Assignment #1: The DAR Model

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Problem 1: The drift criteron

Consider the double autoregressive (DAR) model given by:

$$\Delta x_t = \pi x_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t$$

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2$$

where z_t is an iid $N(0,1)$

1. Find $E(x_t|x_{t-1})$ and $Var(x_t|x_{t-1})$. Be precise about what results you use for the derivations

We can rewrite $\Delta x_t = x_t - x_{t-1}$ into the following:

$$x_t = x_{t-1} + \pi x_{t-1} + \varepsilon_t = (1+\pi)x_{t-1} + \varepsilon_t$$

Thus $E(x_t|x_{t-1})$ can be found:

$$\begin{aligned}
E(x_t|x_{t-1}) &= E((1+\pi)x_{t-1} + \varepsilon_t|x_{t-1}) = (1+\pi)E(x_{t-1}|x_{t-1}) + E(\varepsilon_t|x_{t-1}) \\
&= (1+\pi)x_{t-1} + E(\sigma_t z_t|x_{t-1}) = (1+\pi)x_{t-1} + E((\omega+\alpha x_{t-1}^2)^{1/2} z_t|x_{t-1}) \\
&= (1+\pi)x_{t-1} + (\omega+\alpha x_{t-1}^2)^{1/2}\underbrace{E(z_t|x_{t-1})}_{=0} = (1+\pi)x_{t-1}
\end{aligned}$$

Where we have used that, $E(x_{t-1}|x_{t-1}) = x_{t-1}$ and that $E(z_t|x_{t-1}) = E(z_t) = 0$ as $z_t \sim N(0,1)$

Likewise we can find $Var(x_t|x_{t-1})$:

$$\begin{aligned}
Var(x_t|x_{t-1}) &= E[(x_t - E(x_t|x_{t-1}))^2|x_{t-1}] = E[(x_t-(1+\pi)x_{t-1})^2\,|x_{t-1}] \\
&= E[x_t^2 + (1+\pi)^2x_{t-1}^2 - 2(1+\pi)x_t x_{t-1}|x_{t-1}] \\
&= E(x_t^2|x_{t-1}) + E((1+\pi)^2x_{t-1}^2|x_{t-1}) - E(2(1+\pi)x_t x_{t-1}|x_{t-1})|x_{t-1}] \\
&= E(x_t^2|x_{t-1}) + (1+\pi)^2x_{t-1}^2 - 2(1+\pi)x_{t-1}\underbrace{E(x_t|x_{t-1})|x_{t-1}]}_{=(1+\pi)x_{t-1}} \\
&= E(x_t^2|x_{t-1}) + (1+\pi)^2x_{t-1}^2 - 2(1+\pi)^2x_{t-1}^2 \\
&= E(x_t^2|x_{t-1}) - (1+\pi)^2x_{t-1}^2 \\
&= E[((1+\pi)x_{t-1}+\varepsilon_t)^2\,|x_{t-1}] - (1+\pi)^2x_{t-1}^2 \\
&= E[((1+\pi)x_{t-1}+\sigma_t z_t)^2|x_{t-1}] - (1+\pi)^2x_{t-1}^2 \\
&= E[(1+\pi)^2x_{t-1}^2 + \sigma_t^2 z_t^2 + (1+\pi)x_{t-1}\sigma_t z_t|x_{t-1}] - (1+\pi)^2x_{t-1}^2 \\
&= (1+\pi)^2x_{t-1}^2 + E[\sigma_t^2 z_t^2|x_{t-1}] + (1+\pi)x_{t-1}E[\sigma_t z_t|x_{t-1}] - (1+\pi)^2x_{t-1}^2 \\
&= E[\sigma_t^2 z_t^2|x_{t-1}] + (1+\pi)x_{t-1}E[\sigma_t z_t|x_{t-1}] \\
&= (\omega+\alpha x_{t-1}^2)\underbrace{E[z_t^2|x_{t-1}]}_{=E(z_t^2)=1} + (1+\pi)x_{t-1}(\omega+\alpha x_{t-1}^2)^{1/2}\underbrace{E[z_t|x_{t-1}]}_{=E(z_t)=0} \\
&= \omega + \alpha x_{t-1}^2 = \sigma_t^2
\end{aligned}$$

