Note on 2D FFT Dedispersion for FRB

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1 Basics of Incoherent Dedispersion

The dispersion of the electromagnetic wave pulse cause a delay in arrival time at frequency ν compared with the reference frequency ν_0, which is given by

$$\Delta t(\nu) = -D(\nu^{-2} - \nu_0^{-2}) \tag{1}$$

where D is the dispersion measure. Thus, we may model a burst with a very short intrinsic width as

$$I(t,\nu) = I_0(\nu)\delta_D(t - t_s - \frac{D}{\nu^2}) \tag{2}$$

where δ_D is the Dirac delta function, t_s marks the signal starting time for infinitely high frequency. If the bandwidth is small, we can approximate

$$\frac{D}{\nu^{-2}} \approx \frac{D}{\nu_0^2}(1 - 2\frac{\nu - \nu_0}{\nu_0})$$

denote $\Delta\nu \equiv \nu - \nu_0$, and assume that the spectrum is not too steep such that within the observing band the signal is constant, then

$$\begin{aligned} I(t,\nu) &\approx I_0\delta_D(t - t_s - \frac{D}{\nu_0^2}(1 - 2\frac{\Delta\nu}{\nu_0})) && (3) \\ &= I_0\delta_D(t - t_0 + \frac{2D}{\nu_0^3}\Delta\nu) && (4) \end{aligned}$$

where t_0 is the arrival time of the signal at the reference frequency ν_0.

Now consider an integral of this signal between frequency ν_1 and ν_2, the signal strength would be

$$s = \int \mathrm{d}\nu \int \mathrm{d}t I(t,\nu) = (\nu_2 - \nu_1)I_0 = I_0 B. \tag{5}$$

where $B = \nu_2 - \nu_1$ is the bandwidth. Now consider the noise. Suppose the data is digitized with time interval δt and frequency channel bandwidth $\delta\nu$. For the incoherent dedispersion, the signal within each time interval and frequency channel is

$$I_n = \frac{2kT_{\rm sys}}{A_{\rm eff}\sqrt{\delta\nu\delta t}} \tag{6}$$

Suppose we are observing between ν_1, ν_2 with a total of N_ν channels, and processing a time interval $T = N_t \delta t$ where $T \geq \Delta t(\nu_1) - \Delta t(\nu_2)$, i.e. the whole of the dispersed signal is within the data frame.

For incoherent dedispersion, in the absence of the pulse signal, the whole read out of the data frame is given by

$$\begin{aligned} n &= \int \mathrm{d}\nu \int \mathrm{d}t \; I_n = \frac{2kT_{\rm sys}}{A_{\rm eff}} \frac{(\nu_2 - \nu_1)T}{\sqrt{\delta\nu\delta t}} & (7) \\ &= \frac{2kT_{\rm sys}}{A_{\rm eff}} B^{1/2} T^{1/2} N_\nu^{1/2} N_t^{1/2} & (8) \end{aligned}$$

So the raw signal to noise ratio is given

$$\mathrm{SNR}_{\rm raw} = \frac{I_0 A_{\rm eff}}{2kT_{\rm sys}} \left(\frac{B}{N_\nu N_t T}\right)^{1/2} \tag{9}$$

In a perfect incoherent dedispersion, we sum up all the signal, which is still given by s. However, we compare it with the noise in the same dedispersion $\nu - t$ track, not the whole data frame. The noise along the same track is given by

$$n = \int \mathrm{d}\nu \int \mathrm{d}t \; I_n \delta_D(t - t_0 + \frac{2D}{\nu_0^3}\Delta\nu) = B I_n \tag{10}$$

then

$$\mathrm{SNR}_{opt} = \frac{I_0}{I_n} = \frac{I_0 A_{\rm eff}}{2kT_{\rm sys}} \left(\frac{BT}{N_\nu N_t}\right)^{1/2} \tag{11}$$

Now consider a pulse of finite width. We replace the Dirac δ function by a Gaussian function with the same normalization

$$\delta_D(t - t') \to g(t - t') \equiv \frac{1}{(2\pi)^{1/2}\sigma} \exp[-\frac{(t - t')^2}{2\sigma_t^2}] \tag{12}$$

If the pulse intrinsic width $\sigma > \delta t$, then in a dedispersion along the track only the part of the signal within one time bin would be included, which gives

$$\int_{-\delta t}^{+\delta t} \mathrm{d}\Delta t \frac{1}{\sqrt{2\pi}\sigma} e^{-\Delta t^2/2\sigma^2} = \mathrm{erf}(\frac{\delta t}{\sqrt{2}\sigma}) \approx \sqrt{\frac{2}{\pi}} \frac{\delta t}{\sigma} \tag{13}$$

where the last holds for the case $\delta t \ll \sigma$, so in this case

$$s = I_0 B \sqrt{\frac{2}{\pi}} \frac{\delta t}{\sigma} \tag{14}$$

while the noise is still given by Eq.(10), so in this case

$$\mathrm{SNR}_{fin} = \frac{I_0}{I_n} = \frac{I_0 A_{\mathrm{eff}}}{2kT_{\mathrm{sys}}} \left(\frac{BT}{N_\nu N_t}\right)^{1/2} \sqrt{\frac{2}{\pi}} \frac{\delta t}{\sigma} \tag{15}$$

