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McGill University

Faculty of Science

Department of Mathematics and Statistics

Statistics Part A Comprehensive Exam

Methodology Paper

Date: Friday, May 13, 2016 Time: 13:00 – 17:00

Instructions

- Answer only **two** questions out of Section L. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**
- Answer only **two** questions out of Section G. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**

Questions	Marks
L1	
L2	
L3	
G1	
G2	
G3	

This exam comprises the cover page and xxx pages of questions.

Section L (Linear Regression Models)
Answer only two questions out of L1–L3

Question L1.

(a) Explain the linear modelling procedure known as *ridge regression*, and outline one specific context in which it might be adopted in place of ordinary least squares (OLS) for estimation in a linear model setting.

Using standard notation with $\mathbf{X}$ denoting the $n \times p$ design matrix, $\mathbf{y}$ the $n \times 1$ vector of response data, and λ the ridge penalty parameter, derive the form of the ridge regression estimates of model parameters β. Explain carefully any pre-processing steps that are necessary. 8 MARKS

(b) In the specific case where the columns of $\mathbf{X}$ are *orthogonal*, derive the form of the ridge regression estimator of β, and also the variance of the estimator. 6 MARKS

(c) An alternative penalization approach to estimation for linear models uses the so-called LASSO (or L_1) penalty given by

$$\lambda \sum_{j=1}^{p} |\beta_j|$$

where $|.|$ denotes absolute value.

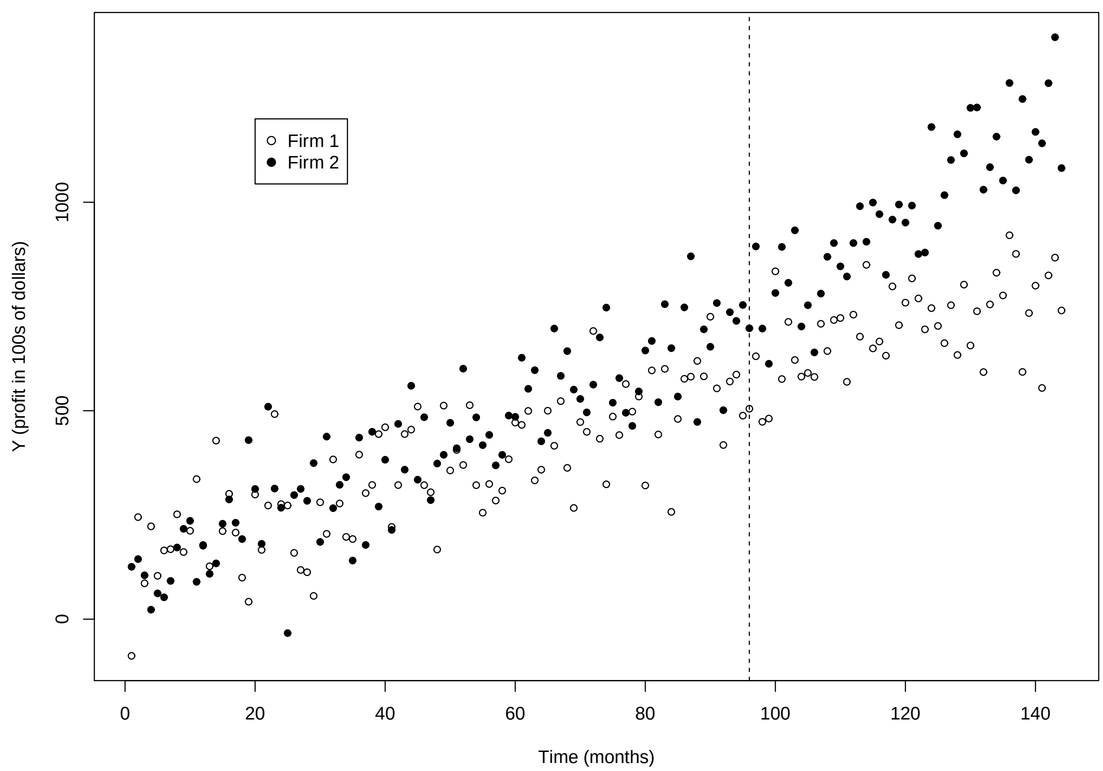
Derive the form of the LASSO estimates of the elements of β in the specific case of orthogonal predictors as in (b). 6 MARKS

Question L2.

