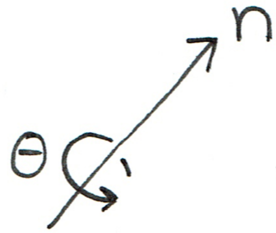
UNIT QUATERION

$$q = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$



$$n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big/ \sin\left(\frac{\theta}{2}\right)$$

$$\theta = 2\cos^{-1}(w)$$

$$R = \begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - zw) & 2(xz + yw) \\ 2(xy + zw) & 1 - 2(x^2 + z^2) & 2(yz - xw) \\ 2(xz - yw) & 2(yz + xw) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

$$q^{-1} = \begin{pmatrix} w \\ -x \\ -y \\ -z \end{pmatrix}$$

$$R^{-1} = R^T$$

QUATERION DELTA

$$q = \begin{pmatrix} q_w \\ q_x \\ q_y \\ q_z \end{pmatrix}$$

CAN BE CONVERTED TO AND FROM EULER ANGLES

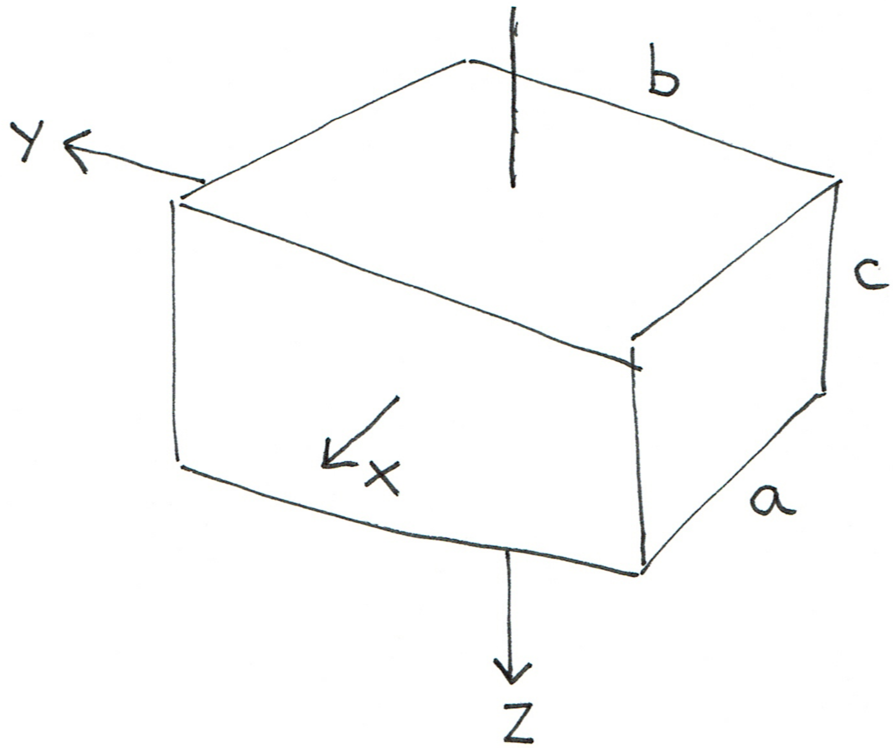
$$\begin{pmatrix} \text{ROLL} \\ \text{PITCH} \\ \text{YAW} \end{pmatrix} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \text{ROTATION } X \\ \text{ROTATION } Y \\ \text{ROTATION } Z \end{pmatrix}$$

$$\Phi = \begin{pmatrix} -q_x & -q_y & -q_z \\ q_w & -q_z & q_y \\ q_z & q_w & -q_x \\ -q_y & q_x & q_w \end{pmatrix}$$

