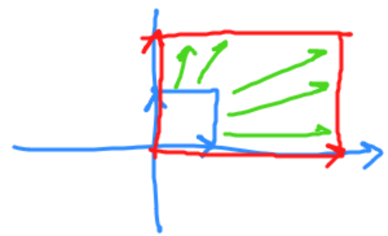
Eigenvalues and diagonalization

$$A = \begin{bmatrix}3 & \\ & 2\end{bmatrix}$$

$$A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3x\\2y\end{bmatrix}$$



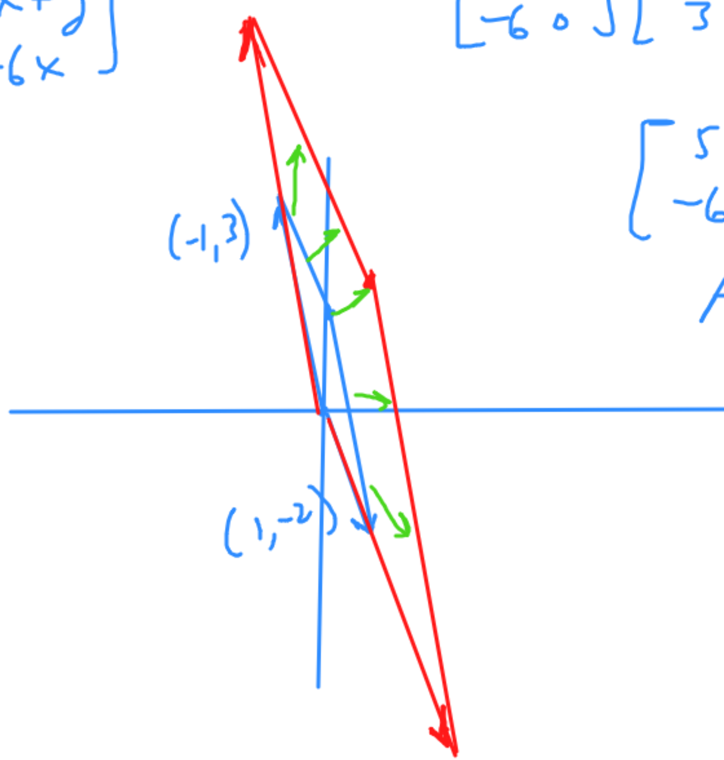
$$A = \begin{bmatrix}5 & 1\\-6 & 0\end{bmatrix}$$

$$A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5x+y\\-6x\end{bmatrix}$$

Notice: $\begin{bmatrix}5 & 1\\-6 & 0\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}3\\-6\end{bmatrix} = 3\begin{bmatrix}1\\-2\end{bmatrix}$

$\begin{bmatrix}5 & 1\\-6 & 0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix} = 2\begin{bmatrix}-1\\3\end{bmatrix}$

$$\underset{A}{\begin{bmatrix}5 & 1\\-6 & 0\end{bmatrix}}\underset{X}{\begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}} = \begin{bmatrix}3 & -2\\-6 & 6\end{bmatrix} = \underset{X}{\begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}}\underset{\Lambda}{\begin{bmatrix}3 & \\ & 2\end{bmatrix}}$$



$$\underline{AX = X\Lambda}$$

- $\begin{bmatrix}5&1\\-6&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}1&\\&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

  $\left(\begin{bmatrix}5&1\\-6&0\end{bmatrix} - 2\begin{bmatrix}1&\\&1\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = 0$

  $\begin{bmatrix}3&1\\-6&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$ $\quad\begin{bmatrix}3&1\\-6&-2\end{bmatrix} = \begin{bmatrix}3&1\\0&0\end{bmatrix}$ $\quad 3x+y=0$, for example $\begin{bmatrix}-1\\3\end{bmatrix}$.

- $\begin{bmatrix}5&1\\-6&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 4\begin{bmatrix}x\\y\end{bmatrix}$

  $\left(\begin{bmatrix}5&1\\-6&0\end{bmatrix} - 4\begin{bmatrix}1&\\&1\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = 0$

  $\begin{bmatrix}1&1\\-6&-4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$ $\quad$ the only soln is $\begin{bmatrix}0\\0\end{bmatrix}$.

Terminology.

Def. Let A be an $n \times n$ matrix. If $Ax = \lambda x$ for some vector x and number λ, then λ is called an eigenvalue (characteristic value) of A, and x is called an eigenvector (characteristic vector) of A.

1) How to find eigenvalues?
2) How to find eigenvectors?
3) How to diagonalize?

