NON-STANDARD INFERENCE FOR AUGMENTED DOUBLE AUTOREGRESSIVE MODELS WITH NULL VOLATILITY COEFFICIENTS

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This paper considers an augmented double autoregressive (DAR) model, which allows null volatility coefficients to circumvent the over-parameterization problem in the DAR model. Since the volatility coefficients might be on the boundary, the statistical inference methods based on the Gaussian quasi-maximum likelihood estimation (GQMLE) become non-standard, and their asymptotics require the data to have a finite sixth moment, which narrows applicable scope in studying heavy-tailed data. To overcome this deficiency, this paper develops a systematic statistical inference procedure based on the self-weighted GQMLE for the augmented DAR model. Except for the Lagrange multiplier test statistic, the Wald, quasi-likelihood ratio and portmanteau test statistics are all shown to have non-standard asymptotics. The entire procedure is valid as long as the data is stationary, and its usefulness is illustrated by simulation studies and one real example.

1. Introduction. Modelling conditional mean and volatility dynamics together is of extreme importance in econometrics and finance. A myriad of specifications have been proposed for the purpose, and among them, the double autoregressive (DAR) model has recently been attracting much attention in the literature, and it is defined as

$$(1.1) \qquad y_t = u + \sum_{i=1}^{p} \phi_i y_{t-i} + \eta_t \sqrt{\omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2}, \quad t = 0, \pm 1, ...,$$

where $u, \phi_i \in \mathbb{R}$, $\omega > 0, \alpha_i > 0$, $\{\eta_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and unit variance, and η_t is independent of $\{y_s; s < t\}$. Model (1.1) was first termed by Ling (2004), and it is a subclass of ARMA-ARCH models in Weiss (1984) and of nonlinear AR models in Cline and Pu (2004), but it is different from Engle's ARCH model if some $\phi_i \neq 0$.

1

As was shown in Ling (2007a), model (1.1) has an important feature that its Gaussian quasi-maximum likelihood estimator (GQMLE) is asymptotically normal as long as y_t has a finite fractional moment, while the ARMA-GARCH model (see, e.g., Ling (2007b) and Zhang and Ling (2015)) does not. This feature makes model (1.1) feasible and convenient to fit the often observed heavy-tailed data in applications, but it relies on a crucial assumption that each volatility coefficient α_i has a positive lower bound, which might result in the over-parameterization problem. Moreover, both the conditional mean and volatility specifications in model (1.1) have the same order p. This could be another shortcoming of model (1.1) and narrow down its applications. Motivated by these facts, this paper considers an augmented DAR (ADAR) model of order (p, q):

$$(1.2) \qquad y_t = u + \sum_{i=1}^{p} \phi_i y_{t-i} + \eta_t \sqrt{\omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2}, \quad t = 0, \pm 1, ...,$$

where all notations are inherited from model (1.1) except that $\alpha_i \geq 0$, and the conditional mean and volatility specifications can have different orders p and q. With these exceptions, we are able to cope with the over-parameterization problem by checking whether some coefficients are significant from zero in model (1.2). However, this makes the statistical inference of model (1.2) non-standard, since the volatility coefficient α_i is allowed to lie on the boundary of the parameter space (see, e.g., Gouriéroux et al. (1982), Andrews (1999, 2001), Francq and Zakoïan (2007, 2009), Iglesias and Linton (2007), Cavaliere et al. (2017) and Pedersen (2017)). Also, when α_i is allowed to be zero, Francq and Zakoïan (2007) has demonstrated that the GQMLE of the ARCH model (i.e., model (1.2) with $u = 0$ and $\phi_i \equiv 0$) requires a finite sixth moment of y_t for its asymptotics, and this makes the GQMLE of model (1.2) deficient to handle the heavy-tailed data with an infinite sixth moment in many circumstances.

This paper contributes to the literature in three aspects. First, a self-weighted GQMLE (S-GQMLE) is proposed for model (1.2) and its limiting distribution is shown to be a projection of a normal vector onto a convex cone by a quadratic approximation. Based on this S-GQMLE, the Wald, Lagrange multiplier and quasi-likelihood ratio tests are constructed to examine the nullity of some coefficients; their limiting distributions are established under both null and local alternative hypotheses, and their power performance is investigated under local alternative hypotheses. As a special interest, testing for the null hypothesis

of one coefficient equaling to zero is also studied. By allowing for the null volatility coefficients, the estimation and testing based on the GQMLE for the conditional variance models have been well known for their non-standard asymptotics (see, e.g., Andrews (1999, 2001), Francq and Zakoïan (2007, 2009) and Pedersen (2017)), but fewer attempts have been made to study their asymptotics in the presence of the conditional mean structure. Our study on the S-GQMLE and its related tests for model (1.2) fills this gap. Interestingly, we find that even when the null volatility coefficients exist, the S-GQMLE of the conditional mean parameter in model (1.2) is always asymptotically normal, and this property generally does not hold for the ARMA-GARCH model. Hence, if we only examine the nullity of the conditional mean coefficients in model (1.2), the Wald, Lagrange multiplier and quasi-likelihood ratio tests can be implemented with standard asymptotics. In contrast, when the volatility coefficients are included for nullity examination, the asymptotics of these three tests become non-standard. In view of this important feature of model (1.2), we can use these three tests to first detect the nullity of the conditional mean coefficients by standard asymptotics, and then detect the nullity of the volatility coefficients by non-standard asymptotics. We shall emphasize that the preceding two-step procedure is not applicable for the ARMA-GARCH model in general, since the distribution of the GQMLE of their conditional mean parameter is indeed non-standard caused by the null volatility coefficients.

Second, motivated by Wong and Ling (2005), we propose a new mixed portmanteau test to check the adequacy of model (1.2). Diagnostic checking for model adequacy is important in time series analysis. The seminal work in Ljung and Box (1978) constructed a portmanteau test for the conditional mean model, and later a similar portmanteau test was developed for the volatility model in Li and Mak (1994). Both portmanteau tests and their many variants have the standard chi-squared limiting null distribution; see, e.g., Zhu (2016) and references therein. When the null volatility coefficients are allowed in model (1.2), we find that our mixed portmanteau test has a non-standard limiting null distribution, which is not the standard chi-squared distribution any more. This result is new to the literature, and it reveals that the null volatility coefficients have a non-ignorable effect on the model diagnostic checking. To implement our mixed portmanteau test in practice, we shall apply the Wald, Lagrange multiplier and quasi-likelihood ratio tests to obtain a reduced ADAR model with all positive volatility coefficients, and then use the

standard chi-squared limiting null distribution for our mixed test.

Third, our entire statistical inference procedure aforementioned is valid as long as y_t is stationary, and hence it can have a wide applicable scope in dealing with the heavy-tailed data. Heavy-tailedness is often observed in many empirical data (see, e.g., Rachev (2003), Hill (2015) and Zhu and Ling (2015)). When the null volatility coefficients exist in ARCH-type models, the statistical inference methods in Francq and Zakoïan (2007, 2009) and Pedersen (2017) require y_t to have a finite sixth moment. In contrast, our entire methodologies have no moment restriction on y_t resulting from the use of the S-GQMLE, which is motivated by the self-weighting technique in Ling (2005). The self-weighting technique is necessary only when y_t has an infinite sixth moment, and its idea is to apply the self-weight functions to reduce the effect of leverage data so that no moment condition of y_t is needed. We emphasize that the ARMA-GARCH model with the S-GQMLE in Ling (2007b) is also applicable to the heavy-tailed data. However, the asymptotics of the S-GQMLE in Ling (2007b) do not allow null volatility coefficients, and hence no statistical inference method is proposed in the presence of the null volatility coefficients. Finally, the importance of our entire methodologies is illustrated by simulation studies and one real example.

The remainder of the paper is organized as follows. Section 2 presents the S-GQMLE and establishes its asymptotics. Section 3 constructs three tests to test for the null coefficients and obtains their asymptotics. Section 4 analyzes the power of these three tests. Section 5 proposes a portmanteau test for the model diagnostic checking. Simulation results are reported in Section 6, and one real example is given in Section 7. Technical proofs of all theorems are relegated to Appendices.

Throughout the paper, A' is the transpose of a matrix A, $\|A\| = (\text{tr}(A'A))^{1/2}$ is the Frobenius norm of a matrix A, $\langle x, y \rangle_A = x'Ay$ for any $x, y \in \mathbb{R}^s$ is the inner product induced by a positive definite matrix $A \in \mathbb{R}^{s \times s}$, $\|x\|_A^2 = \langle x, x \rangle_A$ is the norm of $x \in \mathbb{R}^s$, $\Phi(\cdot)$ is the c.d.f. of standard normal random variable, $I(\cdot)$ is the indicator function, and $\to_{\mathcal{L}}$ denotes the convergence in distribution.

2. Self-weighted Gaussian quasi-maximum likelihood estimation. Let $\theta = (\phi', \alpha')' \in \mathbb{R}^d$ be the unknown parameter of model (1.2), where $\phi = (u, \phi_1, ..., \phi_p)'$, $\alpha = (\omega, \alpha_1, ..., \alpha_q)'$ and $d = p + q + 2$. Let $m = \max(p, q)$. Assume that the observations $\{y_{-m}, ..., y_n\}$ are generated from model (1.2) with the true value $\theta_0 = (\phi_0', \alpha_0')'$, where $\phi_0 = (u_0, \phi_{10}, ..., \phi_{p0})'$ and $\alpha_0 = (\omega_0, \alpha_{10}, ..., \alpha_{q0})'$. Given the observations

$\{y_{-m}, ..., y_n\}$, the self-weighted Gaussian quasi-maximum likelihood estimator (S-GQMLE) of θ_0 is $\hat{\theta}_n :=$ $(\hat{\phi}'_n, \hat{\alpha}'_n)'$, which is defined as

$$(2.1) \qquad \hat{\theta}_n = \arg\min_{\theta \in \Theta} F_n(\theta) := \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} w_t \ell_t(\theta),$$

where Θ is the parameter space, $w_t := w(y_{t-1}, ..., y_{t-m})$ is the self-weighted function with $w(\cdot)$ being a measurable real positive and bounded function on $\mathbb{R}^m$, and

$$(2.2) \qquad \ell_t(\theta) = \frac{1}{2}\left\{\ln(\alpha'\mathbf{x}_{-1}) + \frac{(y_t - \phi'\mathbf{y}_{t-1})^2}{\alpha'\mathbf{x}_{-1}}\right\}$$

with $\mathbf{y}_t = (1, y_t, ..., y_{t-p+1})'$ and $\mathbf{x}_t = (1, y_t^2, ..., y_{t-q+1}^2)'$. Particularly, when $w_t = 1$, the S-GQMLE reduces to the classical GQMLE in Ling (2004).

To obtain the asymptotic properties of $\hat{\theta}_n$, the following four assumptions are needed.

ASSUMPTION 2.1. $\{y_t\}$ is strictly stationary and ergodic.

ASSUMPTION 2.2. The parameter space Θ is compact with $|u| < \bar{u}$, $|\phi_i| < \bar{\phi}$, $i = 1, 2, \cdots, p$, $\underline{\omega} \le \omega \le \bar{\omega}$, and $0 \le \alpha_j \le \bar{\alpha}$, $j = 1, 2, \cdots, q$, where $\bar{u}$, $\bar{\phi}$, $\underline{\omega}$, $\bar{\omega}$, and $\bar{\alpha}$ are all finite positive constants.

ASSUMPTION 2.3. $E\{(w_t + w_t^2)(\|\mathbf{z}_{t-1}\|^2 + \|\mathbf{z}_{t-1}\|^3)\} < \infty$, where $\mathbf{z}_{t-1} = (1, y_{t-1}^2, \cdots, y_{t-m}^2)'$.

ASSUMPTION 2.4. The matrix $D = \begin{pmatrix} 1 & \frac{\kappa_3}{\sqrt{2}} \\ \frac{\kappa_3}{\sqrt{2}} & \frac{\kappa_4-1}{2} \end{pmatrix}$ is positive definite, where $\kappa_3 = E\eta_t^3$ and $\kappa_4 = E\eta_t^4 < \infty$.

We offer some remarks on the aforementioned assumptions. Assumption 2.1 is a mild setting for time series models. When $p = q = 1$, a sufficient and necessary condition for Assumption 2.1 was obtained in Borkovec and Klüppelberg (2001) and Chen et al. (2014). When $p = q > 1$, a sufficient yet complicated condition for Assumption 2.1 is available in Ling (2007a).

Assumption 2.2 allows the volatility coefficient α_i to be zero. In Ling (2004, 2007a), each α_i is required to have a positive lower bound so that the GQMLE only needs a finite fractional moment of y_t for its asymptotic normality. However, the requirement that $\alpha_i > 0$ for each i is stringent and could cause

the trouble of over-parameterization. Under Assumption 2.2, the over-parameterization problem can be solved, but as a trade-off, the asymptotic distribution of the GQMLE becomes non-standard and requires $Ey_t^6 < \infty$ (see, e.g., Francq and Zakoïan (2007) and Pedersen (2017)). In applications, the finiteness of Ey_t^6 could be restrictive for two reasons. First, this moment condition does not allow us to deal with many heavy-tailed data. Second, this moment condition gives us a small admissible parameter space. As a simple illustration, we consider a DAR(1, 1) model:

$$(2.3) \qquad\qquad y_t = \phi y_{t-1} + \eta_t \sqrt{\omega + \alpha y_{t-1}^2}.$$

When $\eta_t \sim \mathcal{N}(0, 1)$, Table 1 gives the constraints on the parameter (ϕ, α) for strict stationarity, the 2nd, 4th, and 6th moments of y_t, and Fig 1 displays these constraints graphically. From this figure, we can see that the region of 6th moment is much smaller than that of strict stationarity. Hence, it is practically important to release the moment condition of y_t so that the admissible parameter space is enlarged as much as possible.

