

# 1 Correctness and Performance Charts

This version of the document is dated 2023-06-13.

The following charts show the correctness of many of the algorithms in “**Bernoulli Factory Algorithms**” and show their performance in terms of the number of bits they use on average. For each algorithm, and for each of 100  $\lambda$  values evenly spaced from 0.0001 to 0.9999:

- 500 runs of the algorithm were done. Then...
- The number of bits used by the runs were averaged, as were the return values of the runs (since the return value is either 0 or 1, the mean return value will be in the interval  $[0, 1]$ ). The number of bits used included the number of bits used to produce each coin flip, assuming the coin flip procedure for  $\lambda$  was generated using the `Bernoulli#coin()` method in *bernoulli.py*, which produces that probability in an optimal or near-optimal way.

For each algorithm, if a single run was detected to use more than 5000 bits for a given  $\lambda$ , the entire data point for that  $\lambda$  was suppressed in the charts below.

In addition, for each algorithm, a chart appears showing the minimum number of input coin flips that any fast Bernoulli factory algorithm will need on average to simulate the given function, based on work by Mendo (2019)[<sup>1</sup>]. Note that some functions require a growing number of coin flips as  $\lambda$  approaches 0 or 1. Note that for the 2014, 2016, and 2019 algorithms—

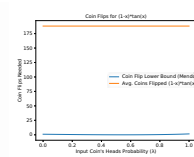
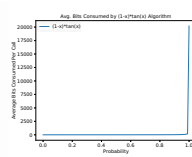
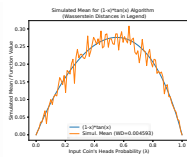
- an  $\epsilon$  of  $1 - (x + c) * 1.001$  was used (or 0.0001 if  $\epsilon$  would be greater than 1), and
- an  $\epsilon$  of  $(x - c) * 0.9995$  for the subtraction variants.

Points with invalid  $\epsilon$  values were suppressed. For the low-mean algorithm, an  $m$  of  $\max(0.49999, xc1.02)$  was used unless noted otherwise.

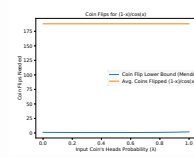
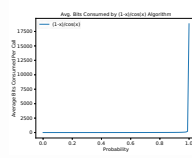
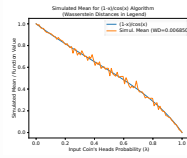
## 1.1 The Charts

| Algorithm | Simulated Mean | Average Bits Consumed | Coin Flips |
|-----------|----------------|-----------------------|------------|
|-----------|----------------|-----------------------|------------|

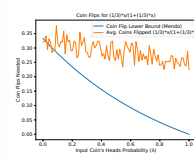
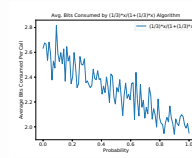
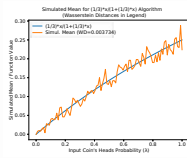
$$(1-x)*\tan(x)$$



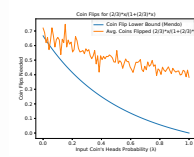
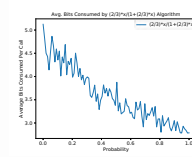
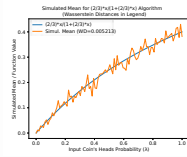
$$(1-x)/\cos(x)$$



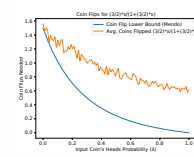
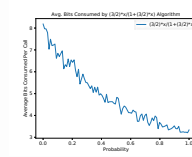
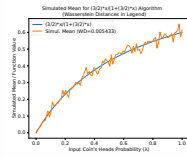
$$(1/3)*x/(1+(1/3)*x)$$



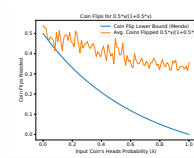
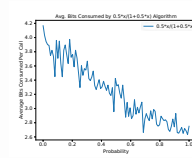
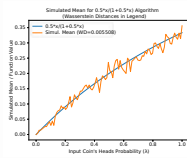
$$(2/3)*x/(1+(2/3)*x)$$



