Determinants.

"Determinant is volume"

When edges of a box are rows of an $n \times n$ matrix A then the signed n-dimensional volume of that box is the determinant of A.

1-dim vol.

Ex. $n=1$

$\mathbb{R}^1$ singned length of $[a]$ is a

2-dim volum

$n=2$, $\mathbb{R}^2$

signed ~~Area~~ of box on (a,b), (c,d)

is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

3-dim vol.

$n=3$

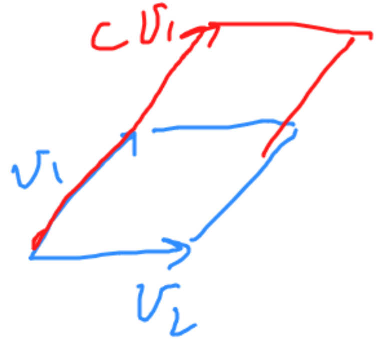
Volume of box is

$$\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{vmatrix}$$

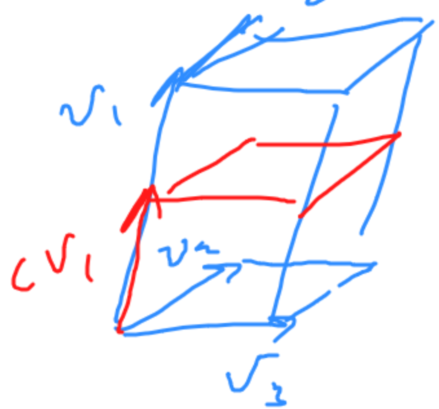
What properties should n-dim volume have?

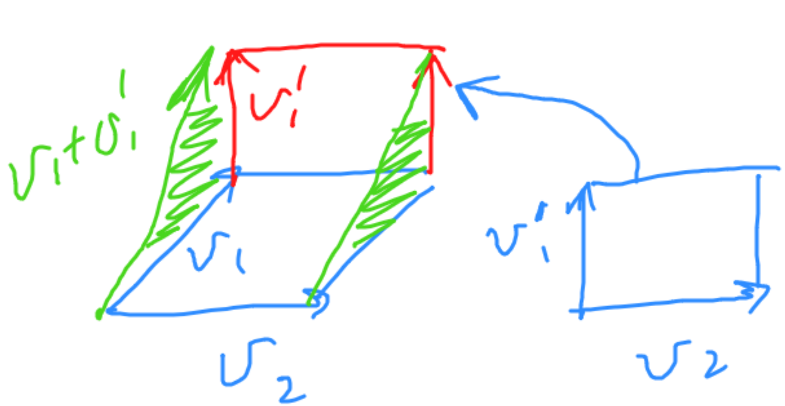
1) Box on $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, $\dots$ $(0,\dots,0,1)$ has volume $1\cdot 1\cdot\dots\cdot 1 = 1$.

2)



Volume on cv_1, v_2 =
= c Volume on v_1, v_2

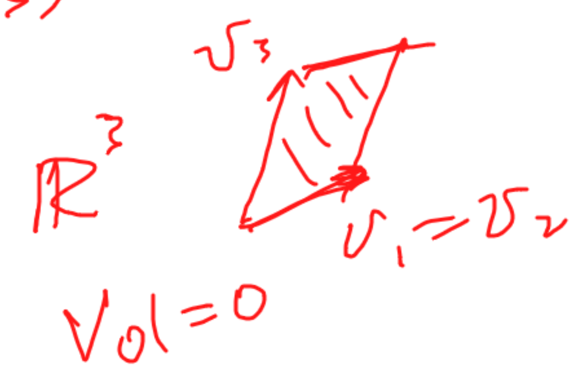




Volume on $v_1+v_1', v_2 =$
$=$ Volume on v_1, v_2 $+$
$+$ Volume on v_1', v_2.

Volume is a linear function of each vector separately.

3) If two vectors are the same then n-dim Vol $= 0$.



Properties of determinant.

determinant: a number associated to <u>square</u> matrix A. Notation: $\det A$, $|A|$.

1) $\det I = 1$

2) det is a linear function of each row separately:

- $\begin{vmatrix} al & bl \\ c & d \end{vmatrix} = l \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

3) det changes sign when two rows are swapped.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}.$$

Consequences:

4) If a row repeats, det $= 0$: $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ a & b \end{vmatrix}$

5) det stays unchanged if subtract a multiple of one row from another row.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} =$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Ex.

$$\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} \xrightarrow[\text{upward}]{\text{eliminat}} \begin{vmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 6 \end{vmatrix} =$$

$$= 1 \cdot 2 \cdot 3 \cdot 6 \underbrace{\begin{vmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{vmatrix}}_{=1} = 36$$