TABLE 1
Parameter constraints for model (2.3)

Condition	Constraint	Region in Fig 1
Strict stationarity	$E \log |\phi + \eta_t \sqrt{\alpha}| < 0$	I+II+III+IV
$Ey_t^2 < \infty$	$\phi^2 + \alpha < 1$	II+III+IV
$Ey_t^4 < \infty$	$\phi^4 + 6\phi^2\alpha + 3\alpha^2 < 1$	III+IV
$Ey_t^6 < \infty$	$\phi^6 + 15\phi^4\alpha + 45\phi^2\alpha^2 + 15\alpha^3 < 1$	IV

Assumption 2.3 plays a key role in releasing the moment condition of y_t. When $Ey_t^6 < \infty$, it is valid without the weight (i.e., $w_t = 1$). When $Ey_t^6 = \infty$, the weight w_t is introduced to reduce the effect of leverage points by shrinking their weights on the objective function $F_n(\theta)$ so that no moment condition of y_t is needed but at the sacrifice of efficiency. This idea was initiated by Ling (2005), and it has been adopted in many studies; see, e.g., Ling (2005, 2007b), Pan et al. (2007), Francq and Zakoïan (2010), Zhu and Ling (2011, 2015) and Yang and Ling (2017). In practice, the selection of w_t is similar to that of the influence function in Huber (1996). For example, we can follow Horváth and Liese (2004) to choose

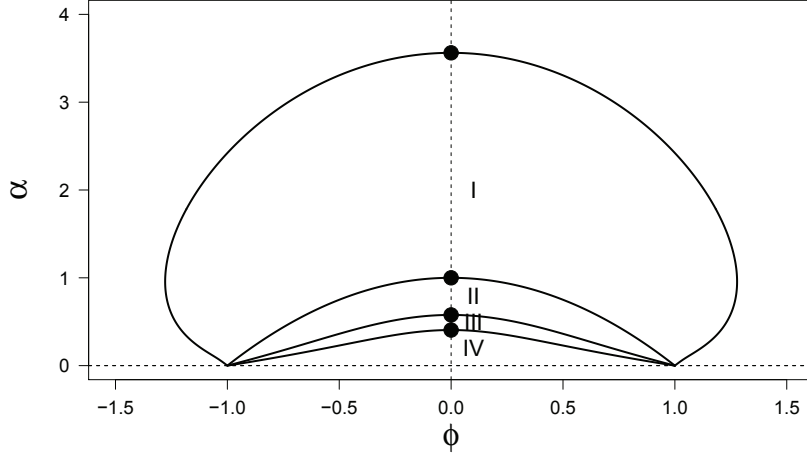$$(2.4) \qquad\qquad w_t = \frac{1}{1 + \sum_{i=1}^{m} y_{t-i}^6}$$

FIG 1. *Regions of strict stationarity (I+II+III+IV), 2nd moment (II+III+IV), 4th moment (III+IV) and 6th moment (IV) of y_t in model (2.3).*

or we can follow Ling (2005) to choose

$$
(2.5) \qquad w_t = \begin{cases} 1, & \text{if } a_t = 0, \\ C_w^3/a_t^3, & \text{if } a_t \neq 0, \end{cases}
$$

where $a_t = \sum_{i=1}^{m} y_{t-i}^2 I(y_{t-i}^2 \geq C_w)$ for some constant $C_w > 0$, and C_w is chosen as the 90% or 95% percentile of $\{y_t^2\}_{t=1}^n$, empirically. However, when the second moment of y_t does not exist, the 95% empirical percentile of $\{y_t^2\}_{t=1}^n$ might be very large, leading to malfunction of (2.5). Therefore, we prefer to use w_t in (2.4) subsequently, but leave the selection of the optimal w_t as an open problem.

Assumption 2.4 is general to derive the asymptotic distribution of $\hat{\theta}_n$. As shown in Wilkins (1944), this assumption is equivalent to that $\mathbb{P}(\eta_t^2 - c\eta_t = 1) < 1$ for any $c \in \mathbb{R}$, which is satisfied for continuous η_t.

Next, let $J = E(w_t\Gamma_t(\theta_0)\Gamma_t(\theta_0)')$ and $\Sigma = E(w_t^2\Gamma_t(\theta_0)D\Gamma_t(\theta_0)')$ with

$$
\Gamma_t(\theta) = \begin{pmatrix} \dfrac{\mathbf{y}_{t-1}}{\sqrt{(\alpha'\mathbf{x}_{t-1})}} & 0_{(p+1)\times 1} \\ 0_{(q+1)\times 1} & \dfrac{\mathbf{x}_{t-1}}{\sqrt{2}(\alpha'\mathbf{x}_{t-1})} \end{pmatrix}.
$$

We are ready to give our first main result on the consistency and asymptotic distribution of $\hat{\theta}_n$.

THEOREM 2.1. *Suppose that Assumptions 2.1-2.3 hold. Then,*

(i) $\hat{\theta}_n \to \theta_0$ *almost surely (a.s.) as $n \to \infty$;*

(ii) *if Assumption 2.4 further holds, $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_{\mathcal{L}} \lambda^\Lambda := \arg\inf_{\lambda \in \Lambda} \|Z - \lambda\|_J$ as $n \to \infty$,*

where $Z \sim \mathcal{N}(0, J^{-1}\Sigma J^{-1})$ and $\Lambda := \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_{p+q+2}$ with $\Lambda_i = \mathbb{R}$ for $i = 1, 2, \cdots, p+2$, and

$$\Lambda_{p+j+2} = \begin{cases} [0, \infty), & \text{if } \alpha_{j0} = 0, \\ \mathbb{R}, & \text{if } \alpha_{j0} \neq 0, \end{cases} \quad \text{for } j = 1, 2, \cdots, q.$$

Theorem 2.1 implies that when θ_0 is not an interior point of Θ (i.e., some of its volatility coefficients are on the boundary), the limiting distribution of $\hat{\theta}_n$ is no longer Gaussian but a projection of a Gaussian random variable Z onto the convex cone Λ with a metric induced by the inner product $\langle \cdot, \cdot \rangle_J$. The uniqueness of such a projection is guaranteed by the convexity of Λ. Particularly, when θ_0 is an interior point of Θ, we have $\Lambda = \mathbb{R}^{p+q+2}$ and then $\lambda^\Lambda = Z \sim \mathcal{N}(0, J^{-1}\Sigma J^{-1})$.

Write $Z = (Z'_\phi, Z'_\alpha)'$ and $\lambda^\Lambda = (\lambda_\phi^{\Lambda'}, \lambda_\alpha^{\Lambda'})'$. In view of that $J = \text{diag}\{J_\phi, J_\alpha\}$ is a block diagonal matrix, by Theorem 4 in Andrews (1999), it is not hard to see that

$$(2.6) \qquad \lambda_\phi^\Lambda = \inf_{\lambda_\phi \in \Lambda_\phi} \|Z_\phi - \lambda_\phi\|_{J_\phi} = Z_\phi \quad \text{and} \quad \lambda_\alpha^\Lambda = \inf_{\lambda_\alpha \in \Lambda_\alpha} \|Z_\alpha - \lambda_\alpha\|_{J_\alpha},$$

where $J_\phi = E\{w_t \mathbf{y}_{t-1} \mathbf{y}'_{t-1}/(\alpha'_0 \mathbf{x}_{t-1})\}$, $J_\alpha = E\{w_t \mathbf{x}_{t-1} \mathbf{x}'_{t-1}/(\alpha'_0 \mathbf{x}_{t-1})^2\}/2$, $\Lambda_\phi = \mathbb{R}^{p+1}$, and $\Lambda_\alpha = \Lambda_{p+2} \times \cdots \times \Lambda_{p+q+2}$. The result (2.6) implies that $\hat{\phi}_n$ is always asymptotically normal, no matter whether the null volatility coefficients exist. This important feature guarantees that we can examine the significance of the conditional mean coefficients by using the standard statistical inference methods, and then implement the statistical inference for the significance of the volatility coefficients. The validity of this two-step procedure is mainly because the matrix J is block diagonal, and it does not need Σ to be block diagonal, allowing $\hat{\phi}_n$ and $\hat{\alpha}_n$ to be asymptotically correlated. Note that the matrix J is the expectation of the Hessian matrix of the objective function. For the ARMA-GARCH model, the corresponding matrix is not diagonal in general, and hence the GQMLE of the conditional mean parameter may not be asymptotically normal if the null volatility coefficients exist.

3. Testing for null coefficients. In this section, we consider the Wald, Lagrange multiplier (LM) and quasi-likelihood ratio (QLR) tests to detect whether some coefficients are equal to zero in model (1.2).

Since the significance of the conditional mean coefficients can be examined ahead, we only focus on the tests for the null volatility coefficients.

We split the true parameter θ_0 into three parts such that $\theta_0 = (\theta_0^{(1)'}, \theta_0^{(2)'}, \theta_0^{(3)'})'$, where $\theta_0^{(i)} \in \mathbb{R}^{d_i}$ for $i = 1, 2, 3$, and $d = d_1 + d_2 + d_3$. Without loss of generality, we assume $\theta_0^{(1)} = \phi_0 \cup \{\alpha_{0i} : \alpha_{0i} > 0\}$, $\theta_0^{(2)} = 0_{d_2 \times 1}$ and $\theta_0^{(3)} = 0_{d_3 \times 1}$. That is, $\theta_0^{(1)}$ contains the conditional mean coefficients as well as volatility coefficients that are strictly larger than zero, and $(\theta_0^{(2)'}, \theta_0^{(3)'})'$ contains all volatility coefficients on the boundary. Note that from now on, the order of components in $\mathbf{x}_t$ and $\mathbf{y}_t$ is changed according to the splitting way of θ_0. Let $K = (0_{(d-d_1) \times d_1}, I_{d-d_1})$ and $K_\alpha = (0_{d_3 \times (d-d_3)}, I_{d_3})$. Our null hypothesis is set as

$$H_0 : \theta_0^{(3)} = 0_{d_3 \times 1} \ \text{(i.e., } K_\alpha \theta_0 = 0_{d_3 \times 1}).$$

Under H_0, the nuisance coefficients vector $(\theta_0^{(1)'}, \theta_0^{(2)'})'$ allows $\theta_0^{(2)}$ on the boundary. This setting is similar to that in Pedersen (2017), and more general than that in Francq and Zakoïan (2009), which only considers the case of $d_2 = 0$.

To construct our test statistics, the following notations are needed:

$$J_n(\theta) = \frac{\partial^2 F_n(\theta)}{\partial \theta \partial \theta'}, \quad \Sigma_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} w_t^2 \Gamma_t(\theta)' D_n(\theta) \Gamma_t(\theta), \quad \text{and}$$

$$D_n(\theta) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}\bar{w}n} \sum_{t=1}^{n} \frac{w_t \epsilon_t^3(\phi)}{(\alpha' \mathbf{x}_t)^{3/2}} \\ \frac{1}{\sqrt{2}\bar{w}n} \sum_{t=1}^{n} \frac{w_t \epsilon_t^3(\phi)}{(\alpha' \mathbf{x}_t)^{3/2}} & \frac{1}{2\bar{w}n} \sum_{t=1}^{n} \frac{w_t \epsilon_t^4(\phi)}{(\alpha' \mathbf{x}_t)^2} - \frac{1}{2} \end{pmatrix},$$

where $\epsilon_t(\phi) = y_t - \phi' \mathbf{y}_{t-1}$ and $\bar{w} = n^{-1} \sum_{t=1}^{n} w_t$. With these notations, we denote

$$\hat{J}_n = J_n(\hat{\theta}_n), \qquad \hat{\Sigma}_n = \Sigma_n(\hat{\theta}_n), \qquad \hat{D}_n = D_n(\hat{\theta}_n),$$

$$\hat{J}_{n|3} = J_n(\hat{\theta}_{n|3}), \quad \hat{\Sigma}_{n|3} = \Sigma_n(\hat{\theta}_{n|3}), \quad \hat{D}_{n|3} = D_n(\hat{\theta}_{n|3}),$$

where $\hat{\theta}_{n|3}$ is the restricted S-GQMLE under H_0. Our Wald, LM and QLR test statistics are defined as

$$W_n = n\hat{\theta}_n^{(3)'} \{K_\alpha \hat{J}_n^{-1} \hat{\Sigma}_n \hat{J}_n^{-1} K_\alpha'\}^{-1} \hat{\theta}_n^{(3)},$$

$$L_n = n \frac{\partial F_n(\hat{\theta}_{n|3})}{\partial \theta'} \hat{J}_{n|3}^{-1} K_\alpha' \{K_\alpha \hat{J}_{n|3}^{-1} \hat{\Sigma}_{n|3} \hat{J}_{n|3}^{-1} K_\alpha'\}^{-1} K_\alpha \hat{J}_{n|3}^{-1} \frac{\partial F_n(\hat{\theta}_{n|3})}{\partial \theta},$$

$$Q_n = 2n [F_n(\hat{\theta}_{n|3}) - F_n(\hat{\theta}_n)],$$

respectively, and their limiting null distributions are given in the following theorem.

