McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

<u>PURE MATHEMATICS</u>
<u>Paper BETA</u>

12 May, 2017
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[ALG. 1]. Let G be a finite group and p be a prime number.

a) Show that the number of elements of order p in G is a multiple (possibly zero) of $p-1$.
b) More generally, if n is a positive integer, what is the number of elements of order n in G a multiple of?

[ALG. 2]. Consider the hermitian space $\mathbb{C}^4$ endowed with its standard basis (e_1, e_2, e_3, e_4). Let S_4 denote the symmetric group on $\{1, 2, 3, 4\}$ and let $\pi\colon S_4 \to \mathrm{GL}_4(\mathbb{C})$ denote the group morphism defined for $s \in S_4$ by

$$\pi(s)(e_i) = e_{s(i)}.$$

a) Show that the hyperplane H orthogonal to $e_1 + e_2 + e_3 + e_4$ in $\mathbb{C}^4$ is invariant under π.

Let $\rho(s)$ denote the restriction of $\pi(s)$ to H.

b) Show that the trace of $\rho(s)$ depends only on the number fixed points of the permutation $s \in S_4$.
c) Describe (without proof) the conjugacy classes in S_4 and compute the function

$$\begin{aligned} f\colon S_4 &\to \mathbb{C} \\ s &\mapsto \mathrm{trace}(\rho(s)) \end{aligned}$$

explicitly.

d) Show that $\sum_{s \in S_4} f(s)^2 = 24$.

[ALG. 3]

a) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{Q}$.
b) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{F}_5$.

[ALG. 4]

Show that for any $a, b \in \mathbb{Q}$ such that $\sqrt{a} + \sqrt{b} \neq 0$, we must have $\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$.

Analysis Module

[AN. 1]

Suppose $X \subset Z$, $Y \subset V$, with X, Y, Z, V Banach spaces and with both inclusions continuous. Suppose that $P : Z \to V$ is a continuous linear operator such that $u \in Z$, $Pu \in Y$ implies $u \in X$. Show that there exists $C > 0$ such that for al $u \in Z$ satisfying $Pu \in Y$, one has

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_Z).$$

(Hint: Consider the map $u \mapsto (Pu, u)$ and its inverse in relevant spaces)

[AN. 2]

Suppose that μ is a σ-finite measure on $(X, \mathcal{A})$, then there is a finite measure ν so that $\nu \ll \mu$ and $\mu \ll \nu$.

[AN. 3]

a) State Arzela-Ascoli theorem.

b) Let $0 < \alpha < 1$. Recall that a function $f(x)$ (defined on $[0,1]$) is called α-Hölder continuous if the quantity

$$N_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

Show that the set of functions

$$\{f \in C([0,1]) : ||f||_u \leq 1 \ \ and \ \ N_\alpha(f) \leq 1\}$$

is compact in $C([0,1])$.

c) Show that for $\alpha > 1$, the only α-Hölder continuous functions are the constant functions.

[AN. 4] Let $X = \mathbb{R}^n$, and let $f \in L^p(X) \cap L^\infty(X)$ (and hence $f \in L^q(X)$ for all $p < q < \infty$). Show that

$$\lim_{q \to \infty} ||f||_q = ||f||_\infty.$$

Geometry and Topology Module

[GT. 1]

Let X be a metric space. A subset $G \subseteq X$ is called G_δ if $G = \bigcap_{n\in\mathbb{N}} U_n$ with U_n open.

(i) Let $f : X \to \mathbb{R}$ be a function (arbitrary). Show that the set $C = \{x \in X : f \text{ is continuous at } x\}$ is G_δ.

(ii) Suppose $Y \subseteq X$ is a subspace and $g : Y \to \mathbb{R}$ is a continuous function. Show that there is a G_δ subset G such that $Y \subseteq G \subseteq X$ and g can be extended to a continuous function $\bar{g} : G \to \mathbb{R}$.

[GT. 2] Find a regular cover of a bouquet of two circles whose group of deck transformations is isomorphic to:

(i) $\mathbb{Z}/6$,

(ii) S_3 (the permutation group of a 3 element set).

Justify your answers.

[GT. 3] Identify $Gl(n, \mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$ through correspondence

$$\forall A = (a_{ij}) \in Gl(n, \mathbb{R}) \to (a_{11}, \cdots, a_{1n}; \cdots; a_{n1}, \cdots, a_{nn}) \in \mathbb{R}^{n^2}.$$

Define $f(A) = \det(A)$, show that $\nabla f(A) \neq 0, \forall A \in Gl(n, \mathbb{R})$. Define a distribution Δ in $Gl(n, \mathbb{R})$ as

$$\Delta_A = \{Y \in T_A(\mathbb{R}^{n^2}) | < \nabla f(A), Y >= 0\}, \quad \forall A \in Gl(n, \mathbb{R}).$$

Show that Δ is involutive and describe the integral manifolds of Δ.

[GT. 4] Let M be a Riemannian manifold and X be a vector field on M. Suppose for each $p \in M$, the local one-parameter group $\phi(t, .)$ generated by X is a local isometry for each fixed t when t small (i.e., there is $\epsilon > 0$ and a neighborhood U of p, such that $\phi(t, .) : U \to M$ is an isometry). Show that

$$< \nabla_Y X, Z > + < \nabla_Z X, Y >= 0, \quad \text{for all vector field } Y, Z \text{ in } M.$$