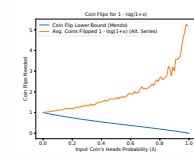
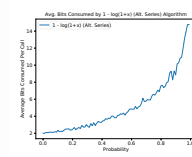
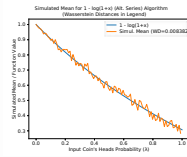
$$(3/2)*x/(1+(3/2)*x)$$



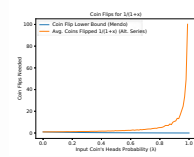
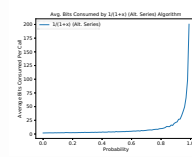
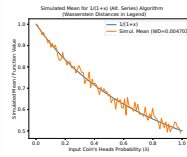
$$0.5*x/(1+0.5*x)$$



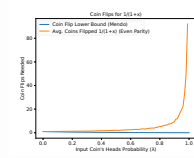
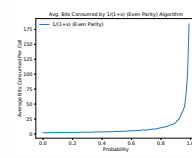
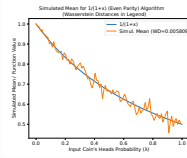
$$1 - \ln(1+x) \text{ (Alt. Series)}$$



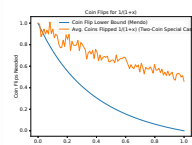
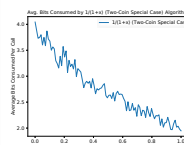
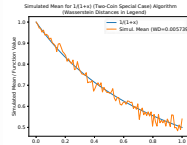
$$1/(1+x) \text{ (Alt. Series)}$$



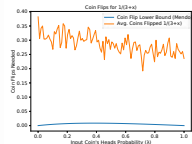
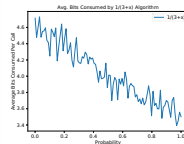
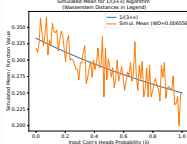
$$1/(1+x) \text{ (Even Parity)}$$



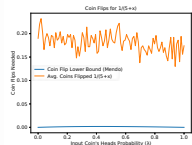
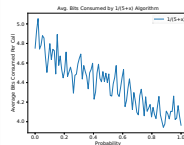
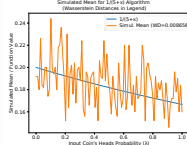
1/(1+x) (Two-Coin Special Case)



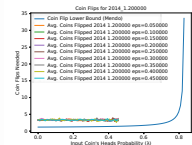
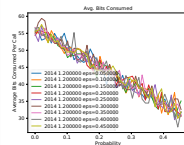
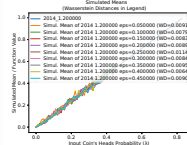
1/(3+x)



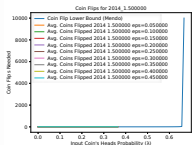
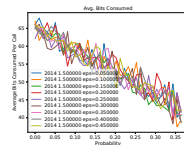
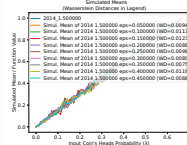
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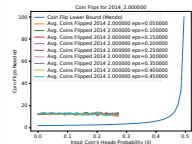
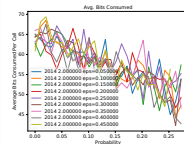
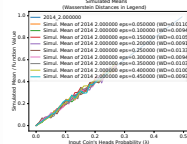
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eps=0.050000



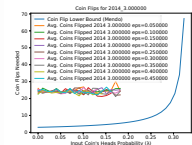
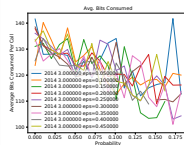
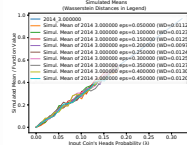
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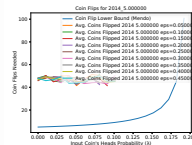
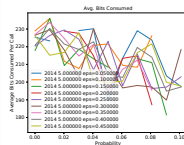
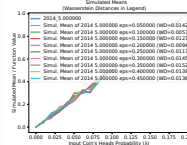
2014 2.000000  
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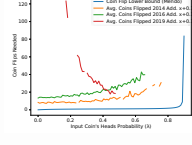
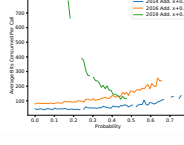
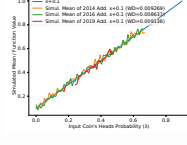
2014 3.000000  
eps=0.050000



