MA 232 Linear Algebra HW Assignment 1 Due: September 16, 2022 by 5:00 pm

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Pledge and Sign: I pledge my honor that I have abided by the Stevens Honor System

Upload solutions to Gradescope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. Consider the vectors $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) [4 pts.] In the xy-plane mark all nine linear combinations $c\vec{\mathbf{v}} + d\vec{\mathbf{w}}$, with $c = -2, 0, 2$, and $d = 0, 1, 2$.

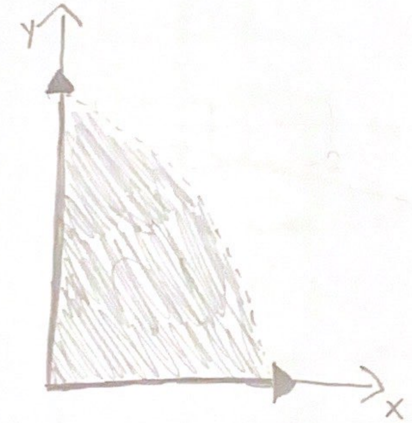
SEE GRAPH ON NEXT PAGE FOR SOLN.

(b) [3 pts.] What shape do all linear combinations $c\vec{\mathbf{v}} + d\vec{\mathbf{w}}$ fill? A line? The whole plane? Are the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ "independent"?

(c) [3 pts.] Restricted to only $c \geq 0$ and $d \geq 0$ draw the "cone" of all linear combinations $c\vec{\mathbf{v}} + d\vec{\mathbf{w}}$.

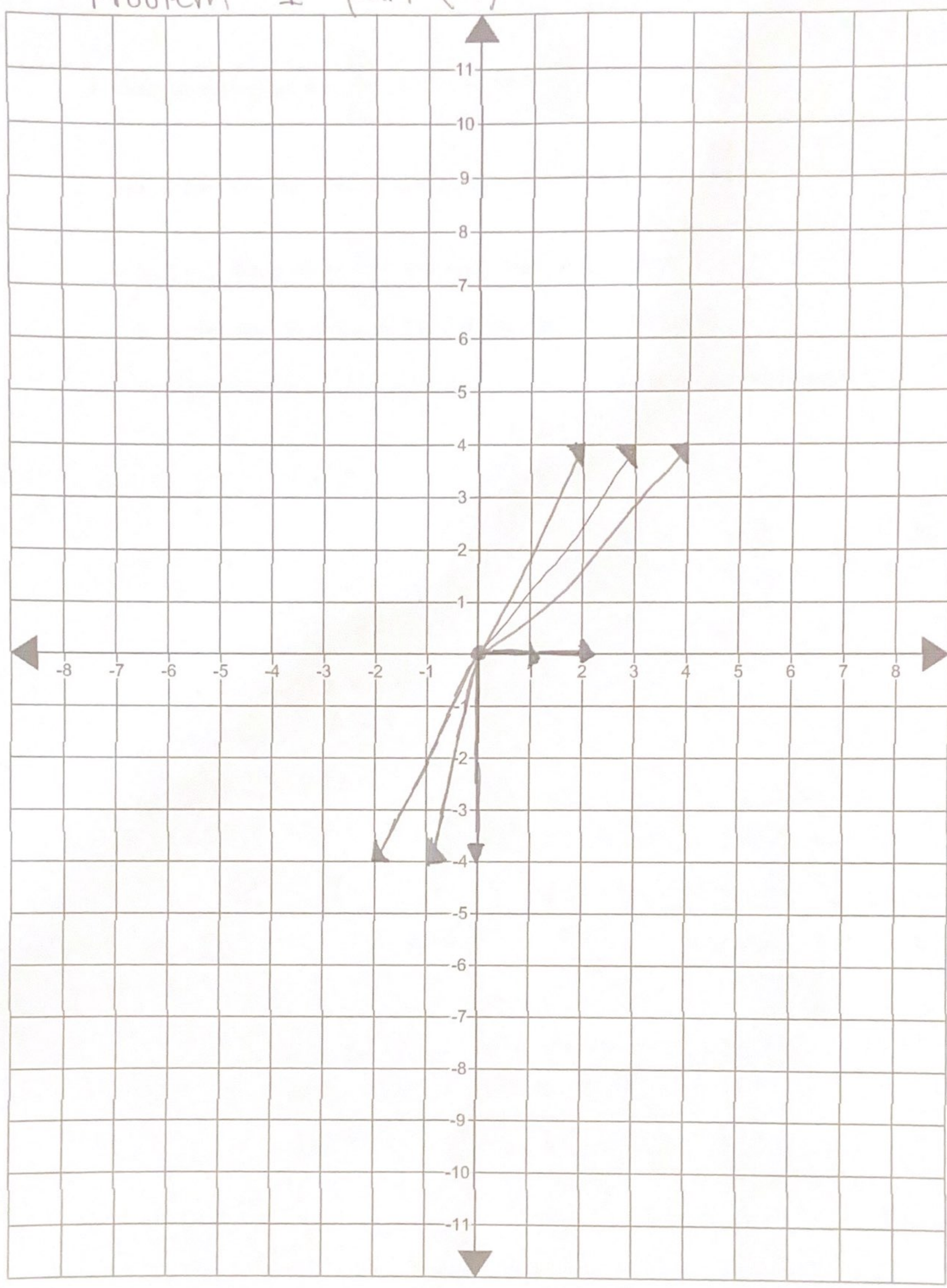
(b) All linear combinations $c\vec{v} + d\vec{w}$ fill the whole xy-plane. Vector $\vec{v}$ is not a scalar multiple of $\vec{w}$, nor vice versa, thus they are independent.

