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Source: *Econometric Theory* , June 2012, Vol. 28, No. 3 (June 2012), pp. 671-679

Published by: Cambridge University Press

Stable URL: https://www.jstor.org/stable/23257647

Econometric Theory, **28**, 2012, 671–679.
doi:10.1017/S0266466611000697

A NECESSARY MOMENT CONDITION FOR THE FRACTIONAL FUNCTIONAL CENTRAL LIMIT THEOREM

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We discuss the moment condition for the fractional functional central limit theorem (FCLT) for partial sums of $x_t = \Delta^{-d} u_t$, where $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ is the fractional integration parameter and u_t is weakly dependent. The classical condition is existence of $q \geq 2$ and $q > \left(d + \frac{1}{2}\right)^{-1}$ moments of the innovation sequence. When d is close to $-\frac{1}{2}$ this moment condition is very strong. Our main result is to show that when $d \in \left(-\frac{1}{2}, 0\right)$ and under some relatively weak conditions on u_t, the existence of $q \geq \left(d + \frac{1}{2}\right)^{-1}$ moments is in fact necessary for the FCLT for fractionally integrated processes and that $q > \left(d + \frac{1}{2}\right)^{-1}$ moments are necessary for more general fractional processes. Davidson and de Jong (2000, *Econometric Theory* 16, 643–666) presented a fractional FCLT where only $q > 2$ finite moments are assumed. As a corollary to our main theorem we show that their moment condition is not sufficient and hence that their result is incorrect.

1. INTRODUCTION

The fractional functional central limit theorem (FCLT) is given in Davydov (1970) for partial sums of the fractionally integrated process $\Delta^{-d}\varepsilon_t$, where $\Delta^{-d} = (1-L)^{-d}$ is the fractional difference operator defined in (3) in our Section 2 and ε_t is independent and identically distributed (i.i.d.) with mean zero. Davydov

We are grateful to Benedikt Pötscher, three anonymous referees, and James Davidson for comments and to the Social Sciences and Humanities Research Council of Canada (SSHRC grant 410-2009-0183) and the Center for Research in Econometric Analysis of Time Series, (CREATES, funded by the Danish National Research Foundation) for financial support. Address correspondence to Morten Ørregaard Nielsen, Department of Economics, Dunning Hall Room 307, Queen's University, 94 University Avenue, Kingston, Ontario K7L 3N6, Canada; e-mail: mon@econ.queensu.ca.

(1970) proved the fractional FCLT under a moment condition of the form $\mathrm{E}|\varepsilon_t|^q < \infty$ for $q \geq 4$ and $q > -4d/\left(d+\frac{1}{2}\right)$, which was subsequently improved by Taqqu (1975) to $q \geq 2$ and $q > q_0 = \left(d+\frac{1}{2}\right)^{-1}$. The standard moment condition from Donsker's theorem is $q \geq 2$ (see Billingsley, 1968, Ch. 2, Sect. 10), and because the condition $q \geq q_0$ is only stronger than $q \geq 2$ when $d < 0$ we consider only $d \in \left(-\frac{1}{2}, 0\right)$.

The fractional FCLT has been extended and generalized in numerous directions. For example, Marinucci and Robinson (2000) replace ε_t by a class of linear processes, assuming the moment condition $q > q_0$. The latter authors proved FCLTs for so-called type II fractional processes, whereas Davydov (1970) and Taqqu (1975) discussed type I fractional processes, but the distinction between type I and type II processes is not relevant for our discussion of the moment condition.

Davidson and de Jong (2000, henceforth DDJ) claim in their Theorem 3.1 that for some near-epoch dependent (NED) processes with uniformly bounded qth moment the fractional FCLT holds, but (incorrectly, as we shall see subsequently) they assume a much weaker moment condition than previous results, namely, $q > 2$. To the best of our knowledge, Theorem 3.1 of DDJ is the only fractional FCLT that claims a moment condition that is weaker than the earlier condition.

