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A DOUBLY MARKOV SWITCHING *AR* MODEL: SOME PROBABILISTIC PROPERTIES AND STRONG CONSISTENCY

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Abstract

In this work, we consider doubly Markov switching *AR* models, where analytic tractability and flexibility are quite simply a competitive advantage, which becomes an attractive tool for modeling economic and financial time series. In these models, the parameters are allowed to depend on an unobservable time-homogeneous Markov chain with finite state space. So, we discuss some basic probabilistic properties of $D-MSAR(p,q)$ model such as conditions ensuring the existence of strict, second-order stationarity solution, causal and ergodic solution and its moments properties. The quasi-maximum likelihood estimator of the parameters in the model is shown to be strongly consistent.

Keywords $D-AR$ model · Stationarity · Quasi-maximum likelihood estimation · Consistency

AMS Subject Classification (2020) Primary 62F12 · Secondary 62M05

Introduction

An autoregressive model with Markov switching has been introduced by Hamilton [16] in econometrics, and was then widely used in econometrics and automatic speech processing (see Krolzig [18] and Douc et al. [11]). Since the paper of Ling [20], the pth-order double AR model ($D-AR(p,p)$, in short) and the pth-order AR model with the pth-order $ARCH$ innovations ($AR-ARCH(p,p)$, in short) are particular cases of $ARMA-ARCH$ models in Weiss [22], who found them to be successful in modeling different *US* macroeconomic time series. This model combines the benefits of an AR model which more sets a goal of the conditional mean and an $ARCH$ model which focuses on the conditional variance. In order to discuss more $D-AR$ models (Motivation, etc.) can be found in Weiss [22] and Ling [19]. So, and to get more flexibility, the novelty of this work, is to suggest to merge $D-AR(p,q)$ with time-varying coefficients. While having this aim, recently we observe an increasing interest of many researchers in the study of periodic parameters. For example, Ghezal [14] studied the periodic log $GARCH$ time series model, the periodic DAR model was studied by Aknouche and Guerbian [1], whereas the periodic time-varying bivariate Poisson $INGARCH(1,1)$ processes were studied by Ghezal [15] and the references therein. In addition, we find Markov switching parameters, since the seminal paper by [16], $MSARCH$ models have received mounting importance later on in macroeconomics for their capability to accurately characterize different observed time series subject to change in regime. Several authors have proposed Markov switching models (for more information, see Francq et al. [13] and Cavicchioli [10] for the univariate Markov switching $GARCH$ models,

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Cavicchioli [6–8] for the multivariate Markov switching *ARMA* models, and Bibi and Ghezal [4] for the Markov switching *BLGARCH*). More accurately, the coefficients are allowed to depend on an unobservable time-homogeneous Markov chain $\left(s_t\right)_{t\in\mathbb{Z}}$ with finite state space $\mathbb{S}=\{1,...,d\}$. The reach model will be referred to as a doubly Markovian-switching $AR(p,q)$ ($D-MSAR(p,q)$, in short) defined by

$$X_t=\sum_{i=1}^{p}\alpha_i\left(s_t\right)X_{t-i}+e_t, \text{(observation equation)} \tag{1}$$

with

$$e_t=\sigma_t\eta_t,\ \sigma_t^2=a_0\left(s_t\right)+\sum_{j=1}^{q}a_j\left(s_t\right)e_{t-j}^2, \tag{2}$$

which slightly differs from the *MSAR* model with *MSARCH* errors of Ling [19, 20] and Weiss [22] when $d=1$, in that the conditional variance process is a function of the innovations not of the observations. In (1)–(2), $\left(\eta_t\right)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variables with zero mean and unit variance, and η_t is independent of $\left(s_t, X_{t-j}; j\geq 1\right)_{t\in\mathbb{Z}}$, and for given $\left\{s_t=k\right\}$, $k\in\mathbb{S}$, $\left(\alpha_i(k)\right)_{1\leq i\leq p}$ take values on $(-\infty,+\infty)$, $\left(a_j(k)\right)_{0\leq j\leq q}$ are non-negative coefficients with $a_0(k)>0$ and the chain $\left(s_t\right)_{t\in\mathbb{Z}}$ satisfies the following assumption:

[A.0] The Markov chain $\left(s_t\right)_{t\in\mathbb{Z}}$ is stationary, irreducible, aperiodic, with $n-$step transition probabilities matrix $\mathbb{P}^{(n)}:=\left(p_{ij}^{(n)}\right)_{(i,j)\in\mathbb{S}^2}$ where $p_{ij}^{(n)}:=P\left(s_t=j|s_{t-n}=i\right)$ with one-step transition probability matrix $\mathbb{P}:=\left(p_{ij}\right)_{(i,j)\in\mathbb{S}^2}$ where $p_{ij}=p_{ij}^{(1)}$ for $(i,j)\in\mathbb{S}^2$, and initial distribution $\underline{\Pi}:=(\pi(1),...,\pi(d))'$ where $\pi(i)=P\left(s_t=i\right)$, $i\in\mathbb{S}$ such that $\underline{\Pi}'=\underline{\Pi}'\mathbb{P}$.

