The Gaussian free field and Liouville quantum gravity

Part III Essay Topic 106

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Contents

Introduction

Motivation. Recall that, by the Karhunen-Loève expansion, one can write Brownian motion as sum of orthonormal basis elements of the Sobolev space $H^1([0,1])$ (choosing elements f with $f(0) = 0$) weighted by independent standard Gaussians. Similarly, one might expect to be able to construct a random *surface*, as opposed to the one dimensional random object described by Brownian motion, by replacing $H^1([0,1])$ with $H^1(D)$ for some $D \subseteq \mathbb{R}^2$ in the above construction. This is made precise in section 1.

While it turns out that this sum does not define a function in the classical sense, at least its action (when formally interpreted as element of the dual space) on elements in $H^1(D)$ is still well-defined. So one takes an approximation by considering circle averages and considers how fast they "typically blow up" when shrinking the circle to a point in order to normalize by that factor to get a well-defined limit. This is done in section 2.

For further applications it is of interest how to sample such a random surface and it turns out that it has interesting connections with SLEs, as shown in section 3.

For certain classes of random sets it is often hard to calculate their expected fractal dimension, but section 4 presents a way of translating the problem to a setting with random geometry which helps doing that in certain cases.

Organisation and references.

Structure of the essay: In section 1 we motivate and define the Gaussian free field (Definition 1.17) and present some of its properties (most notably the Markov property and existence, as well as covariances of circle averages) that will be important to prove existence of Liouville measures.

In section 2 we show why a "random surface" (as defined later in Definition 2.7) is inherently harder to define than a random curve (like Brownian motion) and then go on to prove that the definition is in fact well-posed (Theorem 2.4). The second half of this section is devoted to a change of coordinates analogue (Proposition 2.5) in this setting, which is essential to define certain properties of random surfaces (which, for example, are needed for the proof of the theorem in section 4) and motivates the subsequent result.

Section 3 starts by recalling some properties of SLEs and then goes on to prove the central result (Theorem 3.3), relating SLEs to Liouville quantum gravity.

Section 4 states a result (Theorem 4.7) enabling us to compute the fractal dimension of random sets in a random geometry by computing its fractal dimension in a deterministic geometry and vice versa. We omit the proof which would have taken us too far afield.

References: The material we present here is not original, but rather drawn from different sources, detailed as follows.

We are generally indebted to [6], [2], and [7], which we have referenced throughout, and were extremely important in our study of the subject. We also consulted [1] to much lesser extents.

Section 1 basically presents [6], although our treatment benefitted from other viewpoints, motivated by a lecture on Gaussian processes for the definition, and [4] for some alternative proofs. Throughout we have assumed background knowledge from Gaussian processes, PDEs, SLEs, and stochastic calculus. We attempted, however, to recall results that might not be considered "standard" knowledge for readers with a pure probability theory background, following [5] for results about Gaussian processes and [3] for results about PDEs and Green's functions.

Sections 2 and 4 focus on [2], while section 3 addresses [7].

Notation.

- Abbreviate the Gaussian free field by GFF and the reproducing kernel Hilbert space by RKHS.
- For an operator A we write AX for the image of the space X under A.
- Write $(-\Delta)^{\alpha/2}$ for the fractional Laplacian, defined as the Fourier multiplier with symbol $|\xi|^\alpha$.
- For X being a random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ write $\mathcal{L}(X)$ for the law of a random variable X, i.e. the pushforward measure $\mathcal{L}(X) := \mathbb{P} \circ X^{-1}$ and $\mathcal{L}_{\mathbb{P}}$ if the probability measure is not clear from context.
- For any measure m on some product space $A \times B$ we will abuse notation to denote the evaluation of its B marginal $\int_A m(a, \cdot)da$ at any measurable $\tilde{B} \subseteq B$ simply by $m(\tilde{B})$ (instead of $\int_{A \times \tilde{B}} m(a, b)dadb$).
- For a topological space S denote its Borel-σ-algebra by $\mathcal{B}_S$.
- For a topological vector space V write V^* for its topological/continuous dual space, i.e. the set of all continuous, linear functionals $f : V \to \mathbb{R}$.
- Unless stated otherwise the probability spaces we are working with will implicitly assumed to be $(\Omega, \mathcal{A}, \mathbb{P})$ and $(H, \mathcal{A}, \mathbb{P})$ for the Gaussian free field h.
- Unless stated otherwise D is a fixed, connected planar domain $D \subseteq \mathbb{C}$ with $D \neq \mathbb{C}$. Let $H_s = H_s(D)$ be the set of smooth, compactly supported functions on D, and let $H(D) = H^1(D)$ (if it is clear from context we drop the D) be its Hilbert space closure under the Dirichlet inner product $(f, g)_\nabla = (f, g)_H := (\nabla f, \nabla g)$, where
$$(f, g) := \int_D f(z)g(z)dz$$
is the L^2 inner product.
- By abuse of notation (motivated by Riesz) we view (f, g) as being well-defined not only for $f, g \in L^2$, but also for functionals f acting on the space g is in or, in the case that g is some measure, the integral of f against that measure.
- We also write $\|f\|^2 := (f, f)$ and $\|f\|_\nabla^2 := (f, f)_\nabla$.
- Write $\stackrel{d}{=}$ for equality in distribution.
- Write $\mathcal{M}_{\text{loc}} \subseteq \mathcal{M}_{c,\text{loc}} \subseteq \mathcal{M}_c \subseteq \mathcal{M}$ for the sets of local, continuous local, continuous, and proper martingales, respectively.

1. Gaussian free field

Prior to defining the GFF we will motivate it with some standard results from Gaussian processes. The following is by no means supposed to be a self-contained introduction to Gaussian processes, but should merely recall the most relevant results.

1.1. **Gaussian processes.**

Definition 1.1 (Gaussian process)**.** *A real valued stochastic process $(X(t) : t \in T)$, where T is some index set, whose finite dimensional distributions $\mu_F := \mathcal{L}((X(t) : t \in F))$ are multivariate normal on $\mathbb{R}^F$ for all $F \subseteq T$ finite.*

Remark 1.2. *For historical reasons this is called T like time, but it does not have to be $\mathbb{R}$. In fact, in what follows we will use a Hilbert space for T.*

Given that we did not impose any conditions on T it is natural to ask for conditions under which such a process exists and whether it is unique. To do that we recall the that definition of a covariance on T is a mapping $\Phi : T \times T \to \mathbb{R}$ such that we have for any $t_1, \dots, t_n \in T$ that the matrix $(\Phi(t_i, t_j))_{i,j=1}^n$ is symmetric and non-negative definite for all $n \in \mathbb{N}$.

Lemma 1.3 (Existence)**.** *Let Φ be a covariance on T, $f : T \to \mathbb{R}$ be a measurable function, then there exists a Gaussian process $(X(t) : t \in T)$ such that for all $s, t \in T$ we have*

- $\mathbb{E}X(t) = f(t)$,
- $\mathbb{E}[(X(s) - f(s))(X(t) - f(t))] = \Phi(s, t)$.

This can be shown by extending well-defined finite dimensional distributions given by the covariance matrices $(\Phi(t_i, t_j))_{i,j=1}^n$ and with mean $f(t_i)_{i=1}^n$ to a random variable on the cylindrical σ-algebra with Kolmogorov's consistency theorem.

Often we want to think of a Gaussian process as a random *function* (in some Banach space) or vice versa, i.e. think of a Banach space valued Borel random variable as a Gaussian process.[*] The following remark illustrates the conditions under which this makes sense.

Remark 1.4 (Identification of Gaussian processes and Gaussian random variables)**.** *It can be shown that for sufficiently nice metric spaces T, a Gaussian process $X : \Omega \times T \to \mathbb{R}$ naturally induces a Borel random variable $X' : (\Omega, \mathcal{A}) \to (C_u(T, d), \mathcal{B}_{(C_u(T,d))})$ in the separable Banach space of $C_u(T, d)$, the space of uniformly continuous (w.r.t.*

[*]Think of Brownian motion viewed either random elements in the Wiener space (functions) or a Gaussian process.

the metric d) functions on T, by looking at the law of $X'(\omega) := X(\omega, \cdot) \in \mathbb{R}^T$. *This result formalizes the intuitive notion of a Gaussian process being a "random function". The non-trivial parts of the claim are that it can be realised as a* Borel *random variable (which is the natural topology on the target space; a priori Kolmogorov's extension theorem would only give us that elements of the cylinder-*σ*-algebra are measurable) and that the target space is* separable.

While those assumptions will not hold in our more general setting, it should at least motivate an (informal) correspondence between Gaussian processes and Banach space valued Gaussian random variables as in the following definition, motivated by the characterisation of a Gaussian in $\mathbb{R}^d$ by its projections on any fixed $w \in \mathbb{R}^d$ which can, by duality, also be interpreted as a linear functional:

Definition 1.5 (Banach space valued Gaussian random variables)**.** *For a separable Banach space* $(B, \|\cdot\|_B)$ *we say that* $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (B, \|\cdot\|_B)$ *is Gaussian/normal whenever* $f(X)$ *is normally distributed for all* $f \in B^*$.

We say it is centered if $\mathbb{E}f(X) = 0 \; \forall f \in B^*$.

In the following our T will not be "small" enough for some standard results from Gaussian processes. However, in some sense (made precise is the next remark) the probabilistic information about a Gaussian random variable in a separable Banach space is also encoded in its dual process, defined as follows:

Definition 1.6 (Dual process)**.** *Given some random variable* X *in a separable Banach space* $(B, \|\cdot\|_B)$ *define its dual process* $(\tilde{X}(f) : f \in B^*)$ *by the application* $\tilde{X}(f) := f(X)$.

Remark 1.7. *It can be shown that there exists a countable, dense subset* $D \subset B^*$ *with* $\|X\|_B = \sup_{f \in D} |\tilde{X}(f)|$.

Remark 1.8. *Note that this dual process* $\tilde{X}$ *is actually a Gaussian process (check that its finite dimensional distributions are* jointly *Gaussian by linearity of the functionals).*

Referring to the finite dimensional case again, we would like to generalize the fact that Gaussians in $\mathbb{R}^d$ can be written as $\sum_{i=1}^d e_i g_i$, where $(e_i)_i$ is any orthonormal basis of $\mathbb{R}^d$ and $g_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$, for (separable) Banach space valued Gaussian random variables. To do that we have to find a suitable subspace H of the Banach space B and endow it with the right Hilbert space structure to make sense of an orthonormal basis.

It turns out that reproducing kernel Hilbert spaces (henceforth abbreviated by RKHS) are the right choice:

Definition 1.9 (Reproducing kernel Hilbert space for Gaussian random variables)**.** *Let $X : \Omega \to B$ be a Gaussian random variable in a separable Banach space B, $F := \{f(X) : f \in B^*\} \subseteq L^2(\mathbb{P})$, then the RKHS of X is the Hilbert space*

$$H := \{\phi_h := \mathbb{E}(hX) : h \in \overline{F}^{L^2(\mathbb{P})}\},$$

endowed with the inner product

$$(\phi_h, \phi_g)_H := \mathbb{E}(hg),$$

where the $\mathbb{E}$ in the definition of H is the Bochner integral and $H \subseteq B$ by definition of the Bochner integral.

Remark 1.10. *It can easily be seen that this definition coincides with the usual definition of a RKHS for Gaussian processes (see e.g. [5]) if the Banach space valued random variable is induced by some Gaussian process as described in Remark 1.4.*

Reproducing kernel Hilbert spaces are interesting for multiple reasons. Before presenting an example of how it looks like in a familiar setting, we need to state one of its properties (for later on) that may not seem very intuitive at first. While in the finite dimensional case we can translate any Gaussian X with law μ by $a \in \mathbb{R}^d$ to obtain the shifted measure $\tau_a\mu$ which is absolutely continuous with respect to μ (and vice versa), this does not hold in full generality in the infinite dimensional case. In fact, the Radon-Nikodym derivative $d\tau_a\mu/d\mu$ turns out to comprise a factor $\|a\|_H$, suggesting that the measures are mutually absolutely continuous if and only if $a \in H$. This is indeed the case:

Theorem 1.11 (Cameron-Martin)**.** *Let X be a centered Gaussian random variable taking values in a separable Banach space B with RKHS H and denote its law μ. For any $a \in B$ let $\tau_a\mu$ be the Gaussian probability measure given by $\tau_a\mu(\cdot) := \mu(\cdot - a)$. Then μ and $\tau_a\mu$ are mutually absolutely continuous if and only if $a \in H$.*

Furthermore, for $a \in H$ the Radon-Nikodym derivative is given by

$$\frac{d\tau_a\mu}{d\mu}(X) = e^{(X,a)_H - \frac{1}{2}\|a\|_H^2},$$

where we slightly abused notation by writing $(X, a)_H$ (see [5] for a more detailed treatise).

