vector spaces and subspaces.

add vectors
number · vector

U is a subspace of vector sp. V
if whenever v, w are in U, then
so are $v+w$ and cv [$cv+dw$ in U].

Column space and nullspace of a matrix.

Col. space of an $m \times n$ matrix A = all linear combinations of columns of A.

Notation: $C(A)$.

Note: $C(A)$ is in $\mathbb{R}^m$

$$A\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = c_1 \cdot (\text{col } 1) + c_2 (\text{col } 2) + \dots + c_m (\text{col } m)$$

In other words, the equ $Ax=b$ has a sol'n
if and only if b is in $C(A)$.

If $v_1, v_2, \ldots, v_N$ are N vectors in a vector space V.

Then all lin. combinations

$$c_1 v_1 + c_2 v_2 + \ldots + c_N v_N$$ form a subspace of V spanned by $v_1, v_2, \ldots, v_N$.

$$\frac{+\ d_1 v_1 + \ldots + d_N v_N}{(c_1+d_1) v_1 + \ldots + (c_N+d_N) v_N}$$

Nullspace of A.

$A x = \underline{\underline{0}}$ sol's: $Av = 0$, $Aw = 0$. then $A(v+w) = 0 + 0 = 0$

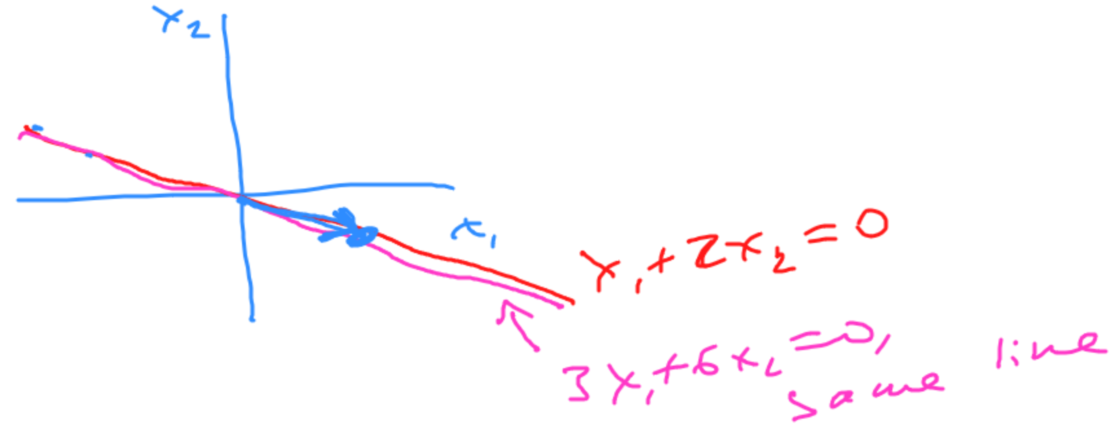
$A(cv) = c \cdot 0 = 0$

Then nullspace $N(A)$ consists of all sol's of $Ax = 0$.

if A is $m \times n$ then $N(A)$ is in $\mathbb{R}^n$.

Ex 1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\quad \begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases} \longleftrightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 0 = 0 \end{cases}$

row picture:



Describe this line:

pick special sol'n $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Nullspace: $c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Ex 2 $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad Ax = 0$ has the only sol'n $x_1 = 0, x_2 = 0$

Nullspace of A: zero subspace.

Ex 3 $A = [1 \quad 2 \quad 3]$ $\qquad x_1 + 2x_2 + 3x_3 = 0$

special sol'ns : $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = s_1$, $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = s_2$

$N(A) : c_1 s_1 + c_2 s_2 = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Ex 4 $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

General procedure to produce special solutions that span the nullspace $N(A)$:

what's this?

1) Bring A to its reduced row echelon form.
2) Find special sol'ns by assigning 1 to one free var and 0 to all others, in all possible ways.

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{R_2/2}$$

$$\to \begin{bmatrix} 1 & \boxed{0} & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

the reduced row echelon form:

(1) upper triangular (in each nonzero row the pivot is to the right of pivots in the preceding rows)

(2) only zeros above each pivot

(3) every pivot $= 1$.

How to bring A to its reduced row echelon form:

(1) Produce upper triangular U, by running Gauss elim.

(2) Produce zeros above pivots
(use each pivot to eliminate upwards)

(3) Make each pivot $=1$
(divide each nonzero row by its pivot)

<u>Ex.</u>

$$U = \begin{bmatrix} 1 & \boxed{*} & * & \boxed{*} & * \\ 0 & 1 & * & \boxed{*} & * \\ 0 & 0 & 0 & 1 & * \\ 0 & - & - & - & 0 \end{bmatrix}$$

second first

Step (2): to do less computation, go from right to left.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \rightsquigarrow U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$Ax = 0$ and $Rx = 0$ have exact same solutions.

Note: If $N(A) = 0 \longrightarrow Ax = 0$ has only one solution $x = 0$

$\longrightarrow$ columns of A are independent.

Special solutions:

$$A \rightsquigarrow R = \begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2, x_4 pivot vars

x_3, x_5 free vars

pivot cols (vars)

free cols (vars)

To find special sols:

1) Identify free variables (correspond to cols without pivots)

2) Set one free variable to be 1, the rest to be 0, find the corresp. solution

3) Repeat step (2) for each free var.

- If $n > m$ (more vars than eqns), then an $m \times n$ matrix will always have free vars
- The rank of A is the number of pivots in its reduced row echelon form. ("size" of A).

$$\begin{bmatrix} 1 & 0 & * & * & * & 0 & * \\ 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1\ x_2\ x_6$ pivot vars

x_3, x_4, x_5, x_7 free vars

4 special sols

rank 3.

$$A=\begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 2 & 2 & 3 & 6 & 10 \\ & & 1 & 2 & 4 \\ & & 1 & 5 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 4 \\ & & 1 & 2 & 4 \\ & & 1 & 5 & 4 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ & & 1 & 2 & 4 \\ & & 0 & 0 & 0 \\ & & 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ & & 1 & 2 & 4 \\ & & 0 & 3 & 0 \\ & & 0 & 0 & 0 \end{bmatrix} = U$$

$$U = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ & & 1 & 2 & 4 \\ & & 0 & 3 & 0 \\ & & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3/3]{R_1/2} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ & & 1 & 2 & 4 \\ & & 0 & 1 & 0 \\ 0 & \cdots & & & 0 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ & & 1 & 0 & 4 \\ & & 0 & 1 & 0 \\ & & & & 0 \end{bmatrix} \longrightarrow$$

free cols

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 0 & 4 \\ & & 0 & 1 & 0 \end{bmatrix} = R$$

pivot cols