McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

APPLIED MATHEMATICS
Paper BETA

18 August, 2017
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the four modules (1) Numerical Analysis, (2) Partial Differential Equations, (3) Probability, (4) Discrete Mathematics, and (5) Optimization - Option A, and (6) Optimization - Option B. Each module is comprises 4 questions. You should answer 7 questions with at most 3 from each of your 3 selected modules.

The two optimization modules differ only in their 2 Discrete Optimization questions; Option A is for students who took Combinatorial Optimization (Math 552) only, and Option B is for student who took Submodular Optimization.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Numerical Analysis Module

[NA. 1]

Let Ω be a bounded smooth domain in $\mathbb{R}^m$ with $m = 1, 2, \ldots$ and let $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^m$ be the exterior normal vector to the boundary $\partial\Omega$. Suppose $f, g \in L^2(\Omega)$ and $a, b \in L^\infty(\Omega)$ such that $a, b \geq 0$ almost everywhere in Ω. Consider the boundary value problem:

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ a(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) + b(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{0.1}$$

(a) Formulate a weak formulation of (0.1) over $H^1(\Omega)$ functions and prove that it is well-posed.
(b) Give an example of a H^1-conforming finite element space and formulate the corresponding discrete problem.
(c) Is the resulting discrete problem always well-posed? Explain.

[NA. 2]

Let V, W be real Banach spaces and let $a : V \times W \to \mathbb{R}$ be a continuous bilinear form with continuity constant $C > 0$. Let V_h and W_h be conforming finite element spaces to V and W, respectively. Assume both the weak formulation (WF) and discrete problem (DP),

$$\begin{aligned} &\text{(WF)} \quad \text{Find } u \in V \text{ such that } a(u, w) = L(w), \text{ for any } w \in W. \\ &\text{(DP)} \quad \text{Find } u_h \in V_h \text{ such that } a(u_h, w_h) = L(w_h), \text{ for any } w_h \in W_h. \end{aligned}$$

have unique solutions $u \in V$ and $u_h \in V_h$ for any continuous linear functional $L : W \to \mathbb{R}$.

(a) Suppose there exists a constant $\alpha_h > 0$ such that for all $v_h \in V_h$,

$$\alpha_h \|v_h\|_V \leq \sup_{0 \neq w_h \in W_h} \frac{a(v_h, w_h)}{\|w_h\|_W}.$$

Prove the following abstract error estimate holds:

$$\|u - u_h\|_V \leq \left(1 + \frac{C}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V \tag{0.2}$$

Hint: Write $u - u_h = (u - v_h) + (v_h - u_h)$ and recall Galerkin orthogonality.

(b) Under what condition(s) does (0.2) imply convergence of the finite element solution u_h as $h \to 0$?

[NA. 3] THE HEAT EQUATION

Consider the following scheme for the 1D heat equation ($u_t = u_{xx}$)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = (1 - \theta) \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \theta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2},$$

for some parameter $\theta \in [0, 1]$, and $u(j\Delta x, n\Delta t) \simeq u_j^n$.

(a) Compute the order of local accuracy in space and time of the scheme as a function of θ.
(b) Compute the stability restriction as a function of θ.
(c) Is this scheme implicit or explicit? Discuss the meaning of these terms.
(d) What is the name of this numerical method in the case $\theta = 1/2$? Why is $\theta = 1/2$ a popular choice?

[NA. 4] INTERPOLATION.

Recall the following about Lagrange interpolation:

THEOREM:

Suppose $f \in C^n([a, b])$. Let $p(x)$ be the unique polynomial inP_{n-1} interpolating $f(x)$ at the nodes $\{x_j\}$ with $a = x_1 < x_2 < \cdots < x_n = b$. Then, for each $x \in [a, b]$ there is a $\xi \in [a, b]$, such that,

$$f(x) - p(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1) \cdots (x - x_n),$$

and if $h = \max_j (x_{j+1} - x_j)$,

$$\|f - p\|_\infty = \sup_{a \leq x \leq b} |f(x) - p(x)| \leq \frac{h^n}{4n} \|f^n\|_\infty.$$

Lagrange Interpolation

Given $N+1$ distinct, and equally spaced points (with spacing h), $\{(x_0, u_0), (x_1, u_1), \cdots, (x_N, u_N)\}$, consider the polynomial $p(x) \in \mathbb{P}_N$ which interpolate a function $f(x)$ at all points. Also, consider the error which is given by $E = \|f - p\|_{L^\infty(\Omega)}$, $\Omega = [-\pi, \pi]$. For each function below, if we were to plot E versus h in *loglog*, what would be the slope (asymptotically as $h \to 0$) of the line $E(h)$.