Actually we will use this theorem "the other way round"; more precisely, we will identify a measure (given its Radon-Nikodym derivative with respect to a Gaussian measure) as a translation of the original measure.

The following example should illustrate how a RKHS "typically" looks like and will motivate the definition of the GFF, which can be thought of as a generalisation of Brownian motion to arbitrary dimensions.

Remark 1.12 (RKHS of Brownian motion)**.** *Considering Brownian motion as a Banach space valued random variable as in remark 1.4, it can be shown that the RKHS H of the Wiener process on $T = [0, 1]$ is given by*

$$H = \{h : [0, 1] \to \mathbb{R}, h' \in L^2([0, 1]), h(0) = 0\},$$

with inner product induced by the usual Sobolev space norm

$$\|h\|_H := \|h'\|_{L^2([0,1])}.$$

Note that this is indeed a norm because of the fixed boundary condition $h(0) = 0$.

Taking, for example, the trigonometric basis of $L^2([0, 1])$, we get (by taking antiderivatives) an orthonormal basis $h_0(t) := t, h_k(t) := \frac{\sqrt{2}}{\pi k} \sin(\pi k t)$ of $(H, (\cdot, \cdot)_H)$. Knowing that one can write Brownian motion as

$$B(\cdot) = \sum_{k=0}^{\infty} h_k(\cdot) g_k,$$

with $g_k \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$ one might expect this to hold in more general cases. Indeed we have the following theorem:

Theorem 1.13 (Karhunen-Loève expansion)**.** *Let X be a centered Gaussian random variable in a separable Banach space B, $(h_k)_k$ an orthonormal basis of the RKHS $(H, (\cdot, \cdot)_H)$ of X, $(g_k)_k \overset{iid}{\sim} \mathcal{N}(0, 1)$, then*

$$X = \sum_k h_k g_k,$$

almost surely, with convergence of the sum in B.

Remark 1.14 (X is in its RKHS if and only if $\dim H < \infty$)**.** *Note that $\mathbb{E}\|X\|_H^2 = \sum_{k=1}^{dim\ H} \mathbb{E}h_k^2$ by the previous theorem and absolute convergence of sum and $\mathbb{E}h_k^2 = 1$ for all k. By a standard result for Gaussian processes we have $\mathbb{E}\|X\|_H^2 = \infty \Rightarrow \exists \Omega_0 \subseteq \Omega : \mathbb{P}(\Omega_0) > 0$ on which $\|X(\omega)\|_H^2 = \infty$), since H is closed (hence measurable* in B, we see by the $0 - 1$ law for Gaussian processes that*

$$\mathbb{P}(X \in H) = 0$$

if and only if $\dim H = \infty$.

*Closed subsets C of a topological space equipped with its Borel-σ-algebra $\mathcal{B}$ are $\mathcal{B}$-measurable.

Now given the RKHS H, it is not immediate in what space B the random variable X (induced by the Karhunen-Loève expansion, i.e. $X := \sum_k h_k g_k$ using the notation from the theorem) will be. Indeed, this choice is rather arbitrary in general, as can be seen in the first section of [6] about abstract Wiener spaces.

Remark 1.15 (X as a linear functional on H)**.** *After identifying H with ℓ^2 by fixing an orthonormal basis, we may identify B with a subset of $(\ell^2)^*$ as seen by the following calculation (by linearity it follows easily for differences), where $h^n := \sum_{k=1}^n h_k e_k$ and $g_k \overset{iid}{\sim} \mathcal{N}(0,1)$:*

L^2 convergence:

$$\mathbb{E}(h^n, f)_H^2 \overset{(def)}{=} \mathbb{E}\left(\sum_{k=1}^n g_k f_k\right)^2 = \sum_{k=1}^n f_k^2 \tag{1.1}$$

converges against $\|f\|_H^2$. The last step holds since the g_k are independent, centered, and of variance one.

Almost sure convergence: L^2 convergence implies convergence in probability, which in turn, by Levy's equivalence theorem (the g_k are independent), implies almost sure convergence of the partial sums *given a fixed ordering. So (by linearity) $(h, f)_H := \lim_n (h^n, f)_H$ is almost surely well-defined after fixing an ordered basis.*

This restriction is not artificially introduced by a lack of means to show the stronger result of $X \overset{(def)}{=} \sum_k h_k g_k$ being in the topological dual space H^* since this would, by Riesz, imply that X is actually in H - a contradiction to Remark 1.14 in our infinite dimensional setting.

Another, less abstract, reason is that the sums $(h^n, f) = \sum_{k=1}^n g_k f_k$ do not have to converge absolutely, so, by the Riemann rearrangement theorem, the limit would not be well-defined if the ordering of the basis were not fixed.

Remark 1.16 (X as distribution)**.** *However, it can be shown (as in [6]) that X is indeed a distribution, i.e. a continuous (with respect to the topology of uniform convergence of all derivatives) linear functional on the space of smooth compactly supported functions H_s. By the above remark its continuity cannot be preserved when extending it to the larger space H^1, where it is merely a linear functional.*

Note that for $D \subseteq \mathbb{R}^2$ there is in fact a unique extension since h is not only a distribution, but, in a sense, "arbitrarily close" to being in L^2. To be precise, we have $h \in (-\Delta)^{\varepsilon} L^2(D)$ for all $\varepsilon > 0$. Naturally, continuity on a dense subset (here H_s dense in H^1) alone does not suffice for a well-defined extension to $(H^1)^*$, the continuous, *linear functionals on that space; take for example $\nabla \delta_0$ and see that it cannot be well defined for $f \in H$ since the derivative of f only needs to be in L^2 which is not defined pointwise.*

While we *could* define the Gaussian free field that way, i.e. as a sum of orthonormal basis elements weighted by independent normals, it has the aesthetic drawback of having to choose a somewhat arbitrary ambient space B and a priori not being well defined for general $f \in H(D)$, but only for $f \in H_s(D)$. Instead, we will be slightly less explicit and define it to be the dual process of X, similar to what was described in Remark 1.15, with the following adaption of the RKHS of the Wiener process: Noting that Brownian motion on $[0,1]$ is induced by the Sobolev space $H^1([0,1])$ it seems natural to investigate the random variable induced by the RKHS $H^1(D)$ for some domain $D \subseteq \mathbb{R}^d$. An easy (formal) calculation (just like (1.1), but for $\mathbb{E}(h^n, f)_H(h^n, g)_H)$ for X as above motivates the covariances for the dual process as given in the following definition:

Definition 1.17 (Gaussian free field)**.** *The Gaussian free field on a domain $D \subseteq \mathbb{R}^d$ is the centered Gaussian process*

$$h := ((h, f)_\nabla : f \in H(D)),$$

with covariance structure

$$\mathbb{E}[(h, a)_\nabla (h, b)_\nabla] := (a, b)_\nabla.$$

Note that this is well-defined by Lemma 1.3.

Remark 1.18. *It follows easily from the definition that $f \mapsto (h, f)_\nabla$ is linear and that for every $f \in H(D)$ we have that $(h, f)_\nabla \sim \mathcal{N}(0, \|f\|_{H(D)})$.*

Remark 1.19. *Note that Dirichlet boundary conditions are encoded in the vanishing boundary conditions on elements in $H(D)$.*

From now on we will restrict ourselves to the case $D \subseteq \mathbb{R}^2$. For what follows it will be useful to define the Green's function G_D of a domain D and stating some of its basic properties:

Definition 1.20 (Green's function)**.** *For $x \in D$ fixed we let*

$$G_D^x(\cdot) := -\log|x - \cdot| - \tilde{G}_D^x(\cdot),$$

where $\tilde{G}_D^x$ is the unique harmonic function chosen such that $G_D^x(y) = 0$ for all $y \in \partial D$. If D is clear from the context we will drop it and simply write G^x for G_D^x. To emphasise symmetry we will also sometimes write $G(x, y)$ for $G^x(y)$.

We also define the corresponding integral operator

$$[-\Delta^{-1}\rho](x) := \int_D G(x, y)\rho(y)dy,$$

which is the inverse of $-\Delta$.

Remark 1.21 (Importance of Green's function to define the standard inner product)**.** *One of the reasons the Green's function is so important is that it can be seen as the distributional solution of* $-\Delta G^x = \delta_x$*. Noting that by viewing* h *as a distribution we would immediately get*[*] $(h,\rho) \overset{(def)}{=} -(h,\Delta^{-1}\rho)_\nabla$ *for smooth, compactly supported* ρ*. So we just take (equivalently to viewing* $(h,f)_\nabla$ *as well defined for all* $f \in H$*)* $(h,\rho) := -(h,\Delta^{-1}\rho)_\nabla$ *to be well-defined for all* $\rho \in -\Delta^{-1}H(D)$*, i.e.* ρ *that can be written as* $\rho = -\Delta f$ *for some* $f \in H(D)$*.*

Remark 1.22 (Qualitative difference of GFF in different dimensions)**.** *One might wonder why a Wiener process, which can be seen as a Gaussian free field on* $[0,1]$*, is a Hölder continuous function while the Gaussian free field on any domain* $D \subseteq \mathbb{R}^2$ *is not even a function (i.e. does not have values at points), but merely a distribution.*

Firstly, we note that this is not as surprising as it may seem if we (equivalently!) chose to introduce Gaussian free fields as in Remark 1.15 since Hölder continuous functions are not in the RKHS H^1 *as well.*

Furthermore Remark 1.21 suggests that the different behaviour of the Green's function in different dimensions is the reason[†] *we cannot define the GFF pointwise for dimensions* $d \geq 2$*: Assume we could, then make the ansatz:* $h(x) = (h,\delta_x) \overset{(def)}{=} -(h,\Delta^{-1}\delta_x)_\nabla = -(h,G^x)_\nabla$*. Now it is easy to see*[‡] *that while* $G^x \in H([a,b])$*, it fails to be in* $H(D)$ *in dimensions* $d \geq 2$ *(in particular for* $D \subseteq \mathbb{R}^2$*).*

We recall the following properties of Green's functions in $\mathbb{C}$ which will be needed later for some calculations:

Proposition 1.23 (Properties of G^x_D)**.** *For* $D \subseteq \mathbb{C}$ *fixed and* $\tilde{G}^x$ *as in the definition we have*

(1) $G(x,y) = G(y,x)$ *for all* $x,y \in D \setminus \{x\}$*. (symmetry)*
(2) $\Delta G^x_D(y) = 0$ *for all* $y \in D\setminus\{x\}$*, i.e. it is harmonic in* $D\setminus\{x\}$*. (harmonicity)*
(3) $\tilde{G}^x(x) = \log C(x;D)$ *for all* $x \in D$*.*

Here (and henceforth) $C(x;D)$ is the conformal radius of D viewed from $z \in D$ which is defined as $C(z;D) := \phi'(z)^{-1}$, where $\phi : D \to \mathbb{D}$ is the unique[§] conformal map from D to $\mathbb{D}$ mapping z to 0 such that $\phi'(z) \geq 0$.

[*]Remember that the derivative of a distribution (motivated by integration by parts) is defined by $(\nabla h,\rho) := (h,-\nabla\rho)$ for smooth test functions ρ.

[†]One might also present a different argument, as in [6], arguing with the qualitatively different decays of eigenvalues of the Laplacian in different dimensions.

[‡]Note that we only defined the Green's function in dimension two, but it can be defined more generally in arbitrary dimensions as the distributional solution of $-\Delta G^x = \delta_x$ so that the above ansatz makes sense.