The impact of a change in economic policy on the monthly profits of two firms is to be investigated using data over a twelve year period (144 months of data). The policy was changed after eight years (96 months).

In the data set of $n = 288$ observations the monthly profit, `Y`, and month since start of study, `Time`, are recorded, along with a label `Firm`, taking values 1 and 2, indicating the firm concerned.

A data plot, and `R` code, related to the data are included below:




```
> table(Firm)
> n<-288
> X1<-rep(1,n)
> X2<-rep(c(0,1),each=n/2)
> X3<-Time
> X4<-(Time-96)*(Time > 96)
> X5<-X2*X3
> X6<-X2*X4
X<-cbind(X1,X2,X3,X4,X5,X6);colnames(X)<-c('1','2','3','4','5','6')
```

```
> fit1<-lm(Y ~ Time)
> fit2<-lm(Y ~ Time+Firm)
> fit3<-lm(Y ~ Time*Firm)
> fit4<-lm(Y ~ -1+X)

> summary(fit1)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  93.1020    16.1223   5.775 2.01e-08 ***
Time          6.1295     0.1929  31.773  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 136.1 on 286 degrees of freedom
Multiple R-squared:  0.7792,    Adjusted R-squared:  0.7785
F-statistic:  1010 on 1 and 286 DF,  p-value: < 2.2e-16

> summary(fit2)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.5346    15.6399   1.633    0.104
Time          6.1295     0.1676  36.581   <2e-16 ***
Firm2       135.1348    13.9305   9.701   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 118.2 on 285 degrees of freedom
Multiple R-squared:  0.834,     Adjusted R-squared:  0.8329
F-statistic: 716.1 on 2 and 285 DF,  p-value: < 2.2e-16

> summary(fit3)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 130.2287    17.0588   7.634 3.48e-13 ***
Time          4.6854     0.2041  22.954  < 2e-16 ***
Firm2       -74.2533    24.1248  -3.078  0.00229 **
Time:Firm2    2.8881     0.2887  10.005  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 101.8 on 284 degrees of freedom
Multiple R-squared:  0.8773,    Adjusted R-squared:  0.876
F-statistic: 676.8 on 3 and 284 DF,  p-value: < 2.2e-16
```

```
> summary(fit4)
Coefficients:
    Estimate Std. Error t value Pr(>|t|)
X1 129.88853   19.42175   6.688 1.21e-10 ***
X2 -26.16417   27.46650  -0.953 0.341616
X3   4.69363    0.31595  14.856  < 2e-16 ***
X4  -0.03102    0.93485  -0.033 0.973552
X5   1.73079    0.44681   3.874 0.000133 ***
X6   4.38568    1.32208   3.317 0.001028 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 98.46 on 282 degrees of freedom
Multiple R-squared:  0.9745,    Adjusted R-squared:  0.974
F-statistic:  1796 on 6 and 282 DF,  p-value: < 2.2e-16

> anova(fit1,fit2,fit3,fit4,test='F')
Analysis of Variance Table

Model 1: Y ~ Time
Model 2: Y ~ Time + Firm
Model 3: Y ~ Time * Firm
Model 4: Y ~ -1 + X
  Res.Df     RSS Df Sum of Sq      F    Pr(>F)
1    286 5296890
2    285 3982068  1   1314823 135.62 < 2.2e-16 ***
3    284 2944340  1   1037728 107.04 < 2.2e-16 ***
4    282 2733967  2    210373  10.85 2.888e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Four models have been fitted, and the results of the analyses are stored in `fit1`, `fit2`, `fit3` and `fit4`.

(a) Write down the mathematical forms of the four models fitted and explain how the variation in response is being represented by each model. 6 MARKS

(b) On the basis of the results presented, summarize the impact of the policy change at 96 months on the profits of the two companies. 6 MARKS

A further model was fitted and the results stored in `fit5`:

```
> fit5<-lm(Y~X[,3]+X[,4])
>
> anova(fit4,test='F')
Analysis of Variance Table