The $D-MSAR(p,q)$ model contains several models as particular cases which have been investigated in the literature, actually, Standard $D-AR(p,q)$ obtained by assuming that the chain has a single regime (see [19, 20, 24]), a double periodic $AR(p,q)$ ($D-PAR(p,q)$, in short) (see Aknouche and Guerbyenne [1]) and mixture $D-AR(p,q)$ if $\left(s_t\right)_{t\in\mathbb{Z}}$ is identically and independently distributed (i.i.d.) across different dates (see Cavicchioli [9]).
Before we begin, we need some basic notations:

Algebraic notation

Some notations used in this work

- $I_{(n)}$ is the $n\times n$ identity matrix and $\mathbb{I}':=\left(\underbrace{I_{(r^2)}\vdots\ldots\vdots I_{(r^2)}}_{d-\text{block}}\right)$ is the $r^2\times r^2d$ matrix and $\underline{1}$ denotes the vector whose entries are all ones.
- $O_{(n,m)}$ denotes the matrix of order $n\times m$ whose entries are zeros, for simplicity we set $\underline{O}_{(n)}:=O_{(n,1)}$.
- The spectral radius of squared matrix M is denoted by $\rho(M)$.
- $\otimes$ is the usual Kronecker product of matrices, $vec(M)$ is the vector obtained from a matrix M by setting down the columns of M underneath each other. If $\left(M_i, i\in I\subset\mathbb{Z}\right)$ is $n\times n$ matrices sequence, we shall write for any integer l and j, $\prod_{i=l}^{j}M_i=M_lM_{l+1}...M_j$ if $l\leq j$ and $I_{(n)}$ otherwise.
- For any integers n and m such that $n\leq m$ let $\underline{X}_{n:m}:=\left(X_n,X_{n+1},...,X_m\right)'$ and $\underline{\Xi}_{n:m}:=\left(\left(X_n,e_n^2\right),...,\left(X_m,e_m^2\right)\right)'$ with $n\to-\infty$ we shall write $\underline{X}_m$and $\underline{\Xi}_m$, respectively.
- $P_{\underline{\theta}}(.,...,.)$ denotes the density with respect to probability measure on $\mathcal{B}_{(\mathbb{S}\otimes R)^T}$ and $\bar{P}_{\underline{\theta}}(.,...,.)$ its logarithm, $f_{\underline{\theta}}(.)$ (resp. $h_{\underline{\theta}}(.)$, $g_{\underline{\theta}}(.)$) denotes the density function of observation (resp. innovations).
- For any set of non random matrices $C:=\{C(i);i\in\mathbb{S}\}$, we shall note

$$\mathbb{P}(C) := \begin{pmatrix} p_{11}C(1) & \cdots & p_{d1}C(1) \\ \vdots & \ddots & \vdots \\ p_{1d}C(d) & \cdots & p_{dd}C(d) \end{pmatrix}, \ \Pi(C) := \begin{pmatrix} \pi(1)C(1) & \cdots & \pi(1)C(1) \\ \vdots & \ddots & \vdots \\ \pi(d)C(d) & \cdots & \pi(d)C(d) \end{pmatrix}, \ \underline{\Pi} := \begin{pmatrix} \pi(1)C(1) \\ \vdots \\ \pi(d)C(d) \end{pmatrix}.$$

A general view about this work is as follows. The model to be considered is described and that we define under some appropriate assumptions to a given $D-MSAR$ model. The next section presents sufficient conditions ensuring the existence of a strictly stationary solution and second (resp. higher)-order stationarity. The "3'' section deals with the strong consistency of *QMLE*.

Some probabilistic properties

We had talked before about $D-MSAR$, while we mentioned the first *MSAR* observation equation. Now we are trying to show the second *MSAR*, i.e., the innovation equation. However, using the inspiration contained in the Aknouche and Guerbyenne ([1] and the references therein), we get the following equation:

$$e_t = \sum_{j=1}^{q} \beta_{j,t}(s_t)e_{t-j} + \sqrt{a_0(s_t)}\eta_t, \ \text{(innovation equation)} \tag{3}$$

where $\beta_{j,t}(s_t), 1 \leq j \leq q$ are components of a vector $\underline{\beta}_t(s_t) := (\beta_{1,t}(s_t), ..., \beta_{q,t}(s_t))'$ which are related to those of (2). Currently, we consider the following regularity conditions.

[A.1] For $s_t = k \in \mathbb{S}$, the sequence $\left(\underline{\beta}_t(k)\right)_{t\in\mathbb{Z}}$ is i.i.d. and independent of $(\eta_t)_{t\in\mathbb{Z}}$.

[A.2] For $s_t = k \in \mathbb{S}$, the sequence $\left(\underline{\beta}_t(k)\right)_{t\in\mathbb{Z}}$ is $\mathcal{F}_t := \sigma(\eta_{t-v}; v \geq 0)$–measurable, X_t and $\left(\underline{\beta}_u(k), \eta_u\right)$ are independent for $u > t$.

[A.3] For $s_t = k \in \mathbb{S}$, the sequence $\left(\underline{\beta}_t(k)\right)_{t\in\mathbb{Z}}$ and $(\eta_t)_{t\in\mathbb{Z}}$ are Gaussian processes.

