# High-dimensional Landscapes and Random Matrices

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December 8, 2015

## Abstract

# Contents

# Introduction

## 0.1 Motivation

Given some Hamiltonian $H_{N,p} : S^{N-1} \to \mathbb{R}$ given by

$$H_{N,p}(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p},$$

where the $J_{i_1,\ldots,i_p}$ are independent centered standard Gaussian random variables and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ referred to as states, one is interested in finding the expected number of critical points $Crt_N(u)$ and the restriction to some index $k$, denoted by $Crt_{N,k}(u)$, in some region in $(-\infty, Nu]$.

## 0.2 Main Results

**Theorem 0.2.1** (Large deviations for $\mathbb{E}[\mathrm{Crt}_{N,k}(u)]$)**.**

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Crt_{N,k}(u)] = \Theta_{k,p}(u) \tag{1}$$

*and*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Crt_N(u)] = \Theta_p(u), \tag{2}$$

*for*

$$\Theta_p(u) = \begin{cases} \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2 - I_1(u), & \textit{if } u \leq -E_\infty \\ \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2, & \textit{if } -E_\infty \leq u \leq 0 \\ \frac{1}{2}\log(p-1), & \textit{if } 0 \leq u \end{cases}$$

*and*

$$\Theta_{k,p}(u) = \begin{cases} \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2 - (k+1)I_1(u), & \textit{if } u \leq -E_\infty \\ \frac{1}{2}\log(p-1) - \frac{p-2}{p}, & \textit{if } u \geq E_\infty \end{cases},$$

*where* $E_\infty = E_\infty(p) = 2\sqrt{\frac{p-1}{p}}$ *and* $I_1 : (-\infty, E_\infty] \to \mathbb{R}$ *is given by*

$$I_1(u) = \frac{2}{E_\infty^2} \int_u^{-E_\infty} \sqrt{z^2 - E_\infty^2} dz$$

*and is the the rate function of the LDP for the smallest eigenvalue of the GOE.*

**Theorem 0.2.2** (Layered structure)**.** *For all* $k \geq 0$ *and* $\varepsilon > 0$ *we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\left\{\sum_{i=k}^{\infty} Crt_{N,i}(-E_k - \varepsilon) > 0\right\}\right) < 0, \tag{3}$$

*where* $E_k = E_k(p)$ *is chosen (uniquely because of strict monotonicity) such that* $\Theta_{k,p}(-E_k) = 0$*.*

mention 2.12 and that I don't prove it

# Chapter 1

# Prerequisites

To be able to prove the above results we first need to prove some other theorems which require some

## 1.1 Gaussian Random Fields

Adler and Taylor (2007) p.121 for

$$\mathbb{E}\left\{\frac{\partial^k f(s)}{\partial s_{i_1}\dots\partial s_{i_k}}\frac{\partial^k f(t)}{\partial t_{i_1}\dots\partial t_{i_k}}\right\} = \frac{\partial^{2k} C(s,t)}{\partial s_{i_1} t_{i_1}\dots\partial s_{i_k} t_{i_k}} \tag{1.1}$$

## 1.2 GOE

**Definition 1.1.** *Symmetric $N \times N$ matrices $H = H_N$ with $\mathbb{E}H_{ij} = 0$ and $\mathbb{E}H_{ij} = 1 + \delta_{ij}$.*

**Remark.** *Their density is given by the Gaussian measure*

$$d\mathbb{P}(H) = Z_N^{-1} \exp\{-\frac{1}{4} trH^2\} \tag{1.2}$$

*with normalization constant $Z_N = \int d\mathbb{P}(H) \prod_{1\le i\le j\le N} dH_{ij}$ which is a shorter way of writing $Z_N^{-1} \exp\left\{-\frac{1}{4}\left(2\sum_{i<j}^N H_{ij}^2 + \sum_{i=j}^N H_{ij}^2\right)\right\}$.*

## 1.3 LDP

## 1.4 Morse Theory

## 1.5 Spin Glass Models

## 1.6 Intermediary Results

To prove our main results we first need some refinements for the expected values of critical values.

define E_GOE, rho and B Borel sets

**Theorem 1.6.1** (Refinement of $\mathbb{E}[Crt_{N,k}(B)]$)**.** *For all $N$, $p \geq 2$ and $k \in \{0, \dots, N-1\}$ we have*

