MA 232 Linear Algebra **HW 4** **Due: Friday Oct 14 by 5:00 pm**

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Pledge and Sign: I pledge my honor that I have abided by the Stevens Honor System. Peter Rauscher

Upload solutions to Gradescope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 7 \\ 9 \\ 1 \\ -1 \end{bmatrix}$. Check whether $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are independent. If they are not, find a basis for the subspace of $\mathbb{R}^4$ spanned by these vectors.

1) $x_1\begin{bmatrix}2\\1\\1\\-1\end{bmatrix}+x_2\begin{bmatrix}0\\3\\0\\1\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\2\\0\end{bmatrix}+x_4\begin{bmatrix}7\\9\\1\\-1\end{bmatrix}=0$

$$\begin{bmatrix}2&0&-1&7\\1&3&0&9\\1&0&2&1\\-1&1&0&-1\end{bmatrix}\xrightarrow{\frac{1}{2}R_1}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\1&3&0&9\\1&0&2&1\\-1&1&0&-1\end{bmatrix}\xrightarrow{R_2-R_1}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&3&\frac{1}{2}&\frac{11}{2}\\1&0&2&1\\-1&1&0&-1\end{bmatrix}\xrightarrow{R_3-R_1}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&3&\frac{1}{2}&\frac{11}{2}\\0&0&\frac{5}{2}&-\frac{5}{2}\\-1&1&0&-1\end{bmatrix}\xrightarrow{R_4+R_1}$$

$$\longrightarrow\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&3&\frac{1}{2}&\frac{11}{2}\\0&0&\frac{5}{2}&-\frac{5}{2}\\0&1&-\frac{1}{2}&\frac{5}{2}\end{bmatrix}\xrightarrow{\frac{1}{3}R_2}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&\frac{5}{2}&-\frac{5}{2}\\0&1&-\frac{1}{2}&\frac{5}{2}\end{bmatrix}\xrightarrow{R_4-R_2}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&\frac{5}{2}&-\frac{5}{2}\\0&0&-\frac{4}{6}&\frac{4}{6}\end{bmatrix}\xrightarrow{\frac{2}{5}R_3}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&1&-1\\0&0&-\frac{4}{6}&\frac{4}{6}\end{bmatrix}\longrightarrow$$

$$\xrightarrow{\frac{6}{4}R_4}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&1&-1\\0&0&-1&1\end{bmatrix}\xrightarrow{R_4+R_3}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&1&-1\\0&0&0&0\end{bmatrix}\quad \begin{matrix}r=3\\n=4\end{matrix}\quad r\neq n$$

The vectors are linearly dependent

$$\left\langle\begin{bmatrix}2\\1\\1\\-1\end{bmatrix},\begin{bmatrix}0\\3\\0\\1\end{bmatrix},\begin{bmatrix}-1\\0\\2\\0\end{bmatrix},\begin{bmatrix}7\\9\\1\\-1\end{bmatrix}\right\rangle$$

$$\text{RREF}\rightarrow\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&\frac{1}{6}&\frac{11}{6}\\0&0&1&-1\\0&0&0&0\end{bmatrix}\xrightarrow{R_2-\frac{1}{6}R_1}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{7}{2}\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{bmatrix}\xrightarrow{R_1+\frac{1}{2}R_3}\begin{bmatrix}1&0&0&3\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

So, the basis for the subspace of $\mathbb{R}^4$ spanned by these vectors is

$$\begin{bmatrix}2\\1\\1\\-1\end{bmatrix},\begin{bmatrix}0\\3\\0\\1\end{bmatrix},\begin{bmatrix}-1\\0\\2\\0\end{bmatrix}$$

2) $V = \langle (1,1,1), (2,1,0) \rangle$

a) The row space of a matrix A is a subspace in $\mathbb{R}^n$ which is spanned by the row vectors of A.