Now, define the $r = (p+q)$–state vector $\underline{X}_t := (X_t, ..., X_{t-p+1}, e_t, ..., e_{t-q+1})'$, $\underline{\zeta}_{s_t}(\eta_t) := \sqrt{a_0(s_t)}\eta_t\left(1, \underline{O}'_{(p-1)}, 1, \underline{O}'_{(q-1)}\right)'$, $\underline{H} := \left(1, \underline{O}'_{(r-1)}\right)'$ and the $r \times r$ matrix $(\Gamma_t(s_t))_{t\in\mathbb{Z}}$,

$$\Gamma_t(s_t) := \begin{pmatrix} \alpha_1(s_t) & \cdots & \alpha_{p-1}(s_t) & \alpha_p(s_t) & \beta_{1,t}(s_t) & \cdots & \beta_{q-1,t}(s_t) & \beta_{q,t}(s_t) \\ & I_{(p-1)} & & \underline{O}_{(p-1)} & & O_{(p-1,q)} & & \\ 0 & \cdots & & 0 & \beta_{1,t}(s_t) & \cdots & \beta_{q-1,t}(s_t) & \beta_{q,t}(s_t) \\ & O_{(q-1,p)} & & & & I_{(q-1)} & & \underline{O}_{(q-1)} \end{pmatrix}.$$

For this purpose, (1) and (3) can be expressed in the following state space representation $X_t = \underline{H}'\underline{X}_t$ and

$$\underline{X}_t = \Gamma_t(s_t)\underline{X}_{t-1} + \underline{\zeta}_{s_t}(\eta_t), \ t \in \mathbb{Z}. \tag{4}$$

Remark 1 In (4), for $s_t = k \in \mathbb{S}$, the sequence $\left(\Gamma_t(k), \underline{\zeta}_k(\eta_t)\right)_{t\in\mathbb{Z}}$ is i.i.d. and $(\Gamma_t(k))_{t\in\mathbb{Z}}$ and $\left(\underline{\zeta}_k(\eta_t)\right)_{t\in\mathbb{Z}}$ are independent. So, the process $\left((\underline{X}'_t, s_t)'\right)_{t\in\mathbb{Z}}$ is a Markov chain on $\mathbb{R}^r \times \mathbb{S}$.

Strict stationarity

In order to prove this result, we need the top Lyapunov exponent, defined as

$$\gamma_L(\Gamma) := \inf_{t>0}\left\{E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\|^{\frac{1}{t}}\right\}\right\} \overset{a.s.}{=} \lim_{t\to\infty}\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\|^{\frac{1}{t}}\right\}.$$

Since $\left(s_t\right)_{t\in\mathbb{Z}}$ is a stationary and ergodic process and under [A.3] we show that both $E\left\{\log^+\left\|\Gamma_t\left(s_t\right)\right\|\right\}$ and $E\left\{\log^+\left\|\underline{\zeta}_{s_t}\left(\eta_t\right)\right\|\right\}$ are finite, where $\log^+ x = \max(\log x, 0)$ for $x > 0$. Instantly, we get the following theorem

Theorem 1 *Consider the $D-MSAR(p,q)$ model with state space representation (4) and suppose that*

$$\gamma_L(\Gamma) < 0. \tag{5}$$

Then the series

$$\underline{X}_t = \sum_{l\geq 0}\left\{\prod_{j=0}^{l-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\}\underline{\zeta}_{s_{t-l}}\left(\eta_{t-l}\right), \tag{6}$$

converges a.s. and the process $\left(\underline{H}'\underline{X}_t\right)$ constitutes the unique, strictly stationary, ergodic and causal solution of (1) and (3).

Proof Proof deduced from Bougerol and Picard [5]. □

The Lyapunov exponent criterion $\gamma_L(\Gamma)$ seems tricky to obtain explicitly even in the simplest models. So, we need to seek for conditions ensuring the existence of moments for the strict stationary solutions.

Second-order stationarity

In this subsection, we give necessary and sufficient conditions for the existence of a unique second-order stationary solution of the $D-MSAR$ process which is also ergodic.

Theorem 2 *Let $\Gamma^{(2)} := \left\{\Gamma^{(2)}(i) = E\left\{\Gamma_t^{\otimes 2}(i)\right\}, i\in\mathbb{S}\right\}$. For (4) has a unique second-order stationary solution given by the series (6) which converges in mean square, it is sufficient that*

$$\lambda_{(2)} := \rho\left(\mathbb{P}\left(\Gamma^{(2)}\right)\right) < 1, \tag{7}$$

and necessary that

$$\left\|\mathbb{I}'\mathbb{P}^l\left(\Gamma^{(2)}\right)\underline{\Pi}\left(\underline{\Sigma}^{(2)}\right)\right\| \underset{l\uparrow\infty}{\longrightarrow} 0. \tag{8}$$

Moreover, this solution is strictly stationary and ergodic.