(a) $f(x) = sign(x)$,
(b) $f(x) = \sin(x)$,
(c) $f(x) = |x|$,
(d) $f(x) = x^5$.

Chebyshev Interpolation

(a) Discuss briefly the idea behind Chebyshev interpolation. (hint: look at the Theorem stated above)
(b) What is the distribution function for points in Chebyshev interpolation.
(c) Consider the function $f(x) = \frac{1}{1+25x^2}$ on the interval $[-1, 1]$. Discuss what happens in the case this function is interpolated with a *large* number of equidistributed points versus its Chebyshev interpolation with the same (large) number of points. You may use a sketch to help answer this question.
(d) What is the name of the phenomenon which happens when such a function (previous question) is interpolated with equally spaced points? What is the root cause of this phenomenon?

Partial Differential Equations Module

[PDE. 1] By using the method of characteristics, find two solutions of the problem

$$\begin{cases} x\partial_x u + y\partial_y u + \frac{1}{2}\left((\partial_x u)^2 + (\partial_y u)^2\right) = u & \text{in } \mathbb{R}\times(0,+\infty) \\ u(x,0) = \frac{1}{2}\left(1-x^2\right) & \text{on } \mathbb{R}\times\{0\}. \end{cases}$$

[PDE. 2] Let U be an open and bounded subset of $\mathbb{R}^n$ with smooth boundary. Let $u \in C^2\left(\overline{U}\times[0,\infty)\right)$ be a solution of the initial value problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } U\times(0,\infty) \\ u = 0 & \text{on } \partial U\times(0,\infty) \\ u = h & \text{on } U\times\{0\} \end{cases}$$

where $f \in C^0\left(\overline{U}\times[0,\infty)\right)$ and $h \in C^0\left(\overline{U}\right)$. Show that

$$\|u(\cdot,t)\|_{L^2(U)} \le \|h\|_{L^2(U)} + \int_0^t \|f(\cdot,s)\|_{L^2(U)}\,ds \qquad \forall t>0.$$

[PDE. 3] Let U be an open and bounded subset of $\mathbb{R}^n$, $n\ge 1$, $f\in L^2(U)$, and L be an operator of the form

$$Lu = -\sum_{i,j=1}^n \partial_{x_j}\left(a_{ij}(x)\,\partial_{x_i}u\right) + c(x)\,u.$$

Assume that $a_{ij}, c \in L^\infty(U)$, $c\ge 0$ a.e. in U, and L is uniformly elliptic in U.

(a) Justify why the problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{P}$$

has a unique weak solution.

(b) Show that

$$\|\nabla u\|_{H^1(U)} \le C\,\|f\|_{L^2(U)}$$

for some constant C independent of u and f.

Hint: As a first step, show that $\|\nabla u\|^2_{L^2(U)} \le C\,\|f\|_{L^2(U)}\,\|u\|_{L^2(U)}$.

[PDE. 4] Let $u\in C^2\left(\mathbb{R}^3\times[0,\infty)\right)$ be a solution of the initial value problem

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } \mathbb{R}^3\times(0,\infty) \\ u = h_0 & \text{on } \mathbb{R}^3\times\{0\} \\ \partial_t u = h_1 & \text{on } \mathbb{R}^3\times\{0\} \end{cases}$$

where $h_0, h_1 \in C^0\left(\mathbb{R}^3\right)$. For any fixed $(x_0,t_0)\in\mathbb{R}^3\times(0,\infty)$, we define

$$V(x) := F\left(x, t_0 - |x-x_0|\right) \qquad \forall x\in B(x_0,t_0)$$

where

$$F(x,t) := \frac{\nabla_x u(x,t)}{|x-x_0|} + \frac{x-x_0}{|x-x_0|^3}u(x,t) + \frac{x-x_0}{|x-x_0|^2}\partial_t u(x,t).$$

Show that div $V(x)=0$ for all $x\in B(x_0,t_0)$ and use this formula to obtain an expression of $u(x_0,t_0)$ in terms of h_0 and h_1.

Hint: Integrate div V *in* $B(x_0,t_0)\setminus B(x_0,\varepsilon)$ *for small* $\varepsilon>0$.

Probability Module

Instructions:

- You can use any result that is known to you, but you must state the name of the result (law / theorem / lemma / formula / inequality) that you are using, and show the work of verifying the condition(s) for that result to apply.
- For the problems with multiple parts, you can always assume the conclusion from the preceding part in order to solve the following part.