Response: Y
           Df    Sum Sq  Mean Sq F value    Pr(>F)
X           6 104461730 17410288  1795.8 < 2.2e-16 ***
Residuals 282   2733967     9695
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> anova(fit5,test='F')
Analysis of Variance Table

Response: Y
           Df   Sum Sq  Mean Sq   F value  Pr(>F)
X[, 3]      1 18696706 18696706 1026.0646 < 2e-16 ***
X[, 4]      1   103688   103688    5.6903 0.01771 *
Residuals 285  5193202    18222
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(c) Define the model underlying `fit5`. 2 MARKS

(d) Is a model comparison comparing `fit4` and `fit5` using an F test possible ? If so, construct the test statistic and complete the test as far as possible using the information given; if not, explain why not. 4 MARKS

(e) In a final analysis, the following `R` code was implemented:

```
> cbind(AIC(fit1,fit2,fit3,fit4,fit5),BIC(fit1,fit2,fit3,fit4,fit5))
     df      AIC df      BIC
fit1  3 3651.374  3 3662.362
fit2  4 3571.202  4 3585.854
fit3  5 3486.250  5 3504.565
fit4  7 3468.900  7 3494.541
fit5  4 3647.680  4 3662.332
```

What is the conclusion of this final analysis ? Justify your answer. 2 MARKS

Question L3.

(a) For a linear model defined in terms of $n \times p$ design matrix $\mathbf{X}$, define the *hat matrix*, $\mathbf{H}$, and show that it is symmetric. 4 MARKS

(b) Suppose the true (data-generating) linear model denoted in the usual notation

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

is to be written in terms of two blocks of predictors $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, $\mathbf{X} = \left[\mathbf{X}^{(1)}\ \mathbf{X}^{(2)}\right]$, so that

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon = \mathbf{X}^{(1)}\beta^{(1)} + \mathbf{X}^{(2)}\beta^{(2)} + \epsilon$$

where

$$\beta = \begin{bmatrix} \beta^{(1)} \\ \beta^{(2)} \end{bmatrix}$$

are the p, $p - r$ and r dimensional parameter vector and sub-vectors respectively.

Let $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{Y}}^{(1)}$ (both $n \times 1$ vectors) denote the predictions from fits of the full data-generating model and the reduced model where $\beta^{(2)}$ is assumed to be zero, respectively. Show that the vectors

$$\widehat{\mathbf{Y}} - \widehat{\mathbf{Y}}^{(1)} \qquad \text{and} \qquad \widehat{\mathbf{Y}}^{(1)}$$

are uncorrelated, conditional on $\mathbf{X}$. 8 MARKS

(c) In light of the result in (b), comment on the effect on predictions made from a linear model if *spurious* predictors that are unrelated to the response are mistakenly included in a fitted model. 4 MARKS

(d) Compute the bias of the ordinary least squares estimator for $\beta^{(1)}$ under the reduced model where vector $\beta^{(2)}$ is incorrectly assumed to be zero. 4 MARKS

Section G (Generalized Linear Models)
Answer only two questions out of G1–G3

Question G1. Consider log-linear modelling of a $2 \times 2 \times 2$ contingency table that summarizes a study with three categorical variables A, B and C.

(a) Explain the dependence pattern described by the model AB+C. Explain why, in this model, the fitted conditional odds ratios are the same as the fitted marginal odds ratios. Recall that marginal odds ratios are odds ratios of the form, e.g.,

$$\frac{\Pr(A = a, B = b) \Pr(A = a + 1, B = b + 1)}{\Pr(A = a + 1, B = b) \Pr(A = a, B = b + 1)}.$$

5 MARKS

(b) Explain the dependence pattern described by the model AB + BC + AC. Explain why, in this model, the fitted marginal odds ratios must equal the empirical marginal odds ratios. 4 MARKS

The table below summarizes a study with variables age of mother A, length of gestation G in days, infant survival I, and number of cigarettes smoked per day during the prenatal period, S (N. Wermuth, *Proc. 9th International Biometrics Conference*, 1976).

