

1 Correctness and Performance Charts

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The following charts show the correctness of many of the algorithms in “[Bernoulli Factory Algorithms¹](#)” and show their performance in terms of the number of bits they use on average. For each algorithm, and for each of 100 λ values evenly spaced from 0.0001 to 0.9999:

- 500 runs of the algorithm were done. Then...
- The number of bits used by the runs were averaged, as were the return values of the runs (since the return value is either 0 or 1, the mean return value will be in the interval $[0, 1]$). The number of bits used included the number of bits used to produce each coin flip, assuming the coin flip procedure for λ was generated using the `Bernoulli#coin()` method in *bernoulli.py*, which produces that probability in an optimal or near-optimal way.

For each algorithm, if a single run was detected to use more than 5000 bits for a given λ , the entire data point for that λ was suppressed in the charts below.

In addition, for each algorithm, a chart appears showing the minimum number of input coin flips that any fast Bernoulli factory algorithm will need on average to simulate the given function, based on work by Mendo (2019)[¹]. Note that some functions require a growing number of coin flips as λ approaches 0 or 1. Note that for the 2014, 2016, and 2019 algorithms—

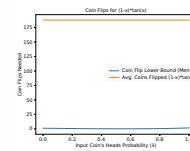
- an ϵ of $1 - (x + c) * 1.001$ was used (or 0.0001 if ϵ would be greater than 1), and
- an ϵ of $(x - c) * 0.9995$ for the subtraction variants.

Points with invalid ϵ values were suppressed. For the low-mean algorithm, an m of $\max(0.49999, xc1.02)$ was used unless noted otherwise.

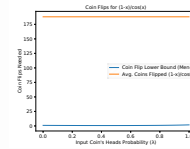
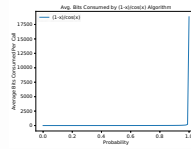
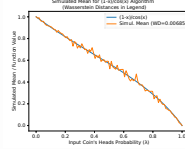
1.1 The Charts

| Algorithm | Simulated Mean | Average Bits Consumed | Coin Flips |
|-----------|----------------|-----------------------|------------|
|-----------|----------------|-----------------------|------------|

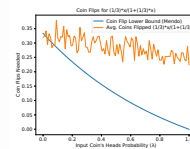
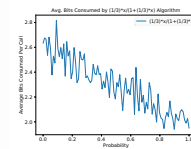
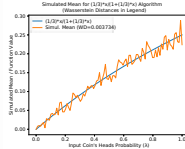
$$(1-x)*\tan(x)$$



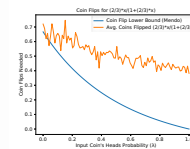
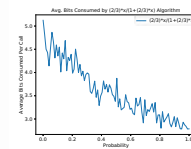
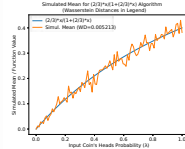
$$(1-x)/\cos(x)$$



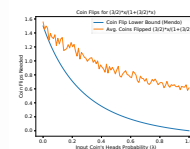
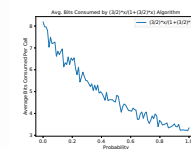
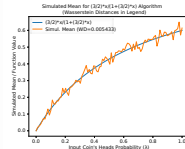
$$(1/3)*x/(1+(1/3)*x)$$



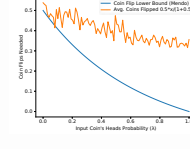
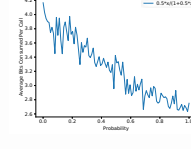
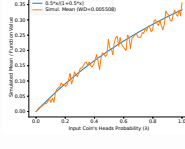
$$(2/3)*x/(1+(2/3)*x)$$



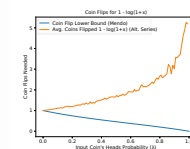
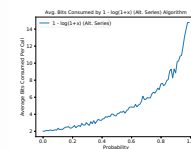
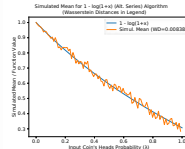
$$(3/2)*x/(1+(3/2)*x)$$



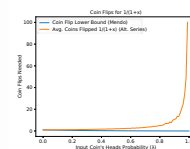
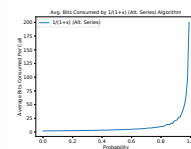
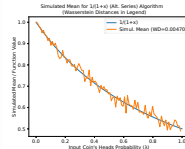
$$0.5*x/(1+0.5*x)$$



