Inverse matrix and Gauss-Jordan Elimination.

First: matrix operations.

A is $m \times n$ matrix : m rows, n cols

- If A, B are of the same size, then we can add $A+B$ (add corresponding entries)
- If A is a matrix, c is a number, then we can multiply cA (multiply every entry by c).
- when can we multiply AB?

Exactly when # cols of A = # rows of B

$$(m \times n)(n \times p) = (m \times p)$$

If $AB = C$, then $C_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$.

Also: • every col of AB is a linear combination of cols of A, with coeffs that come from col's of B.

$$\underset{m\times n}{\begin{bmatrix} \cdots & \cdots \\ & \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} \bullet \\ \bullet \\ \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} \cdot \\ \\ \end{bmatrix}}$$

• every row of AB is a linear combination of rows of B, with coeffs that come from rows of A.

$$\underset{1\times n}{\begin{bmatrix} \bullet & \bullet & \cdots \end{bmatrix}} \underset{n\times p}{\begin{bmatrix} & & \\ & & \end{bmatrix}} = \underset{1\times p}{\begin{bmatrix} & & \end{bmatrix}}$$

Ex.

$$\underset{2\times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 7 & 8 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 2\cdot 7 & 2\cdot 8 \\ 3\cdot 7 & 3\cdot 8 \end{bmatrix}} = \begin{bmatrix} 14 & 16 \\ 21 & 24 \end{bmatrix}$$

$$\underset{1\times 2}{\begin{bmatrix} 7 & 8 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} 7\cdot 2 + 8\cdot 3 \end{bmatrix}} = \begin{bmatrix} 38 \end{bmatrix}$$

AB and BA are not the same!

$(3\times 4)(4\times 5) = (3\times 5)$

$(4\times 5)(3\times 4)$ is not defined

even if A, B are square,

$AB \neq BA$ (unless coincidentally)

<u>But:</u>

- $A(BC) = (AB)C$. In particular, $A(\underset{\substack{\uparrow \\ \text{column}}}{Bx}) = (AB)x$.

  (associative law)

- $A(B+C) = AB + AC$

  $(A+B)C = AC + BC$

  (distributive law)

A, B, C matrices

Inverse Matrix.

<u>Def.</u> A square ($n\times n$) matrix A is invertible if there is a matrix A^{-1} s.t. $AA^{-1} = A^{-1}A = I$.

- not all matrices are invertible.
- if A is invertible then it only has <u>one</u> inverse matrix.

suppose A^{-1}, $\tilde{A}^{-1}$ are both inverses of A.

$$A^{-1}A\tilde{A}^{-1} = (A^{-1}A)\tilde{A}^{-1} = I\tilde{A}^{-1} = \tilde{A}^{-1}$$
$$= A^{-1}(A\tilde{A}^{-1}) = A^{-1}I = A^{-1}$$

- $(AB)^{-1} = B^{-1}A^{-1}$

why? check: $(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

By def'n, $B^{-1}A^{-1}$ is the inverse of AB.

Also: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$!

Which matrices have an inverse?

- Suppose A is invertible
  $(n \times n)$

"therefore" $\Downarrow$

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

- $Ax=b$ has a <u>unique</u> sol for every b.

$\Downarrow$

take $b=0$

- $Ax=0$ has only the zero solution $x=0$.

$\Downarrow$

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \\ 0 & & 0 \ \ddots \ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

unknowns w/o pivots are free, so we get infinitely many sol's.

- n pivots in Gauss Elimination

$\Downarrow$ → Gauss-Jordan Elimination.

- A is invertible

Gauss-Jordan Elimination is a procedure to invert a matrix.

0) Write $[A \mid I]$

1) Apply Gauss Elim. to $[A \mid I]$

2) If you get $< n$ pivots, A is not invertible.
If you get n pivots, use the pivots to eliminate entries <u>above</u> them.

3) Divide each row by its pivot.

The result of this process is a matrix $[I \mid A^{-1}]$.

P.S. (2), (3) can be swapped.

Ex. $A = \begin{bmatrix} 2 & 3 \\ 10 & 11 \end{bmatrix}$

product of pivots = determinant of A

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 10 & 11 & 0 & 1 \end{array}\right] \xrightarrow[(1)]{R_2 \to R_2 - 5R_1} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{array}\right] \xrightarrow[R_2 = \frac{1}{-4} R_2 \ (2)]{R_1 \to \frac{1}{2} R_1}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{4} & \frac{-1}{4} \end{array}\right] \xrightarrow[(2)]{R_1 \to R_1 - \frac{3}{2} R_2} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{11}{8} & \frac{3}{8} \\ 0 & 1 & \frac{5}{4} & \frac{-1}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{-11}{8} & 3/8 \\ 5/4 & -1/4 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 11 & -3 \\ -10 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex. $\begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix} = A$. Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 7 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -5 & -3 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2} R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

go bottom to top, eliminate these.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 + 4R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -9 & -6 & 2 \\ 0 & 1 & 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & 1 & -5/2 & -3/2 & \frac{1}{2} \end{array}\right] \leftarrow A^{-1}$$

Why this works?

Row operations can be performed by left multiplication by special matrices E (elementary) (see last class)

left half of G-J process:

$$A \to E_1 A \to E_2 E_1 A \to \dots \to \overbrace{E_k \cdots E_1}^{A^{-1}} A = I$$

right half of G-J process:

$$I \to E_1 I \to E_2 E_1 I \to \dots \to \underbrace{E_k \cdots E_1 I}_{A^{-1} I = A^{-1}}$$