Name: ***Solutions***

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1. A system of linear equations can't have exactly two solutions. Why?

(a) Indeed, given two distinct solutions $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ of $A\mathbf{x} = \mathbf{b}$, find a third solution.

(b) If four planes meet at two points, where else do they meet?

Solution: (a) Assume the system is $Ax = b$. Then $Av = b$ and $Aw = b$.

Consider the vector $\frac{v+w}{2}$. For this vector, we have

$$A\frac{v+w}{2} = \frac{1}{2}A(v+w) = \frac{1}{2}(Av + Aw) = \frac{1}{2}(b+b) = b,$$

So $\frac{v+w}{2}$ is a third solution. More generally, we can take any combination $tv + (1-t)w$ (where t is a real number). For example, for $\frac{v+w}{2}$ we take $t = \frac{1}{2}$. Do the computation for $tv + (1-t)w$:

$$A(tv + (1-t)w) = A(tv) + A((1-t)v) = tAv + (1-t)Aw = tb + (1-t)b = b.$$

Notice that $tv + (1-t)w$ is a parameterization of a straight line along the vector $v - w$. Indeed:

$$tv + (1-t)w = w + t(v-w),$$

which is a straight line through w in the direction of $v - w$.

(b) With item (a) in mind, if 4 planes (or indeed any number of planes) meet at two points, they must also meet on the straight line connecting those two points.

2. (a) Use Gaussian elimination to find all solutions.

$$\begin{array}{cccc} 2x & -3y & & = 3 \\ 4x & -5y & +z & = 7 \\ 2x & -y & +2z & = 5 \end{array}$$

(b) Write down all the elimination matrices. Check the first two row operations versus elimination by matrices.

Solution: (a) We have:

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & 2 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{array}\right].$$

From this we get $y = 1 - z$ and $2x = 3 + 3y$, so $x = 3 - \frac{3}{2}z$. This gets us the general form of a solution:

$$\begin{bmatrix} 3 - \frac{3}{2}z \\ 1 - z \\ z \end{bmatrix}.$$

(b)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}(-2), \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_{31}(-1), \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{32}(-2).$$

To check multiplication, compute the product:

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & 2 & 5 \end{array}\right]$$

and make sure it comes out equal to

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array}\right].$$

(Computation is omitted.)

3. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 3 & 2 \\ 0 & 3 & 3 \end{bmatrix}$.

(a) Find LU and LDU factorizations of A.

(b) Find A^{-1}.

Solution: (a) We start by reducing A to triangular form by row subtractions and additions (no swaps and no multiplication by a scalar):

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 3 & 2 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = U.$$

To find L, we inspect the transformations we performed along the way:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}(-2) \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = E_{32}(-3).$$

We have equality

$$E_{32}(-3)E_{21}(-2)A = U,$$

which means

$$L = (E_{32}(-3)E_{21}(-2))^{-1} = E_{21}(-2)^{-1}E_{32}(-3)^{-1} = E_{21}(2)E_{32}(3)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Putting everything together, we get

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}.$$

Notice that L, U inherited (some of) zeros of A.

For an LDU-decomposition, we bring out pivots from each row of the upper-triangular matrix:

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) To invert A, we run the usual Gauss–Jordan process:

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 6 & 3 & 2 & 0 & 1 & 0 \\ 0 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 6 & -3 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3/(-3)} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{3} \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{3} \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 1 & -\frac{2}{3} \\ 0 & 1 & 0 & 2 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{3} \end{array}\right] \xrightarrow{R_1 \to R_1/3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{9} \\ 0 & 1 & 0 & 2 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{3} \end{array}\right].$$

This gets us

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{2}{9} \\ 2 & -1 & \frac{2}{3} \\ -2 & 1 & -\frac{1}{3} \end{bmatrix}.$$

Notice that A^{-1} does not have any zeros, even though A did. LU decomposition behaves nicer in the presence of multiple zero entries (see (a)) than the A^{-1} does.