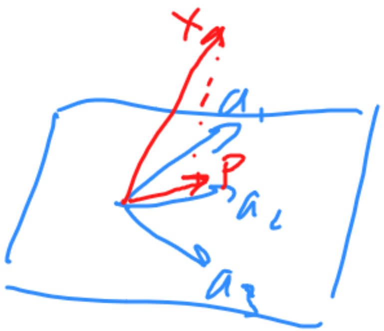
Correction to last time:



$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

$a_1, \dots, a_n$ indep.

Then $p = Px$, where

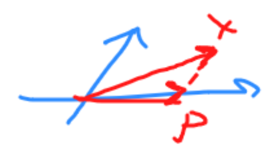
$$P = A(A^TA)^{-1}A^T$$

$$P^T = P$$

$$P^2 = P$$

last time: If $\underline{P^2 = P}$ & $\underline{P^T = P}$ then P is a projection matrix: onto $C(P)$, along $C(P)^{\perp} = N(P^T) = N(P)$.

If $P^2 = P$ (without $P^T = P$) then P is still a projection matrix but not for an orth. projection



Orthonormal bases and Gram-Schmidt process.

Vectors $q_1, q_2, \ldots, q_n$ are orthogonal if $q_i^T q_j = 0$ when $i \neq j$

Vectors $q_1, q_2, \ldots, q_n$ are orthonormal if $q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

"orthogonal and normalized"

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \qquad Q^T Q = I \quad \text{whenever } q_1 \ldots q_n \text{ are orthonormal.}$$

$m \times n$ matrix

$$\underbrace{\begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}}_{n \times m} \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{m \times n} = \underbrace{\begin{array}{c|c} & j \\ \hline i & \left[\begin{array}{ccc} & & \\ & q_i^T q_j & \\ & & \end{array}\right] \end{array}}_{n \times n}$$

such Q do not change lengths and dot products:

$$\|Qx\| = \|x\|, \quad (Qx)\cdot(Qy) = x\cdot y.$$

$$(Qx)\cdot(Qy) = (Qx)^T(Qy) = x^TQ^TQy = x^TIy = x^Ty = x\cdot y.$$

so $\|x\|^2 = x\cdot x = Qx\cdot Qx = \|Qx\|^2$.

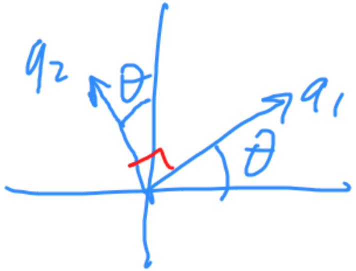
Special case: $n=m$ (square matrix Q):

$$Q^TQ = I \qquad Q^{-1} = Q^T$$

such matrices are called orthogonal. (square + columns make an orthonormal system)

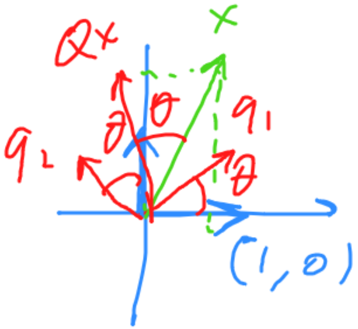
Ex. $\mathbb{R}^2$, $\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, $\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$. What does Q do?

$\quad\; q_1 \qquad\qquad q_2$



$$Q = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix},$$ rotation matrix.

Compute $Q\begin{bmatrix}1\\0\end{bmatrix}$, $Q\begin{bmatrix}0\\1\end{bmatrix}$



$$Q\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}.$$

$$Q\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

Back to general case (m is not necessarily $= n$):

$P = A(A^TA)^{-1}A^T$. What if $A = Q$? (cols of A are orthonormal).

$$P = Q(Q^TQ)^{-1}Q^T = QIQ^T = QQ^T$$

so $p = Px = QQ^Tx$, $\quad P = QQ^T$.

Note: $QQ^T \neq I$ unless Q is square!

__Ex.__ $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \neq I.$$

where do we get orthonormal bases?
(now)

Answer: Gram-Schmidt process.