			Infant Surival	
Age	Smoking	Gestation	No	Yes
< 30	< 5	< 260	50	315
		$260+$	24	4012
	$5+$	< 260	9	40
		$260+$	6	459
$30+$	< 5	< 260	41	147
		$260+$	14	1594
	$5+$	< 260	4	11
		$260+$	1	124

This question continues overleaf.

(c) On the basis of the `R` output on pages 3–4, decide which one of the fitted models is most appropriate for the data at hand. Comment on the overall fit of the model and interpret it. 5 MARKS

(d) In the model `mod2`, compute the conditional odds ratio describing the association between infant survival and the age of the mother. How does this odds ratio change if `mod4` is used? 4 MARKS

(e) Which log-linear model is equivalent to the logistic model $A + G$ using the variable infant survival I as a response? 2 MARKS

R Code and Output for Question G1

```
>mod1 <- glm(counts~A*S+G+I,family=poisson)
>mod2 <- glm(counts~A*I + A*S+ G*I,family=poisson)
>mod3 <- glm(counts~A*I*S+ G+I,family=poisson)
>mod4 <- glm(counts~A*I*S+ G*I,family=poisson)
>mod5 <- glm(counts~A*G + A*I + A*S+ G*I+G*S+I*S,family=poisson)
>mod6 <- glm(counts~A*G*I+A*I*S+A*G*S+G*I*S,family=poisson)
>anova(mod6,mod5,mod4,mod3,mod2,mod1)

Analysis of Deviance Table

Model 1: counts ~ A * S + G + I
Model 2: counts ~ A * I + A * S + G * I
Model 3: counts ~ A * I * S + G + I
Model 4: counts ~ A * I * S + G * I
Model 5: counts ~ A * G + A * I + A * S + G * I + G * S + I * S
Model 6: counts ~ A * G * I + A * I * S + A * G * S + G * I * S
  Resid. Df Resid. Dev Df Deviance
1        10     360.18
2         8       7.72  2   352.46
3         7     346.53  1  -338.81
4         6       4.19  1   342.33
5         5       1.72  1     2.47
6         1       0.36  4     1.36

>coef(summary(mod2))
              Estimate Std. Error    z value      Pr(>|z|)
(Intercept)  4.0186210 0.11901131  33.766715 6.077584e-250
A30+        -0.3588939 0.16723231  -2.146080  3.186662e-02
IY           1.7828069 0.12707183  14.029914  1.022748e-44
S5+         -2.1473638 0.04661273 -46.068186  0.000000e+00
G260+       -0.8377284 0.17843094  -4.694973  2.666423e-06
A30+:IY     -0.5505843 0.16924207  -3.253235  1.140989e-03
A30+:S5+    -0.4043110 0.09935993  -4.069156  4.718384e-05
IY:G260+     3.3279814 0.18425118  18.062199  6.325238e-73

>coef(summary(mod3))
              Estimate Std. Error   z value      Pr(>|z|)
(Intercept)  1.8967842 0.12242660 15.493237  3.853900e-54
A30+        -0.2967319 0.17803073 -1.666745  9.556503e-02
IY           4.0685646 0.11723735 34.703655 6.939108e-264
S5+         -1.5960149 0.28316100 -5.636422  1.736196e-08
G260+        2.3129044 0.04220374 54.803304  0.000000e+00
A30+:IY     -0.6136829 0.18027876 -3.404078  6.638787e-04
A30+:S5+    -0.8018804 0.54621009 -1.468080  1.420824e-01
IY:S5+      -0.5640088 0.28708059 -1.964636  4.945643e-02
A30+:IY:S5+  0.4049639 0.55548390  0.729029  4.659839e-01
```

