

1 Correctness and Performance Charts

This version of the document is dated 2023-06-13.

The following charts show the correctness of many of the algorithms in “**Bernoulli Factory Algorithms**” and show their performance in terms of the number of bits they use on average. For each algorithm, and for each of 100 λ values evenly spaced from 0.0001 to 0.9999:

- 500 runs of the algorithm were done. Then...
- The number of bits used by the runs were averaged, as were the return values of the runs (since the return value is either 0 or 1, the mean return value will be in the interval $[0, 1]$). The number of bits used included the number of bits used to produce each coin flip, assuming the coin flip procedure for λ was generated using the `Bernoulli#coin()` method in *bernoulli.py*, which produces that probability in an optimal or near-optimal way.

For each algorithm, if a single run was detected to use more than 5000 bits for a given λ , the entire data point for that λ was suppressed in the charts below.

In addition, for each algorithm, a chart appears showing the minimum number of input coin flips that any fast Bernoulli factory algorithm will need on average to simulate the given function, based on work by Mendo (2019)[¹]. Note that some functions require a growing number of coin flips as λ approaches 0 or 1. Note that for the 2014, 2016, and 2019 algorithms—

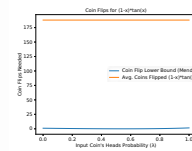
- an ϵ of $1 - (x + c) * 1.001$ was used (or 0.0001 if ϵ would be greater than 1), and
- an ϵ of $(x - c) * 0.9995$ for the subtraction variants.

Points with invalid ϵ values were suppressed. For the low-mean algorithm, an m of $\max(0.49999, xc1.02)$ was used unless noted otherwise.

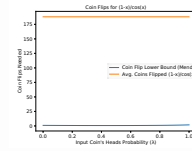
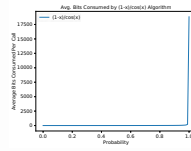
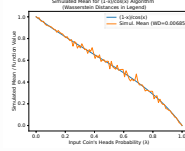
1.1 The Charts

Algorithm	Simulated Mean	Average Bits Consumed	Coin Flips
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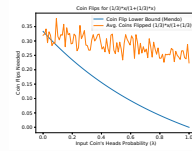
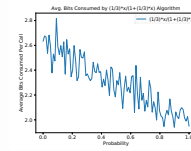
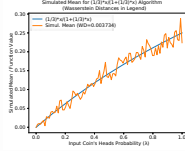
$$(1-x)*\tan(x)$$



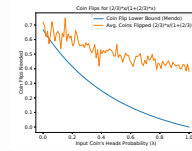
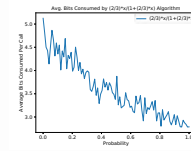
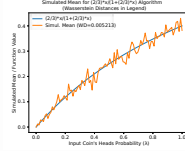
$$(1-x)/\cos(x)$$



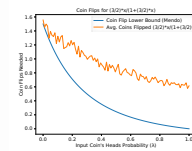
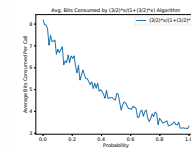
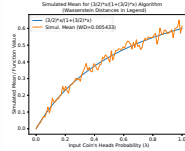
$$(1/3)*x/(1+(1/3)*x)$$



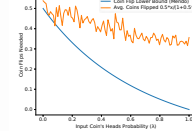
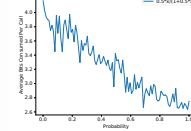
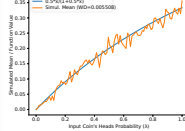
$$(2/3)*x/(1+(2/3)*x)$$



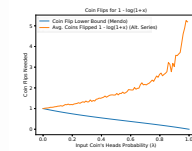
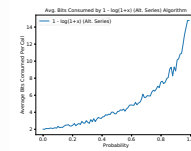
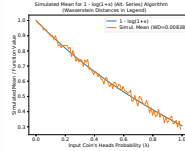
$$(3/2)*x/(1+(3/2)*x)$$



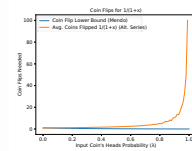
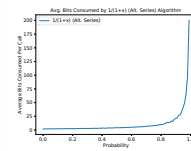
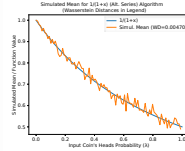
$$0.5*x/(1+0.5*x)$$



