MA 232 **Exam 1** **Sep 22, 2022**

Name: ***Solutions***

Check your Section: ☐ A – 12:30pm ☐ B – 3:30pm

Closed book: only 1 one-sided handwritten lettersize "cheat-sheet" allowed.
Answers must include supporting work, unless stated otherwise in the question.
Calculators, cell phones, all other electronics out of sight. *75 minutes to complete.*

Check that all 5 pages are present in this booklet. **Put solutions on the page where the problem is stated. If you run out of room, use the reverse and clearly indicate that. Do not submit anything other than this booklet.**

1. [4 pt] Assume $\mathbf{v}$ and $\mathbf{w}$ are **unit** vectors, that is, their length is 1. What is the value of the dot product $(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})$?
For this question, you are not required to show your work. There is only one correct answer.

   - ○ -2
   - ○ -1
   - ○ 0
   - ○ 1
   - ○ 2
   - ○ Impossible to tell without further information about $\mathbf{v}$ and $\mathbf{w}$.

   Solution: Answer: (c) 0.

   Expand brackets:

$$\begin{aligned}(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) &= \mathbf{v}\cdot(\mathbf{v}-\mathbf{w})+\mathbf{w}\cdot(\mathbf{v}-\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}\end{aligned}$$

   Now recall that $\mathbf{v}\cdot\mathbf{v}=\|v\|^2=1$ and the same with $\mathbf{w}\cdot\mathbf{w}=1$; and that $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$. This gives $1-\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}-1=0$.

   By the way, one may notice that the question should still work when $\mathbf{v}=\mathbf{w}$, so the real choice is between answers 0 and "Impossible to tell".

2. [4 pt] Which of the following best describes the angle between the vectors $(1,1-a^2,1,a^2)$ and $(-1,1,a^2,-1)$?
For this question, you are not required to show your work. There is only one correct answer.

   - ○ It is an acute angle ($<\frac{\pi}{2}$).
   - ○ It is a right angle ($\frac{\pi}{2}$).
   - ○ It is an obtuse angle ($>\frac{\pi}{2}$).
   - ○ It can be acute ($<\frac{\pi}{2}$) or right ($\frac{\pi}{2}$), depending on a.
   - ○ It can be obtuse ($>\frac{\pi}{2}$) or right ($\frac{\pi}{2}$), depending on a.
   - ○ It can be acute, right, or obtuse, depending on a.

Solution: Answer: (e) obtuse or right.

Compute the dot product:

$$(1, 1-a^2, 1, a^2) \cdot (-1, 1, a^2, -1) = -1 + (1-a^2) + a^2 - a^2 = -a^2.$$

This can only be negative (corresponding to negative cosine, therefore an obtuse angle) or zero (zero cosine, therefore a right angle).

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3. [4 pt] Consider the system of equations $\begin{cases} ax + by = 0, \\ cx + dy = 0, \end{cases}$ where the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is known to be **invertible**. Which statement best describes the solution(s) of the system?
For this question, you are not required to show your work. There is only one correct answer.

   - ○ There are no solutions.
   - ○ There might be infinitely many or no solutions.
   - ○ There are infinitely many solutions.
   - ○ There is a unique solution, but we don't know that solution unless we know a, b, c, d.
   - ○ There is a unique solution: $x = 0, y = 0$.

Solution: Answer: (e) There is a unique solution: $x = 0, y = 0$.

Recall that when the matrix of a system of linear equations is invertible, there is a unique solution.

Also note that $x = 0$ and $y = 0$ is clearly a solution, regardless of particular values of a, b, c, d (we can see this, for example, buy plugging in 0 for x and y).

4. [4 pt] Given two matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, what is the $(2, 2)$ entry (row 2, column 2) of AB?
For this question, you are not required to show your work. There is only one correct answer.

   - ○ $3c + 4d$
   - ○ $3d + 4c$
   - ○ $3b + 4d$
   - ○ $3d + 4b$
   - ○ $2c + 4d$
   - ○ $2d + 4c$
   - ○ $2b + 4d$
   - ○ $2b + 4c$
   - ○ None of the above, since AB is not even defined.

Solution: Answer: (c) $3b + 4d$. We need to multiply second row of A by second column of B: $\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 3b + 4d.$

5. [4 pt] If $A = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix}$ and X is some 3×100 matrix, then computing AX has the same effect as applying the following row transformations to matrix X:
For this question, you are not required to show your work. There is only one correct answer.

- ○ First $R_2 \to R_2 - 7R_1$, then $R_3 \to R_3 + 5R_2$.
- ○ First $R_3 \to R_3 + 5R_2$, then $R_2 \to R_2 - 7R_1$.
- ○ First $R_2 \to R_2 + 7R_1$, then $R_3 \to R_3 - 5R_2$.
- ○ First $R_3 \to R_3 - 5R_2$, then $R_2 \to R_2 + 7R_1$.
- ○ None of the above.