[§]This is well-defined by the Riemannian mapping theorem.

1.2. **Properties.**

1.2.1. *Markov Property.*

Remark 1.24 (Motivation by Brownian motion)**.** *The Markov property of Brownian motion can also be stated as follows: Given a Brownian motion $B = (B_t)_{t\geq 0}$ then B is equal in distribution to $B^{a,b}$, which is obtained by sampling a Brownian motion B, an independent Brownian bridge $\tilde{B}$ on (a,b) (i.e. a Brownian motion conditioned to be zero on $\mathbb{R}^+ \setminus (a,b)$), and taking*

$$B_t^{a,b} := \begin{cases} B_t, & \text{if } t \in \mathbb{R}^+ \setminus (a,b) \\ b_t + \tilde{B}_t, & \text{if } t \in (a,b) \end{cases}$$

where $(b_t)_{t\in(a,b)}$ is the linear interpolation between B_a and B_b.

This can be rephrased in a slightly more abstract (and less rigorous) way as "given the values of a Brownian motion B outside some open set $U = (a,b)$, the values in U are equal in distribution to the projection on the harmonic part plus an independently sampled Brownian motion with zero boundary conditions on (a,b)".*

We hope for something similar to hold for Gaussian free fields in two dimensions (with zero boundary conditions) as well. To make this rigorous, we first have to define what we mean by the "projection on the harmonic part", so we define the following subspaces of $H = H(D)$:

Definition 1.25. *For $U \subseteq D$ open we write*

- *H_U for the closure (with respect to $(\cdot,\cdot)_\nabla$) of the set of smooth functions supported in some compact subset of U and*
- *$H_U^\perp := \{b \in H : \Delta b = 0 \text{ on } U\}$, the functions which are harmonic on U.*

Remark 1.26 (Justification of the definition)**.** *Since U is assumed to be open, integration by parts† implies that for $a \in H_U, b \in H_U^\perp$ we have*

$$\mathbb{E}[(h,a)_\nabla (h,b)_\nabla] \overset{(def)}{=} (a,b)_\nabla = -(a,\Delta b) = 0,$$

so the two spaces are indeed orthogonal.

Remark 1.27 (Independence of orthogonal subspaces)**.** *Note that, by definition of the covariance function for the Gaussian free field $\mathbb{E}[(h,a)_\nabla (h,b)_\nabla] \overset{(def)}{=} (a,b)_\nabla$*

*Note that in one dimension the harmonic functions are just linear functions; hence the linear interpolation.

†We need U open since otherwise it could be that the intersection of a compact subset of U and ∂U is not empty, so $a \in H_U$ would not imply that a has zero boundary conditions - hence introducing an additional (potentially non-zero) term when integrating by parts.

and the fact that vanishing covariance implies independence for Gaussians, we have that the random variables $(h,\cdot)_\nabla$ restricted to orthogonal subspaces $H_1(D), H_2(D) \subseteq H(D)$ are independent of one another. Moreover, if $H_1(D), H_2(D)$ span $H(D)$ then $\mathcal{F}_{H_1}, \mathcal{F}_{H_2}$, the smallest σ-algebras such that $(h,f)_\nabla$ are measurable for all f in H_1,H_2 respectively, generate $\mathcal{F}$, the smallest σ-algebra making $(h,\cdot)_\nabla$ measurable.

Theorem 1.28 (Markov property of the GFF)**.** *The two subspaces $H_U(D), H_U^\perp(D)$, as defined above, span $H(D)$.*

This theorem tells us that we can write every $f \in H(D)$ as a unique sum of elements from H_U and $H_U^\perp$, which is, by the previous remark, analogous to the alternative statement of Brownian motion given in Remark 1.24, justifying it being called the Markov property.

Proof. By a density argument it suffices to show that every $f \in H_s(D)$, the space of smooth functions compactly supported in D, can be written as $a+b$, with $a \in H_U(D), b \in H_U^\perp$.

The idea: Intuitively we would like to set b to be the unique continuous function which equals f outside of U and is harmonic inside of U, and then set $a = f - b$.

The problem: However, if for example U is the complement of a single point $z \in D$, then there is no unique b which coincides with f on $\{z\}$ and is harmonic on U. This is similar to trying to solve the Laplace equation in one dimension given only the initial value, but no derivatives which would be defined if we were "allowed" to know the values of f in a neighbourhood of z since $f \in H_s(D)$. This motivates the following approach:

The solution: We take the δ-neighbourhood of the complement of U (so that every point in the complement of U gets a "padding" of δ), take its complement $U_\delta \subseteq U$, and use the fact that the expectation of a function evaluated at the first exit point of Brownian motion of a set is harmonic in that set and (by definition) equal to the original function outside of that set.

More precisely, we take

$$b_\delta(x) := \mathbb{E}_x(f(B_{\tau(\delta)})),$$

where $\tau(\delta)$ is the first time t the Brownian motion B_t exits U_δ. Then $a_\delta := f - b_\delta$ is supported on a compact subset of U and is in H_{U_δ}, which is increasing (as $\delta \searrow 0$) family of subspaces of H_U. Now a_δ converges to some $a \in H_U$. Thus the b_δ must converge to some b which has to be in $H_U^\perp$ since the $\|\cdot\|_\nabla$ limit of harmonic functions is harmonic (just like the L^2 limit of constant, smooth functions is constant). We still have $f = a + b$ which finishes the proof. □

1.2.2. *Circle Averages.* Although h is not a function (i.e. it does not have values at points), we shall see circle averages are well-defined by duality. To see this, first introduce the ε-regularized Green's function:

Definition 1.29. *We let*

$$G^x_{D,\varepsilon}(\cdot) := -\log[|x - \cdot| \vee \varepsilon] - \tilde{G}^x_{D,\varepsilon},$$

where, as before, $\tilde{G}^x_{D,\varepsilon}$ is the unique harmonic (in D) function such that $G^x_{D,\varepsilon}(y) = 0$ for all $y \in \partial D$. In case D is clear from context, we will also drop it and simply write G^x_ε for $G^x_{D,\varepsilon}$.

Now this is harmonic in $B_\varepsilon(x)$ (since it is and does not have a singularity anymore) and in $\overline{B_\varepsilon(x)}^c$ (as a sum of two harmonic functions) so its Laplacian will be supported on $\partial B_\varepsilon(x)$. Hence one might expect that $-\Delta G^x_\varepsilon$ is a measure supported on $\partial B_\varepsilon(x)$.

Indeed, using the rotational symmetry and lack of differentiability at $\partial B_\varepsilon(x)$ of the derivative, the following is easy to see:

Proposition 1.30. *For any $x \in D, \varepsilon > 0$ with ε small enough[*] we have that*

(1) $G^x_\varepsilon \in H^1(D)$
(2) $-\Delta[G^x_\varepsilon] = \nu^x_\varepsilon$, *where ν^x_ε is the uniform measure of total mass one on $\partial B_\varepsilon(x)$.*

While (1) ensures that $(h, G^x_\varepsilon)_\nabla$ is well defined, (2) states that $(h, G^x_\varepsilon)_\nabla = (h, \nu^x_\varepsilon)$ can be interpreted as a circle average of h with radius ε around x.

Proof. Write it in polar coordinates and use that $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, where the last term vanishes by rotational symmetry. □

Thus the following definition is well-posed and can be interpreted as circle average:

Definition 1.31 (ε-regularized GFF h_ε)**.** *The ε-regularized GFF h_ε, sometimes also referred to simply as "circle average", is defined as follows:*

$$h_\varepsilon(z) := (h, G^z_\varepsilon)_\nabla$$

Since working directly with h is not possible, in what is to come we will want to approximate h with h_ε by letting ε tend to zero. In those calculations it will turn out to be helpful to see that for any fixed $z \in D$ and suitably chosen t_0 the process $(h_{e^{-t}}(z))_{t \geq t_0}$ is a standard Brownian motion.

Lemma 1.32 (Circle averages are Brownian motion as radius decreases)**.** *Let h be a Gaussian free field in D, $z \in D$ fixed, $t_0 := \inf\{t \geq 0 : B_{e^{-t}} \subseteq D\}$, and*

$$\mathcal{B}_t := h_{e^{-(t+t_0)}}(z) - h_{e^{-t_0}}(z).$$

[*] ε such that $\varepsilon < d(x, \partial D)$

Then $(\mathcal{B}_t)_{t\geq 0}$ is a standard Brownian motion.

Before we start with the proof of the lemma, we first express the covariances of the ε-regularized Gaussian free field h_ε in terms of the conformal radius $C(z;D)$.

Proposition 1.33 (Covariance structure of h_ε)**.** *Writing*

$$G_{\varepsilon_1,\varepsilon_2}(z_1,z_2) := \mathbb{E}[h_{\varepsilon_1}(z_1)h_{\varepsilon_2}(z_2)] = \int_{D\times D} G^x(y)d\nu_{\varepsilon_1}^{z_1}(x)d\nu_{\varepsilon_2}^{z_1}(y),$$

for ν_ε^z the uniform measure of total mass one on $\partial B_\varepsilon(z)$ as before, we have

(1) *$G_{\varepsilon_1,\varepsilon_2}(z_1,z_2) = (G_{\varepsilon_1}^{z_1}, G_{\varepsilon_2}^{z_2})_\nabla = (G_{\varepsilon_1}^{z_1}, \nu_{\varepsilon_2}^{z_2})$, the mean value of $G_{\varepsilon_1}^{z_1}$ on $\partial B_{e_2}(z_2)$.*

(2) *If $B_{\varepsilon_1}(z_1)$ and $B_{\varepsilon_2}(z_2)$ are disjoint and in D, then*

$$G_{\varepsilon_1,\varepsilon_2}(z_1,z_2) = G(z_1,z_2),$$

the usual (i.e. not regularized) Green's function.

(3) *If $B_{\varepsilon_1}(z)\subseteq D$ and $\varepsilon_1\geq\varepsilon_2$ then*

$$G_{\varepsilon_1,\varepsilon_2}(z,z) = -\log\varepsilon_1 + \log C(z;D).$$

Proof of proposition. (1) follows directly by definition and partial integration, using Proposition 1.30.

(2) follows by the circle average property of harmonic functions, using that G^x is harmonic in $D\setminus\{x\}$ and coincides with $\tilde{G}_\varepsilon^x$ in $D\setminus B_\varepsilon(x)$.

(3) holds by the following calculation:

$$\begin{aligned} G_{\varepsilon_1,\varepsilon_2}(z,z) &\overset{(1)}{=} (G_{\varepsilon_1}^{z_1}, \nu_{\varepsilon_2}^{z_2}) \\ &\overset{(\text{def})}{=} \int_{\mathbb{C}} (-\log(\varepsilon_1 \vee |z-y|) - \tilde{G}_{\varepsilon_1}^z(y))d\nu_{\varepsilon_2}^z(y), \end{aligned}$$

where the first summand equals $-\log\varepsilon_1$ since, by definition of $\nu_{\varepsilon_2}^z$, we integrate only over $y\in\partial B_{\varepsilon_2}(z)$ and $\int_{\mathbb{C}} d\nu_{\varepsilon_2}^z(y) = 1$, and the second summand evaluates to $\tilde{G}^z(z)$ by the circle average property of harmonic functions and because $\varepsilon_1\leq \operatorname{dist}(z,\partial D)$ (so that the harmonic correction term does not "see" the change around z). Using Proposition 1.23(3) this is exactly what we wanted to show. □

Proof of lemma. Clearly $(\mathcal{B}_t)_{t\geq 0}$ is a centered Gaussian process. Hence it suffices to show that for any $0\leq s\leq t$ we have that $\mathbb{E}[\mathcal{B}_s\mathcal{B}_t] = s$.

By definition we have

$$\mathbb{E}[\mathcal{B}_s\mathcal{B}_t] \overset{(\text{def})}{=} \mathbb{E}[(h_{e^{-(s+t_0)}}(z) - h_{e^{-t_0}}(z))(h_{e^{-(t+t_0)}}(z) - h_{e^{-t_0}}(z))],$$

so using linearity of expectation and Proposition 1.33(3) repeatedly one easily arrives at $\mathbb{E}[\mathcal{B}_s\mathcal{B}_t] = s$. □

Note that one could have done the same calculations with the ε parameterization to get some logarithmic terms from the regularized Green's function in the end (instead of the correct s) - this way it would become obvious why we chose the e^{-t} parameterization.

