ODE

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Contents

Chapter 1

Special Cases

1.1 Finding implicit solutions

1.1.1 The idea

Given an autonomous differential equation $\dot{x} = f(x)$, $x(0) = x_0$, $f \in \mathcal{C}(\mathbb{R})$, suppose $f(x_0) \neq 0$, then

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))} ds = t$$

Substitution yields

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))} ds = \int_{x_0}^x \frac{dy}{f(y)} =: F(x)$$

Since $f(x_0) \neq 0$ (in the following let w.l.o.g. $f(x_0) > 0$) F is strictly monotone near x_0 and can thus be inverted, leading to $F^{-1}(t) = x =: \phi(t)$.

1.1.2 Maximal interval of definition

Let (x_1, x_2) be a maximal interval s.t. $x_1 \leq x_0 \leq x_2$ and $f(x) > 0$[1] $\forall x \in (x_1, x_2)$.

Now let $T_+ := \lim_{x \to x_2} F(x)$ and $T_- := \lim_{x \to x_1} F(x)$.

If $T_+ < \infty$ either $x_2 = \infty$ (in this case the solution diverges to $+\infty$) or $x_2 < \infty$ (here the solution reaches x_2 after finite time T_+ and as x_2 was

[1] because we assumed w.l.o.g. $f(x_0) > 0$

chosen s.t. $f(x_2) = 0$ there are (potentially) multiple ways of extending the solution).[2]

1.2 Finding explicit solutions

1.2.1 Homogeneous equations

$\dot{x} = f(\frac{x}{t})$ is called a (nonlinear) homogeneous DE.

Now let $y = \frac{x}{t}$ [3] and see that $dy = \underbrace{\frac{\partial y}{\partial t}}_{-\frac{x}{t^2}} + \underbrace{\frac{\partial y}{\partial x}}_{\frac{1}{t}} \dot{x} = \frac{f(y)-y}{t}$.[4] This is separable.

1.3 Qualitative Analysis of First-Order Equations

[2]note that the fact that the DE is autonomous is crucial as otherwise $f(x_2) = 0$ does not imply that the constant function (starting at x_2) is a solution for all t.

[3]$y = y(x, t)$

[4]partial derivative lets x fixed even though it depends on t as well