Symmetric and positive definite matrices.

- eigenvalues > 0
- pivots > 0
- Sylvester's criterion: top left corner dets > 0

- $x^T S x > 0$ for every nonzero vector x
- $A^T A = S$ where A has indep. cols

(Cholesky decomposition is a particular case of this)

Ex. $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix} =$

$$= x(ax + by) + y(bx + cy)$$

$$= ax^2 + bxy + byx + cy^2$$

$$= ax^2 + 2bxy + cy^2$$

$$[x_1\ x_2 \cdots x_n]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$$

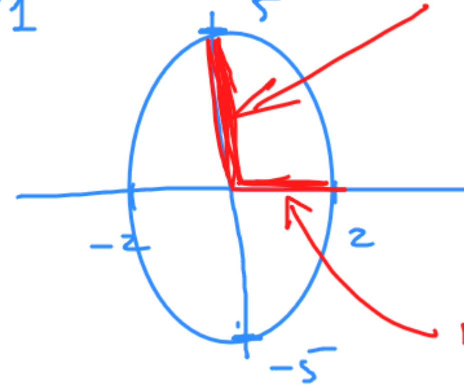
$$= \sum_{i,j=1}^{n} a_{ij} x_i x_j = a_{11} x_1^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + \cdots$$

$$+ \cdots + a_{nn} x_n^2 .$$

what is the curve $5x^2-4xy+8y^2=1$?

$$x^2+y^2=1$$

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{5}\right)^2=1$$



major semiaxis = 5

$x=0$

$\left(\frac{y}{5}\right)^2=1 \quad y=\pm 5$

minor semiaxis = 2

$$\lambda_1 x^2+\lambda_2 y^2=1$$

semiaxes: $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}$.

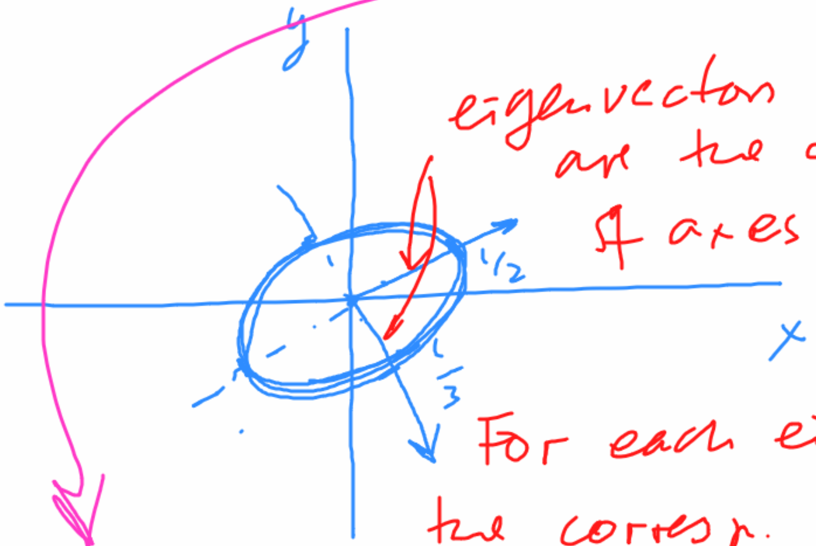
$$5x^2-4xy+8y^2=[x\ \ y]\,G\begin{bmatrix}x\\y\end{bmatrix}=\left([x\ \ y]\,Q\right)\Lambda\left(Q^T\begin{bmatrix}x\\y\end{bmatrix}\right)=[u\ \ v]\,\Lambda\begin{bmatrix}u\\v\end{bmatrix}.$$

(e) Use the above decomposition to identify major an minor semi-axis of the ellipse and their direction.

$$u=\frac{2x+y}{\sqrt5}\qquad v=\frac{x-2y}{\sqrt5}$$

$$5x^2-4xy+8y^2=[u\ \ v]\begin{bmatrix}4&\\&9\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=4u^2+9v^2,$$

$u=\frac12$ $\quad$ $v=\frac13$



eigenvectors are the directions of axes of ellipse

For each eigenvalue λ, the corresp. semiaxis is $\frac{1}{\sqrt\lambda}$.

• "u-axis" is $v=0$

$$\frac{x-2y}{\sqrt5}=0 \qquad \begin{bmatrix}x\\y\end{bmatrix}\perp\begin{bmatrix}1\\-2\end{bmatrix}$$

that is, $\begin{bmatrix}2\\1\end{bmatrix}$, the eigenvector for $\lambda=4$

• "v-axis" is $u=0$

$$\frac{2x+y}{\sqrt5}=0 \qquad \begin{bmatrix}x\\y\end{bmatrix}\perp\begin{bmatrix}2\\1\end{bmatrix}$$

$\begin{bmatrix}1\\-2\end{bmatrix}$

$$4u^2+9v^2=1$$

$v=0$, $u=\frac12$ $\qquad$ $u=0$, $v=\frac13$

oint at $(0,0,0,0)$. If the

either or

$$H = \begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & -4 & 1 & -1 \\ -2 & 1 & -6 & 2 \\ 0 & -1 & 2 & -4 \end{bmatrix}$$

(for example, $\frac{\partial^2 f}{\partial x \partial z}(0,0,0,0) = -2$), then is the point $(0,0,0,0)$ a point of local maximum, a point of local minimum, or not a point of local extremum at all?

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\cdot x + \frac{\partial f}{\partial y}(0,0)\cdot y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x \partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2\right) + \dots$$

at $(0,0)$

$(0,0)$ is critical:

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

$$= f(0,0) + \frac{1}{2}[x \ y]\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \dots$$

$$= f(0,0) + \frac{1}{2}[x \ y]\, H \begin{bmatrix} x \\ y \end{bmatrix} + \dots$$

Hessian

$$[x \ y]\, H \begin{bmatrix} x \\ y \end{bmatrix} > 0 \,? \quad < 0\,?$$