

1 Correctness and Performance Charts

This version of the document is dated 2023-06-13.

The following charts show the correctness of many of the algorithms in “[Bernoulli Factory Algorithms¹](#)” and show their performance in terms of the number of bits they use on average. For each algorithm, and for each of 100 λ values evenly spaced from 0.0001 to 0.9999:

- 500 runs of the algorithm were done. Then...
- The number of bits used by the runs were averaged, as were the return values of the runs (since the return value is either 0 or 1, the mean return value will be in the interval $[0, 1]$). The number of bits used included the number of bits used to produce each coin flip, assuming the coin flip procedure for λ was generated using the `Bernoulli#coin()` method in *bernoulli.py*, which produces that probability in an optimal or near-optimal way.

For each algorithm, if a single run was detected to use more than 5000 bits for a given λ , the entire data point for that λ was suppressed in the charts below.

In addition, for each algorithm, a chart appears showing the minimum number of input coin flips that any fast Bernoulli factory algorithm will need on average to simulate the given function, based on work by Mendo (2019)[¹]. Note that some functions require a growing number of coin flips as λ approaches 0 or 1. Note that for the 2014, 2016, and 2019 algorithms—

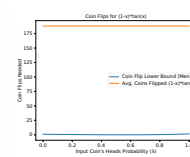
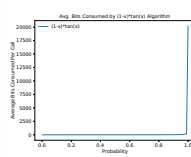
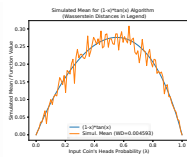
- an ϵ of $1 - (x + c) * 1.001$ was used (or 0.0001 if ϵ would be greater than 1), and
- an ϵ of $(x - c) * 0.9995$ for the subtraction variants.

Points with invalid ϵ values were suppressed. For the low-mean algorithm, an m of $\max(0.49999, xc1.02)$ was used unless noted otherwise.

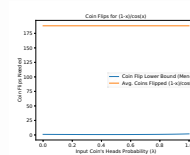
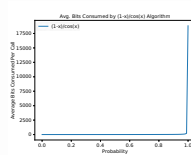
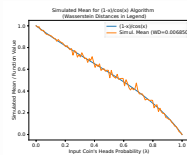
1.1 The Charts

Algorithm	Simulated Mean	Average Bits Consumed	Coin Flips
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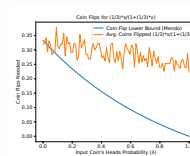
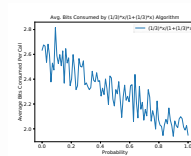
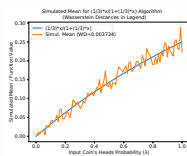
$$(1-x)*\tan(x)$$



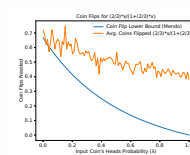
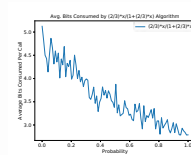
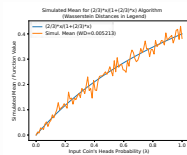
$$(1-x)/\cos(x)$$



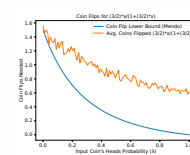
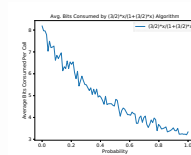
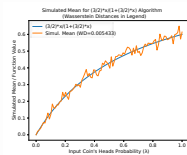
$$(1/3)*x/(1+(1/3)*x)$$



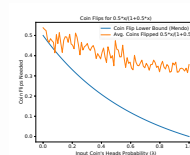
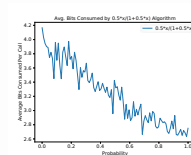
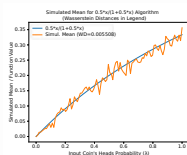
$$(2/3)*x/(1+(2/3)*x)$$



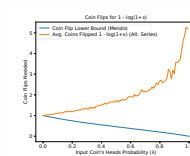
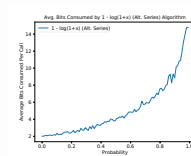
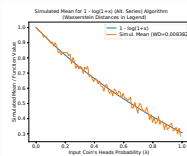
$$(3/2)*x/(1+(3/2)*x)$$



