Change of basis.

Terminology: If $b_1, \dots, b_n$ are a basis of a vector sp. V then every $v \in V$ can expressed as

$$v = x_1 b_1 + x_2 b_2 + \dots + x_n b_n \quad \text{in a unique way.}$$

The column $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ are called <u>coordinates</u> of v in the basis $b_1, \dots, b_n$.

<u>Ex.</u> $V = \mathbb{R}^2$, $b_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

what are coordinates of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in this basis?

In other words:

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has coords $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

want: coords $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in basis $\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} y_1 + \begin{bmatrix} 3 \\ 5 \end{bmatrix} y_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad y = \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1}}_{B^{-1}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

In general: to switch to basis $b=(b_1,\dots,b_n)$ of $\mathbb{R}^n$:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y_1 b_1 + \dots + y_n b_n = \underbrace{\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}}_{B} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

then $$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = B^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

"new" coords (in basis b)

"old" coords (in standard basis)

change of basis matrix from standard basis to basis b.

<u>Ex.</u> For $\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, change of basis matrix is

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}.$$

How to change from basis $c=(c_1,\dots,c_n)$ to a basis $b=(b_1,\dots,b_n)$:

$$v = x_1c_1 + x_2c_2 + \dots + x_nc_n =$$
$$= y_1b_1 + y_2b_2 + \dots + y_nb_n$$

coord in basis b.

coords in basis c

$$\begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$Cx = By$, then $y = B^{-1}Cx$

and $x = C^{-1}By$

So
- $B^{-1}C$ is the change of basis matrix from c to b,
- $C^{-1}B$ is the change of basis matrix from b to c.

What about transformations.

Suppose a linear transformation T from U to V has matrix A_{st} in [the standard] basis $e_1, \ldots, e_n$.

some

What is the matrix of T in another basis b?

$T(v) = v'$

$v = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$ $\qquad$ $v' = x_1' e_1 + \ldots + x_n' e_n$

$$A_{st} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}, \qquad A_{st} x = x'$$

Supp. $v = y_1 b_1 + \ldots + y_n b_n$, $\quad v' = y_1' b_1 + \ldots + y_n' b_n$

then $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | & & | \\ b_1 & \ldots & b_n \\ | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\quad x = By$, $\quad x' = By'$

$A_{st} B y = B y' \qquad B^{-1} A_{st} B y = y'$

If $T(v) = v'$ and
- v has coords x in basis $e_1 \dots e_n$
- v' has coords x' in basis $e_1 \dots e_n$
- change of basis matrix from $e_1 \dots e_n$ to $b_1 \dots b_n$ is B^{-1}
- v has coords y in $b_1 \dots b_n$
- v' has coords y' in $b_1 \dots b_n$
- T has matrix A_{st} in $e_1 \dots e_n$

then:

$$\underbrace{B^{-1} A_{st} B}\, y = y'$$

A_b = matrix of T in the basis $b_1 \dots b_n$

$$A_b = B^{-1} A_{st} B$$

$$\underbrace{x' \rightsquigarrow y' \quad x \rightsquigarrow x' \quad y \rightsquigarrow x}_{y \rightsquigarrow y'}$$

So: A and $B^{-1}AB$ are matrices of the same linear transformation in different bases.

The matrices A, $B^{-1}AB$ are called similar (conjugate).

$T(v) = cv$ $\leftarrow$ eigenvector v of T with eigenvalue c

<u>Ex.</u> e^{5x} is eigenvector of $\frac{d}{dx}$ with eigenvalue 5.

All properties of a linear transformation itself are shared by A and $B^{-1}AB$.

<u>Ex.</u> If A has eigenvalue c, then so does $B^{-1}AB$.

If A has eigenvector $\begin{bmatrix} x_1 \\ x_n \end{bmatrix}$ then $B^{-1}AB$ has eig. $B^{-1}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Ex. Diagonalization:

$$A = X \Lambda X^{-1},$$

$$\Lambda = X^{-1} A X$$

diagonal

column = eigenvectors

X^{-1} = change of basis matrix from standard basis to an eigenvector basis

Terminology:

Kernel of lin. transform. T: vectors v s.t. $T(v) = 0$ $\longleftrightarrow$ nullspace of A.

Range of T: vectors w s.t. $w = T(v)$ $\longleftrightarrow$ column space of A

Change of basis when input and output bases are different.

$e_1 \dots e_n$ $\qquad$ $d_1 \dots d_m$

$$\underset{m\times n}{A}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}z_1\\ \vdots\\ z_m\end{bmatrix}$$

new bases: $b_1 \dots b_n$ for V

$c_1 \dots c_m$ for W

$$\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = B\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$$

$$\begin{bmatrix}z_1\\ \vdots\\ z_m\end{bmatrix} = C\begin{bmatrix}t_1\\ \vdots\\ t_m\end{bmatrix}$$

$$ABy = Ct$$

$$C^{-1}ABy = t$$

SVD:

$$A = U \Sigma V^T$$

$$\left(\begin{array}{l} V^T = V^{-1} \\ U^T = U^{-1} \end{array} \right)$$

$$\Sigma = U^T A V$$

"new" matrix, in bases $u_1 \dots u_m$ and $v_1 \dots v_n$

"old" matrix

C^{-1}

B

what is a good basis for $n \times n$ matrix A that is not diagonalizable?
what does A look like in that basis?

Jordan Canonical Form (Jordan Normal Form):
over complex numbers, there is a basis in which A looks like:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & J_3 & \\ & & & J_4 \end{bmatrix} \quad \text{where } J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Jordan cells

Ex. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$.