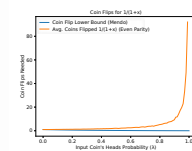
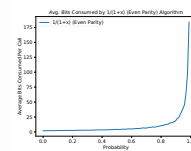
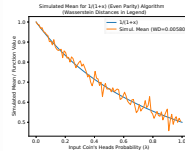
$$1 - \ln(1+x) \text{ (Alt. Series)}$$



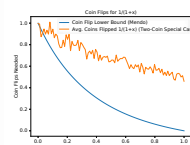
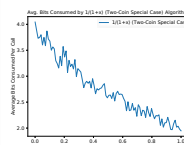
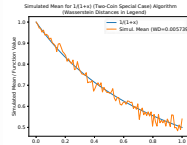
$$1/(1+x) \text{ (Alt. Series)}$$



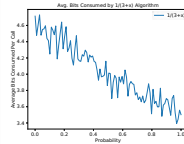
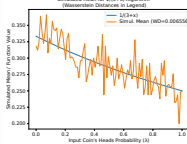
$$1/(1+x) \text{ (Even Parity)}$$



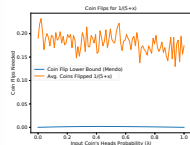
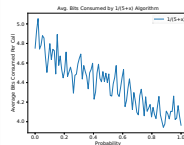
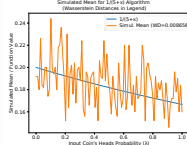
1/(1+x) (Two-Coin Special Case)



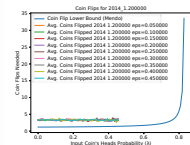
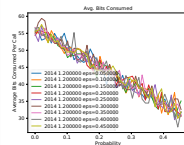
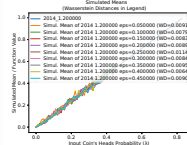
1/(3+x)



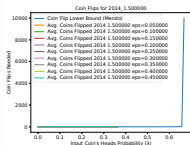
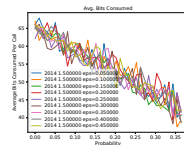
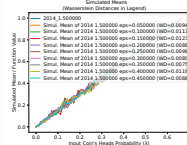
1/(5+x)



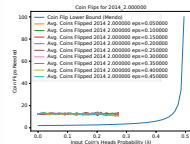
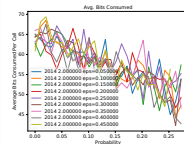
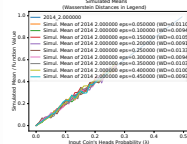
2014 1.200000
eps=0.050000



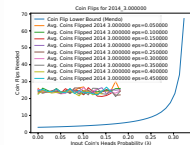
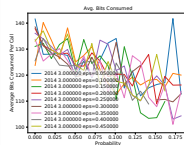
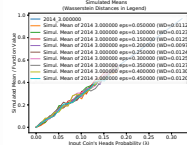
2014 1.500000
eps=0.050000



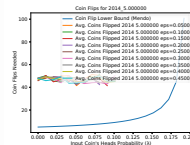
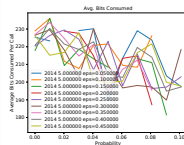
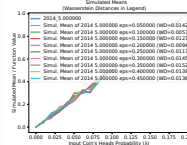
2014 2.000000
eps=0.050000



2014 3.000000
eps=0.050000



2014 5.000000
eps=0.050000



2014 Add. x+0.1

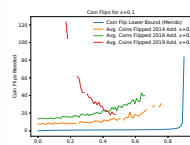
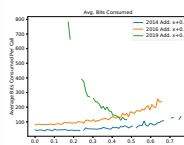
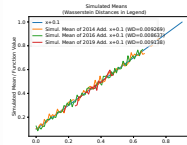


