Additional tutorial problems for Induction

Induction-Like Implications[1]

Do not *use induction for this problem, but think about how this problem relates to the mechanics of induction.*

For each statement S below, answer the following two prompts:

- Find a predicate $P(n)$ on natural numbers where $\forall k, P(k) \rightarrow P(k+2)$ and also S is true, or explain why no such predicate exists.
- Find a predicate $P(n)$ on natural numbers where $\forall k, P(k) \rightarrow P(k+2)$ but S is false, or explain why no such predicate exists.

For example, if S is the statement "$\forall n, P(n)$", then:

- For the first prompt, we could define $P(n)$ to be "n is non-negative" (or anything else that is true for every natural number). This clearly makes S true, and it also makes the required induction-like implication true since for every k, $P(k) \rightarrow P(k+2) \equiv T \rightarrow T \equiv T$).
- For the second prompt, we could define $P(n)$ to be "n is even". This makes S false (as not all natural numbers are even), but our induction-like implication is still true: on even k it's $T \rightarrow T \equiv T$, and on odd k it's $F \rightarrow F \equiv T$.

c) $S =$ "$\forall n \geq 0, \neg P(n)$"

d) $S =$ "$(\forall n \leq 100, P(n)) \wedge (\forall n > 100, \neg P(n))$"

e) $S =$ "$(\forall n \leq 100, \neg P(n)) \wedge (\forall n > 100, P(n))$"

f) $S =$ "$P(0) \rightarrow \forall n, P(n+2)$"

h) $S =$ "$P(1) \rightarrow \forall n, P(2n+1)$"

[1] This problem was adapted from "Mathematics for Computer Science" by Lehman et al. problem 5.16. `https://courses.grainger.illinois.edu/cs173/fa2020/Textbook/MITMathCS.pdf`

The Diagonal Robot

A robot is walking around on the 2D integer grid. It starts at $(1, 1)$, and at each step it moves to one of the closest diagonal grid points - e.g. its first step can take it to any of $(2, 2)$, $(2, 0)$, $(0, 0)$, or $(0, 2)$. Prove that the robot can never reach the point $(0, 1)$.

Hint:

1. *First, draw a picture to make sure the problem statement makes sense, and experiment with what the robot can reach in a few steps. For example, find a sequence of steps that allows the robot to reach* $(-3, 3)$.

2. *Based on patterns you see in step 1, guess some property which is true of all points the robot can reach - as a (wrong) example, you might guess that every* (x, y) *point the robot can reach will satisfy* $x \leq y$.

3. *Prove by induction that your guess from the previous step is correct. Your induction variable should be the number of steps the robot takes.*

4. *Conclude by showing that* $(0, 1)$ *does not have your proven property, so it must not be reachable.*