McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

<u>PURE AND APPLIED MATHEMATICS</u>
<u>Paper ALPHA</u>

10 May 2016 13:00 - 17:00

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

Problem 1. Prove that the function $\sin x^2$ is not uniformly continuous on $\mathbb{R}$.

Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that

$$f(x) \leq \int_a^x f(t)dt$$

for all $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Problem 3. Let $x_1 > 0$ and $x_{n+1} = \dfrac{1}{4x_n + 3}$ for all $n \in \mathbb{N}$. Show that (x_n) is a contractive sequence and find $\lim (x_n)$.

Problem 4. Let $S := \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $f : [0, 1] \to \mathbb{R}$, $f(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

Use the squeeze theorem to prove that f is Riemann integrable on $[0, 1]$ and determine $\displaystyle\int_0^1 f$.

Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

Problem 5. Let $\mathcal{C}[0,1]$ be the vector space of continuous real-valued functions on the unit interval and let $f_1, f_2 \in \mathcal{C}[0,1]$. Prove that f_1 and f_2 are linearly independent if and only if there exist points $a_1, a_2 \in [0,1]$ such that the vectors $\big(f_1(a_1), f_1(a_2)\big)$ and $\big(f_2(a_1), f_2(a_2)\big)$ in $\mathbb{R}^2$ are linearly independent.

Problem 6. Let $T, R \colon V \to V$ be two linear maps of rank 1 with the same kernel and the same image. Prove that $T \circ R = R \circ T$.

Problem 7. Let V be a finite dimensional inner product space and U a subspace of dimension 2. Let $(v_1, \dots, v_n)$ be an orthonormal basis for V. Compute the sum of the squares of the norms of the orthogonal projections of $v_1, \dots, v_n$ onto U.

Problem 8. Let $A = [a_{ij}]$ and let B be the matrix obtained from A by replacing the ith row of A by the row vector $(b_{i1}, ..., b_{in})$. Show that:

$$|B| = b_{i1}A_{i1} + b_{i2}A_{i2} + \cdots + b_{in}A_{in}.$$

Furthermore, show that, for $j \neq i$,

$$a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + a_{jn}A_{in} = 0 \text{ and } a_{1j}A_{1i} + a_{2j}A_{2i} + \cdots + a_{nj}A_{ni} = 0.$$

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

Problem 9. Let $\mathbf{F} = (P, Q)$ be the vector field in $\mathbb{R}^2$ defined as

$$P(x,y) = \begin{cases} \frac{x+y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases} \qquad Q(x,y) = \begin{cases} \frac{-x+y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(i) Show that $\mathbf{F}$ is a gradient vector field in $\mathbb{R}^2 \setminus \{0 \leq x < \infty, y = 0\}$.

(ii) Let $D = \{x^{2016} + y^{\frac{1}{2016}} \leq 1\}$, compute the line integral $\int_{\partial D} P dx + Q dy$ where the positive orientation is given for ∂D.

Problem 10. Denote $B_r = \{x^2 + y^2 + z^2 \leq r\}$, and suppose f is a C^2 function in $B_2 \setminus \{0\}$ satisfying

$$|\nabla f(x,y,z)| \leq \frac{1}{\sqrt{x^2+y^2+z^2}}, \quad \text{and}$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}, \quad \forall (x,y,z) \in B_2 \setminus \{0\}.$$

Evaluate

$$\int\int_{\partial B_{\frac{\pi}{2}}} \nabla f \cdot d\mathbf{S}.$$

Note that f *is not assumed to be defined at the origin.*

Problem 11. Let f be a holomorphic function in the whole complex plane. Show that the following two statements are equivalent:

(a) There exist two real numbers C and a such that $|f(z)| \leq Ce^{a|z|}$ for all $z \in \mathbb{C}$.

(b) There exists a real number M such that $|f^{(n)}(0)| \leq M^{n+1}$ for all $n \in \mathbb{N}$.

Problem 12. Evaluate the integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}, dx$$

by using the residue formula.