So, $R(A) = V$ when

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

b) $Null(B) = V$ $\qquad$ $rank(B) = \#\,cols - dim(V) = 3 - 2 = 1$

Let $B_1 = [b_1 \ b_2 \ b_3]$, solve $Bx = 0$

$$[b_1 \ b_2 \ b_3]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \qquad [b_1 \ b_2 \ b_3]\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$b_1 + b_2 + b_3 = 0 \qquad 2b_1 + b_2 = 0$$

$$b_1 - 2b_1 + b_3 = 0 \qquad b_2 = -2b_1$$

$$b_3 - b_1 = 0$$

$$b_1 = b_3$$

So B is of the form $[b_1 \ -2b_1 \ b_1]$ for any value of b_1. The smallest nontrivial matrix B would then be

$$B = [1 \ -2 \ 1]$$

c) $$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{bmatrix} \xrightarrow[REF]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $AB = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{bmatrix}$ and $rank(AB) = 1$

3) $A=\begin{bmatrix}1&2&-3&-2\\2&5&-8&-1\\1&4&-7&5\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}2&5&-8&-1\\1&2&-3&-2\\1&4&-7&5\end{bmatrix}\xrightarrow{\frac{1}{2}R_1}\begin{bmatrix}1&\frac{5}{2}&-4&-\frac{1}{2}\\1&2&-3&-2\\1&4&-7&5\end{bmatrix}$

$\xrightarrow{R_2-R_1}\begin{bmatrix}1&\frac{5}{2}&-4&-\frac{1}{2}\\0&-\frac{1}{2}&1&-\frac{3}{2}\\1&4&-7&5\end{bmatrix}\xrightarrow{R_3-R_1}\begin{bmatrix}1&\frac{5}{2}&-4&-\frac{1}{2}\\0&-\frac{1}{2}&1&-\frac{3}{2}\\0&\frac{3}{2}&-3&\frac{11}{2}\end{bmatrix}\xrightarrow[2R_3]{-2R_2}\begin{bmatrix}1&\frac{5}{2}&-4&-\frac{1}{2}\\0&1&-2&3\\0&3&-6&11\end{bmatrix}$

$\xrightarrow{R_3-3R_2}\begin{bmatrix}1&\frac{5}{2}&-4&-\frac{1}{2}\\0&1&-2&3\\0&0&0&2\end{bmatrix}\xrightarrow[\frac{1}{2}R_3]{R_1-\frac{5}{2}R_2}\begin{bmatrix}1&0&1&-8\\0&1&-2&3\\0&0&0&1\end{bmatrix}\xrightarrow[R_1+8R_3]{R_2-3R_3}\begin{bmatrix}1&0&1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}$

So, the basis of $C(A)$ is $\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\5\\4\end{bmatrix},\begin{bmatrix}-2\\-1\\5\end{bmatrix}$

The pivot columns are 1, 2 and 4, and the free column is 3

$Rx=0$

$\begin{bmatrix}1&0&1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

x_3 is free variable

$\begin{bmatrix}-1x_3\\2x_3\\x_3\\0\end{bmatrix}=x_3\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}$

$x_1+x_3=0$
$x_2-2x_3=0$
$x_4=0$

$x_1=-1x_3$
$x_2=2x_3$
$x_3=x_3$
$x_4=0$

So, the basis of $N(A)$ is $\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}$

$A^T=\begin{bmatrix}1&2&1\\2&5&4\\-3&-8&-7\\-2&-1&5\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}2&5&4\\1&2&1\\-3&-8&-7\\-2&-1&5\end{bmatrix}\xrightarrow{\frac{1}{2}R_1}\begin{bmatrix}1&\frac{5}{2}&2\\1&2&1\\-3&-8&-7\\-2&-1&5\end{bmatrix}\xrightarrow[\substack{R_3+3R_1\\R_4+2R_1}]{R_2-R_1}\begin{bmatrix}1&\frac{5}{2}&2\\0&-\frac{1}{2}&-1\\0&-\frac{1}{2}&-1\\0&4&9\end{bmatrix}$

$\xrightarrow[-2R_2]{R_3-R_2}\begin{bmatrix}1&\frac{5}{2}&2\\0&1&2\\0&0&0\\0&4&9\end{bmatrix}\xrightarrow[R_3\leftrightarrow R_4]{R_4-4R_2}\begin{bmatrix}1&\frac{5}{2}&2\\0&1&2\\0&0&1\\0&0&0\end{bmatrix}\xrightarrow{R_1-R_2}\begin{bmatrix}1&\frac{3}{2}&0\\0&1&2\\0&0&1\\0&0&0\end{bmatrix}\xrightarrow{\substack{R_2-2R_3\\R_1-\frac{3}{2}R_2}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

Since all columns are pivots, the basis of $C(A^T)$ is $\begin{bmatrix}1\\2\\-3\\-2\end{bmatrix},\begin{bmatrix}2\\5\\-8\\-1\end{bmatrix},\begin{bmatrix}1\\4\\-7\\5\end{bmatrix}$

Since there are no free columns, the nullspace of A^T is zero, and the basis of $N(A^T)$ is empty