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A comment on the drift criterion and ARCH(2)

1 Introduction

The aim of this note is to "bridge" the discussion of ARCH(1), ARCH(2) and the drift criterion, and supplements the discussion of Part II of the lecture notes.

Details of the derivations and theory can be found in e.g. Nielsen and Rahbek (2014) and Francq and Zakoian (2019).

2 ARCH(1) Stationarity discussion

Consider the ARCH(1) as given by:

$$x_t = \sigma_t z_t, \qquad \sigma_t^2 = \omega + \alpha x_{t-1}^2$$

and z_t i.i.d.N$(0,1)$. In order to find the "minimal" condition for stationarity and ergdodicity (weakly mixing) consider the drift finction

$$\delta\left(x\right) = 1 + |x^2|^\kappa$$

for some positive small κ, $\kappa < 1$. We find the following lemma:

Lemma 1 *The ARCH(1) process with z_t i.i.d.N$(0,1)$, is stationary and (geometrically) ergodic, or weakly mixing, provided*

$$E\log\left(\alpha z_t^2\right) < 0.$$

Proof: It follows (using $|x+y|^\kappa \leq |x|^\kappa + |y|^\kappa$) that

$$\begin{aligned} E\left(\delta\left(x_t\right)|x_{t-1} = x\right) &= 1 + E\left(|\sigma_t^2 z_t^2|^\kappa | x_{t-1} = x\right) = 1 + |\omega + \alpha x^2|^\kappa E\left[\left(z_t^2\right)^\kappa\right] \\ &\leq 1 + \left[\omega^\kappa + \alpha^\kappa\left(x^2\right)^\kappa\right] E\left[\left(z_t^2\right)^\kappa\right] \end{aligned}$$

Hence we need,

$$h(\kappa) = \alpha^{\kappa} E\left[\left(z_t^2\right)^{\kappa}\right] = E\left[\left(\alpha z_t^2\right)^{\kappa}\right] < 1.$$

Now $h(0) = 1$ and hence $h(\kappa) < 1$ is equivalent to

$$\frac{h(\kappa) - h(0)}{\kappa} < 0.$$

Next, $\partial h(\kappa)/\partial\kappa|_{\kappa=0} = E[\log(\alpha z_t^2)]$, so

$$\partial h(\kappa)/\partial\kappa|_{\kappa=0} = \lim_{\kappa\to 0}\left[\frac{h(\kappa)-h(0)}{\kappa}\right] = E\left[\log\left(\alpha z_t^2\right)\right] < 0.$$

$$\boxed{E\log(\alpha z_t^2) = \log(\alpha) + E\log(z_t^2) < 0}$$

□

The condition $E\log(\alpha z_t^2) < 0$ can be re-stated as $\log(\alpha) < -E[\log(z_t^2)]$, or

$$\alpha < \exp\left(-E\left[\log\left(z_t^2\right)\right]\right). \tag{1}$$

For z_t $N(0,1)$ distributed we find

$$E\left[\log\left(z_t^2\right)\right] = \int_{-\infty}^{\infty} \log\left(z^2\right) \overbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)}^{\text{Density of } z_t} dz \simeq -1.2704$$

such that $\alpha \lesssim \exp(-1.2704) = 3.56$.

Remark 2 *The same arguments carry over for to the case of z_t i.i.d. $t_v(0,1)$ distributed, that is, x_t is stationary provided (1) holds. And again (given some value of $v, v > 2$), one can find $E[\log(z_t^2)]$.*

3 GARCH(1,1)

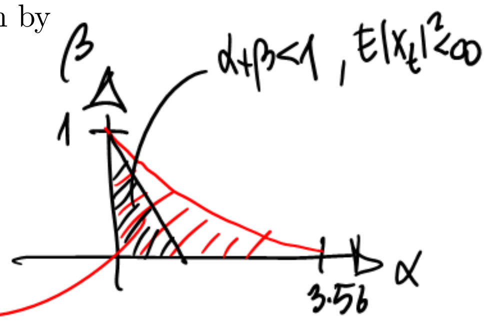
For $x_t = \sigma_t z_t$, with $\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta\sigma_{t-1}^2$ it follows that σ_t^2 is a Markov chain,

$$\sigma_t^2 = \omega + \left(\alpha z_t^2 + \beta\right)\sigma_{t-1}^2.$$

As for the ARCH(1) case, one can show that the condition for stationarity and geometric ergodicity in the case of GARCH(1,1) is given by

$$E\log\left(\alpha z_t^2 + \beta\right) < 0,$$

which trivially reduces to the ARCH(1) condition for $\beta = 0$.