2. Liouville measures

In this section we want to study two dimensional random surfaces. Similarly to how Brownian motion (the one dimensional Gaussian free field) is a "random line", a (regularized) Gaussian free field in some bounded domain $D \subseteq \mathbb{C}$ can be identified with a random Riemann surface via the correspondence suggested by the following theorem:

Theorem 2.1 (Riemann uniformization theorem)**.** *For every smooth, simply connected Riemannian manifold*[*] $(\mathcal{M}, \mu)$ *there exists a real valued function* $\lambda : D \to \mathbb{R}$ *for* $D \in \{\mathbb{C}, \mathbb{C} \cup \{\infty\}, \mathbb{D}\}$ *such that the Radon Nikodym derivative of* μ *at* $z \in D$ *is given by* $e^{\lambda(z)}$*.*

2.1. **Existence of weak limits.** Now we would like to take $\lambda = \gamma h$ (for some fixed parameter $\gamma \geq 0$) to get two dimensional random surfaces, but since h does not have values at points this is not well-defined. Instead, we will take $\lambda = \gamma h_\varepsilon$ and see "in which order of magnitude this approximation typically explodes" as ε goes to zero - or to be more precise: Find c, α such that $\mathbb{E}e^{\gamma h_\varepsilon}(z) \to c\varepsilon^\alpha$.

Normalising by this factor, i.e. taking $\lambda = \gamma h_\varepsilon - \alpha \log \varepsilon$, so that $\mathbb{E}e^{\gamma h_\varepsilon(z) - \alpha \log \varepsilon} = O(1)$, will be our best shot at finding a well defined limiting measure.

Remark 2.2 (Asymptotics of $\mathbb{E}e^{\gamma h_\varepsilon(z)}$)**.** *Noting that* $\mathbb{E}e^{\mathcal{N}(a,b)} = e^{a+b/2}$ *and* $h_\varepsilon(z) \sim \mathcal{N}(0, -\log \varepsilon + \log C(z; D))$ *by 1.33(3) we get that*

$$\mathbb{E}e^{\gamma h_\varepsilon(z)} = e^{\gamma^2/2(-\log \varepsilon + \log C(z;D))} = \left(\frac{C(z;D)}{\varepsilon}\right)^{\gamma^2/2} = c\varepsilon^{-\gamma^2/2}.$$

So it seems natural to normalise by a factor of $\varepsilon^{\gamma^2/2}$*:*

Definition 2.3 (Regularized h_ε)**.** *Write* $\overline{h}_\varepsilon := \gamma h_\varepsilon + \frac{\gamma^2}{2} \log \varepsilon$*.*

Intuitively a weak limit should be strong enough to preserve information about area since (at least for probability measures) we have $\mu_n \to \mu$ weakly implies that $\mu_n(B) \to \mu(B)$ for all B in the Borel-σ algebra with $\mu(\partial B) = 0$.

[*]Note that a surface can be described by several quantities like the area of any fixed measurable subset, its geodesics, the length of any fixed, smooth curve, etc. Here we chose to describe it by its area.

Theorem 2.4 (Weak limit of $\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon}$)**.** *Let h be a Gaussian free field in some bounded domain $D \subseteq \mathbb{C}$, $\gamma \in [0,2)$ fixed, and $\mu_\varepsilon := \varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}dz = e^{\overline{h}_\varepsilon(z)}dz$. Then μ_ε converges weakly almost surely against some $\mu = \mu_h$ for $\varepsilon = 2^{-k} \to 0$ as $k \to \infty$.*

Proof. Reduction to squares: It suffices to show that $\mu_{2^{-k}}(S)$ converges almost surely against some finite limit $\mu(S)$ for every diadic square compactly supported in D.

This is immediate from the definition of weak convergence used here: μ_n converges weakly against μ if for any compactly supported, continuous function with support contained in D, we have that $\int f(x)\mu_n(dx) \to \int f(x)\mu(dx)$.

Reduction to exponential decay: To show that $\mu_{2^{-k}}(S) \to \mu(S) < \infty$ with probability one, we show that $\mathbb{E}|\mu_{2^{-(k+1)}}(S) - \mu_{2^{-k}}(S)|$ decays exponentially in k.

Writing $X_k := \mu_{2^{-k}}(S)$, a real valued random variable, the exponential decay implies that $(X_k)_k$ is a Cauchy sequence in $L^1(\mathbb{P})$, but L^1 is complete and hence the limit $X := \mu(S)$ is in L^1 again. Since its expectation is finite it implies that $\mu(S) < \infty$ with probability one.

Strategy to prove exponential decay: To show that $\mathbb{E}|\mu_{2^{-(k+1)}}(S) - \mu_{2^{-k}}(S)|$ decays exponentially in k we assume without loss of generality* that $S = [0,1]^2$, so that $\mu_\varepsilon(S)$ is just the mean value of $e^{\overline{h}_\varepsilon}$ on S and calculate $\mathbb{E}(|\mu_{2^{-(k+1)}}(S) - \mu_{2^{-k}}(S)|^2)$.

We would like to formalize the intuition that, by the Markov property, $h_\varepsilon(z)$ and $h_\varepsilon(z')$, conditioned on $h_{2\varepsilon}(z)$ and $h_{2\varepsilon}(z')$, respectively, are independent (for ε small enough) which simplifies calculations. In the following we will take the average values of $\overline{h}_{2^{-(k+1)}}$ and $\overline{h}_{2^{-(k+2)}}$, conditioned on $\overline{h}_{2^{-(k+1)}}$, over some lattice S_k^y chosen such that all those values are independent. Then we will average over all lattices $(S_k^y)_{y\in S}$ and take another expectation to get rid of the conditioning.

To make that rigorous we define

- $S_k^y := (y + 2^{-k}\mathbb{Z}) \cap D$
- $A_k^y := \frac{1}{|S_k^y|}\sum_{z\in S_k^y} exp(\overline{h}_{2^{-(k+1)}}(z))$
- $B_k^y := \frac{1}{|S_k^y|}\sum_{z\in S_k^y} exp(\overline{h}_{2^{-(k+2)}}(z))$

the average values of $h_{2^{-(k+1)}}$ and $h_{2^{-(k+2)}}$ over S_k^y, respectively.

*Otherwise we would have to keep writing an additional factor given by the (Lebesgue) area of S.

Reduction to disjoint balls: Writing $\mathbb{E}^y$ for the expectation over $y \in S$ (with respect to the Lebesgue measure) we have

$$\begin{aligned}\mathbb{E}|\mu_{2^{-k}}(S) - \mu_{2^{-(k+1)}}(S)| &= \mathbb{E}|\mathbb{E}^y(exp(\overline{h}_{2^{-k}}) - exp(\overline{h}_{2^{-(k+1)}}))| \\ &= \mathbb{E}|\mathbb{E}^y(A_k^y - B_k^y) \\ &\overset{\text{Jensen}}{\leq} \mathbb{E}\mathbb{E}^y|A_k^y - B_k^y| \\ &\overset{\text{Fubini}}{=} \mathbb{E}^y\mathbb{E}|A_k^y - B_k^y|,\end{aligned}$$

so it suffices to show that $\mathbb{E}|A_k^y - B_k^y|$ decays exponentially in k, uniformly for all $y \in S$. (Note that we can apply Jensen since $\mathbb{E}^y(1) = \int_S ds = 1$ since we assumed that $S = [0,1]^2$. Otherwise we would have had to introduce a normalising factor.)

Getting a conditional expectation: Now by the Markov property (Theorem 1.28) of the Gaussian free field we have that the random variables $h_{2^{-(k+2)}}(z)$ conditioned on $h_{2^{-(k+1)}}(z)$ are independent of one another for different $z \in S_k^y$. Moreover, by Lemma 1.32, we have that they are $\mathcal{N}(h_{2^{-(k+1)}}(z), \log 2)$. Using these results an elementary (but tedious) calculation shows that

$$\begin{aligned}\mathbb{E}[|A_k^y - B_k^y|^2|\{h_{2^{-(k+1)}}(z)\}_{z\in S_k^y}] &\overset{\text{(def)}}{=} 2^{-4k}\sum_{z\in S_k^y}\mathbb{E}[|e^{\overline{h}_{2^{-(k+1)}}(z)} - e^{\overline{h}_{2^{-(k+2)}}(z)}|^2|\{h_{2^{-(k+1)}}(z)\}_{z\in S_k^y}] \\ &= 2^{-4k}C\sum_{z\in S_k^y}\left(e^{\overline{h}_{2^{-(k+1)}}(z)}\right)^2, \qquad (2.1)\end{aligned}$$

where $C > 0$ is some constant only depending on γ.

Showing exponential decay for the unconditional expectation: Now we can take expectations to get the unconditional expectation $\mathbb{E}|A_k^y - B_k^y|^2$, but it turns out that the naive approach of bounding the expectation by the second moment only works for $\gamma < \sqrt{2}$ since

$$\begin{aligned}\mathbb{E}[|A_k^y - B_k^y|^2] &\overset{(2.1)}{=} 2^{-4k}C\sum_{z\in S_k^y}\mathbb{E}\left[(e^{\overline{h}_{2^{-(k+1)}}(z)})^2\right] \\ &= O\left(|S_k^y|2^{-4k}2^{-k\gamma^2}\mathbb{E}[e^{2\gamma h_{2^{-(k+1)}}}]\right) \\ &= O(2^{2k}2^{-4k}2^{-k\gamma^2}2^{2k\gamma^2}) \\ &= O(2^{-k(2-\gamma^2)}),\end{aligned}$$

which finishes the proof for $\gamma < \sqrt{2}$ (because we have exponential decay in that regime).

The more general result for $2 \leq \gamma^2 < 4$ can be obtained by showing that the second moment is only made large by rare occurrences of $h_\varepsilon(z)$ being much larger

than expected and that their contribution to the first moment (the expectation) vanishes exponentially. □

2.2. **Parameterizing a surface with different domains.** We will start by stating the main result of this subsection (mainly to introduce notation), then state some immediate applications, and conclude by proving some other results needed to show it.

Proposition 2.5 (Change of coordinates)**.** *Let h be a Gaussian free field on a domain D, ψ a conformal map from a domain $\tilde{D}$ to D, and $\tilde{h} := h \circ \psi + Q \log |\psi'|$ a distribution on $\tilde{D}$, where $Q := \frac{2}{\gamma} + \frac{\gamma}{2}$.*

Then

$$\mu_{\tilde{h}} = \mu_h \circ \psi,$$

i.e. $\mu_{\tilde{h}}(A) = \mu_h(\psi(A))$ for all Borel measurable $A \subseteq \tilde{D}$.

Remark 2.6 (Motivation)**.** *For some calculations we would like to parameterize the same surface given by the Gaussian free field on D with a different domain $\tilde{D}$ since some quantities are easier to define and calculate in one domain than in another. It also leads to a definition of a class of random surfaces henceforth referred to as quantum surfaces. They will be shown to have connections to SLEs later on.*

Definition 2.7 (Quantum surfaces)**.** *A quantum surface is an equivalence class of pairs (D, h) under the equivalence transformation*

$$(D, h) \to \psi^{-1}(D, h) := (\psi^{-1}(D), h \circ \psi + Q \log |\psi'|) = (\tilde{D}, \tilde{h}),$$

for any conformal $\psi : \tilde{D} \to D$.

This definition admits the following interpretation: Pick a random surface (D, h) induced (or "parameterized") by the Gaussian free field h on D and let $\tilde{D}$ be another domain with some conformal $\psi : \tilde{D} \to D$. Now if we wanted to consider the same surface parameterized by $\tilde{D}$ instead of D, Proposition 2.5 tells us to consider $\tilde{h} := h \circ \psi + Q \log |\psi'|$, the Gaussian free field on $\tilde{D}$ (by conformal invariance) plus some correction term to account for the local changes of ψ.

