McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

<u>PURE MATHEMATICS</u>
<u>Paper BETA</u>

13 May, 2016
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[**ALG. 1**] Classify the finite groups having exactly two conjugacy classes.

[**ALG. 2**] Let **Set** denote the category of sets, and $\mathbf{Set}^*$ denote the groupoid whose objects are sets, and whose arrows are bijections between sets. Let $F\colon \mathbf{Set}^* \to \mathbf{Set}$ denote the covariant powerset functor. Recall that the functor F takes a set X to the set of subsets of X, and a bijection $f\colon X \to Y$ to the arrow f_* given by $f_*(A) := \{f(x),\ x \in A\}$ for $A \subset X$.

Consider also $G\colon \mathbf{Set}^* \to \mathbf{Set}$ defined by $G(X) := \{\text{all maps } g\colon X \to X\}$ and $G(f)(g) := fgf^{-1}$ where X is a set and $f\colon X \to Y$ is a bijection.

(i) Show that G is a functor.
(ii) If $g\colon X \to X$ is a map let $\eta(g) := \{x \in X,\ g(x) = x\}$ denote the set of fixed points of g. Show that η defines a natural transformation $G \Rightarrow F$.

[**ALG. 3**] State the primitive element theorem. If $K = F_p(x)$ and α is a p-root of x, show that $K(\alpha)/K$ is a primitive algebraic extension which is not separable. Give an example of a finite extension which is not primitive.

[**ALG. 4**] Compute the Galois group of $X^4 - 2$.

Analysis Module

[AN. 1]

(a) Let (X, μ) be a σ-finite measure space, and let $1 \leq p < \infty$. Describe the set of bounded linear functionals Λ on $L^p(X, \mu)$ (you don't need to prove that result). Give an expression for the norm of a bounded linear functional on L^p. [6 marks]

(b) State Banach-Steinhaus theorem (you don't need to prove it). [6 marks]

(c) Let $\{f_n\}$ be a sequence of functions in $L^4(X, \mu)$ such that

$$\lim_{n\to\infty} ||f_n||_4 = \infty.$$

Show that there exists a function $g \in L^{4/3}(X, \mu)$ such that

$$\lim_{n\to\infty} |\int_X f_n g\mu| = \infty.$$

Hint: use parts (a) and (b). [8 marks]

[AN. 2]

(a) State Riemann-Lebesgue lemma for Fourier series.

(b) Let $f \in C^1(\mathbb{T})$. Express the Fourier coefficients of f' through the Fourier coefficients of f.

(c) Let $f \in C^k(\mathbb{T}^n)$ (f is k times continuously differentiable). Show that

$$\lim_{n\to\infty} \widehat{f}(n)/n^k = 0.$$

Hint: use parts (a) and (b).

[AN. 3] Let $0 < \epsilon < 1$ and m be the Lesbesgue measure. Construct an open set $E \subset [0, 1]$ such that E is dense in $[0, 1]$ and $m(E) = \varepsilon$.

[AN. 4] Construct a sequence of continous functions f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

but such that the sequence $\{f_n(x)\}$ converges for no x.

Geometry and Topology Module

[GT. 1] Denote $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ the upper half planeUse complex variable expression $z = x + iy$, define

$$Az = \frac{az + b}{cz + d}, \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{R}).$$

Show that $Az \in \mathbb{H}, \forall z \in \mathbb{H}$. Let $\Gamma = \{A \in Sl(2, \mathbb{R}) | a, b, c, d \in \mathbb{Z}\}$, prove that Γ is a discrete group acting properly discontinuously on $\mathbb{H}$.

[GT. 2] Let G be a compact Lie group with bi-invariant metric g, denote $\mathcal{G}$ its Lie algebra.

(i) Show that for $h \in G$ fixed, $Ad_h(x) = hxh^{-1}$ is a Riemannian isometry, and its differential at $x = e$, $Ad_h' : \mathcal{G} \to \mathcal{G}$ is a linear isometry with respect to g.

(ii) Show that $\forall Z \in \mathcal{G}$, the adjoint action $ad_Z : \mathcal{G} \to \mathcal{G}$ defined by $ad_Z(X) = [Z, X]$ is skew-symmetric, i.e.,

$$g([Z, X], Y) = -g(X, [Z, Y]), \forall X, Y \in \mathcal{G}.$$

[GT. 3] Let $X = \{f : [0, 1] \to [0, 1] : f \text{ is continuous}\}$. Let d be the following metric on X:

$$d(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

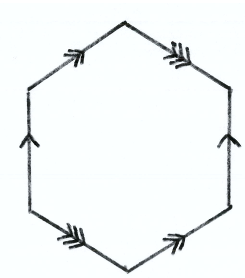
(i) Is d a complete metric on X?

Consider X with the topology induced by d.

(ii) Is X a compact space?
(iii) Is X contractible?
(iv) Show that X is not homeomorphic to a subspace of the real line.

Justify your answers.

[GT. 4] Consider the surface S obtained from the hexagon by identifying its opposite sides via translations (see picture).



Is S homeomorphic to:
(i) the sphere?
(ii) the torus?
(iii) the projective plane?
(iv) other surface?

Justify your answers.