Week 14 Tutorial Solutions

19.1 Which Kind of Infinity?

A common fast way to show that a set *is* countable is to note that every element in the set has a finite representation. Be careful trying to use this to show that a set is *uncountable* because even if the representations aren't finite, there may be alternate representations that are.

a) **Countably infinite**. In fact it's basically the definition of countably infinite - the bijection mapping it to $\mathbb{N}$ is $id_{\mathbb{N}}$.

b) **Uncountable**. The powerset of a set always has a (strictly) larger cardinality than that set. *(Thinking with representations: these do not appear to all have finite representations - if I have an infinite set of naturals with no pattern, how would I possibly write down that set?)*

c) **Uncountable**. We know $\mathbb{R}$ is uncountable, and $\mathbb{R} \subseteq \mathbb{C}$.

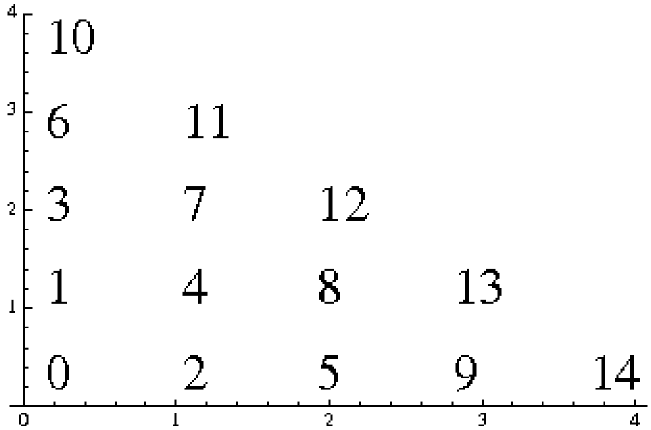
d) **Countably infinite**. We can provide a one-to-one function f mapping these to the (finite) bit strings: given S with maximum element n, return the bit string of length $n+1$ with a 1 in (0-indexed) position i iff $i \in S$. For example, $f(\{0,3,4\}) = 10011$. And we know the set bit strings (or any other strings with a finite alphabet) are countable. *(Thinking with representations: each $S \in X$ has a roster notation which is finite - e.g. $\{0,3,4\}$.)*

e) **Countably infinite**. Each book is just one finite string using a finite alphabet. *(You may be tempted to think of a book as a* list *of strings separated by spaces, but that's making it more complicated than necessary.)*

f) **Countably infinite**. We know $\mathbb{Q}$ is countable, and this set is a subset of $\mathbb{Q}$. *(Thinking with representations: these are reals specifically chosen to have expansions that end - i.e. representations that are finite.)*

19.2 A Curious Bijection

a)



b) Consider the values of x, y satisfying $x + y = k$.

Because we are in $\mathbb{N}$, for any such values of x and y we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.

Thus, x ranges from 0 to k, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k+1)}{2}$, we can also write this as:

$$\frac{k(k+1)}{2} \leq f(x, y) \leq \frac{k(k+1)}{2} + k$$

c) The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.

We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all x, y, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of x and y shows that $(2, 3)$ is the only element in the pre-image of 17.

It would also have been sufficient to prove that f is one-to-one, but this takes considerably more effort.

d) Let $k = x + y$, $l = p + q$. We are assuming that $k \neq l$. So, without loss of generality, assume that $k < l$. (If k was bigger than l, we could just swap the names of the two variables.) We aim to show that $f(x, y) < f(p, q)$.

From the solution to b), we know that the sums of the coordinates k and l restrict the output values to very limited ranges. So, $f(x, y)$ has to be no bigger than the upper end of the range of outputs for the sum $k = x + y$. That is:

$$f(x, y) \leq \frac{k(k+1)}{2} + k$$

Similarly, $f(p, q)$ has to be at least as big as the lower end of the range of outputs for the sum $l = p + q$. That is:

$$\frac{l(l+1)}{2} \leq f(p,q)$$

Thus, to show that $f(x,y) < f(p,q)$, it suffices to show that $\frac{k(k+1)}{2} + k < \frac{l(l+1)}{2}$. Since $k < l$, we have that $k+1 \leq l$ and therefore, substituting $k+1$ for l, we have that $\frac{(k+1)(k+2)}{2} \leq \frac{l(l+1)}{2}$. It therefore suffices to show that $\frac{k(k+1)}{2} + k < \frac{(k+1)(k+2)}{2}$, which we do as follows:

$$\frac{k(k+1)}{2} + k = \frac{k(k+1)+2k}{2} = \frac{k^2+3k}{2} < \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

e) Assume the contrary, that $f(x,y) = f(p,q)$. Further, let $k = x+y = p+q$. Then:

$$\begin{aligned} f(x,y) &= f(p,q) \\ s(x+y)+x &= s(p+q)+p \\ s(k)+x &= s(k)+p \\ x &= p \end{aligned}$$

Since $x = p$ and $x+y = p+q$, we have that $y = q$. But we assumed that $(x,y) \neq (p,q)$, contradiction.

Additional problem

Lemma: For (non-empty) sets A and B, there exists a one-to-one function $f : A \to B$ if and only if there exists an onto function $g : B \to A$.

Proof: See solution to week 5's additional tutorial problem - the only difference is that now we are working with arbitrary sets instead of subsets of $\mathbb{N}$, so where that solution uses the function *minimum* (which can choose a representative from a set of naturals), we instead have to use the choice function h from the hint. □

We know that by definition, there exists a one-to-one function $f : A \to B$ if and only if $|A| \leq |B|$. So now by the lemma, we've established that there exists an onto function $g : B \to A$ if and only if $|A| \leq |B|$.