Transpose of a matrix A is denoted by A^T

Cols of A^T are rows of A.

<u>Ex.</u> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $(A^T)_{ij} = A_{ji}$

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T \leftarrow$ to see this
  - look at defn of AB
  - $(Ax)^T$ first.
- $(A^{-1})^T = (A^T)^{-1} \leftarrow I = AA^{-1}$

$$I = I^T = (AA^{-1})^T = \underbrace{(A^{-1})^T}_{\text{must be } (A^T)^{-1}} \cdot A^T$$

$$(ABC)^T = C^T B^T A^T$$

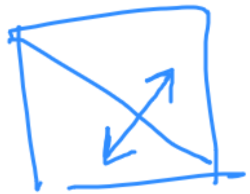
In particular: $A = LDU$, $A^T = U^T D^T L^T$

Note: if x, y are col. vectors then the dot product (inner product) $x \cdot y$ can be written as: $x^T y = x \cdot y$

$$\underbrace{\begin{bmatrix} & & & \end{bmatrix}}_{x^T}\underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{y} = x \cdot y$$

$$\underline{(Ax)} \cdot \underline{y} = (Ax)^T y = x^T A^T y = x^T (A^T y) = \underline{x} \cdot \underline{A^T y}.$$

Symmetric matrices



S is symmetric if

$$S^T = S, \quad S_{ij} = S_{ji}$$

Ex. $\begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$

Inverse of a symm. matrix is symmetric:

$$(S^{-1})^T = (S^T)^{-1} = S^{-1}.$$

For every matrix $\underset{n\times m}{A}$, the products

$$\overset{n\times n}{\underset{n\times m}{A}\,\underset{m\times n}{A^T}}, \quad \underset{m\times m}{A^T A}$$

are symmetric

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

$$S=\begin{bmatrix}1&4&5\\4&2&6\\5&6&3\end{bmatrix}\xrightarrow[R_3\to R_3-5R_1]{R_2\to R_2-4R_1}\begin{bmatrix}1&4&5\\0&-14&-14\\0&-14&-22\end{bmatrix}\xrightarrow{R_3\to R_3-R_2}$$

$$\to\begin{bmatrix}1&4&5\\0&-14&-14\\0&0&-8\end{bmatrix}=DU$$

$$S=\begin{bmatrix}1&&\\4&1&\\5&1&1\end{bmatrix}\begin{bmatrix}1&4&5\\0&-14&-14\\0&0&-8\end{bmatrix}=$$

$$=\underbrace{\begin{bmatrix}1&&\\4&1&\\5&1&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&&\\&-14&\\&&-8\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1&4&5\\0&1&1\\0&0&1\end{bmatrix}}_{U}$$

coincidence???

Why this happens?

For an invertible A, if $A = LDU$ where
L is lower triang. with 1's on the diag.,
D is diag.,
U is upper triang. with 1's on the diag.,
then such decomp. is unique!

$$A = LDU = L_1 D_1 U_1$$

$$I = L_1^{-1} L = D_1 U_1 U^{-1} D^{-1} \longrightarrow \begin{matrix} U_1 = U \\ D_1 = D \end{matrix}$$

lower triang

upper triang.

so $L = L_1$

$S = LDU$ so $L = U^T$ and $U = L^T$

$S = S^T = U^T D^T L^T$

$= U^T D L^T$

To find LDU decomp. of a symm. S:

1) Do row elimination to find DU (D diag., U upper triang with 1's on the diag)

2) L is automatically $= U^T$

For symm. matrices, instead of LDU-decomp., we usually say LDL^T-decomp.

Not all invertible matrices have an LU-decomp.!

Ex. $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 100 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P$

- do all swaps first: $PA = LU$
- do all swaps afterwards: $A = LPU$

what P look like?

$$P = \underbrace{\begin{bmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & 1 & & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}}_{n \times n}$$

n 1's, one in each col, in each row

P is called a permutation matrix.

feature: $P^{-1} = P^T$.

why do we want LU-decomp.?

$A = LU$

solve $Ax = b$

$L(Ux) = b$

1) solve $Ly = b$, get some $y = c$

2) solve $Ux = c$

triangular, easy to solve

Vector spaces and subspaces

A real vector space is a set of elements (called vectors) with rules for vector addition and for multiplication by a real number.

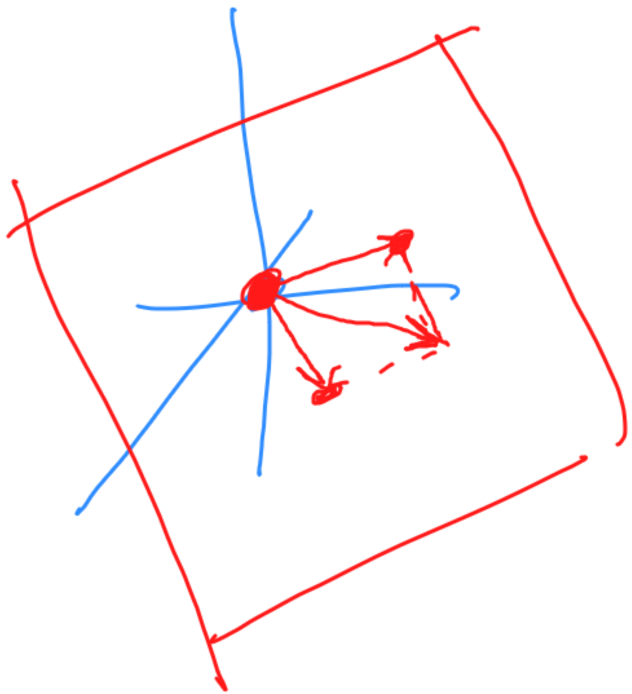
<u>Ex.</u> M vector sp. of 2×2 real matrices

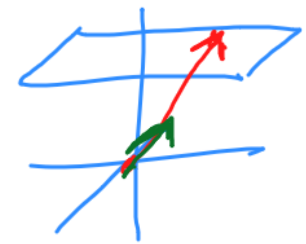
F vector sp. of all real functions $f(x)$.

$0, Z$ vector sp. that consists only of the zero vector

$\mathbb{R}^n$ vector sp. of all col. vectors with components.

Ex. A plane through $(0,0,0)$





Def A subspace of a vector space is a set of vectors (including 0) that satisfies the two reqs:

If v, w are vectors in the subspace, c any real number

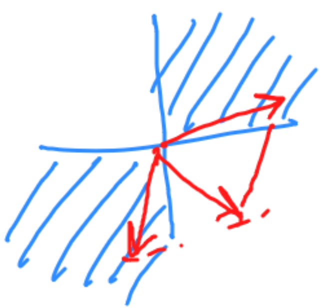
then (i) $v+w$ is in the subspace

(ii) cv is in the subspace.

In short: linear combinations stay in the subspace.

Ex.

$c = -1$ not a subspace



$(10,5)+(-6,-6)=(4,-1)$

not a subspace

All possible subspaces of $\mathbb{R}^3$:

L line through $(0,0,0)$

P plane through $(0,0,0)$

$\mathbb{R}^3$ whole space

0 zero subspace