Name: ***Solutions***

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find the full SVD for A. Find the pseudoinverse A^+ of A.

Solution:

Step 1. The first step is to compute A^TA and find its eigenvalues and eigenvectors.

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

To compute the eigenvalues, we find the characteristic polynomial (note that it will have root 0, since is matrix is singular, since it has the same rank as A, that is 2).

$$\begin{aligned} \det(A^TA - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \text{(cofactors in first row)} \\ &= (1-\lambda)((2-\lambda)(1-\lambda)-1) - 1((1-\lambda)-0) \\ &= (1-\lambda)(\lambda^2 - 3\lambda + 2 - 1 - 1) = (1-\lambda)(3-\lambda)\lambda. \end{aligned}$$

Nonzero eigenvalues therefore are $\lambda_1 = 3$ and $\lambda_2 = 1$. Respectively, the singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = 1$. Now we find the respective eigenvectors. They are going to be singular vectors v_1, v_2 for A.

For $\lambda_1 = 3$, we get

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix} v_1 = 0,$$
$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} v_1 = 0,$$

solving which we get v_1 proportional to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Since we want a vector of unit length, we take $v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

For $\lambda_1 = 1$, we get

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} v_2 = 0,$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} v_2 = 0,$$

solving which we get v_2 proportional to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Since we want a vector of unit length, we take $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Step 2. Now u_1, u_2 are computed from $Av_i = \sigma_i u_i$.

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

At this point we now have equality

$$\begin{aligned} A &= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \\ &= \sqrt{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}. \end{aligned}$$

If we want the full SVD $A = U\Sigma V^T$ (which we do), then we proceed to step 2.

Step 3. We have to complete u_1, u_2 to an orthonormal basis of $\mathbb{R}^2$ (2 is the respective dimension of A). However, we already have two vectors u_1, u_2, so there is nothing to further to do with u_i's.

Step 4. We have to complete v_1, v_2 to an orthonormal basis of $\mathbb{R}^3$ (3 is the respective dimension of A). Solve system given by $v \cdot v_1 = 0$, $v \cdot v_2 = 0$ (we omit square roots since they don't change equality of the respective dot products to 0):

$$\begin{aligned} x + 2y + z &= 0, \\ x - z &= 0. \end{aligned}$$

Generally, we have to solve this and construct an orthonormal basis of the solution space. In this case, since the solution space is 1-dimensional, we only have to normalize a solution. Solving the system, we get solution space $\begin{bmatrix} z \\ -z \\ z \end{bmatrix}$.

For a length 1 vector v_3 we can take $v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Organizing vectors u_1, u_2 into columns of a matrix U, and vectors v_1, v_2, v_3 into columns of a matrix V, we obtain full SVD:

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$

To find the pseudoinverse A^+, we find Σ^+ (we invert singular values and transpose A):

$$\Sigma^+ = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and obtain the product:

$$\begin{aligned} A^+ &= V\Sigma^+U^T = \\ &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}. \end{aligned}$$

Out of curiosity, actually perform the multiplication.

$$\begin{aligned} A^+ &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1/3\sqrt{2} & 1/\sqrt{2} \\ 2/3\sqrt{2} & 0 \\ 1/3\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6}+\frac{1}{2} & \frac{1}{6}-\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6}-\frac{1}{2} & \frac{1}{6}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}. \end{aligned}$$

Remark 1. In Steps 1 and 2, we could hold off on normalizing: for example, take $v_1' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $u_1' = Av_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, and then normalize after that: $v_1 = \frac{v_1'}{\|v_1'\|}$ and $u_1 = \frac{u_1'}{\|u_1'\|}$. The result is the same, but that way we avoid dragging square roots around.

Remark 2. We also could have worked with AA^T in Step 1, obtaining u_1, u_2; in Step 2 we would then get v_i by applying A^T to u_i. For illustration purposes, let's the first two steps that way now.

Step 1. $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

$$\det(AA^T - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1).$$

(Note the nonzero eigenvalues of A^TA and AA^T are the same.) We get eigenvectors:

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} u_1 = 0, \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u_1 = 0, \quad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} u_2 = 0, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u_2 = 0, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Notice the same result in Step 2 in the original solution.

Step 2. Now find v_i from $A^T u_i = \sigma_i v_i$:

$$v_1 = \frac{1}{\sigma_1} A^T u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$v_2 = \frac{1}{\sigma_2} A^T u_2 = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Again, note the same result as in Step 1 in the original solution. From this point, Step 3 and Step 4 are the same.