McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

14 August, 2018
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Question 1.

Let A be a square complex matrix. Show that A and A^t have the same characteristic and the same minimal polynomials.

Question 2.

Let A be a complex matrix. Show that the row rank of A is equal to the column rank of A.

Question 3.

Compute the determinant of a Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

Question 4.

Denote by $M_n(\mathbb{C})$ the complex vector space of all $n \times n$ complex matrices. Fix $A \in M_n(\mathbb{C})$, and denote by $T_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ the linear transformation given by

$$T_A(X) = AX, \qquad X \in M_n(\mathbb{C}).$$

(a) Show that T_A and A have the same eigenvalues.

(b) Express the characteristic polynomial of T_A in terms of the characteristic polynomial of A.

Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

Question 5.

If a sequence $\{a_n\}$ of positive numbers converges to a finite number A, show that

$$\lim_{n\to\infty} (a_1 a_2 \cdot ... \cdot a_n)^{1/n} = A.$$

Question 6.

Let $a_n \geq 0$. Suppose $\sum_n a_n$ converges. Show that

$$\sum_n \frac{\sqrt{a_n}}{n^{3/4}}$$

also converges.

Question 7. Define a sequence of functions $f_n : [0, \pi] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \sin(nx), & x \in [0, \pi/n], \\ 0, & otherwise. \end{cases}$$

Determine whether $\{f_n\}$ converges (a) pointwise; (b) uniformly.

Question 8.

Let $a_0 = 1/2$ and consider the sequence defined recursively by

$$a_n = \ln(1 + a_{n-1}).$$

(a) Show that $\lim_{n\to\infty} a_n = 0$.

(b) Determine the radius of convergence of the power series $\sum_n a_n x^n$.

Solve any three out of the four questions 9,10,11,12.

Question 9.

Use the method of Frobenius to find a general formula for the coefficient a_n in a series expansion about $x = 0$ for a solution to the equation $xw''(x) - w'(x) - xw(x) = 0$.

Question 10.

Solve an initial value problem:

$$y'''(x) - 4y''(x) + 7y'(x) - 6y(x) = 0; \qquad y(0) = 1, y'(0) = 0, y''(0) = 0.$$

Question 11.

Let C be the triangular boundary of the plane $6x + 2y + 3z = 6$ in the first octant. Compute

$$\int_C F \cdot dS$$

where the vector field F is given by $F(x, y, z) = (yz, -xz, xy)$.

Question 12.

Find the volume between the paraboloids $z = 10 - x^2 - y^2$ and $z = x^2/9 + y^2/4$.