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McGill University

Faculty of Science

Department of Mathematics and Statistics

Statistics Part A Comprehensive Exam
Methodology Paper

Date: Friday, May 17, 2013 Time: 13:00 – 17:00

Instructions

- Answer only **two** questions out of Section L. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**
- Answer only **two** questions out of Section G. If you answer more than two questions, then only the **FIRST TWO questions will be marked.**

Questions	Marks
L1	
L2	
L3	
G1	
G2	
G3	

This exam comprises the cover page and 16 pages of questions.

Section L (Linear Regression Models)
Answer only two questions out of L1–L3

L1.

In a medical study, the plasma colloid osmotic pressures (COP) of three healthy infants from each of the three age groups *1-4 months*, *5-8 months* and *9-12 months*, are measured. The data are given in the following table:

1-4 months	5-8 months	9-12 months
24	26	26
23	26	28
26	28	22

In this study, **age** is considered as a categorical variable with the three levels given in the above table. The goal is to determine whether the COP of healthy infants is related to **age**. Denote:

* μ_0: grand mean of the COP variable for a healthy infant.
* τ_i: the effect of the i-th age group on the COP variable, for $i = 1, 2, 3$.
* $\mu_i = \mu_0 + \tau_i$: mean of the COP variable for a healthy infant in the i-th age group, for $i = 1, 2, 3$.

(**a**) Write down the *one-way analysis of variance* model that is used to analyze such data. State what assumptions one typically makes when using this model.

(3 Marks)

(**b**) Suppose the indicator variables x_1 and x_2 are defined as:

$$x_1 = \begin{cases} 1 & \text{if observation is from the age group 1-4 months,} \\ -1 & \text{if observation is from the age group 9-12 months,} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_2 = \begin{cases} 1 & \text{if observation is from the age group 5-8 months,} \\ -1 & \text{if observation is from the age group 9-12 months,} \\ 0 & \text{otherwise.} \end{cases}$$

Using x_1 and x_2, write down a multiple linear regression model that may be used to analyze the effect of age on COP (y). Use β_0, β_1 and β_2 to represent the regression parameters in your regression model. **(3 Marks)**

(c) Using the data in the table on the previous page, write down the design matrix $\boldsymbol{X}$ and the vector $\boldsymbol{Y}$ for the regression model in part **(b)**. Find the value of the least-squares estimate of $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)^\top$ by solving the normal equations $(\boldsymbol{X}^\top \boldsymbol{X})\, \boldsymbol{\beta} = \boldsymbol{X}^\top \boldsymbol{Y}$ *directly*. **(3 Marks)**

(d) Using the least-squares estimates $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ from part **(c)**, find the value of the least-squares estimates of the parameters $(\mu_0, \tau_1, \tau_2, \tau_3)$ of the model in part **(a)**.

(3 Marks)

(e) A data analysts wants to test the null hypothesis

$$H_0 : \tau_1 = \tau_2 = \tau_3 = 0 \ \ (\text{NO AGE EFFECT ON COP})$$

in the analysis of variance model in part **(a)**.

Using the regression model in part **(b)**, derive an appropriate test statistic that is used to test H_0 at the significance level α.

In addition, perform the test using the data in the table on the previous page, at the significance level $\alpha = 5\%$. **(8 Marks)**

L2.

The quality of Pinot Noir wine is thought to be related to the following variables:

x_1: aroma (pleasant smell); x_2: body; x_3: flavour; x_4: oakiness.

The goal is to study the relationship between the wine **quality** (y) and the covariates (or regressors) x_1, x_2, x_3, x_4 through a multiple linear regression model.

Data for 38 wines, on their quality and the four variables x_1–x_4, are collected. A portion of the data is given in the following table:

Table 1: Wine quality data.

Obs.	Aroma	Body	Flavour	Oakiness	Quality
1	3.30	2.80	3.10	4.10	9.80
2	4.40	4.90	3.50	3.90	12.60
⋮	⋮	⋮	⋮	⋮	⋮
37	4.30	5.50	3.50	5.80	10.30
38	5.20	4.80	5.70	3.50	13.20

To begin, we fit the following multiple linear regression model to the data:

$$\textbf{Model (1):}\quad y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon_i\ ;\ i = 1, 2, \ldots, 38.$$

The R-output for **Model (1)** is given on page 7. Use the significance level 5% in your analysis.