Kac-Rice formulae, AT07

$$\mathbb{E}[Crt_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\mathbb{E}^N_{GOE}\left[e^{-N\frac{p-2}{2p}(\lambda^N_k)^2}\mathbf{1}\left\{\lambda^N_k \in \sqrt{\frac{p}{2(p-1)}}B\right\}\right] \tag{1.3}$$

*and*

sum up previous eq.

$$\mathbb{E}[Crt_N(B)] = 2N\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\int_{\sqrt{\frac{p}{2(p-1)}}B} exp\left\{-\frac{N(p-2)x^2}{2p}\right\}\rho_N(x)dx \tag{1.4}$$

**Theorem 1.6.2** (LDP for $k$-th largest eigenvalue)**.** *The $k$-th largest eigenvalue $\lambda_{N-k+1}$ of the GOE of dimension $N$ with variance $\sigma^2 N^{-1}(1+\delta_{ij})$ satisfies an LDP with speed $N$ and a good rate function*

define GOE

define good rate function

$$I_k(x;\sigma) = kI_1(x;\sigma) = \begin{cases} k\int_{2\sigma}^{x}\sigma^{-1}\sqrt{(\frac{z}{2\sigma})^2-1}dz, & \textit{if } x \geq 2\sigma \\ \infty, & \textit{otherwise} \end{cases}. \tag{1.5}$$

# Appendix A

# Wigner's Semicircle Law

## A.1 Wigner's Semicircle Law

**Definition A.1** (Wigner matrix)**.** *A Wigner matrix $M$ is a complex, Hermitian matrix with independent and identically distributed entries $M_{ij}$ for $i \geq j$ and with mean 0 and variance 1 for $i > j$. The diagonal entries $M_{ii}$ have bounded mean and variance.*

If $M_n$ is an $n$-dimensional Wigner matrix we know that the operator norm $\|M_n\|_{OP}$ is typically of size $\mathcal{O}(\sqrt{n})$, so it natural to define the empirical spectral distribution (ESD) as follows:

prove it

**Definition A.2** (ESD)**.**

$$\mu_{\frac{1}{\sqrt{n}}M_n} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(M_n)/\sqrt{n}},$$

*where $\lambda_1(M_n) \leq \cdots \leq \lambda_n(M_n)$ are the ordered, real eigenvalues of $M_n$.*

Since we are considering random matrices the ESDs will be random as well and thus it is interesting to ask if there is a measure on the real line $\mu$ such that it is the weak limit $\mu_{M_n/\sqrt{n}} \rightharpoonup \mu$ of $\mu_{M_n/\sqrt{n}}$, that is $\int_{\mathbb{R}} \varphi d\mu_{M_n/\sqrt{n}}$ converges in probability (or almost surely) against $\int_{\mathbb{R}} \varphi d\mu$ for all $\varphi \in C_c(\mathbb{R})$. This can also be derived from the more general definition of convergence in probability or almost surely, but we will not do that here.

which do we want to show here?

Surprisingly such a limit $\mu$ exists and is even deterministic.

**Theorem A.1.1** (Wigner's semicircle law)**.** *Let $M_n$ be the top left $n \times n$ minors of an infinite Wigner matrix, then the ESDs $\mu_{M_n/\sqrt{n}}$ converge almost* *surely (and thus in probability) to the Wigner semicircle distribution given by*

$$\mu_{sc} := \begin{cases} \frac{1}{2\pi}\sqrt{4-|x|^2}dx, & \textit{if } |x| \leq 2 \\ 0, & \textit{else} \end{cases}.$$

A rough outline of the proof is given by this list of intermediary results that will be shown:

preliminary reductions - look up which one we need

1. 

2. To show that $\mu_n \rightharpoonup \mu$ almost surely, it suffices to show that the respective Stieltjes transforms converge almost surely, pointwise in the upper half plane, i.e. $\mu_n \rightharpoonup \mu \Leftrightarrow \forall z \in \mathbb{C} : Im(z) > 0 s_{\mu_n}(z) \to s_\mu(z)$ almost surely.

3. The Stieltjes transform $s_n := s_{\mu_{M_n/\sqrt{n}}}$ is "stable in $n$", i.e. $s_n(z) = s_{n-1}(z) + \mathcal{O}(\frac{1}{n})$, where $\mathcal{O}$ can depend on $z$ and even $s_n(z) - \mathbb{E}s_n(z) \to 0$ almost surely.

