McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper BETA

17 August, 2018
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the the following modules [AL] Algebra; [AN] Analysis; [GT] Geometry & Topology; [NA] Numerical Analysis; [PDE] Partial Differential Equations, each of which comprises 4 questions. You should answer 7 questions from 3 modules, with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 5 pages of questions.

Algebra Module

[ALG. 1]

Show that there is no simple group of order 56. Show that there is no simple group of order 351.

[ALG. 2]

Let R be a commutative ring with 1. We say that $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.

(a) Show that the set of all nilpotent elements in R is an ideal. We will write $N(R)$ for this set. It is called the *nilradical* of R.

(b) Find the nilradicals $N(\mathbb{Z})$ and $N(\mathbb{Z}/n\mathbb{Z})$ for $n > 1$ an integer.

[ALG. 3]

Let f be an irreducible polynomial with rational coefficients of degree 4. Let G be the Galois group of f (namely, if K is a splitting field for f then $G = Gal(K/\mathbb{Q})$). Explain how one may view G as a subgroup of S_4. Which subgroups can arise this way?

[ALG. 4]

Let R be a commutative ring and M, N finitely generated projective R-modules. Prove that the following modules are projective as well:

$$M \otimes_R N, \quad M \oplus N, \quad \mathrm{Hom}_R(M, N).$$

Analysis Module

[AN. 1]

Consider the Dirichlet kernel

$$D_m(x) := \sum_{k=-m}^{m} e^{2\pi kix}.$$

Show that there exists a constant $C > 0$ such that for every $[a, b] \subset [-1/2, 1/2]$ and for every $m \geq 0$,

$$\Big| \int_a^b D_m(x)\, dx \Big| \leq C.$$

[AN. 2]

Let $\alpha \in (0, 1)$ and consider the integral operator given by

$$Tf(x) = \int_I |x - y|^{-\alpha} f(y)\, dy,$$

where $I = [0, 1]$ and dy denotes Lebesque measure. Show that for any $1 \leq p < \infty$, the operator $T : L^p(I) \to L^p(I)$ is bounded and estimate $\|T\|$ from above. Please carefully justify your answer.

[AN. 3]

Let $f \in L^2(\mathbb{R})$ and let

$$g(x) = \int_x^{x+1} f(t)\ dt.$$

Prove that

$$\lim_{x\to\infty} g(x) = 0.$$

[AN. 4]

Let μ be a finite Borel measure on $\mathbb{R}$ and

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{itx} d\mu(x)$$

its Fourier transform.

(1) Prove that the function $t \to \widehat{\mu}(t) \in \mathbb{C}$ is uniformly continuous for $t \in \mathbb{R}$.

(2) Suppose that the measure μ has no atoms, namely that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Prove that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\widehat{\mu}(t)|^2 dt = 0.$$

Geometry and Topology Module

[GT. 1]

Consider the subgroup G of $GL(3,\mathbb{R})$ consisting of matrices of the form

$$A = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & cd & \frac{c}{a} \end{pmatrix}, \tag{1}$$

where $a, c \in \mathbb{R} \setminus \{0\}$.

a) Show that G is a 4-dimensional Lie group, that is a group endowed with the structure of a C^ω 4-manifold for which the multiplication and inversion maps are C^ω.

b) Show that the 1-forms

$$\omega = \frac{da}{a}, \quad \theta = \frac{db}{c} - \frac{b}{c}\frac{da}{a},$$

are right-invariant 1-forms on G, meaning that they satisfy

$$R_B^*\omega = \omega, \quad R_B^*\theta = \theta,$$

for all $B \in G$, where R_B denotes the right-multiplication map $R_B(A) = A.B$.

[GT. 2]

For this question, you may assume without proof that the de Rham cohomology of $P_n(\mathbb{C})$ is generated by the cohomology classes defined by the by the exterior powers of a 2-form Ω on $P_n(\mathbb{C})$ which is closed, but not exact.

Show that the manifolds $M := P_3(\mathbb{C})$ and $N := S^2 \times S^4$ have equal Betti numbers, that is

$$\dim H^i_{dR}(M) = \dim H^i_{dR}(N), \quad \forall\, 0 \le i \le 6.$$

Show that M and N are not diffeomorphic by proving that their de Rham cohomology rings are not isomorphic.

[GT. 3]

Let B denote a bouquet of two circles.

(1) Find two degree 4 covering spaces $\widehat{B}_1 \to B$ and $\widehat{B}_2 \to B$ such that there is no isomorphism between these two covering spaces, but such that the graphs $\widehat{B}_1, \widehat{B}_2$ are homeomorphic.
(2) How many connected regular degree 4 covering spaces of B are there up to isomorphism?
(3) Find a degree 4 connected cover $\widehat{B} \to B$ such that $\mathrm{Aut}(\widehat{B} \to B) \cong \mathbb{Z}_2$.

