Name: ***Solutions***

**1.** Let

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}.$$

(a) Find eigenvalues and eigenvectors of $A$.

(b) Diagonalize $A$.

(c) Use diagonalization to find a formula for arbitrary powers $A^n$ of $A$.

***Solution***:

(a) $\begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = (-1-\lambda)(4-\lambda) - 2(-3) = \lambda^2 - 3\lambda - 4 + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$, so the eigenvalues are 1 and 2. It remains to find the eigenvectors.

For $\lambda = 1$, we get:

$$\begin{bmatrix} -1-1 & 2 \\ -3 & 4-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$
$$\begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

We see that the solutions space has dimension 1 (we knew that anyway, since $\lambda = 1$ is not a multiple eigenvalue). Therefore, there is only one independent eigenvector, which we can get by picking any nonzero solution, for example $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, we get:

$$\begin{bmatrix} -1-2 & 2 \\ -3 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$
$$\begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Same as in the case of the other eigenvalue, we get one independent eigenvector, for example $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) We have $A = X\Lambda X^{-1}$, where $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and $X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. We only need to find

$$X^{-1} = \frac{1}{\det X} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

We get

$$\underbrace{\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}}_{X^{-1}}.$$

(c) We get

$$
\begin{aligned}
A^n &= (X\Lambda X^{-1})^n = X\Lambda^n X^{-1} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^n \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2^n & 2^n \end{bmatrix} \\
&= \begin{bmatrix} 3 - 2^{n+1} & -2 + 2^{n+1} \\ 3 - 3 \cdot 2^n & -2 + 3 \cdot 2^n \end{bmatrix}.
\end{aligned}
$$

**2.** Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

The characteristic polynomial is $\det(A - \lambda I) = (\lambda + 1)^2(5 - \lambda)$.

(a) Find all eigenvalues and independent eigenvectors.

(b) Is $A$ diagonalizable? Why?

***Solution***: From the given characteristic polynomial, we get an eigenvalue $\lambda = 5$, and a double eigenvalue $\lambda = -1$.

For $\lambda = 5$, we write the matrix of the resulting linear system and row reduce it:

$$\begin{bmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{bmatrix} \xrightarrow{R_{1,2,3}/2} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2+R_1}$$
$$\rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow[R_2/(-3)]{R_3-R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the latter matrix we see that we get one independent eigenvector, for example $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we do the same:

$$\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

from which we immediately see that there are two independent eigenvectors, for example $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$A$ is diagonalizable, since it has 3 linearly independent eigenvectors: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

**3.** Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Is $A$ diagonalizable?

***Solution***: Find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (-\lambda+1)(-\lambda-1)+1 = \lambda^2 - 1 + 1 = \lambda^2.$$

We have the only (double) eigenvalue $\lambda = 0$. Find the corresponding eigenvectors:

$$\begin{bmatrix} 1-0 & -1 \\ 1 & -1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

The solution space is one-dimensional, it is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since there are no two independent eigenvectors, the matrix is not diagonalizable.

**4.** Two square matrices $A$ and $C$ are said to be **similar**, if there exists an invertible matrix $B$, such $BAB^{-1} = C$. Show that the matrices $A$ and $C$ have the same eigenvalues, and if $\mathbf{x}$ is an eigenvector of $A$, then $B\mathbf{x}$ is an eigenvector of $C$.
***Solution***: We'll show the second part and get the first part as a byproduct.

Suppose $\mathbf{x}$ is an eigenvector of $A$, that is $A\mathbf{x} = \lambda\mathbf{x}$. Then compute

$$C(B\mathbf{x}) = BAB^{-1}(B\mathbf{x}) = BA\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x},$$

so $C(B\mathbf{x}) = \lambda\mathbf{Bx}$, that is, $B\mathbf{x}$ is a eigenvector of $C$ with the same $\lambda$.

In particular, note that the eigenvalues of $A$ and $C$ are the same. In fact, even stronger statement is true: $A$ and $C$ have the same exact characteristic polynomial! Indeed, let's look at the characteristic polynomial of $C = BAB^{-1}$:

$$\begin{aligned} \det(C - \lambda I) &= \det(BAB^{-1} - \lambda I) = [\text{since } \lambda I = B\lambda I B^{-1}] \\ &= \det(BAB^{-1} - B\lambda I B^{-1}) \\ &= \det(B(A - \lambda I)B^{-1}) \\ &= \det(B)\det(A - \lambda I)\det(B^{-1}) \\ &= \det(B)\det(B^{-1})\det(A - \lambda I) \\ &= \det(A - \lambda I). \end{aligned}$$

**5.** (a) Show that if $\mathbf{q}_1, \ldots, \mathbf{q}_k$ is an orthonormal system of vectors in $\mathbb{R}^n$, then the projection of a vector $\mathbf{b}$ onto the space spanned by $\mathbf{q}_1, \ldots, \mathbf{q}_k$ can be found as

$$\mathbf{p} = c_1\mathbf{q}_1 + \ldots + c_k\mathbf{q}_k,$$

where $c_1 = \mathbf{b} \cdot \mathbf{q}_1$, $c_2 = \mathbf{b} \cdot \mathbf{q}_2$, and so on.

***Solution***: Write

$$\mathbf{b} = \underbrace{c_1\mathbf{q}_1 + \ldots + c_k\mathbf{q}_k}_{\mathbf{p}} + \mathbf{e}$$

and take dot product with $\mathbf{q}_1$ (remember that $\mathbf{q}_1, \ldots, \mathbf{q}_k$ and that all of them are orthogonal to $\mathbf{e}$):

$$\mathbf{b} \cdot \mathbf{q}_1 = c_1 + c_2 0 + \ldots + c_k 0 + 0,$$

which immediately tells us what $c_1$ is. The other coefficients are found similarly.

(b) Follow (a) to find the projection of $(1, 2, 3, 4)$ onto the space spanned by $\frac{1}{2}(1, 1, -1, -1)$ and $\frac{1}{2}(-1, 1, -1, 1)$.

***Solution***: Notice that $\frac{1}{2}(1, 1, -1, -1)$ and $\frac{1}{2}(-1, 1, -1, 1)$ make an orthonormal system. With item (a) in mind, we get that

$$\mathbf{p} = c_1 \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + c_2 \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix},$$

where $c_1 = (1, 2, 3, 4) \cdot \frac{1}{2}(1, 1, -1, -1) = \frac{1}{2}(1 + 2 - 3 - 4) = -2$, and $c_2 = (1, 2, 3, 4) \cdot \frac{1}{2}(-1, 1, -1, 1) = \frac{1}{2}(-1 + 2 - 3 + 4) = 1$, so

$$\mathbf{p} = \frac{-2}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}.$$