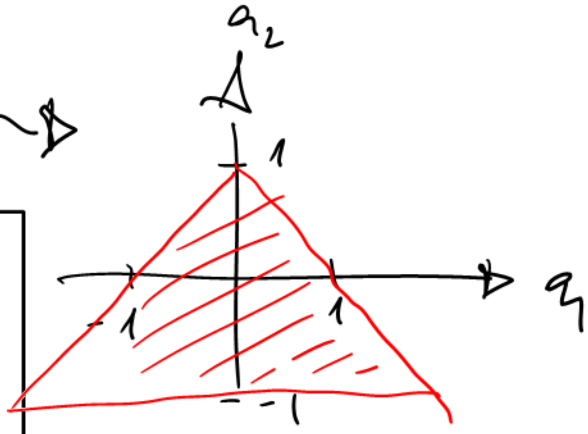


AR(2): $X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$

$$\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix}$$

(MARKOV CHAIN)

$$\boxed{\begin{array}{l} (*)\ \varrho\left(\begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix}\right) < 1 \\ \varrho(A) = \max|\lambda_i|,\ \lambda_i \text{ eigenvalue of } A. \end{array}}$$



USE: $\|A^m\|^{1/m} \to \varrho(A) \quad m \to \infty$

$$\|A\| = \left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\| = \begin{cases} \max|a_{ij}| \\ |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}| \\ \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2} \end{cases} \Bigg\}$$

EQUIVALENT NORMS

$\|A+B\| \le \|A\| + \|B\|$

$\|AB\| \le \|A\|\|B\|$!

Illustration:

$$A = \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} \qquad \|A\| = 10 \qquad \|A^2\| = \left\| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0$$

$$\rho(A) = 0 \quad \text{as} \quad |\lambda I - A| = 0 \Rightarrow \lambda = 0$$

(*) $$X_t = \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ x_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} = A X_{t-1} + e_t$$

(*) $$X_t = A^m X_{t-m} + \sum_0^{m-1} A^i e_{t-i}$$

(*) $$\delta(X) = 1 + X'X = 1 + \|X\|^2$$

$$\boxed{E(\delta(X_t) \mid X_{t-m} = X) \approx \|A^m\|'}$$

ARCH(2):

$$x_t = \sigma_t z_t \qquad \sigma_t^2 = \omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2$$

(*) $$x_t^2 = (\omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2) z_t^2$$

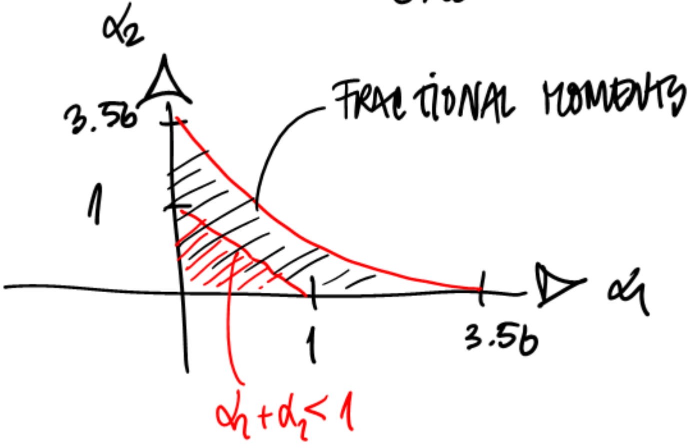
(*) $$X_t = \begin{pmatrix} x_t^2 \\ x_{t-1}^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 z_t^2 & \alpha_2 z_t^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1}^2 \\ x_{t-2}^2 \end{pmatrix} + \begin{pmatrix} \omega z_t^2 \\ 0 \end{pmatrix}$$

$$\boxed{X_t = A_t X_{t-1} + e_t}$$

$$(A^m X_{t-m}): \quad = \underbrace{(A_t A_{t-1} A_{t-2} \cdots A_1)}_{A^m \text{ in the VAR}} X_{t-m} + \cdots$$

