**McGill University**
**Department of Mathematics and Statistics**

**Ph.D. preliminary examination, PART A**

**APPLIED MATHEMATICS**
**Paper BETA**

19 August, 2016
1:00 p.m. - 5:00 p.m.

**INSTRUCTIONS:**

**(i) This paper consists of the three modules (1) Analysis (2) Numerical Analysis and (3) Partial Differential Equations, each of which comprises 4 questions. You should answer 7 questions with at most 3 from each module. If you exceed any of these limits, then clearly identify which questions should be graded.**

**(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.**

**(iii) Leave enough margins on all sides of the page to permit easy scanning and copying. Each question must be answered on consecutive pages without interruption, and only on the pages on the right side.**

This exam comprises this cover and 4 pages of questions.

**Analysis Module**

**[AN. 1]**
Let $\delta > 0$ be fixed. Show that the set of all real numbers $x \in [0,1]$ such that there exist infinitely many pairs $p, q \in \mathbf{N}$ such that $|x - p/q| < 1/q^{2+\delta}$ has Lebesgue measure 0.

**[AN. 2]**
Let $f$ be a uniformly continuous function on $\mathbb{R}$. Suppose that $f \in L^p$ for some $p$, $1 \leq p < \infty$. Prove that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

**[AN. 3]**
(a) Give a definition of $||f||_\infty$ of a measurable complex function $f$.

(b) Recall that the essential range of a function $f \in L^\infty(\mu, \mathbb{C})$ is the set consisting of complex numbers $w$ such that

$$\mu(\{x : |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that $R_f$ is compact.

(c) Show that $||f||_\infty = \sup_{w \in R_f} |w|$.

**[AN. 4]**
(a) Give a definition of a locally compact topological space.

(b) Give an example of a Borel measure $\mu$ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}, \mu)$ is locally compact and explain why it is so.

(c) Give an example of a Borel measure $\mu$ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}, \mu)$ is not locally compact and explain why it is so.

**Numerical Analysis module**

**[NA. 1]** *Quadrature and Newton's Method*

Let $f(x) = \dfrac{1}{4}(x-5)^4 + x$.

(a) Compute $f'(x), f''(x)$. Is $f$ convex? Explain your answer.
(b) Find the minimizer of $f(x)$.
(c) Write out the formula for Newton's method for function minimization.
(d) Compute two Newton iterations, for $x^0 = 4.5$. Are the values approaching the minimum?
(e) Approximate the integral $\int_0^3 \frac{1}{x^2+2}\,dx$ using four equally spaced points and the composite trapezoidal rule. (Small penalty if you don't simplify your answer).

**[NA. 2]** *Convexity, Polyhedra and Polytopes*

(a) Suppose $f(x)$ and $g(x)$ are convex functions. Prove, using the definition of convexity, that $h(x) = \max(f(x), g(x))$ is convex.
(b) Consider the quadrilateral $Q$ in the plane with vertices

$$v_1 = (-4,0), \quad v_2 = (4,0), \quad v_3 = (0,2), \quad v_4 = (0,-2).$$

Find a matrix $A$ and a vector $b$ so that

$$Q = \{x \mid Ax \leq b\}$$

(c) Write out the representations for the $n$-dimensional hypercube, $C$, as (i) an intersection of halfspaces. (ii) the convex hull of the vertices. Recall that the polar of a set $A$ is given by

$$A^\circ = \{y \mid y^T x \leq 1 \text{ for all } x \in A\}$$

Find the polar of $C$, and write it in both representations above.

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**[NA. 3]** The conserved quantity $q$ with flux function $F$ satisfies the conservation law

$$\frac{\partial}{\partial t}q(x,t) + \frac{d}{dx}F(q,x,t) = 0, \qquad \text{for } x \in [0,1] \tag{1}$$

along with no-flux boundary conditions

$$F(q,x,t) = 0, \qquad \text{for } x = 0, 1.$$

(a) Show that the total mass of $q$ is conserved.

(b) Assume that Fick's law of diffusion holds, so that $F(q(\cdot),x,t) = -\beta(x)q_x(x,t)$. The energy is $E(t) = \frac{1}{2}\int_0^1 q^2(x,t)dx$. Prove that the energy is non-increasing.

(c) Let $G = [0, h, \dots, 1]$ be the finite difference grid, where $h = 1/(n-1)$. Let $\partial_x^h$ be the forward difference operator on the grid. Let $Q = (Q_0, \dots, Q_n)$ be a grid function. Write down the matrix $M$ which maps the grid function $Q$ to the grid function $\partial_x^h Q$, and includes the boundary conditions.

(d) Let $Q(t) = (Q_0(t), \dots, Q_n(t))$ be a time-dependent grid function. Consider the method of lines for the PDE,

$$\frac{d}{dt}Q + M^{\intercal}(diag(\beta)MQ)$$

Prove that mass is conserved, and that the discrete energy $E^h(t) = \frac{h}{2}\langle Q(\cdot,t), Q(\cdot,t)\rangle$ is non-increasing.

