Matrix of a linear transformation.

$$T(\underset{\substack{\uparrow \\ \text{from } V}}{v}) = \underset{\substack{\uparrow \\ \text{from } W}}{w} \qquad T(c_1v_1+c_2v_2) = c_1T(v_1) + c_2T(v_2)$$

$$V = \{Dx+E\} \qquad \{ax^2+bx+c\} = W$$

$$\frac{d}{dx}(ax^2+bx+c) = 2ax+b$$

$$\underbrace{\boxed{\begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ b \end{bmatrix}$$

$$\int_0^x (Dx+E)\,dx = \frac{1}{2}Dx^2 + Ex + 0\cdot 1$$

$$\underbrace{\boxed{\begin{matrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}}_{B} \begin{bmatrix} D \\ E \end{bmatrix} = \begin{bmatrix} \frac{1}{2}D \\ E \\ 0 \end{bmatrix}$$

$$B = A^+$$

integration is pseudoinverse of differentiation

Terminology:

Let $v_1, v_2, \dots, v_n$ be a basis for V.

Then every v in V can be written as

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$ uniquely.

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ are called coordinates of v in the basis $v_1, v_2, \dots, v_n$

Ex. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are a basis for $\mathbb{R}^2$

coords of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in this basis?

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{cases} c_1 + 2c_2 = 3 \\ 2c_1 + c_2 = 4 \end{cases} \rightsquigarrow \text{solve}$$

$$\frac{5}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$ are coords of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Ex. $\underset{v_1}{\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}}$ ← basis for $\mathbb{R}^2$

coords of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in that basis?

$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1 v_1 + c_2 v_2 \quad \Big| \cdot v_1$

$v \cdot v_1 = c_1 v_1 \cdot v_1 + c_2 v_2 \cdot v_1$

$v \cdot v_1 = c_1 \cdot 1 + 0$

$c_1 = \underline{v \cdot v_1} = 3 \cdot \frac{1}{2} + 4 \cdot \frac{\sqrt{3}}{2}$

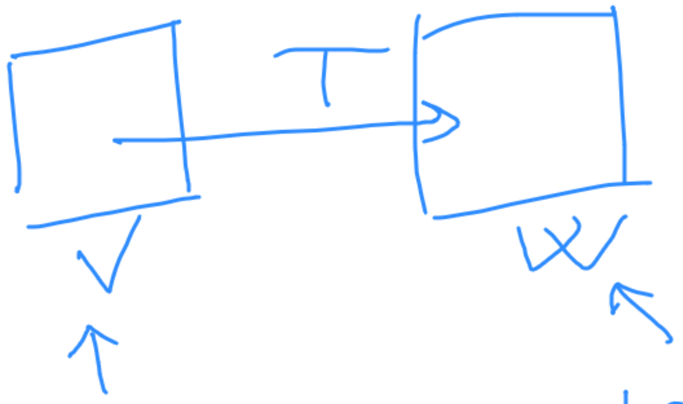
$\Big| \cdot v_2$

$v \cdot v_2 = c_1 \cdot 0 + c_2$

$c_2 = \underline{v \cdot v_2} = 3 \cdot \frac{-\sqrt{3}}{2} + 4 \cdot \frac{1}{2}$

In general: if $v_1 \dots v_n$ is orthonormal then coords of v are $\begin{bmatrix} v \cdot v_1 \\ \vdots \\ v \cdot v_n \end{bmatrix}$.

Matrix of T.



basis $v_1 \dots v_n$

basis $w_1 \dots w_m$

Then if $T(v) = w$ where $v = c_1 v_1 + \dots + c_n v_n$ and $w = d_1 w_1 + \dots + d_m w_m$ then:

$$m \overbrace{\begin{bmatrix} | & | & & | \\ T(v_1) & T(v_2) & \dots & T(v_n) \\ | & | & & | \end{bmatrix}}^{n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

coords in $w_1 \dots w_m$

$$T(c_1 v_1 + \dots + c_n v_n) =$$

$$= c_1 T(v_1) + \dots + c_n T(v_n)$$

Ex. $V = \{ax^2+bx+c\} = W$

basis: $x^2, x, 1$

$x^2+10x+7$

$-x^2+9x$

$2023x-5$

$$T(f) = x^2 \frac{d^2}{dx^2} f$$

$$T(x^2) = x^2\cdot(x^2)'' = 2x^2 \longleftarrow \begin{bmatrix}2\\0\\0\end{bmatrix}$$

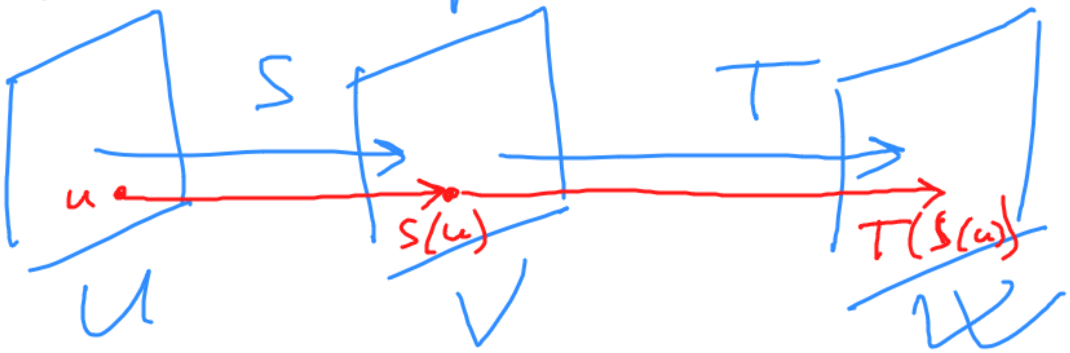
$$T(x) = x^2\cdot(x)'' = 0 \longleftarrow \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$T(1) = x^2\cdot(1)'' = 0 \longleftarrow \begin{bmatrix}0\\0\\0\end{bmatrix}$$

so matrix is:

$$\begin{bmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Matrix multiplication : WHY.



what is the matrix of TS?

bases: $u_1 \dots u_n$ $v_1 \dots v_m$ $w_1 \dots w_k$

matrix of S : $\begin{bmatrix} s_{11} & \dots & s_{1n} \\ \vdots & & \\ s_{m1} & \dots & s_{mn} \end{bmatrix}$ of T: $\begin{bmatrix} t_{11} & \dots & t_{1m} \\ \vdots & & \\ t_{k1} & \dots & t_{km} \end{bmatrix}$

Find $TS(u_1) = T(S(u_1))$

$$S(u_1) = s_{11} v_1 + s_{21} v_2 + \dots + s_{m1} v_m$$

$$T(S(u_1)) = T(\boxed{s_{11} v_1} + \boxed{s_{21} v_1} + \dots + \boxed{s_{m1} v_m}) =$$

$$s_{11}(t_{11} w_1 + \dots + t_{k1} w_k) +$$

$$+ s_{21}(t_{12} w_1 + \dots + t_{k2} w_k) + \dots$$

$$\dots + s_{m1}(t_{1m} w_1 + \dots + t_{km} w_k) =$$

$$= (\underbrace{t_{11} s_{11} + t_{12} s_{21} + \dots + t_{1m} s_{m1}}_{\text{rule for } (TS)_{11}}) w_1 + \dots + (t_{k1} s_{11} + t_{k2} s_{21} + \dots + t_{km} s_{m1}) w_k$$

$\uparrow (TS)_{k1}$