THEOREM 3.1. *Suppose that Assumptions 2.1-2.4 hold. Then, under H_0, as $n \to \infty$,*

(i) $W_n \to_{\mathcal{L}} W := \lambda^{\Lambda'} \Omega \lambda^{\Lambda}$;

(ii) $L_n \to_{\mathcal{L}} L := \chi^2_{d_3}$;

(iii) $Q_n \to_{\mathcal{L}} Q := \|Z - \lambda^{\Lambda}_{|3}\|^2_J - \|Z - \lambda^{\Lambda}\|^2_J = \lambda^{\Lambda'} \Xi \lambda^{\Lambda} - \lambda^{\Lambda'}_{|3} \Xi \lambda^{\Lambda'}_{|3}$,

where λ^{Λ} and Z are defined as in Theorem 2.1,

$$(3.1) \quad \Omega = K'_{\alpha} \{K_{\alpha} J^{-1} \Sigma J^{-1} K'_{\alpha}\}^{-1} K_{\alpha}, \quad \Xi = K'(KJ^{-1}K')^{-1}K, \quad and \quad \lambda^{\Lambda}_{|3} = \arg \inf_{\lambda_{|3} \in \Lambda_{|3}} \|Z - \lambda_{|3}\|^2_J$$

with $\Lambda_{|3} := \mathbb{R}^{d_1} \times [0, \infty)^{d_2} \times \{0\}^{d_3}$. Particularly, when $d_2 = 0$, $Q = \lambda^{\Lambda'} \Xi \lambda^{\Lambda}$.

Theorem 3.1 shows that except for L_n, the limiting null distributions of W_n and Q_n are not intuitive. The test L_n has standard chi-squared limiting null distribution, since its limiting distribution depends on that of the score $\frac{\partial F_n(\theta_0)}{\partial \theta}$, which is asymptotically normal under H_0 (see (A.6) in Appendix A). On the contrary, the tests W_n and Q_n have non-standard limiting null distributions, since their limiting distributions rely on that of $\hat{\theta}^{(3)}_n$, however, $\hat{\theta}^{(3)}_n$ is not asymptotically normal under H_0 as shown in Theorem 2.1(ii).

By letting

$$(3.2) \qquad \qquad \hat{\Omega}_n = K'_{\alpha} \{K_{\alpha} \hat{J}^{-1}_n \hat{\Sigma}_n \hat{J}^{-1}_n K'_{\alpha}\}^{-1} K_{\alpha} \quad and \quad \hat{\Xi}_n = K'(K\hat{J}^{-1}_n K')^{-1}K$$

be the estimators of Ω and Ξ in (3.1), we propose an algorithm, which is similar to Algorithm 1 in Pedersen (2017), to calculate the critical values of W_n and Q_n in practice.

ALGORITHM 3.1. (**Simulated critical values of W_n and Q_n**)

1. *Draw ϵ_* from $\mathcal{N}(0, I_d)$ and then compute $Z_* = [\hat{J}^{-1}_n \hat{\Sigma}_n \hat{J}^{-1}_n]^{1/2} \epsilon_*$.*

2. *Find λ^{Λ}_* that minimizes $(Z_* - \lambda_*)' \hat{J}_n (Z_* - \lambda_*)$ for $\lambda_* \in \Lambda$ and $\lambda^{\Lambda}_{*|3}$ that minimizes $(Z_* - \lambda_{*|3})' \hat{J}_n (Z_* - \lambda_{*|3})$ for $\lambda_{*|3} \in \Lambda_{|3}$.*

3. *Calculate $w_* = \lambda^{\Lambda'}_* \hat{\Omega}_n \lambda^{\Lambda}_*$ and $q_* = \lambda^{\Lambda'}_* \hat{\Xi}_n \lambda^{\Lambda}_* - \lambda^{\Lambda'}_{*|3} \hat{\Xi}_n \lambda^{\Lambda'}_{*|3}$.*

4. *Repeat steps 1-3 N times to get $\{w_{i,*}\}_{i=1}^N$ and $\{q_{i,*}\}_{i=1}^N$, where $w_{i,*}$ and $q_{i,*}$ are the realizations of w_* and q_* in i^{th} time, respectively. At the level $\beta \in (0,1)$, the critical values of W_n and Q_n are the empirical $100(1-\beta)\%$ sample percentiles of w_* and q_* based on $\{w_{i,*}\}_{i=1}^N$ and $\{q_{i,*}\}_{i=1}^N$, respectively.*

To implement Algorithm 3.1, we need know Λ and $\Lambda_{|3}$. Hence, Algorithm 3.1 is only applicable when either $d_2 = 0$ (i.e., no nuisance parameters on the boundary exist under H_0) or $d_2 \neq 0$ with a known $\theta_0^{(2)}$ (i.e., the true nuisance parameters on the boundary are known under H_0). If $d_2 \neq 0$ with an unknown $\theta_0^{(2)}$, so far it is unclear how to obtain the critical values of W_n and Q_n. In practice, we recommend to use both W_n and Q_n by presuming $d_2 = 0$. If H_0 is rejected in this case, we then get a reduced ADAR model with coefficients $(\theta_0^{(1)'}, \theta_0^{(2)'})'$, from which we could get the supportive evidence of $d_2 = 0$ if our tests imply that all coefficients $(\theta_0^{(1)'}, \theta_0^{(2)'})'$ are non-zeros.

Besides the simulation method in Algorithm 3.1, one may use the "m out of n bootstrap" method as in Politis and Romano (1994) and Andrews (2000) to compute the critical values of W_n and Q_n. Although this bootstrap method is valid in theory, its performance could be sensitive to the choice of the subsampling size m (as demonstrated by our un-reported simulation results), while how to choose the optimal m remains unsolved in our time series setting. Based on this argument, we recommend to use Algorithm 3.1 for convenience.

In many situations, we are mostly interested in testing the nullity of one volatility coefficient:

$$H_0' : \alpha_{i0} = 0 \quad \text{for some given } i \in \{1, 2, \cdots, q\}.$$

Besides the Wald, LM and QLR tests, a t-type test statistic t_n is often used in practice to detect H_0', where

$$t_n = \frac{\hat{\alpha}_{in}}{\hat{\sigma}_{\hat{\alpha}_{in}}},$$

and $\hat{\sigma}_{\hat{\alpha}_{in}}$ is the square root of the ith diagonal element of $\hat{J}_n^{-1}\hat{\Sigma}_n\hat{J}_n^{-1}$. Particularly, when $d_3 = 1$, we have $t_n^2 = W_n$.

When $d_2 > 0$ with a known $\theta_0^{(2)}$, we can apply Algorithm 3.1 to find the critical values of t_n, W_n and Q_n. When $d_2 = 0$, t_n, W_n and Q_n have simpler critical regions, which are closely related to standard normal and chi-squared distributions.

THEOREM 3.2.   *Suppose that Assumptions 2.1-2.4 hold and $d_2 = 0$. Then, the t-type, Wald, LM and QLR tests of asymptotic significance level $\beta \in (0, 1/2)$ for H_0' are defined by the critical regions*

$$\{t_n > \Phi^{-1}(1 - \beta)\}, \qquad \{W_n > \chi^2_{1,1-2\beta}\}, \qquad \{L_n > \chi^2_{1,1-\beta}\}, \qquad \{\hat{\xi}_n Q_n > \chi^2_{1,1-2\beta}\},$$

respectively, where $\hat{\xi}_n = \hat{\Omega}_n / \hat{\Xi}_n$, and $\hat{\Omega}_n$ and $\hat{\Xi}_n$ are defined in (3.2).

The preceding theorem demonstrates that when the coefficient α_{i0} lies on the boundary of parameter space, the standard critical regions of t_n, W_n and Q_n are not correct, except L_n. In other words, if we follow L_n to use the conventional critical regions for t_n, W_n and Q_n, we would encounter a distorted size problem for the latter three tests.

4. Power analysis.   This section studies the efficiency of the Wald, LM and QLR tests via the Pitman analysis.

First, we need the asymptotic distribution of the S-GQMLE under sequences of local alternatives to the true parameter θ_0. Let $\theta_n = \theta_0 + h/\sqrt{n}$, where $h = (h_1, \cdots, h_{p+q+2})'$ with $(h_1, \cdots, h_{p+1})' \in \mathbb{R}^{p+1}$ and $(h_{p+2}, \cdots, h_{p+q+2}) \in (0, \infty)^{q+1}$ such that $\theta_n \in \Theta$ for sufficiently large n. When n is sufficiently large, we can define a strictly stationary solution to

$$y_{t,n} = \left(u_0 + \frac{h_1}{\sqrt{n}}\right) + \sum_{i=1}^{p}\left(\phi_{0i} + \frac{h_{i+1}}{\sqrt{n}}\right)y_{t-i,n} + \eta_t\sqrt{\left(\omega_0 + \frac{h_{p+2}}{\sqrt{n}}\right) + \sum_{j=1}^{q}\left(\alpha_{0j} + \frac{h_{p+j+2}}{\sqrt{n}}\right)y_{t-j,n}^2},$$

where η_t is defined as in model (1.2). Based on $\{y_{-m,n}, \cdots, y_{n,n}\}$, the S-GQMLE is

(4.1)
$$\hat{\theta}_{n,h} = \arg\min_{\theta \in \Theta} \frac{1}{n}\sum_{t=1}^{n} w_{t,n} l_{t,n}(\theta),$$

where

$$l_{t,n}(\theta) = \frac{1}{2}\left\{\ln(\alpha' \mathbf{x}_{t-1,n}) + \frac{\epsilon_{t,n}^2(\phi)}{\alpha' \mathbf{x}_{t-1,n}}\right\}$$

with $\epsilon_{t,n}(\phi) = y_{t,n} - \phi' \mathbf{y}_{t-1,n}$ and $\mathbf{y}_{t,n} = (1, y_{t,n}, ..., y_{t-p+1,n})'$. Below, we impose a stronger sufficient assumption on the self-weighted function $w_{t,n}$ to derive the asymptotics of $\hat{\theta}_{n,h}$.

ASSUMPTION 2.3'. $(w_{t,n} + w_{t,n}^2)(\|\mathbf{z}_{t-1,n}\|^2 + \|\mathbf{z}_{t-1,n}\|^{3+\delta}) < \infty$ *for some* $\delta \geq 0$, *where* $\mathbf{z}_{t-1,n} = (1, y_{t-1,n}^2, \cdots, y_{t-m,n}^2)'$.

Denote by $\mathbb{P}_{n,h}$ the law of $y_{t,n}$. Then, we have the following result.

THEOREM 4.1. *Suppose that Assumptions 2.1-2.2 and 2.4 hold. Then, under $\mathbb{P}_{n,h}$,*

(i) *if Assumption 2.3′ holds with $\delta = 0$, $\hat{\theta}_{n,h} \to \theta_0$ in probability as $n \to \infty$;*

(ii) *if Assumption 2.3′ holds with $\delta = 2$, $\sqrt{n}(\hat{\theta}_{n,h} - \theta_0) \to_{\mathcal{L}} \lambda^\Lambda(h)$ as $n \to \infty$, where*

$$\lambda^\Lambda(h) = \arg \inf_{\lambda \in \Lambda} \|Z + h - \lambda\|_J$$

with Z being defined as in Theorem 2.1.

We emphasize that since the log-likelihood ratio is not asymptotically normal in our setting, it seems difficult to apply the classical Le Cam's third lemma to prove Theorem 4.1(ii); see Francq and Zakoïan (2009) for more discussions. In this paper, we show Theorem 4.1(ii) in a direct way.

Let $\chi_s^2(c)$ be the noncentral chi-squared distribution with noncentrality parameter c and degrees of freedom s. The asymptotic distributions of all three test statistics under the local alternatives are given as follows.

THEOREM 4.2. *Suppose that the conditions in Theorem 4.1(ii) hold and $\theta_0^{(3)} = 0_{d_3 \times 1}$. Then, under $\mathbb{P}_{n,h}$, as $n \to \infty$,*

(i) $W_n \to_{\mathcal{L}} W(h) := \lambda^\Lambda(h)'\Omega\lambda^\Lambda(h)$;

(ii) $L_n \to_{\mathcal{L}} L(h) := \chi_{d_3}^2(h'\Omega h)$;

(iii) $Q_n \to_{\mathcal{L}} Q(h) := \|Z + h - \lambda_{|3}^\Lambda\|_J^2 - \|Z + h - \lambda^\Lambda\|_J^2 = \lambda^{\Lambda'}(h)\Xi\lambda^\Lambda(h) - \lambda_{|3}^{\Lambda'}(h)\Xi\lambda_{|3}^{\Lambda'}(h)$,

where Ω and Ξ are defined in (3.1), $\lambda^\Lambda(h)$ is defined as in Theorem 4.1, and

$$\lambda_{|3}^\Lambda(h) = \arg \inf_{\lambda \in \Lambda_{|3}} \|Z + h - \lambda\|_J$$

with Z being defined as in Theorem 2.1.

For all of four tests in Theorem 3.2, the following theorem shows that the local asymptotic power of the t-type, Wald and QLR tests is the same, and it is higher than the one of LM test.

