Peter Rauscher HW 7

I pledge my honor that I have abided by the Stevens Honor System.

1)

a) $Av = \lambda v$

$\det(A - I\lambda) = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - (4)(2) = 0$

$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda+1)(\lambda-5) = 0$

$\lambda = 5$ or $\lambda = -1$

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} 5$

$x + 4y = 5x$

$2x + 3y = 5y$

$y = x$

$\lambda = 5 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$

$x + 4y = -x$

$2x + 3y = -y$

$2x = -4y$

$x = -2y$

$\lambda = -1$ and $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$(I + A)v = \lambda v$

$(I + A)v - I\lambda v = 0$

$\det((I + A) - I\lambda) = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(4-\lambda) - 8 = 0$

$8 - 4\lambda - 2\lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 6\lambda = 0$

$\lambda = 6$ or $\lambda = 0$

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 6\begin{bmatrix} x \\ y \end{bmatrix}$

$2x + 4y = 6x$

$2x + 4y = 6y$

$x = y$

$\lambda = 6 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0\begin{bmatrix} x \\ y \end{bmatrix}$

$2x + 4y = 0$

$2x + 4y = 0 \quad x = -2y$

$\lambda = 0 \quad v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

b) The eigenvalues of $I+A$ are equivalent to 1 plus the eigenvalues of A,

$\lambda_A + 1 = \lambda_{I+A}$

And the eigenvectors of $I+A$ and A are equivalent

$$V_A = V_{I+A}$$

c) We solve for the eigenvalue of an arbitrary $n \times n$ matrix A by solving for λ in the equation

$\det(A - I\lambda) = 0$

So, to solve for λ for $I+A$ with the same matrix A, we use

$\det(I + A - I\lambda) = 0$

$\det(I) + \det(A - I\lambda) = 0$

$\det(A - I\lambda) + 1 = 0$

Thus, in solving for λ_{I+A}, it will always be λ_A plus one, for any arbitrary $n \times n$ matrix.

(cont.)

For an arbitrary $n \times n$ matrix A,

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

The eigenvector V_A can be found using the equation

$$A V_A = \lambda_A V_A \qquad \text{where } V_A = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + a_{12} + \dots + a_{1n} \\ a_{21} + a_{22} + \dots + a_{2n} \\ \dots \\ a_{n1} + a_{n2} + \dots + a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda_A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

And for the matrix $I + A$ using the A above, and

$$\lambda_{I+A} = \lambda_A + 1$$

$$I + A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix} + \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = \begin{bmatrix} a_{11}+1 & & \\ & \ddots & \\ & & a_{nn}+1 \end{bmatrix}$$

Use the equation $\qquad$ where $V_{I+A} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$(I + A) V_{I+A} = \lambda_{I+A} V_{I+A}$$

$$\begin{bmatrix} a_{11} + 1 + a_{12} + \dots + a_{1n} \\ a_{21} + 1 + a_{22} + \dots + a_{2n} \\ \dots \\ a_{n1} + 1 + a_{n2} + \dots + a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \lambda_A + 1$$

C (cont.)

Cancelling out the $+I$ on each side of the equation leaves us with the same equation we used to solve for V_A, and thus $V_A = V_{I+A}$ for any arbitrary matrix A.

In conclusion, we have proven for any arbitrary matrix A of $n \times n$, the eigenvalues of $I+A$ are $1+\lambda_A$, and the eigenvectors of A and $I+A$ are equivalent.

2)

a) $A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ $\begin{vmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{vmatrix} = (0.6-\lambda)(0.8-\lambda) - (0.2)(0.4)$

$= 0.48 - 0.8\lambda - 0.6\lambda + \lambda^2 - 0.08$

$= \lambda^2 - 1.4\lambda + 0.4$

$\lambda_1 = 1 \quad \lambda_2 = 0.4$

$Av_i = \lambda_i v_i$

$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix}$

$0.6x + 0.2y = x$
$0.4x + 0.8y = y$
$2x = y$

$V_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0.4\begin{bmatrix} x \\ y \end{bmatrix}$

$0.6x + 0.2y = 0.4x$
$0.4x + 0.8y = 0.4y$
$-y = x$

$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}$

$X^{-1} = \frac{1}{(2)(1)-(1)(-1)}\begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 3/5 & 1/5 \\ 2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2/5 \end{bmatrix}\begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}}$

b) $A^K = X\Lambda^K X^{-1}$

$= \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2/5 \end{bmatrix}^K \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -\frac{2}{5}^K \\ 1 & \frac{2}{5}^K \end{bmatrix}\begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

$\boxed{A^K = \begin{bmatrix} \frac{1}{3} + \frac{2}{3}\left(\frac{2}{5}\right)^K & \frac{1}{3} - \frac{1}{3}\left(\frac{2}{5}\right)^K \\ \frac{2}{3} - \frac{2}{3}\left(\frac{2}{5}\right)^K & \frac{2}{3} + \frac{1}{3}\left(\frac{2}{5}\right)^K \end{bmatrix}}$

2c) $\lim_{K\to\infty} \begin{bmatrix} \frac{1}{3}+\frac{2}{3}\left(\frac{2}{5}\right)^K & \frac{1}{3}-\frac{1}{3}\left(\frac{2}{5}\right)^K \\ \frac{2}{3}-\frac{2}{3}\left(\frac{2}{5}\right)^K & \frac{2}{3}+\frac{1}{3}\left(\frac{2}{5}\right)^K \end{bmatrix} =$

$$\boxed{= \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}}$$

3a) $A = \begin{bmatrix} a & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $\det(A - I\lambda) = 0$

$\begin{vmatrix} a-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$ $(a-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$

$= (a-\lambda)((1-\lambda)(2-\lambda) - 0(1)) - 0 + (0(0) - (0)(2-\lambda))$

$= (a-\lambda)(1-\lambda)(2-\lambda) = 0$

$\lambda_1 = a \quad \lambda_2 = 1 \quad \lambda_3 = 2$

b, c, and d)

If $a \neq 1, 2$ then there are three unique eigenvalues for which we can solve the equation $Av = \lambda v$, meaning there will be three linearly independent eigenvectors with which we can build the matrix X for digonalization. If $a=1$ or $a=2$, there would be only two eigenvectors for the matrix X, which would make it a 2×2 matrix, making factoring A into $X\Lambda X^{-1}$ impossible. So, if $a=1$, A is not digonalizable, and if $a=2$, A is not diagonalizable.

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