$$q \rightarrow q + \frac{1}{2} \Phi \omega \Delta t$$

$$q \rightarrow \frac{q}{|q|}$$

MOMENT OF INERTIA

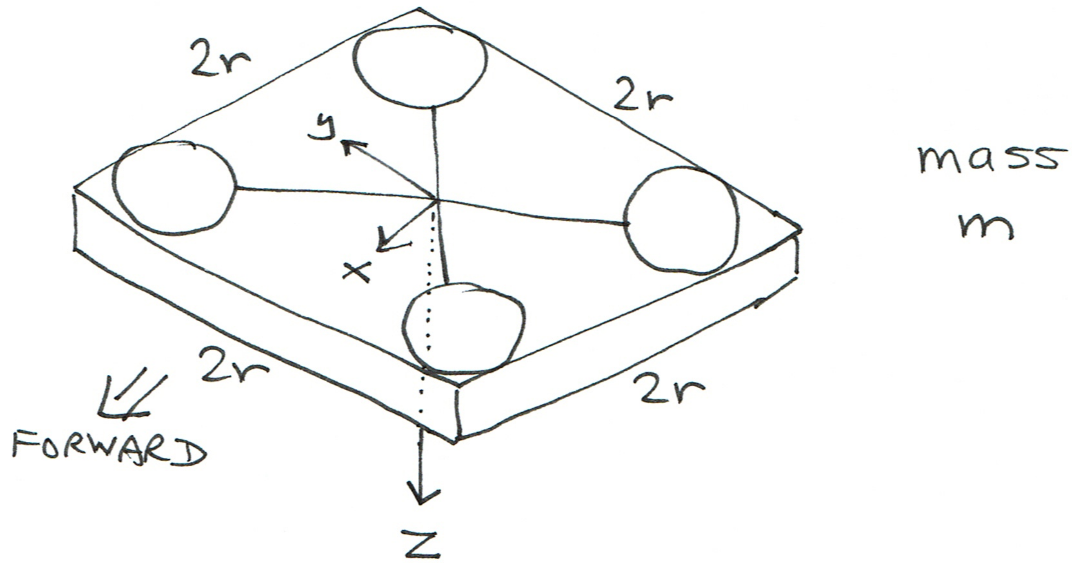


$$I_x = \frac{1}{12} m \left(b^2 + c^2 \right)$$

$$I_y = \frac{1}{12} m \left(a^2 + c^2 \right)$$

$$I_z = \frac{1}{12} m \left(a^2 + b^2 \right)$$

MOMENT OF INERTIA

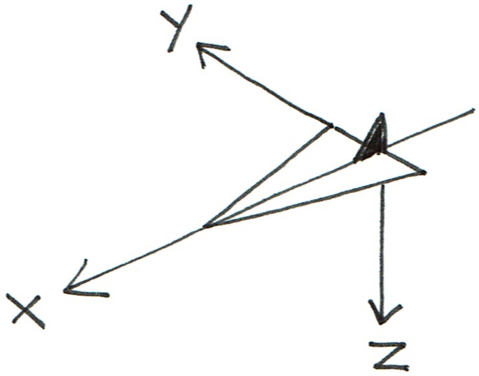


$$I_x = \frac{1}{3} m r^2$$

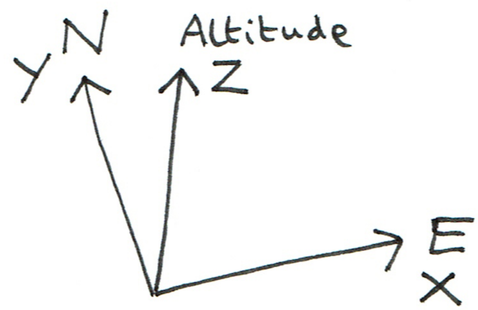
$$I_y = \frac{1}{3} m r^2 = I_x$$

$$I_z = \frac{2}{3} m r^2 = 2 I_x$$

FRAME OF REFERENCE



BODY FRAME



EARTH FRAME

ROTATION MATRIX $R(q)$

POSITION $p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

VELOCITY $v = \dot{p}$

$$V_B = R(q) \cdot v$$

$$\Delta p = v \Delta t$$

MAGNETIC NORTH VECTOR M

Re RADIUS OF EARTH

$$Re = 6{,}371{,}000 \, m$$

STICK (BODY FRAME)