THEOREM 4.3.   *Suppose that the conditions in Theorem 4.1(ii) hold and $d_2 = 0$. Then, the local asymptotic power of the t-type, Wald and QLR tests is*

$$\lim_{n\to\infty} \mathbb{P}_{n,h}(t_n > \Phi^{-1}(1-\beta)) = \lim_{n\to\infty} \mathbb{P}_{n,h}(W_n > \chi^2_{1,1-2\beta}) = \lim_{n\to\infty} \mathbb{P}_{n,h}(\hat{\xi}Q_n > \chi^2_{1,1-2\beta}) = 1 - \Phi(c_1 - h^*),$$

and the local asymptotic power of LM test is

$$\lim_{n\to\infty} \mathbb{P}_{n,h}(L_n > \chi^2_{1,1-\alpha}) = 1 - \Phi(c_2 - h^*) + \Phi(-c_2 - h^*) < 1 - \Phi(c_1 - h^*),$$

where $h^* = h_d/\sigma_d$, $c_1 = \Phi^{-1}(1-\beta)$ and $c_2 = \Phi^{-1}(1-\beta/2)$.

In addition to the Pitman analysis, the Bahadur slopes as in Bahadur (1960) under fixed alternatives are also established for three test statistics in the supplementary material (Jiang et al. (2019)). However, as in Francq and Zakoïan (2009), a formal comparison of Bahadur slopes for all considered tests is not easy, since J, $J_{0|3}$, Σ and $\Sigma_{0|3}$ are unknown in closed form, particularly when the self-weighted function w_t is included.

5. Model checking.   This section proposes a new portmanteau test to check the adequacy of model (1.2). Define the self-weighted innovation $\zeta_t = w_t \eta_t$ and the self-weighted squared innovation $\xi_t = w_t^2(\eta_t^2 - 1)$. Accordingly, define the self-weighted residual $\hat{\zeta}_t = w_t \hat{\eta}_t$ and the self-weighted squared residual $\hat{\xi}_t = w_t^2(\hat{\eta}_t^2 - 1)$, where $\hat{\eta}_t = \epsilon_t(\hat{\phi}_n)/\sqrt{\hat{\alpha}'_n \mathbf{x}_{t-1}}$ is the residual of model (1.2). The idea of our mixed portmanteau test is based on the fact that $\{\zeta_t\}$ (or $\{\xi_t\}$) is a sequence of uncorrelated random variables. Hence, if model (1.2) is correctly specified, it is expected that the sample autocorrelation function of $\{\hat{\zeta}_t\}$ (or $\{\hat{\xi}_t\}$) at lag k, denoted by $\hat{\rho}_{nk}$ (or $\hat{r}_{nk}$), is close to zero, where

$$\hat{\rho}_{nk} = \frac{\sum_{t=k+1}^{n}(\hat{\zeta}_t - \bar{\zeta})(\hat{\zeta}_{t-k} - \bar{\zeta})}{\sum_{t=1}^{n}(\hat{\zeta}_t - \bar{\zeta})^2} \quad \text{and} \quad \hat{r}_{nk} = \frac{\sum_{t=k+1}^{n}(\hat{\xi}_t - \bar{\xi})(\hat{\xi}_{t-k} - \bar{\xi})}{\sum_{t=1}^{n}(\hat{\xi}_t - \bar{\xi})^2},$$

with $\bar{\zeta} = \sum_{t=1}^{n} \hat{\zeta}_t/n$ and $\bar{\xi} = \sum_{t=1}^{n} \hat{\xi}_t/n$. On the other hand, if the value of $\hat{\rho}_{nk}$ (or $\hat{r}_{nk}$) deviates from zero significantly, it implies that the conditional mean (or variance) structure in model (1.2) is misspecified. Motivated by this, our new mixed portmanteau test takes both $\hat{\rho}_{nk}$ and $\hat{r}_{nk}$ into account, and this new test can detect the misspecification in conditional mean and variance simultaneously.

Let $M \geq 1$ be a given integer, $\bar{\sigma}^2 = (\kappa_4 - 1)Ew_t^4$, $\hat{\rho}_n = (\hat{\rho}_{n1}, \cdots, \hat{\rho}_{nM})'$, $\hat{r}_n = (\hat{r}_{n1}, \cdots, \hat{r}_{nM})'$, $U_\rho = (U_{\rho 1}', \cdots, U_{\rho M}')'$, and $U_r = (U_{r1}', \cdots, U_{rM}')'$, where

$$U_{\rho k} = -\left(E\left[\frac{w_t w_{t-k}\eta_{t-k}\mathbf{y}_{t-1}'}{\sqrt{\alpha_0' \mathbf{x}_{t-1}}}\right], 0_{1\times(q+1)}\right) \quad \text{and} \quad U_{rk} = -\left(0_{1\times(p+1)}, E\left[\frac{w_t^2 w_{t-k}^2(\eta_{t-k}^2 - 1)\mathbf{x}_{t-1}'}{\alpha_0' \mathbf{x}_{t-1}}\right]\right).$$

Meanwhile, let $\mathcal{G}$ be a mapping: $\mathbb{R}^{2M+p+q+2} \to \mathbb{R}^{2M} \times \Lambda$, defined as

$$\mathcal{G}(a) = \begin{pmatrix} a_1 \\ a_2 \\ \arg\inf_{\lambda\in\Lambda} \|a_3 - \lambda\|_J \end{pmatrix}$$

for any $a = (a_1', a_2', a_3')' \in \mathbb{R}^{2M+d}$, where $a_1, a_2 \in \mathbb{R}^M$ and $a_3 \in \mathbb{R}^d$. Then, we have the following result on the limiting joint distribution of $(\hat{\rho}_n', \hat{r}_n')'$.

THEOREM 5.1. *Suppose that Assumptions 2.1-2.4 hold. Then, if model (1.2) is correctly specified, as $n \to \infty$,*

$$(5.1) \qquad \sqrt{n}\begin{pmatrix} \hat{\rho}_n \\ \hat{r}_n \end{pmatrix} \to_{\mathcal{L}} V\mathcal{G}(g)$$

for some random vector $g \sim \mathcal{N}(0, G)$, where

$$V = \begin{pmatrix} I_M & 0 & (Ew_t^2)^{-1}U_\rho \\ 0 & I_M & (\bar{\sigma}^2)^{-1}U_r \end{pmatrix} \quad \text{and} \quad G = Ev_t v_t'$$

with $v_t = \left(\frac{\zeta_t\zeta_{t-1}}{Ew_t^2}, \cdots, \frac{\zeta_t\zeta_{t-M}}{Ew_t^2}, \frac{\xi_t\xi_{t-1}}{\bar{\sigma}^2}, \cdots, \frac{\xi_t\xi_{t-M}}{\bar{\sigma}^2}, J^{-1}w_t\frac{\partial \ell_t(\theta_0)}{\partial \theta}\right)$.

Based on Theorem 5.1, we propose a new mixed portmanteau test statistic defined by

$$(5.2) \qquad Q_M = n\begin{pmatrix} \hat{\rho}_n \\ \hat{r}_n \end{pmatrix}' (\hat{V}_n \hat{G}_n \hat{V}_n')^{-1} \begin{pmatrix} \hat{\rho}_n \\ \hat{r}_n \end{pmatrix},$$

where $\hat{V}_n$ and $\hat{G}_n$ are the consistent sample counterparts of V and G, respectively. By Theorem 5.1 and the continuous mapping theorem, we have

$$Q_M \to_{\mathcal{L}} \mathcal{G}(g)'V'(VGV')^{-1}V\mathcal{G}(g) \quad \text{as } n \to \infty.$$

Hence, when some coefficients of θ_0 are on the boundary of parameter space, $\mathcal{G}(g) \neq g$ and our portmanteau test Q_m in (5.2) has a non-standard limiting distribution; when none of the coefficients of θ_0 is on the boundary of parameter space, $\mathcal{G}(g) = g$ and Q_m has limiting distribution χ^2_{2M}, which is the standard result as many existing portmanteau tests. Note that the non-standard limiting distribution of Q_M depends on Λ, which is determined by the location of all null volatility coefficients but is hard to be correctly specified. To implement Q_M in practice, we need first apply our Wald, LM and QLR tests to obtain a reduced ADAR model with all positive volatility coefficients, and then use the standard chi-squared limiting null distribution for Q_M. Even though this reduced ADAR model has no boundary effect, the self-weighted function w_t is still needed in general. This is because w_t is used to make sure Lemma B.1 in Appendix B always holds and so the result (5.1) is valid. Clearly, if either (i) $Ey_t^2 < \infty$ or (ii) there exists a positive volatility coefficient α_i for any i such that $\phi_i \neq 0$, Lemma B.1 holds without w_t, and consequently, w_t is not needed for Q_M. Otherwise, the use of w_t seems necessary.

6. Simulation studies. In this section, Monte Carlo experiments are conducted to assess the finite-sample performance of our Wald (W_n), LM (L_n) and QLR (Q_n) tests.

We generate 1000 replications of sample size $n = 1000$ and 5000 from the following three different data generating procedures (DGPs):

$$\text{DGP } 1 : y_t = \phi_1 y_{t-1} + \eta_t \sqrt{\omega + \alpha y_{t-1}^2 + k y_{t-2}^2};$$
$$\text{DGP } 2 : y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \eta_t \sqrt{\omega + \alpha y_{t-1}^2 + k y_{t-2}^2 + k y_{t-3}^2};$$
$$\text{DGP } 3 : y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \eta_t \sqrt{\omega + \alpha y_{t-2}^2 + k y_{t-3}^2},$$

where $\phi_1 = 0.5$, $\phi_2 = -0.3$, $\omega = 1$, $\alpha = 0.1$ or 0.6, and $k = h/\sqrt{n}$ with $h \in \{0, 1, \cdots, 10\}$. Here, three different distributions of the innovation η_t are considered:

- $\eta_t \sim \mathcal{N}(0, 1)$;
- $\eta_t \sim \text{st}_{10}$, where st_ν is the standardized student-t distribution with density

$$g(x) = \frac{\Gamma[(\nu + 1)/2]}{\Gamma(\nu/2)\sqrt{(\nu - 2)\pi}} \Big(1 + \frac{x^2}{\nu - 2}\Big)^{-(\nu+1)/2};$$

- $\eta_t \sim \mathrm{sst}_{10,2}$, where $\mathrm{sst}_{\nu,\xi}$ is the standardized skewed student-t distribution in Fernández and Steel (1998) with the density

$$f(x) = \begin{cases} \frac{2\rho}{\xi+1/\xi} g(\xi(\rho x + \bar{\omega})), & \text{if} \quad x < -\bar{\omega}/\rho, \\[2mm] \frac{2\rho}{\xi+1/\xi} g((\rho x + \bar{\omega})/\xi), & \text{if} \quad x \geq \bar{\omega}/\rho, \end{cases}$$

where

$$\bar{\omega} = \frac{\sqrt{\nu-2}\,\Gamma[(\nu-1)/2]}{\sqrt{\pi}\,\Gamma(\nu/2)}(\xi - \xi^{-1}) \quad \text{and} \quad \rho^2 = (\xi^2 + \xi^{-2} - 1) - \bar{\omega}^2.$$

For each replication, we apply W_n, L_n and Q_n tests for the hypotheses:

(6.1) $$H_0 : k = 0 \quad \text{v.s.} \quad H_a : k > 0.$$

Under H_0 in (6.1), there is only one coefficient on the boundary of parameter space in DGP 1, and hence the results in Theorem 3.2 are directly applicable, while Algorithm 3.1 is needed to calculate the critical values of W_n and Q_n in GDPs 2-3, since there exist two coefficients on the boundary of parameter space in DGP 2, and a known nuisance coefficient with respect to α_1 exists in DGP 3. To implement Algorithm 3.1, we take $N = 50,000$ in all calculations.

To justify the necessity of the self-weighting technique, all three tests are constructed based on either the GQMLE (i.e., $w_t = 1$) or the S-GQMLE, where w_t is selected as in (2.4) with $m = 2, 3$ and 3 for DGP 1, DGP 2 and DGP 3, respectively, and the values of α are chosen according to two moment situations on y_t:

- Case I: $Ey_t^6 < \infty$ (with respect to $\alpha = 0.1$);
- Case II: $Ey_t^6 = \infty$ and $Ey_t^2 < \infty$ (with respect to $\alpha = 0.6$).

Under Case I, all three tests are applicable no matter whether the self-weighted function w_t is used, while under Case II, all three tests may have poor performance without using w_t.

6.1. *Size comparison.* We first examine the size performance of W_n, L_n and Q_n in finite samples at the level $\beta = 1\%$, 5% and 10%.

Table 2 reports the sizes of all three tests in DGP1. From this table, we can see that under Case I, (i) W_n based on the GQMLE is always under-sized even when $n = 5,000$, and W_n based on the S-GQMLE

has a much more accurate size; (ii) both L_n and Q_n have a satisfactory size performance based on either the GQMLE or the S-GQMLE. In comparison, under Case II, all three tests based on the GQMLE are seriously under-sized, while their sizes are much more accurate based on the S-GQMLE.