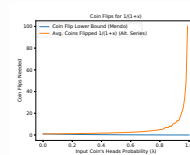
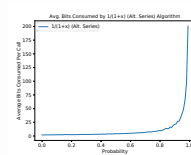
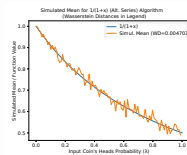
$$0.5*x/(1+0.5*x)$$



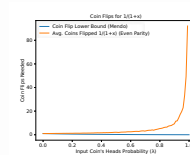
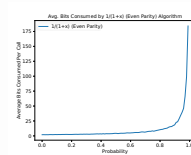
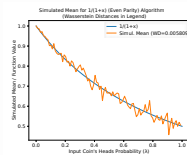
$$1 - \ln(1+x) \text{ (Alt. Series)}$$

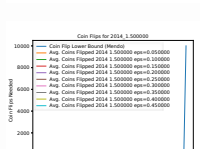
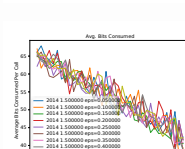
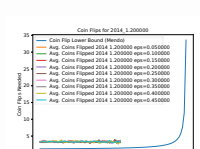
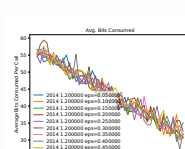
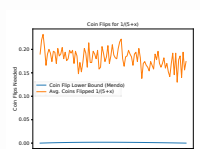
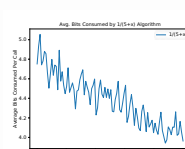
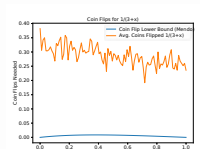
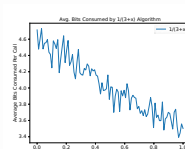
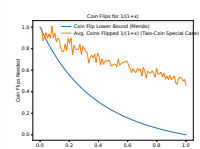
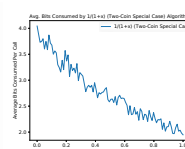


$$1/(1+x) \text{ (Alt. Series)}$$



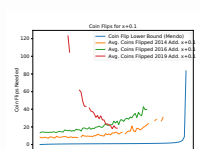
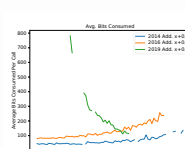
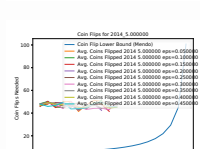
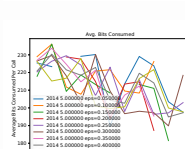
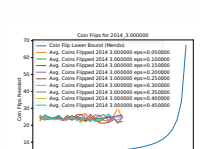
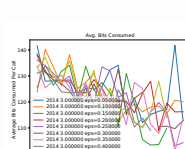
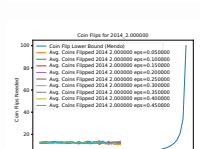
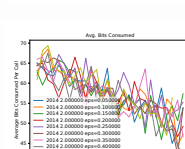
$$1/(1+x) \text{ (Even Parity)}$$



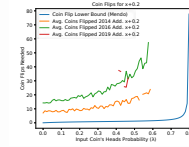
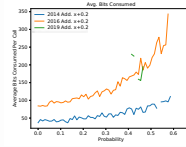
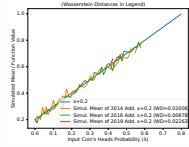


Simulated Means
(Wasserman Distances in Legend)

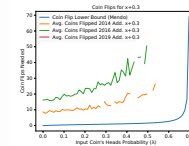
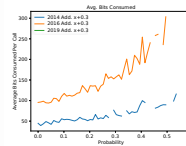
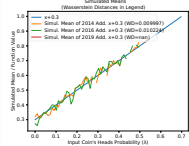
Line Color	Year	n	μ	RSD
Blue	2014	200000	-	-
Orange	Simul. Mean of 2014	2.000000	$\mu = 0.050000$	(RSD = 0.0110)
Green	Simul. Mean of 2014	2.000000	$\mu = 0.100000$	(RSD = 0.0094)
Purple	Simul. Mean of 2014	2.000000	$\mu = 0.150000$	(RSD = 0.0105)
Brown	Simul. Mean of 2014	2.000000	$\mu = 0.200000$	(RSD = 0.0093)
Pink	Simul. Mean of 2014	2.000000	$\mu = 0.250000$	(RSD = 0.0131)
Red	Simul. Mean of 2014	2.000000	$\mu = 0.300000$	(RSD = 0.0094)
Grey	Simul. Mean of 2014	2.000000	$\mu = 0.350000$	(RSD = 0.0101)
Cyan	Simul. Mean of 2014	2.000000	$\mu = 0.400000$	(RSD = 0.0100)
Light Blue	Simul. Mean of 2014	2.000000	$\mu = 0.450000$	(RSD = 0.0092)