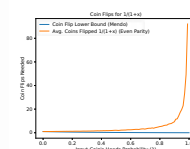
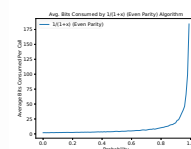
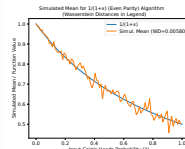
$$1 - \ln(1+x) \text{ (Alt. Series)}$$



$$1/(1+x) \text{ (Alt. Series)}$$



$$1/(1+x) \text{ (Even Parity)}$$



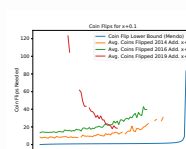
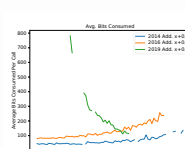
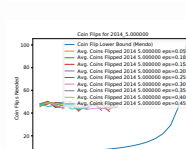
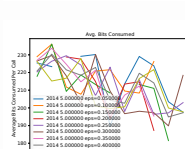
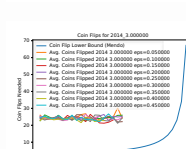
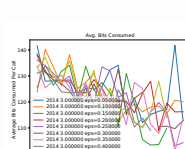
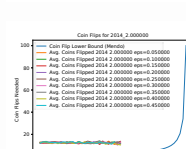
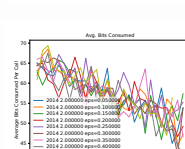
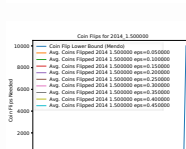
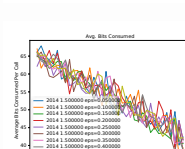
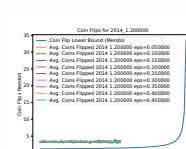
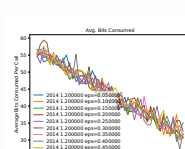
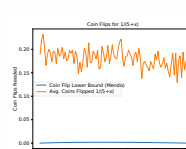
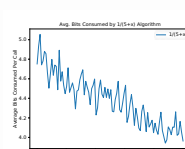
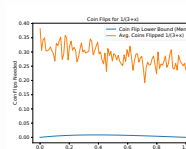
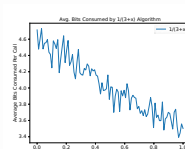
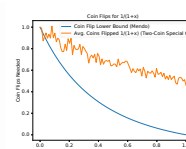
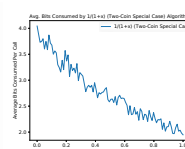


Figure 1 consists of three subplots. The left subplot shows the 'Displacement Rate / Critical Displacement' on the y-axis (ranging from 0.0 to 1.0) versus the 'Avg. Displacement / Average Displacement' on the x-axis (ranging from 0.0 to 1.0). It includes a blue line for the theoretical solution and several data series for different mesh sizes: $h=0.2$ (green), $h=0.1$ (orange), $h=0.05$ (red), $h=0.025$ (purple), and $h=0.0125$ (brown). The middle subplot shows the 'Average Displacement Error' on the y-axis (ranging from 0 to 10) versus the same x-axis. It includes a blue line for the theoretical solution and several data series for different mesh sizes: $h=0.2$ (green), $h=0.1$ (orange), $h=0.05$ (red), $h=0.025$ (purple), and $h=0.0125$ (brown). The right subplot shows the 'CPU Time (s)' on the y-axis (ranging from 0 to 80) versus the same x-axis. It includes a blue line for the theoretical solution and several data series for different mesh sizes: $h=0.2$ (green), $h=0.1$ (orange), $h=0.05$ (red), $h=0.025$ (purple), and $h=0.0125$ (brown). All plots show a sharp increase in error and CPU time as the displacement ratio approaches 1.0.

[illegible]

Figure 1 consists of three subplots illustrating the performance of the proposed algorithm across different input data correlations (0.0 to 0.5).

Left Subplot: Size of the Feature Space (log scale)

This plot shows the size of the feature space (log scale) versus input data correlation. The proposed algorithm (blue line) maintains a low feature space size, while the baseline methods (orange, green, red lines) show a significant increase in feature space size as correlation increases. The black line represents the theoretical bound.

