McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper BETA

11 May, 2018
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the the following modules [AL] Algebra; [AN] Analysis; [GT] Geometry & Topology; [CO] Continuous Optimization; [NA] Numerical Analysis; [PDE] Partial Differential Equations; [PR] Probability; [DM] Discrete Mathematics, each of which comprises 4 questions. You should answer 7 questions from 3 modules, with at least 2 from each module. All questions are worth 10 points.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[**ALG. 1**]
- (a) Let k be a field of characteristic p and let K be a cyclic Galois extension of k of degree p. If σ is a generator of $\mathrm{Gal}(K/k)$, show that the extension K is generated by an element α satisfying $\sigma(\alpha) = \alpha + 1$.
- (b) Show that K is the splitting field of a polynomial of the form $x^p - x - a \in k[x]$.

[**ALG. 2**]
- (a) State the classification theorem for finite-dimensional central simple algebras over a field k.
- (b) The double centralizer theorem asserts that if B is a simple k-subalgebra of a finite dimensional central simple algebra A over k, then the centralizer of B in A (i.e., the set of elements of A that commute with all the elements of B), denoted C, is also a simple k-algebra and that $\dim_k(B) \cdot \dim_K(C) = \dim_k(A)$. Use this to show that any finite dimensional division algebra over k contains a maximal (commutative) subfield and that its dimension is the square of an integer.
- (c) Give an example of a central simple algebra of dimension n^2 over $\mathbb{Q}$ which contains all possible (isomorphism classes of) degree n extensions of k. Give an example of a central simple algebra over $\mathbb{Q}$ of rank 4 which contains no field of the form $\mathbb{Q}(\sqrt{d})$ with $d > 0$.

[**ALG. 3**]
- (a) Write the element $11 + 2i$ as a product of irreducible elements in the ring of Gaussian integers.
- (b) Show that the element $3 + \sqrt{-5}$ is irreducible in the ring $\mathbb{Z}[\sqrt{-5}]$ but that the ideal it generates is not maximal, or even prime. Express the principal ideal generated by $3 + \sqrt{-5}$ as a product of prime ideals.

[**ALG. 4**]
- (a) Give two examples you know of non-commutative groups of order 8, and show that they are not isomorphic to each other.
- (b) Write down the character tables for each of the groups constructed in part (a).

Analysis Module

[AN. 1]

Let μ be a finite Borel measure on $\mathbb{R}$ and

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx} d\mu(x)$$

its Fourier transform.

(1) Prove that the function $\mathbb{R} \ni t \mapsto \hat{\mu}(t) \in \mathbb{C}$ is uniformly continuous.

(2) Suppose that the measure μ has no atoms, namely that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Prove that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 dt = 0.$$

[AN. 2]

Let $(X, \mathcal{F}, \mu)$ be a measure space, $f \in L^1(\mu)$, and $f_n \in L^1(\mu)$ a sequence such that

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0.$$

Show that there exists a subsequence f_{n_k} such that for μ-a.e. $x \in X$,

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

[AN. 3]

Let $1 \le p < \infty$. Given $g \in L^p(\mathbb{R}^n, dx)$ and $f \in L^1(\mathbb{R}^n, dx)$ where dx is Lebesgue measure, one defines the convolution by the formula

$$f * g(x) := \int_{\mathbb{R}^n} f(x - t) \, g(t) \, dt.$$

Show that $f * g \in L^p(\mathbb{R}^n, dx)$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Please carefully justify your answer.

[AN. 4]

Let $\alpha \in (0, 1)$ and consider the integral operator given by

$$Tf(x) = \int_I |\sin(x - y)|^{-\alpha} f(y) \, dy,$$

where $I = [0, 1]$ and dy denotes Lebesgue measure. Show that for any $1 \le p < \infty$, the operator $T : L^p(I) \to L^p(I)$ is bounded and estimate $\|T\|$ from above. Please carefully justify your answer.

