McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS
Paper ALPHA

12 August, 2014 13:00 - 17:00

INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

Problem 1 Assume that x is a positive real number. Prove the following inequalities:

(a) $\ln(1+x) < \frac{x}{\sqrt{1+x}}$;

(b) $(x-1)^2 \geq x\ln^2(x)$.

Problem 2

(a) Let f_n be a sequence of functions on the interval $[0,1]$. Prove that f_n converges uniformly on $[0,1]$ to a function f if and only if

$$\lim_{n\to\infty}\left(\sup_{x\in[0,1]} r_n(x)\right) = 0,$$

where $r_n(x) = |f_n(x) - f(x)|$.

(b) Prove or disprove: The sequence of functions $f_n(x) = x^n - x^{n+1}$ converges to 0 on $[0,1]$.

Problem 3. Suppose that f and g are continuous functions defined on the closed bounded interval $[a,b]$ (with $a<b$) such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove that there is at least one $c \in [a,b]$ with $f(c) = g(c)$.

Problem 4. Let $0 < x_0 < 2$ and define a sequence $\{x_n\}$ inductively by

$$x_{n+1} = \frac{x_n}{3-x_n},\ \ n = 1,2,...$$

Find $\lim_{n\to\infty} x_n$ and prove your result.

Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

Problem 5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with the property that Lv is orthogonal to v for every $v \in \mathbb{R}^3$. Prove that L cannot be invertible. Is a similar assertion true for a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$?

Problem 6. Suppose that T_1 and T_2 are linear operators on the space V (over the field F) and that $p_1(X)$ and $p_2(X)$ are polynomials with coefficients in F. Suppose also that $T_1T_2 = T_2T_1$. Show that each of the spaces $ker(p_2(T_2))$ – the kernel of $p_2(T_2)$ – and $im(p_2(T_2))$ – the image of $p_2(T_2)$ – is invariant under the operator $p_1(T_1)$.

Problem 7. Suppose that A is an $m \times n$ matrix with $(i, j)-$entry $a_{i,j}$. Whenever $1 \le i_1 < i_2 < \cdots < i_k \le m$ and $1 \le j_1 < j_2 < \cdots < j_k \le n$, we call

$$\det \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\ \cdots & \cdots & \cdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{pmatrix}$$

a $k \times k$ *subdeterminant* of A.

Show that the rank of A is the biggest number k such that there is a nonzero $k \times k$ subdeterminant of A.

Problem 8. Let $M_n(\mathbb{C})$ be the vector space (over the complex numbers) of $n \times n$ matrices, and for any $n \times n$ matrix A, let $Z(A)$ be the subspace

$$\{X \in M_n(\mathbb{C}) : XA = AX\}.$$

(a) Show that if A is similar to B, then the dimension of $Z(A)$ is the same as that of $Z(B)$.

(b) Suppose that $n = 3$. Find all possible dimensions for $Z(A)$. Justify that you have all possible answers. *Hint:* Consider the possible Jordan forms for A.

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

Problem 9.

(a) State the divergence theorem.
(b) Use the divergence theorem or otherwise evaluate

$$\int\int_{\Sigma} \mathbf{F} \cdot d\mathbf{s},$$

where

(i) $\mathbf{F} = (\sin z + x^3 + y, e^2 + y^3 + 2y, z^3 + z^2)$; and

(ii) Σ consists of the part of the sphere $x^2+y^2+z^2 = 4z$ outside the paraboloid P given by $z = x^2+y^2$ and the part of P inside S; this surface is oriented with the normal pointing away from the region enclosed between P and S.

Problem 10. Consider the following initial value problem:

$$\frac{d\mathbf{x}(t)}{dt} = -\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \mathbf{x}(t) + \mathbf{b},\ t \in [0, \infty],$$

where $\mathbf{x}(t), \mathbf{b} \in \mathbb{R}^2$. Find all $\mathbf{b} \in \mathbb{R}^2$ such that

$$\sup_{t\in[0,\infty]} ||\mathbf{x}(t)||_2 < \infty$$

for any choice of $\mathbf{x}(0) \in \mathbb{R}^2$.

Problem 11. Consider the second-order linear inhomogeneous ODE

$$x'' + \lambda x' + x = f(t),$$

where $f(t)$ is any function on $\mathbb{R}$. For what values of λ will any two solutions of this ODE converge to each other as $t \to \infty$? (By definition, two functions $x_1(t), x_2(t)$ are said to converge to each other as $t \to \infty$ if $\lim_{t\to\infty}(x_1(t) - x_2(t)) = 0$.)

Problem 12. For each of the following series of complex numbers, either find the sum in the form $A + iB$ where A and B are real and simplified as far as possible, or explain why the series fails to converge.

$$\text{(a)} \sum_{n=1}^{\infty} nz^n \text{ where } z = \frac{1+i}{2}$$

$$\text{(b)} \sum_{n=0}^{\infty} \frac{z^n}{2n+1} \text{ where } z = \frac{1+i\sqrt{3}}{2}$$