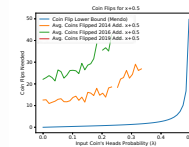
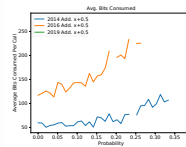
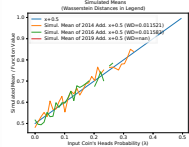
2014 Add. $x+0.2$



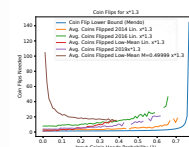
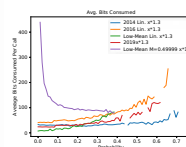
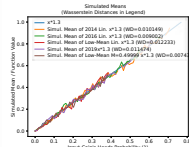
2014 Add. $x+0.3$



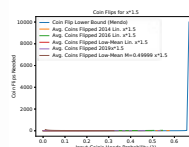
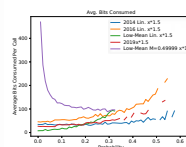
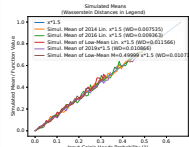
2014 Add. $x+0.5$



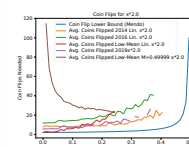
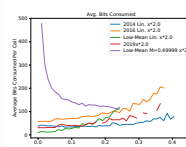
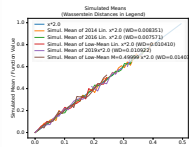
2014 Lin. $x*1.3$



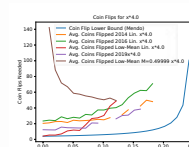
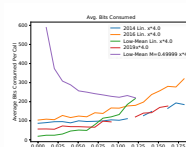
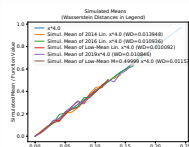
2014 Lin. $x*1.5$



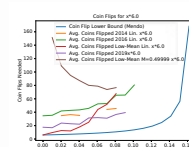
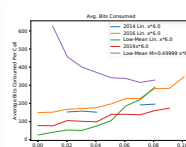
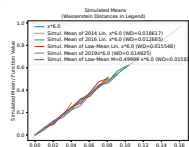
2014 Lin. $x*2.0$



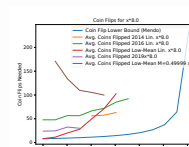
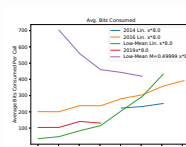
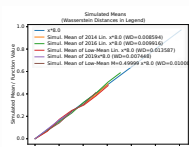
2014 Lin. $x*4.0$



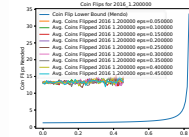
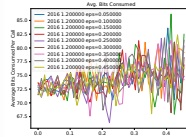
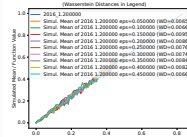
2014 Lin. $x*6.0$



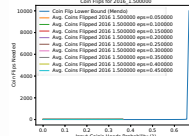
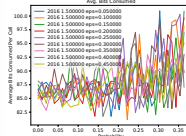
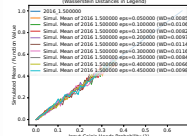
2014 Lin. $x*8.0$



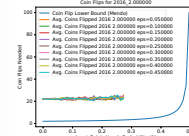
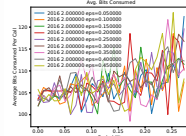
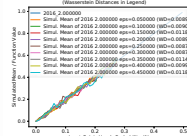
2016 1.200000
eps=0.050000



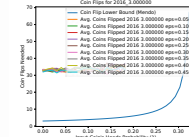
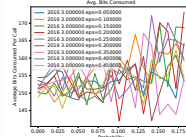
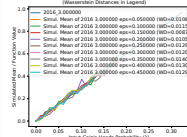
2016 1.500000
eps=0.050000