Geometry and Topology Module

[GT. 1]

Let U be an open subset of $\mathbf{R}^n$ and let $\omega^1, \ldots, \omega^n$ be C^∞ 1-forms on U such that the cotangent vectors $\omega^1|_x, \ldots, \omega^n|_x$ are linearly independent at every point $x \in U$. (Such a set of 1-forms is called a *coframe* on U). Show that

$$d\omega^i = \sum_{1 \leq j < k \leq n} C^i_{jk} \omega^j \wedge \omega^k, \quad 1 \leq i \leq n,$$

for some C^∞ functions C^i_{jk}, $1 \leq i \leq n$, $1 \leq j < k \leq n$. Let now $f : U \to U$ be a diffeomorphism of class C^∞ and let $\bar{\omega}^1, \ldots, \bar{\omega}^n$ be 1-forms defined by

$$\bar{\omega}^i|_x = f^*(\omega^i|_{f(x)}), \quad 1 \leq i \leq n.$$

Show that the cotangent vectors $\bar{\omega}^1|_x, \ldots, \bar{\omega}^n|_x$ are linearly independent at very $x \in U$. Show that if

$$d\bar{\omega}^i = \sum_{1 \leq j < k \leq n} \bar{C}^i_{jk} \bar{\omega}^j \wedge \bar{\omega}^k, \quad 1 \leq i \leq n,$$

then

$$C^i_{jk} \circ f = \bar{C}^i_{jk},$$

for all $1 \leq i \leq n$, $1 \leq j < k \leq n$.

[GT. 2]

Compute the de Rham cohomology of the group $U(2)$ of two-by-two unitary matrices. (Hint: Show that the group $SU(2)$ of two-by-two unimodular unitary matrices is diffeomorphic to S^3.)

[GT. 3]

Let S be a surface. Let $C \hookrightarrow S$ and $C' \hookrightarrow S$ be embedded circles. Call C and C' *equivalent* if there exists a homeomorphism $f : S \to S$ with $f(C) = C'$.

(1) Let S_3 be the orientable surface of genus 3. Describe a collection of 3 pairwise nonequivalent embedded circles in S_3. Justify your example.

(2) Let N_3 be a genus 3 non-orientable surface. Describe a collection of 5 nonequivalent embedded circles in N_3. (Find as many as you can.)

[GT. 4]

Let $X_1 = \mathbb{R}^3 - (P_1 \cup P_2 \cup P_3)$, where:

$$P_1 = \{(t, 0, 0) : t \geq 0\}, \qquad P_2 = \{(0, t, 0) : t \geq 0\}, \qquad P_3 = \{(0, 0, t) : t \geq 0\}.$$

Let $X_2 = \mathbb{R}^3 - (C_1 \cup C_2)$ where $C_r = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = r^2\}$.

(1) Show that X_1 deformation retracts to a surface with three boundary circles.

(2) Show that X_2 deformation retracts to the wedge $S^2 \vee S^1 \vee S^1$ of a 2-sphere and two circles.

(3) Show that $\pi_1 X_1 \cong \pi_1 X_2$ but X_1 and X_2 do not have the same homotopy type.

Continuous Optimization Module

[CO. 1] (Convex optimization)
    Let $X \subset \mathbb{R}^n$ be nonempty, convex and let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and consider

(1) $$\min \ f(x) \quad \text{s.t.} \quad x \in X.$$

    a) Show that the solution set $\operatorname{argmin}_X f$ of (1) is convex (possibly empty).
    b) Show that every local minimizer of (1) is a global minimizer.
    c) Suppose that X is closed and f is given by $f(x) = \frac{1}{2}(x - \bar{x})^T Q(x - \bar{x})$, where $Q \in \mathbb{R}^{n \times n}$ is (symmetric) positive definite and $\bar{x} \in \mathbb{R}^n$. Show that

$$\operatorname{argmin}_X f = \{P_{S(X)}(S\bar{x})\}$$

    where S is the *square root* of Q (i.e. $S = S^T$ and $S^2 = Q$) and

$$S(X) := \{Sx \mid x \in X\}.$$

(3+3+4 P.)

[CO. 2] (Constraint qualifications) Consider the nonlinear program

(2) $$\min_{x \in \mathbb{R}^2} \ x_1^2 + (x_2 + 1)^2 \quad \text{s.t.} \quad x_2 - x_1^2 \leq 0, \ -x_2 \leq 0.$$

    a) Sketch the feasible set X and some level sets of the objective function to show that $\bar{x} = (0,0)^T$ is the global minimizer of (2).
    b) Show that the Abadie constraint qualification (ACQ) holds at $\bar{x}$.
    c) Do the Mangasarian-Fromovitz constraint qualification (MFCQ) or the linear independence constraint qualification (LICQ) hold at $\bar{x}$?

