Name: ***Solutions***

1. (a) Compute $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) In the next items we identify and sketch the curve given by $5x^2 - 4xy + 8y^2 = 1$. Start by identifying a matrix S that delivers the equality:

$$5x^2 - 4xy + 8y^2 = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{S} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) Find $Q\Lambda Q^T$-decomposition of S.

(d) Compute $Q^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ and write

$$5x^2 - 4xy + 8y^2 = \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix} = (\begin{bmatrix} x & y \end{bmatrix} Q)\Lambda(Q^T \begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} u & v \end{bmatrix} \Lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

(e) Use the above decomposition to identify major an minor semi-axis of the ellipse and their direction.

Solution:

(a) $ax^2 + 2bxy + cy^2$.

(b) $S = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$.

(c) Find eigenvalues:
$\det(S - \lambda I) = (5-\lambda)(8-\lambda) - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9)$, so eigenvalues are 4 and 9.
Now we find an orthonormal system of eigenvectors:

$$\begin{bmatrix} 5-4 & -2 \\ -2 & 8-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\lambda = 4$ (we normalized so it's length 1).

$$\begin{bmatrix} 5-9 & -2 \\ -2 & 8-9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for $\lambda = 9$ (we normalized so it's length 1).
We therefore get the decomposition

$$\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}_{Q^T}.$$

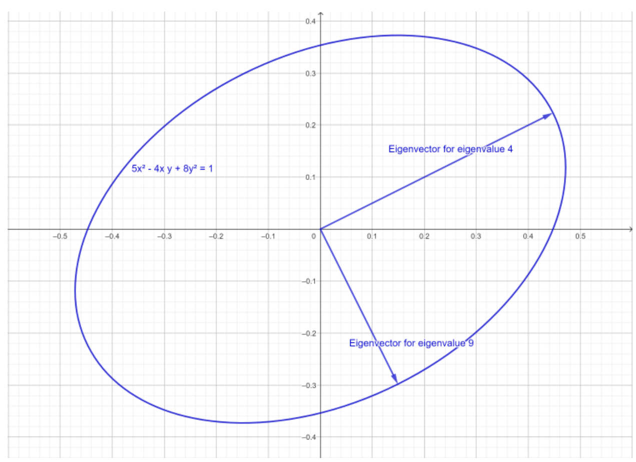
(d) We get

$$Q^T \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2x+y}{\sqrt{5}} \\ \frac{-x+2y}{\sqrt{5}} \end{bmatrix},$$

so $u = \frac{2x+y}{\sqrt{5}}$ and $v = \frac{-x+2y}{\sqrt{5}}$. Then

$$5x^2 - 4xy + 8y^2 = \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix} = (\begin{bmatrix} x & y \end{bmatrix} Q)\Lambda(Q^T \begin{bmatrix} x \\ y \end{bmatrix}) = 4\left(\frac{2x+y}{\sqrt{5}}\right)^2 + 9\left(\frac{-x+2y}{\sqrt{5}}\right)^2.$$

(e) The minor (smaller) semiaxis is $\sqrt{1/9} = 1/3$. Its direction is where $u = 0$, that is $2x + y = 0$, that is the direction of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$—the eigenvector that corresponds to $\lambda = 9$. The major (larger) semiaxis is $\sqrt{1/4} = 1/2$. Its direction, similarly, is the eigenvector for $\lambda = 4$, that is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.



2. Suppose a differentiable function $f(x, y, z, t)$ has a critical point at $(0, 0, 0, 0)$. If the matrix of second derivatives at that point is

$$H = \begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & -4 & 1 & -1 \\ -2 & 1 & -6 & 2 \\ 0 & -1 & 2 & -4 \end{bmatrix}$$

(for example, $\frac{\partial^2 f}{\partial x \partial z}(0, 0, 0, 0) = -2$), then is the point $(0, 0, 0, 0)$ a point of local maximum, a point of local minimum, or not a point of local extremum at all?

Solution: We can test positive/negative definiteness in several ways. Two practical ones in this instance are Sylvester criterion (signs of top left corner determinants), or performing row elimination (only with row subtractions) to find pivots.

Through determinants:

- $M_1 = -1 < 0$. At this point we know that the matrix is not positive definite (so the point is not a point of local min), but it may be negative definite.

- $M_2 = \begin{vmatrix} -1 & 1 \\ 1 & -4 \end{vmatrix} = 4 - 1 = 3 > 0.$
- $M_3 = \begin{vmatrix} -1 & 1 & -2 \\ 1 & -4 & 1 \\ -2 & 1 & -6 \end{vmatrix} = (-1)(-4)(-6) + 1 \cdot 1 \cdot (-2) + 1 \cdot 1 \cdot (-2) -$
  $(-2)(-4)(-2) - 1 \cdot 1 \cdot (-6) - (-1) \cdot 1 \cdot 1 = -24 - 2 - 2 + 16 + 6 + 1 = -5.$
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$$M_4 = \begin{vmatrix} -1 & 1 & -2 & 0 \\ 1 & -4 & 1 & -1 \\ -2 & 1 & -6 & 2 \\ 0 & -1 & 2 & -4 \end{vmatrix} = \begin{vmatrix} -1 & 1 & -2 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & -2 & 2 \\ 0 & -1 & 2 & -4 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & -2 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & 2 & -4 \end{vmatrix}$$
$$= - \begin{vmatrix} -1 & 1 & -2 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & 2 & -4 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & -2 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 4 & -6 \end{vmatrix}$$
$$= -(-1)(-1)(5 \cdot (-6) - 4 \cdot (-7)) = -(-30 + 28) = 2 > 0.$$

The sign of determinants alternates starting with < 0, so the matrix is negative definite, and therefore the critical point is a point of local maximum.

Through pivots: Here we perform row elimination (no row scaling and no row swaps):

$$\begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & -4 & 1 & -1 \\ -2 & 1 & -6 & 2 \\ 0 & -1 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -2 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & -2 & 2 \\ 0 & -1 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -2 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -5/3 & 7/3 \\ 0 & 0 & 7/3 & -11/3 \end{bmatrix} \xrightarrow[\frac{-11}{3}+\frac{49}{15}=\frac{2}{5}]{R_3+(7/5)R_2}$$
$$\to \begin{bmatrix} -1 & 1 & -2 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -5/3 & 7/3 \\ 0 & 0 & 0 & -2/5 \end{bmatrix}.$$

We see that all pivots are negative, so the matrix is negative definite.

Comment. Look how products of pivots are equal to the respective corner determinants:

$$-1 = -1, \quad (-1)(-3) = 3, \quad (-1)(-3)(-3/5) = -5, \quad (-1)(-3)(-3/5)(-2/5) = 2.$$

Nice.

3. Find the Cholesky factorization of

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 3 & 8 & 16 \end{bmatrix}.$$

Solution: We start by finding LDL^T-decomposition in the usual way:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 3 & 8 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U.$$

From this, we get that $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $L^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The Cholesky factor is then

$$C = \sqrt{D}L^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{5} \end{bmatrix},$$

so

$$S = C^T C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & \sqrt{2} & 0 \\ 3 & \sqrt{2} & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{5} \end{bmatrix}.$$