TABLE 2
Sizes of all three tests in DGP 1

n	Tests	Case I: $\alpha = 0.1$						Case II: $\alpha = 0.6$					
		with weight			no weight			with weight			no weight		
		1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
		$\eta_t \sim \mathcal{N}(0,1)$											
1000	W_n	0.009	0.048	0.105	0.003	0.038	0.077	0.011	0.051	0.100	0.001	0.013	0.042
	L_n	0.010	0.047	0.102	0.010	0.052	0.094	0.011	0.047	0.098	0.017	0.042	0.079
	Q_n	0.011	0.051	0.105	0.008	0.051	0.087	0.012	0.057	0.098	0.005	0.037	0.076
5000	W_n	0.008	0.051	0.101	0.008	0.033	0.083	0.010	0.053	0.100	0.002	0.025	0.057
	L_n	0.008	0.050	0.103	0.011	0.044	0.087	0.008	0.049	0.107	0.020	0.046	0.085
	Q_n	0.009	0.051	0.101	0.010	0.037	0.090	0.009	0.051	0.100	0.012	0.044	0.097
		$\eta_t \sim \mathrm{st}_{10}$											
1000	W_n	0.010	0.062	0.111	0.004	0.031	0.068	0.008	0.061	0.119	0.002	0.021	0.049
	L_n	0.008	0.053	0.111	0.013	0.048	0.091	0.011	0.050	0.104	0.019	0.040	0.064
	Q_n	0.011	0.061	0.116	0.010	0.046	0.094	0.010	0.066	0.123	0.011	0.043	0.087
5000	W_n	0.010	0.055	0.110	0.005	0.038	0.086	0.010	0.052	0.098	0.003	0.021	0.052
	L_n	0.010	0.058	0.104	0.014	0.055	0.106	0.010	0.055	0.109	0.020	0.044	0.074
	Q_n	0.011	0.059	0.110	0.012	0.054	0.100	0.010	0.053	0.099	0.007	0.046	0.084
		$\eta_t \sim \mathrm{sst}_{10,2}$											
1000	W_n	0.010	0.049	0.107	0.002	0.031	0.073	0.012	0.062	0.105	0.003	0.024	0.053
	L_n	0.012	0.048	0.101	0.015	0.039	0.067	0.013	0.051	0.096	0.022	0.042	0.064
	Q_n	0.013	0.054	0.111	0.013	0.052	0.091	0.016	0.068	0.113	0.018	0.050	0.086
5000	W_n	0.011	0.059	0.110	0.006	0.033	0.077	0.013	0.059	0.114	0.004	0.024	0.056
	L_n	0.011	0.054	0.112	0.014	0.044	0.089	0.012	0.050	0.102	0.023	0.043	0.064
	Q_n	0.013	0.061	0.111	0.014	0.054	0.092	0.014	0.060	0.113	0.016	0.049	0.089

Table 3 reports the sizes of all three tests in DGP 2. From this table, we have similar findings as those in Table 2, except that (i) the sizes of W_n and L_n based on the GQMLE in the cases of $\eta_t \sim \mathrm{st}_{10}$ and $\mathrm{sst}_{10,2}$ are worse than those in the case of $\eta_t \sim \mathcal{N}(0,1)$; (ii) both W_n and Q_n based on the S-GQMLE are slightly oversized, especially when n is small.

Table 4 reports the sizes of all three tests in DGP 3. In this case, we can reach the similar conclusions as in Table 2, except that all tests based on the GQMLE under Case II are undersized even when n is large.

Overall, all three tests based on the S-GQMLE have accurate sizes in all examined cases, while their size performance based on the GQMLE is not satisfactory under Case II, indicating the necessity of the self-weighting technique when y_t has an infinite sixth moment.

TABLE 3

Sizes of all three tests in DGP 2

n	Tests	Case I: $\alpha = 0.1$						Case II: $\alpha = 0.6$					
		with weight			no weight			with weight			no weight		
		1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
		$\eta_t \sim \mathcal{N}(0,1)$											
1000	W_n	0.016	0.065	0.118	0.001	0.022	0.055	0.017	0.075	0.129	0.000	0.010	0.026
	L_n	0.017	0.056	0.119	0.011	0.037	0.090	0.011	0.054	0.101	0.019	0.048	0.075
	Q_n	0.017	0.070	0.127	0.010	0.036	0.082	0.017	0.074	0.133	0.005	0.031	0.066
5000	W_n	0.013	0.055	0.109	0.005	0.031	0.074	0.015	0.060	0.111	0.001	0.017	0.052
	L_n	0.012	0.051	0.096	0.016	0.056	0.106	0.009	0.052	0.102	0.023	0.057	0.099
	Q_n	0.015	0.054	0.113	0.011	0.047	0.092	0.014	0.065	0.111	0.008	0.045	0.091
		$\eta_t \sim \mathrm{st}_{10}$											
1000	W_n	0.016	0.066	0.121	0.004	0.026	0.062	0.019	0.073	0.132	0.002	0.013	0.035
	L_n	0.014	0.050	0.112	0.022	0.054	0.100	0.015	0.053	0.106	0.025	0.053	0.075
	Q_n	0.019	0.073	0.136	0.013	0.053	0.089	0.024	0.084	0.145	0.014	0.040	0.071
5000	W_n	0.011	0.054	0.101	0.003	0.028	0.065	0.016	0.057	0.109	0.002	0.019	0.044
	L_n	0.009	0.052	0.100	0.010	0.040	0.086	0.010	0.051	0.103	0.022	0.050	0.082
	Q_n	0.011	0.058	0.106	0.009	0.044	0.087	0.017	0.060	0.112	0.011	0.042	0.086
		$\eta_t \sim \mathrm{sst}_{10,2}$											
1000	W_n	0.022	0.076	0.139	0.001	0.022	0.048	0.021	0.084	0.154	0.002	0.014	0.043
	L_n	0.017	0.061	0.109	0.026	0.058	0.080	0.016	0.056	0.116	0.021	0.046	0.062
	Q_n	0.032	0.089	0.146	0.019	0.051	0.084	0.031	0.098	0.167	0.014	0.044	0.083
5000	W_n	0.015	0.061	0.111	0.002	0.022	0.060	0.012	0.066	0.126	0.006	0.019	0.052
	L_n	0.011	0.053	0.108	0.025	0.053	0.086	0.009	0.050	0.102	0.031	0.051	0.070
	Q_n	0.018	0.061	0.113	0.017	0.048	0.092	0.013	0.070	0.128	0.017	0.049	0.092

TABLE 4

Sizes of all three tests in DGP 3

n	Tests	Case I: $\alpha = 0.1$						Case II: $\alpha = 0.6$					
		with weight			no weight			with weight			no weight		
		1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
		$\eta_t \sim \mathcal{N}(0,1)$											
1000	W_n	0.016	0.060	0.112	0.007	0.034	0.076	0.016	0.071	0.135	0.002	0.037	0.049
	L_n	0.018	0.061	0.116	0.013	0.050	0.102	0.009	0.043	0.089	0.014	0.034	0.077
	Q_n	0.018	0.063	0.114	0.013	0.045	0.098	0.015	0.064	0.120	0.006	0.050	0.065
5000	W_n	0.011	0.056	0.104	0.006	0.036	0.084	0.015	0.077	0.131	0.002	0.024	0.063
	L_n	0.010	0.051	0.102	0.012	0.046	0.094	0.009	0.046	0.105	0.014	0.041	0.077
	Q_n	0.012	0.056	0.105	0.009	0.045	0.096	0.014	0.071	0.120	0.005	0.035	0.079
		$\eta_t \sim \mathrm{st}_{10}$											
1000	W_n	0.010	0.057	0.119	0.002	0.037	0.089	0.013	0.074	0.136	0.003	0.024	0.054
	L_n	0.013	0.060	0.118	0.010	0.054	0.103	0.010	0.052	0.103	0.021	0.048	0.079
	Q_n	0.014	0.065	0.125	0.012	0.058	0.109	0.015	0.073	0.132	0.011	0.041	0.078
5000	W_n	0.010	0.056	0.117	0.007	0.046	0.104	0.013	0.056	0.116	0.005	0.028	0.059
	L_n	0.013	0.058	0.114	0.014	0.058	0.103	0.010	0.050	0.093	0.014	0.044	0.091
	Q_n	0.013	0.061	0.122	0.013	0.059	0.116	0.013	0.053	0.113	0.011	0.039	0.071
		$\eta_t \sim \mathrm{sst}_{10,2}$											
1000	W_n	0.017	0.071	0.130	0.005	0.024	0.068	0.020	0.080	0.143	0.035	0.022	0.056
	L_n	0.015	0.065	0.124	0.014	0.040	0.082	0.013	0.054	0.101	0.017	0.046	0.071
	Q_n	0.021	0.084	0.131	0.013	0.044	0.095	0.024	0.079	0.139	0.009	0.038	0.072
5000	W_n	0.013	0.055	0.109	0.007	0.036	0.079	0.011	0.060	0.123	0.002	0.024	0.061
	L_n	0.013	0.059	0.108	0.016	0.049	0.082	0.008	0.038	0.089	0.013	0.040	0.072
	Q_n	0.016	0.058	0.113	0.013	0.047	0.095	0.013	0.058	0.121	0.007	0.041	0.082

6.2. *Power comparison.* We next compare the local power of all three tests in finite samples at the level $\beta = 5\%$. Under Case I, the size-adjusted local power is computed to do a better power comparison, while under Case II, the size-adjusted local power is only calculated based on the S-GQMLE, since the sizes of all three tests based on the GQMLE are not accurate in this case. Also, we only show the power plot for $\eta_t \sim \mathcal{N}(0, 1)$, since the power plots for $\eta_t \sim \mathrm{st}_{10}$ and $\eta_t \sim \mathrm{sst}_{10,2}$ are similar, and they are available upon request but not depicted here for saving the space.
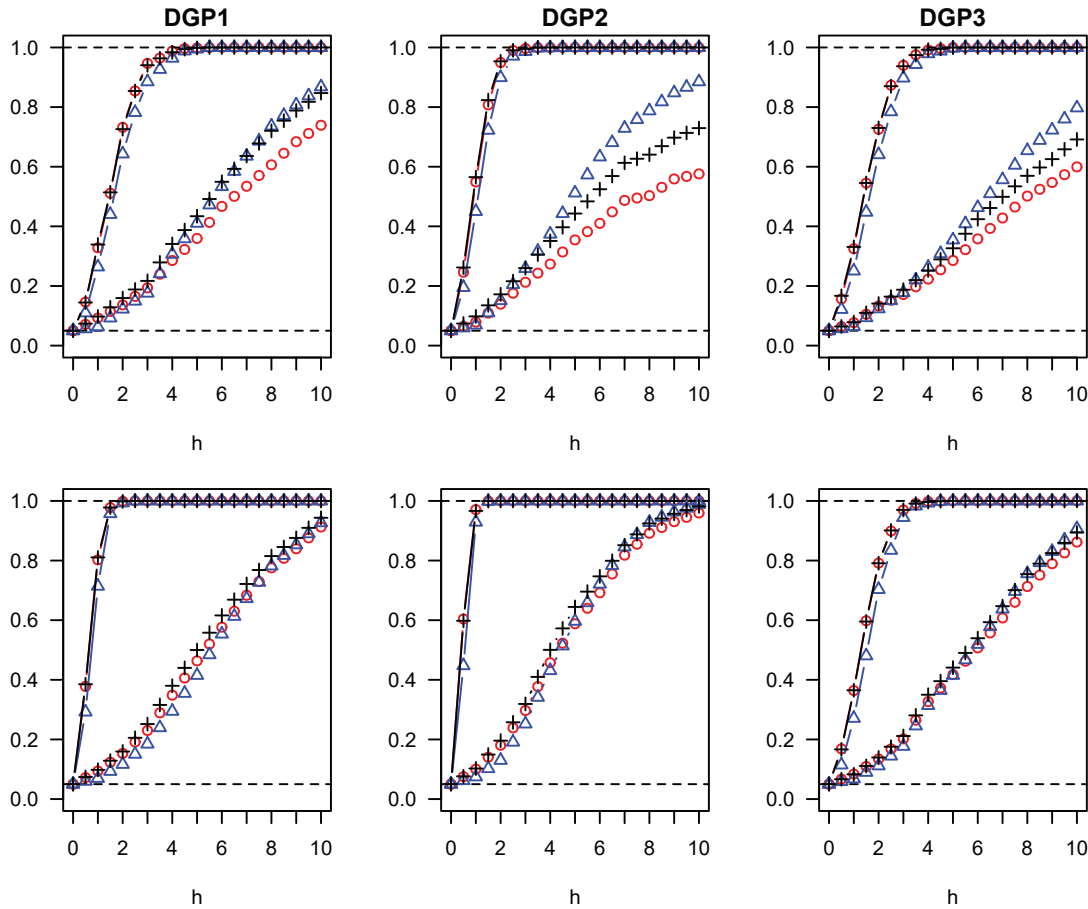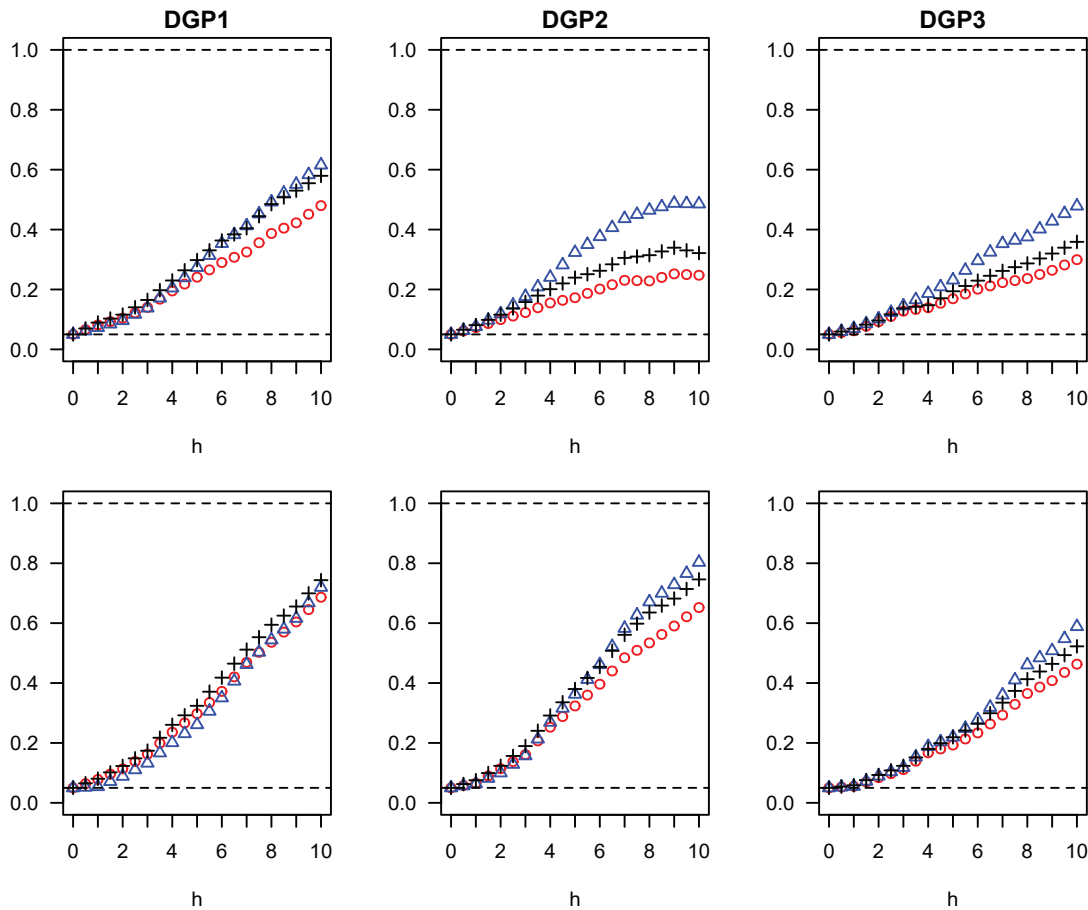

