Name: ***Solutions***

1. Suppose $\begin{bmatrix}1\\3\\1\end{bmatrix}$ is a particular solution of $A\mathbf{x}=\mathbf{b}$ and $N(A)=\langle\begin{bmatrix}-2\\0\\1\end{bmatrix}\rangle$. Find two additional solutions to $A\mathbf{x}=\mathbf{b}$.

Solution: We are given that the equation $Ax=b$ has a particular solution $x_p=\begin{bmatrix}1\\3\\1\end{bmatrix}$, and that nullspace vectors have the form $x_n=t\begin{bmatrix}-2\\0\\1\end{bmatrix}$. Therefore, general solution is

$$x_p+x_n=\begin{bmatrix}1\\3\\1\end{bmatrix}+t\begin{bmatrix}-2\\0\\1\end{bmatrix}.$$

We can get specific solutions by taking any specific values of t. For example:

$$\text{for } t=1,\ \text{we get } x=\begin{bmatrix}1\\3\\1\end{bmatrix}+1\begin{bmatrix}-2\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\3\\2\end{bmatrix},$$

$$\text{for } t=2,\ \text{we get } x=\begin{bmatrix}1\\3\\1\end{bmatrix}+2\begin{bmatrix}-2\\0\\1\end{bmatrix}=\begin{bmatrix}-3\\3\\3\end{bmatrix}.$$

2. Determine whether the following vectors are independent $\begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$.

Solution: We organize these column vectors into a matrix and perform Gauss elimination:

$$\begin{bmatrix} 6 & 1 & 2 \\ 12 & 1 & 5 \\ 6 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 6 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We observe that there are only 2 pivots (which is < 3), which means that the vectors are dependent.

We were not asked to find a specific linear dependence, but we can do it looking at the obtained matrix. Indeed, we can express the free column through the pivot columns, by matching the second components and then the first components:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}, \text{ or } 1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = 0.$$

The same dependence then takes place for the respective original vectors:

$$1 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} = 0.$$

3. Find bases of the four fundamental spaces of $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 1 \end{bmatrix}$.

Solution: For $C(A)$ and $N(A)$, we bring the matrix to RREF (reduced row echelon form):

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3/3} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 1/3 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 13/3 \\ 0 & 1 & 1/3 \end{bmatrix} = R.$$

First and second are the pivot columns (which we actually knew before we even started, because of the 0 in the $(2,1)$ position), third is a free column. We therefore get that first two columns are a basis for $C(A)$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Looking at R, we see x_3 is a free variable, and the general solution to $Ax = 0$ has the form

$$\begin{bmatrix} -\frac{13}{3}x_3 \\ \frac{-1}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{13}{3} \\ \frac{-1}{3} \\ 1 \end{bmatrix},$$

which gives basis for $N(A)$: $\begin{bmatrix} -13/3 \\ -1/3 \\ 1 \end{bmatrix}$.

To find bases for $C(A^T)$ and $N(A^T)$, we perform the same elimination for the transposed matrix A^T:

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 5 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_2]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since both columns are pivot columns, we conclude that both columns of the original matrix A^T form a basis for $C(A^T)$ (to be fair, we knew that before the elimination, since there are only two columns and they are not proportional): $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

For the left nullspace $N(A^T)$, we note that there are not free columns and therefore the nullspace of A^T is zero. The basis for $N(A^T)$ is therefore empty (note that it's **not** the zero vector).