McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

13 May, 2014
13:00 - 17:00

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

Problem 1, Prove that function $g(x) = \sqrt{|x|}\sin(\sqrt{|x|})$ is uniformly continuous in $\mathbb{R}$.

Problem 2, Let $\{f_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$. Suppose $f_n(1) = 1$ and $|f_n^{'}(x)| \leq \frac{1}{\sqrt{x}}$ for all n and for all $0 < x \leq 1$. Show that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is convergent uniformly on $[0,1]$.

Problem 3. Find the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{10^{\nu(n)}}{n} x^n,$$

where $\nu(n)$ is the number of digits of n (for example, $\nu(2011) = 4$). Does the series converge for $x = R$ and $x = -R$?

Problem 4. Let X be a set and d the metric defined by $d(x,y) = 0$ if $x = y$, $d(x,y) = 1$ if $x \neq y$. Describe all compact subsets of X.

Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

Problem 5. Let F be a field, and let V be the F-vector space of all infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$ with values $a_i \in F$. Show that V does not admit a countable basis. (Hint: given any nested sequence

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots$$

of subspaces of V with

$$\dim(V_j) = j$$

for all j, produce an element of V that does not belong to any of the V_j by a variation on Cantor's diagonal argument.)

Problem 6. Let M be a real symmetric $n \times n$ matrix. Show that M has an eigenvalue in $\mathbf{R}$. Use this to show that M is always diagonalisable over $\mathbf{R}$.

Problem 7. Let T be a linear transformation from a finite-dimensional vector space V to itself, and let W be a T-stable subspace of V. Is it always possible to find a T-stable subspace W' for which $V = W \oplus W'$? If yes, prove it, and if no, give a counterexample.

Problem 8. Let V be a vector space over a field F. Let T be a linear transformation from an n-dimensional vector space to itself, and let U be a linear transformation that commutes with T. Show that U is diagonalisable if T admits n distinct eigenvalues in F. Illustrate by an example the fact that this conclusion may fail if T is merely assumed to be diagonalisable.

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

Problem 9. Let $I = (a, b)$ be a nonempty open interval, and let $y \in C^2(I)$ be a real-valued solution of the differential equation

$$y'' + py' + qy = 0,$$

on I, where p and q are real constants. Let $k = -\frac{p}{2}$, $D = (\frac{p}{2})^2 - q$, $\omega = \sqrt{|D|}$, $r_1 = k + \omega$, and $r_2 = k - \omega$. Depending on the sign of D, we set

$$y_1(x) = \begin{cases} e^{r_1 x}, & \text{if} \quad D \geq 0, \\ e^{kx} \cos \omega x, & \text{if} \quad D < 0, \end{cases} \qquad \text{and} \qquad y_2(x) = \begin{cases} e^{r_2 x}, & \text{if} \quad D > 0, \\ x e^{r_2 x}, & \text{if} \quad D = 0, \\ e^{kx} \sin \omega x, & \text{if} \quad D < 0. \end{cases}$$

Show that there exist real constants A and B such that

$$y(x) = Ay_1(x) + By_2(x), \qquad x \in I.$$

Problem 10. Find the general solution of the equation

$$y'' + 3y' + 2y = \frac{1}{4 + e^x}.$$

Problem 11. Let f be analytic on a closed disc $\bar{D}$ of radius $b > 0$, centered at z_0. Show that

$$\frac{1}{\pi b^2} \iint_D f(x + iy) dy dx = f(z_0).$$

(Hint: Show that you may assume $z_0 = 0$ without loss of generality. Then use polar coordinates and Cauchy's formula.)

Problem 12. Determine the poles and find the residues of the following functions:

(a) $\dfrac{1}{\sin z}$

(b) $\dfrac{1}{1 - e^z}$

(c) $\dfrac{z}{1 - \cos z}$