MA 232 Linear Algebra **HW 9** **Due: Wednesday Dec 14, 2022 by 11:59 pm**

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Pledge and Sign: I pledge my honor that I have abided by the Stevens Honor System

Upload solutions to Gradescope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. [10 points] For the matrix $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, find its full singular value decomposition (SVD): $A = U\Sigma V^T$.

$$AA^T = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 41-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(41-\lambda) - (12^2) = 0$$

$$\lambda^2 - 50\lambda + 225 = 0$$

$$(\lambda-45)(\lambda-5) = 0$$

$$\lambda_1 = 45 \quad \lambda_2 = 5 \quad \sigma_1 = 3\sqrt{5} \quad \sigma_2 = \sqrt{5}$$

$$\begin{bmatrix} -36 & 12 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 12 \\ 12 & 36 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-36x_1 + 12x_2 = 0 \qquad 4x_1 + 12x_2 = 0$$

$$12x_1 - 4x_2 = 0 \qquad 12x_1 + 36x_2 = 0$$

$$4x_2 = 12x_1 \qquad x_1 = -3x_2$$

$$x_2 = 3x_1$$

$$u_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$v_1 = \frac{1}{\sigma_1}A^T u_1 = \frac{1}{3\sqrt{5}}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{15\sqrt{2}}\begin{bmatrix} 15 \\ 15 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2}A^T u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

2. [10 points]

(a) For the matrix $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$ from the previous problem, compute its pseudoinverse $A^+ = V\Sigma^+U^T$.

(b) Compute A^{-1} by Gauss–Jordan elimination or by cofactor formula and verify that $A^{-1} = A^+$ (which should be the case whenever A is invertible).

a) $\Sigma^+ = \begin{bmatrix} 1/3\sqrt{5} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix}$

$$A^+ = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1/3\sqrt{5} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix}}_{\Sigma^+} \underbrace{\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}}_{U^T}$$

b) $A^{-1} = \frac{\text{Adj}}{|A|} = \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix} \frac{1}{15} = \begin{bmatrix} 1/3 & 0 \\ -4/15 & 1/5 \end{bmatrix}$

$$A^+ = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/3\sqrt{5} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3\sqrt{10} & -1/\sqrt{10} \\ 1/3\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1/30 + 3/10 & 1/10 - 1/10 \\ 1/30 - 3/10 & 1/10 + 1/10 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -4/15 & 1/5 \end{bmatrix} = A^{-1}$$

3. [10 points] Let V be the vector space of polynomials of degree at most 2, $V = \{a + bx + cx^2\}$, with basis $1, x, x^2$. Let W be the vector space of polynomials of degree at most 4, $W = \{a + bx + cx^2 + dx^3 + ex^4\}$, with basis $1, x, x^2, x^3, x^4$.

(a) Find the matrix B of linear transformation S from V to W (in the given bases), if $S(f(x)) = x^2 f(x)$.

(b) Find the matrix A of linear transformation T from W to V (in the given bases), if $T(f(x)) = f''(x)$.

(c) Find the matrix C of linear transformation R from V to V (in the given bases), if $S(f(x)) = (x^2 f(x))''$.

(d) Verify by a direct computation that $C = AB$. (This is because $R(f) = T(S(f))$.)

a) $S(f(x)) = x^2 f(x)$

$S(x^2) = x^2 \cdot x^2 = x^4 \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$S(x) = x^2 \cdot x = x^3 \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$S(1) = x^2 \cdot 1 = x^2 \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $T(x^4) = (x^4)'' = 12x^2$

$T(x^3) = (x^3)'' = 6x$

$T(x^2) = (x^2)'' = 2 \longrightarrow A = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$

$T(x) = (x)'' = 0$

$T(1)'' = (1)'' = 0$

c) $R(x^2) = (x^2 \cdot x^2)'' = 12x^2$

$R(x) = (x^2 \cdot x)'' = 6x \longrightarrow C = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$R(1) = (x^2 \cdot 1)'' = 2$

d) $AB = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} = C$