FIG 2. *Local power comparison in Case I. (Upper: $n = 1,000$; bottom: $n = 5,000$. Circle ($\circ$): W_n ; triangle ($\triangle$): L_n; cross (+): Q_n. With weight: points; without weight: points and lines)*

Fig 2 plots the size-adjusted local power (across h) for all three tests under Case I. From this figure, we can find that (i) the power of all three tests based on the GQMLE is always higher than that based on

the S-GQMLE; (ii) based on the S-GQMLE, L_n is most powerful when n is small (i.e., $n = 1,000$), while the power advantage of L_n over other two tests becomes vague when n is large (i.e., $n = 5,000$); (iii) all three tests are consistent in all examined cases; (iv) when $n = 5,000$, W_n and Q_n are more powerful than L_n in DGP 1, and this is consistent to the result in Theorem 4.3.

Fig 3 plots the size-adjusted local power (across h) for all three tests under Case II. From this figure, we have the similar findings as those in Fig 2, except that (i) when n is small, the local power for all three tests is relatively small even when h is large; (ii) L_n becomes more powerful than W_n in DGP 1, especially for large h.

FIG 3. *Local power comparison in Case II with weight.(Upper: $n = 1,000$; bottom: $n = 5,000$. Circle (○): W_n ; triangle (△): L_n; cross (+): Q_n)*

Overall, when y_t has a finite sixth moment, the power comparison suggests us to use all three tests based on the GQMLE instead of the S-GQMLE. On the other hand, when y_t has an infinite sixth moment, all three tests based on the S-GQMLE have a satisfactory power.

7. A real example. In this section, we study the weekly 3-Month Treasury Bill rate of second market in U.S. from January, 1970 to December, 1989, which has 1044 observations in total. Denote this data set by $\{x_t\}_{t=0}^{1043}$, and its difference (in percentage) by $\{y_t\}_{t=1}^{1043}$, where $y_t = 100(x_t - x_{t-1})$, and both $\{x_t\}_{t=0}^{1043}$ and $\{y_t\}_{t=1}^{1043}$ are plotted in Fig 4. To begin with, we apply the Phillips-Perron test $Z(\hat{\alpha})$ in Phillips and Perron (1988, p.341) to $\{x_t\}_{t=0}^{1043}$ and $\{y_t\}_{t=1}^{1043}$, and find that the corresponding p-values are 0.3474 and 0.0010, respectively. These results imply that y_t is stationary while x_t is not. Hence, we consider y_t instead of x_t in the sequel.
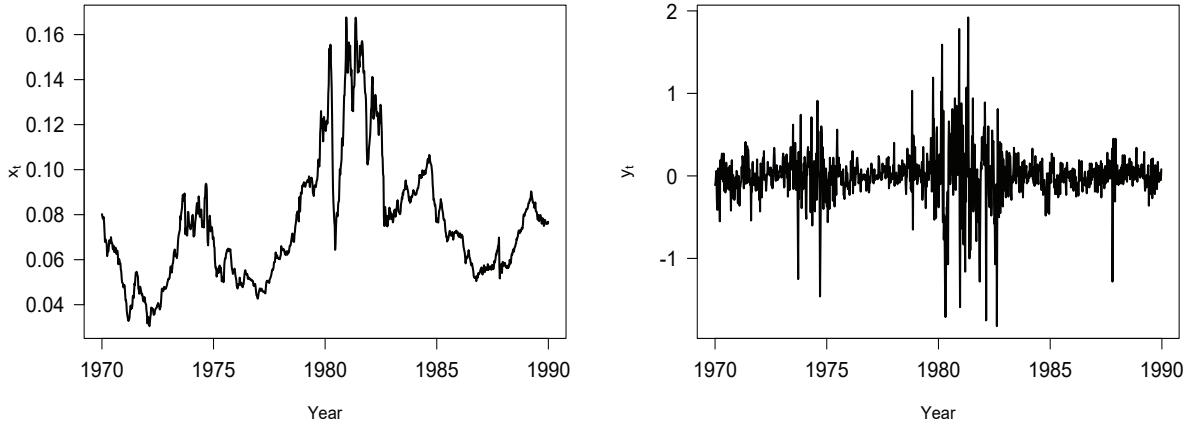

FIG 4. *Plots of $\{x_t\}$ and $\{y_t\}$.*

Next, based on $\{y_t\}_{t=1}^{1043}$, we use the Hill estimator to estimate the tail index of y_t. Fig 5 plots the right-tail and left-tail Hill estimators of y_t given by

$$H_{1k} = \left\{ \frac{1}{k} \sum_{i=1}^{k} \log\left(\frac{y_{(i)}}{y_{(k+1)}}\right) \right\}^{-1} \qquad \text{and} \qquad H_{2k} = \left\{ \frac{1}{k} \sum_{i=1}^{k} \log\left(\frac{y_{(n-i+1)}}{y_{(n-k)}}\right) \right\}^{-1},$$

respectively, where $\{y_{(t)}\}_{t=1}^{1043}$ are the ascending order statistics of $\{y_t\}_{t=1}^{1043}$. From Fig 5, we can see that both the right-tail and left-tail indices of y_t are most likely less than 2. Hence, y_t seems to be heavy-tailed with an infinite second moment, indicating the use of self-weighting technique to model $\{y_t\}_{t=1}^{1043}$.
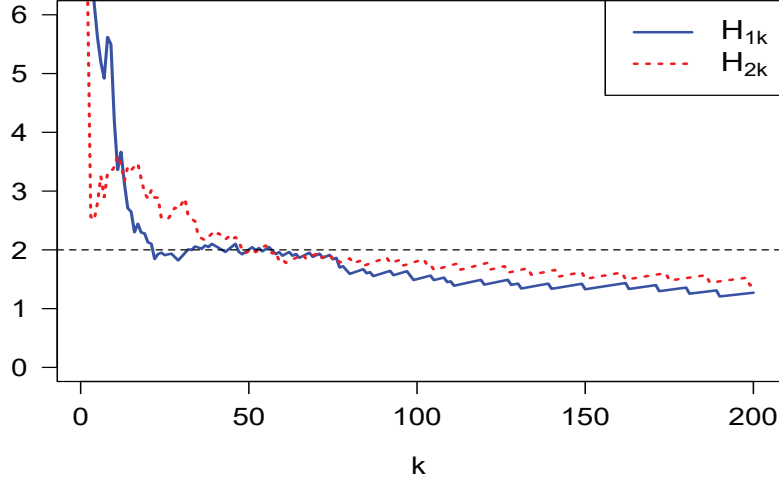

FIG 5. *Right-tail Hill estimators $H_{1k}(-)$ and left-tail Hill estimators $H_{2k}(- -)$.*

Based on these facts, we fit $\{y_t\}_{t=1}^{1043}$ by an ADAR model:

$$(7.1) \quad \begin{cases} y_t = -0.0002_{(-0.0137)} + 0.2733_{(6.4160)}\, y_{t-1} - 0.0097_{(0.2506)}\, y_{t-2} + 0.0405_{(1.0887)}\, y_{t-3} \\ \quad + 0.1689_{(4.7712)}\, y_{t-4} + 0.0303_{(0.8535)}\, y_{t-5} + 0.0186_{(0.4526)}\, y_{t-6} - 0.0317_{(0.8661)}\, y_{t-7} + \eta_t\sqrt{h_t}, \\ h_t = 0.0062_{(3.6471)} + 0.3560_{(3.8280)}\, y_{t-1}^2 + 0.1274_{(1.9969)}\, y_{t-2}^2 + 0.1024_{(1.7595)}\, y_{t-3}^2 \\ \quad + 0.0537_{(1.0783)}\, y_{t-4}^2 + 0.0322_{(0.6880)}\, y_{t-5}^2 + 0.2871_{(3.4466)}\, y_{t-6}^2 + 0.0791_{(1.4785)}\, y_{t-7}^2, \end{cases}$$

where model (7.1) is estimated by the S-GQMLE with the self-weighted function w_t defined in (2.4) with $m = 7$, and the corresponding values of t_n are given in parentheses.

In the following, we use a two-step test procedure to examine the significance of coefficients in model (7.1). First, we apply the tests W_n, L_n and Q_n to detect the null hypothesis $H_0^{(m)} : u_0 = \phi_{20} = \phi_{30} = \phi_{50} = \phi_{60} = \phi_{70} = 0$. Since $H_0^{(m)}$ is designed for the conditional mean coefficients, all W_n, L_n and Q_n

tests have standard asymptotics, and their corresponding p-values are 0.831, 0.829 and 0.907, respectively. Hence, we can not reject $H_0^{(m)}$ at the level 5%, and then fit $\{y_t\}_{t=1}^{1043}$ by the following reduced ADAR model:

$$(7.2) \quad \begin{cases} y_t = 0.2699_{(6.4569)}\, y_{t-1} + 0.1765_{(5.1912)}\, y_{t-4} + \eta_t \sqrt{h_t}, \\ h_t = 0.0063_{(3.7059)} + 0.3679_{(3.9773)}\, y_{t-1}^2 + 0.1254_{(2.0096)}\, y_{t-2}^2 + 0.1109_{(1.8990)}\, y_{t-3}^2 \\ \qquad + 0.0480_{(1.0063)}\, y_{t-4}^2 + 0.0305_{(0.6489)}\, y_{t-5}^2 + 0.2897_{(3.5372)}\, y_{t-6}^2 + 0.0746_{(1.4458)}\, y_{t-7}^2, \end{cases}$$

where the preceding model is estimated in the same way as model (7.1). Second, since the values of t_n for the coefficients α_4 and α_5 are relatively small in model (7.2), we further apply the tests W_n, L_n and Q_n to detect the null hypothesis $H_0^{(v)'} : \alpha_{40} = \alpha_{50} = 0$. In this case, the p-values of W_n, L_n and Q_n are 0.197, 0.391 and 0.181, respectively, where the p-value of L_n is computed by Theorem 3.1(ii), and the p-values of W_n and Q_n are computed by Algorithm 3.1. Here, we take $d_1 = 8$, $d_2 = 0$, $d_3 = 2$, $\theta_0^{(1)} = (\phi_{10}, \phi_{40}, \omega_0, \alpha_{10}, \alpha_{20}, \alpha_{30}, \alpha_{60}, \alpha_{70})'$, $\theta_0^{(3)} = (\alpha_{40}, \alpha_{50})'$, $\Lambda = \mathbb{R}^{d_1} \times [0, \infty]^{d_3}$, and $\Lambda_{|3} = \mathbb{R}^{d_1} \times \{0\}^{d_3}$ in the implementation of Algorithm 3.1. Since $H_0^{(v)'}$ can not be rejected at the level 5% as indicated by all three tests, we refit $\{y_t\}_{t=1}^{1043}$ by a simpler reduced ADAR model:

$$(7.3) \quad \begin{cases} y_t = 0.2704_{(6.4381)}\, y_{t-1} + 0.1778_{(5.6444)}\, y_{t-4} + \eta_t \sqrt{h_t}, \\ h_t = 0.0069_{(4.0588)} + 0.3943_{(4.2627)}\, y_{t-1}^2 + 0.1351_{(2.1790)}\, y_{t-2}^2 + 0.1326_{(2.2784)}\, y_{t-3}^2 \\ \qquad + 0.2960_{(3.7374)}\, y_{t-6}^2 + 0.0787_{(1.5835)}\, y_{t-7}^2, \end{cases}$$

where the preceding model is estimated in the same way as model (7.1). For model (7.3), the tests W_n, L_n and Q_n (with p-values close to zero) imply that the null hypothesis $H_0^{(v)''} : \omega_0 = \phi_{10} = \phi_{20} = \phi_{30} = \phi_{60} = \phi_{70} = 0$ is rejected. Meanwhile, by using the results in Theorem 3.2, the p-values of the tests t_n, W_n, L_n and Q_n for the null hypothesis $H_0^{(v)'''} : \phi_{70} = 0$ are 0.056, 0.056, 0.005 and 0.008, respectively, indicating that ϕ_{70} is not zero. These results may suggest that our presumption of $d_2 = 0$ above is appropriate. Moreover, we find that model (7.3) is adequate at the level 5%, since its p-values of portmanteau tests Q_6, Q_{12} and Q_{18} calculated from the standard chi-squared limiting null distribution are 0.984, 0.840 and 0.760, respectively.