It takes: $a_1, a_2, \ldots, a_n$ independent vectors

Produces: $q_1, q_2, \ldots, q_n$ orthonormal vectors s.t.:

$$\langle a_1 \rangle = \langle q_1 \rangle$$

$$\langle a_1, a_2 \rangle = \langle q_1, q_2 \rangle$$

$$\cdots$$

$$\langle a_1, \ldots, a_n \rangle = \langle q_1 \ldots q_n \rangle$$

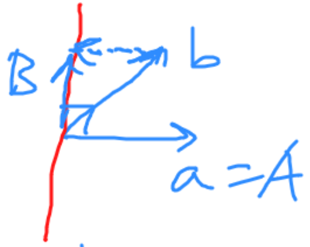
two parts:

1) make orthogonal vectors
2) normalize them.

Take indep. vectors a, b, c. We will make orthogonal vectors A, B, C, then normalize: q_1, q_2, q_3.

1) • $A = a$

• $B = b - xA \perp A$



$$A^T(b - xA) = 0$$

$$A^T b - x A^T A = 0, \quad x = \frac{A^T b}{A^T A}.$$

$$B = b - \frac{A^T b}{A^T A} A.$$

• $C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B \perp A, B$

2) $q_1 = \frac{A}{\|A\|} = \frac{A}{\sqrt{A^T A}}$ $\quad q_2 = \frac{B}{\|B\|} = \frac{B}{\sqrt{B^T B}}$ $\quad q_3 = \frac{C}{\|C\|} = \frac{C}{\sqrt{C^T C}}$.

Gram-Schmidt process in matrix form:

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$m \times 3$ / $m \times n$; $m \times 3$ / $m \times n$ orthonormal cols ; 3×3 / $n \times n$ upper triang

$$A = Q \cdot R$$

QR-decomposition!
Q orthonormal cols
R upper triangular.

why QR? Projection / least squares:

$$A^T A \hat{x} = A^T b \quad , \quad A = QR$$

$$\cancel{R^T} \underbrace{\cancel{Q^T Q}}_{I} R \hat{x} = \cancel{R^T} Q^T b$$

$$R\hat{x} = Q^T b, \quad \hat{x} = R^{-1} Q^T b$$

Example $\underset{a}{\begin{bmatrix}1\\1\\-1\end{bmatrix}}, \underset{b}{\begin{bmatrix}2\\0\\1\end{bmatrix}}, \underset{c}{\begin{bmatrix}1\\1\\0\end{bmatrix}}$

- $A = a = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \qquad q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$
- $B = b - \frac{A^Tb}{A^TA}A = \begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{2+0-1}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}5/3\\-1/3\\4/3\end{bmatrix}. \qquad q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{42}}\begin{bmatrix}5\\-1\\4\end{bmatrix}$

  Note: $A \cdot B = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}5/3\\-1/3\\4/3\end{bmatrix} = \frac{5}{3} - \frac{1}{3} - \frac{4}{3} = 0.$
- $C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B =$

$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} - \frac{5/3 - 1/3}{\left(\frac{5}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{4}{3}\right)^2}\begin{bmatrix}5/3\\-1/3\\4/3\end{bmatrix} =$$

$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} - \frac{2}{21}\begin{bmatrix}5\\-1\\4\end{bmatrix} = \begin{bmatrix}-1/7\\3/7\\2/7\end{bmatrix}.$$

Note: $\begin{bmatrix}-1/7\\3/7\\2/7\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix} = 0$

$\begin{bmatrix}-1/7\\3/7\\2/7\end{bmatrix} \cdot \begin{bmatrix}5/3\\-1/3\\4/3\end{bmatrix} = \frac{-5-3+8}{21} = 0$

$$q_3 = \frac{c}{\|c\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Matrix form:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \\ \begin{matrix} a & \quad b & \quad c \end{matrix} \end{array} = \begin{array}{c} \begin{bmatrix} 1/\sqrt{3} & 5/\sqrt{42} & -1/\sqrt{14} \\ 1/\sqrt{3} & -1/\sqrt{42} & 3/\sqrt{14} \\ -1/\sqrt{3} & 4/\sqrt{42} & 2/\sqrt{14} \end{bmatrix} \\ \begin{matrix} q_1 & \qquad q_2 & \qquad q_3 \end{matrix} \end{array} \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} & 2/\sqrt{3} \\ 0 & 14/\sqrt{42} & 4/\sqrt{42} \\ 0 & 0 & 2/\sqrt{14} \end{bmatrix}.$$