Middle Subplot: Average Bits Component

This plot shows the average bits component versus input data correlation. The proposed algorithm (blue line) maintains a low average bits component, while the baseline methods (orange, green, red lines) show a significant increase in average bits component as correlation increases. The black line represents the theoretical bound.

Right Subplot: Gain Ratio for $n=5$

This plot shows the gain ratio for $n=5$ versus input data correlation. The proposed algorithm (blue line) maintains a high gain ratio, while the baseline methods (orange, green, red lines) show a significant decrease in gain ratio as correlation increases. The black line represents the theoretical bound.

[illegible][illegible][illegible][illegible]

Figure 1 consists of two subplots, (a) and (b), showing the performance of the proposed algorithm across different dimensionless frequencies (ω).

Subplot (a) is titled "Structural Errors (Dimensionless Frequency ω (rad))". The x-axis ranges from 0.0 to 0.4, and the y-axis is "Structural Errors (log scale)" ranging from 10^{-1} to 10^1 . The legend includes:

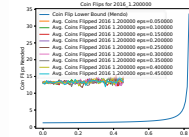
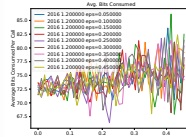
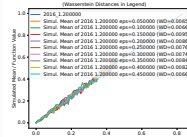
- $\omega = 0$ (black line)
- $\omega = 0.05$ (red line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.05) (blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.1) (green line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.15) (purple line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.2) (orange line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.3) (brown line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.4) (pink line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.5) (light blue line)

Subplot (b) is titled "Avg. Bits Consumed (Dimensionless Frequency ω (rad))". The x-axis ranges from 0.0 to 0.4, and the y-axis is "Average Bits Consumed per Cell" ranging from 0 to 160. The legend includes:

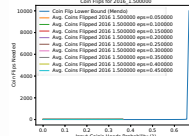
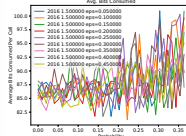
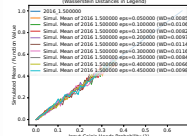
- 2014 Lin. w/ $\omega = 0$ (black line)
- 2014 Lin. w/ $\omega = 0.05$ (red line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.05) (blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.1) (green line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.15) (purple line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.2) (orange line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.3) (brown line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.4) (pink line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.5) (light blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.6) (dark blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.7) (dark purple line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.8) (dark green line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-0.9) (dark brown line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.0) (dark pink line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.1) (dark light blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.2) (dark purple line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.3) (dark green line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.4) (dark brown line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.5) (dark pink line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-1.6) (dark light blue line)
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- 2014 Lin. w/ $\omega = 0.05$ (0.05-2.4) (dark brown line)
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- 2014 Lin. w/ $\omega = 0.05$ (0.05-5.9) (dark brown line)
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- 2014 Lin. w/ $\omega = 0.05$ (0.05-7.8) (dark green line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-7.9) (dark brown line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-8.0) (dark pink line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-8.1) (dark light blue line)
- 2014 Lin. w/ $\omega = 0.05$ (0.05-8.2) (dark purple line)
- 2014 Lin

Figure 10 consists of three line plots showing the performance of different models on the CIFAR-100 dataset. The left plot shows the 'Distorted Mean (Covariance Matrix is diagonal)' on the y-axis (ranging from 0 to 1) versus the 'Distorted Mean (Covariance Matrix)' on the x-axis (ranging from 0 to 1). The middle plot shows the 'Avg. City Block' on the y-axis (ranging from 0 to 100) versus the 'Avg. City Block' on the x-axis (ranging from 0 to 1). The right plot shows the 'City-Block for P+L' on the y-axis (ranging from 0 to 200) versus the 'City-Block' on the x-axis (ranging from 0 to 1). The legend for all plots includes: P+L (blue), P+L+D (orange), Distorted Mean of 2014 Lin (red), Distorted Mean of 2014 Lin + P+L (green), Distorted Mean of 2014 Lin + P+L + D (purple), Distorted Mean of 2014 Lin + P+L + D + D (brown), and Distorted Mean of City-Block ResNet (grey).

