

McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE MATHEMATICS
Paper BETA

15 August, 2014
13:00 – 17:00

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module. If you answer more than 7 questions, then clearly identify which 7 questions should be graded.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[**ALG. 1**] Let K be a field in which $2 \neq 0$. Let L be a quadratic extension of K .

- (a) Prove that L/K is a Galois extension and denote by σ its non-trivial automorphism.
- (b) Prove that the map $N : L^\times \rightarrow K^\times$, $N(x) = x \cdot \sigma(x)$, is a group homomorphism. Give an example where N is not surjective, but show that if K is a finite field then N is surjective.
- (c) Prove that the map $T : L \rightarrow K$, $T(x) = x + \sigma(x)$, is a surjective K -linear transformation. Determine the kernel of T (referring back to your solution to (a)).

[**ALG. 2**] Let R be a commutative ring and P an R -module.

- (a) Define what it means for P to be projective over R .
- (b) If P and Q are projective over R show that $P \oplus Q$ and $P \otimes_R Q$ are projective over R as well. (Remark: you may want to choose in (a) the definition of projective making the proof of (b) as easy as possible.)
- (c) Let R be a PID with fraction field K and let $P \subseteq K$ be a finitely generated R module. Prove that P is projective (in fact, P is isomorphic to R).

[**ALG. 3**] Let G be the group of 3×3 matrices with entries in the field with 4 elements.

- (a) What is the cardinality of G ?
- (b) What is the cardinality of Sylow 2-subgroup of G ?
- (c) Describe such a subgroup precisely.

[**ALG. 4**] Let R be an integral domain. A polynomial $f \in R[x]$ is called primitive if there is no non-unit element which divides all the coefficients of f . We say R satisfies GL (Gauss Lemma) if the product of any two primitive polynomials is primitive.

- (a) Prove that any Unique Factorization Domain satisfies GL.
- (b) Show that if R satisfies GL, then any irreducible element in R is prime.
- (c) Give an example of a ring R that does not satisfy GL.
- (d) Give an example of a domain R and an irreducible $f \in R[x]$ such that f is reducible in $K[x]$, where K is the fraction field of R .

Analysis Module

[AN. 1] Let F and G , real, be in $\mathbb{L}_1([0, 1])$ and suppose that $\int_0^1 F(x)\varphi(x)dx = \int_0^1 G(x)\varphi(x)dx$ for **every** $\varphi \in \mathbb{C}([0, 1])$. Show that $F(x) = G(x)$ a.e. on $[0, 1]$. *Hint: Given K compact $\subseteq [0, 1]$, take the $\varphi_n(x) \in \mathbb{C}([0, 1])$ equal to $[1 - \text{dist}(x, K)]^n$.*

[AN. 2] Given the real quantities a_n and $\gamma_n, n = 0, 1, 2, \dots$, suppose that the series

$$(*) \sum_{n=0}^{\infty} a_n \cos(nx + \gamma_n)$$

(N.B. **not** necessarily a Fourier series!) converges to a finite limit for each $x \in E$, a Lebesgue measurable subset of $[-\pi, \pi]$ with $|E| > 0$.

- (a) Show that there is a measurable subset E_0 of E with $|E_0| > 0$ such that the convergence of $(*)$ is **uniform** on E_0 . Note: $|E|$ and $|E_0|$ denote the Lebesgue measures of E and E_0 respectively.
- (b) Show that

$$\int_{E_0} \cos^2(nx + \gamma_n) dx \rightarrow_n \frac{1}{2}|E_0|.$$

Hint: Start by using the half angle formula and then the addition formula, for cosine.

- (c) Hence show that, under the given condition on $(*)$, we have $a_n \rightarrow_n 0$. *Hint: Refer to the result in (a).*

[AN. 3] Suppose that the f_n and f are in $\mathbb{L}_1([0, 1])$, that $\int_0^1 |f_n(x)|^2 dx \leq c < \infty$, and that $\int_0^1 |f(x) - f_n(x)| dx \rightarrow_n 0$.

- (a) Show that $\int_0^1 |f(x)|^2 dx \leq c$. *Hint: Extract an appropriate subsequence from $\{f_n\}$.*
- (b) Show that $\int_0^1 |f(x) - f_n(x)|^2 dx$ **need not** $\rightarrow_n 0$. *Hint: With $f(x) = 0$, take each f_n to be an appropriate multiple of an appropriate subset of $[0, 1]$.*

[AN. 4] Let μ be a finite regular Borel measure on \mathbb{R} and suppose that for all x , without exception,

$$\liminf_{h \downarrow 0} \frac{\mu((x - h, x + h))}{2h}$$

is finite.

Show that there is then an $F \geq 0$ in $\mathbb{L}_1(\mathbb{R})$ with

$$\mu(E) = \int_E F(x) dx$$

for Borel sets E .

Hint: If E_0 is a Borel subset of \mathbb{R} with $|E_0| = 0$ but $\mu(E_0) > 0$, consider the new Borel measure $\sigma(A) = \mu(E_0 \cap A)$, A Borel. Apply a known result.

Geometry and Topology Module

[GT. 1] Let M be a moebius strip with boundary circle denoted by ∂M , and let $m \in \partial M$ be a basepoint.

- (a) Compute $\pi_1(M, m)$ and describe generators.
- (b) Let $i : (\partial M, m) \rightarrow (M, m)$ denote the inclusion map. Describe the homomorphism $i_* : \pi_1(\partial M, m) \rightarrow \pi_1(M, m)$.
- (c) Show that there is no retraction $r : M \rightarrow \partial M$ such that $r(M) = \partial M$. Recall that a *retraction* f is a continuous map $f : X \rightarrow X$ such that $f^2 = f$.
- (d) Find a homeomorphism $\phi : M \rightarrow M$ that is not homotopic to $i : M \rightarrow M$. Compare ϕ_* with i_* to prove that ϕ is not homotopic to i .

[GT. 2] Let J be the quotient space of \mathbb{C} obtained by identifying 1 with $1 + i$ and identifying 2 with $2 - i$.

- (a) Describe a connected degree 2 covering space $\hat{J} \rightarrow J$.
- (b) Describe a deformation retraction of J to a graph with a single vertex.
- (c) How many connected degree 2 covering spaces $\hat{J} \rightarrow J$ are there up to isomorphism?
- (d) Let J^+ be the one-point compactification of J . Draw a picture of J^+ .
- (e) Determine the homology $H_n(J^+)$ for each n .

[GT. 3]

- (a) Consider the group $U(n)$ of $n \times n$ unitary matrices ($MM^* = \mathbf{I}$). Show that this is a manifold, and a Lie group.
- (b) What is its Lie algebra?
- (c) What is its top dimensional de Rham cohomology group? in which dimension?
- (d) Describe a surjective group homomorphism $U(n) \rightarrow U(1)$. Is $H^1(U, \mathbb{R})$ zero?

[GT. 4]

- (a) Consider the two vector fields on \mathbb{R}^3 :

$$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y},$$

Show that they span a rank two sub-bundle E of $T\mathbb{R}^3$. Is there a two dimensional submanifold V of \mathbb{R}^3 with $E|_V = TV$, under the embedding of V into \mathbb{R}^3 ?

- (b) Consider the vector field $R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$. Let $\omega = dx \wedge dy \wedge dz$ be the volume form. Compute the Lie derivative $\mathcal{L}_R(\omega)$. Find a vector field, different from R or its multiples, whose Lie bracket with R is zero.