Orthogonality.

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

$$C(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle \qquad N(A^T) = \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

$$C(A^T) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\rangle \qquad N(A) = \left\langle \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$$

coincidence?

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

terminology: if $v \cdot w = 0$ the we say v, w are perpendicular, or orthogonal to each other.

- V is orthogonal to W if $v \cdot w = 0$ for <u>all</u> v from V and <u>all</u> w from W.

<u>Ex.</u>

perpendicular but not orthogonal.

$v \cdot w = 0$

$v = u, w = u$ then if planes were orth., $u \cdot u = 0$.

- If V, W are orthogonal then their intersection is zero! $V \cap W = 0$.

$N(A)$ is orthogonal to $C(A^T)$

why:

- nullspace of A: all vectors x s.t. $Ax=0$

$$0 = Ax = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } m \end{bmatrix} x = \begin{bmatrix} (\text{row } 1)\cdot x \\ (\text{row } 2)\cdot x \\ \vdots \\ (\text{row } m)\cdot x \end{bmatrix} \begin{matrix} = 0 \\ = 0 \\ \vdots \\ = 0 \end{matrix}$$

x is perp. to every row, so to the whole row space.

- another notation for the same reasoning:

recall: if v, w are column vectors then $v\cdot w = v^T w$

recall: $C(A^T)$ consists of all $b = A^T y$ for some y.

supp. x is in $N(A)$. Consider the dot product:

$$(A^T y)\cdot x = (A^T y)^T x = y^T A x = y^T (Ax) = y^T 0 = 0.$$

So: $N(A)$ orth. to $C(A^T)$, $N(A^T)$ orth. to $C(A)$.

More terminology: Orthogonal complement of V contains <u>every</u> vector perpendicular to V.

Notation: $V^\perp$ ("V-perp")

So: $N(A) = C(A^T)^\perp$, and $N(A^T) = C(A)^\perp$

Fundamental Thm of Linear Algebra, Part 2:

$$m\ \overset{n}{\boxed{A}}$$

- $N(A) = C(A^T)^\perp$
- $N(A^T) = C(A)^\perp$

reminder: Part 1 says

$$\dim \overset{n-r}{N(A)} + \dim \overset{r}{C(A^T)} = n$$
$$\dim \underset{m-r}{N(A^T)} + \dim \underset{r}{C(A)} = m$$

0) $V^\perp$ is a subspace.

suppose w_1, w_2 are in $V^\perp$

claim: $c_1 w_1 + c_2 w_2$ is still in $V^\perp$

take any v from V, compute dot product:

$$v \cdot (c_1 w_1 + c_2 w_2) = c_1 v \cdot w_1 + c_2 v \cdot w_2 = c_1 0 + c_2 0 = 0.$$

1) $V \cap W = 0$ if V, W are orthogonal to each other $(u \cdot u = 0)$

2) If V, W orthogonal to each other, and if:

$v_1, v_2, \dots, v_k$ are lin. indep. in V

$w_1, w_2, \dots, w_\ell$ are lin. indep. in W

then $v_1, v_2, \dots, v_k, w_1, \dots, w_\ell$ are lin. indep.:

sp. $c_1 v_1 + \dots + c_k v_k + d_1 w_1 + \dots + d_\ell w_\ell = 0$

$$\underbrace{c_1 v_1 + \dots + c_k v_k}_{\text{in } V} = \underbrace{-d_1 w_1 - \dots - d_\ell w_\ell}_{\text{in } W}$$

so both sides $= \vec{0}$

so $c_1 = c_2 = \dots = c_k = 0$

$d_1 = \dots = d_\ell = 0$

$\dim C(A^T) + \dim N(A) = n$

$\{$basis for $C(A^T)$, basis for $N(A)\}$ is linearly indep., and has n vectors.

so this is a basis for $\mathbb{R}^n$.

so every x in $\mathbb{R}^n$ can be expressed as

$x = x_r + x_n$ where x_r is in $C(A^T)$

x_n is in $N(A)$

In general: V, $V^\perp$ span the whole space $\mathbb{R}^n$.

that is: every x in $\mathbb{R}^n$ can written as

$x = v + v'$ where v in V, v' in $V^\perp$.

why:

$V \longrightarrow \begin{bmatrix} \text{basis vector } 1 \\ \vdots \\ \text{basis vector } r \end{bmatrix} = A$, $C(A^T) = V$