As a comparison, we also fit $\{y_t\}_{t=1}^{1043}$ by an AR-GARCH model:

$$(7.4) \quad \begin{cases} y_t = 0.2185_{(5.6901)}\, y_{t-1} + 0.1183_{(3.1547)}\, y_{t-4} + \epsilon_t, \quad \epsilon_t = \eta_t \sqrt{h_t}, \\ h_t = 0.0014_{(3.5012)} + 0.3039_{(6.3445)}\, \epsilon_{t-1}^2 + 0.7328_{(22.0723)}\, h_{t-1}, \end{cases}$$

where model (7.4) is estimated by the S-GQMLE in Ling (2007b) with the self-weighted function w_t defined in (2.4), and the corresponding values of t_n are given in parentheses. The values of AIC and BIC of model (7.4) are -1799.5 and -1774.8, respectively, while these of model (7.3) are -1841.5 and -1801.9, respectively. In view of this, model (7.3) is more suitable than model (7.4) to fit $\{y_t\}_{t=1}^{1043}$.

APPENDIX A: PROOFS OF THEOREMS

Recall that $J_n(\theta) = \frac{\partial^2 F_n(\theta)}{\partial\theta\partial\theta'}$. In what follows, we let $I_n(\theta) = \frac{\partial F_n(\theta)}{\partial\theta}$ and C be a positive generic constant. As for $\theta_0 = (\theta_0^{(1)'}, \theta_0^{(2)'}, \theta_0^{(3)'})'$, we write $Z = (Z_{(1)}', Z_{(2)}', Z_{(3)}')'$ and $G := JZ = (G_{(1)}', G_{(2)}', G_{(3)}')'$, where $Z_{(i)}$ or $G_{(i)}$ is of dimension d_i for $i = 1, 2, 3$. Meanwhile, we let $J_{(ij)}$, $i, j = 1, 2, 3$, be the corresponding blocks of the matrix J, and $\Lambda_{(i)}$, $i = 1, 2, 3$, be the corresponding blocks of the space Λ. Further, we let $Z_{(2,3)} = (Z_{(2)}', Z_{(3)}')'$, $\Lambda_{(2,3)} = \Lambda_{(2)} \times \Lambda_{(3)}$, $\lambda_{(2,3)}^\Lambda$ be the last $(d_2 + d_3)$ components of λ^Λ, and

$$\lambda_{(2,3)}^{\Lambda_{(2,3)}} = \arg \inf_{\lambda_{(2,3)} \in \Lambda_{(2,3)}} \left\| Z_{(2,3)} - \lambda_{(2,3)} \right\|_{(KJ^{-1}K')^{-1}}^2 .$$

Proof of Theorem 2.1. (i) By Lemma B.2 and the similar arguments as for Theorem 2.1 in Zhu and Ling (2011), we can show that (i) holds.

(ii) Let

$$(A.1) \qquad \theta_{J_n}(Z_n) = \arg \inf_{\theta \in \Theta} \|Z_n - \sqrt{n}(\theta - \theta_0)\|_{J_n} \quad \text{and} \quad \lambda_n^\Lambda = \arg \inf_{\lambda \in \Lambda} \|Z_n - \lambda\|_{J_n},$$

where

$$(A.2) \qquad J_n = J_n(\theta_0) \quad \text{and} \quad Z_n = -J_n^{-1}\sqrt{n}I_n(\theta_0).$$

By Lemma B.4(i)-(ii), J is positive definite and $J_n - J = o_p(1)$. Then, by Lemma 2.3 in Knopov and Korkhin (2011), it is not hard to show that J_n is a positive definite matrix (a.s.) for sufficiently large n. Since Θ contains a hypercube, a similar argument as Francq and Zakoïan (2007) implies that

$$
\text{(A.3)} \qquad \sqrt{n}\big(\theta_{J_n}(Z_n) - \theta_0\big) = \lambda_n^\Lambda
$$

for sufficiently large n. Using Taylor's expansion for a function with right partial derivatives we get for all θ and $\theta_0 \in \Theta$,

$$
\begin{aligned}
F_n(\theta) =& F_n(\theta_0) + I_n(\theta_0)'(\theta - \theta_0) + \frac{1}{2}(\theta - \theta_0)' J_n(\theta_0)(\theta - \theta_0) + R_n(\theta) \\
=& F_n(\theta_0) - \frac{1}{2n} Z_n' J_n \sqrt{n}(\theta - \theta_0) - \frac{1}{2n}\sqrt{n}(\theta - \theta_0)' J_n Z_n + \frac{1}{2}(\theta - \theta_0)' J_n(\theta - \theta_0) + R_n(\theta) \\
\text{(A.4)} \qquad =& F_n(\theta_0) + \frac{1}{2n}\|Z_n - \sqrt{n}(\theta - \theta_0)\|_{J_n}^2 - \frac{1}{2n}\|Z_n\|_{J_n} + R_n(\theta),
\end{aligned}
$$

where $R_n(\theta)$ is the remainder. Next, by Lemma B.4(ii) and the similar arguments as for Theorem 2 in Francq and Zakoïan (2007), we can show that

(ai) $\sqrt{n}\big(\theta_{J_n}(Z_n) - \theta_0\big) = O_p(1)$;

(aii) $\|\sqrt{n}(\hat{\theta}_n - \theta_0)\|_{J_n} = O_p(1)$;

(aiii) for any sequence $(\tilde{\theta}_n)$ such that $\sqrt{n}(\tilde{\theta}_n - \theta_0) = O_p(1)$, $R_n(\tilde{\theta}_n) = o_p(n^{-1})$;

(aiv) $\|Z_n - \sqrt{n}(\hat{\theta}_n - \theta_0)\|_{J_n}^2 - \|Z_n - \lambda_n^\Lambda\|_{J_n}^2 = o_p(1)$;

(av) $\sqrt{n}(\hat{\theta}_n - \theta_0) - \lambda_n^\Lambda = o_p(1)$;

(avi) $\lambda_n^\Lambda \to_\mathcal{L} \lambda^\Lambda$.

Now, the conclusion in (ii) holds by (av) and (avi). This completes all of the proofs. $\qquad \square$

Proof of Theorem 3.1. By Lemma B.4(ii), we have that under H_0,

$$
\hat{\Sigma}_n = \Sigma + o_p(1), \ \ \hat{J}_n = J + o_p(1), \ \ \hat{\Sigma}_{n|3} = \Sigma + o_p(1) \ \text{ and } \ \hat{J}_{n|3} = J + o_p(1).
$$

(i) Note that $\sqrt{n}(\hat{\theta}_n^{(3)} - \theta_0^{(3)}) = \sqrt{n} K_\alpha(\hat{\theta}_n - \theta_0) \to_\mathcal{L} K_\alpha \lambda^\Lambda$ by Theorem 2.1. Thus, it follows by the continuous mapping theorem that $W_n \to_\mathcal{L} W = \lambda^{\Lambda'} \Omega \lambda^\Lambda$.

(ii) Rewrite the Lagrangian as $F_n(\theta) + (K_\alpha\theta)'\gamma$ for a Lagrange multiplier γ. Then, the solutions $\hat{\theta}_{n|3}$ and $\gamma_{n|3}$ satisfy the saddle-point condition:

$$I_n(\hat{\theta}_{n|3}) + K_\alpha'\gamma_{n|3} = 0.$$

By Taylor's expansion, it follows that

$$0 = \sqrt{n}I_n(\theta_0) + \sqrt{n}J_n(\theta^*)(\hat{\theta}_{n|3} - \theta_0) + \sqrt{n}K_\alpha'\gamma_{n|3} + o_p(1),$$

where θ^* lies between $\hat{\theta}_{n|3}$ and θ_0. Since $J_n(\theta^*) = J + o_p(1)$ under H_0 by Lemma B.4(ii), the preceding equation entails that

(A.5) $$\sqrt{n}I_n(\theta_0) + \sqrt{n}J(\hat{\theta}_{n|3} - \theta_0) + \sqrt{n}K_\alpha'\gamma_{n|3} = o_p(1).$$

Multiplying both sides of (A.5) by $K_\alpha J^{-1}$, we have $\sqrt{n}\gamma_{n|3} = -(K_\alpha J^{-1}K_\alpha')^{-1}K_\alpha J^{-1}[\sqrt{n}I_n(\theta_0)] + o_p(1)$ by noting the restriction $K_\alpha(\hat{\theta}_{n|3} - \theta_0) = 0$. Thus, since

(A.6) $$\sqrt{n}I_n(\theta_0) \to_{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

by Lemma B.1(ii) and the martingale central limit theorem in Brown (1971), we can conclude that

$$\sqrt{n}\gamma_{n|3} \to_{\mathcal{L}} \mathcal{N}(0, (K_\alpha J^{-1}K_\alpha')^{-1}K_\alpha J^{-1}\Sigma J^{-1}K_\alpha'(K_\alpha J^{-1}K_\alpha')^{-1}),$$

which implies that $L_n \to_{\mathcal{L}} \chi_{d_3}^2$ by the continuous mapping theorem, and the facts that both Σ and J are positive definite by Lemma B.4(i).

(iii) By Taylor's expansion and the similar proof as for Theorem 2.1, we can get

$$nF_n(\hat{\theta}_n) = nF_n(\theta_0) + nI_n(\theta_0)(\hat{\theta}_n - \theta_0) + \frac{n}{2}(\hat{\theta}_n - \theta_0)'J(\hat{\theta}_n - \theta_0) + o_p(1)$$
$$= nF_n(\theta_0) + \frac{1}{2}(\left\|Z - \lambda^\Lambda\right\|_J^2 - \|Z\|_J^2) + o_p(1)$$

and

$$nF_n(\hat{\theta}_{n|3}) = nF_n(\theta_0) + nI_n(\theta_0)(\hat{\theta}_{n|3} - \theta_0) + \frac{n}{2}(\hat{\theta}_{n|3} - \theta_0)'J(\hat{\theta}_{n|3} - \theta_0) + o_p(1)$$
$$= nF_n(\theta_0) + \frac{1}{2}(\left\|Z - \lambda_{|3}^\Lambda\right\|_J^2 - \|Z\|_J^2) + o_p(1).$$

Hence, it follows that $Q_n \to_{\mathcal{L}} Q = \left\| Z - \lambda_{|3}^{\Lambda} \right\|_J^2 - \left\| Z - \lambda^{\Lambda} \right\|_J^2$.

Note that

$$
\begin{aligned}
\left\| Z - \lambda^{\Lambda} \right\|_J^2 - \|Z\|_J^2 = & - G'_{(1)} J_{(11)}^{-1} G_{(1)} - 2\lambda_{(2,3)}^{\Lambda'} (KJ^{-1}K')^{-1} Z_{(2,3)} + \lambda_{(2,3)}^{\Lambda'} (KJ^{-1}K')^{-1} \lambda_{(2,3)}^{\Lambda} \\
= & - G'_{(1)} J_{(11)}^{-1} G_{(1)} - 2\lambda_{(2,3)}^{\Lambda_{(2,3)}'} (KJ^{-1}K')^{-1} Z_{(2,3)} + \lambda_{(2,3)}^{\Lambda'} (KJ^{-1}K')^{-1} \lambda_{(2,3)}^{\Lambda_{(2,3)}} \\
= & - G'_{(1)} J_{(11)}^{-1} G_{(1)} - \lambda_{(2,3)}^{\Lambda_{(2,3)}'} (KJ^{-1}K')^{-1} \lambda_{(2,3)}^{\Lambda_{(2,3)}} + 2\lambda_{(2,3)}^{\Lambda_{(2,3)}'} (KJ^{-1}K')^{-1} (\lambda_{(2,3)}^{\Lambda_{(2,3)}} - Z_{(2,3)}) \\
= & - G'_{(1)} J_{(11)}^{-1} G_{(1)} - \lambda^{\Lambda'} \Xi \lambda^{\Lambda},
\end{aligned}
$$

where the first equation follows by Lemma B.5(i)-(ii), the second equation follows by Lemma B.5(iii), and the third equation holds since $\lambda_{(2,3)}^{\Lambda_{(2,3)}} = K\lambda^{\Lambda}$ and $\lambda_{(2,3)}^{\Lambda_{(2,3)}}$ is orthogonal to $(\lambda_{(2,3)}^{\Lambda_{(2,3)}} - Z_{(2,3)})$. Using the same technique to $\left\| Z - \lambda_{|3}^{\Lambda} \right\|_J^2 - \|Z\|_J^2$, we can obtain that $Q = \lambda^{\Lambda'} \Xi \lambda^{\Lambda} - \lambda_{|3}^{\Lambda'} \Xi \lambda_{|3}^{\Lambda}$. This completes all of the proofs. $\qquad \square$

Proof of Theorem 3.2. When H_0' holds and $d_2 = 0$, $\Lambda = R^{d-1} \times [0, \infty)$ and $K = K_\alpha = (0, \cdots, 0, 1)$. In this case,

$$
\lambda^{\Lambda} = ZI(Z_d \geq 0) + PZI(Z_d < 0) = Z - Z_d^- c^*,
$$

where $P = I_d - J^{-1}K'(KJ^{-1}K')^{-1}K$, $Z_d^- = Z_d I(Z_d < 0)$, and the vector c^* is the last column of J^{-1} divided by the (d, d) element of J^{-1}.