(4+4+2 P.)

[CO. 3] (KKT conditions for NLP) Consider the nonlinear program

(3) $$\min \ x_1^2 - x_2^2 - x_3^2 \quad \text{s.t.} \quad x_1^4 + x_2^4 + x_3^4 \leq 1.$$

    a) Find all KKT points of (3).
    b) Find the optimal solution of (3).

(6+4 P.)

[CO. 4] (Linear programming and duality) For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ consider

(4) $$\min_{x \in \mathbb{R}^n} \ c^T x \quad \text{s.t.} \quad Ax = b.$$

    a) State the KKT conditions for (4). Are they necessary and sufficient optimality conditions, respectively?
    b) Determine the Lagrangian dual problem of (4).

(4+6 P.)

Numerical Analysis Module

[NA. 1] (10 points)

In this problem we celebrate the 50th anniversary of the Strang splitting $(1968 - 2018)$. Consider the initial value problem: $u_t = Au + Bu$, where A and B are two differential operators. We define the Lie fractional time step $(t \to t + \Delta t)$ as:

$$\begin{array}{ll} (1) & u_t = Au \\ (2) & u_t = Bu \end{array},$$

and the Strang fractional time step $(t \to t + \Delta t)$ as:

$$\begin{array}{ll} (1) & u_t = \frac{1}{2}Au \\ (2) & u_t = Bu \\ (3) & u_t = \frac{1}{2}Au \end{array}.$$

(1) Compute the Local Truncation Error (LTE) for each splitting (Lie and Strang).
(2) Which splitting would you use for the convection-diffusion equation: $u_t + cu_x = du_{xx}$ (c and d are two positive real constants)? (detail your argument)
(3) Consider now the 2D convection equation: $u_t + au_x + bu_y = 0$ (a and b are two positive real constants.)
  (a) Show that the Lie splitting is exact.
  (b) Now, discretize this equation using forward Euler in time and upwind first-order differences in space. Derive both the unsplit and the Lie splitting schemes.
  (c) Compute the Lie splitting error for this scheme.
  (d) Deduce the modified equation for the Lie splitting version of the scheme.

[NA. 2] (10 points)

Use finite differences to solve Airy's equation $u_t = u_{xxx}$ on $x \in [-1, 1)$, using periodic boundary conditions, and starting with smooth enough initial conditions.

(1) Use forward Euler in time, and approximate u_{xxx} by a four-point stencil. Investigate the four possible placements of the stencil (completely upwind, two points upwind–one point downwind, one point upwind–two points downwind, completely downwind) for stability, using von Neumann stability analysis.
(2) Derive a centered symmetric five-point stencil that approximates u_{xxx}. Show that it leads to an unstable method. Recover stability by adding a numerical diffusion term cu_{xx}. Show that the new method is stable, given the constant c is chosen large enough.

[NA. 3] (10 points)

Consider the PDE

$$-(a(x)u_x)_x = f(x), \qquad \text{for } x \in (0, 1)$$

along with boundary conditions

$$u(0) = u_0, \qquad u(1) = u_1$$

where $a(x) \in C^1[0, 1]$, $a(x) \geq a_0 > 0$, and $f \in L^2(a, b)$.

  (a) Define the bilinear form $\mathcal{A}$ and write down the Galerkin formulation in $H_E^1(0, 1)$.
  (b) Prove uniqueness of weak solutions in $H_E^1(0, 1)$.
  (c) Let S_E^h be a finite dimensional subspace of $H_E^1(0, 1)$, let u be the weak solution of the PDE, and let u^h be the Galerkin approximation. Write down (but don't prove) Galerkin orthogonality, and briefly explain/interpret the formula.

[NA. 4] (10 points)

Consider the initial value problem for the variable coefficient parabolic equation on the real line

$$\begin{cases} u_t(x, t) + f_0 u_x(x, t) = \frac{1}{2}\sigma(x)u_{xx}, & \text{for } 0 < t \leq T, \\ u(x, 0) = u_0(x) \end{cases}$$

where $\bar{\sigma} \geq \sigma(x) \geq \sigma_0 > 0$ and $|f(x)| \leq \bar{f}$ for all $x \in \mathbb{R} \times [0, T]$. Approximate the PDE using the Forward Euler method in time, centred differences for u_{xx} and centered finite differences for u_x. Write down the discretization in the explicit form, and give conditions on h, dt so that it is an averaging finite difference method.

