MA 232 Linear Algebra **HW 5** **Due: Oct 31, 2022 by 11:59 pm**

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Pledge and Sign: I pledge my honor that I have abided by the Stevens Honor System

Peter Rauscher

Upload solutions to Gradescope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. Consider the vectors $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) [6 pts.] Find the projection $\mathbf{p}$ of $\mathbf{b}$ onto the subspace spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$.

(b) [4 pts.] Find the error vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$, and verify that it is orthogonal to both $\mathbf{a}_1$ and $\mathbf{a}_2$ by computing the corresponding dot products.

HW 5

1a) $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$P = A(A^TA)^{-1}A^T$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$

$A^TA\hat{x} = A^Tb$

$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

$2\hat{x}_1 + 2\hat{x}_2 = 6$

$2\hat{x}_1 + 3\hat{x}_2 = 12$

$\hat{x}_1 = -3, \hat{x}_2 = 6$

$\hat{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$

$$\boxed{p = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}}$$

1b) $e = b - p$

$e = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$e \cdot a_1 =$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1(1) + 1 \cdot 1 + 0 \cdot 0 = 0$

$e \cdot a_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= (-1)(1) + (1)(1) + 0(1)$

$= 0$

Thus, $e \perp a_1$ and $e \perp a_2$

2)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \overbrace{\begin{bmatrix} C \\ D \end{bmatrix}}^{=\hat{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{=b}$$

$$A^T A \hat{x} = A^T \cdot b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 37 \\ 84 \end{bmatrix}$$

So, $\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 37 \\ 84 \end{bmatrix}$

$$4\hat{C} + 6\hat{D} = 37 \qquad 6\hat{C} + 14\hat{D} = 84$$

$$\hat{D} = \frac{37 - 4\hat{C}}{6} \qquad 6\hat{C} + 14\left(\frac{37 - 4\hat{C}}{6}\right) = 84$$

$$\hat{D} = \frac{37 - 4\left(\frac{7}{10}\right)}{6} \qquad 6\hat{C} + 7\left(\frac{37 - 4\hat{C}}{3}\right) = 84$$

$$18\hat{C} + 7(37 - 4\hat{C}) = 252$$

$$259 - 10\hat{C} = 252$$

$$\hat{D} = \frac{57}{10} \qquad \hat{C} = \frac{7}{10}$$

So the best-fitting line is

$$\boxed{y = \frac{7}{10} + \frac{57}{10}x}$$

3) $$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = a = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$B = b - \frac{A^Tb}{A^TA} A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1+0+0+1}{\underbrace{(1^2+2^2+0^2+1^2)}_{\to (\frac{1}{3})}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 2/3 \end{bmatrix} \quad q_2 = \frac{\sqrt{3}}{2} \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 2/3 \end{bmatrix}$$

$$C = c - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$q_3 = \sqrt{2} \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$D = d - \frac{A^Td}{A^TA} A - \frac{B^Td}{B^TB} B - \frac{C^Td}{C^TC} C$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 2/3 \end{bmatrix} - \frac{0}{1/2} \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad q_4 = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad q_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix} \quad q_3 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \quad q_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 \cdot q_1 = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right) + 0 + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)$$
$$= 0$$

$$q_2 \cdot q_2 = \left(\frac{1}{\sqrt{3}}\right)^2 \left(-\frac{1}{\sqrt{3}}\right)^2 + 0 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1$$

$$q_2 \cdot q_3 = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{\sqrt{2}}{2}\right) + 0 + 0 + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$q_2 \cdot q_4 = 0 + 0 + 0 + 0 = 0$$