Remark 2.8 (Preliminaries for the proof)**.** *First, we recall that $(\cdot, \cdot)_\nabla$ is conformally invariant in two dimensions and that, by the Cauchy-Riemann equations, we have that the determinant of a Jacobian for a conformal coordinate change ψ is given by $|\psi'|^2$.*

Proof of the proposition. Reduction to continuous approximations: Since it is hard to show the desired result $\mu_h \circ \psi = \mu_{\tilde{h}}$ directly we will work with approximations μ_h^n to μ_h as given by Theorem 2.9 (note that they are well-defined functions, as opposed

to distributions) and show that $\mu_h^n \circ \psi = \mu_{\tilde{h}}^n$ for all $n \in \mathbb{N}$ and hence for their (weak) limit.

Show it for the approximation:

$$\begin{aligned}
\mu_{\tilde{h}}^n &\overset{(\text{def})}{=} \exp\left(\gamma(h^n \circ \psi + Q\log|\psi'|) - \frac{\gamma^2}{2}\text{var}(h^n \circ \psi + Q\log|\psi'|) + \frac{\gamma^2}{2}\log C(z;D)\right) \\
&= \exp\left(\gamma h^n \circ \psi + \frac{\gamma^2}{2}\log|\psi'| + 2\log|\psi'| - \frac{\gamma^2}{2}\text{var}(h^n \circ \psi) + \frac{\gamma^2}{2}\log C(z;D)\right) \\
&= \exp\left(\gamma h^n \circ \psi + \frac{\gamma^2}{2}\log(|\psi'|C(z;D)) - \frac{\gamma^2}{2}\text{var}(h^n \circ \psi)\right)|\psi'|^2 \\
&= \mu_h^n \circ \psi,
\end{aligned}$$

where the last step holds since

- $|\psi'(z)| = \frac{|\psi'(z)|C(z,\tilde{D})}{C(z,\tilde{D})} = \frac{C(\psi(z);D)}{C(z;\tilde{D})}$, so the factor $\frac{\gamma}{2}\log|\psi'|$ is here to adjust the conformal radius, and
- $|\psi'(z)|^2 = |\det(D\psi)(z)|$ is the factor we get when using the integration by substitution formula, integrating against the Radon-Nikodym derivative $\frac{d\mu^n}{dz}$, and going from $\tilde{D}$ to D by ψ.

□

This proof relied heavily on the following theorem which is also of independent interest since it shows the non-triviality of the limiting measure μ_h as a by-product.

Theorem 2.9. *Write $h := \overline{h} + h^0$ where $\overline{h}$ is the zero boundary Gaussian free field on some domain D and h^0 is a deterministic continuous function on D. Let $(f_i)_{i\in\mathbb{N}}$ be an orthonormal basis of continuous functions for $H(D)$ and let $h^n := h^0 + \sum_{i=1}^n (h, f_i)_\nabla f_i$. Then $\mu = \mu_h$ is almost surely the weak limit of*

$$\mu_h^n = \mu^n := exp\left(\gamma h^n(z) - \frac{\gamma^2}{2}Varh^n(z) + \frac{\gamma^2}{2}\log C(z;D)\right)dz$$

as $n \to \infty$.

Furthermore, for each measurable $A \subseteq D$ and for each $n \geq 0$ we have

$$\mathbb{E}[\mu(A)|h^n] = \mu^n(A).$$

Before we get to the proof of this theorem we have to define rooted random measures and record some of their properties.

Definition 2.10 (Rooted random measures)**.** *Let $\theta_\varepsilon := Z_\varepsilon^{-1} e^{\gamma h_\varepsilon(z)} dz dh$, where Z_ε^{-1} is some normalising constant chosen such that θ_ε is a probability measure and dh is the law of h.*

Remark 2.11 (θ_ε is well-defined)**.** *By the Radon-Nikodym theorem θ_ε is well defined because the Radon-Nikodym derivative of θ_ε with respect to $dzdh$ is $Z_\varepsilon^{-1}e^{\gamma h_\varepsilon(z)}$, a non-negative, continuous (hence measurable) function on a measurable space $H \times D$ endowed with the product σ-algebra of their two respective σ-algebras and $dzdh$ and θ_ε are both σ-finite measures.*

While we could just define θ as the (e.g. weak) limit of θ_ε, we would like to have something more explicit. We will achieve this by explicitly writing down a way to sample from θ_ε (in particular we will first sample z from its marginal distribution $\int_H \theta_\varepsilon dh$ and then sample h from its conditional distribution) and notice that this enables us to give a "more explicit" definition of θ. To do this we first state the relevant marginal laws and conditional distributions of θ_ε:

Remark 2.12 (Marginal laws and conditional distributions)**.** .

(1) *The θ_ε marginal distribution of z is given by*

$$\int_H \theta_\varepsilon dh = Z_\varepsilon^{-1}\left(\mathbb{E}_h e^{\gamma h_\varepsilon(z)}\right) dz,$$

where Z_ε^{-1} is some normalising constant to make it a probability measure. Remember that $\mathbb{E}_h e^{\gamma h_\varepsilon(z)}$ is proportional to $C(z;D)^{\gamma^2/2}$.

(2) *The θ_ε law conditional on z is given by*

$$d\mathbb{P}_{\theta_\varepsilon}(h|z) = \frac{e^{\gamma h_\varepsilon(z)}}{\mathbb{E}_h e^{\gamma h_\varepsilon(z)}} dh.$$

(3) *For later we also record the θ_ε marginal distribution of h:*

$$\int_D \theta_\varepsilon dz = Z_\varepsilon^{-1}\left(\int_D e^{\gamma h_\varepsilon(z)} dz\right) dh.$$

Now we may formally state the two step sampling procedure hinted at above:

Remark 2.13 (Two step procedure)**.** *Sampling a pair (z,h) with law given by θ_ε is equivalent to*

(1) *first sampling $\tilde{z}$ from the θ_ε marginal distribution of z, i.e. $\tilde{z} \overset{d}{=} Z_\varepsilon^{-1}\mathbb{E}_h e^{\gamma h_\varepsilon(z)}$*
(2) *and then sampling $\tilde{h}$ from the conditional distribution $\frac{e^{\gamma h_\varepsilon(\tilde{z})}}{\mathbb{E}_h e^{\gamma h_\varepsilon(\tilde{z})}} dh$.*

Note that the first step does not depend on ε since $\mathbb{E}_h e^{\gamma h_\varepsilon(z)}$ is proportional to $C(z;D)^{\gamma^2/2}$. The second step still seems to depend on ε in a non-trivial way, but writing $\gamma h_\varepsilon(z) \overset{(def)}{=} (h, \gamma G_\varepsilon^z)_\nabla$ and recalling that dh is the law of a Gaussian it follows easily from Cameron-Martin (Theorem 1.11) that

$$\tilde{h} \overset{d}{=} h + \gamma G_\varepsilon^z. \tag{2.2}$$

This has several implications: Firstly, it provides a direct way to define θ by replacing the truncated Green's function G^z_ε with the real one G^z in the second step of our sampling procedure.

Definition 2.14. *Let θ be the probability measure on pairs (z,h) obtained by first sampling z as in step one in Remark 2.13, i.e. $d\mathbb{P}(z)$ is proportional to $C(z;D)^{\gamma^2/2}$, and then, given z, the conditional law of h is that of the original GFF plus γG^z.*[*]

Furthermore for $\tilde{D}$ compact subsets of D we set $\theta^{\tilde{D}}$ and $\theta^{\tilde{D}}_\varepsilon$ to be the respective measures conditioned on z being in $\tilde{D}$, where the latter is well-defined only for ε small enough, i.e. $\varepsilon < dist(\tilde{D}, \partial D)$.

Secondly, the following calculation shows that, up to an additive constant, the θ law of $h_{e^{-t}}(z)$ conditioned on z is that of a Brownian motion with drift term γt:

Remark 2.15 (Conditioned on z the θ law of $h_{e^{-t}}(z)$ is that of Brownian plus a drift term)**.**

$$\begin{aligned}\mathcal{L}_\theta(h_{e^{-t}}(z)|z) &\overset{(def)}{=} \mathcal{L}_\theta((h, G^z_{e^{-t}})_\nabla|z)\\ &\overset{(2.2)}{=} \mathcal{L}_h((h+\gamma G^z_{e^{-t}}, G^z_{e^{-t}})_\nabla|z)\\ &= \mathcal{L}_h(h_{e^{-t}}(z) + \gamma(G^z_{e^{-t}}, G^z_{e^{-t}})_\nabla|z),\end{aligned}$$

where $(G^z_{e^{-t}}, G^z_{e^{-t}})_\nabla = (\nu^z_{e^{-t}}, G^z_{e^{-t}})$, the circle average of $G^z_{e^{-t}} \overset{1.29}{=} -\log(e^{-t})$ plus a harmonic correction term, which, by the circle averaging property of harmonic functions does not change in t. Now Lemma 1.32 this implies the claim.

We are now ready to prove the theorem.

Proof of the theorem. To show equality we first argue why μ^n almost surely has a weak limit, say $\tilde{\mu}$, and then show that this limit actually coincides with $\mu \overset{(\text{def})}{=} \lim_\varepsilon e^{\gamma \overline{h_\varepsilon}}$.

μ^n has a weak limit $\tilde{\mu}$ almost surely: To prove that a weak limit exists it suffices, just as in the proof of Theorem 2.4, to show that $\tilde{\mu}(S) := \lim_n \mu^n(S)$ exists almost surely for all diadic squares S compactly supported in D. Since $\mu^n(S)$ is non-negative this would follow if we could show that $\mu^n(S)$ is a martingale. This, in turn, follows after showing that the Radon-Nikodym derivative $d\mu^n/dz = exp\left(\gamma h^n(z) - \frac{\gamma^2}{2}\text{Var}h^n(z) + \frac{\gamma^2}{2}\log C(z;D)\right)$ is a martingale with respect to $\sigma(h^n)$,

[*]The difference to before being that we replaced the truncated Green's function with the actual Green's function G^z.

the σ-algebra generated by h^n, and applying Fubini. Note that, indeed, we have

$$\begin{aligned}\mathbb{E}[d\mu^n/dz|\sigma(h^{n-1})] &\overset{(\text{def})}{=} \mathbb{E}\left[\exp\left(\gamma(h^{n-1}+(h,f_n)_\nabla)-\frac{\gamma^2}{2}(\mathrm{Var}h^{n-1}+\mathrm{Var}(h,f_n)_\nabla)+\text{const}\right)|\sigma(h^{n-1})\right]\\ &= d\mu^{n-1}/dz\;\mathbb{E}\left[\exp\left(\gamma(h,f_n)_\nabla-\frac{\gamma^2}{2}\right)|\sigma(h^{n-1})\right]\\ &= d\mu^{n-1}/dz.\end{aligned}$$

Show that $\tilde{\mu}=\mu$: We would like to translate this problem into something "more tangible", like a statement about the rooted random measures θ_ε for which we have a surprisingly nice (i.e. explicit) representation of its limit θ. To this end we have the following chain of implications to reduce it to the problem of L^1 convergence of random variables:

$\mu=\tilde{\mu}$
if $\mu(S)=\tilde{\mu}(S)$ for every diadic square compactly supported in D
if $\mathbb{E}[\mu(S)|h]=\mathbb{E}[\tilde{\mu}(S)|h]$ since μ and $\tilde{\mu}$ are functions of h
if $\mathbb{E}[\mu(S)|h^n]=\mathbb{E}[\tilde{\mu}(S)|h^n]\overset{(\text{def})}{=}\mu^n(S)$ for all n
if $\mathbb{E}[\lim_\varepsilon \mu_\varepsilon(S)|h^n]=\lim_\varepsilon \mathbb{E}[\mu_\varepsilon(S)|h^n]$ for all n, since
- $\mathbb{E}[\lim_\varepsilon \mu_\varepsilon(S)|h^n]\overset{(\text{def})}{=}\mathbb{E}[\mu(S)|h^n]$ and
- $\lim_\varepsilon \mathbb{E}[\mu_\varepsilon(S)|h^n]=\mu^n(S)$,

where the latter equality holds by the following calculation:

$$\begin{aligned}\mathbb{E}\left[\mu_\varepsilon(z)|h^n\right] &\overset{(\text{def})}{=} \exp\left(\frac{\gamma^2}{2}\log\varepsilon\right)\exp\left(\mathbb{E}\left[e^{\gamma h_\varepsilon(z)}|h^n\right]\right)\\ &= \exp\left(\frac{\gamma^2}{2}\log\varepsilon+\mathbb{E}[\gamma h_\varepsilon(z)|h^n]+\frac{\gamma^2}{2}\mathrm{Var}[h_\varepsilon(z)|h^n]\right)\\ &= \exp\left(\frac{\gamma^2}{2}\log\varepsilon+\gamma h_\varepsilon^n(z)+\frac{\gamma^2}{2}\mathbb{E}\left[(h_\varepsilon(z)-\mathbb{E}[h_\varepsilon(z)|h^n])^2|h^n\right]\right)\\ &= \exp\left(\frac{\gamma^2}{2}\log\varepsilon+\gamma h_\varepsilon^n(z)+\frac{\gamma^2}{2}\left(\mathbb{E}\left[h_\varepsilon(z)^2|h^n\right]-\mathbb{E}h_\varepsilon^n(z)^2\right)\right)\\ &\overset{1.33}{=} \exp\left(\gamma h_\varepsilon^n(z)-\frac{\gamma^2}{2}\mathrm{Var}h_\varepsilon^n(z)+\frac{\gamma^2}{2}\log C(z;D)\right),\end{aligned}$$

where $h_\varepsilon^n(z)$ is the expectation of $h_\varepsilon(z)$ given the projection of h onto the span of $f_1,f_2,\dots,f_n$. This can be seen to converge against μ^n as $\varepsilon\searrow 0$ by continuity, which implies the claim after integrating over S.

