VARIABLES

LINEAR	DISPLACEMENT	d
	VELOCITY	v
	ACCELERATION	a
	MOMENTUM	p
	MASS	m
	FORCE	F
ANGULAR	DISPLACEMENT	e/q/r/R
	VELOCITY	ω
	ACCELERATION	α
	MOMENTUM	L
	MASS	I
	TORQUE	τ

BODY FRAME

LINEAR

$$d = R^T d'$$

$$v = R^T v'$$

$$a = R^T a'$$

$$p = R^T p'$$

$$m = m'$$

$$F = R^T F'$$

ANGULAR

$$\omega = R^T \omega'$$

$$\alpha = R^T \alpha'$$

$$L = R^T L'$$

$$I = R^T M R$$

$$M = I'$$

$$\tau = R^T \tau'$$

NEWTON - EULER EQUATIONS

$$F' = m a'$$

$$\tau' = M\alpha' + \omega' \times (M\omega')$$

ANGULAR VELOCITY

$$\dot{e} = E_{123}(e)^{-1}\omega$$ P9(39)

P24(295)

$$\dot{e} = E'_{123}(e)^{-1}\omega'$$ P9(40)

P24(296)

$$\dot{q} = \tfrac{1}{2}\, q \circ \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$ P16(156)

$$= \tfrac{1}{2}\, Q(q) \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

$$\dot{q} = \tfrac{1}{2} \begin{pmatrix} 0 \\ \omega' \end{pmatrix} \circ q$$ P16(157)

$$= \tfrac{1}{2}\, \hat{Q}(q) \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

$$\dot{r} = \tfrac{1}{2}\, V(r)^{-1}\omega$$ P21(264)

$$\dot{r} = \tfrac{1}{2}\, V(r)^{-T}\omega'$$ P21(265)

$$V(r) = W_y(r)\, G(r)$$

P20(237) P19(214)

RIGID BODY SIMULATION

LINEAR

$$p \leftarrow p + F \Delta t$$

$$v = p / m$$

$$d \leftarrow d + v \Delta t$$

ANGULAR

$$L \leftarrow L + \tau \Delta t$$

$$I^{-1} = R^T M^{-1} R$$

$$M = \begin{pmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_z \end{pmatrix}$$

$$\omega = I^{-1} L$$

RIGID BODY ROTATION

$$R \leftarrow R(\omega \Delta t)\, R$$

$$q \leftarrow q \circ q_v(\omega \Delta t)$$

$$q \leftarrow q + \tfrac{1}{2}\, q \circ \begin{pmatrix} 0 \\ \omega \Delta t \end{pmatrix}$$

$$r \leftarrow v_q\big(q_v(r) \circ q_v(\omega \Delta t)\big)$$

$$r \leftarrow r + \tfrac{1}{2}\, V(r)^{-1} \omega \Delta t$$

$$e \leftarrow e + E_{123}(e)^{-1} \omega \Delta t$$

NORMALIZATION & CANONICALIZATION

ANGLE $\quad \pm \quad [-\pi, \pi) \quad + \quad [0, 2\pi)$

NORMALISE $\rightarrow$ CANONICALIZE

EULER ANGLES

$$\phi \in [-\pi, \pi)$$

$$\theta \in [-\pi, \pi) \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\psi \in [-\pi, \pi)$$