(**a**) Using the R-output on page 7, comment on:

(i) The individual contribution of each covariate in the fitted model. (**1 Mark**)

(ii) Overall significance and also performance of the fitted model. (**2 Marks**)

Provide concrete statistical reasoning for your answers.

(**b**) Residual analysis:

(i) Assume that **Model (1)** is fitted to the data using the least-squares method. Denote the vector of residuals and fitted values as $\boldsymbol{e}_{n\times 1} = (e_1, e_2, \ldots, e_n)^\top$ and $\hat{\boldsymbol{y}}_{n\times 1} = (\hat{y}_1, \hat{y}_2, \ldots, y_n)$, respectively. Show that

$$\sum_{i=1}^{n} e_i = 0 \quad (1)$$

$$\sum_{i=1}^{n} e_i \hat{y}_i = 0 \quad (2)$$

$$\sum_{i=1}^{n} e_i x_{ij} = 0\ ,\ j = 1, 2, 3, 4 \quad (3)$$

where n is the sample size. (**3 Marks**)

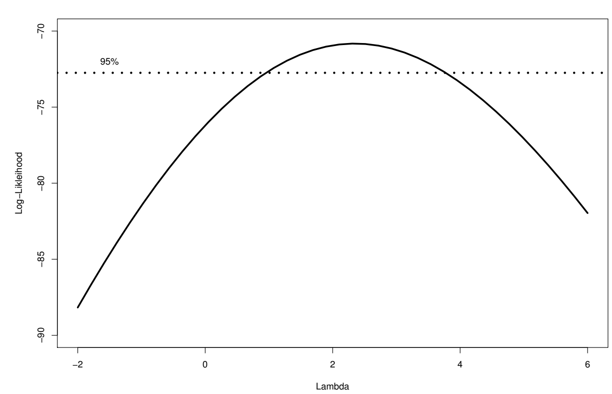
(ii) In the ideal situation, what pattern do you expect to see in the first *five* residual plots in Figure 1 on page 8. Explain your answer *carefully* using the results (1), (2) and (3) above. (**3 Marks**)

(iii) Comment on all the residual plots given in Figure 1 on page 8. (**2 Marks**)

(iv) Show that in the *partial regression plot* corresponding to a covariate x_j, if x_j enters the regression model in a linear fashion the plot will have a slope equal to β_j. (**3 Marks**)

(v) Partial regression plots and partial residual plots for **Model (1)** are given in Figure 2 and Figure 3 on page 9. Comment on these plots. (**2 Marks**)

(**c**) We have decided to investigate a possible Box-Cox transformation on the response variable y. The following plot is created in R.



Explain in full detail how this plot in constructed. Also, based on the plot, what is your suggested Box-Cox transformation on the response variable y? (**4 Marks**)

L3.

Consider the multiple linear regression model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_k)$ is the unknown vector of regression parameters, $\boldsymbol{Y}$ is the $n \times 1$ dimensional vector of observations of the response variable y, $\boldsymbol{X}$ is the $n \times (k+1)$ dimensional design matrix; $\boldsymbol{\varepsilon}$ is the $n \times 1$ dimensional vector of errors ε_i, and $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I}_n)$, where $\boldsymbol{I}_n$ is the identity matrix and σ^2 is unknown.

Consider the partitioning $\boldsymbol{X} = (\boldsymbol{X}_p | \boldsymbol{X}_r)$, where $\boldsymbol{X}_p$ is an $(n \times p)$ matrix (including the intercept column) and $\boldsymbol{X}_r$ is an $(n \times r)$ matrix (and $\boldsymbol{X}_r$ does not include the intercept column), where $p + r = k + 1$. We can also partition $\boldsymbol{\beta}$ into $\boldsymbol{\beta} = (\boldsymbol{\beta}_p | \boldsymbol{\beta}_r)$, where $\boldsymbol{\beta}_p$ is $(p \times 1)$ and $\boldsymbol{\beta}_r$ is $(r \times 1)$ dimensional. Thus, the model can be rewritten as

$$\boldsymbol{Y} = \boldsymbol{X}_p \boldsymbol{\beta}_p + \boldsymbol{X}_r \boldsymbol{\beta}_r + \boldsymbol{\varepsilon} \tag{4}$$

which we call it the *full* model from now on.

