Cramer's Rule (Cramer's Formula)

$$\underset{\substack{\uparrow\\ 3\times 3}}{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Notice:

B_1 is A where we replaced col. 1 by b (RHS).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}^{B_1}$$

take det:

$$\det A \cdot \det \begin{bmatrix} x_1 & & \\ x_2 & 1 & \\ x_3 & & 1 \end{bmatrix} = \det B_1$$

$$\det A \cdot x_1 = \det B_1 \quad \text{so} \quad x_1 = \frac{\det B_1}{\det A}$$

what about x_2? $A \cdot \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} = B_2$

$$\det A \cdot x_2 = \det B_2 \quad \text{so} \quad x_2 = \det B_2 / \det A.$$

In general: Cramer's Rule:

$$\underset{\substack{\uparrow\\ n\times n,\\ \text{invertible}}}{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$ has a solution $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where

$$x_i = \frac{\det B_i}{\det A}$$

where B_i is the matrix obtained from A by replacing column i with b.

Note: computationally, Gauss elimination is more efficient.

Ex. $\begin{cases} 3x - 5y = 10 \\ 4x + 12y = -7 \end{cases}$

$A = \begin{bmatrix} 3 & -5 \\ 4 & 12 \end{bmatrix}$ $B_1 = \begin{bmatrix} 10 & -5 \\ -7 & 12 \end{bmatrix}$ $B_2 = \begin{bmatrix} 3 & 10 \\ 4 & -7 \end{bmatrix}$

$|A| = 36 + 20 = 56$ $|B_1| = 120 \cancel{+} 35$
$= 85$

$B_2 = -21 - 40$
$= -61$

$x = \frac{|B_1|}{|A|} = \frac{85}{56}$ $y = \frac{-61}{56}$

Ex. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$ guarantees that $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$

has integer sol'n when e, f are integer.

$\begin{cases} 7x + 3y = e \\ 5x + 2y = f \end{cases}$

Cofactor formula for matrix inversion.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^{-1} = ?$$

cofactors

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$0 = a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} \qquad \text{(wrong row)}$$

$$= \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

when we find A^{-1} we solving systems

$$Ax = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \dots, \ Ax = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Take cofactor matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

and compute

$$AC^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \cdot \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

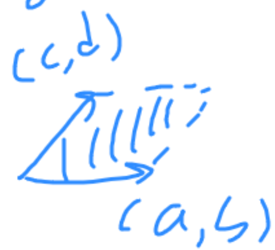
$$A \cdot \frac{C^T}{\det A} = I \qquad \text{so} \qquad A^{-1} = \frac{C^T}{\det A}.$$

Cor. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}, \quad C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Volumes

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = signed area of para on (a,b), (c,d)

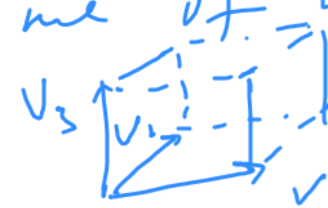


$\det \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{bmatrix}$ = signed volume of box on v_1, v_2, v_3

3×3

+ pos. triple

− neg. triple



One more thing:

$$(v_1, v_2, v_3) \times (w_1, w_2, w_3) = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

vector cross product in $\mathbb{R}^3$.