MA 232 Linear Algebra **HW 2** **Due: September 23, by 5:00 pm**

Name (Printed): Peter Rauscher

Pledge and Sign: I pledge my honor that I have abided by the Stevens Honor System

Upload solutions to Gradescope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. (a) [6 pts.] Find all matrices $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, which commute with the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, that is, find all 2 by 2 matrices B, such that $AB = BA$.
[Hint: Write 4 equations, one for each entry of $AB = BA$.]

(b) [4 pts.] For nonzero matrices A, B and C, is it true that $AC = BC$ implies $A = B$? Either prove the statement, or find an example, where the statement fails.
[Hint: Consider matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$].

1a) $AB = BA$

$$\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}x&y\\z&t\end{bmatrix} = \begin{bmatrix}x&y\\z&t\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1\cdot x+2\cdot z & 1\cdot y+2t\\ 0\cdot x+1\cdot z & 0\cdot y+t\cdot 1\end{bmatrix} = \begin{bmatrix}x\cdot 1+y\cdot 0 & x\cdot 2+y\cdot 1\\ z\cdot 1+t\cdot 0 & z\cdot 2+t\cdot 1\end{bmatrix}$$

$$\begin{bmatrix}x+2z & y+2t\\ z & t\end{bmatrix} = \begin{bmatrix}x & 2x+y\\ z & 2z+t\end{bmatrix}$$

$$\begin{bmatrix}x+2z-x & y+2t-2x+y\\ z-z & 2z-t\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$x+2z-x=0$ $\quad 2y+2t-2x=0$ $\quad z-z=0$ $\quad 2z-t=0$

$2z=0$ $\quad 2y+2x=0$ $\quad 0-t=0$

$z=0$ $\quad y+x=0$ $\quad t=0$

$$\boxed{B=\begin{bmatrix}-y&-x\\0&0\end{bmatrix} \text{ for any } x, y}$$

1b) $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\quad B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $\quad C=\begin{bmatrix}2&2\\2&2\end{bmatrix}$

$$AC=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}2&2\\2&2\end{bmatrix}=\begin{bmatrix}2&2\\0&0\end{bmatrix}$$

So $AC=BC$

$$BC=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}2&2\\2&2\end{bmatrix}=\begin{bmatrix}2&2\\0&0\end{bmatrix}$$

Here with $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, and $C=\begin{bmatrix}2&2\\2&2\end{bmatrix}$ we can see that $AC=BC$ but $A\neq B$, therefore $AC=BC$ does not imply $A=B$.

2)
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & -3 & 0 & 1 & 0 & 0 \\ 1 & 4 & 5 & 5 & 0 & 0 & 1 & 0 \\ 3 & 6 & 9 & 14 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 4 & 5 & 5 & 0 & 0 & 1 & 0 \\ 3 & 6 & 9 & 14 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 6 & 9 & 14 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4/2} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \frac{5}{2} & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \frac{9}{2} & 2 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_1 \to R_1 - 4R_4} \begin{bmatrix} 1 & 2 & 3 & 0 & 7 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & \frac{9}{2} & 2 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - (3)R_3} \begin{bmatrix} 1 & 2 & 0 & 0 & 13 & 3 & -3 & -2 \\ 0 & 2 & 0 & 0 & \frac{9}{2} & 2 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{17}{2} & 1 & -2 & -\frac{3}{2} \\ 0 & 2 & 0 & 0 & \frac{9}{2} & 2 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2/2} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{17}{2} & 1 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{9}{4} & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{17}{2} & 1 & -2 & -\frac{3}{2} \\ \frac{9}{4} & 1 & -\frac{1}{2} & -\frac{1}{4} \\ -2 & -1 & 1 & 0 \\ -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$

3)

$A = LU$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\boxed{A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

$A = LDU$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$