(C)



The shaded area represents all linear combinations for $c, d \geq 0$. They span the entirety of the first quadrant in the xy-plane, including the positive x-axis, positive y-axis, and the origin.

Problem 1 part (a)



2. Consider the vectors $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(a) [3 pts.] Can you solve the system $x\vec{\mathbf{u}} + y\vec{\mathbf{v}} + z\vec{\mathbf{w}} = \vec{\mathbf{b}}$, if $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

(b) [3 pts.] What if $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? How many solutions are there?

(c) [2 pts.] Are the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ dependent or independent?

(d) [2 pts.] Use parts (a)-(c) to decide if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is an invertible matrix or not.

(a) $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$x + 2z = 0$

$-y + z = 0 \rightarrow y = z$

$x + y = 1$ $\quad y = -1$

$x + z = 1 \rightarrow x = 1 - z$

$x + 2z = 0$ $\quad x = 1 - (-1)$

$\boxed{x = 2}$

$(1 - z) + 2z = 0$

$1 + z = 0$

$\boxed{z = -1}$

$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Yes, a solution exists where $x = 2$, $y = -1$, and $z = -1$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$ $\begin{cases} x + 2z = 0 \\ z - y = 0 \rightarrow z = y \\ x + y = 0 \end{cases}$

$x + 2y = 0$ $\quad x = -2y$

$x + y = 0$ $\quad x = -y$

$x = 0$ $\quad -2y = -y$

$z = 0$ $\quad y = 0$

There is only one solution when $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where $x = y = z = 0$

(c) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Since we found above that the only solution to $Ax = 0$ is when x is a zero vector, the vectors u, v, and w are linearly independent

(d) Since the vectors are linearly independent, their coefficient vector A must be invertible

3. Consider the linear system for some constants b and g:

$$\begin{aligned} x \; -2y \; +3z &= 3 \\ 2x \; +y \; +bz &= -4 \\ x \; +0y \; +1z &= g \end{aligned}$$

(a) [4 pts.] Use Gauss elimination to figure out what constant b makes the system singular (fewer pivots than unknowns).

(b) [4 pts.] For the value of b found in Part (a), for which values of g, the system has infinitely many solutions?

(c) [2 pts.] Find two distinct solutions of the system for that g.

(a) $x-2y+3z=3$
$2x+y+bz=-4$
$x+z=g$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & b & -4 \\ 1 & 0 & 1 & g \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 1 & 0 & 1 & g \\ 2 & 1 & b & -4 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 2 & -2 & g-3 \\ 2 & 1 & b & -4 \end{array}\right] \xrightarrow{R_3-2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 2 & -2 & g-3 \\ 0 & 5 & b-6 & -10 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -1 & \frac{g-3}{2} \\ 0 & 5 & b-6 & -10 \end{array}\right] \xrightarrow{R_1+2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & g+2 \\ 0 & 1 & -1 & \frac{g-3}{2} \\ 0 & 5 & b-6 & -10 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & g+2 \\ 0 & 1 & -1 & \frac{g-3}{2} \\ 0 & 0 & b-1 & -10 \end{array}\right]$$

$b-1=-10$

$\boxed{b=-9}$

(b)