McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

15 August, 2017
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Problem 1.

Suppose that V is an n-dimensional vector space over $\mathbb{R}$, and T_1 and T_2 are linear operators on V which commute. Suppose also that T_1 has n distinct eigenvalues in $\mathbb{R}$. Prove that T_2 is diagonalizable over $\mathbb{R}$.

Problem 2.

Let B, C be matrices over $\mathbb{C}$ (B is $k \times k$ and C is $\ell \times \ell$), $*$ stands for any $k \times \ell$ matrix and 0 for a zero matrix. Say that A has block form $\begin{pmatrix} B & * \\ 0 & C \end{pmatrix}$. Suppose that B and C have no eigenvalues in common. Show that A is similar to $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$.

Problem 3.

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Suppose that $char_A(\lambda)$ (the characteristic polynomial of A) factors as $\Pi_{j=1}^n(\lambda - \lambda_j)$ ($\lambda_j \in \mathbb{C}$). Let $f(\lambda)$ be any polynomial with coefficients in $\mathbb{C}$. Prove that

$$char_{f(A)}(\lambda) = \Pi_{j=1}^n(\lambda - f(\lambda_j)).$$

Problem 4.

Let A be an $n \times m$ matrix with real entries, with transpose A^t. Prove the following:

(1) $A^tA\vec{x} = A^t\vec{v}$ has a solution $\vec{x}$ for any $\vec{v} \in V$.

(2) For any solution $\vec{x}$ as in part (a), $A\vec{x}$ is uniquely determined. That is, if $A^tA\vec{x} = A^tA\vec{y} = A^t\vec{v}$, then $A\vec{x} = A\vec{y}$.

(3) The solution $\vec{x}$ in part (a) is unique if and only if A has rank m.

(4) For a solution $\vec{x}$ as in part (a), $A\vec{x} = \vec{v}$ if and only if $\vec{v}$ is in the column space of A.

Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

Question 5.

Let $a_n \geq 0, \sum a_n < \infty$. For each of the following statements, either give a proof or provide a counterexample.

a) $\sum n a_n^2 < \infty$.

b) $\liminf_{n\to\infty}(n a_n) = 0$.

Question 6.

Consider the power series

$$x - \frac{3}{4}x^4 + \frac{3^2}{4\cdot 7}x^7 - \frac{3^3}{4\cdot 7\cdot 10}x^{10} \pm \dots$$

a) What is the radius ρ of convergence?

b) Show that $f'(x) + 3x^2 f(x) = 1$ for $|x| < \rho$.

Question 7. If G_n is an open and dense subsets of $\mathbf{R}$ for $n = 1, 2, \dots$, prove that $\cap_{n=1}^{\infty} G_n \neq \emptyset$.

Question 8.

a) State Taylor's Theorem with Lagrange's form of the remainder term.

b) Show that $\left|\ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^n}{n}\right)\right| < \frac{x^{n+1}}{n+1}$ in the case $x > 0$.

Solve any three out of the four questions 9,10,11,12.

Question 9.

The point $x = 0$ is a regular singular point of the equation $4xy'' + 2y' + y = 0$. Find two independent solutions using the method of Frobenius (solve the indicial equation and compute the first three nonzero terms in the power series expansion assuming that the first nonzero coefficient is equal to 1).

Question 10.

Find $d^2x/dz^2, dy/dz$ for $(x, y, z) = (1, -1, 2)$ and

$$\begin{cases} x^2 + y^2 = z^2/2, \\ x + y + z = 2. \end{cases}$$

Question 11.

Prove that the vector field $\overrightarrow{F} = (yze^{xyz}, xze^{xyz} + 4y^3z, xye^{xyz} + y^4)$ is conservative. Find the potential for $\overrightarrow{F}$ and evaluate the integral

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r},$$

where C is the straight line segment from $(-1, 1, 0)$ to $(2, -2, 0)$, followed by the curve $x = 2, y = -2\cos t, z = 2\sin t, 0 \le t \le (\pi/2)$.

Question 12.

Find the volume between the paraboloids $z = 10 - x^2 - y^2$ and $z = x^2/9 + y^2/4$.