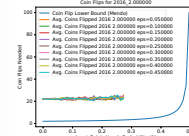
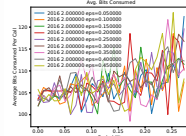
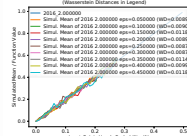
2016 1.200000
eps=0.050000



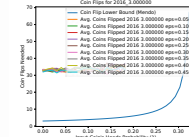
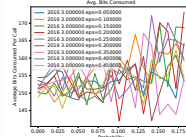
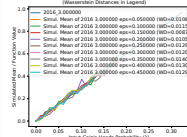
2016 1.500000
eps=0.050000



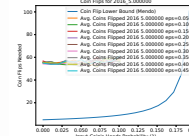
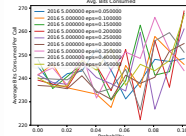
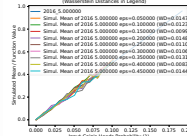
2016 2.000000
eps=0.050000



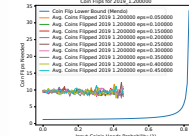
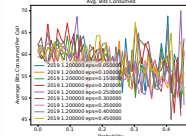
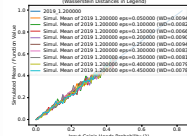
2016 3.000000
eps=0.050000



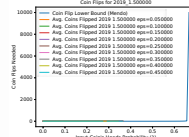
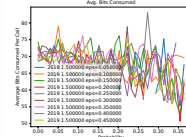
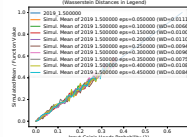
2016 5.000000
eps=0.050000



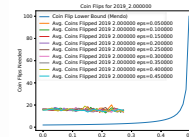
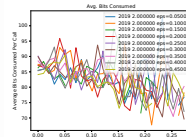
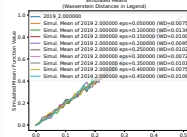
2019 1.200000
eps=0.050000



2019 1.500000
eps=0.050000

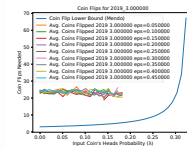


2019 2.000000
eps=0.050000

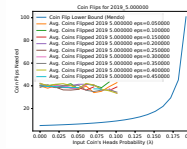


2019 3.000000

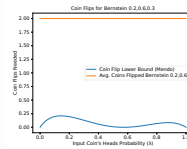
$\epsilon=0.050000$



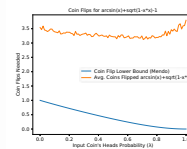
2019 5.000000
 $\epsilon=0.050000$



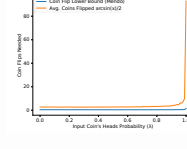
Bernstein
0.2,0.6,0.3



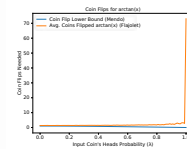
$\arcsin(x)+\sqrt{1-x^2}-1$



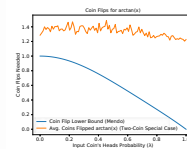
$\arcsin(x)/2$



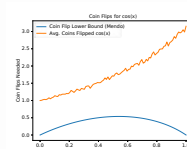
$\arctan(x)$
(Flajolet)



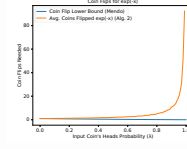
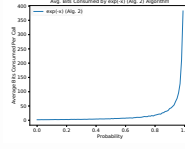
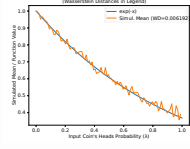
$\arctan(x)$ (Two-Coin Special Case)



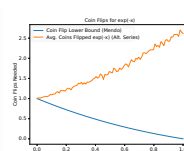
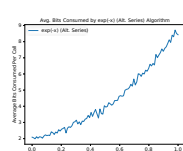
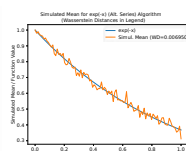
$\cos(x)$



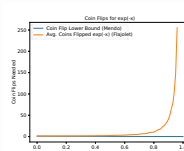
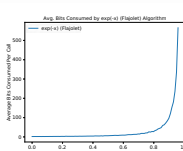
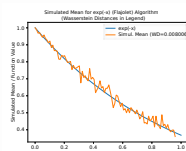
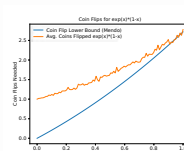
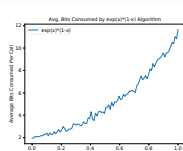
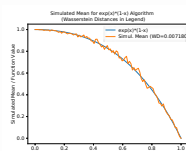
$\exp(-x)$ (Alg. 2)



