McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper BETA

10 May, 2019
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the the following modules [AL] Algebra; [AN] Analysis; [GT] Geometry & Topology; [NA] Numerical Analysis; [PDE] Partial Differential Equations; [PR] Probability; [DM] Discrete Mathematics, each of which comprises 4 questions. The Optimization module comprises 2 questions in Continuous Optimization [CO]; and 2 questions in Discrete Optimization [DO].

You should answer 7 questions from 3 modules, with at least 2 from each module. All questions are worth 10 points.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 8 pages of questions.

Algebra Module

[ALG. 1]

Let k be a field and $k[x]$ the ring of polynomials over k.

Consider the ideal $I = xk[x]$. Prove that the projective limit $lim_n\ k[x]/I^n$ is isomorphic to $k[[x]]$, the power series ring over k.

Prove that $k[[x]]$ is a PID all whose non-zero ideals are of the form J^n for some $n \geq 0$ where $J = xk[[x]]$.

Consider $k[[x,y]]/(xy)$ as a $k[[x]]$-module. What is its torsion submodule?

[ALG. 2]

Let q be a prime. You may assume that there is a prime p such that $q|(p-1)$. Prove, using results about cyclotomic fields, that the field of rational numbers $\mathbf{Q}$ has a cyclic Galois extension of order q.

[ALG. 3]

Let R be a commutative ring. Recall that a projective module P over R is an R module such that there is some R module Q for which $P \oplus Q$ is a free R module.

Prove that the tensor product of two projective modules is a projective module.

Let I be a non-zero ideal of R. Can R/I ever be a projective R module?

[ALG. 4]

Let G be a finite group and let p be a prime dividing the cardinality of G. Recall that a Sylow p-subgroup of G is a subgroup of G whose cardinality is equal to the exact power of p dividing the cardinality of G.

(a) Show that G can be realized as a subgroup of the group $\Gamma := GL_n(F_p)$ of invertible $n \times n$ matrices with entries in the field with p elements, for a suitable n.

(b) Describe an explicit Sylow p-subgroup S of Γ.

(c) Let $X := \Gamma/S$ be the set of cosets for S in Γ, endowed with its natural left action of G. Show that there is an element x of X whose stabiliser is divisible by the same power of p as G is. Show that this stabiliser is a Sylow p-subgroup of G.

Analysis Module

[AN. 1]

Let $\mu_L^* : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ denote Lebesgue outer measure given by

$$\mu_L^*(A) := \inf \Big\{ \sum_{k=1}^{\infty} |I_k|;\ A \subset \cup_{k=1}^{\infty} I_k,\ I_k \textit{ open interval } \Big\}.$$

Let μ_L denote the induced Lebesgue measure on the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$. Suppose $E \subset \mathbf{R}$ is Lebesgue measurable with $\mu_L(E) = 0$. Show that E^c is dense in $\mathbf{R}$. Please carefully justify your answer.

[AN. 2]

Let $f \in L^1(\mathbf{R})$ with respect to Lebesgue measure. Compute

$$\lim_{n \to \infty} \int_{\mathbf{R}} e^{-nx^2} f(x)\, dx$$

and carefully justify your answer.

[AN. 3]

Let $1 \le p < \infty$. Given $g \in L^p(R^n, dx)$ and $f \in L^1(R^n, dx)$ where dx is Lebesgue measure, one defines the convolution by the formula

$$f * g(x) := \int_{R^n} f(x-t)\, g(t)\, dt.$$

Show that $f * g \in L^p(R^n, dx)$ and

$$\|f * g\|_p \le \|f\|_1\, \|g\|_p.$$

Please carefully justify your answer.

[AN. 4]

Consider the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{r>0} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r) = \{y \in \mathbf{R}^n;\ |y-x| < r\}$. Does there exist a finite constant $C > 0$ such that $\|Mf\|_1 \le C\|f\|_1$ for all $f \in L^1(\mathbf{R}^n)$.? Please carefully justify your answer.

Geometry and Topology Module

[GT. 1]

Let M be a surface obtained by removing 3 points from a projective plane.

(1) Compute $\pi_1 M$.
(2) List all compact surfaces with the same homotopy type as M. And explain why.
(3) Prove that M does not have the same homotopy type as a closed genus 2 surface.

