Name: ***Solutions***

1. If a 4×4 matrix has $\det(A) = \frac{1}{4}$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$ and $\det(A^{-1})$.

Solution: Since A is 4×4, $\det(cA) = c^4 \det(A)$. In particular, $\det(2A) = 2^4 \det(A) = 16 \cdot \frac{1}{4} = \mathbf{4}$ and $\det(-A) = (-1)^4 \det(A) = \mathbf{\frac{1}{4}}$.

Since $\det(AB) = \det(A)\det(B)$, we get $\det(A^2) = \det(A)\det(A) = (\det(A))^2 = \mathbf{\frac{1}{16}}$.

Since $\det(AB) = \det(A)\det(B)$, and therefore $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$, we get $\det(A^{-1}) = (\det(A))^{-1} = \mathbf{4}$.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$.

(a) Find $\det(A)$ by reducing A to its upper-triangular form U.

(b) Find $\det(A)$ using the cofactor formula.

Solution:

(a) Recall that row subtractions do not change the determinant and swaps (which we are not doing in this instance) change the sign of determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix} \xlongequal[R_3-3R_1]{R_2-2R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{vmatrix} \xlongequal{R_3-\frac{3}{2}R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{vmatrix} = 1 \cdot (-2) \cdot (-\frac{3}{2}) = 3.$$

(b) Use cofactors in row 1:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} + 3 \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = -3 + 2 \cdot 3 + 0 = 3.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

(a) Find A^{-1} using the cofactor formula $A^{-1} = C^T / \det(A)$.

(b) Use Cramer's rule to solve $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. [You already found $\mathbf{x}$ in part (a)!]

Solution:

(a) Compute the cofactors:

$$\begin{aligned}
C_{11} &= +\begin{vmatrix} 3 & 0 \\ 7 & 1 \end{vmatrix} = 3, & C_{12} &= -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, & C_{13} &= +\begin{vmatrix} 0 & 0 \\ 3 & 7 \end{vmatrix} = 0, \\
C_{21} &= -\begin{vmatrix} 2 & 0 \\ 7 & 1 \end{vmatrix} = -2, & C_{22} &= +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, & C_{23} &= -\begin{vmatrix} 1 & 2 \\ 0 & 7 \end{vmatrix} = -7, \\
C_{31} &= +\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0, & C_{32} &= -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, & C_{33} &= +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3.
\end{aligned}$$

Therefore,

$$C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix}.$$

Finally, we are going to need $\det(A)$. We can either notice that a matrix where only one column has off-diagonal nonzero entries has determinant equal to the product of diagonal entries (why?), in our case $\det A = 1 \cdot 3 \cdot 1 = 3$; or we can use the cofactor formula, especially since we already did the work computing cofactors:

$$\det A = 1C_{11} - 2C_{12} + 0C_{13} = 3 - 0 + 0 = 3.$$

This gets us (don't forget to transpose C!)

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}.$$

(b) We already solved it: the answer is the middle column of A^{-1}, since in the product with A it will produce the middle column of I, that is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the answer is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ -7/3 \end{bmatrix}$. (Don't forget the 1/3 in the entries!)

But even if we didn't notice this, we can just use Cramer's formula.

$$\begin{aligned}
x_1 &= \frac{\det B_1}{\det A} = \frac{C_{21}}{\det A} = \frac{-2}{3}, \\
x_2 &= \frac{\det B_2}{\det A} = \frac{C_{22}}{\det A} = \frac{1}{3}, \\
x_3 &= \frac{\det B_3}{\det A} = \frac{C_{23}}{\det A} = \frac{-7}{3}.
\end{aligned}$$