More about singular value decomposition.

SVD of $\underset{m\times n}{A} = \underset{\substack{m\times m\\ \text{orth.}}}{U}\ \underset{\substack{m\times n\\ \text{diag.}}}{\Sigma}\ \underset{\substack{n\times n\\ \text{orth.}}}{V^T} = U \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \\ & & & & & 0 \end{bmatrix}}_{\Sigma} V^T$

1) Orthogonal diagonalization of A^TA ← positive semi-definite

$v_1, \dots, v_r$ are eigenvectors for eigenvalues of A^TA, $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_r^2 > 0$

$\phantom{v_1, \dots, v_r \text{ are eigenvectors for eigenvalues of } A^TA,\ } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$

2) Compute $\boxed{u_i = \dfrac{Av_i}{\sigma_i}}$ for $i = 1, \dots, r$

$Av_i = \sigma_i u_i$

3) Take orthonormal basis $u_{r+1}, \dots, u_m$ for $N(A^T)$

4) Take orthonormal basis $v_{r+1}, \dots, v_n$ for $N(A)$

Then

$$A \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}}_{\Sigma}$$

SVD:

$$A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} =$$

$$\quad U \qquad\qquad \Sigma \qquad\qquad V$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & \sigma_2 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \sigma_r & \\ & & & & 0 \; \ddots \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} =$$

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \sigma_1 v_1 & - \\ - & 0 & - \\ & \vdots & \\ - & 0 & - \end{bmatrix} = \sigma_1 \underbrace{\underset{m \times 1}{u_1} \; \underset{1 \times n}{v_1^T}}_{\substack{\text{rank } 1 \\ m \times n}}$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

Ex 1. $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

1) $\lambda^2 - (\text{trace})\lambda + \det = \lambda^2 - 50\lambda + (25^2 - 20^2) =$

$= \lambda^2 - 50\lambda + (25+20)(25-20) =$

$= (\lambda - 45)(\lambda - 5)$

$\sigma_1 = \sqrt{45} \quad \sigma_2 = \sqrt{5}$

eigenvectors of A^TA: $\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 45\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \begin{bmatrix} -20 & 20 \\ 20 & -20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

unit length eigenvector: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1$

$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \begin{bmatrix} 20 & 20 \\ 20 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = v_2$

2) Find u_1, u_2: $\quad Av_1 = \sigma_1 u_1, \quad Av_2 = \sigma_2 u_2$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{10}}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\overset{\shortparallel}{U\Sigma V^T}$

$$u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

3) $\underline{m} = n = \underline{r = 2}$ no more vectors needed for U

4) $m = \underline{n} = \underline{r} = \underline{2}$ no more vectors needed for V.

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{V^T}$$

Ex 2 $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

1) Find $v_1 \dots v_r$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Trick AB and BA have the same nonzero eigenvalues!

$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

eigenvalues : 25, 9

$\sigma_1 = 5 \quad \sigma_2 = 3$.

$\lambda = 25$: $\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ unit eigenvector: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$

$\lambda = 9$: $\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$ unit eigenvector:

$$\frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = v_2$$

2) Find u_1, u_2:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = u_1$$

$$u_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{3 \cdot 3\sqrt{2}}\begin{bmatrix} 3-2+8 \\ 2-3-8 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = u_2$$

3) $m = 2$ $r = 2$ so no more u_i's.

4) $n = 3$ $r = 2$ so one more v_3:

v_3 is orthonormal basis for $N(A)$:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = v_3$$

$v_3 \perp v_1, v_2$, $\|v_3\| = 1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = v_3$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{bmatrix}}_{V^T}.$$

Ex 3

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & 0 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 4 & \\ & & & 9 \end{bmatrix} \qquad \sigma_1 = 3 > \sigma_2 = 2 > \sigma_3 = 1$$

eigvalue 0.

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3 & & & \\ & 2 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ex 3'

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ \frac{1}{60000} & & & 0 \end{bmatrix} \qquad A^TA = \begin{bmatrix} \frac{1}{(60000)^2} & & & \\ & 1 & & \\ & & 4 & \\ & & & 9 \end{bmatrix}$$

eigenvalues $\frac{\pm 1}{10}$, $\frac{\pm i}{10}$

$$U = \begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3 & & & \\ & 2 & & \\ & & 1 & \\ & & & \frac{1}{60000} \end{bmatrix} \qquad V = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

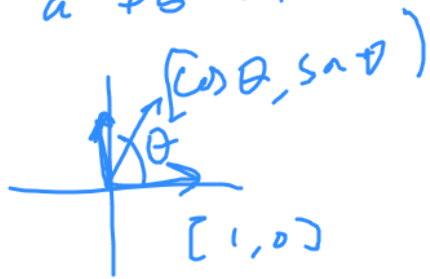
Geometry of SVD.

2×2: $A = U\Sigma V^T = (\text{rotate by } \theta)\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix}(\text{rot. by } -\varphi)$

$U = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ matrix of rotation by θ

$\left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\| = 1$ so $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$a^2+b^2=1$



$V = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ rot. by φ, V^T rotate by $-\varphi$