Figure 1 consists of three subplots. The left subplot shows the 'Displacement Rate / Critical Displacement' on the y-axis (ranging from 0.0 to 1.0) versus the 'Avg. Displacement / Average Displacement' on the x-axis (ranging from 0.0 to 1.0). It includes five data series for different mesh sizes: $h=0.2$ (blue), $h=0.1$ (orange), $h=0.05$ (green), $h=0.025$ (red), and $h=0.0125$ (purple). All series follow a similar trend, starting at (0,0) and increasing linearly until they reach a sharp increase near $x=1.0$. The right subplot shows the 'Average Displacement Error' on the y-axis (ranging from 0 to 10) versus the 'Avg. Displacement / Average Displacement' on the x-axis (ranging from 0.0 to 1.0). It uses the same color scheme for mesh sizes. The error remains low (below 5) for most of the range but spikes significantly as the displacement ratio approaches 1.0. The left subplot shows the 'CPU Time (s)' on the y-axis (ranging from 0 to 80) versus the 'Avg. Displacement / Average Displacement' on the x-axis (ranging from 0.0 to 1.0). It uses the same color scheme for mesh sizes. The CPU time is relatively constant (around 10-20 seconds) for most of the range but spikes sharply as the displacement ratio approaches 1.0.

Figure 10 consists of three subplots comparing the proposed model with existing models (2014, 2016, 2018, and 2019) across different input C:N ratio probabilities (U).

Top Subplot: Deviated Water (m³/ha) vs Input C:N Ratio Probability (U)

- X-axis:** Input C:N Ratio Probability (U), ranging from 0.0 to 0.7.
- Y-axis:** Deviated Water (m³/ha), ranging from 0 to 14.
- Legend:**
 - 2014 Reg. $\alpha = 0.2$ (Blue line)
 - 2016 Reg. $\alpha = 0.2$ (Orange line)
 - 2018 Reg. $\alpha = 0.2$ (Green line)
 - 2019 Reg. $\alpha = 0.2$ (Red line)
 - 2014 Reg. $\alpha = 0.3$ (Blue dashed line)
 - 2016 Reg. $\alpha = 0.3$ (Orange dashed line)
 - 2018 Reg. $\alpha = 0.3$ (Green dashed line)
 - 2019 Reg. $\alpha = 0.3$ (Red dashed line)
- Observations:** The proposed model (solid lines) shows a linear increase in deviated water with increasing input C:N ratio probability. The existing models (dashed lines) show a non-linear, increasing trend, with the 2019 model showing the highest deviation at higher probabilities.

Bottom-Left Subplot: Average Soil C:N Concentration, C:N vs Probability

- X-axis:** Probability, ranging from 0.0 to 0.5.
- Y-axis:** Average Soil C:N Concentration, C:N, ranging from 10 to 20.
- Legend:** Same as the top subplot.
- Observations:** The proposed model (solid lines) shows a relatively stable average soil C:N concentration, while the existing models (dashed lines) show a significant increase in concentration as the input C:N ratio probability increases.

Bottom-Right Subplot: Cumulative Error vs Input C:N Ratio Probability (U)

- X-axis:** Input C:N Ratio Probability (U), ranging from 0.0 to 0.7.
- Y-axis:** Cumulative Error, ranging from 0 to 70.
- Legend:** Same as the top subplot.
- Observations:** The proposed model (solid lines) shows a very low cumulative error, while the existing models (dashed lines) show a significant increase in cumulative error as the input C:N ratio probability increases.

[illegible]

Figure 1 consists of two subplots. Subplot (a) is a scatter plot titled 'Performance of the proposed model' showing 'True Labels' on the x-axis and 'Predicted Labels' on the y-axis, both ranging from 0.0 to 1.0. It compares several models: 'Proposed' (red line), '2018 Liu et al. [13]' (blue line), '2018 Liu et al. [13] + 2018 Liu et al. [13]' (green line), '2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13]' (orange line), '2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13]' (purple line), and '2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13] + 2018 Liu et al. [13]' (brown line). The 'Proposed' model shows the highest performance, closely following the diagonal line. Subplot (b) is a line plot titled 'Average Bit Losses per CoS' on the y-axis (ranging from 0 to 140) versus 'Channel SNR' on the x-axis (ranging from 0.0 to 1.0). It compares the same models as in (a). The 'Proposed' model (red line) shows the lowest average bit losses across the entire SNR range, while the other models show significantly higher losses, especially at higher SNR values.