First, we will show this for $n=0, h^0=0$, i.e. $\mathbb{E}[\lim_\varepsilon \mu_\varepsilon(S)]=\lim_\varepsilon \mathbb{E}[\mu_\varepsilon(S)]$. Fixing a diadic square S, compactly supported in D and setting $M_\varepsilon:=\mu_\varepsilon(S)$, this is equivalent to saying that $M_\varepsilon\to\mu(S)$ in $L^1(\mathbb{P})$ as $\varepsilon\searrow 0$. Similarly to before, using Fubini, we see that $(M_\varepsilon)_\varepsilon$ is a martingale with almost sure limit $\mu(S)$, so by

the L^1 martingale convergence theorem it suffices to show that $(M_\varepsilon)_\varepsilon$ is uniformly integrable.

Reduction to θ_ε^S: Note that $(M_\varepsilon)_\varepsilon$ being uniformly integrable means that for all $\delta > 0$ there is a $C > 0$ such that

$$\begin{aligned}\mathbb{E}_h[M_\varepsilon 1\{M_\varepsilon > C\}] &= \int_{M_\varepsilon^{-1}([C,\infty))} M_\varepsilon(h)dh \\ &= c\int_{h:M_\varepsilon(h)>C} d\theta_\varepsilon^S(h) \\ &= c\,\theta_\varepsilon^S(h : M_\varepsilon(h) > C) < \delta \end{aligned} \tag{2.3}$$

for all $\varepsilon > 0$ and some $c > 0$ since M_ε is non-negative and uniquely determined by h, i.e. it can be seen as a random variable on (H, dh). The step before the last holds because M_ε is, up to a constant, equal to the Radon-Nikodym derivative of the h marginal of θ_ε^S, i.e. $\frac{d\theta_\varepsilon^S}{dh} = cM_\varepsilon$ (by abuse of notation, denoting the marginal by θ_ε^S again).

By (2.3) it suffices to show that for all $\delta > 0$ we can find a $C > 0$ such that

$$\theta_\varepsilon^S(h : M_\varepsilon(h) > C) < \delta, \tag{2.4}$$

for all $\varepsilon > 0$.

For the sake of brevity we will only sketch the rest of the proof which is considerably more technical.

Sketch of the rest of the proof: All probabilities and expectations will be, unless stated otherwise, taken with respect to θ_ε^S. Similar to the proof of the existence of the weak limit we will also show it for a conditional expectation and then generalise. The general idea is to translate the problem to a statement about Brownian motion which we know more about.

To that end define

$$\tilde{h} := h - \gamma G_\varepsilon^z,$$

which, under θ_ε^S, given z, has the law of a GFF on D (remember how we used Cameron-Martin in (2.2)). Hence, by Lemma 1.32,

$$B_t := \tilde{h}_{e^{-t}\varepsilon_0}(z) - \tilde{h}_{\varepsilon_0}(z),$$

for $t \in [0, -\log(\varepsilon/\varepsilon_0)]$ is a Brownian motion for $\varepsilon_0 := \sup\{\varepsilon : B_\varepsilon(S) \subseteq D\}$.

Similarly to the two step procedure described in Remark 2.13 we will first sample a $z \in S$. Then we will sample a Brownian motion on the interval $[0, -\log(\varepsilon/\varepsilon_0)]$ and interpret it as knowing the differences between the circle averages $\{\tilde{h}'_\varepsilon(z)\}_{\varepsilon'=\varepsilon}^{\varepsilon_0}$ and $\tilde{h}_{\varepsilon_0}(z)$. Intuitively, if the Brownian motion is negative at time t, this means, by definition, that $h_{e^{-t}\varepsilon_0}(z) < \tilde{h}_{\varepsilon_0}(z)$, i.e. that the circle average with the smaller radius is less than the one with the bigger radius. So conditioned on that, we expect

h to be negative (in a distributional sense) in a small area around any w with $|z-w| \approx e^{-t}\varepsilon_0$. More precisely (and more generally), we have that

$$\tilde{h}'(w) := \mathbb{E}[\tilde{h}(w)|z,(B_t)_t] = \begin{cases} \tilde{h}_{|z-w|\vee\varepsilon}(z) - \tilde{h}_{\varepsilon_0}(z) = B_{u(w)} & \text{if } |z-w| < \varepsilon_0 \\ 0 & \text{if } |z-w| \geq \varepsilon_0 \end{cases},$$

for $u(w) := -\log \frac{|z-w|\vee\varepsilon}{\varepsilon_0}$.

One goes on to show that for $\tilde{h}'_\varepsilon(w)$, the mean value of $\tilde{h}'(w)$ on $\partial B_\varepsilon(w)$, we have

$$|\mathrm{Var}\tilde{h}'_\varepsilon(w) - u(w)| < \log 2.$$

Since $\mathbb{E}[\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w)] = \tilde{h}'_\varepsilon(w)$ an easy calculation shows that $\mathrm{Var}\tilde{h}'_\varepsilon(w) = \mathrm{Var}(\tilde{h}'_\varepsilon(w)) - \mathbb{E}\mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w))$. But, as linear functionals of h, $\tilde{h}_\varepsilon(w)$ and $\tilde{h}'_\varepsilon(w)$ are jointly Gaussian and hence the conditional variance is in fact independent of $\tilde{h}'_\varepsilon(w)$ (i.e. $\mathbb{E}\mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w)) = \mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w))$), giving

$$|\mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w)) - (\mathrm{Var}(\tilde{h}_\varepsilon(w)) - u(w))| < \log 2,$$

almost surely. Roughly speaking, this allows us to replace $\mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w))$ with the unconditioned $\mathrm{Var}(\tilde{h}_\varepsilon(w)) - u(w)$ at the cost of at most some universal constant (in particular bounded between positive constants for varying z and ε).

This gives

$$\begin{aligned}\mathbb{E}\left[\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(w)}|z,(B_t)_t\right] &= \varepsilon^{\gamma^2/2}\exp\left(\gamma\mathbb{E}[\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w)] + \frac{\gamma^2}{2}\mathrm{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}'_\varepsilon(w))\right) \\ &\asymp \exp\left(\gamma\tilde{h}'_\varepsilon(w) + \frac{\gamma^2}{2}u(w)\right),\end{aligned}$$

where $\asymp$ denotes equality up to a multiplicative factor bounded between positive constants uniformly in ε and z. The last (asymptotic) equality holds by Proposition 1.33 and noting that $G^z_\varepsilon(w) = u(w) - \log\varepsilon_0 - \tilde{G}^z(w)$.

Remembering that $\tilde{h}'_\varepsilon(w)$ is just a weighted average over values of the Brownian motion it is not far to seek a bound on $\mathbb{E}\left[\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(w)}|z,(B_t)_t\right]$ by conditioning further on the event $A = \{\gamma B_t < a + bt \text{ for all } t\}$ for some a, b - this has a positive probability which can be made as close to one as possible by taking a sufficiently large for a fixed b.

Choosing a and b appropriately (in particular $b + \gamma^2/2 < 2$ and a sufficiently large) one can show that $\mathbb{E}\left[\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(w)}|z,(B_t)_t,A\right] \leq \text{const } e^a|z-w|^{-(b+\gamma^2/2)}$ for $|z-w| < \varepsilon_0$, which, after integrating, gives

$$\mathbb{E}[\mu_\varepsilon(S)|z,(B_t)_t,A] \leq \text{const}\int_{B_{\varepsilon_0}(z)} e^a|z-w|^{-(b+\gamma^2/2)}dw < C_1(a), \tag{2.5}$$

for some $C_1(a) < \infty$ independent of ε.

For every fixed b and $\delta > 0$ we can choose a large enough so that $\mathbb{P}(A) > 1 - \delta/2$. Now assume for contradiction that $\mathbb{P}\left(\mu_\varepsilon(S) > \frac{C_1(a)}{\delta/2}\right) > \delta/2$. This implies that $\mathbb{P}\left(A, \mu_\varepsilon(S) > \frac{C_1(a)}{\delta/2}\right) > \delta/2$, but by Markov and (2.5) we have

$$\mathbb{P}\left(A, \mu_\varepsilon(S) > \frac{C_1(a)}{\delta/2}\right) \leq \frac{\delta/2}{C_1(a)} \mathbb{E}[\mu_\varepsilon(S), A] \overset{(2.5)}{\leq} \delta/2,$$

a contradiction. This implies (2.4) for $n = 0$ and $h^0 = 0$.

Extending to the general case of $n \in \mathbb{N}$ and $h^0 \neq 0$ is not too hard, since $\mu_\varepsilon(S)$ converges $dzdh$ (as opposed to θ_ε^S) almost surely to a limit and it is easy for h^0 that are constant on diadic squares, but for a full proof we refer to the original paper [2]. □

3. Connection to SLEs

Before we state the main result of this section we first have to fix notation and introduce some new objects we will be working with.

3.1. **Preliminaries.**

3.1.1. *SLEs.* Naturally, the object of central interest in this section is the SLE. In this subsection we shall merely fix notation.

Definition 3.1 (Chordal SLE_κ in $\mathbb{H}$ from 0 to ∞)**.** *It is well known that for each $\kappa \geq 0$ and instance of a Brownian motion $B = (B_t)_{t \geq 0}$ there is almost surely a unique continuous curve η in $\mathbb{H}$ from 0 to ∞ (parameterized by $[0, \infty)$) for which*

$$dg_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t} dt, \tag{3.1}$$

where $g_t : \mathbb{H} \setminus \eta([0,t]) \to \mathbb{H}$ are the unique conformal maps satisfying the normalisation $\lim_{z \to \infty} |g_t(z) - z| = 0$.

Equation (3.1) is usually called the Loewner flow. The resulting (random) object η is said to be an instance of chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ with half-plane capacity t. We set $\eta_T := \eta([0,T])$ and denote by K_T the complement of the unbounded component of $\mathbb{H} \setminus \eta_T$.

SLE_κ is known to exhibit different behaviour for different choices of κ which is stated in the following proposition:

Proposition 3.2 (Properties of SLE_κ)**.** *We distinguish between the following cases:*

- *$\kappa \in [0,4]$: SLE_κ is almost surely a simple curve.*
- *$\kappa \in (4,8)$: SLE_κ is almost surely a self-intersecting, but non-space-filling curve.*

- $\kappa \geq 8$*: SLE*$_\kappa$ *is almost surely a space-filling curve.*

So for $\kappa \leq 4$ we get that $K_T = \eta_T$ almost surely.