Answer to 1): we are looking for λ s.t. $A - \lambda I$ is singular. In other words, we are looking for λ s.t.

$$\boxed{\det(A - \lambda I) = 0.}$$

Ex.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad 0 = \det \begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix}.$$

$$(0.8-\lambda)(0.7-\lambda) - 0.3 \cdot 0.2 =$$

$$= \lambda^2 - \lambda(0.8+0.7) + 0.8 \cdot 0.7 - 0.3 \cdot 0.2 =$$

$$= \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\tfrac{1}{2})$$

solve

Eigenvalues: 1 and $\frac{1}{2}$.

In general: $\det(A-\lambda I)$ is a degree n polynomial, so it has $\leq n$ roots (possibly repeating). This polynomial is called the characteristic polynomial of A.

Ex.

$(\lambda-1)(\lambda+1)(\lambda+3)$ — 3 roots: $1, -1, -3$

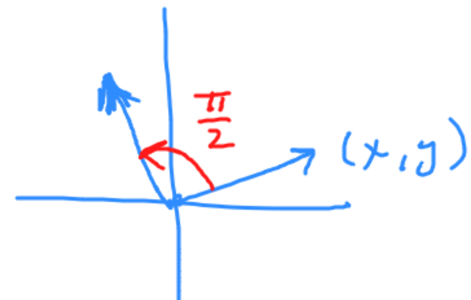
$(\lambda-1)^3$ — 3 roots: $1, 1, 1$

$(\lambda-1)(\lambda^2+1)$ — 1 root: 1

Ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

$\lambda^2 + 1 = 0$ no (real) solutions.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



Answer to 2): once we know eigenvalues, take each of them and solve $(A - \lambda I)x = 0$.

Ex. $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ $\quad \lambda = 1, \frac{1}{2}$.

A

$\lambda = \frac{1}{2}$ $\quad A - \frac{1}{2} I = \begin{bmatrix} 0.8 - 0.5 & 0.3 \\ 0.2 & 0.7 - 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$

$\begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ideally, we hope to get n linearly independent eigenvectors.

Problems: 1) we may not have enough eigenvalues
(ex. $(\lambda-1)(\lambda^2+1)$)

2) even if we do have n eigenvalues,
we still may have $< n$ eigenvectors
(can happen only if eigenvalues repeat).

Ex. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\det\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2$. two eigenvalues $\lambda = 2$.

$(A-2I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $y = 0$

solution space has dim $= 1$ (one free variable)

so there isn't two indep. eigenvectors,

only one: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Steps to find eigenvectors:

1) compute $\det(A-\lambda I)$
2) solve $\det(A-\lambda I)=0$ (get eigenvalues)
3) for each eigenvalue λ, solve $(A-\lambda I)x=0$.

<u>Notes</u>. If a matrix A has n eigenvalues then:

- the product of all eigenvalues is $\det A$
- the sum of all eigenvalues is the sum of diagonal entries of A

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix}$$

$$(-\lambda+\lambda_1)(-\lambda+\lambda_2)\cdots(-\lambda+\lambda_n)$$

$$\pm\lambda^{n-1}(\lambda_1+\lambda_2+\cdots+\lambda_n)$$

Why do we want to have n independent eigenvectors?
Because diagonalization! A is called diagonalizable

Suppose an $n \times n$ matrix has n lin. indep. eigenvectors

$x_1, x_2, \ldots, x_n$ with respective eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Put eigenvectors into columns of a matrix X. Then

$X^{-1}AX$ is the eigenvalue matrix of A:

$$X^{-1}AX = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \Lambda.$$

why: $AX = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix} = X\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$= X\Lambda$

$$AX = X\Lambda$$

$$X^{-1}AX = \Lambda \qquad A = X\Lambda X^{-1}$$

automatic if there n different eigenvalues

Ex. $\begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}^{100} =$

$$A = \underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 3 & \\ & 2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}}_{X^{-1}}$$

$$= (X \Lambda X^{-1})^{100} = (X \Lambda \cancel{X^{-1}})(\cancel{X} \Lambda \cancel{X^{-1}})(\cancel{X} \Lambda \cancel{X^{-1}}) \cdots (\cancel{X} \Lambda X^{-1})$$

$$= X \Lambda^{100} X^{-1} =$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3^{100} & \\ & 2^{100} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3^{101} & 3^{100} \\ 2^{101} & 2^{100} \end{bmatrix} =$$

$$= \begin{bmatrix} 3^{101} - 2^{101} & 3^{100} - 2^{100} \\ -2 \cdot 3^{101} + 3 \cdot 2^{101} & -2 \cdot 3^{100} + 3 \cdot 2^{100} \end{bmatrix}$$