Figure 1 consists of three subplots labeled (a), (b), and (c). Subplot (a) is an ROC curve showing the Area Under the Curve (AUC) on the y-axis (ranging from 0.0 to 1.0) against the Input Class Weights Probability on the x-axis (ranging from 0.0 to 0.6). The legend for (a) includes:

- Blue line: Mean of 2018 Data, $\alpha = 0.15$
- Orange line: Mean of 2018 Data, $\alpha = 0.3$
- Green line: Mean of 2018 Data, $\alpha = 0.5$
- Red line: Mean of 2018 Data, $\alpha = 1.0$
- Purple line: Mean of 2018 Data, $\alpha = 1.5$
- Black line: Mean of Case Weight, $\alpha = 0.15$
- Dark Blue line: Mean of Case Weight, $\alpha = 0.3$
- Dark Green line: Mean of Case Weight, $\alpha = 0.5$
- Dark Red line: Mean of Case Weight, $\alpha = 1.0$
- Dark Purple line: Mean of Case Weight, $\alpha = 1.5$

 Subplot (b) shows the Average F1 Score on the y-axis (ranging from 0.0 to 0.6) against the Probability on the x-axis (ranging from 0.0 to 0.6). The legend for (b) includes:

- Blue line: 2018 Data, $\alpha = 1$
- Orange line: 2018 Data, $\alpha = 0.5$
- Green line: 2018 Data, $\alpha = 0.3$
- Red line: 2018 Data, $\alpha = 0.15$
- Purple line: Case Weight, $\alpha = 1$
- Dark Blue line: Case Weight, $\alpha = 0.5$
- Dark Green line: Case Weight, $\alpha = 0.3$
- Dark Red line: Case Weight, $\alpha = 0.15$
- Dark Purple line: Case Weight, $\alpha = 0.05$

 Subplot (c) is a confusion matrix showing the Count on the y-axis (ranging from 0 to 2000) against the Input Class Weights Probability on the x-axis (ranging from 0.0 to 0.6). The legend for (c) includes:

- Blue line: Confusion Matrix of 2018 Data, $\alpha = 1$
- Orange line: Confusion Matrix of 2018 Data, $\alpha = 0.5$
- Green line: Confusion Matrix of 2018 Data, $\alpha = 0.3$
- Red line: Confusion Matrix of 2018 Data, $\alpha = 0.15$
- Purple line: Confusion Matrix of Case Weight, $\alpha = 1$
- Dark Blue line: Confusion Matrix of Case Weight, $\alpha = 0.5$
- Dark Green line: Confusion Matrix of Case Weight, $\alpha = 0.3$
- Dark Red line: Confusion Matrix of Case Weight, $\alpha = 0.15$
- Dark Purple line: Confusion Matrix of Case Weight, $\alpha = 0.05$

Figure 1 consists of three subplots labeled (a), (b), and (c).

Subplot (a) is a line graph titled "Divergence Rate vs. Input Coin's Heads Probability". The x-axis is "input Coin's Heads Probability (x)" ranging from 0.0 to 1.0. The y-axis is "Divergence Rate (log base 2)" ranging from 0.0 to 1.0. There are five data series:

- OTD (blue line with circles): A straight line from (0,0) to (1,1).
- Mean of 2048 coin, $\mu = 0.5$ (0.0000000000000000) (red line with circles): A curve starting at (0,0), peaking at (0.5, 1.0), and ending at (1,1).
- Mean of 2048 coin, $\mu = 0.5$ (0.0000000000000000) (green line with circles): A curve starting at (0,0), peaking at (0.5, 1.0), and ending at (1,1).
- Mean of 2048 coin, $\mu = 0.5$ (0.0000000000000000) (orange line with circles): A curve starting at (0,0), peaking at (0.5, 1.0), and ending at (1,1).
- Mean of 2048 coin, $\mu = 0.5$ (0.0000000000000000) (purple line with circles): A curve starting at (0,0), peaking at (0.5, 1.0), and ending at (1,1).