In what follows we will mainly deal with the denominator in (3.1) which we will henceforth refer to as

$$f_t(z) := g_t(z) - \sqrt{\kappa} B_t,$$

satisfying

$$df_t(z) = \frac{2}{f_t(z)} dt - \sqrt{\kappa} dB_t,$$

and $f_t(\eta(t)) = 0$.

3.1.2. *Free Boundary GFF.* While there is a lot to say about the free boundary Gaussian free field as well, we will restrict ourselves to the bare minimum needed in the following proof, focusing primarily on the differences to the zero boundary GFF we introduced in the section 1.

The free boundary GFF is defined just like the zero boundary GFF, except that we do not take the Hilbert space closure of $H_s(D)$ with respect to $(\cdot,\cdot)_\nabla$, but rather the one of $\{f : \nabla f \in L^2, \int_D f = 0\}$ with respect to the same inner product $(\cdot,\cdot)_\nabla$, which induces a norm on this space since the additive constant that is "not seen" by the Dirichlet inner product is fixed by the requirement of having mean zero.

We define the Green's function just as before, but without the harmonic correction term. So the free boundary Green's function of $\mathbb{H}$ will be $-\log|x-y| - \log|x-\overline{y}|$ (by a reflection argument).

Again, similarly to before, we can thus define (h,ρ) by integration by parts and the inverse Laplacian given by integrating against the Green's function. However, one needs to be careful since integration by parts relied on the compact support in order not to introduce any correction terms. For $g = -\Delta^{-1}\rho$ it is easy to see that $(f, -\Delta^{-1}\rho)_\nabla = (f,\rho)$ (and hence well defined for a any fixed f) if and only if $\int_{\partial D} f \nabla g = 0$, where ∂D is to be interpreted by viewing $\mathbb{C}$ as a Riemann sphere. For example, in the case of $D = \mathbb{C}$ we have that $\int_{\partial D} f\nabla g = 0$ is equivalent to $\lim_{z\to\infty} \nabla g(z) = 0$, i.e. the gradient of g has no point mass at infinity.

3.2. **Sampling quantum surfaces with SLEs.** The next theorem provides another way of sampling a quantum surface $(\mathbb{H}, h)$ as defined in 2.7, where $h := \tilde{h} + \frac{2}{\sqrt{\kappa}} \log|\cdot|$ and $\tilde{h}$ is a free GFF on $\mathbb{H}$. Informally, it may be rephrased as: The law of the quantum surface $(\mathbb{H}, h)$ is invariant under the following operation:

(1) sample a Brownian motion $(B_t)_{t\in[0,T]}$ inducing f_T (indepedently of h)
(2) cut out K_T

(3) use $f_T^{-1} : \mathbb{H} \setminus K_T \to \mathbb{H}$ in the place of ψ in Definition 2.7 to get

$$\begin{aligned} f_T^{-1}(\mathbb{H} \setminus K_T, h) &\overset{(\text{def})}{=} (f_T^{-1}(\mathbb{H} \setminus K_T), h \circ f_T + Q \log |f_T'|)) \\ &= (\mathbb{H}, h \circ f_T + Q \log |f_T'|)) \end{aligned}$$

Note that the definition of a quantum surface gives us that even for deterministic ψ we have that the induced surfaces are invariant under the ψ^{-1} operation as in the third step above. The non-trivial part is that we can neglect the information in K_T but "regain" it (in law) by the randomness of K_T. This cannot be possible for any deterministic ψ.

While the following theorem can be shown for general $\kappa > 0$ we will restrict ourselves to $\kappa \leq 4$ to avoid technical difficulties introduced by K_T not being a curve.

Theorem 3.3. *Fix $\kappa \in (0, 4]$ and let η_T be the segment of SLE_κ generated by a reverse Loewner flow*

$$df_t(z) = -\frac{2}{f_t(z)} dt - \sqrt{\kappa} dB_t, \quad f_0(z) = z \tag{3.2}$$

up to a fixed time $T > 0$. Furthermore let

- $Q := \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}$ *(as in Proposition 2.5 with $\gamma = \sqrt{\kappa}$),*
- $\mathfrak{h}_0(z) := \frac{2}{\sqrt{\kappa}} \log |z|$,
- $\mathfrak{h}_t(z) := \mathfrak{h}_0(f_t(z)) + Q \log |f_t'(z)|$,
- $\tilde{h}$ *an instance of the free boundary Gaussian free field on $\mathbb{H}$, independent of $B = (B_t)_{t \geq 0}$.*

Then the following two random distributions (modulo additive constants since $\tilde{h}$ is a free boundary *GFF) on $\mathbb{H}$ agree in law:*

$$h := \mathfrak{h}_0 + \tilde{h},$$

$$h \circ f_T + Q \log |f_T'| \overset{(def)}{=} \mathfrak{h}_T + \tilde{h} \circ f_T,$$

where f_t' refers to the spatial derivative of $f_t(\cdot)$.

A similar result holds for the forward SLE (see [7], Theorem 1.1) and can be proved almost analogously. However, it would require introducing a notion of level lines for (sufficiently well-behaved) distributions from which we will refrain here.

Remark 3.4. *First, we notice that the statement is that h and $h \circ f_T + Q \log |f_T'|$ agree as distributions on $\mathbb{H}$. The statement itself (without proving it) is already non-trivial since our definition of a Gaussian free field is not a random distribution, but a Gaussian process indexed by $H(D)$. However, as stated (without proof) in Remark 1.16, it turns out that the GFF (as a random distribution) is sufficiently*

nice to have a unique extension to the larger space of (non-continuous) functionals on $H(D)$, which is compatible with our definition of the GFF. Henceforth we shall use the definition of a GFF as random distribution, knowing that it can be shown to be equivalent to ours.

Proof. Two deterministic distributions on D are equal if their evaluations at every $\rho \in H_s(D)$ coincide. To show that the two random distributions $h \overset{(\text{def})}{=} \mathfrak{h}_0 + \tilde{h}$ and $\mathfrak{h}_T + \tilde{h} \circ f_T$ are equal in distribution we will thus show that

$$(\mathfrak{h}_0, \rho) + (\tilde{h}, \rho) \overset{d}{=} (\mathfrak{h}_T, \rho) + (\tilde{h} \circ f_T, \rho). \tag{3.3}$$

Let G_T be the Green's function of $\mathbb{H} \setminus K_T$ and $E_T := \int \rho(x) G_T(x,y) \rho(y) dxdy$, then we have that

- $(\mathfrak{h}_0, \rho)$ is deterministic.
- $(\tilde{h}, \rho) \sim \mathcal{N}(0, E_0)$.
- $(\tilde{h} \circ f_T, \rho) \sim \mathcal{N}(0, E_T)$ and it is independent of $(\mathfrak{h}_T, \rho)$ since $\tilde{h}$ was chosen independently of the Brownian motion inducing f_T.

Now assume we had shown that $X_T := (\mathfrak{h}_T, \rho)$ was a continuous local martingale with quadratic variation $[X]_T = E_0 - E_T$. Then, by Dubins-Schwarz we would have that the recentered process $M_T := (\mathfrak{h}_T, \rho) - (\mathfrak{h}_0, \rho)$ (chosen such that $M_0 = 0$) satisfies $M_t = \tilde{B}_{E_0 - E_T}$, where $\tilde{B}$ is a Brownian motion, so in particular: $(\mathfrak{h}_T, \rho) \sim \mathcal{N}((\mathfrak{h}_0, \rho), E_0 - E_T)$.

As a result we could rewrite (3.3) as

$$\mathcal{N}((\mathfrak{h}_0, \rho), E_0) \overset{d}{=} \mathcal{N}((\mathfrak{h}_0, \rho), E_0 - E_T) + \mathcal{N}(0, E_T),$$

keeping in mind that the two normals on the r.h.s. are independent so that their variances add up and we would get the desired result.

Hence it suffices to prove the assumption that $(\mathfrak{h}_T, \rho)$ is a continuous local martingale with quadratic variation $E_0 - E_T$. The strategy will be as follows:

(1) Calculate $d\mathfrak{h}_T(z)$ to see that it is a continuous, local martingale.*
(2) Calculate (the derivative of) the covariation $[\mathfrak{h}_t(y), \mathfrak{h}_t(z)]$ so that we can later use Fubini to get the quadratic variation $[(\mathfrak{h}_t, \rho), (\mathfrak{h}_t, \rho)]$.
(3) Show that $\mathfrak{h}_t(z)$ is actually a proper martingale for some technical reasons (like being able to apply Doob's inequality).
(4) Show that $\mathfrak{h}_t(z)$ is almost surely continuous and that $(\mathfrak{h}_t, \rho)$ is a continuous martingale so that both can be seen as a randomly stopped Brownian motion.
(5) Then we calculate $d[\mathfrak{h}_t(z), \mathfrak{h}_t(y)]$ and $d[(\mathfrak{h}_t(z), \rho), (\mathfrak{h}_t(z), \rho)]$.

*At this point we might already expect that $(\mathfrak{h}_T, \rho)$, as a "sufficiently nice" linear combination of local martingales, is a local martingale as well.

Calculating $d\mathfrak{h}_t(z)$: It will be helpful to view $\mathfrak{h}_t(z)$ as the real part of some function $\mathfrak{h}_t^*(z) := \frac{2}{\sqrt{\kappa}} \log f_t(z) + Q \log f_t'(z)$ and calculating its derivative instead. To do that we will use linearity of d and calculate one term at a time. Starting with $d \log f_t(z)$ we get by Ito's formula (writing f_t instead of $f_t(z)$ for the sake of brevity):

$$\begin{aligned} d \log f_t &= \frac{1}{f_t} df_t + \frac{1}{2}\left(-\frac{1}{f_t^2}\right) df_t df_t \\ &= -\frac{2}{f_t^2} dt - \frac{\sqrt{\kappa}}{f_t} dB_t - \frac{1}{2f_t^2}\left(\frac{2}{f_t} dt + \sqrt{\kappa} dB_t\right)^2 \\ &= -\frac{4+\kappa}{2f_t^2} dt - \frac{\sqrt{\kappa}}{f_t} dB_t, \end{aligned}$$

where we used that $df_t = -\frac{2}{f_t(z)} dt - \sqrt{\kappa} dB_t$ (by its definition in (3.2)), that the quadratic variation of dt with any semimartingale vanishes, and that $d[B_t, B_t] = dt$. Now an elementary calculation gives that $df_t' = \frac{2f_t'}{f_t^2} dt$ and hence (again by Ito's formula):

$$d \log f_t' = \frac{1}{f_t'} df_t' + 0 = \frac{2}{f_t^2} dt.$$

Putting it all together we get that the derivative of $\mathfrak{h}_t(z) \overset{(\text{def})}{=} \mathsf{Re}\, \mathfrak{h}_t^*(z)$ is given by

$$\begin{aligned} d\mathfrak{h}_t(z) &= \mathsf{Re}\left(\frac{2}{\sqrt{\kappa}} d \log f_t + Q d \log f_t'\right) \\ &= \mathsf{Re}\left(-\frac{4+\kappa}{\sqrt{\kappa} f_t^2} dt - \frac{2}{f_t} dB_t + \frac{2Q}{f_t^2} dt\right) = -\mathsf{Re}\, \frac{2}{f_t} dB_t. \end{aligned}$$

Now it is clear that $\mathfrak{h}_t \in \mathcal{M}_{c,\text{loc}}$.

Calculating $d[\mathfrak{h}_t(y), \mathfrak{h}_t(z)]$: To calculate the quadratic variation of a linear combination of local martingales like $(\mathfrak{h}_t, \rho)$ it is instructive* to calculate the covariations $[\mathfrak{h}_t(x), \mathfrak{h}_t(y)]$. We will do so by recalling that, by conformal invariance, the (free boundary) Green's function of $\mathbb{H} \setminus K_t$ is given by

$$G_t(x, y) := G(f_t(x), f_t(y)),$$

where $G(x, y) := -\log|x - y| - \log|x - \overline{y}|$ is the Green's function of $\mathbb{H}$ (which can be seen by a reflection argument).