Also, consider the *subset* model or *p-term* model

$$\boldsymbol{Y} = \boldsymbol{X}_p \boldsymbol{\beta}_p + \boldsymbol{\varepsilon} \tag{5}$$

where $\boldsymbol{\varepsilon}$ has the same distributional properties as the error term in the full model (4).

(**a**) (i) Write down the least-squares estimator of $\boldsymbol{\beta}_p$ using the p-term model (5). Call your estimator $\hat{\boldsymbol{\beta}}_p$. Using the estimator $\hat{\boldsymbol{\beta}}_p$ provide a model-based estimator for the error variance σ^2 in model (5). Call it $\hat{\sigma}_p^2$. (**4 Marks**)

(ii) Under the assumption that the full model (4) is the true model underlying the data, find the bias of $\hat{\boldsymbol{\beta}}_p$ and $\hat{\sigma}_p^2$ in estimating β_p and σ^2, respectively. Under what conditions are the biases zero? (**6 Marks**)

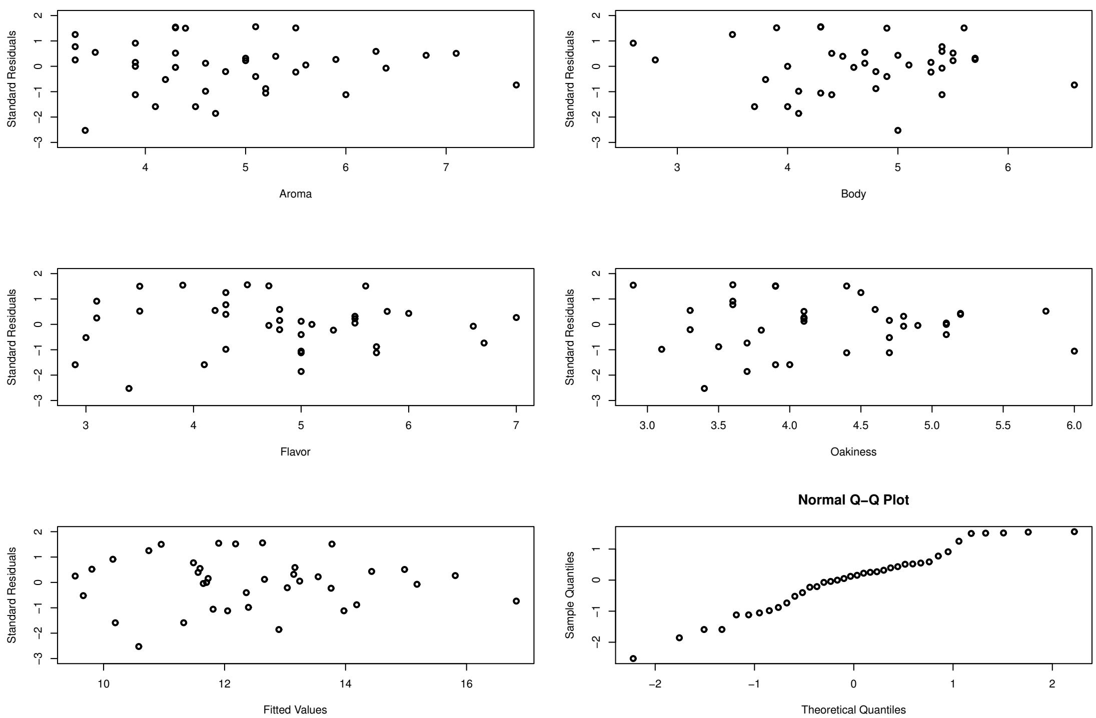
(**b**) Derive *Mallows's C_p* statistic. Explain the usage of this statistic in multiple linear regression. (**10 Marks**)

