Name: ***Solutions***

1. Let $A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$. Find the nullspace and column space of A.

Solution: Both for the nullspace and for the column space, we first bring the matrix to the reduced row echelon form.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To write the nullspace, we identify pivot variables (x_1, x_3) and free variables (x_2, x_4, x_5). From the first row we get $x_1 = -3x_2 - 2x_4 + x_5$ and from the second row we get $x_3 = -4x_4 + 3x_5$. Given a row-reduced row-echelon matrix above, we can easily express the complete solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 - 2x_4 + x_5 \\ x_2 \\ -4x_4 + 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that we would get the same result by listing special solutions as usual, by setting one free variable to 1 and the rest to 0:

$$\begin{matrix} x_2 = 1 \\ x_4 = 0 \\ x_5 = 0 \end{matrix} \rightarrow s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} x_2 = 0 \\ x_4 = 1 \\ x_5 = 0 \end{matrix} \rightarrow s_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{matrix} x_2 = 0 \\ x_4 = 0 \\ x_5 = 1 \end{matrix} \rightarrow s_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix},$$

which then gives the same

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

For column space, we recall that $C(A)$ is spanned by the columns of A in pivot positions, in this case by columns 1 and 3:

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

2. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$, and whose nullspace contains $(1, 1, 2)$.

Solution: To easily satisfy the first condition, we include $(1, 1, 5)$ and $(0, 3, 1)$ as columns in our yet unknown matrix A.

Now we recall that the condition for $(1, 1, 2)$ to be in the nullspace $N(A)$ is:

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0.$$

In order for this multiplication to even make sense, the matrix A must have 3 columns. Now we can write that

$$A = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix},$$

and that a, b, c must satisfy the condition

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0,$$

that is:

$$\begin{aligned} &1 + 2a = 0, \\ &1 + 3 + 2b = 0, \\ &5 + 1 + 2c = 0. \end{aligned}$$

We easily solve this to get $a = -1/2$, $b = -2$, $c = -3$, which gives us the answer:

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}.$$

3. Under what condition on b_1, b_2, b_3 is this system solvable? Find the general solution $\mathbf{x}$, when that condition holds. Express it as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_n}$.

$$\begin{aligned} x &\ +2y &\ -2z &\ = b_1 \\ 2x &\ +5y &\ -4z &\ = b_2 \\ 4x &\ +9y &\ -8z &\ = b_3 \end{aligned}$$

Solution: We start by running Gauss elimination process:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right] \xrightarrow{R_2-2R_1, R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 1 & 0 & b_3-4b_1 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right].$$

A triangular system has solution when and only when every zero row has 0 in the right hand side, so, in this case, when $b_3 - b_2 - 2b_1 = 0$.

Under this condition, we can find the general solution. For that, we bring the matrix to its reduced row echelon form (we omit the zero row in the subsequent computation). In the case of this matrix, we just have to eliminate one entry above second pivot:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & b_1-2(b_2-2b_1) \\ 0 & 1 & 0 & b_2-2b_1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1-2b_2 \\ 0 & 1 & 0 & b_2-2b_1 \end{array}\right].$$

Now we have two slightly different ways to arrive at the same result. Either suffices by itself, but we do both for the sake of completeness. It is highly recommended that you figure out why these two ways will always give the same result.

- We can express everything through a particular and special solutions.

  We can take any particular solution as x_p. Easiest is to assign free variables (in this case, z) to 0:

  $$x_p = \begin{bmatrix} 5b_1 - 2b_1 \\ b_2 - b_1 \\ 0 \end{bmatrix}.$$

  For the x_n, we first build all special solutions. In this case, there is only one since there is only one free variable, and we get this special solution by assigning $z = 1$ (and remember that this is a solution the system with 0 in the RHS, $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$):

  $$\begin{aligned} x + 0y - 2 \cdot 1 &= 0, \\ 0x + y + 0 \cdot 1 &= 0, \end{aligned} \qquad \text{which gives the special solution } \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

  We can now write the general solution as

  $$x = x_p + x_n = \underbrace{\begin{bmatrix} 5b_1 - 2b_1 \\ b_2 - b_1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{x_n}.$$

- Or we can just write out a general solution and manipulate it algebraically.
  The row reduced matrix corresponds to the system

$$\begin{array}{cccl} x & & -2z & = 5b_1 - 2b_2 \\ & +y & & = b_2 - 2b_1 \end{array},$$

  which gets us

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 + 2z \\ b_2 - 2b_1 \\ z \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

  (The above manipulation just splits the constants and terms with variables, in this case just z, into separate vectors.)
  Notice that this is exactly the same as in the first computation.