# Gauss Elimination

(Gaussian)

Ex. $\begin{cases} x + 2y + z = 3 \\ 2x + 5y - z = -4 \\ 3x - 2y - z = 5 \end{cases} \xrightarrow{\text{something}} \begin{cases} x + 2y + z = 3 \\ y - 3z = -10 \\ z = 3 \end{cases}$

upper triangular system

$x = 3 - 2y - z = 3 + 2 - 3 = 2$

$y = -10 + 3z = -1$

$z = 3$

Back substitution

something → Elementary row operations

1) swap two equations (rows)
2) multiply an eqn by a nonzero number
3) add a multiple of a row to another row.

pivot = 1st nonzero coeff. in a row that does elimination

$\begin{cases} 1x + 2y + z = 3 \\ 2x + 5y - z = -4 \\ 3x - 2y - z = 5 \end{cases} \xrightarrow{R_2 \to R_2 + (-2)R_1} \begin{cases} 1x + 2y + z = 3 \\ 0x + y - 3z = -10 \\ 3x - 2y - z = 5 \end{cases} \xrightarrow{R_3 \to R_3 + (-3)R_1} \begin{cases} x + 2y + z = 3 \\ 1y - 3z = -10 \\ -8y - 4z = -4 \end{cases} \longrightarrow$

pivot

pivot

$\xrightarrow{R_3 \to R_3 + 8R_2} \begin{cases} x + 2y + z = 3 \\ y - 3z = -10 \\ -28z = -84 \end{cases} \xrightarrow{R_3 \to \frac{R_3}{28}} \begin{cases} x + 2y + z = 3 \\ y - 3z = -10 \\ z = 3 \end{cases}$

multipliers : $= \dfrac{\text{coeff to eliminate}}{\text{pivot}}$ .

Failures of Gauss Elim.:

- temporary failure: $\begin{cases} 2y = 6 \\ 2x + 3y = -6 \end{cases}$ swap! $R_1 \leftrightarrow R_2$

- permanent failure:

1) $\begin{cases} 2x + 7y = 3 \\ -4x - 14y = -6 \end{cases} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{cases} 2x + 7y = 3 \\ 0x + 0y = 0 \end{cases}$ $x = \dfrac{3 - 7y}{2}$

infinitely many solutions

2) $\begin{cases} 2x + 7y = 3 \\ -4x - 14y = -5 \end{cases} \xrightarrow{\cdots} \begin{cases} 2x + 7y = 3 \\ 0 = 1 \end{cases}$

inconsistent system
no solutions.

Gauss elimination leads to a unique solution if and only if there are as many pivots as there are unknowns.

$$\begin{cases} 3x_1 + 6x_2 - 3x_3 + 9x_4 = 12 \\ 2x_1 + 4x_2 - 2x_3 + 7x_4 = 10 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -6 \end{cases}$$

$$\underbrace{\left[\begin{array}{cccc|c} 3 & 6 & -3 & 9 & 12 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right]}_{\text{augmented matrix}} \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] \longrightarrow$$

matrix of the system of eqns

augmented matrix

variables in pivot columns are called dependent variables

$$\xrightarrow[R_3 \to R_3 + 1R_1]{R_2 \to R_2 + (-2)R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 4 \\ x_4 = 2 \end{cases}$$

$x_1 = 4 - 2x_2 + x_3 - 3x_4 = -2 - 2x_2 + x_3$

$x_4 = 2$

columns with no pivots correspond to free variables

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - 2x_2 + x_3 \\ x_2 \\ x_3 \\ 2 \end{bmatrix}$

← only free variables in the RHS

$\#\overset{\text{unknowns}}{\text{vars}} = \#\text{ pivots} \quad \Longleftrightarrow \quad \text{unique sol}$

$\underset{\text{unknowns}}{\#\text{ vars}} > \#\text{ pivots} \quad \Longleftarrow \quad \text{multiple or no sol} \quad (\text{singular system})$

$$A = \left[\begin{array}{cccc|c} 3 & 6 & -3 & 9 & 12 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1}$$

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$

EA is the result of applying $R_1 \to \frac{R_1}{3}$ to E !

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} 3 & 6 & -3 & 9 & 12 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-2)R_1}$$

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + (-2)R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$. EA is the result of $R_2 \longrightarrow R_2 + (-2)R_1$ :

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right]$$

- $R_1 \longleftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$. Then EA is the result of $R_1 \leftrightarrow R_2$.