UNIT QUATERNION

$$w \in [-1, 1] \longrightarrow [0, 1]$$

$$|q| = 1$$

AXIS ANGLE

$$\alpha \in [0, 2\pi) \rightarrow [0, \pi]$$

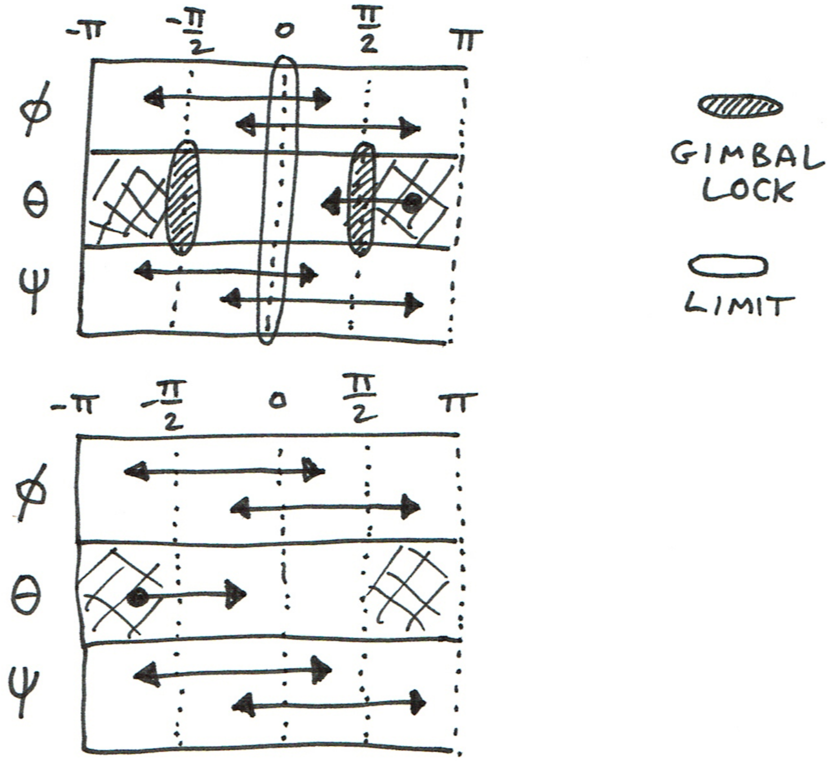
$$|n| = 1$$

ROTATION VECTOR

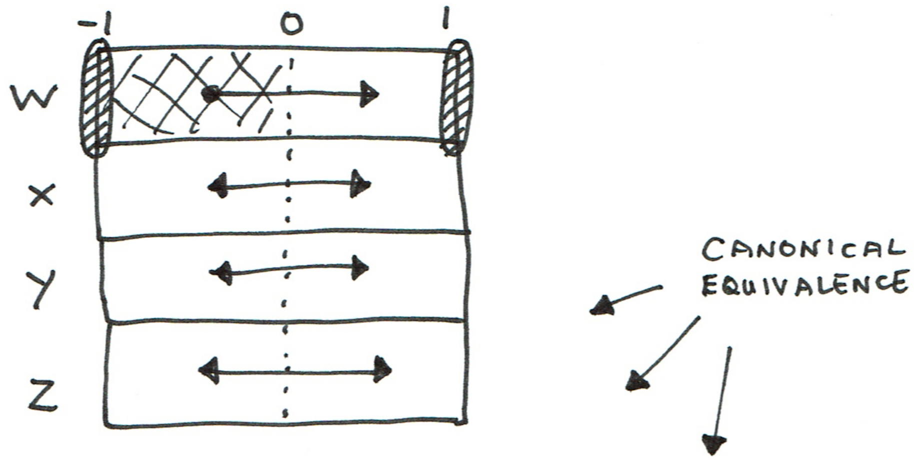
$$|r| \in [0, 2\pi) \rightarrow [0, \pi]$$