That is $Var(x_t|x_{t-1})$ is the conditional variance of ε_t

2. Argue that the process x_t is a Markov chain, with a conditional density of x_t, i.e. $f(x_t|x_{t-1})$, that satisfies Assumption I.1 for the drift criterion

The process x_t satisfies Assumption I.1 for $(x_t)_{t=0,1,2...}$:

i) The conditional distribution of x_t depends only on x_{t-1} as seen directly from $x_t = (1+\pi)x_{t-1} + \varepsilon_t$. Therefore:

$$(x_t|x_{t-1}, x_{t-2}, ..., x_0) \overset{d}{=} (x_t|x_{t-1}) \overset{d}{=} N((1+\pi)x_{t-1}, \sigma_t^2)$$

hence $(x_t)_{t=0,1,2...}$ is a Markov Chain

ii) x_t conditional on x_{t-1} is $N((1+\pi)x_{t-1}, \sigma_t^2)$ distributed, that is

$$f(x_t|x_{t-1}) = \frac{1}{2\pi\sigma_t^2}\exp\left(-\frac{(x_t-(1+\pi)x_{t-1})^2}{2\sigma_t^2}\right) > 0, \quad \sigma_t^2 = \omega + \alpha x_{t-1}^2$$

which is positive (as $\exp\{\} > 0$, $\omega > 0$ and $\alpha \geq 0$) and continous in x_t and x_{t-1}

3. Consider the drift function $\delta(x) = 1 + x^2$, and show that x_t satisfies the drift criterion in this case if $(1+\pi)^2 + \alpha < 1$

We consider the drift function $\delta(x) = 1 + x^2$:

$$\begin{aligned}
E(\delta(x_t)|x_{t-1}) &= E(1+x_t^2|x_{t-1}) = 1 + E(x_t^2|x_{t-1}) = 1 + E[((1+\pi)x_{t-1}+\varepsilon_t)^2|x_{t-1}] \\
&= 1 + (1+\pi)^2x_{t-1}^2 + E(\varepsilon_t^2|x_{t-1}) + 2(1+\pi)x_{t-1}\underbrace{E(\varepsilon_t|x_{t-1})}_{=\sigma_t E(z_t)=0} \\
&= 1 + (1+\pi)^2x_{t-1}^2 + E(\sigma_t^2 z_t^2|x_{t-1}) = 1 + (1+\pi)^2x_{t-1}^2 + E((\omega+\alpha x_{t-1}^2)z_t^2|x_{t-1}) \\
&= 1 + (1+\pi)^2x_{t-1}^2 + (\omega+\alpha x_{t-1}^2)\underbrace{E(z_t^2|x_{t-1}^2)}_{=E(z_t^2)=1} = 1 + (1+\pi)^2x_{t-1}^2 + \omega + \alpha x_{t-1}^2 \\
&= 1 + \omega + ((1+\pi)^2+\alpha)x_{t-1}^2
\end{aligned}$$

For simplicity we can also evaluate, at $x_{t-1} = x$, and thus:

$$\begin{aligned}
E(\delta(x_t)|x_{t-1}=x) &= \left[1+\omega+((1+\pi)^2+\alpha)x^2\right]\frac{\delta(x)}{\delta(x)} \\
&= \left[\frac{1+\omega}{1+x_{t-1}^2} + \frac{((1+\pi)^2+\alpha)x^2}{1+x^2}\right]\delta(x) \\
&= \left[\frac{1+\omega}{1+x_{t-1}^2} + ((1+\pi)^2+\alpha)\frac{x^2}{1+x^2}\right]\delta(x) \\
&= \left[\frac{1+\omega}{1+x_{t-1}^2} + ((1+\pi)^2+\alpha)\frac{x^2}{1+x^2}\right]\delta(x) \leq ((1+\pi)^2+\alpha)\delta(x) < \phi\delta(x)
\end{aligned}$$

There exists $\phi < 1$, such that for x large ($> M$), $E(\delta(x_t)|x_{t-1}=x) < \phi\delta(x)$ if $(1+\pi)^2+\alpha \leq \phi < 1$.

