McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

APPLIED MATHEMATICS
Paper BETA, Version 1

21 August 2015
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the four modules: (1) Numerical Analysis, (2) Optimization, (3) Partial Differential Equations, and (4) Analysis. Each module comprises 4 questions. You should answer 7 questions with at most 3 from each module. You should answer questions from exactly 3 modules. If you exceed any of these limits, then clearly identify which questions should be graded.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 6 pages of questions.

Numerical Analysis Module

[NA. 1] Consider the initial value problem for the constant coefficient parabolic equation

$$u_t(x,t) + f u_x(x,t) = D u_{xx},$$

for $x \in \mathbb{R}$ and $0 < t \leq T$, with $u(x,0) = u_0(x)$.

(a) Approximate the PDE using Forward Euler method in time, and centered finite differences for the spatial terms. Write down the discretization in the explicit form. Give conditions on h, dt in terms of a and D so that it is an averaging finite difference method. **[8 marks]**

(b) Consider the nonlinear version of the PDE

$$u_t(x,t) + f u_x(x,t) = c|u_x| + D u_{xx}$$

for $x \in \mathbb{R}$ and $0 < t \leq T$, with $u(x,0) = u_0(x)$ where now $c < 0$. Approximate the PDE using Forward Euler method in time, and choose appropriate finite differences for the spatial terms. Write down the discretization in the explicit form. Give conditions on h, dt so that it is a monotone finite difference method. **[6 marks]**)

(c) Prove that the maximum principle holds for each of the finite difference schemes you obtained. **[6 marks]**

[NA. 2]

(a) State the Lax-Milgram Theorem. **[4 marks]**

(b) Give the appropriate Hilbert Space (specify the inner product) and verify the assumptions of the Lax-Milgram Theorem for Poisson's equation with source term

$$-\Delta u + u = f, \text{ in } \Omega$$

with Neumann boundary conditions, $\partial_n u = 0$, on $\partial\Omega$. **[6 marks]**

(c) Suppose u is the solution of Poisson's equation and U is the Galerkin finite element approximation in V_h. Prove, using Galerkin orthogonality, that

$$\|\nabla(u - U)\| \leq \|\nabla(u - V)\|, \quad \text{for all } V \in V_h.$$

[10 marks]

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[NA. 3] Consider the advection-diffusion equation

$$u_t + cu_x = du_{xx}$$

(c and d are positive constants) and the following finite difference scheme to solve it

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2h}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{d\Delta t}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

where h is the mesh size, and Δt is the time step.

(a) Determine the restriction on Δt so that the scheme is stable.

(b) Determine the local and global truncation error.

(c) If $d = 0$, propose a semi-Lagrangian scheme (characteristics tracing) which is second order in space and first order in time. Show your work.

(d) Propose a simple modification of the scheme derived in the previous question for the case $d > 0$, and check its stability.

[NA. 4]

1D Case: Consider the following PDE,

$$\left(\beta\left(x\right)u_x\right)_x = f\left(x\right)$$

where, $u : [0, 2\pi] \to \mathbb{R}$, $x \in [0, 2\pi]$, $\beta : [0, 2\pi] \to \mathbb{R}$, $f : [0, 2\pi] \to \mathbb{R}$, all are periodic, $\beta > 0$.

Assume $\beta\left(x\right)$ is a non-constant, periodic function on $[0, 2\pi]$ with n continuous derivatives. Assume $f\left(x\right) = \sin\left(x\right)$.

(a) Write the Fourier spectral discretization for this problem.

(b) Write the sequence of steps (FFT,..., IFFT,...).

(c) What convergence rate for the solution do you expect as a function of n?

2D Case: Consider the following equation on the 2-torus$\mathbb{T}^2$,

$$\nabla \cdot \left(\beta\left(x, y\right)\nabla u\right) = f\left(x, y\right),$$

$u : [0, 2\pi]^2 \to \mathbb{R}$, $f : [0, 2\pi]^2 \to \mathbb{R}$, $\beta : [0, 2\pi]^2 \to \mathbb{R}$, $\beta > 0$.

(a) Write the sequence of steps (FFT,..., IFFT,...) for the case $\beta \equiv 1$.

