Determinants.

1) $\det I = 1$

2) det changes sign when two rows are swapped.

3) det is a linear function of each row separately

$$\det\begin{bmatrix} R_1 \\ \vdots \\ cR_i + dR_i' \\ \vdots \\ R_n \end{bmatrix} = c\det\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix} + d\cdot\det\begin{bmatrix} R_1 \\ \vdots \\ R_i' \\ \vdots \\ R_n \end{bmatrix}.$$

$$\det\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} = -\det\begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}$$

Corollaries of 1)-3):

4) If two rows are equal then $\det = 0$.

5) Subtracting a multiple of a row from another row leaves det unchanged.

6) A matrix with a zero row has $\det = 0$.

$$\det\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ 0 \\ \vdots \\ R_n \end{bmatrix} = \det\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_1 \\ \vdots \\ R_n \end{bmatrix} = 0$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} \xrightarrow[\text{upward}]{\text{eliminate}} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

A

$$\det \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} = a \det \begin{bmatrix} 1 & & \\ & b & \\ & & c \end{bmatrix} = ab \det \begin{bmatrix} 1 & & \\ & 1 & \\ & & c \end{bmatrix} = abc \det \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$= abc.$$

7) A triangular matrix has $\det A = a_{11} a_{22} \cdots a_{nn}$ (product of diag. entries)

8) A is singular (non-invertible) if and only if $\det A = 0$

A is invertible if and only if $\det A \neq 0$.

9) $\det(AB) = \det A \cdot \det B$

(super wrong: $\det(A+B) = \det A + \det B$)

Sketch:

$$D(A) = \frac{\det(AB)}{\det B}$$

satisfies 1)–3)

so $D(A) = \det A$

Split A into product of elem. matrices, check the case when A = elem.

10) $\det A^T = \det A$

$$PA = LU \qquad A^T P^T = L^T U^T$$

$$|P| \cdot |A| = |L| \cdot |U| \qquad |P^T| \cdot |A^T| = |L^T| \cdot |U^T|$$

P is a product of symmetric P_{ij} so $|P| = |P^T|$

$|L^T| = |L|$ and $|U^T| = |U|$.

$$\begin{vmatrix} a & & 0 \\ & b & \\ * & & c \end{vmatrix} = \begin{vmatrix} a & & * \\ 0 & b & \\ & & c \end{vmatrix}$$

so $\quad |P| \cdot |A| = |L| \cdot |U| = |P^T| \cdot |A^T| = |P| \cdot |A^T|$

so $\quad |A| = |A^T|$

So! rules 2) - 6) apply to columns, too!

Ex. $\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} \xlongequal[R_3-tR_2]{} \begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ 0 & 0 & 1-t^2 \end{vmatrix} \xlongequal[R_2-tR_1]{} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{vmatrix} =$

$= 1\cdot(1-t^2)(1-t^2) = (1-t^2)^2.$

Ex. $|5A| \overset{?}{=} \cancel{5|A|}$ $\det\begin{bmatrix} 5R_1 \\ 5R_2 \\ \vdots \\ 5R_n \end{bmatrix} = \underbrace{5\cdot 5 \cdots 5}_{n}\cdot \det\begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$

$= 5^n |A|$

$|-A| = \begin{cases} -|A| & \text{if odd size} \\ +|A| & \text{if even size} \end{cases}$

Ex. Sp. Q is orthogonal. Then $\det Q = \pm 1$.

$Q^T Q = I$

$|Q|^2 = |Q^T|\cdot|Q| = |I| = 1$ so $|Q| = \pm 1$.

Three ways to compute determinants:

- Use properties 1) - 10) above.
- Use the BIG FORMULA.
- Use the cofactor Formula.

BIG FORMULA.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$[a \ \ b] = [a \ \ 0] + [0 \ \ b]$$

$$= 0 + ad\begin{vmatrix} 1 & \\ & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0$$

$$= ad + bc(-1) = ad - bc$$

3×3:

$$\begin{array}{ccc} & \text{col } 1 \to 2 \to 3 & \text{col } 1 \to 3 \to 2 \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = & \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + & \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{31} & \end{vmatrix} + \end{array}$$

$$\begin{array}{cc} 2 \to 1 \to 3 & \\ + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + & \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} = \end{array}$$

$$+ \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} =$$

$$a \ldots \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

way to rearrange 1, 2, 3!

