McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

18 August 2015 13:00 - 17:00

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

Problem 1.

(a) Let f and g be differentiable functions on $\mathbb{R}$ and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

(b) Now let $f(x) := x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) := 0$ and let $g(x) := \sin x$ for $x \in \mathbb{R}$. Show that $\lim_{x\to 0} f(x)/g(x) = 0$, but that $\lim f'(x)/g'(x)$ does not exist.

Problem 2. Let $A = \{x : x \in [0, 1]\}$ and let $h(0) := 1$. For any irrational number $x > 0$, define $h(x) = 0$. For a rational number in A of the form m/n, with natural numbers m, n having no common factors except 1, define $h(m/n) := 1/n$. (i.e. h is Thomae's function). Now let $\operatorname{sgn}(x)$ be the signum function. Show that the composite function $\operatorname{sgn} \circ h$ is not Riemann integrable on [0,1].

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) = f(x)$ for all $x \in \mathbb{R}$.

(a) Show that there exist real numbers α, β such that $f(x) = \alpha c(x) + \beta s(x)$ for all $x \in \mathbb{R}$.

(b) Let $f_1(x) := e^x$ and $f_2(x) := e^{-x}$ for $x \in \mathbb{R}$. Apply (a) to f_1 and f_2 and show that $c(x) = \frac{1}{2}(e^x + e^{-x})$ and $s(x) = \frac{1}{2}(e^x - e^{-x})$ for $x \in \mathbb{R}$.

Problem 4.

(a) If $a_n := 1$ when n is the square of a natural number and $a_n := 0$ otherwise, find the radius of convergence of $\sum a_n x^n$.

(b) If $b_n := 1$ when $n = m!$ for $m \in \mathbb{N}$ and $b_n := 0$ otherwise, find the radius of convergence of the series $\sum b_n x^n$.

Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

Problem 5. Suppose $V = U \oplus W$ and suppose $T : V \to V$ is linear. Show that U and W are both $T-invariant$ if and only if $TE = ET$ where E is the projection of V into U.

Problem 6. Let $\mathbf{M}$ denote the set of all $n-$square matrices A over a field K. Show that there exists a unique function $D : \mathbf{M} \to K$ such that:

(a) D is multilinear;
(b) D is alternating;
(c) $D(I) = 1$.

Problem 7. Prove that the geometric multiplicity of an eigenvalue λ of T does not exceed its algebraic multiplicity.

Problem 8. Let $T : V \to V$ be a linear operator, and let S be a (finite) basis of V. Then, for any vector v in V, $[T]_S[v]_S = [T(v)]_S$.

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

Problem 9. Use Green's Theorem to evaluate the integral

$$\oint_C (\cos x - 2y^4)dx + (x^2 - e^{-y^3})dy,$$

where C is the boundary of the half-disk $x^2 + y^2 \leq 9$, $y \geq 0$, oriented clockwise.

Problem 10. Suppose $u(x, y)$ and $v(x, y)$ have continuous second partial derivatives and satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that u and v are both harmonic (i.e. that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ and $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$).

Problem 11. Let $B = \{x^2 + y^2 + z^2 \leq 1\}$, and denote $B_{\frac{1}{n}} = \{x^2 + y^2 + z^2 \leq \frac{1}{n}\}, \forall n = 1, 2 \cdots$. Suppose f is a continuous function and $\|\nabla f\| \leq 1$ on B, suppose

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2 + y^2 + z^2}}, \quad in \quad B \setminus \{0\}.$$

(a) Show that

$$\lim_{n\to\infty} \int\int_{\partial B_{\frac{1}{n}}} \nabla f \cdot d\mathbf{S} = 0.$$

(b) Evaluate

$$\int\int_{\partial B} \nabla f \cdot d\mathbf{S}.$$

Problem 12. Use contour integration and residues to show that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin^2 \theta} = \frac{2\pi}{\sqrt{a(a+1)}},$$

for $a > 0$.