Proof At the beginning, we prove the sufficiency. Let $\Lambda_t(l) := \left\{\prod_{j=0}^{l-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\}$ and $\underline{S}_t(n) := \sum_{l=0}^{n}\Lambda_t(l)\underline{\zeta}_{s_{t-l}}\left(\eta_{t-l}\right)$. Then we have $\underline{X}_t = \underline{S}_t(n) + \Lambda_t(l+1)\underline{X}_{t-n-1}$. From Remark 1, we have

$$E\left\{\underline{S}_t^{\otimes 2}(n)\right\} = \sum_{l=0}^{n}E\left\{\Lambda_t^{\otimes 2}(l)\underline{\zeta}_{s_{t-l}}^{\otimes 2}\left(\eta_{t-l}\right)\right\} = \sum_{l=0}^{n}E\left\{\left\{\prod_{j=0}^{l-1}\Gamma_{t-j}^{\otimes 2}\left(s_{t-j}\right)\right\}\underline{\zeta}_{s_{t-l}}^{\otimes 2}\left(\eta_{t-l}\right)\right\}$$
$$= \mathbb{I}'\left(\sum_{l=0}^{n}\mathbb{P}^l\left(\Gamma^{(2)}\right)\right)\underline{\Pi}\left(\underline{\Sigma}^{(2)}\right),$$

where $\underline{\Sigma}^{(2)} := \left\{\underline{\Sigma}^{(2)}(i) = E\left\{\underline{\zeta}^{\otimes 2}_{s_t=i}\left(\eta_t\right)\right\}; i \in \mathbb{S}\right\}$, and, therefore, $\lim_{n\to\infty} E\left\{\underline{S}^{\otimes 2}_t(n)\right\} = \mathbb{I}'\left(\sum_{l\geq 0}\mathbb{P}^l\left(\Gamma^{(2)}\right)\right)\underline{\Pi}\left(\underline{\Sigma}^{(2)}\right)$. The rest follows immediately as in Bibi and Ghezal [2].

We now proceed to prove the necessity. From Nicholls and Quinn [21] the stationarity of model (4) implies that

$$\lim_{n\to\infty} E\left\{\sum_{l=0}^{n}\left(\underline{x}^{\otimes 2}\right)'\Lambda_t^{\otimes 2}(l)\underline{\zeta}^{\otimes 2}_{s_{t-l}}\left(\eta_{t-l}\right)\right\} < \infty,$$

for all fixed $r-$vector $\underline{x}$. According to $\underline{S}_t(n) \perp \Lambda_t(l+1)\underline{X}_{t-n-1}$, we have

$$E\left\{\underline{X}_t^{\otimes 2}\right\} = \sum_{l=0}^{n} E\left\{\Lambda_t^{\otimes 2}(l)\underline{\zeta}^{\otimes 2}_{s_{t-l}}\left(\eta_{t-l}\right)\right\} + E\left\{\Lambda_t^{\otimes 2}(l+1)\underline{X}^{\otimes 2}_{t-n-1}\right\}.$$

Furthermore, we define for $l \geq 1$

$$\begin{aligned} V_0(t) : &= E\left\{\underline{X}_t\underline{X}_t'\right\},\ V_l(t) := E\left\{\Gamma_t\left(s_t\right)V_{l-1}(t)\Gamma_t'\left(s_t\right)\right\}, \\ G_0(t) : &= \underline{\zeta}_{s_t}\left(\eta_t\right)\underline{\zeta}'_{s_t}\left(\eta_t\right), G_l(t) := E\left\{\Gamma_t\left(s_t\right)G_{l-1}(t)\Gamma_t'\left(s_t\right)\right\}. \end{aligned}$$

Therefore, for $n \geq 1, V_0(t) = \sum_{l=0}^{n} G_l(t) + V_{n+1}(t)$. We can note the following

$$\sum_{l=0}^{n} \underline{x}'G_l(t)\underline{x} \leq \underline{x}'V_0(t)\underline{x} < \infty \text{ for all } n \geq 1 \text{ and } t.$$

So, $\left(\sum_{l=0}^{n}\left(\underline{x}^{\otimes 2}\right)' vec\left(G_l(t)\right)\right)_n$ converges as $n \longrightarrow \infty$ to a non-negative real number for any $\underline{x}$. Moreover, we get

$$\lim_{l\to\infty} E\left\{\Lambda_t^{\otimes 2}(l)\underline{\zeta}^{\otimes 2}_{s_{t-l}}\left(\eta_{t-l}\right)\right\} = 0,$$

for all t and $\lim_{l\to\infty}\left\|\mathbb{I}'\mathbb{P}^l\left(\Gamma^{(2)}\right)\underline{\Pi}\left(\underline{\Sigma}^{(2)}\right)\right\| = 0$ for all t. □

Example 1 In the following table, we summarize the sufficient conditions for the existence of $E\left\{X_t^2\right\}$ for some particular cases (Table 1)

Example 2 In the following table, we summarize the necessary conditions for some particular cases (Table 2)

Remark 2 Using the same arguments from Aknouche and Guerbyenne [1], we have the Condition (7) is equivalent to conditions $\rho\left(\mathbb{P}\left(A^{(2)}\right)\right) < 1$ and $\rho\left(\mathbb{P}\left(B^{(2)}\right)\right) < 1$ where $A^{(2)} := \left\{A^{\otimes 2}(i), i \in \mathbb{S}\right\}$ and $B^{(2)} := \left\{E\left\{B_t^{\otimes 2}(i)\right\}, i \in \mathbb{S}\right\}$, where