F	TRUST FORCE
τ_x	ROLL TORQUE
τ_y	PITCH TORQUE
τ_z	YAW TORQUE

$$I\alpha + \omega \times (I\omega) = \tau$$

$$I_x \alpha_x + (I_z - I_y)\omega_y \omega_z = \tau_x$$

$$I_y \alpha_y + (I_x - I_z)\omega_z \omega_x = \tau_y$$

$$I_z \alpha_z + (I_y - I_x)\omega_x \omega_y = \tau_z$$

$$\omega \longrightarrow \omega + \alpha \Delta t$$

THROTTLE

$$a \longrightarrow R^T(q) \cdot \begin{pmatrix} 0 \\ 0 \\ -\frac{F}{m} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

$$g = 9.8\ ms^{-2}$$

$$v \longrightarrow v + a\Delta t$$

$$p \rightarrow p + v\Delta t$$

$$x_k = f(x_{k-1}, u_k)$$

$$z_k = h(x_k)$$

x_k is 19 DIMENSIONAL VECTOR

u_k is 4 DIMENSIONAL VECTOR

z_k is 13 DIMENSIONAL VECTOR

$$F = J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{19}} \\ \vdots & & \\ \frac{\partial f_{19}}{\partial x_1} & & \frac{\partial f_{19}}{\partial x_{19}} \end{pmatrix}$$

$$H = J_h = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_{19}} \\ \vdots & & \\ \frac{\partial h_{13}}{\partial x_1} & \cdots & \frac{\partial h_{13}}{\partial x_{19}} \end{pmatrix}$$

ESTIMATES

$$X_K = \begin{pmatrix} p \\ v \\ a \\ q \\ \omega \\ \alpha \end{pmatrix}_{19D}$$

EARTH FRAME

BODY FRAME

$$\omega = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$q = \begin{pmatrix} q_w \\ q_x \\ q_y \\ q_z \end{pmatrix}$$

CONTROL

$$U_K = (S)$$

BODY FRAME

$$S = \begin{pmatrix} F \\ \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} \begin{array}{l} \text{THROTTLE OR TRUST} \\ \text{ROLL TORQUE} \\ \text{PITCH TORQUE} \\ \text{YAW TORQUE} \end{array}$$

MEASURE

$$Z_K = \begin{pmatrix} Z_P \\ Z_A \\ Z_G \\ Z_B \\ Z_M \end{pmatrix}_{13D} \begin{array}{l} \text{GPS} \\ \text{ACCELEROMETER} \\ \text{GYROSCOPE} \\ \text{BAROMETER} \\ \text{MAGNETOMETER} \end{array}$$

EARTH FRAME

BODY FRAME

EXTENDED KALMAN FILTER

MODEL

$$x_k = f(x_{k-1}, u_k) + w_k \sim \mathcal{N}(0, Q_k)$$

$$z_k = h(x_k) + v_k \sim \mathcal{N}(0, R_k)$$

PREDICT

$$\hat{x}_k = f(\hat{x}_{k-1}, u_k)$$

$$P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

UPDATE

$$G_k = P_k H_k^T (H_k P_k H_k^T + R_k)^{-1}$$

$$\hat{x}_k \leftarrow \hat{x}_k + G_k (z_k - h(\hat{x}_k))$$

$$P_k \leftarrow (I - G_k H_k) P_k$$

$$F = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

JACOBIAN

Df

J_f

$$H = J_h$$

$$\frac{\partial f}{\partial x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

COMPUTED DERIVATIVE

IS ALTERNATIVE

TO CALCULATION

BY FORMULA

MODEL

$$x_k = A x_{k-1} + B u_k + w_k \sim \mathcal{N}(0, Q)$$

$$z_k = C x_k + v_k \sim \mathcal{N}(0, R)$$

PREDICT

$$\hat{x}_k = A \hat{x}_{k-1} + B u_k$$

$$P_k = A P_{k-1} A^T + Q$$

UPDATE

$$G_k = P_k C^T \left(C P_k C^T + R \right)^{-1}$$

$$\hat{x}_k \leftarrow \hat{x}_k + G_k \left(z_k - C \hat{x}_k \right)$$

$$P_k \leftarrow \left(I - G_k C \right) P_k$$

KALMAN FILTER MODEL

$$x_k = F_k x_{k-1} + B_k u_k + w_k$$

$$z_k = H_k x_k + v_k$$

$$w_k \sim \mathcal{N}(0, Q_k)$$

$$v_k \sim \mathcal{N}(0, R_k)$$

x_k = TRUE STATE

z_k = OBSERVED STATE

$\boxed{\text{PREDICT}}$

$$\hat{x}'_k = F_k \hat{x}_{k-1} + B_k u_k$$

$$P'_k = F_k P_{k-1} F_k^T + Q_k$$

$\boxed{\text{UPDATE}}$

$$y_k = z_k - H_k \hat{x}'_k$$

$$S_k = R_k + H_k P'_k H_k^T$$

$$K_k = P'_k H_k^T S_k^{-1}$$

$$\hat{x}_k = \hat{x}'_k + K_k y_k$$

$$P_k = (I - K_k H_k) P'_k$$

$$x_k = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$$x_k = F x_{k-1} + G a_k$$