Subplot (b) is a line graph titled "Average Run Length vs. Probability". The x-axis is "Probability" ranging from 0.0 to 1.0. The y-axis is "Average Run Length (Coin Tosses)" ranging from 0 to 100. There are five data series:

- 2048 coin, $\mu = 0.5$ (0.0000000000000000) (blue line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- 2048 coin, $\mu = 0.5$ (0.0000000000000000) (red line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- 2048 coin, $\mu = 0.5$ (0.0000000000000000) (green line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- 2048 coin, $\mu = 0.5$ (0.0000000000000000) (orange line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- 2048 coin, $\mu = 0.5$ (0.0000000000000000) (purple line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).

Subplot (c) is a line graph titled "Cost Function vs. Input Coin's Heads Probability". The x-axis is "input Coin's Heads Probability (x)" ranging from 0.0 to 1.0. The y-axis is "Cost Function" ranging from 0 to 120. There are five data series:

- Coin Tosses Required (blue line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- Coin Tosses Required (red line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- Coin Tosses Required (green line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- Coin Tosses Required (orange line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).
- Coin Tosses Required (purple line with circles): A curve starting at (0,0), peaking at (0.5, 100), and ending at (1,1).

Figure 1 consists of two plots. The left plot shows the Area Under the Curve (AUC) versus the input class's weak probability (X). The right plot shows the Average Mis-Coverage Rate (Mis-Coverage Rate) versus the input class's weak probability (X). Both plots compare the proposed model (Prop.) with the baseline model (Baseline) for various input class weak probabilities (X) ranging from 0.00 to 0.15. The proposed model consistently outperforms the baseline model, achieving higher AUC and lower Mis-Coverage Rate.

Left Plot: AUC vs. Input Class's Weak Probability (X)

Input Class's Weak Probability (X)	Prop. (X=0.0)	Prop. (X=0.05)	Prop. (X=0.1)	Prop. (X=0.15)	Baseline (X=0.0)	Baseline (X=0.05)	Baseline (X=0.1)	Baseline (X=0.15)
0.00	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
0.05	0.65	0.65	0.65	0.65	0.55	0.55	0.55	0.55
0.10	0.75	0.75	0.75	0.75	0.65	0.65	0.65	0.65
0.15	0.85	0.85	0.85	0.85	0.75	0.75	0.75	0.75

Right Plot: Average Mis-Coverage Rate vs. Input Class's Weak Probability (X)

Input Class's Weak Probability (X)	Prop. (X=0.0)	Prop. (X=0.05)	Prop. (X=0.1)	Prop. (X=0.15)	Baseline (X=0.0)	Baseline (X=0.05)	Baseline (X=0.1)	Baseline (X=0.15)
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.05	0.05	0.05	0.05	0.05	0.10	0.10	0.10	0.10
0.10	0.10	0.10	0.10	0.10	0.20	0.20	0.20	0.20
0.15	0.15	0.15	0.15	0.15	0.30	0.30	0.30	0.30

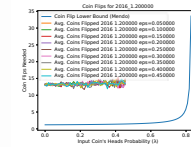
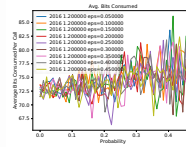
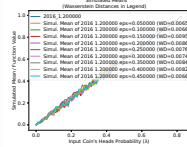
[illegible]

Figure 1 consists of three subplots labeled (a), (b), and (c), showing the performance of the proposed algorithm under various conditions.

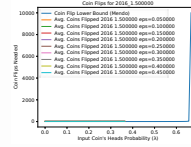
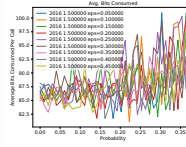
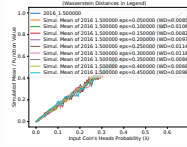
(a) Shuffled filter SNR (dB) vs. input SNR (dB). The plot shows a linear relationship between the input SNR and the shuffled filter SNR. The legend indicates the following conditions:

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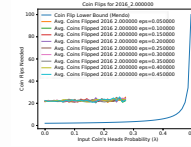
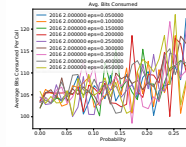
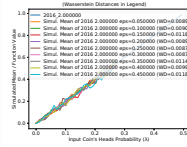
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eps=0.050000