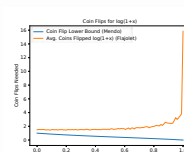
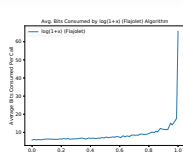
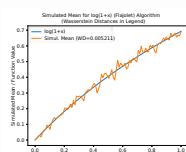
exp(-x) (Alt.
Series)



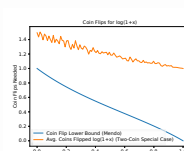
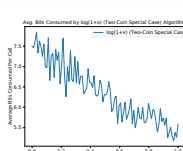
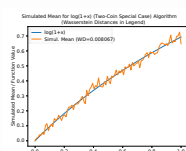
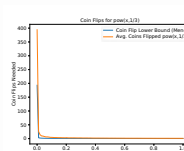
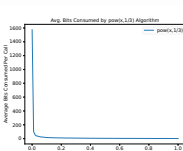
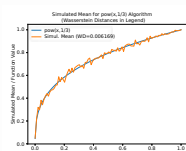
exp(-x) (Flajolet)


$$\exp(x) \cdot (1-x)$$


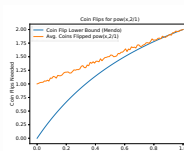
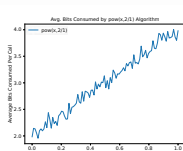
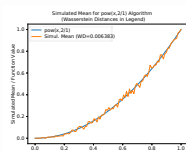
$\ln(1+x)$ (Flajolet)



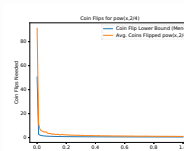
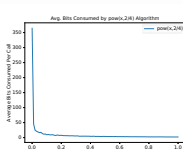
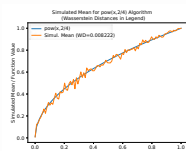
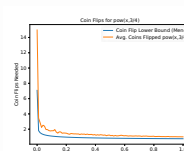
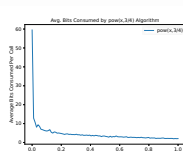
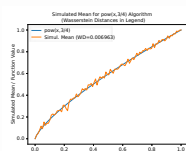
$\ln(1+x)$ (Two-Coin Special Case)

 $\text{pow}(x, 1/3)$ 

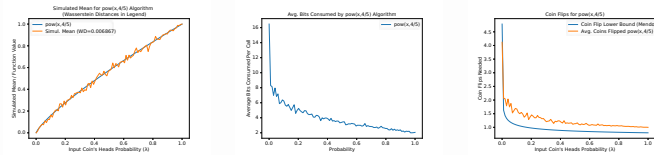
pow(x,2/1)



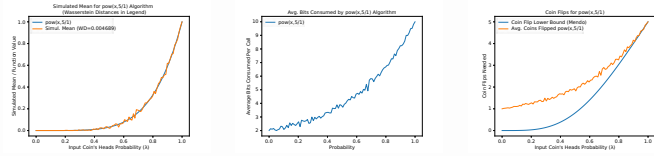
pow(x,2/4)

 $\text{pow}(x, 3/4)$ 

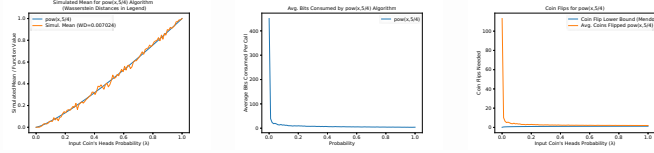
$\text{pow}(x, 4/5)$



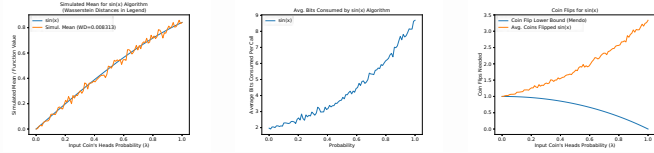
$\text{pow}(x, 5/1)$



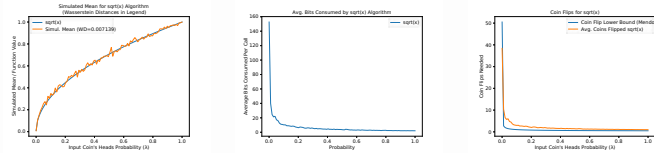
$\text{pow}(x, 5/4)$



$\sin(x)$



$\text{sqrt}(x)$



1. <https://peteroupc.github.io/bernoulli.md>