R-Code for Question L2, part (a).

```
> ### Regression output for part (a)
>
> # Model (1)
>
> Model1<-lm(Quality~Aroma+Body+Flavour+Oakiness,data=Data)
> summary(Model1)

Call:
lm(formula = Quality ~ Aroma + Body + Flavour + Oakiness, data = Data)

Residuals:
    Min      1Q  Median      3Q     Max
-2.6814 -0.6830  0.1559  0.5860  1.7725

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   6.2670     1.4833   4.225 0.000177 ***
Aroma         0.5682     0.2682   2.118 0.041769 *
Body          0.1016     0.3110   0.327 0.746074
Flavour        1.1567     0.3081   3.754 0.000672 ***
Oakiness     -0.6053     0.2681  -2.258 0.030706 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.177 on 33 degrees of freedom
Multiple R-squared: 0.7047,Adjusted R-squared: 0.6689
F-statistic: 19.69 on 4 and 33 DF,  p-value: 2.292e-08

> anova(Model1)
Analysis of Variance Table

Response: Quality
          Df Sum Sq Mean Sq F value    Pr(>F)
Aroma      1 77.442  77.442 55.9138 1.351e-08 ***
Body       1  5.703   5.703  4.1176 0.0505691 .
Flavour     1 18.878  18.878 13.6304 0.0007992 ***
Oakiness   1  7.060   7.060  5.0970 0.0307055 *
Residuals 33 45.706   1.385
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Figures for Question L2, part (b).

Figure 1: Residuals plots for **Model (1)**.



Figures for Question L2, part (b).

Figure 2: Partial regression plots for **Model (1)**.

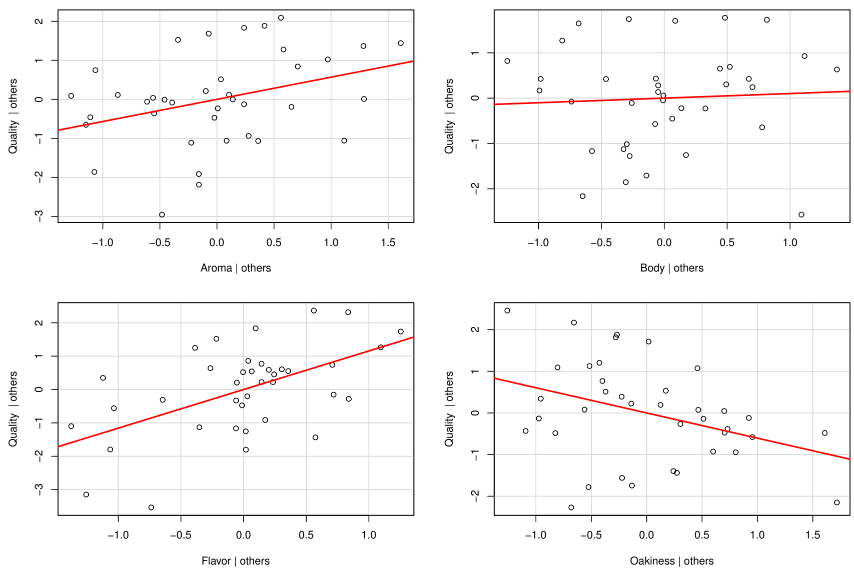
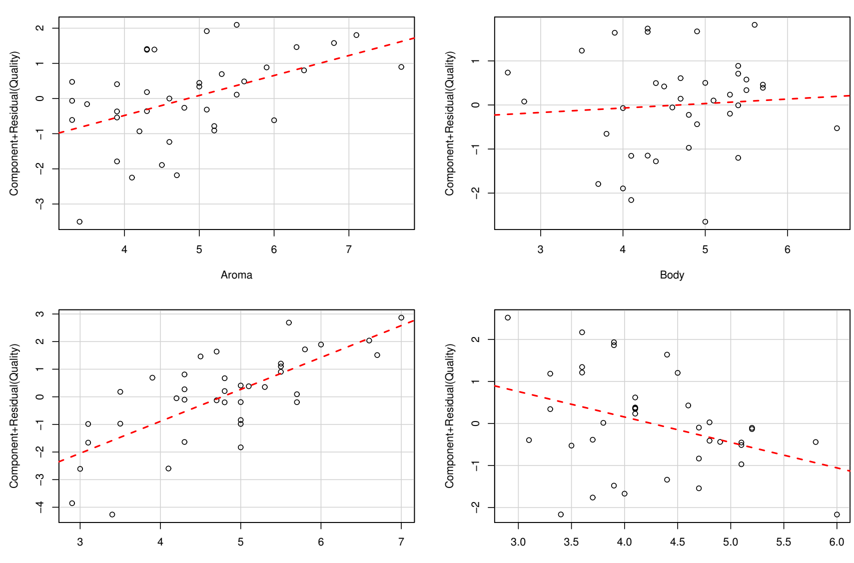


Figure 3: Partial residual plots for **Model (1)**.



Section G (Generalized Linear Models)
Answer only two questions out of G1–G3

G1.

(a) Show, by establishing links between the joint, marginal and conditional likelihoods involved, that inferences for parameters of interest in a multinomial generalized linear model (GLM) can be obtained by using a likelihood based on the Poisson model. State precisely the conditions under which this equivalence holds.

(**8 Marks**)

(b) Consider a Poisson log-linear model for a contingency table defined by three factors A, B and C with J, K and L levels respectively. Denote the entries in the table by $\{y_{jkl}, j = 1, \ldots, J, k = 1, \ldots, K, l = 1, \ldots, L\}$.

Consider the *marginal totals* defined for the contingency table: for example, the marginal totals for the (A, B) margin are defined by

$$y_{jk.} = \sum_{l=1}^{L} y_{jkl} \qquad j = 1, \ldots, J, k = 1, \ldots, K.$$

Which margins are preserved by the model defined (in the usual notation) by

$$A + B + C + A.B$$

that is, for which margins do the *fitted* values match the observed values ? Justify your answer. (**8 Marks**)

(c) In an epidemiological study of n districts of different population sizes, the *standardized mortality rate* (SMR) for a particular disease is denoted for region i by λ_i: this statement is interpreted to mean that the count of the number of disease instances in region i, Y_i, can be modelled by assuming

$$Y_i \sim \text{Poisson}(E_i \lambda_i) \qquad i = 1, \ldots, n$$

where E_i is a known constant that differs for each i. A model for λ_i based on predictor values x_i can be constructed using a link function and linear predictor.

Describe how to carry out a GLM-based analysis of such data, and explain how to estimate the *rate ratio* comparing the incidence rate in two districts having different levels of a given binary factor that is measured. (**4 Marks**)

G2.

(a) Consider the logistic regression model for n binary response data (that is, binomial data with $m_i = 1, i = 1, \ldots, n$) with linear predictor $\eta_i = x_i\beta$.