2014 5.000000  
eps=0.050000



2014 Add. x+0.1



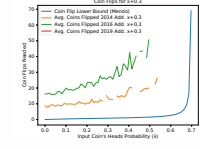
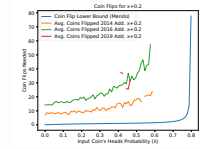
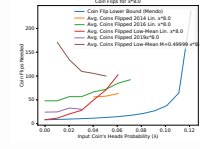
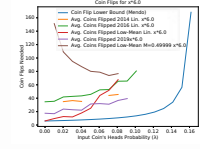
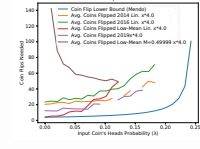
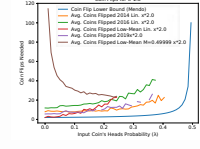
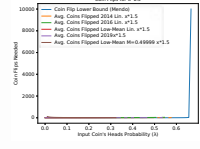
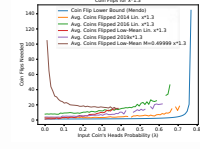
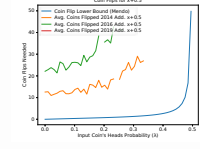


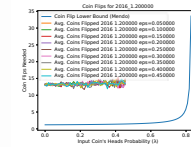
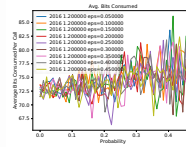
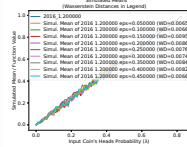
Figure 1 is a line graph titled "Simulated Mean Function Value" on the y-axis and "Input Coin's Heads Probability (x)" on the x-axis. The x-axis ranges from 0.0 to 0.5, and the y-axis ranges from 0.5 to 1.0. There are four data series plotted:

- x=0.5**: A solid blue line representing the theoretical function  $f(x) = 2x$ , which is a straight line from (0, 0) to (0.5, 1.0).
- Simul. Mean of 2014 Add. x=0.5 (MD=0.01525)**: An orange line representing the simulated mean function value for the 2014 dataset. It follows the blue line closely with small fluctuations.
- Simul. Mean of 2016 Add. x=0.5 (MD=0.01163)**: A green line representing the simulated mean function value for the 2016 dataset. It also follows the blue line closely.
- Simul. Mean of 2019 Add. x=0.5 (MD=nan)**: A red line representing the simulated mean function value for the 2019 dataset. It shows much larger fluctuations and deviates significantly from the blue line, especially in the middle range of x.

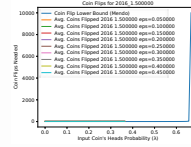
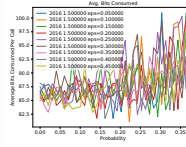
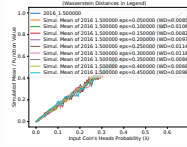
The legend at the top of the graph identifies these four series.



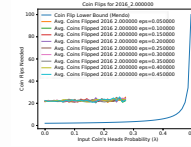
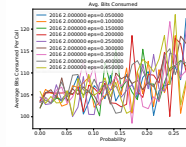
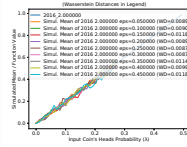
2016 1.200000  
eps=0.050000



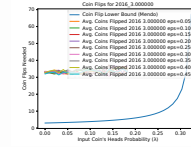
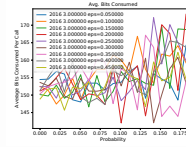
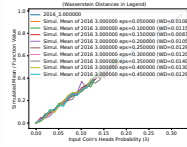
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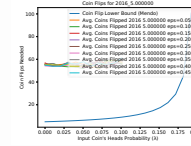
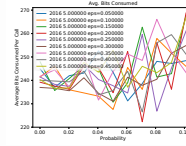
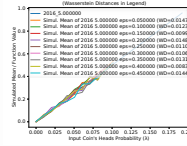
2016 2.000000  
eps=0.050000