4. Derive the semicircle law by deriving the recursion $\mathbb{E}s_n(z) = -\frac{1}{z+\mathbb{E}s_n(z)} + o(1)$, where, again, $o(1)$ will depend on $z$ and "inverting" the Stieltjes transform.

**Remark.** *Note that instead of step 4 one could have plugged in the semicircle distribution and simplified the proof by just checking that this is indeed the limit. This is not done here because we want to see how the Stieltjes transform method can be used to derive such a conclusion without knowing about it beforehand.*

*Also, there are other proofs (e.g. Boutet de Monvel and Khorunzhy (2015)) specifically for the GOE/GUE (instead of the more general Wigner matrices) which exploit their symmetries to shorten the proof considerably.*

### A.1.1 Preliminary Reductions

### A.1.2 Stieltjes Transform

**Definition A.3** (Stieltjes transform)**.** *For a probability measure $\mu$ we write $s_\mu$ for its Stieltjes transform*

$$\int_{\mathbb{R}} \frac{1}{x-z} d\mu(x).$$

As mentioned above, $s_n$ will be a shorthand for $s_{\mu_{M_n/\sqrt{n}}}$.

**Lemma A.1.2** (Properties of the Stieltjes transform)**.** *In the following let $\mu$ be some probability measure.*

1. *For $z = a + ib$ we have $Im\frac{1}{x-z} = \frac{b}{(x-a)^2+b^2} > 0$.*
2. *$s_\mu$ is analytic in $\mathbb{C} \setminus supp(\mu) \supset \mathbb{C}_+$.*
3. *We can bound the absolute value as well as the derivatives by $|\frac{d^j}{dz^j} s_\mu(z)| \leq \mathcal{O}(|Im(z)|^{-(j+1)})$ for all $j \in \{0, 1, \dots\}$.*
4. 

insert them here as one uses them

*Proof.* The first property is trivial, the second one can be seen by integrating $s_\mu$ over any contour not containing the support of $\mu$, interchanging the order of integration and noting that integrating $\frac{1}{x-z}$ gives 0 by Cauchy's integral formula ($\frac{1}{x-z}$ being holomorphic outside the support of $\mu$). The Stieltjes transform being holomorphic (and thus analytic) follows by Morera's theorem.

The third property can be obtained by using $\frac{1}{x-z} \leq \frac{1}{Im(z)}$ and using Cauchy's integral formula integrating this inequality. □

**Corollary A.1.2.1.** *From A.1.2.1 it follows that $s_\mu$ is a Herglotz function and thus (e.g. Teschl (2009)) $Im(s_\mu(. + ib)) \rightharpoonup \pi\mu$ as $b \to 0^+$ in the vague topology or equivalently (by $\overline{s_\mu(z)} = s_\mu(\overline{z})$)*

$$\frac{s_\mu(. + ib) - s_\mu(. - ib)}{2\pi i} \rightharpoonup \mu. \tag{A.1}$$

*Note that this can also be seen by writing $Im(s_\mu)$ as the convolution $\pi\mu * P_b(a)$ with the Poisson kernels $P_b(x) := \frac{1}{\pi}\frac{b}{x^2+b^2} = \frac{1}{b}P_1(\frac{x}{b})$ which form a family of approximations to the identity.*

**Theorem A.1.3** (Stieltjes continuity theorem)**.** *For $\mu_n$ random measures and $\mu$ a deterministic measure the following statement holds:*

*$\mu_n \rightharpoonup \mu$ almost surely in the vague topology if and only if $s_{\mu_n}(z) \to s_\mu(z)$ almost surely for every $z \in \mathbb{C}_+$.*

*Proof.*

$$\mathbb{P}(\{\limsup_{n\to\infty} d_v(\mu_n, \mu) = 0\}) = 1$$

$$\mathbb{P}(\{\forall \phi \in C_c(\mathbb{R}) : \lim_{n\to\infty} \int_{\mathbb{R}} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}} \phi \, \mathrm{d}\mu\}) = 1$$

$$\forall z \in \mathbb{C}_+ : \mathbb{P}(\{\lim_{n\to\infty} s_{\mu_n}(z) = s_\mu(z)\}) = 1$$