[GT. 4]

Let S be a compact surface with boundary with the same homotopy type as a bouquet of two circles B.

(1) Describe all four possible such surfaces (up to homeomorphism).
(2) In each case find an embedding $B \subset S$ such that S deformation retracts to B.

Numerical Analysis Module

[NA. 1]

Consider the ODE $x'(t) = x(t)^{1/4}$.

(a) Find a solution of the form $x(t) = (Ct)^\alpha$ where $C, \alpha > 0$ are positive constants.

(b) Compute the solution of the Forward Euler method for the ODE, x_n^{FE}, with time step dt, starting from $x_0^{FE} = 0$. Does is converge to the solution above at $h \to 0$? Explain your results.

(c) Write down the equation for the backward Euler method for the ODE with time step dt, x_n^{BE}, and find x_1^{BE} when $x_0^{BE} = 0$.

[NA. 2]

Consider the PDE

$$-(a(x)u_x)_x = f(x), \qquad \text{for } x \in (0,1)$$

along with boundary conditions

$$u(0) = u_0, \qquad u(1) = u_1$$

where $a(x) \in C^1[0,1]$, $a(x) \geq a_0 > 0$, and $f \in L^2(a,b)$.

(a) Define the bilinear form $\mathcal{A}$ and write down the Galerkin formulation in $H_E^1(0,1)$.

(b) Prove uniqueness of weak solutions in $H_E^1(0,1)$.

(c) Let S_E^h be a finite dimensional subspace of $H_E^1(0,1)$, let u be the weak solution of the PDE, and let u^h be the Galerkin approximation. Write down (but don't prove) Galerkin orthogonality, and briefly explain/interpret the formula.

[NA. 3] Consider the 1D advection-diffusion equation $u_t = cu_x + \theta u_{xx} (\theta > 0)$.

(a) Write the finite-difference approximation to this equation using centered difference in space (i.e. only u_{i-1}, u_i and u_{i+1} should appear) and forward Euler in time (with the time step Δt and grid size Δx).

(b) What is the local truncation error for this scheme?

(c) Show that for $\theta = \frac{1}{2}c^2\Delta t$, the finite-difference approximation above is a second order (locally) in both space and time approximation to the linear advection equation $u_t = cu_x$. What is the name of this method?

[NA. 4] Suppose $A \in \mathbb{C}^{m\times n}$ has rank r.

(a) State the singular value decomposition (SVD) of A.

(b) Show that the SVD can be used to find all solutions to

$$\min_x ||b - Ax||_2.$$

(c) Under which condition is there a unique solution in part (b)?

PDE Module

[PDE. 1]

a) Let $a, b \in \mathbb{R}$ and $\phi(t)$ be a smooth function. Suppose

$$u(x,t) \;=\; \begin{cases} a & \text{if } x \geq \phi(t) \\ b & \text{if } x < \phi(t), \end{cases}$$

is a weak (distributional) solution to

$$u_t \,+\, u\,u_x \;=\; 0.$$

Derive (using integration by parts) the required jump conditions on a, b and ϕ.

b) Solve for $u(x,y)$ where

$$u_x\,u_y \;=\; u \quad \text{on} \quad \{(x,y)\,|\,x>0\}, \qquad \text{with} \quad u(0,y) = y^2.$$

[PDE. 2]

a) In four space dimensions, find the Fundamental solution $\Phi(\mathbf{x})$ of the Laplacian operator $-\Delta$. That is, find a solution to

$$-\Delta\Phi \;=\; \delta_{\mathbf{0}} \qquad \text{in the sense of distributions.}$$

Hint: look for radially solutions of the $\Delta u \,=\, 0$.

b) Use the fundamental solution to write down an explicit formula for the Poisson equation

$$\Delta u \;=\; f,$$

in $\mathbb{R}^4$ where $f \in C_c(\mathbb{R}^4)$.

[PDE. 3]

Let the space dimension be 5. For what values of $\alpha > 0$ and $p \geq 1$ is the function

$$\frac{1}{|\mathbf{x}|^{\alpha}}$$

in

a) $W^{1,p}\big(B(\mathbf{0},1)\big)$
b) $W^{1,p}\big(B^c(\mathbf{0},1)\big)$
c) $W^{1,p}\big(\mathbb{R}^5\big)$

Show all your steps.

[PDE. 4]

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let $f \in L^2(\Omega)$. Consider the Poisson equation

$$\triangle\, u = f \quad \text{in } \Omega \qquad\qquad u = 0 \quad \text{on } \partial\Omega.$$

a) What does it mean for $u \in H_0^1(\Omega)$ to be a weak solution? (recall that $H_0^1(\Omega)$ is the closure of $C_c^{\infty}(\Omega) \subset H^1(\Omega)$ with respect to the H^1 norm).

b) Prove that there exists a unique weak solution.