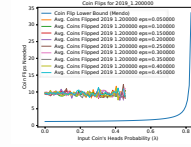
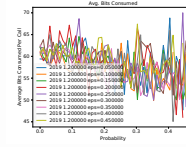
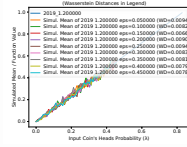
2016 3.000000  
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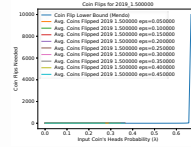
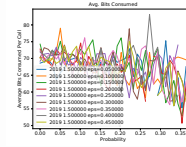
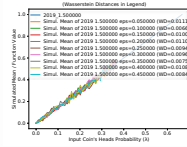
2016 5.000000  
eps=0.050000



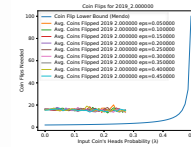
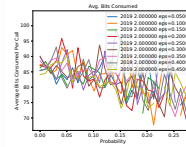
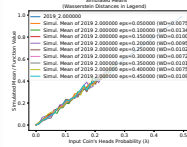
2019 1.200000  
eps=0.050000



2019 1.500000  
eps=0.050000

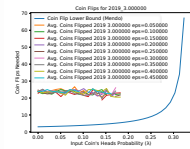


2019 2.000000  
eps=0.050000

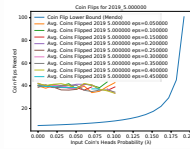


2019 3.000000

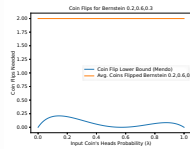
$\epsilon=0.050000$



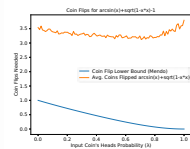
2019 5.000000  
 $\epsilon=0.050000$



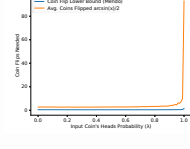
Bernstein  
0.2,0.6,0.3



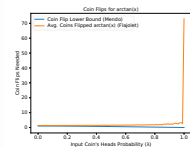
$\arcsin(x)+\sqrt{1-x^2}-1$



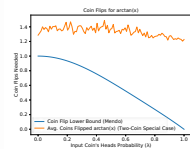
$\arcsin(x)/2$



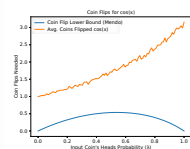
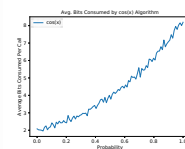
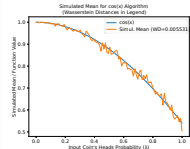
$\arctan(x)$   
(Flajolet)



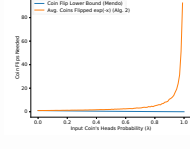
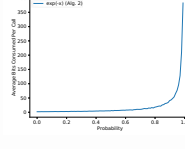
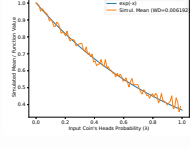
$\arctan(x)$  (Two-Coin Special Case)



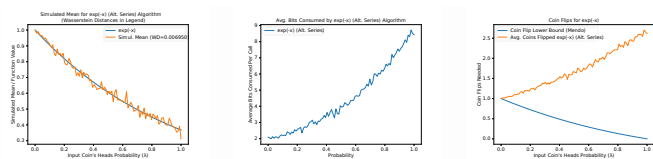
$\cos(x)$



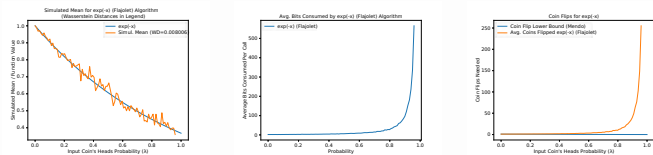
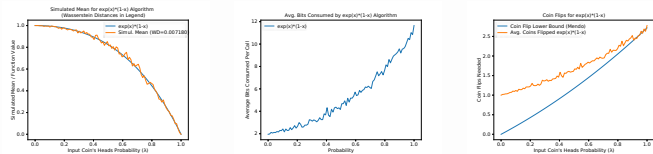
$\exp(-x)$  (Alg. 2)



