**McGill University**
**Department of Mathematics and Statistics**

**Ph.D. preliminary examination, PART A**

**PURE AND APPLIED MATHEMATICS**
**Paper ALPHA**

5 May 2015 13:00 - 17:00

**INSTRUCTIONS:**

**(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.**

**(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.**

This exam comprises this cover and 3 pages of questions.

**Single variable real analysis**

**Solve any three out of the four questions 1, 2, 3, and 4.**

**Problem 1.**

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

**Problem 2.**

(a) Prove that the function given by $f(x) := e^{-1/x^2}$ for $x \neq 0$, $f(0) := 0$, has derivatives of all orders at every point and that all of these derivatives vanish at $x = 0$. Hence, this function is not given by its Taylor expansion about $x = 0$.

(b) Give an example of a function which is equal to its Taylor series expansion about $x = 0$ for $x \geq 0$, but which is not equal to this expansion for $x < 0$.

**Problem 3.** Discuss the following sequences of functions and their integrals on [0,1]. Evaluate the limit of the integrals, when possible.

(a) $\lim_{k\to\infty} \int_0^1 kxe^{-kx} dx$

(b) $\lim_{k\to\infty} \int_0^1 x^k(1+x)^{-2} dx$

(c) $\lim_{k\to\infty} \int_0^1 kx^k(1+x)^{-1} dx$

**Problem 4.** Let $\{r_1, ..., r_n, ...\}$ be an enumeration of the rational numbers in $I ::= [0,1]$, and let $f_n : I \to \mathbb{R}$ be defined to be 1 if $x = r_1, ..., r_n$ and equal to 0 otherwise.

(a) Show that $f_n$ is Riemann integrable for each $n \in \mathbb{N}$ and that $f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$.

(b) Find $f(x) := \lim f_n(x)$.

(c) Is the function $f(x)$ in part (b) is Riemann integrable on $[0,1]$? Prove your answer.

**Linear Algebra**

**Solve any three out of the four questions 5, 6, 7, and 8.**

**Problem 5.** Let $A$ be an $n \times n$ matrix of rank one. Prove that $A^2 = \lambda A$ for some constant $\lambda$.

**Problem 6.** Show that any linear operator $T : V \to V$ satisfying $T^2 = T$ is the projection of V onto some subspace and that there is a a decomposition $V = U \oplus W$ such that $T$ is a projection on to $U$ .

**Problem 7.** Let $T$ be a linear transformation of finite order on a complex vector space, i.e. $T^m = \mathrm{id}_V$ for some positive integer $m$. Then $T$ is diagonalizable.

**Problem 8.** Suppose $P$ is the change-of-basis matrix from a basis $\{u_i\}$ to a basis $\{w_i\}$, and suppose $Q$ is a change-of-basis matrix from the basis $\{w_i\}$ back to $\{u_i\}$. Prove that $P$ is invertible and that $Q = P^{-1}$.

**Vector calculus, ODE, and complex analysis**

**Solve any three out of the four questions 9, 10, 11, and 12.**

**Problem 9.** Let

$$f(x,y)=\begin{cases}\frac{x^4y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases}$$

(a) Prove $f$ is differentiable everywhere.

(b) Is $\frac{\partial f}{\partial y}$ continuous at the origin? Justify.

**Problem 10.** Let $\mathbf{F}=(P,Q)$ be the vector field in $\mathbb{R}^2$ defined as

$$P(x,y)=\begin{cases}\frac{x+y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases} \qquad Q(x,y)=\begin{cases}\frac{-x+y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

(a) Show that $\mathbf{F}$ is a gradient vector field in $\mathbb{R}^2\setminus\{0\le x<\infty, y=0\}$.

(b) Let $D=\{x^{100}+y^{100}\le 1\}$, compute the line integral $\int_{\partial D} Pdx+Qdy$ where the positive orientation is given for $\partial D$.

**Problem 11.** Compute the outward flux of the vector field

$$\mathbf{F}=-\nabla\left(\frac{q_1}{4\pi||\mathbf{r}-\mathbf{r}_1||}\right)-\nabla\left(\frac{q_2}{4\pi||\mathbf{r}-\mathbf{r}_2||}\right)+\nabla(e^x\sin y+z),$$

where $q_1$, $q_2$ are constants and where $\mathbf{r}=(x,y,z)$, $\mathbf{r}_1=(1/4,1/4,1/16)$, $\mathbf{r}_2=(2,2,2)$, through the boundary $S$ of the solid tetrahedron $V$ with vertices at $(0,0,0)$, $(2,0,0)$, $(0,1,0)$ and $(0,0,4)$.

**Problem 12.** Determine the values of $a\in\mathbf{R}$ and $b\in\mathbf{R}$ for which it is true that **every** solution $y(x)$ of the ODE

$$y''+ay'+by=0,$$

remains **bounded** as $x$ varies over the real line.