First, since the last element of c^* is 1, the dth element of λ^{Λ} is $\lambda_d^{\Lambda} = Z_d I(Z_d \geq 0)$. Thus, it follows that

$$
(A.7) \qquad W = \lambda^{\Lambda'} \Omega \lambda^{\Lambda} = [\lambda_d^{\Lambda}]^2 / \mathrm{var}(Z_d) = Z_d^2 I(Z_d \geq 0) / \mathrm{var}(Z_d) \sim U^2 I(U \geq 0),
$$

where $U \sim \mathcal{N}(0, 1)$. By Theorem 3.1(i) and (A.7), we know that W_n has the critical region $\{W_n > \chi_{1,1-2\beta}^2\}$.

Second, t_n has the critical region $\{t_n > \Phi^{-1}(1 - \beta)\}$, since $t_n = \sqrt{W_n}$ has the law of $UI(U \geq 0)$.

Third, since $\xi Q = W$ with $\xi = \Omega / \Xi$, we know that ξQ has the law of W. By Theorem 3.1(iii), it follows that Q_n has the critical region $\{\xi_n Q_n > \chi_{1,1-2\beta}^2\}$, where $\xi_n = \hat{\Omega}_n / \hat{\Xi}_n$ is a consistent estimator of ξ. This completes all of the proofs. $\qquad \square$

Proof of Theorem 4.1. (i) Note that under $\mathbb{P}_{n,h}$, $F_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} w_{t,n} \ell_{t,n}(\theta)$, where $w_{t,n}$ and $\ell_{t,n}(\theta)$ are defined as in (4.1). First, by Assumption 2.3′ with $\delta = 0$ and a similar argument as for Lemma B.1(i), it is not hard to show that

$$(A.8) \qquad E \sup_{\theta \in \Theta} w_{t,n}^2 \ell_{t,n}^2(\theta) < \infty.$$

Next, by Theorem 2.1 in Ling (2007a), $y_{t,n}$ is geometrically ergodic, and hence β-mixing. By Theorem 3.49 in White (2001), it follows that $\ell_{t,n}(\theta)$ is β-mixing. Then, by Lemma B.3 with $s = 2$, we have

$$(A.9) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} [w_{t,n} \ell_{t,n}(\theta) - E w_{t,n} \ell_{t,n}(\theta)] = 0 \quad \text{in probability.}$$

Furthermore, the stationarity of $w_{t,n} \ell_{t,n}(\theta)$ ensures that

$$\frac{1}{n} \sum_{t=1}^{n} E w_{t,n} \ell_{t,n}(\theta) = E w_{t,n} \ell_{t,n}(\theta),$$

and the dominated convergence theorem entails that

$$\lim_{n \to \infty} E w_{t,n} \ell_{t,n}(\theta) = E \lim_{n \to \infty} w_{t,n} \ell_{t,n}(\theta) = E w_t \ell_t(\theta).$$

By (A.9) and the preceding equalities, we know that for any $\theta \in \Theta$,

$$(A.10) \qquad F_n(\theta) - E w_t \ell_t(\theta) = o_p(1).$$

Third, by a similar argument as for Lemma B.1(ii), we can prove that $\sup_{\theta \in \Theta} \|I_n(\theta)\| = O_p(1)$, and hence it follows that Assumption 3A in Newey (1991) holds, and then by (A.10) and Theorem 2.1 in Newey (1991), we have

$$(A.11) \qquad \sup_{\theta \in \Theta} |F_n(\theta) - E w_t \ell_t(\theta)| = o_p(1).$$

Finally, since $E[w_t \ell_t(\theta)]$ attains the global minimum at θ_0 by Lemma B.2(i), (i) holds by (A.11) and Theorem 4.1.1 in Amemiya (1985).

(ii) Recall that $\theta_n = \theta_0 + h/\sqrt{n}$. Let

$$\theta_{J_{n,h}}(Z_{n,h}) = \arg \inf_{\theta \in \Theta} \left\| Z_{n,h} - \sqrt{n}(\theta - \theta_n) \right\|_{J_{n,h}} \quad \text{and} \quad \lambda_{n,h}^{\Lambda} = \arg \inf_{\lambda \in \Lambda} \left\| Z_{n,h} + h - \lambda \right\|_{J_{n,h}},$$

where

(A.12) $J_{n,h} = J_n(\theta_n)$ and $Z_{n,h} = -J_{n,h}^{-1}\sqrt{n}I_n(\theta_n).$

By Lemma B.4(i) and (iii), we can show that $J_{n,h}$ is a positive definite matrix (a.s.) for sufficiently large n. Next, by the similar arguments as for (A.3) and (A.4), we have that

$$\sqrt{n}\big(\theta_{J_{n,h}}(Z_{n,h}) - \theta_n\big) = \lambda_{n,h}^{\Lambda} - h$$

for sufficiently large n, and

$$F_n(\theta) = F_n(\theta_n) + \frac{1}{2n}\big\|Z_{n,h} - \sqrt{n}(\theta - \theta_n)\big\|_{Z_{n,h}}^2 - \frac{1}{2n}\|Z_{n,h}\|_{J_{n,h}} + R_n(\theta),$$

where $R_n(\theta)$ is the remainder.

Furthermore, by Lemma B.4(iii) and the similar arguments as for (ai)-(avi), we can show that

(bi) $\sqrt{n}\big(\theta_{J_{n,h}}(Z_{n,h}) - \theta_n\big) = O_p(1);$

(bii) $\|\sqrt{n}(\hat{\theta}_{n,h} - \theta_n)\|_{J_{n,h}} = O_p(1);$

(biii) for any sequence $(\tilde{\theta}_n)$ such that $\sqrt{n}(\tilde{\theta}_n - \theta_n) = O_p(1),\ \ R_n(\tilde{\theta}_n) = o_p(n^{-1});$

(biv) $\|Z_{n,h} - \sqrt{n}(\hat{\theta}_{n,h} - \theta_n)\|_{J_{n,h}}^2 - \|Z_{n,h} - (\lambda_{n,h}^{\Lambda} - h)\|_{J_{n,h}}^2 = o_p(1);$

(bv) $\sqrt{n}(\hat{\theta}_{n,h} - \theta_n) - (\lambda_{n,h}^{\Lambda} - h) = o_p(1);$

(bvi) $\lambda_{n,h}^{\Lambda} \to_{\mathcal{L}} \lambda^{\Lambda}(h).$

Now, (ii) holds by (bv) and (bvi). This completes the proof.   □

Proof of Theorem 4.2.   The proof is given in the supplementary material (Jiang et al. (2019)).

Proof of Theorem 4.3.   We first consider the Wald test. When $d_2 = 0$ and $d_3 = 1$, $\Omega = 1/\mathrm{var}(Z_d) = 1/\sigma_d^2$ by (3.1), and $\lambda_d^{\Lambda}(h) = (Z_d + h_d)I(Z_d + h_d > 0)$ by the similar arguments as for (A.7). Hence, it follows that

(A.13) $W(h) = \lambda_d^{\Lambda}(h)^2/\mathrm{var}(Z_d) = \left(\dfrac{Z_d}{\sigma_d} + \dfrac{h_d}{\sigma_d}\right)^2 I(Z_d + h_d > 0) \sim (U + h^*)^2 I(U + h^* > 0)$

for $U \sim \mathcal{N}(0,1)$ and $h^* = h_d/\sigma_d$. By (A.13), we know that the result for the Wald test holds. Furthermore, since $\xi Q(h) = W(h)$, it follows that the result for the QLR test holds.

Next, we consider the t-type test. Since $\{(U + h^*) > c_1\} \subset \{(U + h^*) > 0\}$, we have

(A.14) $$\mathbb{P}\{(U + h^*)I(U + h^* > 0) > c_1\} = \mathbb{P}\{(U + h^*) > c_1\}.$$

By noting that $W_n = t_n^2$ and (A.13)-(A.14), we can show that the result for the t-test holds.

Third, we consider the LM test. Since $L(h) \sim \chi_1^2(h'\Omega h) \sim (U + h^*)^2$ and

$$\mathbb{P}\{(U + h^*)^2 > c_2^2\} = \mathbb{P}\{(U + h^*) > c_2\} + \mathbb{P}\{(U + h^*) < -c_2\},$$

it follows that the result for the LM test holds.

Finally, by the similar arguments as in Francq and Zakoïan (2009), we can prove that the asymptotic local power of the t-type, Wald and QLR tests is higher than that of the LM test. The proof is completed. □

Proof of Theorem 5.1. The proof is given in the supplementary material (Jiang et al. (2019)). □

APPENDIX B: LEMMAS

This appendix provides five useful lemmas, and their proofs can be found in the supplementary material (Jiang et al. (2019)).

LEMMA B.1. *Suppose that Assumptions 2.1-2.3 hold. Then,*

(i) $E \sup\limits_{\theta \in \Theta} |w_t \ell_t(\theta)| < \infty$;

(ii) $E \sup\limits_{\theta \in \Theta} \left\| w_t \dfrac{\partial \ell_t(\theta)}{\partial \theta} \right\| < \infty$;

(iii) $E \sup\limits_{\theta \in \Theta} \left\| w_t \dfrac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'} \right\| < \infty$.

LEMMA B.2. *For any $\theta^* \in \Theta$, let $B_\eta(\theta^*) = \{\theta \in \Theta : \|\theta - \theta^*\| < \eta\}$ be an open neighborhood of θ^* with radius $\eta > 0$. Suppose that Assumptions 2.1-2.3 hold. Then,*

(i) $E[w_t \ell_t(\theta)]$ *has a unique minimum at θ_0.*

(ii) $E \left[\sup\limits_{\theta \in B_\eta(\theta^*)} w_t |\ell_t(\theta) - \ell_t(\theta^*)| \right] \to 0$ *as $\eta \to 0$.*

LEMMA B.3. *Let* $\{X_{ni} : 1 \leq i \leq k_n, n = 1, 2, \cdots\}$ *be a mean-zero triangular array of β-mixing sequences that are L^s-bounded for some $s > 1$, and $\mathcal{F}_{ni} = \sigma(X_{n1}, \cdots, X_{ni})$ for $1 \leq i \leq k_n$. Then $\{X_{ni}, \mathcal{F}_{ni}\}$ is a uniformly integrable L^1-mixingale. Furthermore, $E\left|\frac{1}{k_n}\sum_{i=1}^{k_n} X_{ni}\right| \to 0$ as $n \to \infty$, and hence $\frac{1}{k_n}\sum_{i=1}^{k_n} X_{ni} \to 0$ in probability as $n \to \infty$.*

LEMMA B.4. *Let θ_n^* be a sequence such that $\theta_n^* \to \theta_0$ in probability. Suppose that Assumptions 2.1-2.4 hold. Then,*

(i) *both J and Σ are finite, non-singular and positive definite matrices;*

(ii) $\|J_n(\theta_n^*) - J\| = o_p(1)$, $\|\Sigma_n(\theta_n^*) - \Sigma\| = o_p(1)$, $\|D_n(\theta_n^*) - D\| = o_p(1)$, *and $Z_n \to_{\mathcal{L}} Z$, where Z_n is defined in (A.2);*

Furthermore, if Assumption 2.3 is replaced by Assumption 2.3' with $\delta = 2$, under $\mathbb{P}_{n,h}$,

(iii) $\|J_n(\theta_n^*) - J\| = o_p(1)$, $\|\Sigma_n(\theta_n^*) - \Sigma\| = o_p(1)$, $\|D_n(\theta_n^*) - D\| = o_p(1)$, *and $Z_{n,h} \to_{\mathcal{L}} Z$, where $Z_{n,h}$ is defined in (A.12).*

LEMMA B.5. *Suppose that Assumptions 2.1-2.4 hold. Then,*

(i) $Z'JZ = Z'_{(2,3)}(KJ^{-1}K)^{-1}Z_{(2,3)} + G'_{(1)}J_{(11)}^{-1}G_{(1)}$;

(ii) $\left\|Z - \lambda^\Lambda\right\|_J^2 = \left\|Z_{(2,3)} - \lambda^\Lambda_{(2,3)}\right\|_{(KJ^{-1}K')^{-1}}^2 = \left\|Z_{(2,3)} - \lambda^{\Lambda_{(2,3)}}_{(2,3)}\right\|_{(KJ^{-1}K')^{-1}}^2$;

(iii) $\lambda^\Lambda_{(2,3)} = \lambda^{\Lambda_{(2,3)}}_{(2,3)}$.

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