$$a_k \sim \mathcal{N}(0, \sigma_a^2)$$

$$F = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \qquad F \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} x + \Delta t \dot{x} \\ \dot{x} \end{pmatrix}$$

$$G = \begin{pmatrix} \frac{1}{2} \Delta t^2 \\ \Delta t \end{pmatrix}$$

$$x_k = F \; x_{k-1} + w_k$$

$$w_k \sim \mathcal{N}(0, Q)$$

$$Q = G G^T \sigma_a^2 = \begin{pmatrix} \frac{1}{4} \Delta t^4 & \frac{1}{2} \Delta t^3 \\ \frac{1}{2} \Delta t^3 & \Delta t^2 \end{pmatrix} \sigma_a^2$$

OR $$w_k \sim G \cdot \mathcal{N}(0, \sigma_a^2)$$

$$z_k = H x_k + v_k$$

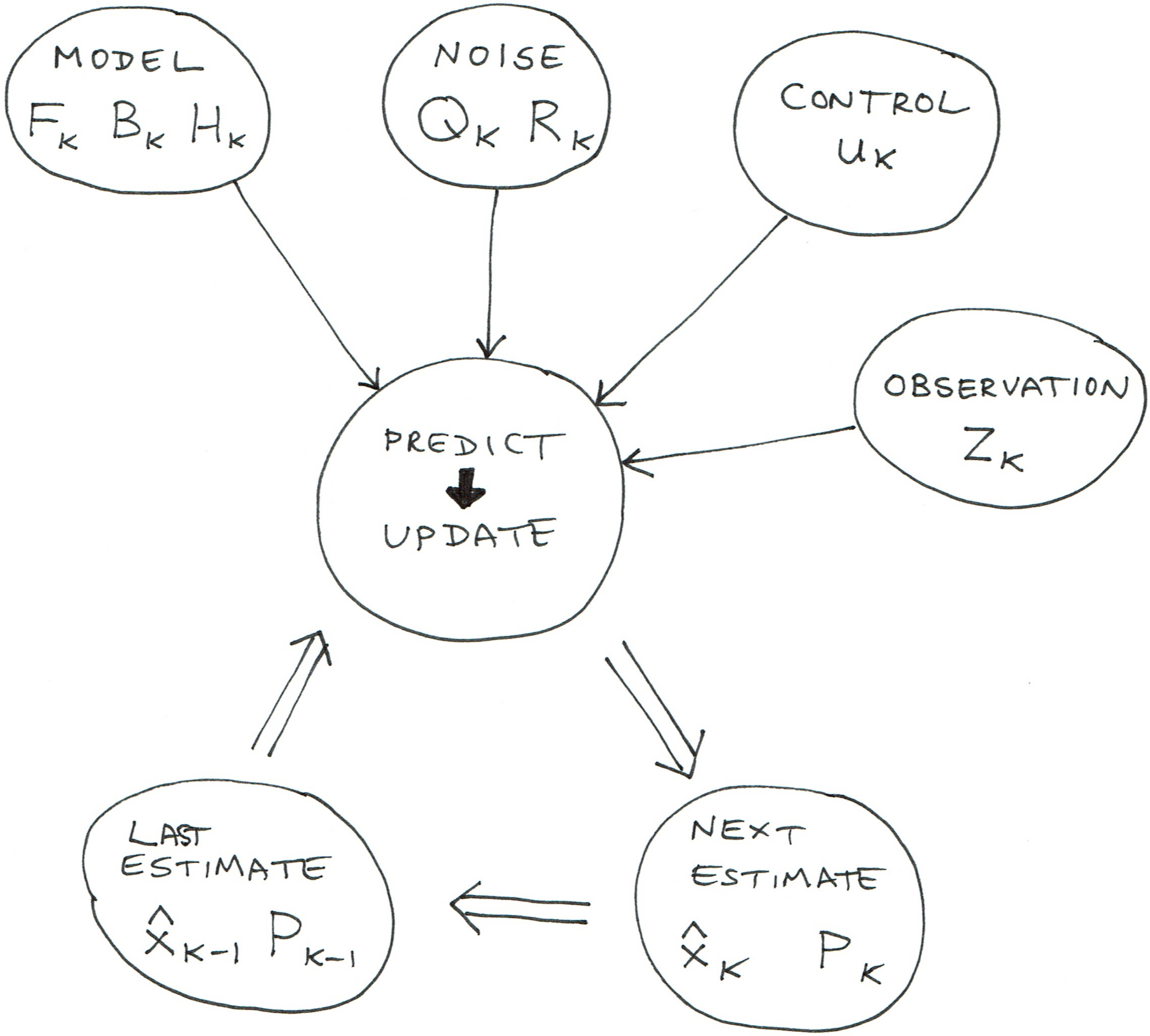
$$v_k \sim \mathcal{N}(0, \sigma_z^2)$$

$$H = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$R = \sigma_z^2$$

$$\hat{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ OR } \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_{\dot{x}}^2 \end{pmatrix}$$

INPUTS / OUTPUTS



ANOTHER EXAMPLE

$$\boxed{\begin{aligned} x_k &= F_k x_{k-1} + B_k u_k + \mathcal{N}(0, Q_k) \\ z_k &= H_k x_k + \mathcal{N}(0, R_k) \end{aligned}}$$

$$x_k = \begin{pmatrix} p_k \\ v_k \end{pmatrix}$$

$$F_k = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}$$

$$u_k = a_k$$

$$B_k = \begin{pmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{pmatrix}$$

$$z_k = \begin{pmatrix} p_k \\ v_k \end{pmatrix}$$

$$H_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q_k = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} ?$$

$$R_k = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} ?$$

CAN PUT $\frac{1}{2}\Delta t^2$

$$x_k = F_k \, x_{k-1}$$

$$x_k = \begin{pmatrix} p_k \\ v_k \\ a_k \end{pmatrix} = \begin{pmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k-1} \\ v_{k-1} \\ a_{k-1} \end{pmatrix}$$