4. Explain why $E(x_t^2) < \infty$ and $E|x_t| < \infty$ for all parameter values which satisfy $(1+\pi_0)^2+\alpha < 1$. Explain how it may be possible for parameter values which satisfy $(1+\pi_0)^2+\alpha < 1$, that $E(x_t^2) < \infty$ and $E(x_t^4) = \infty$

For $(1+\pi_0)^2+\alpha < 1$, then $E(\delta(x_t)) = E(1+x_t^2) < \infty \Rightarrow E(x_t^2) < \infty$. This is a consequence of $\phi < 1$, such that the process is stationary (and doesn't explode). As $E(x_t^2)$ is finite, when parameter values satisfy $(1+\pi_0)^2+\alpha < 1$, then moments of order below this too are finite, i.e. $E|x_t|^k$ for $k \in (0,2)$.

$(1+\pi_0)^2+\alpha < 1$ satisfies the bound for which moments of orders up to $E(x_t^2)$ are finite. This however doesn't ensure that the fourh order moment, $E(x_t^4)$ is finite. To find the condition for this, we need to evaluate the drift function, $\delta(x_t) = 1 + x_t^4$. This would require more restriction on the parameter values.

Problem 2: Strict stationarity and Drift Criterion (Optional)

1. Show that

$$E\left(\delta\left(x_t\right)|x_{t-1}\right) \leq 1 + E\left(|\eta_t|^s\right) + E\left(|\phi_t|^s\right) E\left(|x_{t-1}|^s\right)$$

With drift function,

$$\delta\left(x\right) = 1 + |x|^s$$
$$E\left(\delta\left(x_t\right)|x_{t-1}\right) = E\left(1 + |x_t|^s|x_{t-1}\right) = 1 + E\left(|x_t|^s|x_{t-1}\right) = 1 + E\left(|\phi_t x_{t-1} + \eta_t|^s|x_{t-1}\right)$$

Using the rule that $|\phi_t x_{t-1} + \eta_t|^s \leq |\phi_t x_{t-1}|^s + |\eta_t|^s$,

$$E\left(\delta\left(x_t\right)|x_{t-1}\right) \leq 1 + E\left(|\eta_t|^s|x_{t-1}\right) + E\left(|\phi_t x_{t-1}|^s|x_{t-1}\right)$$

Using that ϕ_t and x_{t-1} are independent and that η_t and ϕ_t do not depend on x_{t-1},

$$E\left(\delta\left(x_t\right)|x_{t-1}\right) \leq 1 + E\left(|\eta_t|^s\right) + E\left(|\phi_t|^s\right) E\left(|x_{t-1}|^s\right)$$

2. Drift criterion for fractional moments

We have that,

$$E\left(\delta\left(x_t\right)|x_{t-1}\right) \leq 1 + E\left(|\eta_t|^s\right) + E\left(|\phi_t|^s\right) E\left(|x_{t-1}|^s\right)$$

Applying the drift function and reducing,

$$E\left(\delta\left(x_t\right)|x_{t-1} = x\right) \leq \frac{1 + E\left(|\eta_t|^s\right) + E\left(|\phi_t|^s\right) E\left(|x|^s\right)}{1 + |x|^s} \delta\left(x\right)$$
$$E\left(\delta\left(x_t\right)|x_{t-1} = x\right) \leq \left(\frac{1 + E\left(|\eta_t|^s\right)}{1 + |x|^s} + \frac{E\left(|\phi_t|^s\right) E\left(|x|^s\right)}{1 + |x|^s}\right) \delta\left(x\right)$$

Inserting $\phi_t = |1 + \pi + \sqrt{\alpha} z_t|^s$,

$$E\left(\delta\left(x_t\right)|x_{t-1} = x\right) \leq \left(\frac{1 + E\left(|\eta_t|^s\right)}{1 + |x|^s} + \frac{E\left(|1 + \pi + \sqrt{\alpha} z_t|^s\right) E\left(|x|^s\right)}{1 + |x|^s}\right) \delta\left(x\right) \leq \delta\left(x\right) E\left(\left|1 + \pi + \sqrt{\alpha} z_t\right|^s\right) < \phi \delta\left(x\right)$$

We see that there exists $\phi < 1$, such that for x large ($> M$), $E\left(\delta\left(x_t\right)|x_{t-1} = x\right) \leq \phi\delta\left(x\right)$ if $E\left(|1 + \pi + \sqrt{\alpha} z_t|^s\right) \leq \phi < 1$.