[GT. 2]

Let a, b, c be distinct points in a projective plane P. Let $Y = P/\{a \sim b \sim c\}$ be the quotient obtained by identifying these three points.

(1) Compute all homology groups of Y with $\mathbb{Z}$ coefficients.
(2) Find two nonhomeomorphic degree 2 connected covering spaces $\widehat{Y}_1$ and $\widehat{Y}_2$ of Y.
(3) Explain why $\widehat{Y}_1$ and $\widehat{Y}_2$ are not homeomorphic.

[GT. 3]

Recall that a C^∞ manifold M is *orientable* if it admits an atlas of C^∞-compatible charts $\{(U_\alpha, \phi_\alpha), \alpha \in A\}$ such that the transition functions

$$\phi_\alpha^{-1} \circ \phi_\beta : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

have everywhere a *positive* Jacobian determinant. Show that M is orientable if and only if there exists a C^∞ n-form $\omega \in \Omega^n(M)$, where $n = \dim M$, such that $\omega_x \neq 0$ for all $x \in M$.

[GT. 4]

A C^∞ manifold M of dimension $2n$ is said to be *symplectic* if it admits a C^∞ 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and the n-th exterior power ω^n of ω satisfies $\omega_x^n \neq 0$ for all $x \in M$. Show that the only even-dimensional sphere S^{2n} which is symplectic is the 2-sphere S^2.

Optimization Module

Continuous Optimization

[CO. 1] (Convex optimization)

Let $X \subset \mathbb{R}^n$ be nonempty, convex and let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and consider

$$\min\ f(x) \quad \text{s.t.} \quad x \in X. \tag{1}$$

a) Show that the solution set $\mathrm{argmin}_X f$ of (1) is convex (possibly empty).
b) Show that every local minimizer of (1) is a global minimizer.
c) Suppose that X is closed and f is given by $f(x) = \frac{1}{2}(x - \bar{x})^T Q(x - \bar{x})$, where $Q \in \mathbb{R}^{n\times n}$ is (symmetric) positive definite and $\bar{x} \in \mathbb{R}^n$. Show that

$$\mathrm{argmin}_X f = \{P_{S(X)}(S\bar{x})\}$$

where S is the *square root* of Q (i.e. $S = S^T$ and $S^2 = Q$) and

$$S(X) := \{Sx \mid x \in X\}.$$

(3+3+4 P.)

[CO. 2] (KKT conditions for NLP)

Consider the nonlinear program

$$\min\ x_1^2 - x_2^2 - x_3^2 \ \ s.t.\ \ x_1^4 + x_2^4 + x_3^4 \le 1. \tag{2}$$

a) Find all KKT points of (2).
b) Find the optimal solution of (2).

(6+4 P.)

Discrete Optimization

[DO. 1]

(a) Define convex set, vertex of a convex set, convex hull and polytope
(b) Consider a polyhedra $Ax \le b$ consisting of m inequalities:

$$\{a_1x \le b_1, a_2x \le b_2, \dots, a_mx \le b_m\}.$$

For a point z in this polyhedra $A_zx = b_z$ is the following system of equations: $\{a_ix = b_i | a_iz = b_i\}$. Show that a point z in the polyhedra is a vertex of the polyhedra precisely if it is the unique solution to $A_zx = b_z$.
(c) Show that every bounded polyhedra is the convex hull of its vertices.
(d) Using (b) and (c) to show that every bounded polyhedra is a polytope (you can do this even if you failed to prove (b) or (c)).

[DO. 2]

(a) Define Matroid, rank function(for a matroid), and Matroid Polytope.
(b) State Edmonds Characterization of the Matroid Polytope
(c) Consider a matroid M with ground set $\{e_1, e_2, e_3, \dots, e_{100}\}$ and the weighting w where e_i has weight $50 - i$. Suppose that $Z = \{e_1, e_8, e_{23}, e_{47}\}$ is a maximum weight independent set for this weighting. We can assign multipliers to the constraints of the polyhedra P_M that Edmonds proved characterize the matching polytope to show that for the characteristic vector x_Z of Z, $wx_z = \max\{wx | x \in P_M\}$. Specify a choice of multipliers which does so.

Numerical Analysis Module

[NA. 1]

Consider the ODE $x'(t) = x(t)^{1/4}$.