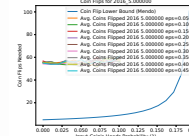
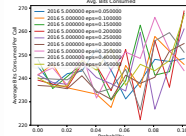
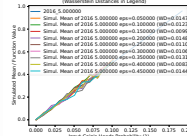
2016 2.000000
eps=0.050000



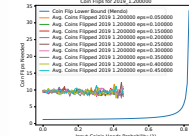
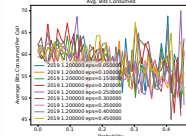
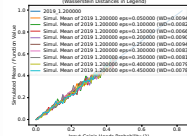
2016 3.000000
eps=0.050000



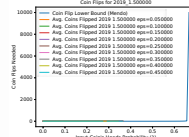
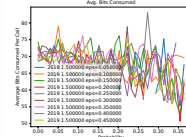
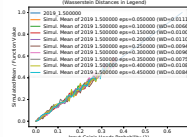
2016 5.000000
eps=0.050000



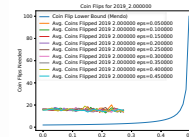
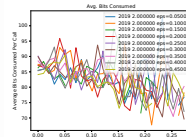
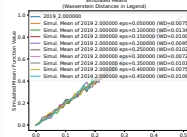
2019 1.200000
eps=0.050000



2019 1.500000
eps=0.050000

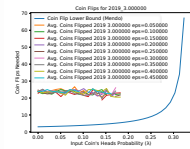


2019 2.000000
eps=0.050000

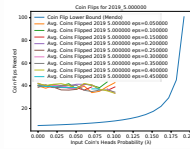


2019 3.000000

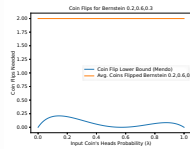
$\epsilon=0.050000$



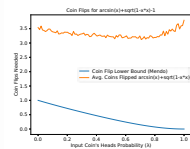
2019 5.000000
 $\epsilon=0.050000$



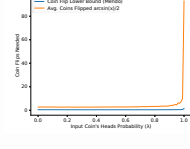
Bernstein
0.2,0.6,0.3



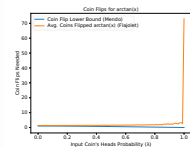
$\arcsin(x)+\sqrt{1-x^2}-1$



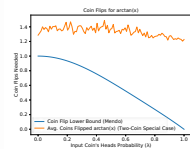
$\arcsin(x)/2$



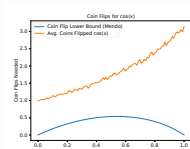
$\arctan(x)$
(Flajolet)



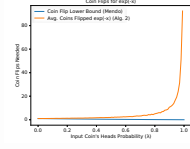
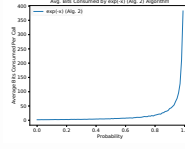
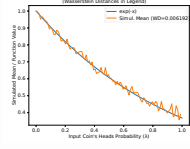
$\arctan(x)$ (Two-Coin Special Case)



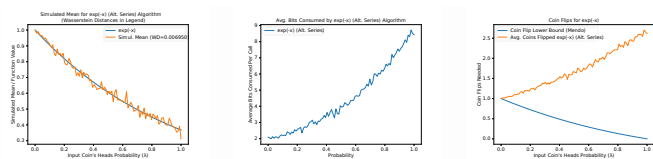
$\cos(x)$



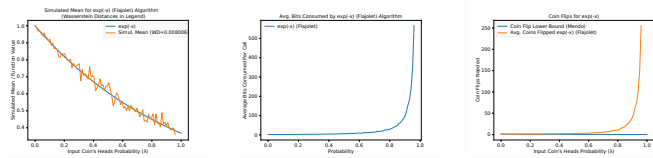
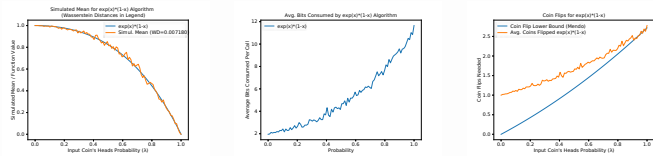
$\exp(-x)$ (Alg. 2)



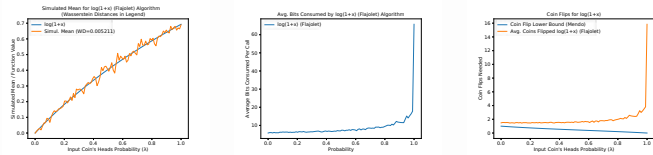
exp(-x) (Alt.
Series)