ESTIMATES JACOBIAN

$$X = \begin{pmatrix} p_x \\ p_y \\ p_z \\ v_x \\ v_y \\ v_z \\ a_x \\ a_y \\ a_z \\ q_w \\ q_x \\ q_y \\ q_z \\ \omega_x \\ \omega_y \\ \omega_z \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$x \leftarrow f(x, u)$$

$$J_f = \begin{pmatrix} \frac{\partial f_{px}}{\partial p_x} & \cdots & \frac{\partial f_{px}}{\partial \alpha_z} \\ \vdots & & \\ \frac{\partial f_{\alpha x}}{\partial p_x} & \cdots & \frac{\partial f_{\alpha x}}{\partial \alpha_z} \end{pmatrix}$$

$$f_{px} = p_x + v_x \Delta t$$
$$f_{py} = p_y + v_y \Delta t$$
$$f_{pz} = p_z + v_z \Delta t$$
$$f_{vx} = v_x + a_x \Delta t$$
$$f_{vy} = v_y + a_y \Delta t$$
$$f_{vz} = v_z + a_z \Delta t$$
$$f_{ax} = 2(q_y q_w - q_x q_z)\frac{F}{m}$$
$$f_{ay} = -2(q_y q_z + q_x q_w)\frac{F}{m}$$
$$f_{az} = \left[2(q_x^2 + q_y^2) - 1\right]\frac{F}{m} - g$$
$$f_{qw} = q_w + \left[-q_x \omega_x - q_y \omega_y - q_z \omega_z\right]\frac{\Delta t}{2}$$
$$f_{qx} = q_x + \left[q_w \omega_x - q_z \omega_y + q_y \omega_z\right]\frac{\Delta t}{2}$$
$$f_{qy} = q_y + \left[q_z \omega_x + q_w \omega_y - q_x \omega_z\right]\frac{\Delta t}{2}$$
$$f_{qz} = q_z + \left[-q_y \omega_x + q_x \omega_y + q_w \omega_z\right]\frac{\Delta t}{2}$$
$$f_{\omega x} = \omega_x + \alpha_x \Delta t$$
$$f_{\omega y} = \omega_y + \alpha_y \Delta t$$
$$f_{\omega z} = \omega_z + \alpha_z \Delta t$$
$$f_{\alpha x} = \left[\tau_x + (I_y - I_z)\omega_y \omega_z\right] / I_x$$
$$f_{\alpha y} = \left[\tau_y + (I_z - I_x)\omega_z \omega_x\right] / I_y$$
$$f_{\alpha z} = \left[\tau_z + (I_x - I_y)\omega_x \omega_y\right] / I_z$$