(a) Find a solution of the form $x(t) = (Ct)^\alpha$ where $C, \alpha > 0$ are positive constants.

(b) Compute the solution of the Forward Euler method for the ODE, x_n^{FE}, with time step dt, starting from $x_0^{FE} = 0$. Does is converge to the solution above at $h \to 0$? Explain your results.

(c) Write down the equation for the backward Euler method for the ODE with time step dt, x_n^{BE}, and find x_1^{BE} when $x_0^{BE} = 0$.

[NA. 2]

(a) Find second order approximations to $u'(x)$ and $u''(x)$ using the values

$$u(x), u(x-h), u(x+h)$$

(b) Find the best Lipschitz constant of the function $f(x) = \min(7x - 4, -2x)$.

[NA. 3]

Given a norm $\|z\|$ for $z \in \mathbb{R}^d$ and a vector $v \in \mathbb{R}^d$, consider the problem

$$\min_{\|z\| \leq 1} v \cdot z$$

(a) Find the minimizer z^* when the norm is given by $\|z\|_p$ for $p = 1, 2, \infty$.

(b) Show that $\|z\| = \max(\|z\|_\infty, .7\|z\|_1)$ is a norm.

[NA. 4]

Consider the functional

$$J[u] = \int_0^1 a(x) u_x^2 \, dx \tag{3}$$

where the coefficient $a(x) \geq a_0 > 0$ is positive.

(a) Find the Euler-Lagrange equation for a critical point $\nabla J[u] = 0$, along with boundary conditions.

(b) Suppose that $u(x,t)$ is a solution of the parabolic PDE, $u_t + \nabla J[u] = 0$, along with the boundary conditions above. Prove that $\frac{d}{dt} J(t) \leq 0$.

PDE Module

[PDE. 1]

Let $\Omega \subset \mathbb{R}^3$ be a domain, and let $\Sigma \subset \Omega$ be a smooth (nontrivial) surface. Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on Σ, where ∂_ν is a normal derivative operator at Σ. Show that u vanishes identically.

[PDE. 2]

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $f \in L^2(\Omega)$. Then consider the problem of finding a function $u \in H_0^1(\Omega)$ and a number $p \in \mathbb{R}$ satisfying

$$-\Delta u + p = f \qquad \text{in} \quad \Omega, \qquad \text{with} \qquad \int_\Omega u = 0.$$

(a) Give a weak formulation of this problem.
(b) Prove that there exists a unique solution $(u, p) \in H_0^1(\Omega) \times \mathbb{R}$.
(c) Show that the solution depends continuously on f in a suitable sense.

[PDE. 3]

Let $\Omega = \mathbb{R}^n \setminus K$ be a domain with C^1 boundary, where $K \subset \mathbb{R}^n$ is a compact set. Show that there exists a unique smooth solution to the initial-boundary value problem

$$\begin{aligned} \Box u &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}, \\ u &= \phi \quad \text{on} \quad \Omega \times \{0\}, \\ \partial_t u &= \psi \quad \text{on} \quad \Omega \times \{0\}, \end{aligned}$$

where ϕ and ψ are smooth functions with compact supports in Ω.

[PDE. 4]

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary, and let $0 < T < \infty$. Suppose that $u \in C^2(\bar{\Omega} \times [0, T])$ satisfies

$$\begin{aligned} \partial_t u - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0, T), \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0, T), \\ u &= 0 \qquad \text{on} \quad \Omega \times \{T\}, \end{aligned}$$

where Δ is the n-dimensional Laplacian, and ∂_t is differentiation along the $(n+1)$-st coordinate $t = x_{n+1}$. In words, the function u is a solution to the heat equation in $\Omega \times (0, T)$, satisfies the homogeneous Dirichlet boundary condition on $\partial\Omega$, and vanishes at the final time T. Prove that $u \equiv 0$ in $\Omega \times (0, T)$.

Probability Module

[PR. 1]

Let $(X_n, n \geq 1)$ and X be random variables defined on a common probability space.

(a) What does it mean for $(X_n, n \geq 1)$ to be uniformly integrable?

(b) Prove that if $(X_n, n \geq 1)$ are uniformly integrable and $X_n \to X$ almost surely, then $\mathbf{E}|X| < \infty$.

(c) Under the assumptions in part (b), show that $\mathbf{E}X_n \to \mathbf{E}X$.