3. Condition for strict stationarity

The question is stated as "argue that $E\left(|1 + \pi + \sqrt{\alpha} z_t|^\kappa\right) < 1$ if $h'\left(0\right) = E\left(\log\left(|1 + \pi + \sqrt{\alpha} z_t|\right)\right) < 0$" but shouldn't it be the opposite way around?

We have that,

$$h\left(0\right) = E\left(\left|1 + \pi + \sqrt{\alpha} z_t\right|^0\right) = 1 \qquad\qquad h\left(\kappa\right) = E\left(\left|1 + \pi + \sqrt{\alpha} z_t\right|^\kappa\right)$$

Inserting the above,

$$h'\left(0\right) = \lim_{\kappa \to 0} \frac{h\left(\kappa\right) - h\left(0\right)}{\kappa} = \lim_{\kappa \to 0} \frac{E\left(|1 + \pi + \sqrt{\alpha} z_t|^\kappa\right) - 1}{\kappa} = E\left(\log\left(\left|1 + \pi + \sqrt{\alpha} z_t\right|\right)\right)$$

As κ approaches zero from the right, the nominator is only negative (and consequently the whole expression), if $E\left(|1 + \pi + \sqrt{\alpha} z_t|^\kappa\right) < 1$. Thus, $E\left(|1 + \pi + \sqrt{\alpha} z_t|^\kappa\right) < 1$ implies that $h'\left(0\right) = E\left(\log\left(|1 + \pi + \sqrt{\alpha} z_t|\right)\right) < 0$.

Problem 3: Strict stationarity condition

1. Simulate strict stationarity

Based on the results from Problem 1 and Problem 2, $\phi^2 + \alpha < 1$ ensures existence of second order moment (finite variance) and $E\left(\log\left(|\phi + \sqrt{\alpha} z_t|\right)\right) < 0$ ensures existence of fractional moment (strict stationarity).

Using Monte Carlo simulation for $\alpha = \phi = 1$ with 1000 draws we find that the strict stationarity conditions is satisfied (existence of fractional moments) with the function evaluated at -0.19. This is also in line with Figure 1, where $\alpha = \phi = 1$ is contained within the area A. However, the second order moment is not finite, as $\alpha = \phi = 1$ does not satisfy $\phi^2 + \alpha < 1$. Rewriting the DAR process using $\phi = 1 + \pi$,

$$x_t = \phi x_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t \sigma_t^2 = \omega + \alpha x_{t-1}^2$$

Inserting $\alpha = \phi = 1$,

$$x_t = x_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t \sigma_t^2 = \omega + x_{t-1}^2$$

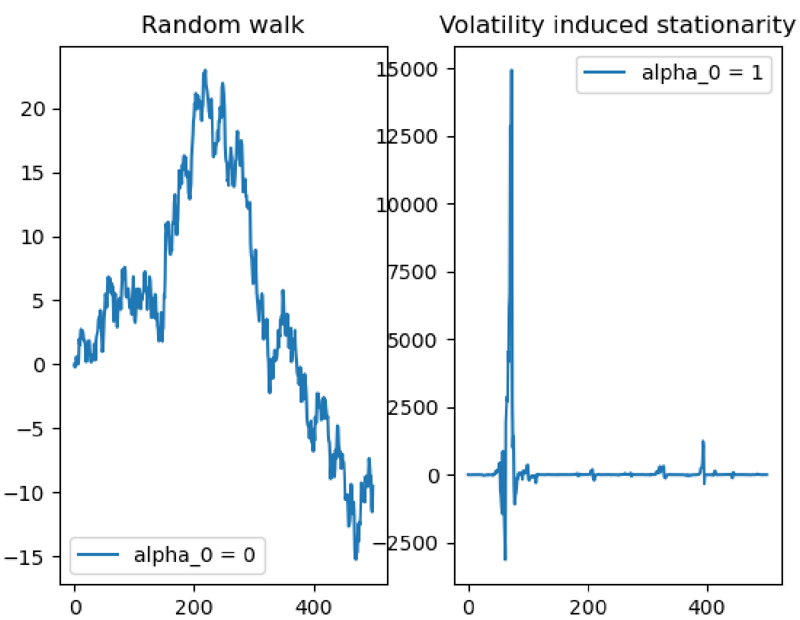
We see that the model reduces to a unit root process, which does not have finite variance.

```
: # question 3.1
phi = 1
alpha = 1

T = 1000
z = np.random.normal(loc = 0, scale = np.sqrt(1), size = T)
result = np.average(np.log(abs(phi + np.sqrt(alpha) * z)))
print(f"For phi = {phi} and alpha = {alpha} with {T} observations, the result is:", resul
```

```
For phi = 1 and alpha = 1 with 1000 observations, the result is: -0.18869302616861527
```