PREDICT JACOBIAN

$$Z = \begin{pmatrix} Z_{PX} \\ Z_{PY} \\ Z_{PZ} \\ Z_{AX} \\ Z_{AY} \\ Z_{AZ} \\ Z_{GX} \\ Z_{GY} \\ Z_{GZ} \\ Z_{B} \\ Z_{MX} \\ Z_{MY} \\ Z_{MZ} \end{pmatrix} \qquad z \leftarrow h(x)$$

$$J_h = \begin{pmatrix} \dfrac{\partial h_{zpx}}{\partial px} & \cdots & \dfrac{\partial h_{zpx}}{\partial \alpha_z} \\ \vdots & & \vdots \\ \dfrac{\partial h_{zmz}}{\partial px} & \cdots & \dfrac{\partial h_{zmz}}{\partial \alpha_z} \end{pmatrix}$$

$$h_{zpx} = \frac{180\, p_x}{\pi R_e \cos(p_y / R_e)}$$

$$h_{zpy} = \frac{180\, p_y}{\pi R_e}$$

$$h_{zA} = R^T(q)\, a$$

$$h_{zpz} = p_z$$

$$\begin{aligned} h_{zAx} = {} & [1 - 2(q_y^2 + q_z^2)]\, a_x + \\ & [2(q_x q_y + q_z q_w)]\, a_y + \\ & [2(q_x q_z - q_y q_w)]\, a_z \end{aligned}$$

$$\begin{aligned} h_{zAy} = {} & [2(q_x q_y - q_y q_w)]\, a_x + \\ & [1 - 2(q_x^2 + q_z^2)]\, a_y + \\ & [2(q_y q_z + q_x q_w)]\, a_z \end{aligned}$$

$$\begin{aligned} h_{zAz} = {} & [2(q_x q_z + q_y q_w)]\, a_x + \\ & [2(q_y q_z - q_x q_w)]\, a_y + \\ & [1 - 2(q_x^2 + q_y^2)]\, a_z \end{aligned}$$

$$h_{zGx} = \alpha_x$$

$$h_{zGy} = \alpha_y$$

$$h_{zGz} = \alpha_z$$

$\mu = 11.3\ \mathrm{Pa\, m^{-1}}$

b_0 = pressure at sea level

$$h_{zB} = b_0 - \mu p_z$$

$$h_{zM} = R^T(q)\,M$$

$$
\begin{aligned}
h_{zMx} = {} & \left[1 - 2\left(q_y^2 + q_z^2\right)\right] M_x + \\
& \left[2\left(q_x q_y + q_z q_w\right)\right] M_y + \\
& \left[2\left(q_x q_z - q_y q_w\right)\right] M_z
\end{aligned}
$$

$$
\begin{aligned}
h_{zMy} = {} & \left[2\left(q_x q_y - q_y q_w\right)\right] M_x + \\
& \left[1 - 2\left(q_x^2 + q_z^2\right)\right] M_y + \\
& \left[2\left(q_y q_z + q_x q_w\right)\right] M_z
\end{aligned}
$$

$$
\begin{aligned}
h_{zMz} = {} & \left[2\left(q_x q_z + q_y q_w\right)\right] M_x + \\
& \left[2\left(q_y q_z - q_x q_w\right)\right] M_y + \\
& \left[1 - 2\left(q_x^2 + q_y^2\right)\right] M_z
\end{aligned}
$$