Table 1 Conditions (7) for the existence of $E\left\{X_t^2\right\}$ for certain specifications

Specification	Condition (7)	Special cases $p = q = 1$
Standard	$\rho\left(E\left\{\Gamma_t^{\otimes 2}(1)\right\}\right) < 1$	$\max\left(\alpha_1^2(1), E\left\{\beta_{1,t}^2(1)\right\}, \alpha_1(1)E\left\{\beta_{1,t}(1)\right\}\right) < 1$
Independent-Switching	$\rho\left(E\left\{\Gamma_t^{\otimes 2}\left(s_t\right)\right\}\right) < 1$	$\max\left(E\left\{\alpha_1^2\left(s_t\right)\right\}, E\left\{\beta_{1,t}^2\left(s_t\right)\right\}, E\left\{\alpha_1\left(s_t\right)\beta_{1,t}\left(s_t\right)\right\}\right) < 1$
Periodic	$\rho\left(\prod_{v=0}^{d-1} E\left\{\Gamma^{\otimes 2}_{dt+d-v}(d-v)\right\}\right) < 1$	$\max\left(\left\{\prod_{v=1}^{d}\alpha_1^2(v)\right\}, \left\{\prod_{v=1}^{d} E\left\{\beta^2_{1,dt+v}(v)\right\}\right\}, \left\{\prod_{v=1}^{d}\alpha_1(v)E\left\{\beta_{1,dt+v}(v)\right\}\right\}\right) < 1$

Table 2 Conditions (8) for certain specifications

Specification	Condition (8)	Special cases $p = q = 1$
Standard	$\left\Vert \left(\Gamma^{(2)}(1)\right)^{l}\underline{\Sigma}^{(2)}(1)\right\Vert \underset{l\uparrow\infty}{\longrightarrow} 0$	$\sum_{k=0}^{l}\left(\alpha_0^{2(l-k)}(1)+2\sum_{j=1}^{l-k}\alpha_0^{2(l-k)-j}(1)\beta_{1,t}^{j}(1)\right)\beta_{1,t}^{2k}(1)\underset{l\uparrow\infty}{\longrightarrow} 0$
Independent-Switching	$\left\Vert \mathbb{I}'\Pi^{l}\left(\Gamma^{(2)}\right)\underline{\Pi}\left(\underline{\Sigma}^{(2)}\right)\right\Vert \underset{l\uparrow\infty}{\longrightarrow} 0$	$\sum_{k=0}^{l}\left(\left(E\left\{\alpha_0^2(s_t)\right\}\right)^{l-k}+2\sum_{j=1}^{l-k}\left(E\left\{\alpha_0^2(s_t)\right\}\right)^{l-k-j}\right.$ $\left.\times\left(E\left\{\alpha_0(s_t)\beta_{1,t}(s_t)\right\}\right)^{j}\right)\left(E\left\{\beta_{1,t}^2(s_t)\right\}\right)^{k}\underset{l\uparrow\infty}{\longrightarrow} 0$

$$A(s_t) := \begin{pmatrix} \alpha_1(s_t) & \cdots & \alpha_{p-1}(s_t) & \alpha_p(s_t) \\ & I_{(p-1)} & & \underline{O}_{(p-1)} \end{pmatrix},\ B_t(s_t) := \begin{pmatrix} \beta_{1,t}(s_t) & \cdots & \beta_{q-1,t}(s_t) & \beta_{q,t}(s_t) \\ & I_{(q-1)} & & \underline{O}_{(q-1)} \end{pmatrix},$$

the first new condition reduces for the particular $MS-AR$ case indicated in Francq and Zakoïan [12], on the other hand, the second-order stationary solution for the particular $MS-ARCH$ model mentioned in Francq and Zakoïan [13] is given by the second new condition.

Higher-order moments

The conditions ensuring the existence of higher-order moments may be carried out upon observing that for any integer $m > 1$, we have

Theorem 3 *Consider the Model (1) and (3) with associated state space representation (4) and assume that* $E\left\{\eta_t^{2m}\right\} < \infty$, $m > 1$ *and*

$$\lambda_{(2m)} := \rho\left(\mathbb{P}\left(\Gamma^{(2m)}\right)\right) < 1, \tag{9}$$

where $\Gamma^{(2m)} := \left\{E\left\{\Gamma_t^{\otimes 2m}(i)\right\}, i \in \mathbb{S}\right\}$. *Then (4) has a unique strictly stationary solution with* $E\left\{X_t^{2m}\right\} < \infty$ *and* $E\left\{e_t^{2m}\right\} < \infty$.