[PT. 1]

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that $\{A_n : n \geq 1\}$ is a sequence of **independent** events such that, if $\alpha_n := \min\{\mathbb{P}(A_n), 1 - \mathbb{P}(A_n)\}$, then $\sum\limits_{n\geq 1} \alpha_n = \infty$. Show that all the singletons of $(\Omega, \mathcal{F}, \mathbb{P})$ are $\mathbb{P}-$null sets, i.e., for every $\omega \in \Omega$ such that $\{\omega\} \in \mathcal{F}$, $\mathbb{P}(\{\omega\}) = 0$.

[PT. 2]

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n : n \geq 1\}$ be a sequence of **independent** and **identically distributed** random variables with the common distribution

$$\mathbb{P}(X_1 = k) = p_k \text{ for } 1 \leq k \leq L \text{where } L \in \mathbb{N}\backslash\{0\},\ p_k \in (0,1) \text{ and } \sum_{k=1}^{L} p_k = 1.$$

For every $n \geq 1$ and $1 \leq k \leq L$, set $S_n := \sum_{j=1}^{n} X_j$, $N_k^{(n)} := \sharp\{j : 1 \leq j \leq n,\ X_j = k\}$ (i.e., $N_k^{(n)}$ is the number of terms among $\{X_1, \cdots, X_n\}$ that take value k), and

$$P(n) := \prod_{k=1}^{L} p_k^{N_k^{(n)}}.$$

First show that

$$\lim_{n\to\infty} \frac{\log(P(n))}{n} \text{ exists a.s.},$$

and then determine what the limit is.
(*Hint:* It might be convenient if you introduce the random variable $Y_{k,j}$ such that $Y_{k,j} = 1$ if $X_j = k$ and $Y_{k,j} = 0$ otherwise.)

[PT. 3]

Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n : n \geq 0\}, \mathbb{P})$, let $\{Y_n : n \geq 1\}$ be an **adapted** sequence of random variables such that for every $n \geq 1$, $Y_n \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[Y_n|\mathcal{F}_{n-1}] = 0$. Further assume that

$$\sum_{n\geq 1} \frac{\mathbb{E}\left[Y_n^2\right]}{n^2} < \infty.$$

Set $X_0 \equiv 0$, and for each $n \geq 1$, set

$$X_n := \sum_{j=1}^{n} \left(\frac{Y_j}{j}\right) \text{ and } S_n := \sum_{j=1}^{n} Y_j.$$

(i) Show that $\{X_n : n \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_n : n \geq 0\}$.

(ii) Prove that as $n \to \infty$, $\frac{S_n}{n} \to 0$ a.s..

[PT. 4]

Let $t > 0$ be fixed. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{X_n : n \geq 1\}$ is a sequence of **independent** and **identically distributed Poisson** random variables with parameter t, i.e.,

$$\mathbb{P}(X_1 = k) = \frac{t^k}{k!} e^{-t} \text{ for each } k \in \mathbb{N}.$$

For each $n \geq 1$, set $S_n := \sum_{k=1}^{n} X_k$. Apply the Central Limit Theorem to show that

$$\lim_{n\to\infty} \mathbb{P}\left(S_n \leq n\right) = \begin{cases} 0, & \text{if } t > 1, \\ \frac{1}{2}, & \text{if } t = 1, \\ 1, & \text{if } 0 < t < 1. \end{cases}$$

Discrete Mathematics Module

Graph Theory

[DM. 1] (Connectivity)

a) State Menger's theorem.

b) Let G be 2-connected, loopless graph, which is not bipartite, and let $u, v \in V(G)$ be distinct. Show there is a path in G withs end u and v and even number of edges.

c) Let $k \geq 3$ be an integer. Show that every k-connected graph G with $|V(G)| \geq 2k$ contains a cycle of length at least $2k$.

[DM. 2] (Planarity)

a) State Kuratowski's theorem.

A graph G is *outerplanar* if it can be drawn in the plane so that every vertex is incident with the unbounded region.

b) Show that a graph G is outerplanar if and only if G has no K_4 or $K_{2,3}$ minor.

c) What is the maximum possible number of edges in a simple outerplanar graph with n vertices?

d) What is the maximum chromatic number of a simple outerplanar graph?

Combinatorics

[DM. 3] (Intersecting Families)

a) State Erdős-Ko-Rado theorem.

b) A set system $\mathcal{F}$ is 2-*intersecting* if $|A \cap B| \geq 2$ for all $A, B \in \mathcal{F}$. What is the maximum possible size of a 2-intersecting hypergraph $\mathcal{F} \subseteq [100]^{(5)}$?

c) Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an intersecting set system. Show that there exists an intersecting set system $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{P}([n])$ and $|\mathcal{F}'| = 2^{n-1}$.