R Code and Output for Question G1

```
>coef(summary(mod4))
                Estimate Std. Error    z value      Pr(>|z|)
(Intercept)  3.9445097  0.1281307 30.7850358 4.156241e-208
A30+        -0.2967319  0.1780318 -1.6667352  9.556708e-02
IY           1.8582346  0.1358260 13.6809928  1.318797e-42
S5+         -1.5960149  0.2831610 -5.6364211  1.736205e-08
G260+       -0.8377284  0.1784310 -4.6949723  2.666428e-06
A30+:IY     -0.6136829  0.1802798 -3.4040572  6.639285e-04
A30+:S5+    -0.8018804  0.5462252 -1.4680398  1.420934e-01
IY:S5+      -0.5640088  0.2870806 -1.9646353  4.945647e-02
IY:G260+     3.3279814  0.1842512 18.0621973  6.325387e-73
A30+:IY:S5+  0.4049639  0.5554988  0.7290095  4.659959e-01

>coef(summary(mod5))
                 Estimate Std. Error   z value      Pr(>|z|)
(Intercept)  3.94097028 0.12653369 31.145619 5.814350e-213
A30+        -0.29141924 0.17034651 -1.710744  8.712842e-02
G260+       -0.76542186 0.18367321 -4.167303  3.082251e-05
IY           1.81173357 0.13477575 13.442579  3.403654e-41
S5+         -1.69870695 0.24722997 -6.870959  6.377180e-12
A30+:G260+  -0.16717590 0.09613072 -1.739048  8.202640e-02
A30+:IY     -0.46385105 0.18004928 -2.576245  9.987984e-03
A30+:S5+    -0.41194190 0.09951748 -4.139392  3.482269e-05
G260+:IY     3.30940173 0.18461642 17.925826  7.414178e-72
G260+:S5+   -0.04726655 0.14907435 -0.317067  7.511928e-01
IY:S5+      -0.41442496 0.26173208 -1.583394  1.133317e-01
```

Question G2.

(a) Consider a binomial GLM with the canonical logit link for a binary response variable. Prove that the deviance satisfies

$$D(\boldsymbol{y}, \widehat{\boldsymbol{\pi}}) = -2\sum_{i=1}^{n}\left\{\widehat{\pi}_i \log\left(\frac{\widehat{\pi}_i}{1-\widehat{\pi}_i}\right) + \log(1-\widehat{\pi}_i)\right\}$$

and explain the implication of this result for model validation. 5 MARKS

Consider the following study on the participation in the U.S. federal food stamp program (Stefanski et al., *Biometrika*, 1986). The response indicates participation in the federal food stamp program (1 for yes, 0 for no); the explanatory variables are tenancy TEN (1 for yes, 0 for no), supplementary income SUP (1 for yes, 0 for no) and the continuous variable LMI, computed from the monthly income INC as $\log(\text{INC} + 1)$.

(b) On the basis of the `R` output on page 6, decide which one of the fitted models is most appropriate for the data at hand. Propose a strategy for validating the overall fit of this model using a statistical test. 5 MARKS

(c) Models `mod4` (TEN*SUP) and `mod5` (TEN+SUP+LMI) are compared graphically using ROC curves in Figure 1 on page 7. Explain how these curves were constructed and decide which of the two models has a greater predictive power. 3 MARKS

(d) Interpret the model `mod6`, TEN*LMI + SUP and explain the meaning of the interaction parameter on line 36. Based on this model, write down a formula for the fitted probability that a person with tenancy and supplemental income will participate in the federal food stamp program. Use a graph to show how this probability depends on LMI. 5 MARKS

(e) Do you think that a quasibinomial model would be suitable for these data? Justify your answer. 2 MARKS

R Code and Output for Question 2

```
> mod1 <- glm(y ~ TEN,family=binomial)
> mod2 <- glm(y ~ SUP,family=binomial)
> mod3 <- glm(y ~ TEN+SUP,family=binomial)
> mod4 <- glm(y ~ TEN*SUP,family=binomial)
> mod5 <- glm(y ~ TEN+SUP+LMI,family=binomial)
> mod6 <- glm(y ~ TEN*LMI+SUP,family=binomial)
> mod7 <- glm(y ~ TEN*SUP+LMI,family=binomial)
> mod8 <- glm(y ~ TEN*SUP*LMI,family=binomial)

Analysis of Deviance Table

Model 1: y ~ TEN
Model 2: y ~ SUP
Model 3: y ~ TEN + SUP
Model 4: y ~ TEN * SUP
Model 5: y ~ TEN + SUP + LMI
Model 6: y ~ TEN * LMI + SUP
Model 7: y ~ TEN * SUP + LMI
Model 8: y ~ TEN * SUP * LMI
  Resid. Df Resid. Dev Df Deviance
1       148     111.49
2       148     125.15  0 -13.6617
3       147     108.02  1  17.1313
4       146     107.53  1   0.4894
5       146     106.40  0   1.1363
6       145     105.62  1   0.7719
7       145     105.99  0  -0.3638
8       142     100.75  3   5.2373

> coef(summary(mod6))
              Estimate Std. Error    z value   Pr(>|z|)
(Intercept)  0.2248635  1.7200978  0.1307272 0.89599114
TEN          1.6015391  4.0273111  0.3976696 0.69087376
LMI         -0.2088196  0.2926234 -0.7136121 0.47546709
SUP          0.8769330  0.4988344  1.7579641 0.07875361
TEN:LMI     -0.5791063  0.6784578 -0.8535627 0.39334733
```