**[NA. 4]**

(a) Consider the initial value problem for the variable coefficient parabolic equation on the real line

$$\begin{cases} u_t(x,t) + f(x,t)u_x(x,t) = \frac{1}{2}\sigma^2(x,t)u_{xx}, \text{ for } 0 < t \le T, \\ u(x,0) = u_0(x) \end{cases}$$

Assume that

$$|f(x,t)| \le \bar{f} \qquad \text{and} \qquad 0 < \sigma_0 \le \sigma(x,t) \le \bar{\sigma}, \qquad \text{for all } (x,t) \in \mathbb{R} \times [0,T)$$

Approximate the PDE using the Forward Euler method in time, and centred differences for $u_{xx}$ and $u_x$. Write down the discretization in the explicit form, and give conditions on $h, dt$ so that it is an averaging finite difference method. **[12 marks]**

(b) Consider the initial value problem for the second order Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, u_x) = \frac{1}{2}\sigma^2 u_{xx}, \quad \text{for } 0 < t \le T, \\ u(x,0) = u_0(x) \end{cases}$$

with $\sigma > 0$ constant. Here

$$H(x,p) = \inf_{|a| \le 1} \{ap + |a|\}$$

Approximate the PDE using Forward Euler method in time, and centred differences for $u_{xx}$ and $u_x$. Write down the discretization in the explicit form, and give conditions on $h, dt$ so that it is a monotone finite difference method. **[8 marks]**

**Partial Differential Equations Module**

**[PDE 1.]** We consider the boundary value problem

$$\begin{cases} \partial_y u + (2x+u)\,\partial_x u = x + 2u & \text{in } U_\lambda \\ u(x, \lambda x) = 1 + \mu x & \text{on } \partial U_\lambda \end{cases} \tag{$P$}$$

where $U_\lambda = \{(x,y) : y > \lambda x\}$ and $\mu, \lambda \in \mathbb{R}$

(a) For which values of $\lambda$ and $\mu$ does the problem (P) satisfy the noncharacteristic boundary condition?
(b) Give all solutions of the problem (P) in case $\lambda = 0$ and $\mu = -1$.
(c) Show that there does not exist any solution of the problem (P) in case $\lambda = 1$ and $\mu \neq 2$.

**[PDE 2.]**

(a) Let $U$ be an open and bounded subset of $\mathbb{R}^n$, $n \geq 1$. Show that for any functions $u, v \in C^2(U) \cap C^0(\overline{U})$ such that $\Delta u \geq \Delta v$ in $U$ and $u \leq v$ on $\partial U$, we have $u \leq v$ in $U$.
(b) Now we assume that $n = 2$ and $U = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$ for some real numbers $R_2 > R_1 > 0$. Show that for any function $u \in C^2(U) \cap C^0(\overline{U})$ such that $\Delta u \geq 0$ in $U$, we have

$$M(r) \leq \frac{M(R_1)\ln(R_2/r) + M(R_2)\ln(r/R_1)}{\ln(R_2/R_1)} \quad \forall r \in (R_1, R_2)$$

where $M(r) = \sup\{u(x) : |x| = r\}$.
Hint: Remember that the function $v(x) = a + b\ln|x|$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ for all $a, b \in \mathbb{R}$.

**[PDE 3.]** Let $U$ be an open and bounded subset of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary. We consider the problem

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } U \times (0,\infty) \\ u = g & \text{on } \Gamma \\ \partial_t u(\cdot, 0) = h & \text{on } U \end{cases} \tag{P}$$

where $\Gamma = (\partial U \times (0,\infty)) \cup (U \times \{0\})$, $g : \Gamma \to \mathbb{R}$, and $h : U \to \mathbb{R}$.
Use energy methods to prove the two following results:

(i) There exists at most one solution $u \in C^2(\overline{U} \times 0, \infty)$ of the problem (P).
(ii) If $u \in C^2(\overline{U} \times 0, \infty)$ is a solution of the problem (P) and $g(\cdot, 0) \equiv h \equiv 0$ on a ball $B(x_0, R_0) \subset U$, then $u(\cdot, t) \equiv 0$ on the ball $B(x_0, R_0 - t)$ for all $t \in (0, R_0)$.

**[PDE 4.]** Let $U$ be an open and bounded subset of $\mathbb{R}^n$, $n \geq 1$. Given $f \in L^2(U)$, we define the functional $I : H_0^1(U) \to \mathbb{R}$ as

$$I(u) = \int_U \left(\frac{1}{2}|Du|^2 - fu\right) dx \quad \forall u \in H_0^1(U).$$

(a) Define what it means for $u \in H_0^1(U)$ to be a weak solution of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{$P$}$$

(b) Show that $u \in H_0^1(U)$ is a weak solution of the problem (P) if and only if

$$I(u) \leq I(v) \quad \forall v \in H_0^1(U).$$

Hint: Write $v = u + w$ and calculate $I(u+w) - I(u)$.