RANGES

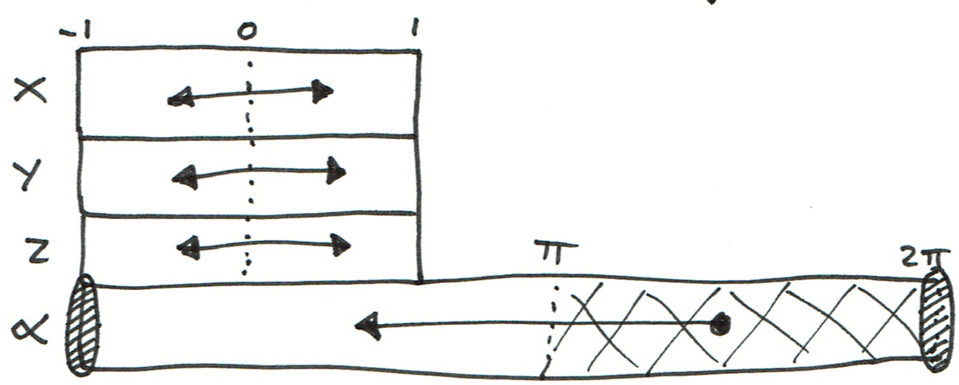
EULER ANGLES



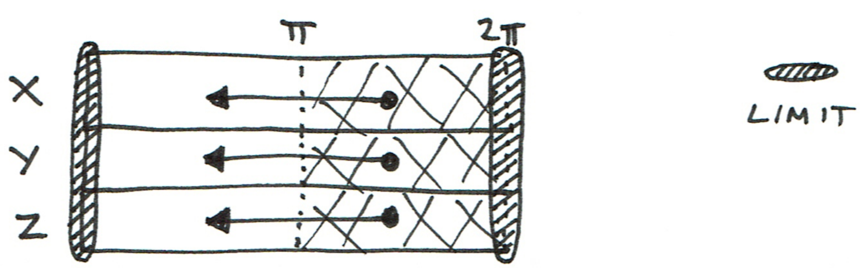
UNIT QUATERNION



AXIS ANGLE



ROTATION VECTOR



$\boxed{\text{LIMIT}}$

$$L = 0.02$$

UNIT QUATERNION	AXIS ANGLE	ROTATION VECTOR
$W > 1 - \frac{L^2}{8}$	$\alpha < L$	$\lvert r \rvert < L$
$W < \frac{L^2}{8} - 1$	$\alpha > 2\pi - L$	$\lvert r \rvert > 2\pi - L$

ROTATION CONVENTION

$$z' = R z$$

$$z = R^T z'$$

NED = NORTH X

EAST Y

DOWN Z

COORDINATE SYSTEM

Z WORLD

Z' BODY-FIXED

ROTATION MATRIX 2D

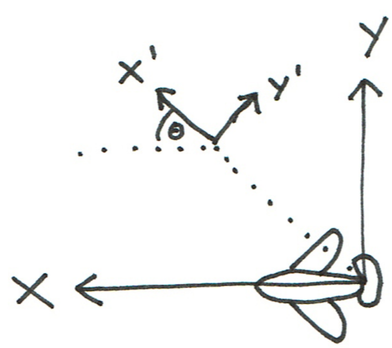
PASSIVE TRANSFORMATION
[CHANGE COORDINATE SYSTEM]

$$R_\theta = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}$$

ACTIVE TRANSFORMATION [CHANGE POSITION]

$$R_\theta^{-1} = R_\theta^{T}$$

$$\frac{\partial R_\theta}{\partial \theta} = \begin{pmatrix} -s\theta & c\theta \\ -c\theta & -s\theta \end{pmatrix}$$



$$d' = R d$$

$$d = R^T d'$$

ROTATION MATRIX

3D

PASSIVE TRANSFORMATION
[CHANGE COORDINATE SYSTEM]

$$R = R_\phi R_\theta R_\psi$$

$$R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix}$$

$$R_\psi = \begin{pmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ACTIVE TRANSFORMATION
[CHANGE POSITION]

$$R^T = R_\psi^T R_\theta^T R_\phi^T$$

TAIT-BRYAN ANGLES
EULER ANGLE SEQUENCE (1, 2, 3)

ϕ ROLL

θ PITCH

ψ YAW

EULER ANGLE EQUATIONS

$$\dot{\theta} = \omega'_y \cos(\phi) - \omega'_z \sin(\phi)$$

$$\dot{\phi} = \omega'_x + \omega'_y \sin(\phi)\tan(\theta) + \omega'_z \cos(\phi)\tan(\theta)$$

$$\dot{\psi} = \omega'_y \frac{\sin(\phi)}{\cos(\theta)} + \omega'_z \frac{\cos(\phi)}{\cos(\theta)}$$

$$u = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$\dot{u} = E'_{123}(u)^{-1}\,\omega'$$

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JACOBIAN

$$J_{ijk} = \frac{\partial F_{ij}}{\partial x_k} =$$

$$\left(\begin{matrix} \left(\frac{\partial F_{11}}{\partial x_1} \cdots \frac{\partial F_{11}}{\partial x_K} \right) & \cdots & \left(\frac{\partial F_{1J}}{\partial x_1} \cdots \frac{\partial F_{1J}}{\partial x_K} \right) \\ \vdots & & \vdots \\ \left(\frac{\partial F_{I1}}{\partial x_1} \cdots \frac{\partial F_{I1}}{\partial x_K} \right) & \cdots & \left(\frac{\partial F_{IJ}}{\partial x_1} \cdots \frac{\partial F_{IJ}}{\partial x_K} \right) \end{matrix} \right)$$

$$J_{ij} = \frac{\partial f_i}{\partial x_j} = \left(\begin{matrix} \left(\frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_J} \right) \\ \vdots \\ \left(\frac{\partial f_I}{\partial x_1} \cdots \frac{\partial f_I}{\partial x_J} \right) \end{matrix} \right)$$

$$F_{ij} = \left(\begin{matrix} \left(F_{11} \cdots F_{1J} \right) \\ \vdots \\ \left(F_{I1} \cdots F_{IJ} \right) \end{matrix} \right)$$