$\dim V = r$

$V^\perp = C(A^T)^\perp = N(A)$, so

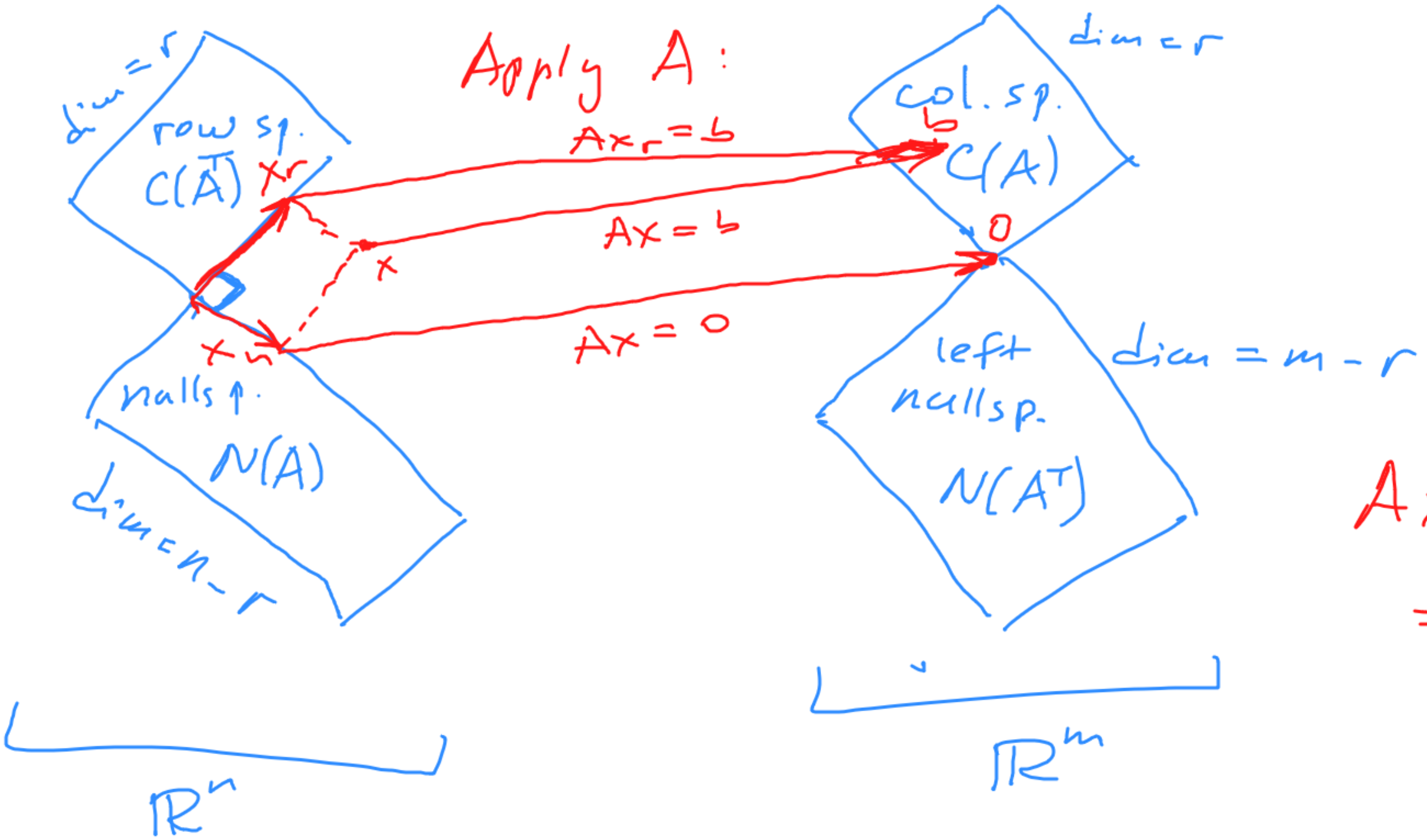
$\dim N(A) + \dim C(A^T) = n$

$\dim V^\perp + \dim V$

Moreover: this representation is unique:

if $v + v' = x = v_1 + v_1' \rightarrow \underbrace{v - v_1}_{V} = \underbrace{v_1' - v'}_{V^\perp}$ so $v - v_1 = 0$, $v' - v_1' = 0$

A $m \times n$ matrix. Draw "big picture".



reminder:
col space = all b s.t. $b = Ax$

$$Ax = A(x_r + x_n) =$$
$$= Ax_r + Ax_n$$
$$= Ax_r$$

Ex. $S = \langle (1,2,2,3), (1,3,3,2) \rangle$

Find basis for $S^\perp$, find $\dim S^\perp$.

$$\begin{cases} (1,2,2,3)\cdot(x_1,x_2,x_3,x_4) = 0 \\ (1,3,3,2)\cdot(x_1,x_2,x_3,x_4) = 0 \end{cases} \qquad \begin{cases} x_1+2x_2+2x_3+3x_4=0 \\ x_1+3x_2+3x_3+2x_4=0 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_1-2R_1} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix} \\ \uparrow\ \uparrow\ \quad \uparrow\ \uparrow \\ \text{pivot} \quad \text{free} \end{array}$$

$x_1 = -5x_4$

$x_2 = -x_3 + x_4$

x_3, x_4 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5x_4 \\ -x_3+x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $S^\perp$: $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\dim S^\perp = 2$ (we knew that: $\dim S^\perp = \dim \mathbb{R}^4 - \dim S = 4 - 2 = 2$)