$3 \cdot 2 \cdot 1 = 3!$

$$= a_{11}a_{22}a_{33} \cdot 1 + a_{11}a_{23}a_{31}(-1) + a_{12}a_{21}a_{33}(-1) +$$

$$+ a_{12}a_{23}a_{31}(+1) + a_{13}a_{21}a_{32}(+1) + a_{13}a_{22}a_{31}(-1)$$

$=$  $-$ .

For 4×4 we would have $4 \cdot 3 \cdot 2 \cdot 1 = 24$ terms

$n \times n$ we would have $n(n-1) \cdots 2 \cdot 1 = n!$

Not a very practical formula.

$$\underset{n \times n}{\det A} = \sum (\det P)\, a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

$n!$ terms.

$n = 6$

720 terms

over all permutation matrices P

where P has 1's in places $1i_1, 2i_2, \ldots, ni_n$

+1 or −1 depending on even or odd # row swaps in P.

$$\begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} = a_{13} a_{21} a_{32} \det \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$3 = i_1 \quad 1 = i_2 \quad 2 = i_3$

Cofactor Formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ - & - & - \\ - & - & - \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ - & \cdots & \\ & & \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ - & - & - \\ - & - & - \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ - & - & - \\ - & - & - \end{vmatrix}$$

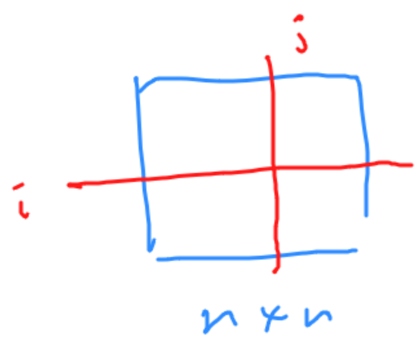
3×3

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{2\times 2}} \\ 0 & & \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ \boxed{\ } & 0 & \boxed{\ } \\ & 0 & \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ \multicolumn{2}{c}{\boxed{2\times 2}} & 0 \\ & & 0 \end{vmatrix}$$

2×2

$\det = C_{11}$ $\quad$ $C_{12} = -\det$ $\quad$ $C_{13} = \det$

Notation:



$\xrightarrow[\text{row } i,\ \text{col } j]{\text{remove}}$ $\boxed{M_{ij}}$

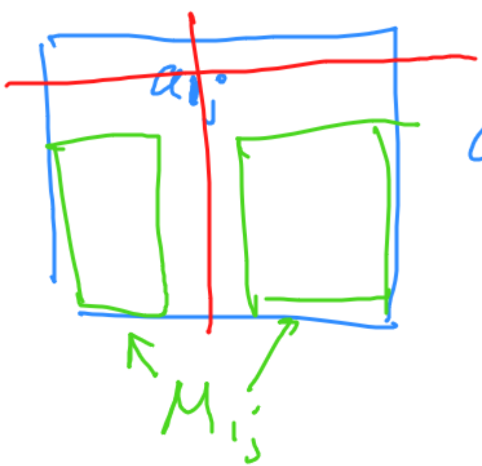
$(n-1)(n-1)$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

where $C_{1j} = (-1)^{j+1}\det M_{1j}$

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j}\det M_{ij}$



$a_{ij}\det \boxed{M_{ij}}$ goes into the sum with a plus or a minus (alternating out from a_{11} which always has a +).

Ex.

$$\begin{vmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} + (-1)(-1)\begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} =$$

cofactor formula in 1st row

$= 5$

signs alternating

$$= 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

cofactor 1st column

$$|D_n| = 2|D_{n-1}| - |D_{n-2}| \rightsquigarrow D_3 = 2\cdot 3 - 2 = 4$$

$n \times n$

$$|D_1| = |2| = 2 \qquad |D_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$D_4 = 2\cdot 4 - 3 = 5$

$D_n = n+1$