JACOBIAN CROSS PRODUCT

$$J(A \times B) = A \times J(B) - B \times J(A)$$

PRODUCT RULE

$$\frac{d}{dt}(a \cdot b) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$$

$$\frac{d}{dt}(a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt}$$

$$\frac{d}{dt}(A \cdot B) = \frac{dA}{dt} \cdot B + A \cdot \frac{dB}{dt}$$

JACOBIAN CHAIN RULE

$$M(x) = N(y) \ ; \ y = f(x)$$

$$\frac{\partial M_{ij}}{\partial x_k} = \sum_l \frac{\partial N_{ij}}{\partial y_l} \frac{\partial y_l}{\partial x_k}$$

$$(I \times J \times K) \sim (I \times J \times L) \cdot (L \times K)$$

EXAMPLE

$$R_v(r) = R_q(q_v(r))$$

$$\frac{\partial R_{vij}}{\partial r_k} = \sum_l \left.\frac{\partial R_{qij}}{\partial q_l}\right|_{q_v(r)} \left.\frac{\partial q_{vl}}{\partial r_k}\right|_r$$

$$(I \times J \times K) \sim (I \times J \times L) \cdot (L \times K)$$

$$(3 \times 3 \times 3) \sim (3 \times 3 \times 4) \cdot (4 \times 3)$$

JACOBIAN MATRIX

$$Y = M(z) \cdot X$$

$$(I \times 1) \sim (I \times J) \cdot (J \times 1)$$

$$\frac{\partial Y}{\partial z} = \frac{\partial M(z)}{\partial z} \cdot X$$

$$(I \times K) \sim (I \times K \times J) \cdot (J \times 1)$$

$$\boxed{\begin{array}{c} \dfrac{\partial M(z)}{\partial z}^{T} \\ J \times K \times I \end{array}}$$

PARTIAL DERIVATIVES

$$Z = R^T z'$$

$$Z_i = \sum_j R_{ji} z'_j$$

$$\frac{\partial z_i}{\partial r_j} = \sum_k \frac{\partial R_{ki}}{\partial r_j} z'_k \qquad (KIJ)$$

$$(3 \times 3) \sim (3 \times 3 \times 3) \cdot (3)$$

$$\frac{\partial z_i}{\partial q_j} = \sum_k \frac{\partial R_{ki}}{\partial q_j} z'_k \qquad (KIJ)$$

$$(3 \times 4) \sim (3 \times 4 \times 3) \cdot (3)$$

$$Z = f(y) \quad \& \quad y = g(x)$$

$$\frac{\partial z_i}{\partial x_j} = \sum_k \frac{\partial f_i(g(x))}{\partial y_k} \frac{\partial g_k(x)}{\partial x_j}$$

MORE PARTIAL DERIVATIVES

$$z' = R\, z$$

$$z'_i = \sum_j R_{ij}\, z_j$$

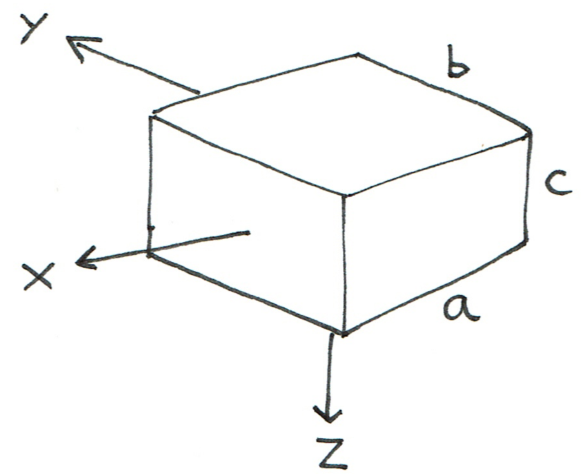
$$\frac{\partial z'_i}{\partial r_j} = \sum_k \frac{\partial R_{ik}}{\partial r_j}\, z_k \qquad (IKJ)$$

