McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

APPLIED MATHEMATICS

Paper BETA, Version 1

12 May 2017
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of three modules: (1) Analysis, (2) Discrete Mathematics, and (3) Probability. Each module comprises 4 questions. You should answer 7 questions with at most 3 from each module. You should answer questions from exactly 3 modules. If you exceed any of these limits, then clearly identify which questions should be graded.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Analysis Module

[AN. 1]

Suppose $X \subset Z$, $Y \subset V$, with X, Y, Z, V Banach spaces and with both inclusions continuous. Suppose that $P : Z \to V$ is a continuous linear operator such that $u \in Z$, $Pu \in Y$ implies $u \in X$. Show that there exists $C > 0$ such that for al $u \in Z$ satisfying $Pu \in Y$, one has

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_Z).$$

(*Hint:* Consider the map $u \mapsto (Pu, u)$ and its inverse in relevant spaces.)

[AN. 2]

Suppose that μ is a σ-finite measure on $(X, \mathcal{A})$, then there is a finite measure ν so that $\nu \ll \mu$ and $\mu \ll \nu$.

[AN. 3]

a) State Arzela-Ascoli theorem.

b) Let $0 < \alpha < 1$. Recall that a function $f(x)$ (defined on $[0, 1]$) is called α-Hölder continuous if the quantity

$$N_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

Show that the set of functions

$$\{f \in C([0, 1]) : ||f||_u \leq 1 \;\; and \;\; N_\alpha(f) \leq 1\}$$

is compact in $C([0, 1])$.

c) Show that for $\alpha > 1$, the only α-Hölder continuous functions are the constant functions.

[AN. 4]

Let $X = \mathbb{R}^n$, and let $f \in L^p(X) \cap L^\infty(X)$ (and hence $f \in L^q(X)$ for all $p < q < \infty$). Show that

$$\lim_{q \to \infty} ||f||_q = ||f||_\infty.$$

Discrete Mathematics Module

Graph Theory

[**DM. 1**] (Matchings)

a) State Hall's theorem and Tutte's theorem.

b) Let G be a simple bipartite graph G with a bipartition (A, B) such that $|A| = |B| = n$ and $\deg(v) \geq \lceil n/2 \rceil$ for every $v \in V(G)$. Show that G has a perfect matching.

c) Let T be a tree. Prove that T has a perfect matching if and only if for every vertex $v \in V(T)$ the induced subgraph by $V(T) \setminus \{v\}$ contains exactly one connected component with odd number of vertices.

[**DM. 2**] (Graph coloring)

a) State Brooks' Theorem and Vizing's Theorem.

b) Let G be a loopless graph. Show that $\chi(G)(\chi(G) - 1) \leq 2|E(G)|$

c) Let G be a loopless graph with n vertices and no K_3 subgraph. Prove that $\chi(G) \leq \sqrt{2n}$.

Combinatorics

[**DM. 3**] (Sperner Families)

a) State Sperner's theorem.

Let n, k be positive integers with $n > 2k$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a Sperner family such that $\mathcal{F}$ contains at least one set of size k, at least one set of size $n - k$, and no sets whose size is strictly between k and $n - k$.

b) Show that $\mathcal{F}$ contains at most $\binom{n}{l} - \binom{n-k}{l}$ sets of size l for every $0 \leq l \leq k$.

c) Deduce from a) and the symmetric statement for $l \geq n - k$, that there exists a constant c_k, dependent on k, but not on n, such that $|\mathcal{F}| \leq c_k n^{k-1}$.

[**DM. 4**] (Convexity)

a) State Helly's theorem.

b) Let $n \geq 3$, and let $z_1, \ldots, z_n \in \mathbb{R}^2$ be points in the plane, such that every three of them can be covered by a circle of radius r. Show that all points can be covered by a circle of radius r.

c) Let $n \geq 2$, and let $X_1, \ldots, X_n \subseteq \mathbb{R}^d$ be compact convex sets such that their union is also convex. Prove that if every $n-1$ of them have non-empty intersection, then the intersection of all of them is non-empty.

Probability Module

[PR. 1]

(a) Show that a monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

(b) Suppose that μ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is (a) translation invariant (i.e. $\mu(A+x) = \mu(A)$ for all $A \in \mathcal{B}(\mathbb{R})$) and (b) finite on bounded sets (i.e. if $A \in \mathcal{B}(\mathbb{R})$ satisfies $A \subset [-x, x]$ for some $0 < x < \infty$, then $\mu(A) < \infty$). Show that μ is a multiple of Lebesgue measure.

[PR. 2]

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n : n \geq 1\}$ be a sequence of **identically distributed** $\mathbb{R}$-valued random variables with $\mathbb{E}\left[X_1^2\right] < \infty$. Show that

(a) for every $\epsilon > 0$, $\lim\limits_{n\to\infty} n \cdot \mathbb{P}\left(|X_1| \geq \epsilon\sqrt{n}\right) = 0$;

(b) $\frac{1}{\sqrt{n}} \max\limits_{1\leq k\leq n} |X_k| \to 0$ in probability as $n \to \infty$.

[PR. 3]

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n : n \geq 1\}$ be a sequence of **independent** and **identically distributed** $N(0,1)$ random variables. Set $Y_0 \equiv 1$,

$$Y_n := \exp\left(\sum_{i=1}^{n} X_i - \frac{n}{2}\right) \quad \text{for } n \geq 1,$$

and let $\mathcal{F}_n := \sigma\left(Y_0, \cdots, Y_n\right)$.

(a) Show that $\{Y_n : n \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n : n \geq 0\}$.

(b) Show that $\lim_{n\to\infty} Y_n = 0$ a.s., but Y_n does **not** converge to 0 in L^1.

[PR. 4]

Fix a non-negative random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a σ-algebra $\mathcal{G} \subset \mathcal{F}$.

(a) Show that almost surely $\mathbb{E}[X \mid \mathcal{G}] = \int_0^\infty \mathbb{P}[X > t \mid \mathcal{G}]\mathrm{d}t$.

(b) Show that almost surely $\mathbb{P}[X \geq a \mid \mathcal{G}] \leq a^{-t}\mathbb{E}[X^t \mid \mathcal{G}]$ for $t \geq 0$ and $a > 0$.