Solution: This question had a typo. Both the answer (e) (technically correct for question as typed) and the answer (b) (correct if the typo is fixed) were accepted for full credit.

Answer: technically, (e) None of the above.

All of the first four answers lead to a lower triangular matrix, but the matrix A has a 1 in the $(2,3)$ position above the main diagonal, so none of those four options can possibly work, leading to answer (e) None of the above.

The question intended to have A with 0 in that position, though:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & \mathbf{0} \\ 0 & 5 & 1 \end{bmatrix}.$$

Then the answer would be (b): First $R_3 \to R_3 + 5R_2$, then $R_2 \to R_2 - 7R_1$.

Indeed, the first option would lead to a different matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 7R_1} \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 5R_2} \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ -35 & 5 & 1 \end{bmatrix} \neq A.$$

The second option does lead to a matrix A (with corrected typo):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 5R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{R_2 - 7R_1} \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = A.$$

We don't need to worry about them now that we found the correct answer, but the other two options have a signs of multipliers and there are incorrect.

6. *In this question, you need to show your work.*
Consider Gauss elimination for the following linear system for some constants b and g:

$$\begin{aligned} 1x + by + 0z &= 1, \\ 1x - 2y - 1z &= 2, \\ 0x + 1y + 1z &= g. \end{aligned}$$

(a) [4 pt] Which number b leads to a row exchange (swap)?

(b) [4 pt] Which number b makes the system singular (missing a third pivot)? (Note: the constant b you find here might be different from the one in part (a).)

(c) [4 pt] For the b you found in part (b), for which value of g the system has infinitely many solutions?

(d) [4 pt] Find the solution of the system with b and g found in part (b) and (c), if $z = 2$.

Solution: (a) We start Gauss Elimination:

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & g \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 0 & -2-b & -1 & 1 \\ 0 & 1 & 1 & g \end{array}\right].$$

At this point we need to swap rows only when the entry $-2-b$ in the second row is 0, that is $b = -2$. If $b \neq -2$, the next row transformation $R_3 \to R_3 + \frac{1}{2+b} R_2$ will bring the matrix to triangular form.

(b) We don't have to, but it is easier to perform the swap regardless of value of b and then do the remaining elimination:

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 0 & -2-b & -1 & 1 \\ 0 & 1 & 1 & g \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 0 & 1 & 1 & g \\ 0 & -2-b & -1 & 1 \end{array}\right] \xrightarrow{R_3 + (b+2)R2} \left[\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 0 & 1 & 1 & g \\ 0 & 0 & -1+(b+2) & 1+(b+2)g \end{array}\right].$$

Therefore, the system becomes singular precisely when $-1 + (b+2) = 0$, i.e. $b = -1$.

We would still arrive at the same conclusion if we didn't perform the row exchange.

(c) The system will have infinitely many solutions when it is singular, so $b = -1$, and the last equation is $0 = 0$, i.e., when $1 + (-1+2)g = 0$, that is, $g = -1$.

(d) When $b = -1$ and $g = -1$, from the last matrix, we get equations $x + by = 1$, $y + z = g$, so $y = g - z = -1 - z$ and $x = 1 - by = 1 - b(g - z) = 1 - bg + bz = -z$, leading to solution:

$$\begin{bmatrix} -z \\ -1-z \\ z \end{bmatrix}. \quad \text{When } z = 2, \text{ this gives: } \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}.$$

Note that we didn't really need to find the general solution. Instead, we could just take the triangular system $x + (-1)y = 1$, $y + 2 = -1$ and solve it for x and y.

7. [15 pt] *In this question, you need to show your work.*
Use Gauss–Jordan elimination to find A^{-1} if

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2/(-1), R_3/(-1)}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 - R_3}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_3}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 2 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right].$$

Since we arrived at an identity matrix in the left half, the right half shows the inverse matrix:

$$\begin{bmatrix} -1 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

8. *In this question, you need to show your work.*
Let
$$A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 1 & -2 \end{bmatrix}.$$

(a) [12 pt] Find the LU decomposition of A.

(b) [3 pt] Find the LDU decomposition of A.

Solution: (a)

$$\begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -7 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -7 \\ 0 & 0 & 5 \end{bmatrix} = U.$$

The two row transformations are delivered by multiplying by $E_{21}(-2)$ first, and by $E_{32}(-1)$ second:
$$E_{32}(-1)E_{21}(-2)A = U.$$
Then $A = (E_{32}(-1)E_{21}(-2))^{-1}U = E_{21}(2)E_{32}(1)U$, so
$$L = E_{21}(2)E_{32}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

This gives the LU-decomposition:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -7 \\ 0 & 0 & 5 \end{bmatrix}.$$

(b) For LDU-decomposition, we need to bring out pivot out of each row:
$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -7 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2/2 & 2/2 & 3/2 \\ 0 & 1 & -7 \\ 0 & 0 & 5/5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$