Proof Let $\underline{X}_t(k) = \sum_{l=0}^{k}\left\{\prod_{j=0}^{l-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\}\underline{\zeta}_{s_{t-l}}\left(\eta_{t-l}\right)$ for $k > 0$, $\underline{X}_t(0) = \underline{\zeta}_{s_t}\left(\eta_t\right)$ and zero otherwise, and let

$$\Omega_k(t) = \underline{X}_t(k) - \underline{X}_t(k-1) = \begin{cases} \left\{\prod_{j=0}^{k-1}\Gamma_{t-j}\left(s_{t-j}\right)\right\}\underline{\zeta}_{s_{t-k}}\left(\eta_{t-k}\right) \text{ if } k > 0 \\ \underline{\zeta}_{s_t}\left(\eta_t\right) \text{ if } k = 0 \\ \underline{O}_{(r)} \text{ if } k < 0 \end{cases}.$$

For any $k > 0, \Omega_k(t) = \Gamma_t\left(s_t\right)\Omega_{k-1}(t-1)$ and for any $m > 1$,

$$\begin{aligned} E\left\{\Omega_k^{\otimes 2m}(t)\right\} &= E\left\{\Gamma_t^{\otimes 2m}\left(s_t\right)\Omega_{k-1}^{\otimes 2m}(t-1)\right\} = E\left\{\left\{\prod_{j=0}^{k-1}\Gamma_{t-j}^{\otimes 2m}\left(s_{t-j}\right)\right\}\underline{\zeta}_{s_{t-k}}^{\otimes 2m}\left(\eta_{t-k}\right)\right\} \\ &= \mathbb{I}'\mathbb{P}^k\left(\Gamma^{(2m)}\right)\underline{\Pi}\left(\underline{\Sigma}^{(2m)}\right), \end{aligned}$$

where $\underline{\Sigma}^{(2m)} := \left\{E\left\{\underline{\zeta}_{s_t=i}^{\otimes 2m}\left(\eta_t\right)\right\}; i \in \mathbb{S}\right\}$. Since $E\left\{\underline{\zeta}_{s_t=i}^{\otimes 2m}\left(\eta_t\right)\right\}$ is finite for all $i \in \mathbb{S}$, a sufficient condition for $E\left\{\Omega_k^{\otimes 2m}(t)\right\}$ is finite as $k \uparrow \infty$ is satisfied under Condition (9). The rest follows immediately. □

Example 3 In the following table, we summarize the sufficient conditions for the existence of $E\left\{X_t^{2m}\right\}$ for some particular cases (Table 3)

Strong consistency

In this section, we consider the quasi-maximum likelihood estimator ($QMLE$) for $D-MSAR$ processes, while were obtained on the conditions ensuring the strong consistency of MLE for MS–model, for example, but not limited $MS-AR$ by Krishnamurthy and Rydén [17], $MS-GARCH$ by Xie [23] and $MS-BL$ by Bibi and Ghezal [3]. Now, the orders p, q and d are assumed to be known and fixed. Let $\underline{\theta} := \left(\underline{p}', \underline{\pi}', \underline{a}', \underline{\alpha}'\right)' \in \Theta \subset \mathbb{R}^{d(r+d+1)-1}$ where $\underline{p}' = \left(\underline{p}_i', i \in \mathbb{S}\right)$, $\underline{\pi}' = (\pi(i), i \in \mathbb{S}\backslash\{d\})$ (due to the constraints $\sum_{i\in\mathbb{S}} \pi(i) = 1$), $\underline{a}' = \left(\underline{a}_i', i \in \mathbb{S}\right)$ and $\underline{\alpha}' = \left(\underline{\alpha}_i', i \in \mathbb{S}\right)$ with $\underline{p}_i' := \left(p_{ij}, ..., p_{id}; i \neq j\right)$ (due to the constraints $\sum_{j\in\mathbb{S}} p_{ij} = 1$ for all $i \in \mathbb{S}$), $\underline{a}_i' := \left(a_0(i), ..., a_q(i)\right)$ and $\underline{\alpha}_i' := \left(\alpha_1(i), ..., \alpha_p(i)\right)$. Let $\underline{X}_{1:T}$ be a realization of length T of the stationary solution of (1) and (3), the problem of interest in this section is the estimation of $\underline{\theta}$. The likelihood function of $\underline{\theta}$ conditional on initial values $\underline{\Xi}_{-\max(p;q):0}$ is given by

$$\begin{aligned} L_T(\underline{\theta}) &= \sum_{\underline{i}_{1:T}\in\mathbb{S}^T} P_{\underline{\theta}}\left(\underline{X}_{1:T} \middle| \underline{\Xi}_{-\max(p;q):0}\right) \\ &= \sum_{\underline{i}_{1:T}\in\mathbb{S}^T} \pi(i_1) f_{\underline{\theta}_{i_1}}\left(X_1 \middle| \underline{\Xi}_{-\max(p;q):0}\right) \prod_{j=2}^{T} p_{i_{j-1},i_j} f_{\underline{\theta}_{i_j}}\left(X_j \middle| \underline{\Xi}_{-\max(p;q):j-1}\right) \\ &= \sum_{\underline{i}_{1:T}\in\mathbb{S}^T} \pi(i_1) h_{\underline{\theta}_{i_1}}\left(e_1(\underline{\theta}) \middle| \underline{\Xi}_{-\max(p;q):0}\right) \prod_{j=2}^{T} p_{i_{j-1},i_j} h_{\underline{\theta}_{i_j}}\left(e_j(\underline{\theta}) \middle| \underline{\Xi}_{-\max(p;q):j-1}\right), \end{aligned}$$