$$(3 \times 3) \sim (3 \times 3 \times 3) \cdot (3)$$

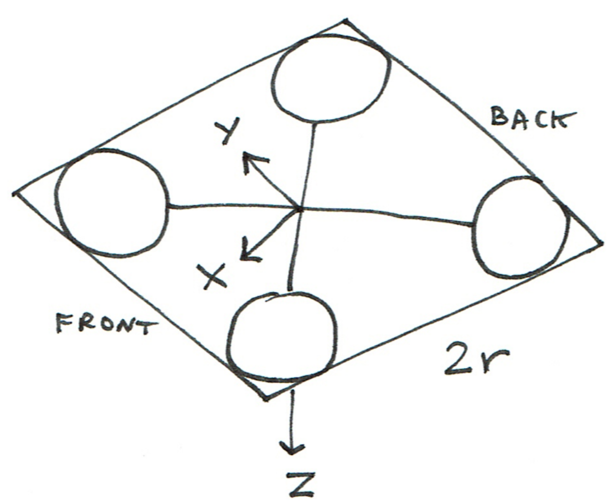
$$\frac{\partial z'_i}{\partial q_j} = \sum_k \frac{\partial R_{ik}}{\partial q_j}\, z_k \qquad (IKJ)$$

$$(3 \times 4) \sim (3 \times 4 \times 3) \cdot (3)$$

MOMENT OF INERTIA



$$I_x = \frac{1}{12} m \left(b^2 + c^2\right)$$
$$I_y = \frac{1}{12} m \left(a^2 + c^2\right)$$
$$I_z = \frac{1}{12} m \left(a^2 + b^2\right)$$



$$I_x = I_y = \frac{1}{3} m r^2$$
$$I_z = \frac{2}{3} m r^2$$

GAUSSIAN

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E[x] = \int_{-\infty}^{\infty} x\, g(x)\, dx = \mu$$

$$\begin{aligned} var(x) &= E\left[\left(x - E[x]\right)^2\right] \\ &= E\left[(x-\mu)^2\right] \\ &= \int_{-\infty}^{\infty} (x-\mu)^2\, g(x)\, dx \\ &= \sigma^2 \end{aligned}$$

$$\int_{-\infty}^{\infty} g(x)\, dx = 1$$

$$E[f(x)] = \int_{-\infty} f(x)\, g(x)\, dx$$

EXPECTATION

$$E[f(x)] = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$f(x) \approx f(a) + \frac{df}{dx}(a)\,(x - a)$$

$$E[f(x)] = f(a) + \frac{df}{dx}(a)\, E[x - a]$$

COVARIANCE MATRIX

$$\mathrm{var}(X) = \mathrm{cov}(X, X)$$

$$\mathrm{cov}(X, Y) = E\Big[(X - E[X])(Y - E[Y])\Big]$$

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{cov}(X, Y) + \mathrm{cov}(Y, X) + \mathrm{var}(Y)$$

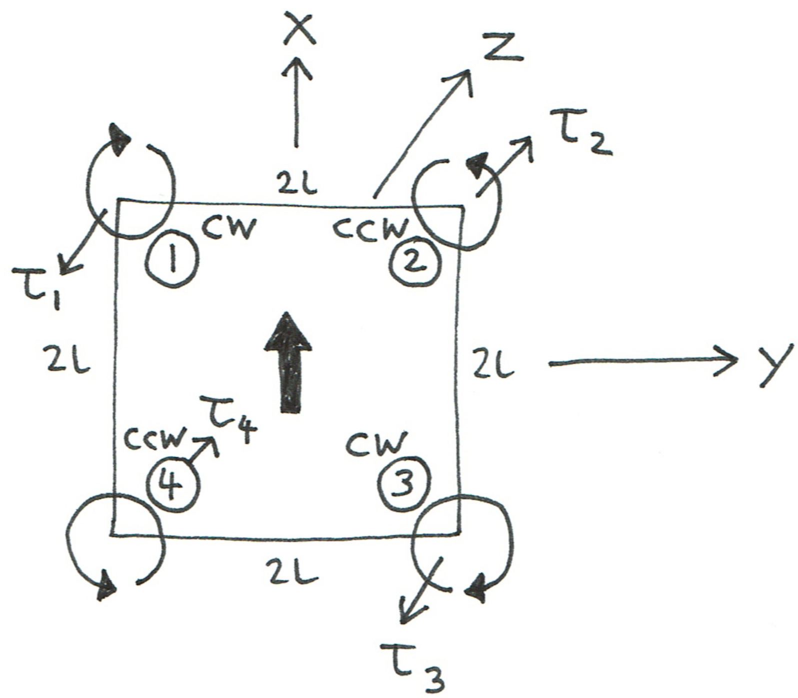
$$\mathrm{var}(X) = \begin{pmatrix} \mathrm{var}(X_1) & \cdots & \mathrm{cov}(X_1, X_N) \\ \vdots & & \vdots \\ \mathrm{cov}(X_N, X_1) & \cdots & \mathrm{var}(X_N) \end{pmatrix}$$