2. Simulate two realizations of the DAR process



The simulated DAR process for $\omega_0 = 1$, $\pi_0 = 0$ and $\alpha_0 = 1$ reduces to,

$$x_t = x_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = 1 + x_{t-1}^2 x_t = x_{t-1} + \sqrt{1 + x_{t-1}^2}$$

The simulated DAR process for $\omega_0 = 1$, $\pi_0 = 0$ and $\alpha_0 = 0$ reduces to,

$$x_t = x_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = 1 + x_{t-1}^2 x_t = x_{t-1} + 1$$

As mentioned in Question 3.1 the parameter value $\pi_0 = 0$ resembles a unit-root process / random walk which upon first inspection does not seem stationary. However, given that $\alpha_0 > 0$, the strict stationarity criterion is satisfied. Looking at the plots for the simulated processes, the case of $\alpha_0 = 1$ seems to follow a stationary process. This is only due to the fact that $\alpha_0 > 0$, which makes it a volatility induced process. On the other hand, $\alpha_0 = 0$ leads to a random walk / unit root process. This is also clear when evaluating $E\left(\log\left(|1 + \pi + \sqrt{\alpha} z_t|\right)\right) < 0$ for $\pi_0 = \alpha_0 = 0$, which gives $E\left(\log\left(|1|\right)\right) = 0$. Thus, strict stationarity is not satisfied for $\pi_0 = \alpha_0 = 0$.

Problem 4: Maximum likelihood estimation

For a realization of the DAR-process $(x_t : t = 0, 1; ...; T)$, the log-likelihood function is given by,

$$L_T(\pi, \omega, \alpha) = \sum_{t=1}^{T} l_t(\pi, \omega, \alpha), \quad l_t(\pi, \omega, \alpha) = -\frac{1}{2}\log[\sigma_t^2(\omega, \alpha)] - \frac{1}{2}\frac{(\Delta x_t - \pi x_{t-1})^2}{\sigma_t^2(\omega, \alpha)}$$
$$\sigma_t^2(\omega, \alpha) = \omega + \alpha x_{t-1}^2$$

$\theta = (\pi, \omega, \alpha)'$ is the maximum likelihood estimator and $\theta_0 = (\pi_0, \omega_0, \alpha_0)'$ denotes the true parameter values

1. Show that

$$\frac{\partial l_t(\theta)}{\partial \alpha} = \frac{1}{2}\frac{x_{t-1}^2}{\omega + \alpha x_{t-1}^2}\left(\frac{(\Delta x_t - \pi x_{t-1})^2}{\omega + \alpha x_{t-1}^2} - 1\right)$$

We start by inserting $\sigma_t^2(\omega, \alpha)$ into $l_t(\theta)$:

$$\begin{aligned} l_t(\theta) &= -\frac{1}{2}\log[\sigma_t^2(\omega, \alpha)] - \frac{1}{2}\frac{(\Delta x_t - \pi x_{t-1})^2}{\sigma_t^2(\omega, \alpha)} \\ &= -\frac{1}{2}\log[\omega + \alpha x_{t-1}^2] - \frac{1}{2}\frac{(\Delta x_t - \pi x_{t-1})^2}{\omega + \alpha x_{t-1}^2} \end{aligned}$$