[PR. 2]

(a) State the continuity theorem for characteristic functions.

(b) Prove that if $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$ all have modulus at most 1 then

$$|z_1 \cdot \ldots \cdot z_n - w_1 \cdot \ldots \cdot w_n| \leq \sum_{k=1}^{n} |z_k - w_k|.$$

(c) Prove using characteristic functions that if $(X_n, n \geq 1)$ are IID and $\mathbf{E}|X_1| < \infty$, then with $S_n = X_1 + \ldots + X_n$, it holds that

$$\frac{S_n}{n} \to \mathbf{E}X_1$$

in distribution.

[PR. 3]

Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. $\mathbb{R}-$valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that X_n's are NOT a.e. constant, i.e., for every $c \in \mathbb{R}$, $\mathbb{P}(X_n = c) < 1$.

(i) Prove that $\mathbb{P}(X_n > X_{n+1}) > 0$.

(ii) Prove that $\mathbb{P}(X_n > X_{n+1} \text{ i.o. }) = 1$.

[PR. 4]

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a constant $p \in (0, 1)$, let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. random variables with the common distribution

$$\mathbb{P}(X_1 = 1) = p \text{ and } \mathbb{P}(X_1 = 0) = 1 - p.$$

Define a mapping $D : \Omega \to [0, 1]$ by

$$D = \sum_{n=1}^{\infty} \frac{X_n}{2^n}.$$

Obviously D is well-defined as a random variable because the series above is convergent almost surely. Denote by μ_p the law of D (therefore, μ_p is a probability measure on $([0, 1], \mathcal{B}([0, 1]))$) and F_p the distribution function of D.

(i) Prove that F_p is a continuous function.

(ii) For each $x \in [0, 1]$ and $n \geq 1$, define

$$E_n(x) := \begin{cases} 1, & \text{if } 2^{n-1}x - \left[2^{n-1}x\right] \geq \frac{1}{2}, \\ 0, & \text{if } 2^{n-1}x - \left[2^{n-1}x\right] < \frac{1}{2}, \end{cases}$$

where $[s]$ denotes the integer part of $s \geq 0$. Prove that

$$\mu_p\left(\left\{x \in [0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} E_n(x) = p\right\}\right) = 1,$$

from where argue that if $p_1 \neq p_2$, then $\mu_{p_1} \perp \mu_{p_2}$, and in particular, if $p \neq \frac{1}{2}$, $\mu_p \perp \lambda$ where λ is the Lebesgue measure on $\mathbb{R}$.

Discrete Mathematics Module

Graph Theory

[DM 1.] Matching.

a) State Tutte's matching theorem.

b) What is the maximum possible number of edges in a simple connected graph with $2n$ vertices and no perfect matching?

c) Show that every 2-connected 3-regular graph has a perfect matching.

[DM 2.] Planarity.

a) State Euler's formula.

b) Let G be a planar graph, such that every cycle in G has length at least six. Show that $\chi(G) \leq 3$.

c) Let G be a simple graph drawn in the plane so that every region in the drawing is bounded by a cycle of length at least six. Is it necessarily true that $\chi(G) \leq 3$?

Combinatorics

[DM 3.] Ramsey theory.

a) State Ramsey's theorem and van der Waerden's theorem.

b) Show that for every positive integer k there exists a positive integer n satisfying the following. For every coloring of $[n]$ with k colors it is possible to find $w, x, y, z \in [n]$ of the same colour, not necessarily distinct, so that $x + y + z = w$.
(*Hint:* Use Ramsey's theorem.)

c) Show that for each $\epsilon > 0$ there exists a positive integer n satisfying the following. For each real $\alpha > 0$ there exist integers $1 \leq q \leq n$ and p such that $|q^2\alpha - p| \leq \epsilon$.
(*Hint:* Color every integer $i \in [n]$ according to the fractional part of $i^2\alpha$ and apply van der Waerden's theorem.)

[DM 4.] Discrete Geometry.

a) State Helly's Theorem.

b) Let $X_1, X_2, \ldots, X_n$ be convex sets in $\mathbb{R}^2$. Suppose that for any three of these sets there exists a circle of radius one intersecting all three. Show that there exists a circle of radius one intersecting all n sets.

c) Does b) necessarily hold if $X_1, X_2, \ldots, X_n$ are not required to be convex?