$$\mathrm{var}(X) = E[XX^T] - E[X]\,E[X]^T$$

$$\mathrm{var}(MX) = M\,\mathrm{var}(X)\,M^T$$

$$\mathrm{var}(f(X)) = \mathrm{var}\left(f(a) + \left.\frac{\partial f}{\partial X}\right|_a (X - a)\right)$$

$$= \left.\frac{\partial f}{\partial X}\right|_a \mathrm{var}(X - a) \left.\frac{\partial f}{\partial X}\right|_a^T$$



$$F_i = K_f \omega_i^2$$

$$\tau_i = K_m \omega_i^2$$

$$F = \begin{pmatrix} 0 \\ 0 \\ -F_1 - F_2 - F_3 - F_4 \end{pmatrix}$$

$$\tau = \begin{pmatrix} L(F_1 - F_2 - F_3 + F_4) \\ L(F_1 + F_2 - F_3 - F_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{pmatrix}$$

IDENTITIES

$$a \times b = a_{\times} \cdot b$$

$$\left(R^T M R\right)^{-1} = R^T M^{-1} R$$

$$R \cdot (a \times b) = (Ra) \times (Rb)$$

$$\frac{\partial}{\partial x}(a \times b) = a \times \frac{\partial b}{\partial x} - b \times \frac{\partial a}{\partial x}$$

$$\int_a^b R_\theta \, d\theta = \frac{\partial R_\theta(a)}{\partial \theta} - \frac{\partial R_\theta(b)}{\partial \theta}$$

FORCE 2D

$$F = ma$$

$$F = R^T(\theta)\, F'$$

$$V = \int_0^t a \, dt$$

$$= \int_0^t R^T(\theta_0 + \omega t)\, \frac{F'}{m} \, dt$$

$$= \int_{\theta_0}^{\theta_0 + \omega t} R^T(\theta)\, \frac{F'}{m\omega} \, d\theta$$

$$= V_0 + S^T K$$

where $K = \dfrac{F'}{m\omega}$

$$S = \int_{\theta_0}^{\theta_0 + \omega t} R(\theta) \, d\theta$$

$$S = \left[\frac{\partial R(\theta)}{\partial \theta} \right]_{\theta_0 + \omega t}^{\theta_0}$$

$$d = \int_0^t v \, dt$$

$$= \int_0^t v_0 + \frac{\partial R^T}{\partial \theta}(\theta_0) K - \frac{\partial R^T}{\partial \theta}(\theta_0 + \omega t) K \, dt$$

$$= d_0 + \left(v_0 + \frac{\partial R^T}{\partial \theta}(\theta_0) K \right) t - \left(\int_0^t \frac{\partial R}{\partial \theta}(\theta_0 + \omega t) \, dt \right)^T K$$

$$= d_0 + \left(v_0 + \frac{\partial R^T}{\partial \theta}(\theta_0) k \right) t - \left(\int_{\theta_0}^{\theta_0 + \omega t} \frac{\partial R(\theta)}{\partial \theta} \, d\theta \right)^T \frac{K}{\omega}$$

$$= d_0 + \left(v_0 + \frac{\partial R^T}{\partial \theta}(\theta_0) k \right) t + \left[R^T(\theta) \right]_{\theta_0 + \omega t}^{\theta_0} \frac{K}{\omega}$$

TORQUE 2D

$$\tau = I \alpha$$

$$\omega = \int_0^t \alpha \, dt$$

$$= \int_0^t \frac{\tau}{I} \, dt$$

$$= \omega_0 + \frac{\tau}{I} t$$

$$\theta = \int_0^t \omega \, dt$$

$$= \int_0^t \omega_0 + \frac{\tau}{I} t \, dt$$

$$= \theta_0 + \omega_0 t + \frac{\tau}{2I} t^2$$