Independence, basis, dimension.

Def The set of vectors $v_1, \dots, v_n$ is linearly independent if the only their linear combination that gives the zero vector is $0v_1 + 0v_2 + \dots + 0v_n$:

$x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0$ only when all x_i's are 0.

Ex 1 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are indep. $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Ex 2 $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ indep.: $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0$ $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Ex 3 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ dependent: $1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Ex 4 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ dependent: $0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 197 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

Ex 5 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ independent! $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$? $x = 0$.

Ex 6 $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}$ are dependent:

$$x_1 \begin{bmatrix} a \\ b \end{bmatrix} + x_2 \begin{bmatrix} c \\ d \end{bmatrix} + x_3 \begin{bmatrix} e \\ f \end{bmatrix} = 0 \longrightarrow \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

3×2, $3 > 2$
there are free vars

Ex 7 Are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$ indep.?

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = 0 \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = 0 \qquad \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

dependent

$3v_1 - 1v_2 = v_3$

Columns of A are independent exactly when $N(A) = 0$.

Reminder: #pivots = rank of A, denoted r.

- Cols of an $m \times n$ matrix A are independent exactly when $r = n$. In that case $N(A) = 0$.
- If $n > m$ then any n vectors in $\mathbb{R}^m$ are dependent.

#pivots $\leq m < n$

Span

Def A set of vectors spans a space if their linear combinations fill the space.

Ex1 $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ span $\mathbb{R}^2$

Ex2 $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ span $\mathbb{R}^2$: $\begin{bmatrix}a\\b\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}3\\4\end{bmatrix}$ $\quad \begin{bmatrix}1&3\\2&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

Ex3 $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}7\\-100\end{bmatrix}$ span $\mathbb{R}^2$

$$\begin{bmatrix}7\\0\end{bmatrix} = 7\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}7\\-100\end{bmatrix}$$
$$= 0\begin{bmatrix}1\\0\end{bmatrix} + 100\begin{bmatrix}0\\1\end{bmatrix} + 1\begin{bmatrix}7\\-100\end{bmatrix}$$

Ex4 $\begin{bmatrix}1\\0\end{bmatrix}$ does not span $\mathbb{R}^2$.

A **basis** of a vector space is a set of vectors with two properties:

1) they are independent
2) they span the space.

Ex: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ form a basis for $\mathbb{R}^2$

$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ form a basis for $\mathbb{R}^2$

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}7\\-100\end{bmatrix}$ is not a basis for $\mathbb{R}^2$, since they are dependent

$\begin{bmatrix}1\\0\end{bmatrix}$ is not a basis for $\mathbb{R}^2$, since it does not span $\mathbb{R}^2$.

Every vector in the vector space is expressed uniquely as a linear combination of basis vectors.

basis $v_1, v_2, \ldots, v_n$

$$v = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$$

$$v = y_1 v_1 + y_2 v_2 + \ldots + y_n v_n$$

$$0 = (x_1 - y_1) v_1 + (x_2 - y_2) v_2 + \ldots + (x_n - y_n) v_n$$

since $v_1, \ldots, v_n$ are indep,

we get $x_1 - y_1 = 0$

$x_2 - y_2 = 0$

$\ldots$

$x_n - y_n = 0$

Ex. $y''(x) + y(x) = 0$

sol's form a vector space ← essentially $\mathbb{R}^2$

with basis $\sin x, \cos x$

$(\sin x)'' + \sin x = 0$

$(\cos x)'' + \cos x = 0$

$(a\sin x + b\cos x)'' + (a\sin x + b\cos x) = 0$

so sol's are encoded by $\begin{bmatrix} a \\ b \end{bmatrix}$.

Ex. Pivot cols form a basis for $C(A)$

$$\begin{bmatrix} 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ & & & 0 & 0 & 0 & 1 \\ 0 & \cdots & & \cdots & & & 0 \end{bmatrix}$$

Ex. Special solutions form a basis for $N(A)$

Ex. figure out why.

Dimension.

Any two bases of a space have the same number of vectors, called the dimension of the space.

Suppose two bases $v_1, v_2, \ldots, v_m$
$w_1, w_2, \ldots, \ldots, w_n$ $\quad n > m$

$$w_1 = a_{11} v_1 + a_{21} v_2 + \ldots + a_{m1} v_m$$
$$w_2 = a_{12} v_1 + \ldots \qquad + a_{m2} v_m$$
$$\ldots$$
$$w_n = a_{1n} v_1 + a_{2n} v_2 + \ldots + a_{mn} v_m$$

$$\boxed{\begin{array}{ccccc} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{array}}$$
$$\quad w_1 \qquad\quad w_2 \qquad\qquad\quad w_n$$

n cols $\quad m$ rows
$n > m$
$\Rightarrow$ less pivots than n
$\Rightarrow$ cols dependent $\Rightarrow$ so are w_i's

How to find basis:

$$\left\langle \begin{bmatrix}1\\1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\2\\3\\2\end{bmatrix} \right\rangle$$

basis

$$\begin{bmatrix}1&1&2\\1&1&2\\3&0&3\\0&2&2\end{bmatrix} \to \begin{bmatrix}1&1&2\\0&0&0\\0&-3&-3\\0&2&2\end{bmatrix} \to$$

$$\to \begin{bmatrix}1&1&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

basis v_1, v_2 and $v_3 = v_1 + v_2$

Notation:

If $v_1, v_2, \ldots, v_n$ are vectors then the subspace spanned by them is denoted

$\langle v_1, v_2, .., v_n \rangle$ or

$\mathrm{Span}(v_1, v_2, .., v_n)$.

The four fundamental spaces.

Let A be an $m \times n$ matrix.

(1) col space $C(A)$:

all linear comb. of A. In other words:

all b s.t. $Ax=b$ has a sol'n. A subspace of $\mathbb{R}^m$.

(2) nullspace $N(A)$:

sol's to $Ax=0$. A subspace of $\mathbb{R}^n$.

(3) row space, $C(A^T)$:

all linear comb. of rows of A. A subspace in $\mathbb{R}^n$.

(4) left nullspace, $N(A^T)$

All sol's to $yA=0$, a subspace in $\mathbb{R}^m$

$(yA)^T=0$ $\quad A^T \overset{x}{y^T}=0$ $\quad A^T x=0$