(b) Write the sequence of steps (FFT,..., IFFT,...) for the case $\beta = \beta\left(x, y\right)$. Hint: you should consider a Fixed Point Iteration (FPI).

(c) Under what condition the FPI in the question above converges?

Optimization Module

[OPT. 1] *Arborescence Polytope.* **[20 marks].**

(a) Define the Arborescence Polytope P_{arb} of a graph $G = (V, E)$.
(b) Give a linear description of P_{arb} and prove that your description is correct.
(c) Show that P_{arb} has the integer decomposition property.

[OPT. 2] *Integer Hulls.*

(a) Define what it means for an $m \times n$ matrix A to be totally unimodular (TUM).**[4 marks]**
(b) Prove that if A is TUM then for any vector $b \in \mathbf{Z}^n$ the following polyhedron is integral. **[16 marks]**

$$P = \{x : Ax \leq b\}.$$

Define all terms used.

[OPT. 3] *Line Search.*

(a) Starting from an initial point $x^{(0)} = (1, 0)$, find the first two iterates $x^{(1)}, x^{(2)}$ of Newton's Method applied to minimizing the function $f(x_1, x_2) = \frac{x_1^4}{4} + x_2^2$. **[12 marks]**
(b) Suppose $f(x) = -b^T x + (1/2)x^T Ax$ where $c \in \mathbf{R}^n$ and A is an $n \times n$ positive definite real matrix. Let x^* be the unique global minimizer of f over $\mathbf{R}^n$. Consider running the steepest descent algorithm (with exact line search) from a starting point $x^{(0)}$. Show that the gradients at successive iterates $x^{(k)}, x^{(k+1)}$ are orthogonal. **[8 marks]**

Hint: Recall that the step length for exact line search is given by:

$$\alpha_k = \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T A \nabla f_k}$$

[OPT. 4] *Trust Region.*

(a) Give pseudo-code for the Trust Region method for minimizing a function $f(x)$. Give a full explanation of the set-up and each step.**[10 marks]**
(b) State the Trust Region Subproblem and explain its significance. **[4 marks]**
(c) Define the Cauchy Point and give its complete derivation, i.e., completely specify the step length.**[6 marks]**

Partial Differential Equations Module

[PDE 1.] Consider the Cauchy problem

$$y\partial_x u + x\partial_y u = 0,$$

with smooth Cauchy data specified on $\Gamma = \{(0,y) : y \in \mathbb{R}\}$. In which region of the xy-plane is the solution uniquely determined? Compute the solution corresponding to the particular Cauchy data $u(0,y) = e^{-y^2}$.

[PDE 2.] Let Ω be an open subset of $\mathbb{R}^n$ $(n \geq 2)$. Suppose that $u \in C^1(\Omega)$ and that for each $y \in \Omega$ there exists $r^* > 0$ such that

$$\int_{\partial B_r(y)} \partial_\nu u = 0, \qquad \text{for all } 0 < r < r^*,$$

where ∂_ν is the normal derivative. Show that u is harmonic in Ω.

[PDE 3.] Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfy the heat equation

$$\partial_t u = \Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

where Δ is the n-dimensional Laplacian, and ∂_t is differentiation along the $(n+1)$-st coordinate $t = x_{n+1}$. In addition, assume that $u(x,0) \leq 0$ for $x \in \mathbb{R}^n$, and that

$$u(x,t) \leq Me^{\alpha x^2} \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0,$$

where M and α are real constants. Show that

$$u(x,t) \leq 0 \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0.$$

You may use the maximim principle for bounded domains, provided you recall a precise statement.

[PDE 4.] In this exercise, we will prove the existence of local isothermal coordinates in two dimensions, under analyticity assumptions. Consider the second order linear operator

$$A = E(x,y)\partial_x^2 + 2F(x,y)\partial_x\partial_y + G(x,y)\partial_y^2,$$

where E, F, and G are real analytic functions in a neighbourhood of $0 \in \mathbb{R}^2$, and the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is positive definite in the same neighbourhood. We want to find a neighbourhood $U \subset \mathbb{R}^2$ of 0, and a mapping $\phi = (u,v) : U \to \mathbb{R}^2$ such that

$$A = \psi\partial_u^2 + \psi\partial_v^2,$$

in the new coordinate system (u,v), with some positive function $\psi : U \to \mathbb{R}$.