We find the derivative of the log-likelihood function with respect to α:

$$\begin{aligned} \frac{\partial l_t(\theta)}{\partial \alpha} &= -\frac{1}{2}\frac{1}{\omega + \alpha x_{t-1}^2} x_{t-1}^2 - \frac{1}{2}\frac{(-(\Delta x_t - \pi x_{t-1})^2) x_{t-1}^2}{\left((\omega + \alpha x_{t-1}^2\right)^2} \\ &= -\frac{1}{2}\frac{x_{t-1}^2}{\omega + \alpha x_{t-1}^2} + \frac{1}{2}\frac{x_{t-1}^2}{\omega + \alpha x_{t-1}^2}\frac{(\Delta x_t - \pi x_{t-1})^2}{\omega + \alpha x_{t-1}^2} \\ &= \frac{1}{2}\frac{x_{t-1}^2}{\omega + \alpha x_{t-1}^2}\left(\frac{(\Delta x_t - \pi x_{t-1})^2}{\omega + \alpha x_{t-1}^2} - 1\right) \end{aligned}$$

2. Use that for the true value $\theta_0 = (\pi_0, \omega_0, \alpha_0)'$

$$\frac{(\Delta x_t - \pi_0 x_{t-1})^2}{\omega_0 + \alpha_0 x_{t-1}^2} = \frac{\varepsilon_t^2}{\omega_0 + \alpha_0 x_{t-1}^2} = z_t^2$$

to show that,

$$\frac{\partial l_t(\theta_0)}{\partial \alpha} := \left.\frac{\partial l_t(\theta)}{\partial \theta}\right|_{\theta = \theta_0} = \frac{1}{2}\frac{x_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2}(z_t^2 - 1)$$

Assume that $\alpha_0 > 0$. State conditions on θ_0 such that $\sum_{t=1}^{T} \partial l_t(\theta_0)/\partial\alpha$ satisfies a CLT from the lecture notes, i.e. state conditions under which

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\theta_0)}{\partial \alpha} \xrightarrow{D} N(0, \Omega)$$

We insert for the true value θ_0

$$
\begin{aligned}
\frac{\partial l_t(\theta_0)}{\partial \alpha} := \left.\frac{\partial l_t(\theta)}{\partial \theta}\right|_{\theta=\theta_0} &= \frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\left(\frac{(\Delta x_t - \pi_0 x_{t-1})^2}{\omega_0+\alpha_0 x_{t-1}^2} - 1\right) \\
&= \frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\left(\frac{\varepsilon_t^2}{\omega_0+\alpha_0 x_{t-1}^2} - 1\right) \\
&= \frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\left(z_t^2 - 1\right) = f(x_t, x_{t-1})
\end{aligned}
$$

The Central Limit Theorem applies for the score if 1. $E(f(x_t, x_{t-1})) = 0$ and 2. $E(f^2(x_t, x_{t-1})) < \infty$. We check for these conditions:

1.

$$
E[f(x_t, x_{t-1}|x_{t-1})] = E\left[\frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\left(z_t^2-1\right)\middle| x_{t-1}\right] = \frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}(\underbrace{E[z_t^2|x_{t-1}]}_{=E(z_t^2)=1}-1) = 0
$$

2.

$$
\begin{aligned}
E[f^2(x_t, x_{t-1}|x_{t-1})] &= E\left[\left(\frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\left(z_t^2-1\right)\right)^2 \middle| x_{t-1}\right] \\
&= \left(\frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2 E[(z_t^2-1)^2|x_{t-1}] \\
&= \left(\frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2 E(z_t^4+1-2z_t^2|x_{t-1}) \\
&= \left(\frac{1}{2}\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2 \left(\underbrace{E(z_t^4|x_{t-1})}_{=E(z_t^4)=3}+1-2\underbrace{E(z_t^2|x_{t-1})}_{=E(z_t^2)=1}\right) \\
&= \frac{1}{4}\left(\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2 2 \\
&= \frac{1}{2}\left(\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2 < \infty
\end{aligned}
$$