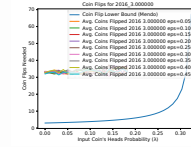
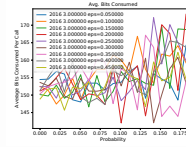
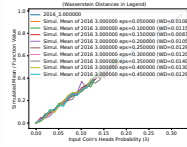
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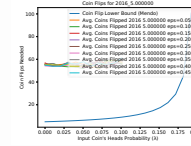
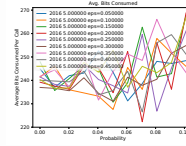
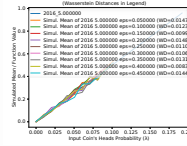
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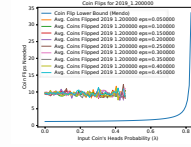
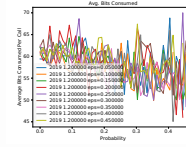
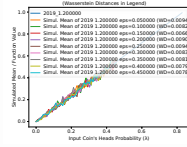
2016 3.000000
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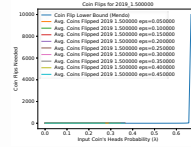
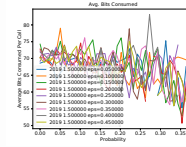
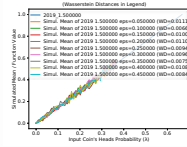
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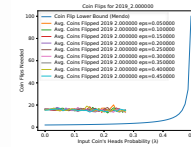
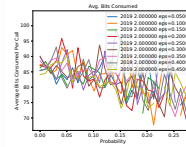
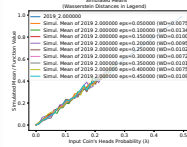
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eps=0.050000



2019 1.500000
eps=0.050000

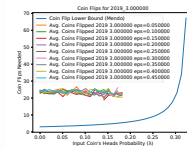


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eps=0.050000

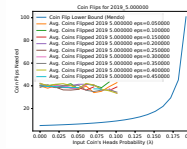


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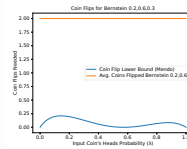
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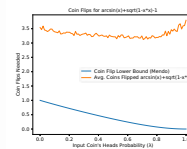
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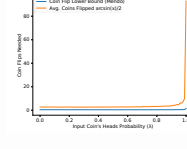
Bernstein
0.2,0.6,0.3



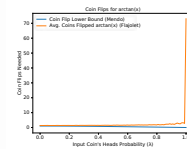
$\arcsin(x)+\sqrt{1-x^2}-1$



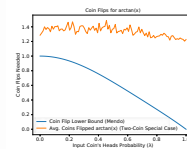
$\arcsin(x)/2$



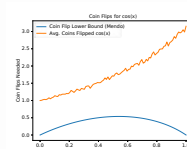
$\arctan(x)$
(Flajolet)



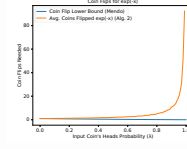
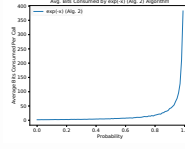
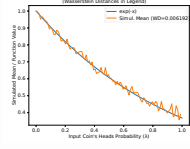
$\arctan(x)$ (Two-Coin Special Case)



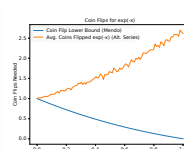
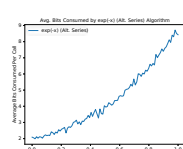
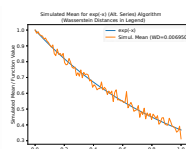
$\cos(x)$



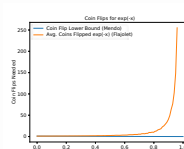
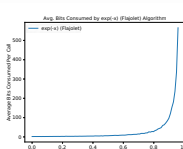
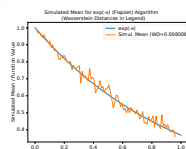
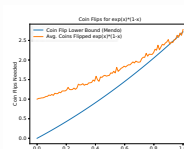
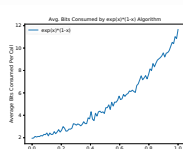
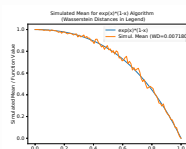
$\exp(-x)$ (Alg. 2)



