McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

7 May, 2019
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Question 1.

Let A be a real $n \times n$ matrix acting on $\mathbf{R}^n$ in the usual way. We think about $\mathbf{R}^n$ as column vectors and equip $\mathbf{R}^n$ with the usual Euclidean norm $||v||$. The norm of A is defined as

$$sup_{v\neq 0}||Av||/||v||.$$

Prove that the norm of A is the square root of the maximal eigenvalue of the matrix $A^t \cdot A$ where A^t is the transpose of A.

Question 2.

Let V_1, V_2 be vector subspaces of $\mathbb{R}^n$. Show that

$$\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - n.$$

Question 3.

Let A be $n \times n$ complex matrix satisfying $A^k = \mathrm{Id}_n$ for some positive integer k (where Id_n is the $n \times n$ identity matrix. Show that the trace of A satisfies

$$|\mathrm{tr}(A)| \leq n.$$

Here $|x + iy| = \sqrt{x^2 + y^2}$ is the absolute value for complex numbers.

Question 4.

Let V be a finite-dimensional vector space over $\mathbb{R}$, equipped with an inner product $\langle -, - \rangle$. Let T be a *self-adjoint* operator on V, i.e. for any vectors $u, v \in V$ we have $\langle Tu, v\rangle = \langle u, Tv\rangle$.

Prove or disprove the following statements:

(a) For any basis $B = (e_1, \dots, e_n)$ of V, the matrix T_B of T in the basis V is symmetric;
(b) If v_1, v_2 be two eigenvectors of T corresponding to *different* eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are orthogonal.

Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

Question 5.

Let $f_n(x) = nx/(1+n^2x^2)$. Determine the pointwise limit of f_n for $x \in [0,\infty)$. Does that sequence of functions converge uniformly?

Question 6.

(a) State Hölder's inequality for sequences.

(b) Let $a_n \geq 0$. Suppose $\sum_n a_n$ converges. Show that

$$\sum_n \frac{a_n^{1/3}}{n}$$

also converges.

Question 7.

Consider the power series

$$f(x) = x - \frac{3}{4}x^4 + \frac{3^2}{4 \cdot 7}x^7 - \frac{3^3}{4 \cdot 7 \cdot 10}x^{10} \pm \dots$$

Find the radius of convergence for $f(x)$.

Question 8.

Suppose that a_n is a bounded sequence of real numbers and b is a real number such that any subsequence of a_n which converges at all has limit b. Prove that $a_n \to b$ as $n \to \infty$.

Solve any three out of the four questions 9,10,11,12.

Question 9.

Find a general solution $y(t)$ of the differential equation

$$y'' - 2y' - 3y = 3t^2 - 5.$$

Question 10.

Find a general solution $y(x)$ of the differential equation

$$y''' - 9y'' + 27y' - 27y = 0.$$

Question 11.

Find the volume of the region lying inside the ellipsoid $x^2 + y^2/4 + z^2/9 = 9$ and outside the ellipsoid $x^2 + y^2/4 + z^2/9 = 3$.

Question 12.

Compute the double integral

$$\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\, dx.$$