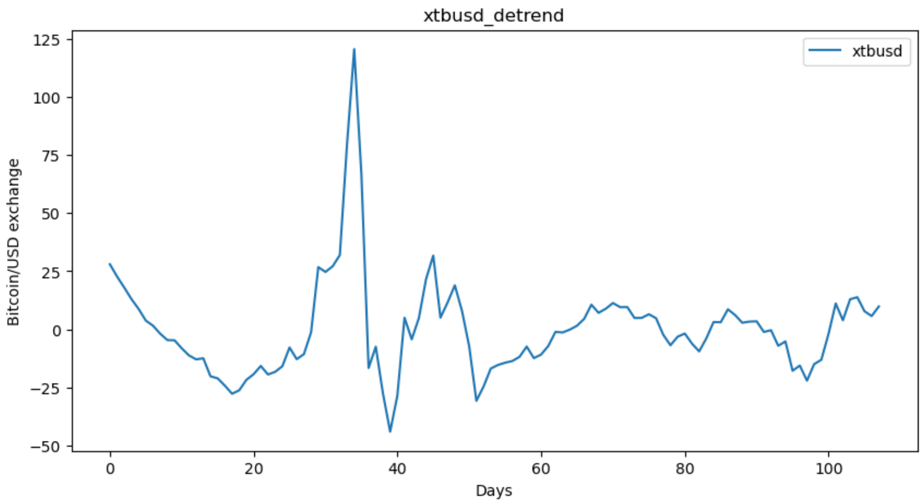
This is finite as $\alpha_0 > 0$

Thus, conditions on θ_0 is that $\alpha_0 > 0$ for the score to satisfy a CLT, $\frac{1}{\sqrt{T}\sum_{t=1}^{T}}\frac{\partial l_t(\theta_0)}{\partial \alpha} \xrightarrow{D} N(0,\Omega) = N(0, f^2(x_t, x_{t-1}))$

Problem 5: "Bubbles"in the Bitcoin/USD exchange rate (Optional)

1.

The below plot shows the detrended Bitcoin/USD exchange rate from February 20 - July 19, 2013:

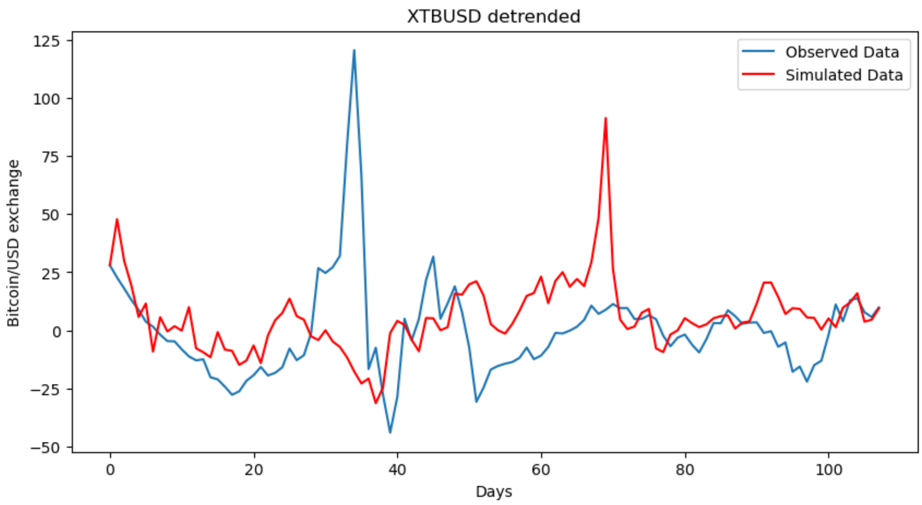


Obvious from the plot above, the exchange rate seem to follow some mostly stationary process, but with large bubble movements.

2.

The estimated DAR model on the detrended Bitcoin/USD exchange gives the point estimates: $(\pi, \omega, \alpha) = (-0.096, 47.665, 0.186)$.

We simulate a realization of the DAR process with the same starting value and sample length as observed data (detrended Bitcoin/USD exchange rate series):



The simulated time series seems to capture the fluctuations/variability of the observed data. That is, the generated model include the bubble movements with large bursts in the exchange rate. Generally, the DAR model seems to be a nice representation of the exchange.

By visual spectation, the simulated data too looks stationary as the series appears to be mean reverting.