McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

<u>PURE MATHEMATICS</u>
<u>Paper BETA</u>

18 August, 2017
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[ALG. 1]

Let G be a finite abelian group.

a) Show that the irreducible representations of G are 1-dimensional.
b) Show that the set $\widehat{G}$ of all such representations can be endowed with the structure of an abelian group, in such a way that G is canonically isomorphic to $\widehat{\widehat{G}}$.

[ALG. 2]

Let K be a field.

a) Show that the natural map $K[X] \otimes_K K[Y] \to K[X, Y]$ is an isomorphism of K-algebras.
b) Show that the natural map $K(X) \otimes_K K(Y) \to K(X, Y)$ is injective. Show that there exists a surjective, non injective, homomorphism of K-algebras $K(X) \otimes_K K(Y) \to K(T)$. Show that the natural map $K(X) \otimes_K K(Y) \to K(X, Y)$ fails to be surjective.

[ALG. 3]

Consider the field $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Compute

a) the degree of $K/\mathbb{Q}$
b) the Galois group of $K/\mathbb{Q}$
c) a primitive element of $K/\mathbb{Q}$
d) an explicit basis of K over $\mathbb{Q}$.

[ALG. 4]

Let q be an odd prime power and let $G = \mathrm{PSL}_2(F_q)$

a) Compute the order of G
b) Let $x \in F_q^*$. Compute the order of the conjugacy class of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.
c) Give a condition on $x \in F_q^*$ ensuring that the two matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are not conjugate.
d) Show that the number of unipotent elements in G is q^2.

Analysis Module

[AN. 1]

Let $\|\cdot\|_p$ be the $L^p(\mathbb{R}^n)$ norm, $1 \le p \le \infty$, and let $C_c^\infty(\mathbb{R})$ be the set of compactly supported C^∞ functions.

a) Show, including the explicit constant, that for $\phi \in C_c^\infty(\mathbb{R})$, $\|\phi\|_\infty \le \frac{1}{2}\int_{-\infty}^{\infty} |\phi'(t)|dt$.

b) Suppose that there is a $C > 0$ such that for all functions $\phi \in C_c^\infty(\mathbb{R}^n)$, there is an inequality of the form $\|\phi\|_q \le C\|\nabla\phi\|_p$. Show that one would necessarily then have $q^{-1} = p^{-1} - n^{-1}$.
(Hint: consider the functions ϕ_t defined by $\phi_t(x) = \phi(tx)$.)

c) When $n = 2$, part (b) suggests that one might have an inequality of the form $\|\phi\|_\infty \le C\|\nabla\phi\|_2$. Show that there is no $C > 0$ such that this inequality holds for all $\phi \in C_c^\infty(\mathbb{R}^2)$.
(Hint: Consider sums whose derivatives are orthogonal and use the fact that if $\{f_n\}_{n=1}^\infty$ are orthogonal, then $\|\sum_{k=1}^N f_n\|_{L^2}^2 = \sum_{n=1}^N \|f_n\|_{L^2}^2$.)

[AN. 2]

Let μ be a non-negative Borel measure on $\mathbb{R}^n$ such that $\mu(A) < \infty$ for each bounded Borel subset $A \subset \mathbb{R}^n$.

a) Setting $\overline{B}_\rho(x) = \{y \in \mathbb{R}^n \mid |y - x| \le \rho\}$, prove that for each $\rho > 0$, $x \mapsto \mu(\overline{B}_\rho(X))$ is an upper semi-continuous function on $\mathbb{R}^n$. (A real-valued function θ on $\mathbb{R}^n$ is upper semi-continuous if for all $x \in \mathbb{R}^n$, $\theta(x) \ge \limsup_{y\to x} \theta(y)$.)

b) Give an example of a Borel measure μ as above and $\rho > 0$ such that $x \mapsto \mu(\overline{B}_\rho(X))$ is not continuous.

[AN. 3]

Let $f \in L^2(\mathbb{R})$ and let

$$g(x) = \int_x^{x+1} f(x)dx.$$

Prove that

$$\lim_{x\to\infty} g(x) = 0.$$

[AN. 4]

Let $p, q, r > 1$; let $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R}), h \in L^r(\mathbb{R})$; and let $1/p + 1/q + 1/r = 1$. Show that $fgh \in L^1(\mathbb{R})$, and that $||fgh||_1 \le ||f||_p||g||_q||h||_r$.

Geometry and Topology Module

[GT. 1]

Consider $\mathbf{R}^{2n}$ $(n \geq 2)$ endowed with a 2-form Ω satisfying

$$\Omega^n \neq 0\,,$$

and

$$d\Omega = 0\,,$$

at every point of $\mathbf{R}^{2n}$. An even-dimensional manifold M_{2k} $(k \geq 1)$ embedded in $\mathbf{R}^{2n}$ is said to be a *symplectic submanifold* if the pull-back $\tilde{\Omega}$ of Ω to M_{2k} satisfies

$$\tilde{\Omega}^k \neq 0$$

at every point of M_{2k}. Show that $\mathbf{R}^{2n}$ admits no *compact* symplectic submanifolds. (Hint: show that $\tilde{\Omega}^k$ is exact and apply Stokes.)

[GT. 2]

Recall that an n-dimensional manifold M is orientable if there exists an atlas such that for all charts (U_α, ϕ_α) and (U_β, ϕ_β), the maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ have *positive* Jacobian determinant.

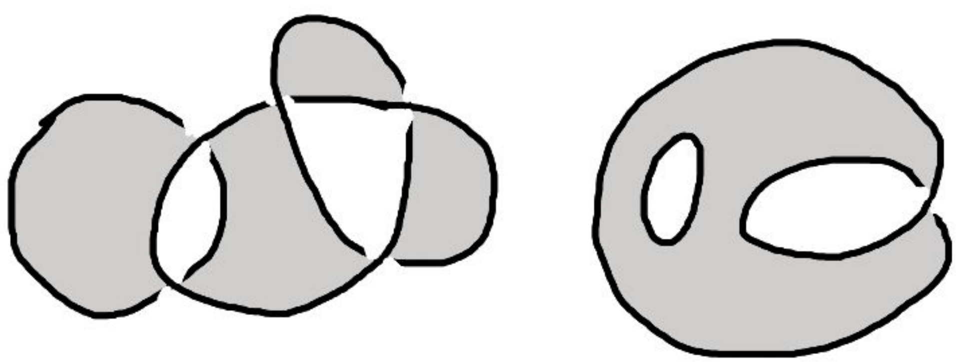
a) Show that M is orientable iff there exists an n-form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$. (Hint: Use partition of unity.)

b) Use item a) to show that every Lie group is orientable. (Hint: Use the fact that right- or left- translation on a Lie group is a diffeomorphism.)

[GT. 3]

Consider the two surfaces with boundary below.

(1) Describe a homeomorphism between them.
(2) Compute their genus.
(3) Find all other compact surfaces that have the same homotopy type as these surfaces.
(4) Draw a picture of each one, and within it, draw a graph that it deformation retracts to.



[GT. 4]

Let S be the surface with boundary obtained from a torus by removing the interior of an embedded open disk. Consider the following two embeddings of a circle $C \hookrightarrow S$ illustrated below. Show that these two embeddings are not homotopic in the following three ways:

(1) Using the fundamental group.
(2) Using Homology.
(3) Exhibiting a covering space where one lifts but the other doesn't.