(i) Derive the score (estimating) equations for this model. (**4 Marks**)

(ii) State the asymptotic properties of the resulting estimators. (**2 Marks**)

(iii) Show how the estimating equations change if the logistic link is replaced by a probit link. (**2 Marks**)

(b) In a study carried out by a credit card company, customers were classified as either being at 'Low' or 'High' risk of missing a credit card payment at least once in the next six months on the basis of three predictors:

- `Age`: age in years
- `Emp`: current employment status (three levels: `U` - Unemployed , `PT` - Part-Time employed, `FT` - Full-Time employed).
- `His`: credit history (two levels: `0` - no missed payments in the last 12 months, `1+` - at least one missed payment in the last twelve months).

A random sample of 758 customers was studied and each of the variables above was recorded, along with an outcome variable which recorded whether at least one missed payment actually occurred in the next six months.

Output related to the analysis of these data in `R` is included on pages 13 to 15. On the basis of the output provided, address the following questions

(i) What is the most appropriate model to predict whether a customer will miss a payment in the next six months ? How do the predictors in this model influence the probability of missing a payment ? (**4 Marks**)

(ii) Using the model in **(b)**(i), compute the estimated probability of a missed payment in the next six months for a 50 year old customer in full-time employment with no missed payments in the last twelve months. (**2 Marks**)

(iii) Explain how to carry out an assessment of model adequacy for the model selected in **(b)**(i). (**2 Marks**)

(iv) Is there any apparent association (or dependence) between employment (either part- or full-time) and a customer's history of missed payments over the last twelve months ? (**4 Marks**)

Justify each answer.

G3.

(a) Describe the *proportional odds* model that is sometimes used to model ordinal categorical response data. **(5 Marks)**

(b) In an observational occupational health study, 371 miners were classified according to the time (in years) for which they had been working in coal mining; eight time categories were used, and the average time t of miners in that category computed. Within each time category, the miners were then questioned as to whether they suffered symptoms associated with the lung condition pneumoconiosis. The following data were recorded

	Symptoms				
t	normal	mild	severe	Total	$\log(t)$
5.8	98	0	0	98	1.758
15.0	51	2	1	54	2.708
21.5	34	6	3	43	3.068
27.5	35	5	8	48	3.314
33.5	32	10	9	51	3.512
39.5	23	7	8	38	3.676
46.0	12	6	10	28	3.829
51.5	4	2	5	11	3.942
Total	289	38	44	371	

These data were analyzed as grouped data in `R`, and the code and results are presented on page 16.

(i) What is the conclusion of the analysis presented ? **(3 Marks)**

(ii) Explain precisely the interpretation of the output on lines 9, 10, 30 and 31. **(2 Marks)**

(iii) Compute the fitted probabilities of a miner who had been working for 20 years presenting with severe symptoms under the fitted two models. **(4 Marks)**

(c) A model to predict, on the basis of the log of time worked in years, the probability that a subject will have non-normal (either mild or severe) symptoms is required. Explain how to use the output on page 16 to obtain a relevant model fit and predictions. **(6 Marks)**

Code and Output for Question G2.