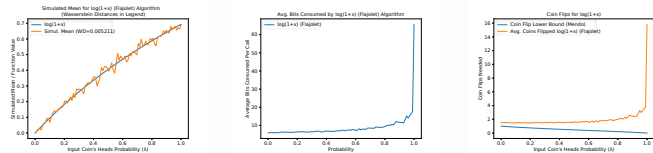
exp(-x) (Alt.  
Series)



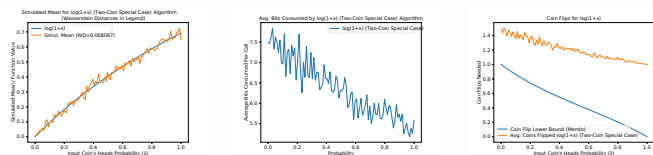
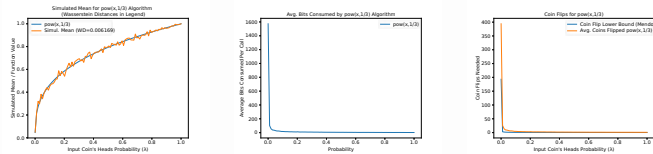
exp(-x) (Flajolet)


$$\exp(x) \cdot (1-x)$$


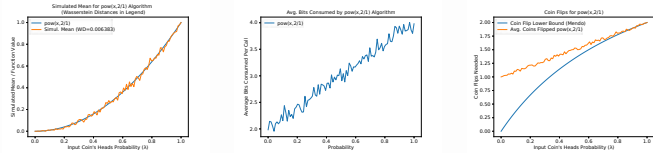
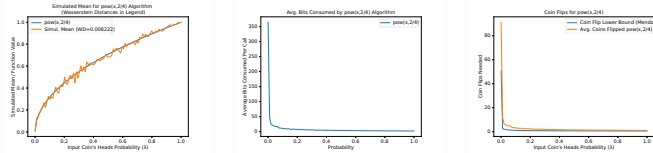
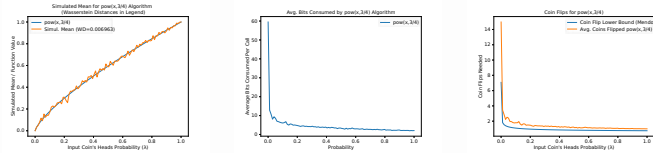
$\ln(1+x)$  (Flajolet)



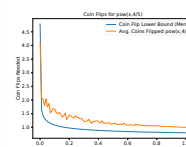
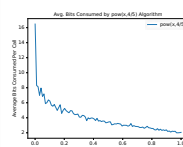
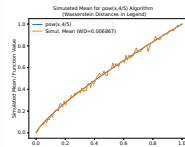
### ln(1+x) (Two-Coin Special Case)

 $\text{pow}(x, 1/3)$ 

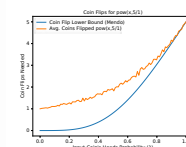
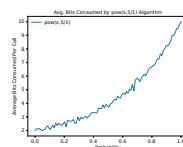
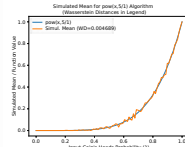
pow(x,2/1)

 $\text{pow}(x, 2/4)$  $\text{pow}(x, 3/4)$ 

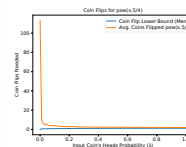
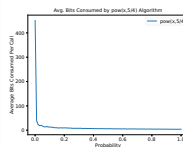
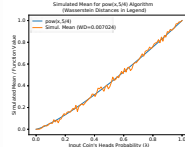
$\text{pow}(x, 4/5)$



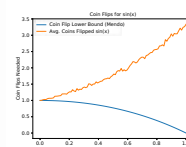
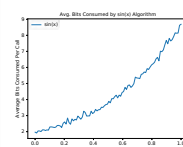
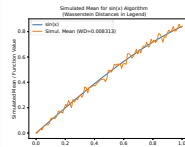
$\text{pow}(x, 5/1)$



$\text{pow}(x, 5/4)$



$\sin(x)$



$\text{sqrt}(x)$

