Name: ***Solutions***

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1. Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. What is the dimension of its orthogonal complement, $S^\perp$? Find a basis for $S^\perp$.

Solution: We need to find vectors x s.t. $(1, 2, 2, 3) \cdot x = 0$ and $(1, 3, 3, 2) \cdot x$. In other words, we need to solve $Ax = 0$, where $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$. We do this in the usual way, by row reduction:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

We see that x_3, x_4 are free variables, and x_1, x_2 are pivot variables. This gets us that every vector x in $S^\perp$ has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

In other words, $S^\perp$ is

$$S^\perp = \langle \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

2. (a) Find the projection matrix P onto the plane $x - y - 2z = 0$.

Solution: The plane given by $x - y - 2z = 0$ is spanned by the special solutions of that equation:

$$\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Since the matrix $\begin{bmatrix} 1 & -1 & -2 \end{bmatrix}$ is already in row reduced echelon form, we get $x = y + 2z$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y + 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

So the space onto which we are projection is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. We organize these two vectors in a matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and compute $P = A(A^T A)^{-1} A^T$. For that, first find $A^T A$ and its inverse.

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

We can invert this matrix by Gauss–Jordan elimination, or by formula for the inverse of a 2×2 matrix. Either way, we get

$$(A^T A)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

Then we only need to perform the matrix multiplications to compute P:

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}.$$

(b) Is the matrix $I - P$ a projection matrix? If it is, which subspace it projects onto? You may answer it for this example or in general!

Solution: If P is a projection matrix onto V, then the matrix $I - P$ is always a projection matrix onto the orthogonal complement of V, in this case, the latter space is $\langle(1, -1, -2)\rangle$—remember that the plane was given by an equation with coefficients $[1 \ -1 \ -2]$ (that is, but its nullspace basis) to begin with. We can also see this directly by computing $I - P$:

$$I - P = \frac{1}{6} \left(\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} \right) = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}.$$

Notice that each column of $I - P$ is a multiple of $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

3. For values $x = -2, -1, 0, 1$ and $y = 1, 0, 0, 3$, find the best parabola $y = C + Dx + Ex^2$ for this set of data.

Solution: Ideally, we would want C, D, E s.t. $C + Dx + Ex^2$ at $x = -2$ gives $y = 1$, at $x = -1$ gives $y = 0$, and so on (for all four data points). In other words, we want to solve the system

$$\begin{aligned} C + D(-2) + E(-2)^2 &= 1 \\ C + D(-1) + E(-1)^2 &= 0 \\ C + D0 + E0^2 &= 0 \\ C + D1 + E1^2 &= 3, \end{aligned}$$

or, in matrix form,

$$\underbrace{\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \\ E \end{bmatrix}}_{\widetilde{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}}_{b}.$$

The equation $A\widetilde{x} = b$ has no solution, so instead we minimize the length of the difference: $\|A\widetilde{x} - b\|$. Recall that this is the same as looking for a vector $A\widetilde{x} - b$ orthogonal to column space of A, in other words, we are looking for $\widetilde{x}$ that produces

$$A^T(A\widetilde{x} - b) = 0,$$

that is,

$$A^T A\widetilde{x} = A^T b.$$

(A mental shortcut is that we simply multiply $A\widetilde{x} = b$ by A^T on the left.) We get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \widetilde{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Computing the matrix products, we get

$$\begin{bmatrix} 4 & -2 & 6 \\ -2 & 6 & -8 \\ 6 & -8 & 18 \end{bmatrix} \widetilde{x} = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}.$$

This system we can solve in the usual way:

$$\left[\begin{array}{ccc|c} 4 & -2 & 6 & 4 \\ -2 & 6 & -8 & 1 \\ 6 & -8 & 18 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ -2 & 6 & -8 & 1 \\ 6 & -8 & 18 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & 3 \\ 0 & -5 & 9 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & 3 \\ 0 & 0 & 4 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 3/5 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

which gives $E = 1$, $D = 3/5 + E = 8/5$, $C = 1 + D/2 - 3E/2 = 3/10$. Therefore, the best fitting parabola is $y = \frac{3}{10} + \frac{8}{5}x + x^2$.

4. Check if the vectors $(2,2,-1)$ and $(-1,2,2)$ are orthogonal. Divide them by their lengths and copy them into columns of a matrix Q. Find both Q^TQ and QQ^T [they are not the same!].

Solution: The dot product is $(2,2,-1)\cdot(-1,2,2) = -2+4-2 = 0$, so the vectors are indeed orthogonal. Length of both vectors is $\sqrt{2^2+2^2+(-1)^2} = 3$. This gives us matrices:

$$Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \quad \text{and} \quad Q^T = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}.$$

Compute the product Q^TQ:

$$Q^TQ = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(Notice that the above matrix product essentially computes the dot products between columns of Q.)

On the other hand, matrix QQ^T is far from an identity matrix:

$$QQ^T = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 2/9 & 0 \\ 2/9 & \vdots & \vdots \\ \vdots & & \end{bmatrix}.$$

Note that if Q was a *square* matrix, the product QQ^T would also be an identity matrix.