The four fundamental subspaces

A an $m \times n$ matrix

(1) col space $C(A)$

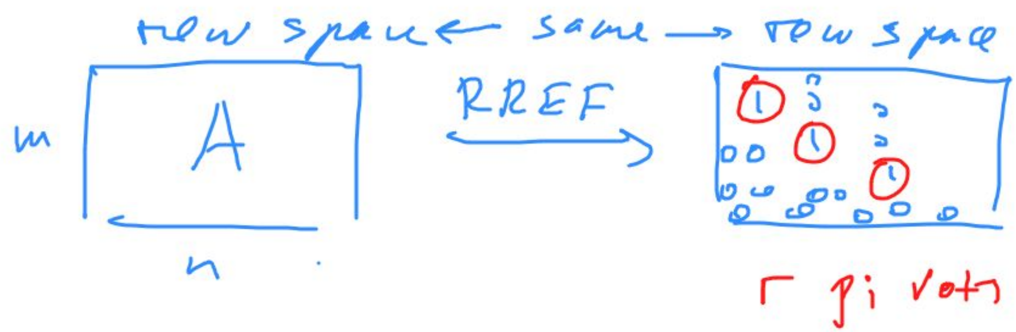
(2) nullspace $N(A)$ $\quad Ax = 0$ $\qquad$ zero space = space that only contains 0.

(3) row space $C(A^T)$

(4) left nullspace $N(A^T)$ $\quad yA = 0$

Reminder! basis: • independent
• spans the space. | largest set of indep. vectors | smallest set that spans the space

dimension = # basis.



terminology:

A has rank r

$\dim C(A) = r$ (# pivot cols)

$\dim N(A) = n - r$ (# free cols)

counting theorem: $\dim C(A) + \dim N(A) = n = \dim \mathbb{R}^n$

$\dim C(A^T) = r$ ← column rank = row rank

$\dim N(A^T) = m - r$

Fundamental Thm of linear algebra.

Ex. $A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$C(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$

$N(A)$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 - 5x_3 - 7x_5 \\ x_2 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_3 \underbrace{\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + x_5 \underbrace{\begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{v_3}$

special sol's

$C(A^T)$: 1) we know $\dim C(A^T) = \dim C(A) = 2$

so any two indep. rows must be a basis:

$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \leftarrow$ basis for $C(A^T)$

2) $R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ pivot rows make a basis for $C(A^T)$: $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

3) $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 5 & 0 & 5 \\ 0 & 1 & 1 \\ 7 & 2 & 9 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ basis in pivot cols of A^T : $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

$N(A^T)$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

basis for $N(A^T)$: $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

feature: (anything from $N(A)$) $\cdot$ (anything from $C(A^T)$) :

$$\begin{bmatrix}-3\\1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\3\\5\\0\\7\end{bmatrix}=0 \quad , \quad \begin{bmatrix}-7\\0\\0\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}1\\3\\5\\0\\7\end{bmatrix}=0 \quad \begin{bmatrix}-7\\0\\0\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}=0$$

$$A x = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

similarly (1 - - - - $N(A^T)$) $\cdot$ (- - - - $C(A)$) $= 0$

ex. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ $\qquad$ $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

$C(A)$ $\qquad$ $N(A^T)$