In order to apply (stochastic) Fubini later we will write $d[\mathfrak{h}_t(x), \mathfrak{h}_t(y)]$ in terms of G_t. To be more precise, we show that $-dG_t(x, y) = \mathsf{Re}\, \frac{2}{f_t(x)} \mathsf{Re}\, \frac{2}{f_t(y)} dt$, which is

*Think of the finite case, i.e. $\rho = \sum_{i=1}^n r_i \delta_{z_i}$, where $(\mathfrak{h}_t, \rho) = \sum_{i=1}^n r_i \mathfrak{h}_t(z_i)$. We cannot expect to get the quadratic variation $[(\mathfrak{h}_t, \rho)]$ without knowing the covariations $[\mathfrak{h}_t(z_i), \mathfrak{h}_t(z_j)]$.

equal to $d[\mathfrak{h}_t(x), \mathfrak{h}_t(y)]$ (by our calculation of $d\mathfrak{h}_t(z)$).

$$\begin{aligned}
-dG_t(x,y) &\overset{(\text{def})}{=} -d\mathsf{Re}\,\log[f_t(x)-f_t(y)] - d\mathsf{Re}\,\log[f_t(x)-\overline{f_t(y)}] \\
&= 2\mathsf{Re}\,\left[\frac{f_t(x)^{-1}-f_t(y)^{-1}}{f_t(x)-f_t(y)} + \frac{f_t(x)^{-1}-\overline{f_t(y)^{-1}}}{f_t(x)-\overline{f_t(y)}}\right] dt \\
&= 2\mathsf{Re}\,\left[f_t(x)^{-1}f_t(y)^{-1} + f_t(x)^{-1}\overline{f_t(y)^{-1}}\right] dt \\
&= 2\mathsf{Re}\,\left[f_t(x)^{-1}\left(2\frac{f_t(y)^{-1}+\overline{f_t(y)^{-1}}}{2}\right)\right] dt \\
&= 4\mathsf{Re}\,\left[f_t(x)^{-1}\mathsf{Re}\,(f_t(y)^{-1})\right] dt,
\end{aligned}$$

where the first equality holds since $d\log[f_t(x)-f_t(y)] = \frac{1}{f_t(x)-f_t(y)}(df_t(x)-df_t(y))+0$ (the quadratic variation of the difference $f_t(x)-f_t(y)$ vanishes since the Brownian motions cancel out each other) and $df_t(x)-df_t(y) = -\frac{2}{f_t(x)}dt - \frac{2}{f_t(y)}$ (the Brownian motions cancel again). The second equality holds by the identity $\frac{\frac{1}{a}-\frac{1}{b}}{a-b} = -\frac{1}{ab}$.

Showing that $\mathfrak{h}_t(z) \in \mathcal{M}_c$: To show that $\mathfrak{h}_t(z)$ is a true (continuous) martingale we note that we already know that it can be parameterized by its quadratic variation so that it essentially becomes a Brownian motion. So one way to see that $\mathfrak{h}_t(z)$ is indeed a martingale is to show that this random stopping time given by the quadratic variation $[\mathfrak{h}_t(z)]$ (which is obviously increasing in t) is bounded by some constant times t uniformly for all z. Actually it suffices to fix any $\rho \in H_s$ and to show that this holds uniformly for all z in the (compact) support of of ρ. This can be checked by noting that, by the Loewner evolution, we have that

$$|\frac{\partial}{\partial t}C_t(z)| \leq \text{const}(\rho),$$

where $C_t(z) := \log \mathsf{Im}\, f_t(z) + \mathsf{Re}\,\log f_t'(z)$. This can be seen to be the quadratic variation $[\mathfrak{h}_t(z)]$ by taking its derivative $dC_t(z)$ (using Ito's formula) and noting that it this is equal to $\lim_{x,y\to z} -dG_t(x,y)$.

Showing that $(\mathfrak{h}_t, \rho) \in \mathcal{M}$: We would like to use conditional Fubini to interchange the conditional expectation and the integral from $(\mathfrak{h}, \rho)$ so that we can use the martingale property of $\mathfrak{h}_t(z)$ to get

$$\begin{aligned}
\mathbb{E}[(\mathfrak{h}_t,\rho)|\mathcal{F}_s] &= (\mathbb{E}[\mathfrak{h}_t|\mathcal{F}_s],\rho) \\
&= (\mathfrak{h}_s,\rho),
\end{aligned}$$

showing that $(\mathfrak{h}_t, \rho)$ is a martingale. To do that we have to check that for $\mathfrak{h}_t(z)$ is $L^1(\text{supp}(\rho)\times\Omega)$ for any fixed ρ and all t. (Recall that Ω denotes the probability space we are taking expectations over.) This is given if the expectation of $\mathfrak{h}_t(z)$ is bounded uniformly in $z \in \text{supp}(\rho)$, which is true since - similar to Brownian motion

(note that it is only rescaled up to a uniformly bounded constant factor - we have $\mathbb{P}(|\mathfrak{h}_t(z)| \geq \lambda) \leq e^{-\lambda c}$, where $c > 0$ is some constant.

Showing that $(\mathfrak{h}_t, \rho)$ is almost surely continuous: We would like to apply dominated convergence to the sequence $(\mathfrak{h}_{t_n}\rho)_n$, where $t_n \to t$, so that we have

$$\lim_n (\mathfrak{h}_{t_n}, \rho) = \lim_n \int \mathfrak{h}_{t_n}(z)\rho(z)dz \overset{DCT}{=} \int \lim_n \mathfrak{h}_{t_n}(z)\rho(z)dz = (\mathfrak{h}_t, \rho),$$

which holds since $\mathfrak{h}_t(z)$ is continuous in t. It remains to check that $(\mathfrak{h}_{t_n}\rho)_n$ is admissible for the dominated convergence theorem, which, since ρ is compactly supported, is given if $\sup_{s\in[0,t]} |\mathfrak{h}_s(z)|$ is in L^1_{loc}. This can easily be seen by noticing that the expected integral of $|\mathfrak{h}_t|$ over any compact set is finite, its probability distribution decays exponentially fast (hence L^1 boundedness implies L^p boundedness), and applying Doob's L^p inequality.

Showing that $d[(\mathfrak{h}_t, \rho)] = -dE_t(\rho)$: Note that $[(\mathfrak{h}_t, \rho)]$ is characterised by the fact that $(\mathfrak{h}_t, \rho)(\mathfrak{h}_t, \rho)^2 - [(\mathfrak{h}_t, \rho)] \in \mathcal{M}_{c,\text{loc}}$. Knowing from previous calculations that the covariation of $\mathfrak{h}_t(x)$ and $\mathfrak{h}_t(y)$ is $-G_t(x, y)$, and that $\mathfrak{h}_t(z) \in \mathcal{M}_c$, we have that

$$\mathfrak{h}_t(x)\mathfrak{h}_t(y) + G_t(x, y)$$

is a continuous martingale for any $x, y \in \mathbb{H}$. Now we would like to use (conditional) Fubini to argue that $(\mathfrak{h}_t, \rho)^2 + \int \rho(x)G_t(x, y)\rho(y)dxdy$ is a continuous martingale. To do that we notice that $G_t(x, y)$ is non-increasing in t and, just as before, $\mathfrak{h}_t(z)$ have laws decaying exponentially fast, uniformly in $z \in \text{supp}(\rho)$. □

4. Connection to the KPZ formula

Since the proofs of the following statements are beyond the scope of a Part III essay we will restrict ourselves to stating the very basic definitions and theorems to give an idea how the KPZ formula translates expected fractal dimensions of random subsets of the plane to fractal dimensions in terms of Liouville quantum gravity.

To define fractal dimensions with respect to a general measure μ we first have to define a ball with respect to μ:

Definition 4.1 (Isothermal quantum ball)**.** *For some fixed measure μ on the domain D (henceforth this will be referred to as* quantum measure*) let $B^\delta(z) := B_\varepsilon(z)$, the Euclidean ball centered at $z \in D$ with radius $\varepsilon := \sup\{\varepsilon' : \mu(B_{\varepsilon'}(z) \leq \delta)\}$. Then $B^\delta(z)$ is called the* isothermal quantum ball *of area δ centered at z.*

Remark 4.2. *In the above definition we would like to pick the Euclidean ball whose radius ε is chosen such that $\mu(B_\varepsilon(z)) = \delta$, but in general such a delta does not need to exist, which is the reason for the less intuitive definition with the supremum.*

Remark 4.3. *Note that if $\gamma = 0$ then μ as in Theorem 2.4 (the limit of $\mu_\varepsilon := e^{\gamma \overline{h}_\varepsilon} dz$) is just the Lebesgue measure and $\delta = \pi\varepsilon^2$.*

Recall the definition of the ε-neighbourhood of $X \subseteq D$ by

$$B_\varepsilon(X) := \{z : B_\varepsilon(z) \cap X \neq \emptyset\}$$

which serves as a motivation for the following definition:

Definition 4.4 (Isothermal quantum δ neighbourhood of X)**.**

$$B^\delta(X) := \{z : B^\delta(z) \cap X \neq \emptyset\}$$

Fix $\gamma \in [0, 2)$. We also recall that a random subset $X \subseteq D$ is said to have (Euclidean expectation) dimension $2(1-x)$, where x is called the Euclidean scaling exponent, if the expected area of $B_\varepsilon(X)$ decays like $(\varepsilon^2)^x$. (Note that $D \subseteq \mathbb{C} \simeq \mathbb{R}^2$.) More precisely,

$$x := \lim_{\varepsilon \to 0} \frac{\log \mathbb{E}\mu_0(B_\varepsilon(X))}{\log \varepsilon^2},$$

where μ_0 denotes the Lebesgue measure on D and the expectation is taken over X.

Remark 4.5 (Examples)**.** *Some easy examples for deterministic X to illuminate the definition:*

- *$X = \{0\}$. We expect x to be one since the dimension of a point is $0 = 2(1-1)$. Indeed we have $\mu_0(B_\varepsilon(0)) = \pi\varepsilon^2$ so that $\frac{\log \varepsilon^2 + \log \pi}{\log \varepsilon^2} \to 1 =: x$.*
- *$X = [0, 1]$. We expect x to be $\frac{1}{2}$ since the dimension of a line is $1 = 2(1 - \frac{1}{2})$. Indeed we have $\mu_0(B_\varepsilon([0, 1])) = O(\varepsilon)$ so that $\frac{\log \varepsilon + c}{\log \varepsilon^2} \to 1/2 =: x$.*

This, in turn, motivates the definition of its "quantum" analogue:

Definition 4.6 (Quantum scaling exponent)**.** *We say a random $X \subseteq D$ has quantum scaling exponent Δ if*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}\mu(B^\delta(X))}{\log \delta} = \Delta,$$

where the expectation $\mathbb{E}$ is with respect to both X and μ which are chosen independently.

Now we can state (without proof) one of the central results of [2]:

Theorem 4.7. *Fix $\gamma \in [0, 2)$ and a compact subset $\tilde{D} \subseteq D$ of some domain D. If a random $X \cap \tilde{D}$ has Euclidean expected scaling exponent $x \geq 0$ then it has quantum scaling exponent Δ, where Δ is the non-negative solution to*

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$

Conclusion

Recapitulation. We introduced the Gaussian free field, showed a selection of its properties needed to define and prove existence and non-triviality of Liouville measures, which lead to a definition of random surfaces, termed "quantum surfaces". We went on to prove a result relating sampling of SLEs and quantum surfaces and finished by mentioning how to compute fractal dimensions of random sets using the KPZ formula.

Outlook and omissions. Most notably we did not address the discrete case where much intuition comes from at all. This includes the discrete Gaussian free field and the construction of Liouville measures as quadrangulations that are glued together.

There are many other properties of the GFF of independent interest like field exploration and how this relates it to Brownian motion and thick points (something we could easily have obtained as a simple corollary from Remark 2.15). Moreover, we did not point out any applications in or intuition from physics, including the motivation of the GFF with Dirichlet energy and its relation to Wick products and harmonic crystals. Generally speaking the GFF can be interpreted in many other ways, not only as a dual Gaussian process.

We only introduced the connections to SLEs and the KPZ relation very superficially; much is left to be said in that respect.

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