Peter Rauscher

HW 8

I pledge my honor that I have abided by the Stevens Honor System

Exercise 8.1

(a) $\langle(3,2)\rangle \in \mathbb{Z}_4 \times \mathbb{Z}_3 =$

$\{(0,0),(3,2),(2,1),(1,0),(0,2),(3,1),(2,0),(1,2),(0,1),(3,0),(2,2),(1,1)\}$

(b) $\langle(3,2)\rangle \in U_5 \times \mathbb{Z}_3 =$

$U_5 = \{1,2,3,4\} \quad \mathbb{Z}_3 = \{0,1,2\}$

$\{(3,2),(1,1),(4,0),(2,2),(3,0),(1,2),(4,1),(2,0),(3,1),(1,0),(4,2),(2,1)\}$

Exercise 8.2

If $\chi(R) \neq 0$, that means there must exist some $n \in \mathbb{N}$ where $n \cdot 1 = 0$

As $n \in \mathbb{N} \Rightarrow n \neq 0$ and $1 \neq 0$, both n and 1 are zero divisors

Consider some a such that $a \in R$

$$n \cdot a = a(n \cdot 1) = \underbrace{(n\cdot 1)+(n\cdot 1)+(n\cdot 1)\ldots+(n\cdot 1)}_{a \text{ times}}$$

$$= \underbrace{0+0+0+\ldots+0}_{a \text{ times}}$$

$$= 0$$

So if $\chi(R) \neq 0$, then for any $a \in R$, $n \cdot a = 0$

Exercise 8.3

We assume $f(x) \in F[x]$ is divisible by some polynomial $g(x) = a_n x^n + \ldots$ of degree n, and so there exists a polynomial $h(x) \in F[x]$ such that

$$f(x) = h(x) \cdot g(x)$$
$$f(x) = h(x) \cdot (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x^0)$$
$$f(x) = h(x) \cdot a_n \cdot (x^n + \tfrac{a_{n-1}}{a_n} x^{n-1} + \ldots + \tfrac{a_0}{a_n} x^0)$$

We can see the polynomial $(x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \ldots + \frac{a_0}{a_n} x^0)$ is a factor in $f(x)$ and thus $f(x)$ is divisible by it. Also note, the leading coefficient is 1, and thus the polynomial is monic.

Exercise 8.4

(a) $x^3 + 2x - 1$ is not factorable, and thus is irreducable

(b) $x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)$

$f(x) = g(x) h(x)$

$g(x) = x + 1$	$h(x) = x^2 + x + 1$
$a_0 = 1 \in \mathbb{Z}_5$	$a_0 = 1 \in \mathbb{Z}_5$
	$a_1 = 1 \in \mathbb{Z}_5$
$a_1 = 1 \neq 0 \in \mathbb{Z}_5$	$a_2 = 1 \in \mathbb{Z}_5 \neq 0$

$f(x)$ is factorable into $g(x) = x + 1$ and $h(x) = x^2 + x + 1$ where $g(x), h(x) \in \mathbb{Z}_5[x]$ and thus it is reducible

Ex 8.4 Continued

(c) $x^4+x^3+x^2+x+1$

$f(x)$ is clearly not factorable into linear factors, so consider irreducible quadratic factors.

$$\deg(f(x)) = 4 = \deg(g(x)) + \deg(h(x))$$

So either $\deg(g(x))=1$ and $\deg(h(x))=3$, or $\deg(g(x))=2$ and $\deg(h(x))=2$

Consider $\deg(g(x))=1$ and $\deg(h(x))=3$

$g(x)=x$ or $x+1$ and $h(x)=x^3+x+1$ or x^3+x^2+1

$(x)(x^3+x+1)=x^4+x^2+x \neq f(x)$
$(x)(x^3+x^2+1)=x^4+x^3+x \neq f(x)$
$(x+1)(x^3+x^2+1)=x^4+2x^3+x^2+x+1 = x^4+x^2+x+1 \neq f(x)$
$(x+1)(x^3+x+1)=x^4+x^3+x^2+2x+1 = x^4+x^3+x^2+1 \neq f(x)$

So, consider $\deg(g(x))=2$ and $\deg(h(x))=2$
There is only 1 irreducible polynomial of degree 2 in $\mathbb{Z}_2[x]$,
$g(x)=h(x)=x^2+x+1$

$(x^2+x+1)(x^2+x+1)=x^4+2x^3+3x^2+2x+1$
$= x^4+x^2+1$ in $\mathbb{Z}_2 \neq f(x)$

Thus, $f(x)$ is irreducible

Ex 8.5

$$\mathbb{Z}_7[x]$$

$$
\begin{array}{r|l}
5x^5 + x^4 + x^3 + 4x^2 + 3x + 4 & 3x^3 + x^2 + 2x + 2 \\ \hline
- \underline{12x^5 + 4x^4 + 8x^3 + 8x^2} & 4x^2 + 6x + 5 \\
4x^4 + 3x^2 + 3x + 4 & \\
- \underline{18x^4 + 6x^3 + 12x^2 + 12x} & \\
x^3 + 5x^2 + 5x + 4 & \\
- \underline{15x^3 + 5x^2 + 10x + 10} & \\
2x + 1 &
\end{array}
$$

So, the quotient $q(x)$ and remainder $r(x)$ of dividing $f(x) = 5x^5 + x^4 + x^3 + 4x^2 + 3x + 4$ by $g(x) = 3x^3 + x^2 + 2x + 2$ in $\mathbb{Z}_7[x]$ are:

$$\boxed{\begin{array}{l} q(x) = 4x^2 + 6x + 5 \\ r(x) = 2x + 1 \end{array}}$$

We can check our answer with

$$f(x) = g(x) \cdot q(x) + r(x)$$

$$
\begin{aligned}
&= (3x^3 + x^2 + 2x + 2)(4x^2 + 6x + 5) + 2x + 1 \\
&= 12x^5 + 22x^4 + 29x^3 + 25x^2 + 24x + 11 \\
&\equiv_7 5x^5 + x^4 + x^3 + 4x^2 + 3x + 4 \\
&= f(x)
\end{aligned}
$$

So, our results are correct

Ex 8.6

(a)
$$\begin{array}{r|l} 2x^4+x & x^3+x^2+1 \\ \cline{2-2} -\underline{2x^4+2x^3+2x} & 2x+1 \\ x^3+2x & \\ -\underline{x^3+x^2+1} & \\ 2x^2+2x+2 & \end{array}$$

$$\begin{array}{r|l} x^3+x^2+1 & 2x^2+2x+2 \\ \cline{2-2} -\underline{4x^3+4x^2+4x} & 2x \\ 2x+1 & \end{array}$$

$$\begin{array}{r|l} 2x^2+2x+2 & 2x+1 \\ \cline{2-2} -\underline{2x^2+x} & x \\ x+2 & \end{array}$$

$$\begin{array}{r|l} 2x+1 & x+2 \\ \cline{2-2} -\underline{2x+4} & 2 \\ 0 & \end{array}$$

By Euclidian algorithm,

$f(x) = (2x+1)g(x) + 2x^2+2x+2 \Rightarrow \gcd(2x^2+2x+2, g(x))$

$g(x) = (2x)(2x^2+2x+2) + 2x+1 \quad = \gcd(2x^2+2x+2, 2x+1)$

$2x^2+2x+2 = x(2x+1) + x+2 \quad = \gcd(x+2, 2x+1)$

$2x+1 = 2(x+2) + 0 \quad = \gcd(x+2, 0) = x+2$

$\gcd(f(x), g(x)) = x+2$