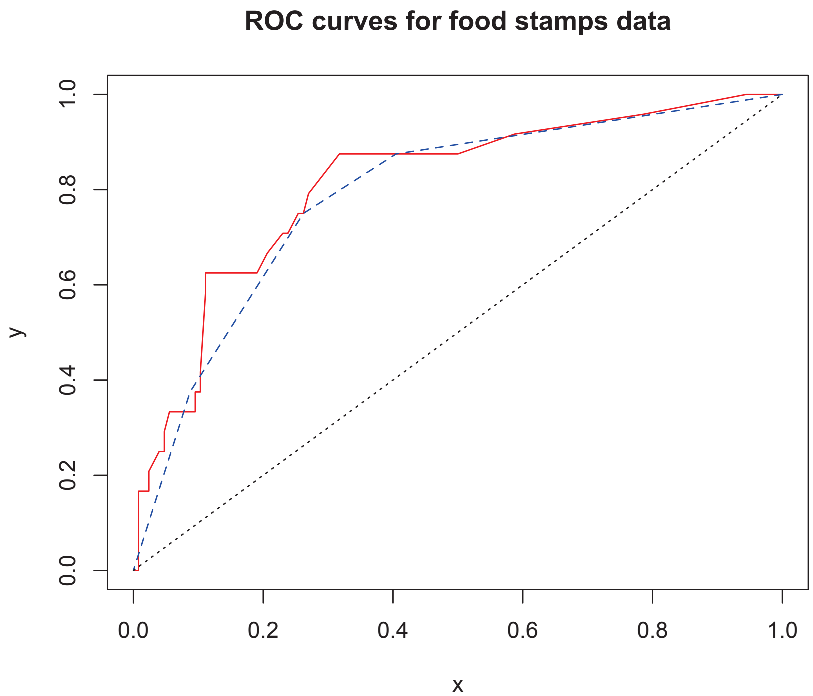


Figure 1: ROC curves for models `mod5` (full line) and `mod4` (dashed line), fitted to the food stamp data in Question G2.

Question G3.

(a) Formulate the proportional odds model for an ordinal response Y. 2 MARKS

(b) Explain how the proportional odds model arises from a regression model for an unobserved latent variable $Z = -X\boldsymbol{\beta} + \varepsilon$. What distributional assumptions on ε must be made? 5 MARKS

The effectiveness of a spray therapy on the reduction of pressure pain in the knee was investigated through a clinical study. After a 10-day treatment following a sports injury, the PAIN level of the patient was assessed; the values were 1 = little pain, 2 = moderate pain, 3 = strong pain, and 4 = severe pain. The explanatory variables collected were TH (1 = therapy, 0 = placebo), AGE (continuous scale), GEN (1 = male, 0 = female). These data were analyzed as ungrouped data in `R` using the `VGAM` library; the output is on pages 9–10.

