1 A brief description of the MATHEMATICA codes

The programs listed are written in such a manner that the relation to the formulas in the manuscript (Axial and polar modes for the ring down of a Schwarzschild black hole with an r dependent mass-function) should be obvious.
Some definitions:

- r: radial distance
- $y = \frac{r}{m_0}$: Dimensionless radial distance
- m_0: mass parameter
- ω (or σ): Frequency, mode
- $\widetilde{\omega} = m_0\omega$: Dimensionless frequency (mode)

2 Eq.(23),(24) and Appendix B, Zerilli-ansatz for Z.nb

This derives the equations (23) and (24) in the manuscript, i.e., it parts from the ansatz for the Z-function, polar modes, and gives the solutions for the functions $\alpha(r)$ and $\beta(r)$.

This program applies the derivative $\frac{d^2}{dy_*^2}$ to an ansatz of $Z^{(+)}$, given by ($y = \frac{r}{m_0}$)

$$Z^{(+)} = \alpha(y)N(y) + \beta(y)V(y) + \gamma(y)(L(y) + X(y)) \tag{1}$$

in terms of the functions $N(y)$, $V(y)$ and $X(y)$, see book of Chandrasekhar and manuscript. It turns out that $\alpha(y) = 0$ and $\beta(y) = y$, see mansucript and further below, such that the Z-function is only a combination of $V(r)$ and $LX = (L(y) + X(y))$.

As a result one obtains that

$$\left(\frac{d^2}{dy^2} + \tilde{\omega}^2\right) Z^{(+)} = V_V^+(y)V(y) + V_{LX}^+(L(y) + X(y)) \quad . \tag{2}$$

The tilde refers to $\tilde{\omega} = m_0\omega$.

More details:
We define

$$LX(r) \quad = \quad L(r) + X(r) \quad . \tag{3}$$

The second derivative of r_* is applied to $Z^{(+)}$ and the factors of $N(r)$, $V(r)$ and $LX(r)$ are determined. In a further step, the routine sets all contributions to zero, except of $\tilde{\omega}^2$ (called in the program σ^2), i.e., the routine selects the term of ω^2. The required result is

$$-\sigma^2 \left(\alpha(r)N(r) + \beta(r)V(r) + \gamma(r)LX(r)\right) \quad . \tag{4}$$

As it turns out, the factor of $N(r)$ does not depend on σ and, thus, by selecting the contribution of σ^2, Eq. (4) is satisfied when

$$\alpha(r) \quad = \quad 0 \quad . \tag{5}$$

Concentrating on the $V(r)$ part, leads to the equations and the result for $\alpha(r)$:

$$n\alpha(r) - \beta(r) \quad = \quad -\beta(r) \;\rightarrow\; \alpha(r) = 0 \quad . \tag{6}$$

Concentrating on the factor of $LX(r)$ and using $\alpha(r) = 0$, leads for $\beta(r)$ to

$$-2\beta(r) + n\gamma(r) + 2r\beta'(r) \quad = \quad n\gamma(r) \;\rightarrow\; \beta'(r) \quad = \quad \frac{1}{r}\beta(r) \quad . \tag{7}$$

the solution of this differential equation is

$$\beta(r) \quad = \quad r \quad . \tag{8}$$

Finally, the factor of $N(r)$, still depending on $\alpha = 0$, $\beta = r$ and the unknown funtion $\gamma(r)$, is set to zero, which leads to the $\gamma(r)$ as given in Eq. (24) of the manuscript.

3 Dif-Equation for gam(r).nb

This routine sets up the differential equation for the $\gamma(r)$ function, equating the functions $rV(r)$ and $\gamma(r)(L(r)+X(r))$.

4 gam(r)-limit GR.nb

Sets the mass-function to a constant and uses the ansatz form Chandrasekhars book of $\gamma(r)$ into the differential equation (27). The result is zero, showing that the limit of GR is satisfied.

5 Figure.6-left hand side.nb and Figure.6-right hand side.nb

Figure.6-left hand side.nb: Plot of the two potentials $V_V^{(+)}$ and $V_{LX}^{(+)}$
Figure.6-right hand side.nb: plots the difference $\left(V_V^{(+)} - V_{LX}^{(+)}\right)$.

All the following programs use implicitly the approximation of $\gamma(r)$!

6 Set up of the Zerilli (polar modes) and Regge-Wheeler (axial modes) equation

The list of programs are:

- **Zerilli-equation-pcGR-axial modes+figures.nb**: Constructs the Regge-Wheeler equation for the axial modes in pc-GR. It also determines the functions λ_0 and s_0 as used in the AIM. At the end, this routine contains some figures of the main text and of the Appendix.
- **Zerilli-QNM-pcGR-axial modes+figures.nb**: Determines the axial modes via the AIM formalism (see the manuscript). At the end, this routine contains some figures of the main text and of the Appendix.

- **Zerilli-equation-pcGR-polar modes+figures.nb**: Constructs the Regge-Wheeler equation for the axial modes in pc-GR. It also determines the functions λ_0 and s_0 as used in the AIM. At the end, this routine contains some figures of the main text and of the Appendix.

- **Zerilli-QNM-pcGR-polar modes+figures.nb**: Determines the polar modes via the AIM formalism (see the manuscript). At the end, this routine contains some figures of the main text and of the Appendix.

- **Zerilli-QNM-GR, axial=polar modes+figures.nb**: Determines the axial and polar modes via the AIM formalism (see the manuscript), within the GR. The coefficient functions λ_0 and s_0 are taken from Cho-2012 (see reference list in the manuscript). At the end, this routine contains some figures of the main text and of the Appendix.

7 Eq. (29) and Figure1, V(r) and VLX(r).nb

This program constructs the potentials $V_V^{(+)}(r)$ and $V_{LX}^{(+)}(r)$, using the ansatz for the Zerilli function with coeffcients determined in "... Zerilli-ansatz for Z.nb" (see Section 2). It compares both functions.

8 Figure1 GR-potential.nb

Constructs the GR-potential. i.e., for a constant mass.

9 Eq.(50),Tortoise coordinate.nb

It determines the analytic solution of the integral (48) in the manuscript for GR and the special ansatz for the mass-function in pc-GR.

The particular mass-function, used in pc-GR is:

$$m(r) = 1 - \frac{27}{32y^3} \quad , \tag{9}$$

This case permits an analytic solution!

10 Eq.(33),(34),EventHorizon-n=4-a=0.nb

Solves the equation (33) in the event horizon, as a function in the parameter value b. The physical solution is finally given in Eq. (34) of the manuscript.

11 plots-300-400-500-polar.nb

This is an example on how to use the calculated list of gravitational modes and create plots, without recurring to repeat the calculations. The example shown is for the polar modes of up to 500 iterations.

The method is to copy the list of modes, naming them in this example as R300, R400 and R500. then make the plots. At the end, all plots are joined.