McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

8 May, 2018
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Question 1.

Let $n \geq 1$ be an integer, and let A_n be the square matrix of size $2n-1$ with entries (a_{ij}) in $\mathbb{R}$ where $a_{ij} = 1$ if $i + j = 2n$ or $i = n$ or $j = n$, and $a_{ij} = 0$ otherwise. For instance, for $n = 4$ we get the matrix[1]

$$A_4 = \begin{pmatrix} & & & 1 & & & 1 \\ & & & 1 & & 1 & \\ & & & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & 1 & & & \\ & 1 & & 1 & & & \\ 1 & & & 1 & & & \end{pmatrix}$$

Find $\det(A_n)$.

Question 2.

Let A be an invertible matrix.

Prove that if rank $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{rank } A$, then $D = CA^{-1}B$.

Question 3.

Let V be the $\mathbb{R}$-vector space of polynomials of degree at most 2 over $\mathbb{R}$. Consider for any real numbers a, b the linear map

$$\phi_{a,b} : V \longrightarrow \mathbb{R}, \quad p(x) \longmapsto \int_a^b p(x)dx$$

Prove that $\{\phi_{0,1}, \phi_{1,2}, \phi_{2,3}\}$ is a basis for the dual space V^*.

Question 4.

Let $V \subset \mathbb{R}^{11}$ be a vector subspace of dimension 4, and consider the family $\mathcal{A}$ of all linear maps $L : \mathbb{R}^{11} \to \mathbb{R}^9$ whose nullspace contains V.

Show that $\mathcal{A}$ is a vector space and compute its dimension.

[1] The zero entries in the matrix are not displayed for clarity.

Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

Question 5.

Let $a_n \geq 0, \sum a_n < \infty$. For each of the following statements, either give a proof or provide a counterexample.

a) $\sum na_n^2 < \infty$.

b) $\sum a_n e^{a_n} < \infty$.

c) $\liminf_{n\to\infty}(na_n) = 0$.

Question 6.

Suppose that $f : (0,1) \to \mathbb{R}$ is bounded but $\lim_{x\to 0} f(x)$ does not exist. Show that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $(0,1)$ which have all of the following three properties:

- $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$;
- both $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist;
- $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

Question 7.

a) Let $K \subset \mathbb{R}$ be compact, and let $x \in \mathbb{R}$. Prove that there exists $y \in K$, such that for every $z \in K$, we have $|x - y| \leq |x - z|$.

b) Give an example of two disjoint sets $C, D \subset \mathbb{R}$ such that

$$\inf_{x\in C, y\in D} |x - y| = 0.$$

Question 8.

If G_n is an open and dense subset of $\mathbb{R}$ for each $n = 1, 2, 3, \ldots$, show that $\cap_{n=1}^{\infty} G_n \neq \emptyset$.

Solve any three out of the four questions 9,10,11,12.

Question 9.

Find a general solution for $x > 0$ of the (Cauchy-Euler) equation

$$x^2y''(x) + 3xy'(x) + 5y(x) = 0.$$

Question 10.

Find a solution to the following initial value problem:

$$y'' - y' - 2y = \cos x - \sin(2x); \qquad y(0) = -7/20, y'(0) = 1/5.$$

You may find it convenient to use the method of undetermined coefficients.

Question 11.

Using polar coordinates, compute the area of the region bounded by the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and lying *outside* the curve $x^2 + y^2 = 1$. You may find the following formulas useful: $\cos(2t) = \cos^2 t - \sin^2 t$, and $\sin(2t) = 2 \sin t \cos t$.

Question 12.

Find the centroid of a region D bounded by the surfaces $x^2 + z^2 = a^2$, $y^2 + z^2 = a^2$, and $z = 0$; here $D \subset \{(x, y, z) : z \geq 0\}$.