**McGill University**
**Department of Mathematics and Statistics**

**Ph.D. preliminary examination, PART A**

**PURE AND APPLIED MATHEMATICS**
**Paper BETA**

21 August 2015 13:00 - 17:00

**INSTRUCTIONS:**

**(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module. If you answer more than 7 questions, then clearly identify which 7 questions should be graded.**

**(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.**

This exam comprises this cover and 3 pages of questions.

**Algebra Module**

**[ALG. 1]** A subgroup $H$ of a group $G$ is subnormal if there exists a chain $H = H_0 < H_1 < ... < H_k = G$ such that $H_i$ is a normal subgroup of $H_{i+1}$ for every $i$. Prove that if $P$ is a Sylow $p$-subgroup of a finite group $G$ then $P$ is a subnormal in $G$ if and only if $P$ is normal in $G$.

**[ALG. 2]** Classify the finite fields.

**[ALG. 3]** Show that the ring $M_n(K)$ of matrices with entries in a field $K$ is simple (as a ring: every two-sided ideal is trivial). Describe the simple left ideals of $M_n(K)$.

**[ALG 4.]** Let $R$ be a ring with unit. Show that a free $R$-modules is projective. Show that an $R$-module is projective if and only if it is a direct summand of a free module. A right module $M$ is said to be flat if the functor $M \otimes_R -$ is exact. Show that a projective $R$-module is flat.

**Analysis Module**

**[AN. 1]**

(a) If $\mathcal{N}$ is a $\sigma$-field of subsets of $Y$ and $\varphi : X \longrightarrow Y$. Let $\mathcal{M} = \{\varphi^{-1}(N); N \in \mathcal{N}\}$. Is $\mathcal{M}$ necessarily a $\sigma$-field of subsets of $X$. Give either a proof or a counterexample.

(b) If $\mathcal{M}$ is a $\sigma$-field of subsets of $X$ and $\varphi : X \longrightarrow Y$. Let $\mathcal{N} = \{N; N \subseteq Y, \varphi^{-1}(N) \in \mathcal{M}\}$. Is $\mathcal{N}$ necessarily a $\sigma$-field of subsets of $Y$. Give either a proof or a counterexample.

(c) If $\mathcal{M}$ is a $\sigma$-field of subsets of $X$ and $\varphi : X \longrightarrow Y$ is a surjection. Let $\mathcal{N} = \{\varphi(M); M \in \mathcal{M}\}$. Is $\mathcal{N}$ necessarily a $\sigma$-field of subsets of $Y$. Give either a proof or a counterexample.

**[AN. 2]** In this question we normalize the Fourier coefficients by $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

(a) Let $f(x) = \operatorname{sgn}(x)$ for $-\pi < x < \pi$. Show that $\hat{f}(n) = \frac{2}{n\pi i}$ for $n$ odd and $\hat{f}(n) = 0$ for $n$ even.

(b) Stating any theorem that you use, deduce that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

(c) Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$ converges to $\frac{\pi}{4}f(x)$ in the $L^2$ norm sense.

(d) Show that the same series converges uniformly on $[\delta, \pi - \delta]$ for $0 < \delta < \frac{\pi}{2}$. Deduce that the sum is $\frac{\pi}{4}$ in the same range.

**[AN. 3]** Let $\rho$ be a function in $\mathbb{C}_c(\mathbb{R})$ such that $\|\rho\|_1 = 1$ and $\rho \geq 0$. Define, for $\lambda > 0$

$$\rho_\lambda(x) = \lambda^{-1}\rho(\frac{x}{\lambda}).$$

(a) Show that $\|\rho_\lambda\|_1 = 1$ and that, for $x \neq 0$, $\rho_\lambda(x) \to 0$.

(b) Show that if $g \in L^\infty$ and $g$ is continuous then

$$\lim_{\lambda \to 0} g \star \rho_\lambda(x) = g(x).$$

(c) Show that

$$\lim_{\lambda \to 0} \|f \star \rho_\lambda - f\|_1 = 0$$

for all $f \in L^1(\mathbb{R})$.

*Remark:* You can use the following statement without proof: For any function $f \in L^p(\mathbb{R})$ and every $y$ in $\mathbb{R}$, set $f_y(x) := f(x - y)$. If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$, the mapping $y \to f_y$ is a uniformly continuous mapping of $\mathbb{R}$ to $L^p(\mathbb{R})$.

**[AN. 4]** Let $\mu$ be a finite regular Borel measure on $[-1, 1]$ such that

$$\lim_{h \to 0} \frac{\mu([x-h, x+h])}{2h} = \begin{cases} e^{-x} & x \neq 0 \\ +\infty & x = 0 \end{cases}.$$

Define $F(x) = \int_{[-1,1]} \mathrm{d}\mu$.

(a) Show that $F$ is differentiable Lesbegue almost everywhere and compute $F'(x)$ where it exists.

(b) Knowing that $\mu[0, 1] = e$, compute $F$.

**Geometry and Topology Module**

**[GT. 1]** Let $S^2$ be a sphere. Let $D_1, D_2, D_3$ be three disjoint subspaces of $S^2$ that are homeomorphic to the unit disk. Let $R$ be the surface obtained by removing the interiors of these disks so:

$$R = S^2 - \big(\mathrm{int}(D_1) \cup \mathrm{int}(D_2) \cup \mathrm{int}(D_3)\big)$$

Let $C_1, C_2, C_3$ be the boundary circles of $R$, so $C_i = \partial D_i$.

(a) Describe two retractions $S \to C_1$ that are not homotopic to each other.
(b) Is there a retraction $S \to C_1 \cup C_2$ ?
(c) Is there a retraction $S \to \partial S$ ?
(d) List all surfaces that have the same homotopy type as $S$.

**[GT. 2]** Let $X = T^2 \vee S^1$ be the wedge of a torus and a circle identified along a point.

(a) Draw a degree 3 finite covering space of $X$ that is not regular.
(b) How many nonisomorphic connected degree 2 covering spaces does $X$ have?
(c) Draw three non-homeomorphic regular covering spaces of $X$ with infinite cyclic automorphism group.

**[GT. 3]** On $\mathbf{R}^2$, let $x, y$ be standard linear coordinates, let $I$ be the tensor field of type $(1,1)$,

$$I = \frac{\partial}{\partial y} \otimes dx - \frac{\partial}{\partial x} \otimes dy$$

and let $X$ be a smooth vector field,

$$X = u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y}$$

Write down the differential equations for the functions $u$ and $v$ that are equivalent to the condition that $I$ be invariant under the flow generated by $X$, *i.e.*, $L_X(I) = 0$.

**[GT. 4]** Let $M_n$ be the space of $n \times n$ real matrices. Let $GL(n, \mathbf{R})$ denote the subset of $M_n$ consisting of invertible matrices.

(a) Show that the map

$$\begin{aligned} \phi : M_n \times M_n &\to M_n \\ (A, B) &\mapsto AB \end{aligned}$$

is *submersive* when restricted to $GL(n, \mathbf{R}) \times M_n$.

(b) Show that this is not the case for the whole space $M_n \times M_n$.

(c) Show that the map $\det : M_n \to \mathbf{R}$ is submersive on $\det^{-1}(1)$.