2 2D FFT dedispersion

The usual Fourier transform is:

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int f(t) e^{-i\omega t} \mathrm{d}t, \qquad f(t) = \int \tilde{f}(\omega) e^{i\omega t} \mathrm{d}\omega \tag{16}$$

For $f(t) = \delta_D(t - t_0)$, $\tilde{f}(\omega) = \frac{1}{2\pi} e^{-i\omega t_0}$. Using the relation

$$\int d\omega e^{i\omega t_0} = 2\pi \delta_D(t_0) \tag{17}$$

we find the above indeed form a Fourier pair. However, here we want to use ν instead of ω, then the Fourier transform pair are

$$\tilde{f}(\nu) = \frac{1}{2\pi} \int f(t) e^{-i2\pi\nu t} \mathrm{d}t, \qquad f(t) = 2\pi \int \tilde{f}(\nu) e^{i2\pi\nu t} \mathrm{d}\nu \tag{18}$$

The 2d transform of the signal $I(\nu, t)$ is

$$\tilde{I}(f, \tau) = \int \mathrm{d}\nu \; e^{2\pi i\nu\tau} \int \mathrm{d}t \; e^{-2\pi i f t} \; I(t, \nu) \tag{19}$$

where we denote the Fourier conjugate variable of ν, t as τ, f to avoid confusion. For the pulse signal given by Eq.(2),

$$\begin{aligned} \tilde{I}(f, \tau) &= \int \mathrm{d}\nu \;\; e^{2\pi i\nu\tau} I_0 e^{-2\pi i f (t_0 - \frac{2D}{\nu_0^3} \Delta\nu)} & (20) \\ &= I_0 e^{-i2\pi f (t_0 + \frac{2D}{\nu_0^2})} \int \mathrm{d}\nu \; e^{i2\pi\nu(\tau + \frac{2Df}{\nu_0^2})} & (21) \\ &= \frac{I_0}{2\pi} \exp[-i2\pi f (t_0 + \frac{2D}{\nu_0^2})] \delta_D(\tau + \frac{2Df}{\nu_0^2}) & (22) \end{aligned}$$

Note $\tilde{I}(\tau, f)$ is non-zero only on the straight line $\tau + \frac{2Df}{\nu_0^2} = 0$, and the value is a complex number whose phase angle gives the arrival time. For the pulse with finite width,

$$\tilde{I}(f,\tau) = \int \mathrm{d}\nu \; e^{2\pi i\nu\tau} \int \mathrm{d}t \; e^{-2\pi i f t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-t')^2}{2\sigma^2}} \tag{23}$$

where $t' = t_0 + \frac{2D}{\nu_0^3}\Delta\nu$. Complete the t integral, we get

$$\begin{aligned}
\tilde{I}(f,\tau) &= \int \mathrm{d}\nu \; e^{2\pi i\nu\tau} I_0 e^{-i2\pi f t'} e^{-\frac{(2\pi f\sigma)^2}{2}} && (24)\\
&= I_0 e^{-i2\pi f(t_0 - \frac{2D}{\nu_0^2})} e^{-\frac{(2\pi f\sigma)^2}{2}} \int \mathrm{d}\nu \exp[i2\pi\nu(\tau + \frac{2Df}{\nu_0^2})] && (25)\\
&= \frac{I_0}{2\pi} e^{-i2\pi f(t_0 - \frac{2D}{\nu_0^2})} e^{-\frac{(2\pi f\sigma)^2}{2}} \delta_D(\tau + \frac{2Df}{\nu_0^2}) && (26)
\end{aligned}$$

Note this is similar to Eq(22) except for the factor $e^{-\frac{(2\pi f\sigma)^2}{2}}$, this limits the usable range of f to$|f| < (2\pi\sigma)^{-1}$.

3 Transform to polar coordinates

We can take $\frac{2f}{\nu_0^2}, \tau$ as the x, y in Cartesian coordinates, then the polar coordinates ρ, θ can be defined as

$$\begin{aligned}
\rho^2 &= \left(\frac{2f}{\nu_0^2}\right)^2 + \tau^2 && (27)\\
\tan\theta &= \frac{\tau}{2f/\nu_0^2} && (28)
\end{aligned}$$

with $\tan\theta = -D$ for the track satisfy Eq. 26. Conversely,

$$\begin{aligned}
f &= \frac{\nu_0^2}{2}\rho\cos\theta && (29)\\
\tau &= \rho\sin\theta && (30)
\end{aligned}$$

Then

$$\tilde{I}(\rho,\theta) = \frac{I_0}{2\pi} e^{-i2\pi(\frac{\nu_0^2 t}{2} - D)\rho\cos\theta} \; e^{-\frac{\pi^2\sigma^2\nu_0^4}{2}\rho^2\cos^2\theta} \rho^{-1} \delta_D(\theta + \arctan D) \tag{31}$$

4 Searching in polar coordinate

Here I haven't think of some analytical method, one needs to try by simulation. This should be studied systematically to gain some intuition on how to choose the step in θ.