(c) Is there statistical evidence that the therapy is effective? Are AGE and GEN significant predictors? 3 MARKS

(d) Consider the model `v2`. Explain why there are three intercepts and calculate the probabilities for experiencing little, moderate, strong and severe pain, respectively, for patients using the treatment and for patients using placebo. 5 MARKS

(e) Consider the model `v2`. Describe the effect of the therapy using the proportional odds ratio. 3 MARKS

(f) Formulate one other generalized linear model that could have been used to analyze these data. 2 MARKS

R Code and Output for Question Q3

```
Call:
vglm(formula = PAIN ~ 1, family = propodds(reverse = F))

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  -1.0033     0.2035  -4.930 8.22e-07 ***
(Intercept):2   0.1466     0.1808   0.811    0.417
(Intercept):3   1.0451     0.2055   5.085 3.67e-07 ***
---
Signif. codes:
0    ***    0.001    **    0.01    *    0.05    .    0.1         1

Number of linear predictors:  3

Names of linear predictors:
logit(P[Y<=1]), logit(P[Y<=2]), logit(P[Y<=3])

Dispersion Parameter for cumulative family:   1

Residual deviance: 339.5075 on 366 degrees of freedom

Log-likelihood: -169.7537 on 366 degrees of freedom


Call:
vglm(formula = PAIN ~ TH, family = propodds(reverse = F))

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  -1.5767     0.2845  -5.543 2.98e-08 ***
(Intercept):2  -0.3769     0.2472  -1.525  0.12734
(Intercept):3   0.5979     0.2519   2.374  0.01761 *
TH              0.9356     0.3321   2.817  0.00484 **
---
Signif. codes:
0    ***    0.001    **    0.01    *    0.05    .    0.1         1

Number of linear predictors:  3

Names of linear predictors:
logit(P[Y<=1]), logit(P[Y<=2]), logit(P[Y<=3])

Dispersion Parameter for cumulative family:   1

Residual deviance: 331.5657 on 365 degrees of freedom

Log-likelihood: -165.7828 on 365 degrees of freedom
```

R Code and Output for Question Q3

```
Call:
vglm(formula = PAIN ~ TH + GEN + AGE, family = propodds(reverse = F))

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept):1 -2.4198157  0.6261102  -3.865 0.000111 ***
(Intercept):2 -1.2048619  0.5965897  -2.020 0.043427 *
(Intercept):3 -0.2183963  0.5883465  -0.371 0.710486
TH             1.0335145  0.3383901   3.054 0.002257 **
GEN           -0.0007686  0.3633596  -0.002 0.998312
AGE            0.0260172  0.0174698   1.489 0.136417
---
Signif. codes:
0    ***    0.001    **    0.01    *    0.05    .    0.1         1

Number of linear predictors:  3

Names of linear predictors:
logit(P[Y<=1]), logit(P[Y<=2]), logit(P[Y<=3])

Dispersion Parameter for cumulative family:   1

Residual deviance: 329.3384 on 363 degrees of freedom

Log-likelihood: -164.6692 on 363 degrees of freedom
```

Table of the Chi-squared distribution

Entries in table are $\chi^2_\alpha(\nu)$: the α tail quantile of Chi-squared(ν) distribution
α given in columns, ν given in rows.