```
> anova(m1,m2,m3,m4,m5,m6,m7,m8,m9)
Analysis of Deviance Table

Model 1: Y ~ Emp * His * Age
Model 2: Y ~ Emp * Age + His * Age
Model 3: Y ~ Emp + His * Age
Model 4: Y ~ Emp * Age + His
Model 5: Y ~ Emp * His + Age
Model 6: Y ~ Emp + His + Age
Model 7: Y ~ Emp * His
Model 8: Y ~ Emp + His
Model 9: Y ~ His + Age
  Resid. Df Resid. Dev Df Deviance
1       746     453.01
2       750     456.95 -4   -3.941
3       752     460.96 -2   -4.003
4       751     457.09  1    3.867
5       751     459.12  0   -2.026
6       753     460.98 -2   -1.868
7       752     489.11  1  -28.125
8       754     491.17 -2   -2.064
9       755     567.87 -1  -76.701

> summary(m2)
Call:glm(formula = Y ~ Emp * Age + His * Age, family = binomial)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -3.49286    2.73944  -1.275  0.20230
EmpPT        -5.43279    3.26820  -1.662  0.09645 .
EmpFT       -11.58621    4.01762  -2.884  0.00393 **
Age           0.07488    0.05711   1.311  0.18978
His1+         1.01171    3.09344   0.327  0.74363
EmpPT:Age     0.06405    0.06782   0.944  0.34493
EmpFT:Age     0.16549    0.08157   2.029  0.04248 *
Age:His1+     0.02347    0.06380   0.368  0.71298
---
    Null deviance: 728.28  on 757  degrees of freedom
Residual deviance: 456.95  on 750  degrees of freedom
AIC: 472.95
```

Code and Output for Question G2.

```
> summary(m3)
Call: glm(formula = Y ~ Emp + His * Age, family = binomial)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -7.11740    1.66869  -4.265 2.00e-05 ***
EmpPT       -2.43009    0.32835  -7.401 1.35e-13 ***
EmpFT       -3.52832    0.37826  -9.328  < 2e-16 ***
His1+        1.67639    2.96613   0.565    0.572
Age          0.15136    0.03408   4.441 8.94e-06 ***
His1+:Age    0.01008    0.06132   0.164    0.869
---
    Null deviance: 728.28  on 757  degrees of freedom
Residual deviance: 460.96  on 752  degrees of freedom
AIC: 472.96

> summary(m4)

Call: glm(formula = Y ~ Emp * Age + His, family = binomial)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -3.70128    2.69952  -1.371  0.17035
EmpPT        -5.67543    3.21689  -1.764  0.07769 .
EmpFT       -11.48537    4.01593  -2.860  0.00424 **
Age           0.07892    0.05641   1.399  0.16182
His1+         2.14650    0.26631   8.060 7.62e-16 ***
EmpPT:Age     0.06912    0.06666   1.037  0.29978
EmpFT:Age     0.16359    0.08155   2.006  0.04485 *
---
    Null deviance: 728.28  on 757  degrees of freedom
Residual deviance: 457.09  on 751  degrees of freedom
AIC: 471.09
```

Code and Output for Question G2.