where $\left(e_t(\underline{\theta})\right)$ is the process invertible and determined recursively by

$$e_t(\underline{\theta}) = X_t - \sum_{i=1}^{p} \alpha_i(s_t) X_{t-i},$$

and also the innovation process $\left(e_t(\underline{\theta})\right)$ is heteroscedastic. Then we have

$$L_T(\underline{\theta}) = \sum_{\underline{i}_{1:T}\in\mathbb{S}^T} \pi(i_1) \frac{1}{\sigma_1(\underline{\theta})} g_{\underline{\theta}_{i_1}}\left(\frac{e_1(\underline{\theta})}{\sigma_1(\underline{\theta})} \middle| \underline{\Xi}_{-\max(p;q):0}\right) \prod_{j=2}^{T} p_{i_{j-1},i_j} \frac{1}{\sigma_j(\underline{\theta})} g_{\underline{\theta}_{i_j}}\left(\frac{e_j(\underline{\theta})}{\sigma_j(\underline{\theta})} \middle| \underline{\Xi}_{-\max(p;q):j-1}\right),$$

or equivalently

Table 3 Conditions (9) for the existence of $E\left\{X_t^{2m}\right\}$ for certain specifications

Specification	Condition (9)	Special cases $p = q = 1$
Standard	$\rho\left(E\left\{\Gamma_t^{\otimes 2m}(1)\right\}\right) < 1$	$\max\left(\alpha_1^{2m}(1), E\left\{\beta_{1,t}^{2m}(1)\right\}, \alpha_1^{2m-u}(1)E\left\{\beta_{1,t}^{u}(1)\right\}\right)^{(a)} < 1$
Independent-Switching	$\rho\left(E\left\{\Gamma_t^{\otimes 2m}(s_t)\right\}\right) < 1$	$\max\left(E\left\{\alpha_1^2(s_t)\right\}, E\left\{\beta_{1,t}^2(s_t)\right\}, E\left\{\alpha_1^{2m-u}(s_t)\beta_{1,t}^{u}(s_t)\right\}\right) < 1$
Periodic	$\rho\left(\prod_{v=0}^{d-1} E\left\{\Gamma_{dt+d-v}^{\otimes 2m}(d-v)\right\}\right) < 1$	$\max\left(\prod_{v=1}^{d} \alpha_1^2(v), \prod_{v=1}^{d} E\left\{\beta_{1,dt+v}^2(v)\right\},\right.$ $\left.\prod_{v=1}^{d} \alpha_1^{2m-u}(v)E\left\{\beta_{1,dt+v}^{u}(v)\right\}\right) < 1$

$^{(a)}$ $1 \leq u \leq 2m-1$

$$L_T(\underline{\theta}) = \underline{1}'\left\{\prod_{j=2}^{T}\mathbb{P}\left(\frac{1}{\sigma_j(\underline{\theta})}g_{\underline{\theta}}\left(\frac{e_j(\underline{\theta})}{\sigma_j(\underline{\theta})}\middle|\underline{\Xi}_{-\max(p;q):j-1}\right)\right)\right\}\underline{\Pi}\left(\frac{1}{\sigma_1(\underline{\theta})}g_{\underline{\theta}}\left(\frac{e_1(\underline{\theta})}{\sigma_1(\underline{\theta})}\middle|\underline{\Xi}_{-\max(p;q):0}\right)\right),$$

where $g_{\underline{\theta}}\left(e_j(\underline{\theta})\middle|\underline{\Xi}_{-\max(p;q):j-1}\right) = \left(g_{\underline{\theta}_k}\left(e_j(\underline{\theta})\middle|\underline{\Xi}_{-\max(p;q):j-1}\right);k \in \mathbb{S}\right)$. So we work with an approximate version $\widetilde{L}_T(\underline{\theta})$ of the likelihood $L_T(\underline{\theta})$, i.e.,

$$\begin{aligned}\widetilde{L}_T(\underline{\theta}) =& \underline{1}'\left\{\prod_{j=2}^{T}\mathbb{P}\left(f_{\underline{\theta}}\left(X_j\middle|\underline{\Xi}_{j-1}\right)\right)\right\}\underline{\Pi}\left(f_{\underline{\theta}}\left(X_1|\underline{\Xi}_0\right)\right)\\ =& \underline{1}'\left\{\prod_{j=2}^{T}\mathbb{P}\left(\frac{1}{\sigma_j(\underline{\theta})}g_{\underline{\theta}}\left(\frac{e_j(\underline{\theta})}{\sigma_j(\underline{\theta})}\middle|\underline{\Xi}_{j-1}\right)\right)\right\}\underline{\Pi}\left(\frac{1}{\sigma_1(\underline{\theta})}g_{\underline{\theta}}\left(\frac{e_1(\underline{\theta})}{\sigma_1(\underline{\theta})}\middle|\underline{\Xi}_0\right)\right).\end{aligned}$$