"$\Rightarrow$": If $\mu_n \rightharpoonup \mu$ in the vague topology almost surely against a deterministic limit $\mu$, then $\forall \phi \in C_c(\mathbb{R}) : \lim_{n\to\infty} \int_{\mathbb{R}} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}} \phi \, \mathrm{d}\mu$ by definition and, by taking the completion, for all bounded, continuous functions vanishing at infinity. The function $x \mapsto \frac{1}{x-z}$ for some $z \in \mathbb{C}$ with $Im(z) > 0$ is bounded and continuous on $\mathbb{R}$ and hence $s_{\mu_n}(z) \to s_\mu(z)$ almost surely.

not sure

"$\Leftarrow$": One can, up to an arbitrary small error $\varepsilon > 0$, approximate $\int_{\mathbb{R}} \phi \, \mathrm{d}\mu$ by $\int_{\mathbb{R}} \phi * P_b \, \mathrm{d}\mu = \frac{1}{\pi} \int_{\mathbb{R}} \phi(a) s_\mu(a+ib) \, \mathrm{d}a$ (and analogously for $\mu_n$). Thus we have $\frac{1}{\pi} \int_{\mathbb{R}} \phi(a)(s_\mu(a+ib) - s_{\mu_n}(a+ib)) \, \mathrm{d}a$ being equal to the difference (we are interested in) $\int_{\mathbb{R}} \phi \, \mathrm{d}\mu - \int_{\mathbb{R}} \phi \, \mathrm{d}\mu_n$ up to an error $\varepsilon$.

In order not to "lose" the almost sure convergence by integration(a summation over uncountable many summands) we approximate it by a Riemann sum (which is possible since we can choose the support $I$ of the test function $\phi$ as the boundaries of integration). The error for the middle sum is proportional to $\max_{x\in I}(|f''(x)|)|I|^3 n^{-2}$, which (for $f$ being the integrand), can be made arbitrarily small. (To be able to control the $\max_{x\in I}(|f''(x)|)$ one may have to approximate the continuous test functions $\phi$ by smooth (or at least twice differentiable) ones like in every partial differential course.)

not sure if that's the reason

This discretized sum now goes to zero almost surely. $\square$

### A.1.3 Stableness and Concentration of Measure

In the following we keep using the notation as defined in A.1.1. To show that $s_n(z) = s_{n-1}(z) + \mathcal{O}_z(1/n)$ we first need to prove the following theorem:

**Theorem A.1.4** (Cauchy's interlacing theorem)**.** *For any $n \times n$ Hermitian matrix $A_n$ with top left minor $A_{n-1}$ we have:*

$$\lambda_i(A_n) \leq \lambda_i(A_{n-1}) \leq \lambda_{i+1}(A_n),$$

*for all $1 \leq i < n$.*

*Proof.* Using the min-max/max-min theorems ($\lambda_i(A) = \inf_{dim(V)=n-i+1} \sup_{v\in V:\|v\|=1} \langle Av, v\rangle$ and $\lambda_i(A) = \sup_{dim(V)=i} \inf_{v\in V:\|v\|=1} \langle Av, v\rangle$ respectively, c.f. Teschl (2009) p.141) and writing $S_{n-i+1}$ for $\{v \in span\{a_i, \ldots, a_n\} : \|v\| = 1\}$, where $A_{n-1}a_j = \lambda_j a_j$ and $P := diag(1, \ldots, 1, 0) \in \mathbb{R}^{n^2}$ we have

$$\lambda_i(A_{n-1}) = \sup_{v\in S_i, \|v\|=1} v^* A_{n-1} v = \sup_{v\in S_i, \|v\|=1} v^* P^* A_n P v \geq \inf_{dim(V)=n-i} \sup_{v\in V, \|v\|=1} v^* A_n v = \lambda_{i+1}(A_n),$$

and

$$\lambda_i(A_n) = \inf_{dim(V)=n-i+1} \sup_{v\in V, \|v\|=1} v^* A_n v \geq \sup_{v\in S_i, \|v\|=1} v^* P^* A_n P v = \sup_{v\in S_i, \|v\|=1} v^* A_{n-1} v = \lambda_i(A_{n-1}),$$

where $S_i := span\{a_i, \ldots, a_{n-1}\}$. $\square$

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