PDE Module
[PDE. 1]

(1a) Find the unique weak entropy solution to

$$u_t + u\, u_x = 0 \qquad u(x,0) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 < x \le 1 \\ 0 & \text{if } x > 1. \end{cases}$$

In other words, find a solution in the sense of distributions which is piecewise smooth and satisfies the entropy condition along its jump discontinuities.

(1b) Consider the 2D vector field

$$\mathbf{u}(x,y) = \begin{cases} \mathbf{a} & \text{if } y \ge f(x) \\ \mathbf{b} & \text{if } y < f(x), \end{cases}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ with $\mathbf{a} \ne \mathbf{b}$ and f is some smooth function. Suppose

$$\text{div } \mathbf{u} = 0 \qquad \text{in the sense of distributions.}$$

Use the divergence theorem (integration by parts) to find a condition on $\mathbf{a}, \mathbf{b}$ and the function f.

[PDE. 2]

(2a) In three space dimensions, find the Fundamental solution $\Phi(\mathbf{x})$ of the operator

$$\Delta\Phi + c\Phi, \qquad c > 0.$$

That is, find a solution to

$$\Delta\Phi + c\Phi = \delta_{\mathbf{0}} \qquad \text{in the sense of distributions.}$$

Hint: look for radially solutions of the $\Delta u + cu = 0$.

(2b) Use the fundamental solution to write down an explicit formula for the solution of

$$\Delta u + cu = f,$$

in $\mathbb{R}^3$ where $f \in C_c(\mathbb{R}^3)$.

[PDE. 3]

(3a) Define the Sobolev space $W^{k,p}(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$.

Is there a discontinuous and unbounded function defined on the unit ball B in $\mathbb{R}^3$ which is in $W^{1,2}(B) = H^1(B)$? If so, give an example and show that it is in $W^{1,2}$. If not, explain why not.

(3b) Let $1 \le p < n$. For what values of q can we find a constant $C = C(q,n)$ such that

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall\, u \in C_c^\infty(\mathbb{R}^n)?$$

You do **not** need to supply a proof.

[PDE. 4]

4) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let $f \in L^2(\Omega)$. Consider the biharmonic equation

$$\triangle^2 u = f \quad \text{in } \Omega \qquad u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ is the outer unit normal to $\partial\Omega$ and $\triangle^2 u = \triangle(\triangle u)$.

(a) What does it mean for $u \in H_0^2(\Omega)$ to be a weak solution? (recall that $H_0^2(\Omega)$ is the closure of $C_c^\infty(\Omega) \subset H^2(\Omega)$ with respect to the H^2 norm).

(b) Prove that there exists a unique weak solution.

Probability Module

[PR. 1] (10 pts)

Show that the Borel $\sigma-$algebra on $\mathbb{R}$ is **not** generated by singletons on $\mathbb{R}$, i.e.,

$$\mathcal{B}\left(\mathbb{R}\right) \neq \sigma\left(\{\{x\} : x \in \mathbb{R}\}\right).$$

[PR. 2] (10 pts)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $X_n, Y_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with $|X_n| \leq Y_n$ a.s. for every $n \geq 1$. Further assume that there exist random variables X and Y such that $X_n \to X$ a.s. and $Y_n \to Y$ a.s.. Prove that, if $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y]$, then $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n \to X$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. (*Hint:* Consider $Y_n - |X_n|$)

[PR. 3] (10 pts)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $\{Y_n : n \geq 1\}$ is a sequence of **independent** and **identically distributed** random variables such that $\mathbb{E}[Y_1] = 1$ and Y_1 is not a.s. constant, i.e., $\mathbb{P}(Y_1 = 1) < 1$. Set $T_0 \equiv 1$ and $T_n := \Pi_{j=1}^{n} Y_j$ for every $n \geq 1$.

(i) (5pts) Show that $\{T_n : n \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_n : n \geq 0\}$, where $\mathcal{F}_n := \sigma\left(\{T_k : 0 \leq k \leq n\}\right)$.