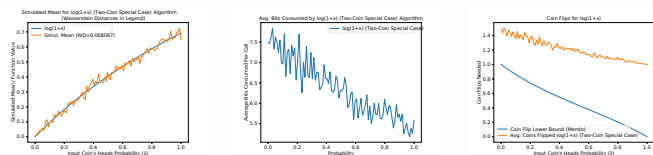
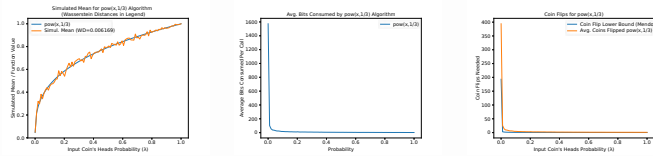
exp(-x) (Flajolet)


$$\exp(x) \cdot (1-x)$$


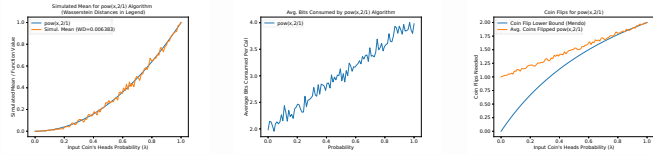
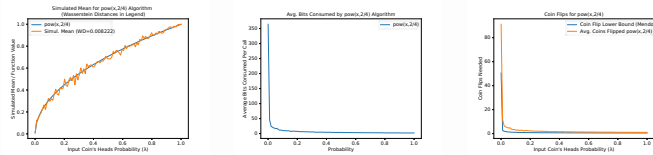
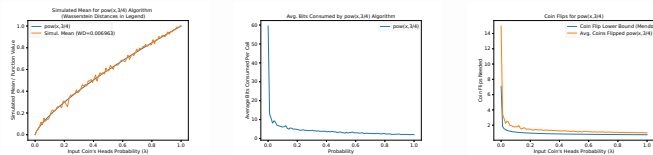
$\ln(1+x)$ (Flajolet)



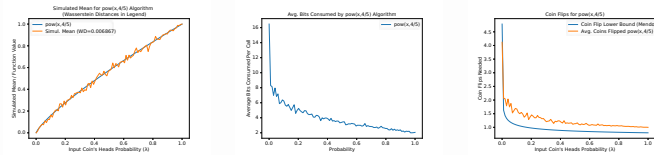
$\ln(1+x)$ (Two-Coin Special Case)

 $\text{pow}(x, 1/3)$ 

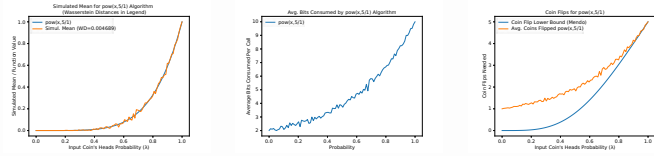
pow(x,2/1)

 $\text{pow}(x, 2/4)$  $\text{pow}(x, 3/4)$ 

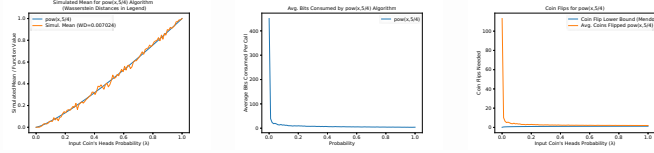
$\text{pow}(x, 4/5)$



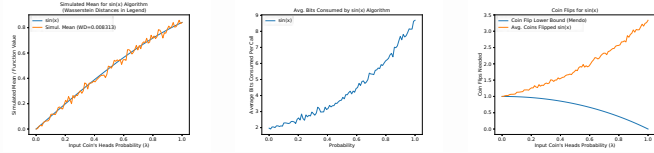
$\text{pow}(x, 5/1)$



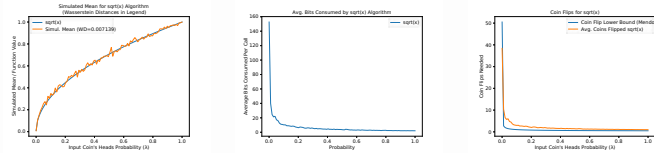
$\text{pow}(x, 5/4)$



$\sin(x)$



$\text{sqrt}(x)$



1. <https://peteroupc.github.io/bernoulli.md>