[DM. 4] (Ramsey theory)

a) State van der Waerden's theorem and Hales-Jewett theorem.

b) Derive van der Waerden's theorem from the Hales-Jewett theorem.

c) Let n, k be positive integers such that $k \geq 2$ and $2^k > n^2$. Show that it is possible to color $[n]$ with two colors so that neither color contains a k-term arithmetic progression.

Optimization Module (Option A)

[O-A. 1] (Continuous)

(a) Consider an unconstrained problem $\min_{x\in\mathbb{R}^n} f(x)$ where $f:\mathbb{R}^n\to\mathbb{R}$ is convex.

I) Show that $\bar{x}$ is a global minimizer of f if and only if $\bar{x}$ is a local minimizer of f.

II) Suppose that, in addition, f is continuously differentiable. Show that $\bar{x}$ is a (global=local) minimizer of f if and only if $\nabla f(\bar{x}) = 0$.

(b) Next consider the specific objective function $f : x \in \mathbb{R}^n \mapsto \frac{1}{2}x^{\mathbb{T}}Ax - b^{\mathbb{T}}x$, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $b \in \mathbb{R}^n$.

I) Show that f has a unique global minimizer $\bar{x}$ and determine it.

II) Show that Newton's method (with constant step size 1) for minimizing f converges to the global minimizer $\bar{x}$ in at most one iteration independent of the starting point.

[O-A. 2] (Continuous)

Consider the constrained minimization problem

$$\begin{array}{ll} \min_{x\in\mathbb{R}^2} & -2(x_1-2)^2 + x_2^2 \\ \text{s.t.} & -x_2^2 - x_1 + 2 \geq 0, \\ & x_2 \geq 0, \\ & x_1 \geq -5. \end{array}$$

(a) Does LICQ hold at $\bar{x} = [2,0]^{\mathbb{T}}$?

(b) Show that the KKT conditions hold at $\bar{x}$ and that the corresponding Lagrange multipliers are unique.

(c) Write down the first-order feasible directions, the tangent and the critical cone at $\bar{x}$.

(d) Do the second order necessary conditions hold at $\bar{x}$?

[O-A. 3] (Discrete)

Let V be a finite set and $\mathcal{I}$ be a family of (so-called independent) subsets of V.

(a) **[4 marks]** Define what it means for $M = (V, \mathcal{I})$ to be a matroid.

(b) **[6 marks]** Prove that if M is a matroid then the greedy algorithm correctly solves the weighted independent set problem:

$$\max w(I) = \sum_{v\in I} w(v) : I \in \mathcal{I}$$

Here $w \in \mathbf{Z}^V$.

[O-A. 4] (Discrete)

Let $P = \{x : Ax \leq b\}$ be a polyhedron in $\mathbf{R}^n$.

(a) **[2 marks]** Define what it means to be an *extreme ray* of P.

(b) **[6 marks]** Give three equivalent definitions for a vertex of P. Justify the equivalences.

(c) **[2 marks]** State the polyhedral decomposition theorem.

Optimization Module (Option B)

[O-B. 1] (Continuous)

(a) Consider an unconstrained problem $\min_{x\in\mathbb{R}^n} f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

I) Show that $\bar{x}$ is a global minimizer of f if and only if $\bar{x}$ is a local minimizer of f.

II) Suppose that, in addition, f is continuously differentiable. Show that $\bar{x}$ is a (global=local) minimizer of f if and only if $\nabla f(\bar{x}) = 0$.

(b) Next consider the specific objective function $f : x \in \mathbb{R}^n \mapsto \frac{1}{2}x^{\mathbb{T}}Ax - b^{\mathbb{T}}x$, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $b \in \mathbb{R}^n$.

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Consider the constrained minimization problem

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(a) Does LICQ hold at $\bar{x} = [2, 0]^{\mathbb{T}}$?

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Let V be a finite set and $\mathcal{I}$ be a family of (so-called independent) subsets of V.

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$$\max w(I) = \sum_{v\in I} w(v) : I \in \mathcal{I}$$

Here $w \in \mathbf{Z}^V$.

[O-B. 4] (Discrete)

Let f be a submodular function over a finite ground set V. Consider the unconstrained problem:

$$\max f(S) : S \subseteq V$$

(a) **[3 marks]** Why should we not expect a good algorithm to solve this problem for general f?

(b) **[7 marks]** Suppose that f is non-negative. Describe a simple randomized algorithm which yields a 4-approximation for the above problem. Briefly outline how to derive the approximation bound.