Drift criterion

(*) $$\log\left[\|A_1 A_2 \cdots A_t\|^{1/t}\right] \xrightarrow[t \to \infty]{} \gamma < 0 \qquad A_t = \begin{pmatrix} \alpha_1 z_t^2 & \alpha_2 z_t^2 \\ 1 & 0 \end{pmatrix}$$



$\rho(A) < 1$

$\gamma = \log[\rho(A)] < 0$

3.1 ARCH(2)

The ARCH(2) process, $x_t = \sigma_t z_t$, with

$$\sigma_t^2 = \omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2,$$

and z_t i.i.d.$(0,1)$ is not not a Markov chain. However, $X_t = (x_t, x_{t-1})'$ is, since

$$X_t = \begin{pmatrix} \left(\omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2\right)^{1/2} z_t \\ x_{t-1} \end{pmatrix} = f\left(X_{t-1}, z_t\right).$$

Next note that, with

$$\dot{X}_t = \begin{pmatrix} x_t^2 \\ x_{t-1}^2 \end{pmatrix}$$

then $\dot{X}_t$ is "similar to a VAR(1)" as,

$$\dot{X}_t = A_t \dot{X}_{t-1} + \epsilon_t$$

with

$$A_t = \begin{pmatrix} \alpha_1 z_t^2 & \alpha_2 z_t^2 \\ 1 & 0 \end{pmatrix} \text{ and } \epsilon_t = \begin{pmatrix} \omega z_t^2 \\ 0 \end{pmatrix}.$$

Using this it can be shown that x_t is stationary provided

$$\gamma = \lim_{t\to\infty} E\left[\log \|A_1 A_2 \cdots A_t\|^{1/t}\right] < 0.$$

Here $\|\cdot\|$ denotes any matrix norm, for example

$$\left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\| = |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}|.$$

The region for the ARCH(2) can be found by simulation.

Remark 3 *Note that for a VAR(1) process*

$$X_t = AX_{t-1} + \epsilon_t$$

with ϵ_t *iid*$N(0,\Omega)$*, the condition for stationarity is*

$$\rho\left(A\right) < 1$$

where $\rho(A)$ *denotes the largest eigenvalue of* A *in absolute value. And from matrix theory it follows that*

$$||A^t||^{1/t} \to \rho\left(A\right) \text{ as } t \to \infty.$$

Remark 4 *For the ARCH(1) we have $\alpha_2 = 0$ and $\alpha_1 = \alpha$, and hence*

$$A_t = \begin{pmatrix} \alpha z_t^2 & 0 \\ 1 & 0 \end{pmatrix}$$

Thus

$$A_1 A_2 \cdots A_t = \begin{pmatrix} \alpha^t z_1^2 \cdots z_t^2 & 0 \\ 1 & 0 \end{pmatrix} = \left[\alpha^{t-1} z_1^2 \cdots z_{t-1}^2\right] A_t,$$

and hence, using $\gamma = \lim_{t\to\infty} \log \|A_1 A_2 \cdots A_t\|^{1/t}$ (a.s.), we find the condition again,

$$\log \|A_1 A_2 \cdots A_t\|^{1/t} = \frac{1}{t} \sum_{i=1}^{t-1} \log\left(\alpha z_i^2\right) + \frac{1}{t} \|A_t\| \rightarrow_{a.s.} E \log\left(\alpha z_t^2\right).$$

References

[1] Francq, C. and J.M. Zakoian, 2019, GARCH Models: Structure, Statistical Inference and Financial Applications, Wiley.

[2] Nielsen, H.B. and A. Rahbek, 2014, Unit root vector autoregression with volatility induced stationarity, *Journal of Empirical Finance,* 144–167.