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MA503: Homework 4

Exercise 4.1. [20pts] Consider a Cartesian product $G = \mathbb{Z} \times \mathbb{Z} = \{ (\alpha, x) \mid \alpha, x \in \mathbb{Z} \}$ and a binary operation $\cdot$ on G defined as follows:

$$(\alpha_1, x_1) \cdot (\alpha_2, x_2) = (\alpha_1 + \alpha_2, (-1)^{\alpha_2} x_1 + x_2)$$

(1) [8pts] Prove that $(G, \cdot)$ is a group.
(2) [2pts] Is $(G, \cdot)$ abelian?
(3) [2pts] Is $(G, \cdot)$ finite?
(4) [2pts] Prove that every cyclic group is abelian. Then use (2) to prove that $(G, \cdot)$ is not cyclic.
(5) [2pts] Does $(G, \cdot)$ have torsion?
(6) [2pts] Is $\pi_1 : G \to \mathbb{Z}$ defined by $(\alpha, x) \overset{\pi_1}{\mapsto} \alpha$ a homomorphism?
[I want to emphasize that G is not the direct product of $\mathbb{Z}$ and $\mathbb{Z}$.]
(7) [2pts] Is $\pi_2 : G \to \mathbb{Z}$ defined by $(\alpha, x) \overset{\pi_2}{\mapsto} x$ a homomorphism?

Exercise 4.2. [5pts] Find $|2|$ in U_{67}.

Exercise 4.3. [5pts] Is 2 a primitive root modulo 31?

Exercise 4.4. [10pts] Consider a set $G = \{x_1, x_2, \ldots, x_8\}$ of eight elements equipped with a binary operation $\cdot$ defined by the multiplication table shown below. $(G, \cdot)$ is a group.

$\cdot$	x_4	x_3	x_7	x_1	x_2	x_6	x_5	x_8
x_4	x_2	x_6	x_5	x_8	x_4	x_3	x_7	x_1
x_3	x_6	x_4	x_8	x_7	x_3	x_2	x_1	x_5
x_7	x_5	x_1	x_4	x_6	x_7	x_8	x_2	x_3
x_1	x_8	x_5	x_3	x_4	x_1	x_7	x_6	x_2
x_2	x_4	x_3	x_7	x_1	x_2	x_6	x_5	x_8
x_6	x_3	x_2	x_1	x_5	x_6	x_4	x_8	x_7
x_5	x_7	x_8	x_2	x_3	x_5	x_1	x_4	x_6
x_8	x_1	x_7	x_6	x_2	x_8	x_5	x_3	x_4

(1) Which element is the identity of G?
(2) Is G abelian? Why?
(3) Find $|x_3|$.
(4) Find $\langle x_4 \rangle$.
(5) Find the coset $x_6 \cdot \langle x_4 \rangle$.
(6) Find x_5^{-1}.
(7) Is x_7 a primitive element?
(8) [3pts] Is G cyclic?