(ii) (5pts) Show that $\lim_{n\to\infty} T_n = 0$ a.s.. (*Hint:* Consider $\ln T_n$.)

[PR. 4] (10 pts)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $\{Y_n : n \geq 1\}$ is a sequence of **independent** and **identically distributed** random variables with the common distribution being $N(0, 1)$. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers. Set $X_0 \equiv 1$ and for every $n \geq 1$,

$$X_n := \exp\left(\sum_{j=1}^{n} a_j Y_j - \frac{1}{2} \sum_{j=1}^{n} a_j^2\right).$$

(i) (5pts) Show that, if $\sum_{n=1}^{\infty} a_n^2 < \infty$, then there exists $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to X$ both a.s. and in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) (5pts) Show that, if $\sum_{n=1}^{\infty} a_n^2 = \infty$, then $X_n \to 0$ a.s.

Discrete Mathematics Module

<div align="center">GRAPH THEORY</div>

[DM. 1] (Connectivity)

    a) : State Menger's theorem.

    b) : For a graph G, let $\kappa(G)$ be the maximum $k \in \mathbb{N}$ such that G is k-connected, and $\kappa'(G)$ the maximum $\ell \in \mathbb{N}$ such that G is ℓ-edge-connected.

        Show that for every $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ with $k \geq \ell$, there is a simple graph G such that $\kappa'(G) = k$ and $\kappa(G) = \ell$.

    c) : For every integer $k \geq 2$, a k-connected graph G and a k-vertex subset $S \subseteq V(G)$, prove that there is a cycle C in G such that $S \subseteq V(C)$.

[DM 2.] (Planarity and Matching.)

    a) : State Euler's formula.

    b) : For a simple graph $G = (V, E)$, let $\nu(G)$ be the size of a maximum matching in G, and $\overline{G}$ the complement of G. (*That is* $\overline{G} = \left(V, \binom{V}{2} \setminus E \right)$.) Let G be a simple triangle-free graph. Prove that $\nu(G) + \chi(\overline{G}) = |V(G)|$.

    c) : For a simple graph G, we define the *girth* of G to be the length of a shortest cycle in G. Prove that if G is a simple planar graph with girth $g \geq 3$, then

$$|E(G)| \leq \frac{g}{g-2} \cdot (|V(G)| - 2).$$

<div align="center">COMBINATORICS</div>

[DM 3.] (Families with prescribed intersections.)

    a) : State the Erdős-Ko-Rado theorem.

    b): Let $1 \leq s < r < n$ and let $\mathcal{F} \subseteq [n]^{(r)}$ be a hypergraph such that $|A \cap B| \leq s$ whenever $A, B \in \mathcal{F}$, $A \neq B$. Show that

$$|\mathcal{F}| \leq \frac{n(n-1)\ldots(n-s)}{r(r-1)\ldots(r-s)}.$$

    c): Show that for all positive integers $1 \leq s < r$ there exists n_0 such that for all $n \geq n_0$ and all $\mathcal{F} \subseteq [n]^{(r)}$ such that $|A \cap B| \geq s$ whenever $A, B \in \mathcal{F}$ we have

$$|\mathcal{F}| \leq \binom{n-s}{r-s}.$$

[DM 4.] Turán densities and Lagrangians.

    For a pair of r-graphs F and G, we say that G is F-*free* if G does not contain a subgraph isomorphic to F. Recall that for an r-graph F we denote by $\mathrm{ex}(n, F)$ the maximum number of edges in an F-free r-graph G on n vertices. The *Turán density* of F is defined as

$$(5) \qquad\qquad \pi(F) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{r}}.$$

The *Lagrangian of an r-graph $G \subseteq [n]^{(r)}$* is defined as

$$\lambda(G) := \max \sum_{e \in G} \prod_{i \in e} x_i,$$

where the maximum is taken over all $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} x_i = 1$.

    a) : Show that the limit in (5) exists.

    b) : Show that for every r-graph F we have $\pi(F) \leq r! \sup \lambda(G)$, where the supremum is taken over all F-free r-graphs G.

    c) : Let r be a positive even integer, and let $F = [r+1]^{(r)}$ be the complete r-graph on $r+1$ vertices. Show that

$$\frac{1}{2} \leq \pi(F) \leq 1 - \frac{1}{r+1}.$$