For instance, the initial values can be chosen as $X_{-\max(p;q)} = e^2_{-\max(p;q)} = \cdots = X_0 = e_0^2 = 0$. A quasi-maximum likelihood estimator of $\underline{\theta}$ is defined as any measurable solution $\widehat{\underline{\theta}}_T$ of

$$\widehat{\underline{\theta}}_T = \arg\max_{\underline{\theta}\in\Theta} L_T(\underline{\theta}).$$

Currently, we consider the following regularity assumptions

[A.4] $\mathring{\underline{\theta}} \in \Theta$ and Θ is a compact subset of $\mathbb{R}^{d(r+d+1)-1}$.
[A.5] $\gamma_L(\mathring{\Gamma}) < 0$ for all $\underline{\theta} \in \Theta$ where $\mathring{\Gamma}$ denotes the sequence (Γ_t) when the parameters $\underline{\theta}_k$ are replaced by their true values $\mathring{\underline{\theta}}_k, k \in \mathbb{S}$.
[A.6] For any $\underline{\theta}, \widetilde{\underline{\theta}} \in \Theta$, if $P_{\underline{\theta}}\left(X_t|\underline{\Xi}_{t-1}\right) = P_{\widetilde{\underline{\theta}}}\left(X_t|\underline{\Xi}_{t-1}\right)$ a.s. then $\underline{\theta} = \widetilde{\underline{\theta}}$.
[A.7] For all $\underline{\theta} \in \Theta$, $0 < \min_{k\in\mathbb{S}}\left\{f_{\underline{\theta}_k}\left(X_t|\underline{\Xi}_{t-1}\right)\right\} < \max_{k\in\mathbb{S}}\left\{f_{\underline{\theta}_k}\left(X_t|\underline{\Xi}_{t-1}\right)\right\} < +\infty$.
[A.8] For all $\underline{\theta} \in \Theta$, there exists a neighborhood $\mathcal{V}_{\underline{\theta}} := \left\{\underline{\theta}' : \|\underline{\theta} - \underline{\theta}'\| < \delta\right\}$ of $\underline{\theta}$ such that $E_{\mathring{\underline{\theta}}}\left\{\sup_{\underline{\theta}'\in\mathcal{V}_{\underline{\theta}}}\left|\bar{P}_{\widetilde{\underline{\theta}}}\left(X_t|\underline{\Xi}_{t-1}\right)\right|\right\} < +\infty$ for some $\delta > 0$.

We have the following intermediate results

Lemma 1 *Under Assumptions* [**A.4**] − [**A.8**], *we have*

i. $\lim_{T\uparrow\infty}\frac{1}{T}\log L_T(\underline{\theta}) = \lim_{T\uparrow\infty}\frac{1}{T}\log\widetilde{L}_T(\underline{\theta}) = \lim_{T\uparrow\infty}\frac{1}{T}\log\left\|\prod_{j=2}^{T}\mathbb{P}\left(f_{\underline{\theta}}\left(X_j\middle|\underline{\Xi}_{j-1}\right)\right)\right\| = E_{\mathring{\underline{\theta}}}\left\{\bar{P}_{\underline{\theta}}\left(X_t|\underline{\Xi}_{t-1}\right)\right\}$;
ii. *Let* $Y_T(\underline{\theta}) = \frac{1}{T}\log\left(\frac{\widetilde{L}_T(\underline{\theta})}{\widetilde{L}_T(\mathring{\underline{\theta}})}\right)$ *for all* $\underline{\theta} \in \Theta$, *then* $\lim_{T\uparrow\infty} Y_T(\underline{\theta}) \le 0$ *a.s. with equality iff* $\underline{\theta} = \mathring{\underline{\theta}}$;
iii. *For all* $\underline{\theta}' \neq \mathring{\underline{\theta}}$, *there exists a neighborhood* $\mathcal{V}_{\underline{\theta}'}$ *of* $\underline{\theta}'$ *such that* $\limsup_{T\uparrow\infty}\sup_{\underline{\theta}\in\mathcal{V}_{\underline{\theta}'}} Y_T(\underline{\theta}) < 0$.

The following result establishes the strong consistency of $\widehat{\underline{\theta}}_T$

Theorem 4 *If Assumptions* [**A.4**] − −[**A.8**] *hold, then* $\widehat{\underline{\theta}}_T \xrightarrow[T\uparrow\infty]{a.s.} \mathring{\underline{\theta}}$.

Proof The proof is similar to that of Bibi and Ghezal [3] for the $MS-BL$ model. □

Conclusion

This work proposes a $D - MSAR(p, q)$ model. This neoteric model is stationary and heteroskedastic. The main aim to introduce this new class of models is to extend $D - PAR(p, q)$ models whose coefficients are allowed to vary according to an unobservable (or hidden) time-homogeneous Markov chain with finite state space. The second goal is that we have suggested sufficient conditions for strict stationarity of a Markov switching $D - AR$. We obtained conditions guaranteeing the existence of second (resp. higher)-order stationary solutions. This problem has been formerly resolved for $D - PAR$ models by [1]. Moreover, this work studies the *QMLE* of the stationary and ergodic $D - MSAR$ process, and finds the consistency of *QMLE*.

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Declarations

Conflict of interest The author declares no competing interests.

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