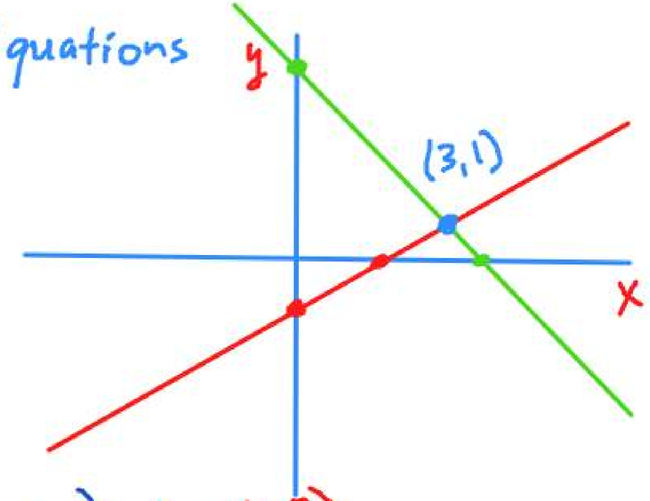
Matrices and Systems of Linear Equations

Ex. $\begin{cases} x-2y=1 \\ 2x+y=7 \end{cases}$

"row picture" $\longrightarrow$



$x-2y=1$

$\hookrightarrow$ $x=0 \quad y=-\frac{1}{2}$

$y=0 \quad x=1$

$2x+y=7$

$\hookrightarrow$ $x=0 \quad y=7$

$y=0 \quad x=7/2$

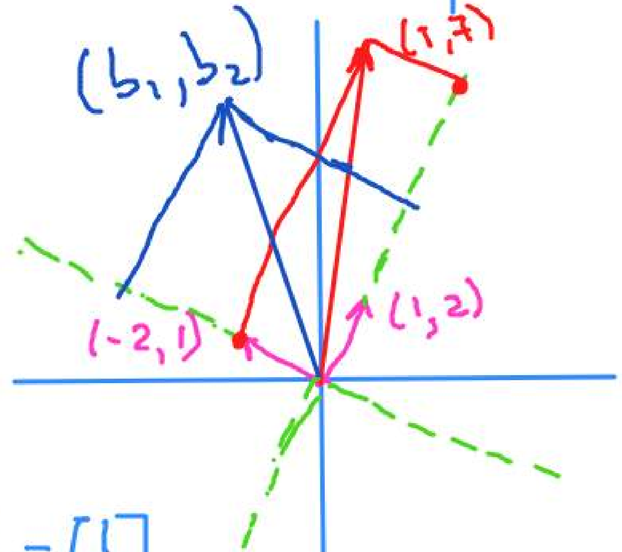
"column picture"

$\hookrightarrow$ $\begin{bmatrix} x-2y \\ 2x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$x=3$
$y=1$

$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$\begin{array}{cc} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\ x \quad y \end{array} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$



$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 6+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (1,-2)\cdot(3,1) \\ (2,1)\cdot(3,1) \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2 \text{ Identity Matrix, } I}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+0y \\ 0x+y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Consider $B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$

$$BA = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} (1,2)\cdot(1,2) \\ (-2,1)\cdot(1,2) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} (1,2)\cdot(-2,1) \\ (-2,1)\cdot(-2,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$2x = 5$$
$$\frac{1}{2}\cdot 2x = \frac{1}{2}\cdot 5$$
$$1x = \frac{5}{2}$$
$$x = \frac{5}{2}$$

$$\underbrace{\overbrace{\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}^{B}\cdot\overbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}^{A}}_{I}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$$

$$\|$$

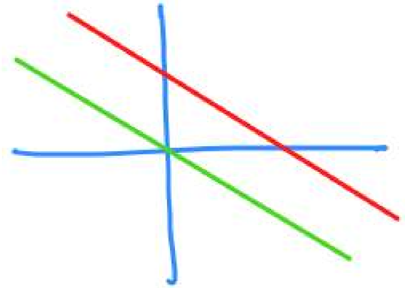
$$\begin{bmatrix} x \\ y \end{bmatrix}$$

B is called the inverse of A, denoted A^{-1}.

A is called invertible

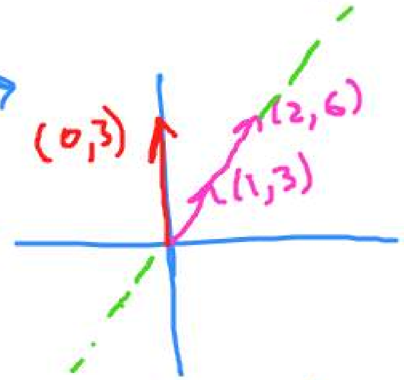
<u>Ex.</u> $\begin{cases} x+2y=0 \\ 3x+6y=3 \end{cases}$

row pic:



col. pic:

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}2\\6\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$



$A=\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$ is singular (non-invertible, degenerate).

How to tell? Columns of A (rows) are "linearly dependent":

$$2\begin{bmatrix}1\\3\end{bmatrix}+(-1)\begin{bmatrix}2\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}2-2\\6-6\end{bmatrix}=0$$

$A=\begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$: columns are linearly independent:

$$x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}-2\\1\end{bmatrix}\neq 0 \quad \text{unless } x=0 \text{ and } y=0$$

Ex. $\begin{cases} x+2y=3 \\ 3x+6y=9 \end{cases}$ (1,1) is a sol'n
(3,0) also is.

$x+2y=3$ every vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3-2t \\ t \end{bmatrix}$ is a sol'n.

$x = 3-2y$

- If A is singular, the corresponding linear system has either 0 or inf. many sol's, depending on RHS.
- If A is invertible, then exactly 1 solution.

Ex

$$M = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\begin{bmatrix} x \\ y-x \\ z-y \end{bmatrix}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{cases} x = 1 = b_1 \\ y-x = 3 = b_2 \\ z-y = 5 = b_3 \end{cases} \qquad \begin{aligned} x &= 1 &&= b_1 \\ y &= 1+3 &&= b_1+b_2 \\ z &= 1+3+5 &&= b_1+b_2+b_3 \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1+b_2 \\ b_1+b_2+b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{M^{-1}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

product of matrices

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

dot product

$$c_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

$$\begin{cases} x - z = 0 \\ y - x = 0 \\ z - y = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

any $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$ is a solution. Therefore, A cannot be invertible.

columns of A are linearly dependent:

$$1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Col's are lin. dep $\iff$ A is not invertible (singular)
(rows)

"if and only if"

Col's are lin. indep. $\iff$ A is invertible
(rows)