	Left-tail					Right-tail				
ν	0.99500	0.99000	0.97500	0.95000	0.90000	0.10000	0.05000	0.02500	0.01000	0.00500
1	0.00004	0.00016	0.00098	0.00393	0.01579	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.01003	0.02010	0.05064	0.10259	0.21072	4.60517	5.99146	7.37776	9.21034	10.59663
3	0.07172	0.11483	0.21580	0.35185	0.58437	6.25139	7.81473	9.34840	11.34487	12.83816
4	0.20699	0.29711	0.48442	0.71072	1.06362	7.77944	9.48773	11.14329	13.27670	14.86026
5	0.41174	0.55430	0.83121	1.14548	1.61031	9.23636	11.07050	12.83250	15.08627	16.74960
6	0.67573	0.87209	1.23734	1.63538	2.20413	10.64464	12.59159	14.44938	16.81189	18.54758
7	0.98926	1.23904	1.68987	2.16735	2.83311	12.01704	14.06714	16.01276	18.47531	20.27774
8	1.34441	1.64650	2.17973	2.73264	3.48954	13.36157	15.50731	17.53455	20.09024	21.95495
9	1.73493	2.08790	2.70039	3.32511	4.16816	14.68366	16.91898	19.02277	21.66599	23.58935
10	2.15586	2.55821	3.24697	3.94030	4.86518	15.98718	18.30704	20.48318	23.20925	25.18818
11	2.60322	3.05348	3.81575	4.57481	5.57778	17.27501	19.67514	21.92005	24.72497	26.75685
12	3.07382	3.57057	4.40379	5.22603	6.30380	18.54935	21.02607	23.33666	26.21697	28.29952
13	3.56503	4.10692	5.00875	5.89186	7.04150	19.81193	22.36203	24.73560	27.68825	29.81947
14	4.07467	4.66043	5.62873	6.57063	7.78953	21.06414	23.68479	26.11895	29.14124	31.31935
15	4.60092	5.22935	6.26214	7.26094	8.54676	22.30713	24.99579	27.48839	30.57791	32.80132
16	5.14221	5.81221	6.90766	7.96165	9.31224	23.54183	26.29623	28.84535	31.99993	34.26719
17	5.69722	6.40776	7.56419	8.67176	10.08519	24.76904	27.58711	30.19101	33.40866	35.71847
18	6.26480	7.01491	8.23075	9.39046	10.86494	25.98942	28.86930	31.52638	34.80531	37.15645
19	6.84397	7.63273	8.90652	10.11701	11.65091	27.20357	30.14353	32.85233	36.19087	38.58226
20	7.43384	8.26040	9.59078	10.85081	12.44261	28.41198	31.41043	34.16961	37.56623	39.99685
21	8.03365	8.89720	10.28290	11.59131	13.23960	29.61509	32.67057	35.47888	38.93217	41.40106
22	8.64272	9.54249	10.98232	12.33801	14.04149	30.81328	33.92444	36.78071	40.28936	42.79565
23	9.26042	10.19572	11.68855	13.09051	14.84796	32.00690	35.17246	38.07563	41.63840	44.18128
24	9.88623	10.85636	12.40115	13.84843	15.65868	33.19624	36.41503	39.36408	42.97982	45.55851
25	10.51965	11.52398	13.11972	14.61141	16.47341	34.38159	37.65248	40.64647	44.31410	46.92789
26	11.16024	12.19815	13.84390	15.37916	17.29188	35.56317	38.88514	41.92317	45.64168	48.28988
27	11.80759	12.87850	14.57338	16.15140	18.11390	36.74122	40.11327	43.19451	46.96294	49.64492
28	12.46134	13.56471	15.30786	16.92788	18.93924	37.91592	41.33714	44.46079	48.27824	50.99338
29	13.12115	14.25645	16.04707	17.70837	19.76774	39.08747	42.55697	45.72229	49.58788	52.33562
30	13.78672	14.95346	16.79077	18.49266	20.59923	40.25602	43.77297	46.97924	50.89218	53.67196
31	14.45777	15.65546	17.53874	19.28057	21.43356	41.42174	44.98534	48.23189	52.19139	55.00270
32	15.13403	16.36222	18.29076	20.07191	22.27059	42.58475	46.19426	49.48044	53.48577	56.32811
33	15.81527	17.07351	19.04666	20.86653	23.11020	43.74518	47.39988	50.72508	54.77554	57.64845
34	16.50127	17.78915	19.80625	21.66428	23.95225	44.90316	48.60237	51.96600	56.06091	58.96393
35	17.19182	18.50893	20.56938	22.46502	24.79665	46.05879	49.80185	53.20335	57.34207	60.27477
36	17.88673	19.23268	21.33588	23.26861	25.64330	47.21217	50.99846	54.43729	58.61921	61.58118
37	18.58581	19.96023	22.10563	24.07494	26.49209	48.36341	52.19232	55.66797	59.89250	62.88334
38	19.28891	20.69144	22.87848	24.88390	27.34295	49.51258	53.38354	56.89552	61.16209	64.18141
39	19.99587	21.42616	23.65432	25.69539	28.19579	50.65977	54.57223	58.12006	62.42812	65.47557
40	20.70654	22.16426	24.43304	26.50930	29.05052	51.80506	55.75848	59.34171	63.69074	66.76596
50	27.99075	29.70668	32.35736	34.76425	37.68865	63.16712	67.50481	71.42020	76.15389	79.48998