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This page is about a mathematical problem of **sampling a probability distribution with unknown parameters**. This problem can be described as sampling from a new distribution using an endless stream of random variates from an incompletely known distribution.

Suppose $(X_0, X_1, X_2, X_3, ...)$ is an endless stream of random variates, or *input values*.

Let `InDist` be the probability distribution of these input values, and let λ be an unknown parameter that determines the distribution `InDist`, such as its expected value. Suppose the problem is to **produce a random variate with a distribution `OutDist` that depends on the unknown parameter** λ. Then, of the algorithms in the section "**Sampling Distributions Using Incomplete Information**[1]"

- In **Algorithm 1** (Jacob and Thiery 2015)[2], `InDist` is arbitrary but must have a known minimum and maximum, λ is the expected value of `InDist`, and `OutDist` is non-negative and has an expected value of $f(\lambda)$.
- In **Algorithm 2** (Duvignau 2015)[3], `InDist` is a fair die with an unknown number of faces, λ is the number of faces, and `OutDist` is a specific distribution that depends on the number of faces.
- In **Algorithm 3** (Lee et al. 2014)[4], `InDist` is arbitrary, λ is the expected value of `InDist`, and `OutDist` is non-negative and has an expected value equal to the mean of $f(X)$, where X is an input value taken.
- In **Algorithm 4** (Jacob and Thiery 2015)[5], `InDist` is arbitrary but must have a known minimum, λ is the expected value of `InDist`, and `OutDist` is non-negative and has an expected value of $f(\lambda)$.
- In **Algorithm 5** (Akahira et al. 1992)[6], `InDist` is Bernoulli, λ is the expected value of `InDist`, and `OutDist` has an expected value of $f(\lambda)$. [7]
- In the **Bernoulli factory problem**[8], `InDist` is Bernoulli, λ is the expected value of `InDist`, and `OutDist` is Bernoulli with an expected value of $f(\lambda)$.

In all cases given above, each input value is independent of everything else.

[1]https://peteroupc.github.io/randmisc.md#Sampling_Distributions_Using_Incomplete_Information

[2]Jacob, P.E., Thiery, A.H., "On nonnegative unbiased estimators", Ann. Statist., Volume 43, Number 2 (2015), 769-784.

[3]Duvignau, R., 2015. Maintenance et simulation de graphes aléatoires dynamiques (Doctoral dissertation, Université de Bordeaux).

[4]Lee, A., Doucet, A. and Łatuszyński, K., 2014. "**Perfect simulation using atomic regeneration with application to Sequential Monte Carlo**", arXiv:1407.5770v1 [stat.CO]. https://arxiv.org/abs/1407.5770v1

[5]Jacob, P.E., Thiery, A.H., "On nonnegative unbiased estimators", Ann. Statist., Volume 43, Number 2 (2015), 769-784.

[6]AKAHIRA, Masafumi, Kei TAKEUCHI, and Ken-ichi KOIKE. "Unbiased estimation in sequential binomial sampling", Rep. Stat. Appl. Res., JUSE 39 1-13, 1992.

[7]Singh (1964, "Existence of unbiased estimates", Sankhyā A 26) claimed that an estimation algorithm with expected value $f(\lambda)$ exists for a more general class of `InDist` distributions than the Bernoulli distribution, as long as there are polynomials that converge pointwise to f, and Bhandari and Bose (1990, "Existence of unbiased estimates in sequential binomial experiments", Sankhyā A 52) claimed necessary conditions for those algorithms. However, Akahira et al. (1992) questioned the claims of both papers, and the latter paper underwent a correction, which I haven't seen (Sankhyā A 55, 1993).

[8]https://peteroupc.github.io/bernoulli.html

There are numerous other cases of interest that are not covered in the algorithms above. An example is the case of **Algorithm 5** except `InDist` is any discrete distribution, not just Bernoulli. An interesting topic is to answer the following: In which cases (and for which functions f) can the problem be solved...

- ...when the number of input values taken is finite with probability 1 (a *sequential unbiased* estimator)?
- ...when only a fixed number n of input values can be taken (a fixed-sample-size unbiased estimator)?
- ...using an algorithm that produces outputs whose expected value *approaches* $f(\lambda)$ as more input values are taken (an *asymptotically unbiased* estimator)?

The answers to these questions will depend on—

- the allowed distributions for `InDist`,
- the allowed distributions for `OutDist`,
- which parameter λ is unknown,
- whether the inputs are independent, and
- whether outside randomness is allowed.

It should be noted that many of these cases have been studied and resolved in academic papers and books (e.g., Keane and O'Brien (1994)[9] for the Bernoulli factory problem) — the problem here is one of bringing all these results together in one place. An additional question is to find lower bounds on the input/output ratio that an algorithm can achieve as the number of inputs taken increases (e.g., Nacu and Peres (2005, Question 2)[10]).

1 Results

The following is an example of results for this problem.

- Suppose `InDist` takes on numbers from a finite set; λ is the expected value of `InDist`; and `OutDist` has an expected value of $f(\lambda)$.
  - Then a fixed-size unbiased estimator exists only if f is a polynomial of degree n or less, where n is the number of inputs taken ((Lehmann 1983, for coin flips)[11], Paninski (2003, proof of Proposition 8, more generally)[12]).
  - The existence of sequential unbiased estimators is claimed by Singh (1964). But see Akahira et al. (1992)[13].

2 Notes

[9]Keane, M. S., and O'Brien, G. L., "A Bernoulli factory", *ACM Transactions on Modeling and Computer Simulation* 4(2), 1994.

[10]Nacu, Şerban, and Yuval Peres. "**Fast simulation of new coins from old**", The Annals of Applied Probability 15, no. 1A (2005): 93-115. https://projecteuclid.org/euclid.aoap/1106922322

[11]Lehmann, E.L., *Theory of Point Estimation*, 1983.

[12]Paninski, Liam. "Estimation of Entropy and Mutual Information." Neural Computation 15 (2003): 1191-1253.

[13]AKAHIRA, Masafumi, Kei TAKEUCHI, and Ken-ichi KOIKE. "Unbiased estimation in sequential binomial sampling", Rep. Stat. Appl. Res., JUSE 39 1-13, 1992.