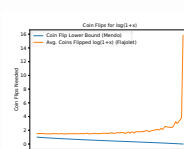
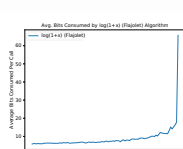
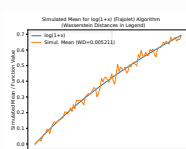
exp(-x) (Alt.
Series)



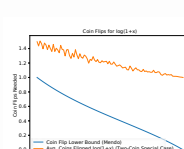
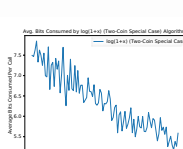
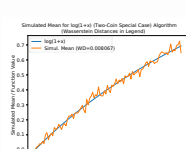
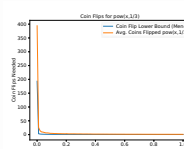
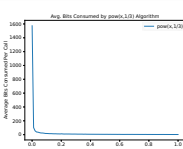
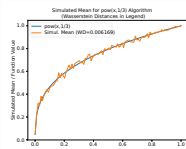
exp(-x) (Flajolet)


$$\exp(x) \cdot (1-x)$$


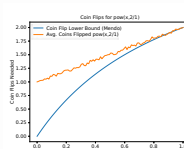
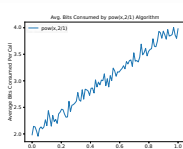
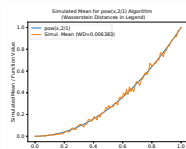
$\ln(1+x)$ (Flajolet)



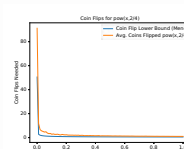
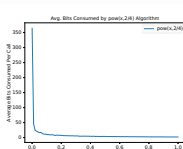
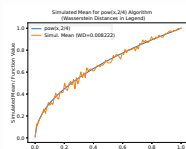
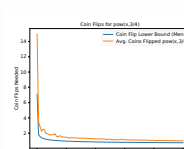
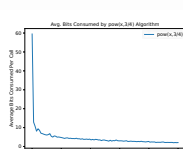
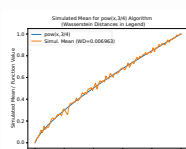
ln(1+x) (Two-Coin Special Case)

 $\text{pow}(x, 1/3)$ 

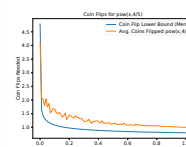
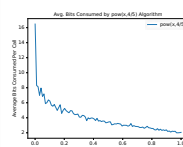
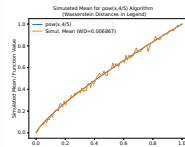
pow(x,2/1)



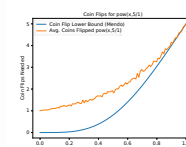
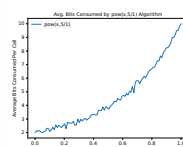
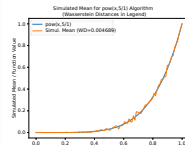
pow(x,2/4)

 $\text{pow}(x, 3/4)$ 

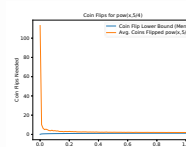
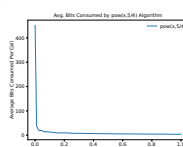
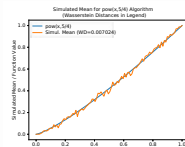
$\text{pow}(x, 4/5)$



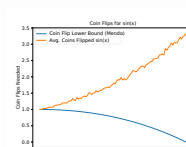
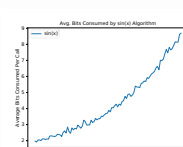
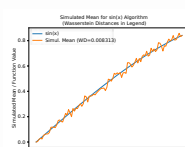
$\text{pow}(x, 5/1)$



$\text{pow}(x, 5/4)$



$\sin(x)$



$\text{sqrt}(x)$

