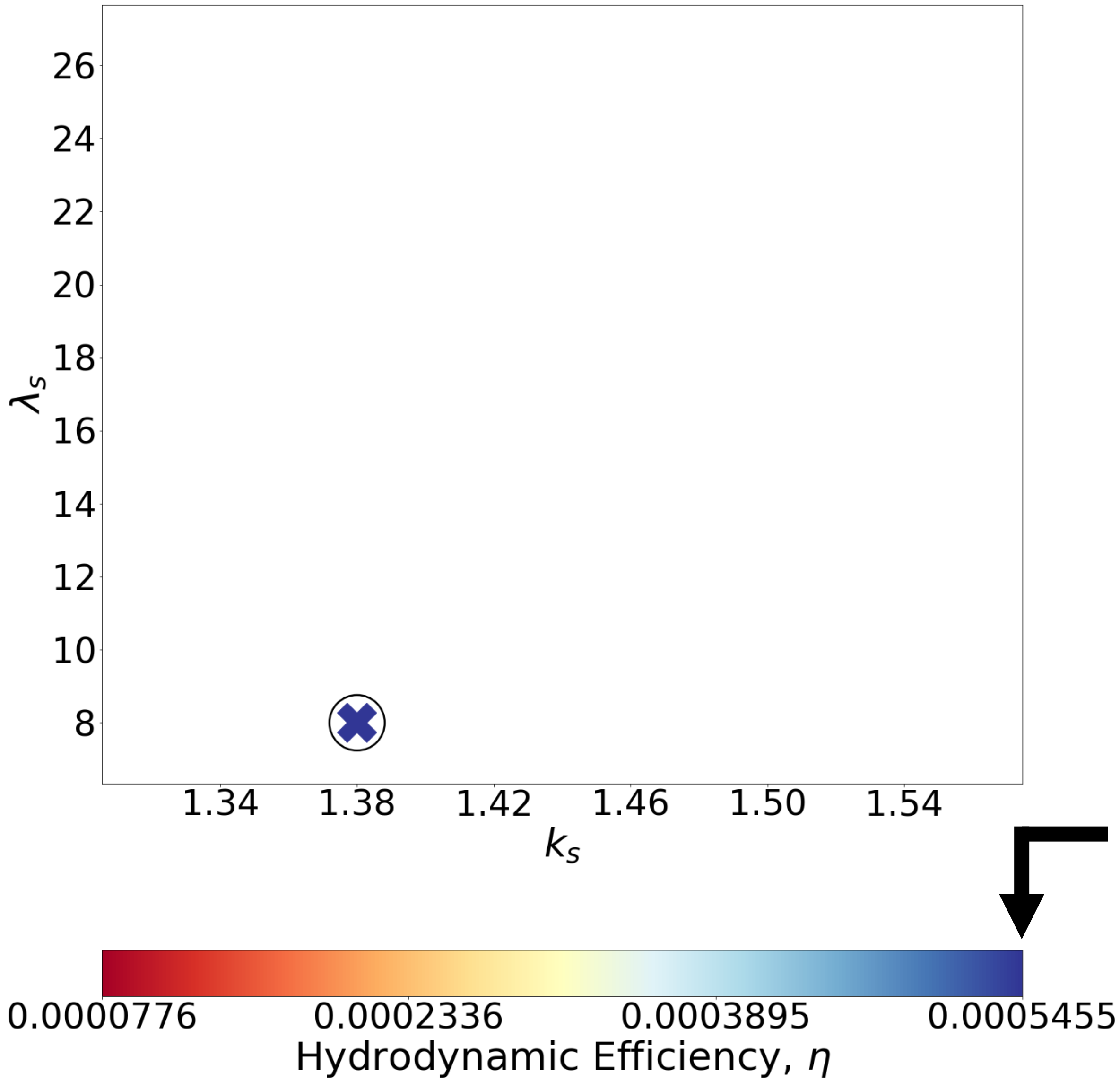
THE BAD

They are **very** inefficient

They are **very** inefficient



They are **very** inefficient



$$\eta < 0.06\%$$

They are **very** inefficient

$$0.002\% < \eta < 0.07\%$$

They are **very** inefficient

$$0.002\% \quad < \eta \quad < 0.07\%$$

$$\boxed{0.1\% \quad < \eta \quad < 14\%}$$

Spagnolie & Lauga (2010), Majmudar et al. (2012)

WHAT’S NEXT?

PART V

What’s next?

What’s next?

- Improve efficiency.

What’s next?

- Improve efficiency.
- Improve biological accuracy.

What's next?

- Improve efficiency.
- Improve biological accuracy.

Depending on goals.

What’s next?

- Improve efficiency.
- Improve biological accuracy.
- Expand to three-dimensions.

What’s next?

- Improve efficiency.
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- Dynamically evolving systems instead of internal clocks.

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www.github.com/petermnhull/MastersProject

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