Symmetric matrices and positive definite matrices

- An $n \times n$ <u>symmetric</u> matrix S has n real eigenvalues and n lin. independent eigenvectors.
- The eigenvectors of S can be picked orthonormal Q $\quad Q^{-1} = Q^T$

$$S = X \Lambda X^{-1}$$

Spectral Theorem: Every symmetric matrix has factorization $S = Q \Lambda Q^T$ with real eigenvalues and orthogonal Q.

- why always n real eigenvalues?
  skip for now (involves some knowledge about complex numbers)
- why n indep. eigenvectors?
  - if all eigenvalues are different, we automatically get n lin.-indep. eigenvectors.
  - if there are multiple eigenvalues, consider $S + \begin{bmatrix} \varepsilon & & & \\ & 2\varepsilon & & \\ & & 3\varepsilon & \\ & & & \ddots \\ & & & & n\varepsilon \end{bmatrix} = S'$

    S' will have n diff. eigenvalues, take $\varepsilon \to 0$.

- Why eigenvectors can be chosen orthogonal (then we can make them orthonormal by rescaling)

Eigenvectors with different λ's are automatically orthogonal (for symmetric matrix)!

Suppose $Sx = \lambda_1 x$, $Sy = \lambda_2 y$, $\lambda_1 \neq \lambda_2$

$$\lambda_1 x^T y = (\lambda_1 x)^T y = (Sx)^T y = x^T S^T y = x^T S y = x^T (Sy)$$
$$= x^T(\lambda_2 y) = \lambda_2 x^T y$$

so! $\lambda_1 x^T y = \lambda_2 x^T y$

since $\lambda_1 \neq \lambda_2$, we get $x^T y = 0$.

Ex. $S = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ eigenvalues $-1, -1, 5$

eigenvectors: for $\lambda = 5$, $a = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = -1$, $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

not $\perp$.

what do? Do Gram-Schmidt.

$C = c = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $B = b - \frac{b \cdot c}{c \cdot c} c = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$

Vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$ are

- eigenvectors with $\lambda = -1$
- orthogonal to each other.

$$q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$S = \begin{bmatrix}1 & -2 & 2\\ -2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} = \underbrace{\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\\ -1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}5 & & \\ & -1 & \\ & & -1\end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\\ -1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6}\end{bmatrix}}_{Q^T = Q^{-1}}.$$

$$= Q \Lambda Q^T$$

"rotation" back — stretch — rigid "rotation"

In general: $\underbrace{x_1, \dots, x_k}_{\lambda_1}, \underbrace{y_1, \dots, y_\ell}_{\lambda_2}, \dots$

Run G-S separately for each group.

Pivots vs eigenvalues.

$$A = LDU \qquad d_1, d_2, \dots, d_n \text{ are pivots}$$

$$L = \begin{bmatrix} 1 & & \\ & 1 & \\ * & \ddots & 1 \end{bmatrix} \quad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad U = \begin{bmatrix} 1 & * \\ & \ddots & \\ & & 1 \end{bmatrix}$$

det = 1 $\qquad$ det = $d_1 \dots d_n$ $\qquad$ det = 1

$\det A = d_1 d_2 \dots d_n$ = product of pivots

$\det A = \lambda_1 \dots \lambda_n$ = product of eigenvalues

$$\left(A = X \Lambda X^{-1}, \quad \det A = \det X \cdot \det \Lambda \cdot (\det X)^{-1} = \det \Lambda = \lambda_1 \dots \lambda_n \right)$$

pivots $\neq$ eigenvalues

Ex: $S = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix}$ $\qquad$ pivots: $1, -8$

eigenvalues: $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$

eigenvalues are $4, -2$

For symmetric matrices,

positive pivots = # positive eigenvalues

(for non-symmetric matrix, this may be false:

$\begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$ pivots 1, 2 eigenvalues -1, -2.)

why: $\underset{S}{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & \\ & -8 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}}$

Important case: all eigenvalues positive.

Positive definite matrices.

A symmetric matrix is called positive definite if all its eigenvalues are positive.

Equivalent condition: all pivots are positive

$$\underbrace{\begin{bmatrix} \boxed{\boxed{\cdot}\ \cdot} & \cdot & \\ & \cdot & \\ & & \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ * & & \ddots & \\ & & & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{L^T}$$

to get this we only need to know the same part of S.

Equivalent condition: all upper left corner determinants (1×1, 2×2, 3×3, ...) are positive.

Example

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$M_1 = 2 > 0$$

$$M_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

$$M_3 = \det S = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2 \cdot 3 + 1 \cdot (-2) = 4 > 0$$

so S is positive definite.

"Energy" definition of positive definiteness.

S is positive definite when and only when

$$\underbrace{x^T}_{1\times n}\underbrace{S}_{n\times n}\underbrace{x}_{n\times 1} > 0 \quad \text{unless } x = 0.$$

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix}^2 = (\sqrt{D})^2$$

$$S = LDL^T = L\sqrt{D}\cdot\sqrt{D}L^T = L\sqrt{D}\,\overbrace{(L\sqrt{D})^T}^{C}$$

$$= C^T C \quad \text{where } C = \sqrt{D}L^T.$$

Cholesky decomposition

Cholesky factor of S.

$$x^T S x = x^T C^T C x = (Cx)^T Cx = \|Cx\|^2 \geq 0$$

and $=0$ only when $Cx = 0$. since C is non-singular,

$Cx \neq 0$ unless $x = 0$.

S being positive definite is equivalent to <u>each</u> of the following:

1) all eigenvalues > 0
2) all pivots > 0
3) all top left corner determinants are > 0
4) $x^T S x > 0$ except for $x = 0$
5) $S = A^T A$ where columns of A are independent

(Cholesky decomp. $S = C^T C$ is a particular case).