Peter Rauscher HW8

I pledge my honor that I have abided by the Stevens Honor System.

1) a) $S^T = \begin{bmatrix} 4 & 8 & 0 \\ 8 & 25 & 9 \\ 0 & 9 & 25 \end{bmatrix} = S$ thus S is symmetric

If all upper left corner determinants are positive, then S is positive definite

$\det[4] = 4 > 0$ ✓

$\det\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = 4(25) - 8(8) = 100 - 64 = 36 > 0$ ✓

$\det\begin{bmatrix} 4 & 8 & 0 \\ 8 & 25 & 9 \\ 0 & 9 & 25 \end{bmatrix} = 4(25^2 - 9^2) - 8(8(25) - 9(0)) + 0$
$= 4(544) - 8(191)$
$= 648 > 0$ ✓

Thus, S is positive definite.

b) First, elimination

$\begin{bmatrix} 4 & 8 & 0 \\ 8 & 25 & 9 \\ 0 & 9 & 25 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 4 & 8 & 0 \\ 0 & 9 & 0 \\ 0 & 9 & 25 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 4 & 8 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} = U$

$LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 9 & \\ & & 25 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$C = \sqrt{D} L^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

$S = C^T C = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

2a) S is positive definite if $x^TSx > 0$ for every nonzero x where

$S=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $\quad x^TSx=\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=ax^2+2bxy+cy^2$

First, we solve for the matrix $S=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with our given quadratic form $f(x,y)$

$$x^2+xy+y^2=ax^2+2bxy+cy^2$$

$$a=1 \qquad 2b=1 \qquad c=1$$
$$b=1/2$$

So, S is a symmetric matrix $S=\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

Recall that $n\times n$ matrix S is positive definite if its n upper-left determinants are positive. So,

$$\det|1| = 1 > 0 \checkmark$$
$$\det\begin{vmatrix} 1 & 1/2 \\ 1/2 & 1 \end{vmatrix} = (1)(1)-(1/2)(1/2) = 3/4 > 0 \checkmark$$

Thus, the symmetric matrix S is positive definite.

2b) $x^2+xy+y^2=1$ $\quad \begin{vmatrix} 1-\lambda & 1/2 \\ 1/2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2-\frac{1}{4} = \lambda^2-2\lambda+3/4$

$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$ $\quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$ $\quad \lambda = 3/2, 1/2$

$x=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad x=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad Q=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$q_1=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad q_2=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$Q\Lambda Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3/2 & \\ & 1/2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\left(\begin{bmatrix} x & y \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 3/2 & \\ & 1/2 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$\begin{bmatrix} \frac{x+y}{\sqrt{2}} & \frac{x-y}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3/2 & \\ & 1/2 \end{bmatrix}\begin{bmatrix} \frac{x+y}{\sqrt{2}} & \frac{x-y}{\sqrt{2}} \end{bmatrix}$$

$$3/2\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1/2\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$$

2b) $\frac{3}{2}\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$

$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 1$ where $u = \frac{x+y}{\sqrt{2}}$ $v = \frac{x-y}{\sqrt{2}}$

u-axis: $v=0$ $\frac{x-y}{\sqrt{2}} = 0$ $\begin{bmatrix} x \\ y \end{bmatrix} \perp \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

v-axis: $u=0$ $\frac{x+y}{\sqrt{2}} = 0$ $\begin{bmatrix} x \\ y \end{bmatrix} \perp \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{1}{\sqrt{3/2}} = \sqrt{2/3}$ $\frac{1}{\sqrt{1/2}} = \sqrt{2}$

The minor axis is $\sqrt{2/3}$ in direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The major axis is $\sqrt{2}$ in direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3) $H = \begin{array}{c} \begin{matrix} M_1 & M_2 & M_3 \end{matrix} \\ \begin{bmatrix} -2 & 1 & a \\ 1 & -2 & 1 \\ a & 1 & -2 \end{bmatrix} \end{array}$

$M_1 = -2 < 0$
$M_2 = 3 > 0$
$M_3 = (3)(-2) - (1)(-2-a) + (a)(1+2a)$
$= -6 + 2 + a + a + 2a^2$
$M_3 = 2a^2 + 2a - 4$
$a = 1, -2$

above 1, M_3 becomes positive, and below -2 it also becomes positive. Thus, for $(0,0,0)$ to be a point of local maximum for f, a must be in the range of

$(-2, 1)$

noninclusive