(a) Show that the aforementioned requirement is equivalent to the differential system

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix} = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix}. \tag{$*$}$$

(b) Show that $(*)$ is satisfied in a neighbourhood of 0 for some function $\psi > 0$, if

$$\begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} F & G \\ -E & -F \end{pmatrix} \begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix}, \tag{$**$}$$

holds in a neighbourhood of 0, where $\Delta = \sqrt{EG - F^2}$. What is ψ thus obtained?

(c) Show that $(**)$ admits a solution $\phi = (u,v)$ in a neighbourhood U of 0. Make sure that ϕ defines a genuine coordinate system in U.

Analysis Module

[AN. 1]

(a) If $\mathcal{N}$ is a σ-field of subsets of Y and $\varphi : X \longrightarrow Y$. Let $\mathcal{M} = \{\varphi^{-1}(N); N \in \mathcal{N}\}$. Is $\mathcal{M}$ necessarily a σ-field of subsets of X. Give either a proof or a counterexample.

(b) If $\mathcal{M}$ is a σ-field of subsets of X and $\varphi : X \longrightarrow Y$. Let $\mathcal{N} = \{N; N \subseteq Y, \varphi^{-1}(N) \in \mathcal{M}\}$. Is $\mathcal{N}$ necessarily a σ-field of subsets of Y. Give either a proof or a counterexample.

(c) If $\mathcal{M}$ is a σ-field of subsets of X and $\varphi : X \longrightarrow Y$ is a surjection. Let $\mathcal{N} = \{\varphi(M); M \in \mathcal{M}\}$. Is $\mathcal{N}$ necessarily a σ-field of subsets of Y. Give either a proof or a counterexample.

[AN. 2] In this question we normalize the Fourier coefficients by $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

(a) Let $f(x) = \text{sgn}(x)$ for $-\pi < x < \pi$. Show that $\hat{f}(n) = \frac{2}{n\pi i}$ for n odd and $\hat{f}(n) = 0$ for n even.

(b) Stating any theorem that you use, deduce that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

(c) Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$ converges to $\frac{\pi}{4}f(x)$ in the L^2 norm sense.

(d) Show that the same series converges uniformly on $[\delta, \pi - \delta]$ for $0 < \delta < \frac{\pi}{2}$. Deduce that the sum is $\frac{\pi}{4}$ in the same range.

[AN. 3] Let ρ be a function in $\mathbb{C}_c(\mathbb{R})$ such that $\|\rho\|_1 = 1$ and $\rho \geq 0$. Define, for $\lambda > 0$

$$\rho_\lambda(x) = \lambda^{-1}\rho(\frac{x}{\lambda}).$$

(a) Show that $\|\rho_\lambda\|_1 = 1$ and that, for $x \neq 0$, $\rho_\lambda(x) \to 0$.

(b) Show that if $g \in L^\infty$ and g is continuous then

$$\lim_{\lambda \to 0} g \star \rho_\lambda(x) = g(x).$$

(c) Show that

$$\lim_{\lambda \to 0} \|f \star \rho_\lambda - f\|_1 = 0$$

for all $f \in L^1(\mathbb{R})$.

Remark: You can use the following statement without proof: For any function $f \in L^p(\mathbb{R})$ and every y in $\mathbb{R}$, set $f_y(x) := f(x - y)$. If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$, the mapping $y \to f_y$ is a uniformly continuous mapping of $\mathbb{R}$ to $L^p(\mathbb{R})$.

[AN. 4] Let μ be a finite regular Borel measure on $[-1, 1]$ such that

$$\lim_{h \to 0} \frac{\mu([x-h, x+h])}{2h} = \begin{cases} e^{-x} & x \neq 0 \\ +\infty & x = 0 \end{cases}.$$

Define $F(x) = \int_{[-1,1]} \mathrm{d}\mu$.

(a) Show that F is differentiable Lesbegue almost everywhere and compute $F'(x)$ where it exists.

(b) Knowing that $\mu[0, 1] = e$, compute F.