Column space of A - how to describe it?

$$\begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 2 & 2 & 3 & 6 & 10 \\ & & 1 & 2 & 4 \\ & & 1 & 5 & 4 \end{bmatrix} = A \longrightarrow R = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

same

why not $x_1 \cdot (\text{col } 1) + x_2 (\text{col } 2) + \dots + x_5 (\text{col } 5)$

in R!

$$\begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 10 \\ 4 \\ 4 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$C(A)$ is spanned by pivot cols of A
(columns of A in the positions as pivot cols of R – the $\underbrace{\text{reduced row echelon form}}_{\text{write: RREF}}$ of A)

Complete solution to $Ax=b$.

$Ax = b$

$Ax' = b$ $\rightarrow$ $A(x-x') = b-b = 0$

so $x-x'$ is a sol'n to $Ax=0$.

$$x' = \underset{\uparrow}{x} + \underbrace{(x'-x)}$$

our favorite sol to $Ax=b$

any sol'n to $Ax=0$ (any vector from $N(A)$)

So: take any one sol'n x_p ($x_{\text{particular}}$), then <u>every</u> sol'n to $Ax=b$ look like

$x = x_p + x_n$ where $x_n = x_{\text{nullspace}}$ is <u>any</u> solution to $Ax=0$.

$x = \underset{\text{one}}{x_p} + \underset{\text{many}}{x_n}$

x particular : one sol'n , solves $Ax=b$

x nullspace : $\underset{\#\text{ vars}}{n} - \underset{\text{rank} = \#\text{ pivot vars}}{r}$ sol's , solve $Ax=0$.

Ex.

$$\left[\begin{array}{cccc|c} 10 & 7 & 0 & 2 & 1 \\ 0 & 0 & 3 & 4 & 6 \\ 10 & 7 & 3 & 6 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 10 & 7 & 0 & 2 & 1 \\ 0 & 0 & 3 & 4 & 6 \\ 0 & 0 & 3 & 4 & 6 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 10 & 7 & 0 & 2 & 1 \\ 0 & 0 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & \frac{7}{10} & 0 & \frac{1}{5} & \frac{1}{10} \\ 0 & 0 & 1 & 4/3 & 2 \\ & & & 0 & 0 \end{array}\right]$$

free

pivot

$x_1 + \frac{7}{10}x_2 + \frac{1}{5}x_4 = 0$

x_p: assign 0 to all free var: $\begin{bmatrix} 1/10 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

x_n: $x_2 = 1$, $x_4 = 0$, $\begin{bmatrix} -7/10 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = 0$, $x_4 = 1 \longrightarrow \begin{bmatrix} -1/5 \\ 0 \\ -4/3 \\ 1 \end{bmatrix}$

complete sol'n!

$$x = x_p + x_n = \begin{bmatrix} 1/10 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -7/10 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/5 \\ 0 \\ -4/3 \\ 1 \end{bmatrix}$$

what sort of complete sol can we get?

rank = # pivots

IF rank = m THEN there guaranteed to be some solutions
$\parallel$
#equs

IF rank = n THEN no pivot vars (nullspace is zero)
$\parallel$
#vars

$r = m = n$ square and inv. A, $Ax = b$ 1 sol'n.

$r = m$, $r < n$ short/wide A, $Ax = b$ inf. many sols

$r < m$, $r = n$ tall/thin A, $Ax = b$ 0 or 1 sol

$r < m$, $r < n$ $Ax = b$ 0 or ∞ sol.