```
> summary(m5)
Call:glm(formula = Y ~ Emp * His + Age, family = binomial)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -7.19583    1.42265  -5.058 4.24e-07 ***
EmpPT       -2.71250    0.41470  -6.541 6.12e-11 ***
EmpFT       -3.57137    0.42973  -8.311  < 2e-16 ***
His1+        1.73056    0.55898   3.096  0.00196 **
Age          0.15576    0.02918   5.338 9.40e-08 ***
EmpPT:His1+  0.69439    0.64317   1.080  0.28031
EmpFT:His1+ -0.09980    0.89299  -0.112  0.91102
---
    Null deviance: 728.28  on 757  degrees of freedom
Residual deviance: 459.12  on 751  degrees of freedom
AIC: 473.12

> summary(m6)
Call: glm(formula = Y ~ Emp + His + Age, family = binomial)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -7.26644    1.40224  -5.182 2.19e-07 ***
EmpPT       -2.42868    0.32894  -7.383 1.54e-13 ***
EmpFT       -3.52671    0.37867  -9.313  < 2e-16 ***
His1+        2.16218    0.26348   8.206 2.29e-16 ***
Age          0.15434    0.02885   5.349 8.83e-08 ***
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1      1
    Null deviance: 728.28  on 757  degrees of freedom
Residual deviance: 460.98  on 753  degrees of freedom
AIC: 470.98

> table(Emp,His)
    His
Emp    0  1+
  U   44  38
  PT 202  94
  FT 361  19
```

Code and Output for Question G3.

```
> fit0<-vglm(cbind(normal, mild, severe)~1, propodds(reverse=F), pneumo)
> summary(fit0)

Call:vglm(formula = cbind(normal, mild, severe)~1,
     family = propodds(reverse = F), data = pneumo)

Coefficients:
              Estimate Std. Error z value
(Intercept):1   1.2597    0.12512  10.068
(Intercept):2   2.0058    0.16058  12.491

Number of linear predictors:  2
Names of linear predictors: logit(P[Y<=1]), logit(P[Y<=2])

Dispersion Parameter for cumulative family:   1

Residual deviance: 101.6406 on 14 degrees of freedom
Log-likelihood: -73.39713 on 14 degrees of freedom


> fit1 <- vglm(cbind(normal, mild, severe)~I(log(exposure.time)),
propodds(reverse=F), pneumo)
> summary(fit1)

Call: vglm(formula = cbind(normal, mild, severe)~ I(log(exposure.time)),
family = propodds(reverse = F), data = pneumo)

Coefficients:
                      Estimate Std. Error z value
(Intercept):1           9.6761     1.3241  7.3078
(Intercept):2          10.5817     1.3454  7.8649
I(log(exposure.time))  -2.5968     0.3811 -6.8139

Number of linear predictors:  2
Names of linear predictors: logit(P[Y<=1]), logit(P[Y<=2])

