McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE MATHEMATICS
Paper BETA

19 August 2016
1:00 p.m. - 5:00 p.m.

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[ALG. 1]

(a) State the primitive element theorem.

(b) If $K = F_p(x)$ and α is a p-root of x, show that $K(\alpha)/K$ is a primitive algebraic extension which is not separable.

(c) Give an example of a finite extension which is not primitive.

[ALG. 2]

A group is said to be *locally free* if every finitely generated subgroup is free.

(a) Show that the additive group $\mathbb{Q}$ of rational numbers is locally free.

(b) Find a proper subgroup of $\mathbb{Q}$ which is not cyclic.

[ALG. 3]

Let G be a group. Consider two elements $s, t \in G$ which commute with their commutator $[s,t] := sts^{-1}t^{-1}$. Show that $(st)^n = t^n s^n [s,t]^m$ for every integer n, and some integer m uniquely determined by n.

[ALG. 4]

(a) State the definition of Noetherian and Artinian modules over a commutative unital ring.

(b) Show that a surjective endomorphism u of a Noetherian module is bijective. [*Hint*: Consider $\ker u^n$.]

(c) Show that an injective endomorphism of an Artinian module is bijective.

Analysis Module

[AN. 1]

Let $\delta > 0$ be fixed. Show that the set of all real numbers $x \in [0,1]$ such that there exist infinitely many pairs $p, q \in \mathbf{N}$ such that $|x - p/q| < 1/q^{2+\delta}$ has Lebesgue measure 0.

[AN. 2]

Let f be a uniformly continuous function on $\mathbb{R}$. Suppose that $f \in L^p$ for some p, $1 \leq p < \infty$. Prove that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

[AN. 3]

(a) Give a definition of $||f||_\infty$ of a measurable complex function f.

(b) Recall that the essential range of a function $f \in L^\infty(\mu, \mathbb{C})$ is the set consisting of complex numbers w such that

$$\mu(\{x : |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that R_f is compact.

(c) Show that $||f||_\infty = \sup_{w \in R_f} |w|$.

[AN. 4]

(a) Give a definition of a locally compact topological space.

(b) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}, \mu)$ is locally compact and explain why it is so.

(c) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}, \mu)$ is not locally compact and explain why it is so.

Geometry and Topology Module

[GT. 1]

(a) Suppose that X is a separable metric space. Show that any subspace of X is separable.

(b) Suppose that X is a compact metric space. Show that X is separable and that any compatible metric on X is complete.

[GT. 2]

(a) Show that the connected sum $T\#P$ of the torus T and the projective plane P is homeomorphic to the connected sum of three copies of the projective plane $P\#P\#P$.

(b) The boundary of the Möbius band is a circle. Which surface do we obtain if we identify antipodal points of that circle? Justify your answer.

[GT. 3]

Let G be a Lie group acting on a manifold M transitively, let H be a connected compact Lie subgroup of G which is an isotropic group of a point $p \in M$. Show that M has a Riemannian metric such that the transformation determined by each element of G is an isometry.

[GT. 4]

Let M be a Riemannian manifold of dimension n and let $p \in M$. Prove that there is a neighborhood U of p and n vector fields $e_1, \cdots, e_n$ in U, such that

$$< e_i, e_j >= \delta_{ij}, \quad \nabla_{e_i} e_j(p) = 0, \quad \forall i, j = 1, \cdots, n.$$