Dispersion Parameter for cumulative family:   1

Residual deviance: 5.02683 on 13 degrees of freedom
Log-likelihood: -25.09026 on 13 degrees of freedom
```

Table of the Chi-squared distribution

Entries in table are $\chi^2_\alpha(\nu)$: the α tail quantile of Chi-squared(ν) distribution
α given in columns, ν given in rows.

	Left-tail					Right-tail				
ν	0.99500	0.99000	0.97500	0.95000	0.90000	0.10000	0.05000	0.02500	0.01000	0.00500
1	0.00004	0.00016	0.00098	0.00393	0.01579	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.01003	0.02010	0.05064	0.10259	0.21072	4.60517	5.99146	7.37776	9.21034	10.59663
3	0.07172	0.11483	0.21580	0.35185	0.58437	6.25139	7.81473	9.34840	11.34487	12.83816
4	0.20699	0.29711	0.48442	0.71072	1.06362	7.77944	9.48773	11.14329	13.27670	14.86026
5	0.41174	0.55430	0.83121	1.14548	1.61031	9.23636	11.07050	12.83250	15.08627	16.74960
6	0.67573	0.87209	1.23734	1.63538	2.20413	10.64464	12.59159	14.44938	16.81189	18.54758
7	0.98926	1.23904	1.68987	2.16735	2.83311	12.01704	14.06714	16.01276	18.47531	20.27774
8	1.34441	1.64650	2.17973	2.73264	3.48954	13.36157	15.50731	17.53455	20.09024	21.95495
9	1.73493	2.08790	2.70039	3.32511	4.16816	14.68366	16.91898	19.02277	21.66599	23.58935
10	2.15586	2.55821	3.24697	3.94030	4.86518	15.98718	18.30704	20.48318	23.20925	25.18818
11	2.60322	3.05348	3.81575	4.57481	5.57778	17.27501	19.67514	21.92005	24.72497	26.75685
12	3.07382	3.57057	4.40379	5.22603	6.30380	18.54935	21.02607	23.33666	26.21697	28.29952
13	3.56503	4.10692	5.00875	5.89186	7.04150	19.81193	22.36203	24.73560	27.68825	29.81947
14	4.07467	4.66043	5.62873	6.57063	7.78953	21.06414	23.68479	26.11895	29.14124	31.31935
15	4.60092	5.22935	6.26214	7.26094	8.54676	22.30713	24.99579	27.48839	30.57791	32.80132
16	5.14221	5.81221	6.90766	7.96165	9.31224	23.54183	26.29623	28.84535	31.99993	34.26719
17	5.69722	6.40776	7.56419	8.67176	10.08519	24.76904	27.58711	30.19101	33.40866	35.71847
18	6.26480	7.01491	8.23075	9.39046	10.86494	25.98942	28.86930	31.52638	34.80531	37.15645
19	6.84397	7.63273	8.90652	10.11701	11.65091	27.20357	30.14353	32.85233	36.19087	38.58226
20	7.43384	8.26040	9.59078	10.85081	12.44261	28.41198	31.41043	34.16961	37.56623	39.99685
21	8.03365	8.89720	10.28290	11.59131	13.23960	29.61509	32.67057	35.47888	38.93217	41.40106
22	8.64272	9.54249	10.98232	12.33801	14.04149	30.81328	33.92444	36.78071	40.28936	42.79565
23	9.26042	10.19572	11.68855	13.09051	14.84796	32.00690	35.17246	38.07563	41.63840	44.18128
24	9.88623	10.85636	12.40115	13.84843	15.65868	33.19624	36.41503	39.36408	42.97982	45.55851
25	10.51965	11.52398	13.11972	14.61141	16.47341	34.38159	37.65248	40.64647	44.31410	46.92789
26	11.16024	12.19815	13.84390	15.37916	17.29188	35.56317	38.88514	41.92317	45.64168	48.28988
27	11.80759	12.87850	14.57338	16.15140	18.11390	36.74122	40.11327	43.19451	46.96294	49.64492
28	12.46134	13.56471	15.30786	16.92788	18.93924	37.91592	41.33714	44.46079	48.27824	50.99338
29	13.12115	14.25645	16.04707	17.70837	19.76774	39.08747	42.55697	45.72229	49.58788	52.33562
30	13.78672	14.95346	16.79077	18.49266	20.59923	40.25602	43.77297	46.97924	50.89218	53.67196
40	20.70654	22.16426	24.43304	26.50930	29.05052	51.80506	55.75848	59.34171	63.69074	66.76596
50	27.99075	29.70668	32.35736	34.76425	37.68865	63.16712	67.50481	71.42020	76.15389	79.48998
60	35.53449	37.48485	40.48175	43.18796	46.45889	74.39701	79.08194	83.29767	88.37942	91.95170
70	43.27518	45.44172	48.75756	51.73928	55.32894	85.52704	90.53123	95.02318	100.42518	104.21490
80	51.17193	53.54008	57.15317	60.39148	64.27784	96.57820	101.87947	106.62857	112.32879	116.32106
90	59.19630	61.75408	65.64662	69.12603	73.29109	107.56501	113.14527	118.13589	124.11632	128.29894
100	67.32756	70.06489	74.22193	77.92947	82.35814	118.49800	124.34211	129.56120	135.80672	140.16949