In the next section we give some definitions and construct an i.i.d. sequence and a fractional linear process that are central to our results. In Section 3 we present our main results, which state that if the fractional FCLT holds for any class of processes $\mathcal{U}(q)$ with uniformly bounded qth moment containing these two processes, then it follows that $q \geq q_0$ if the fractional FCLT is based on fractional coefficients and $q > q_0$ if the coefficients are more general. The proofs of both results are based on counterexamples that are constructed in a similar way as a counterexample in Wu and Shao (2006, Rmk. 4.1). In Section 4 we discuss the results and give two applications. In particular, it follows from our results that if the FCLT holds for NED processes with uniformly bounded q moments, then $q \geq q_0$. Hence DDJ's Theorem 3.1 and all their subsequent results do not hold under the assumptions stated in their theorem.

Throughout, c denotes a generic finite constant, which may take different values in different places.

2. DEFINITIONS

DEFINITION 1. *We define the class $\mathcal{U}_0(q)$ as the class of processes u_t that satisfy the moment condition*

$$\sup_{-\infty<t<\infty} \mathrm{E}|u_t|^q < \infty \quad \textit{for some } q \geq 2 \tag{1}$$

and have long-run variance

$$\sigma_u^2 = \lim_{T\to\infty} T^{-1}\mathrm{E}\left(\sum_{t=1}^{T} u_t\right)^2, \qquad 0 < \sigma_u^2 < \infty. \tag{2}$$

For such processes we define two subclasses of $\mathcal{U}_0(q)$ as follows.

(i) $\mathcal{U}_{LIN}(q) \subset \mathcal{U}_0(q)$ is the class of linear processes $u_t \in \mathcal{U}_0(q)$ satisfying $u_t = \sum_{n=0}^{\infty} \tau_n \varepsilon_{t-n}$, where $\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} \tau_j^2 < \infty$ and ε_t is i.i.d. with mean zero and variance $\sigma_\varepsilon^2 > 0$.

(ii) $\mathcal{U}_{NED}(q) \subset \mathcal{U}_0(q)$ is the class of zero mean covariance stationary processes $u_t \in \mathcal{U}_0(q)$ that are $\mathcal{L}_2$-NED of size $-\frac{1}{2}$ on v_t with $d_t = 1$, where v_t is either an α-mixing sequence of size $-q/(1-q)$ or a ϕ-mixing sequence of size $-q/(2(1-q))$; see Assumption 1 of DDJ.

Note that the condition $\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} \tau_j^2 < \infty$ in the definition of the class $\mathcal{U}_{LIN}(q)$ neither implies nor is implied by (2). Also note that if $u_t \in \mathcal{U}_{LIN}(q)$ then u_t is zero mean and stationary.

Next we define the fractional and general fractional processes.

DEFINITION 2. *For any $u_t \in \mathcal{U}_0(q)$ we we define the fractional process*

$$x_t = \Delta^{-d} u_t = \sum_{j=0}^{\infty} b_j(d) u_{t-j} \quad \textit{for } -\frac{1}{2} < d < 0, \tag{3}$$

where $b_j(d) = (-1)^j \binom{-d}{j} = d(d+1)\ldots(d+j-1)/j! \sim cj^{d-1}$ are the fractional coefficients, i.e., the coefficients in the binomial expansion of $(1-z)^{-d}$, and $\sim$ means that the ratio of the left- and right-hand sides converges to one. The general fractional process is defined by

$$x_t = \sum_{j=0}^{\infty} a_j(d) u_{t-j} \quad \textit{for } -\frac{1}{2} < d < 0, \tag{4}$$

where $a_j(d) \sim c\ell(j) j^{d-1}$ and $\ell(j)$ is a (normalized) slowly varying function; see Bingham, Goldie, and Teugels (1989, p. 15).

Note that the $b_j(d)$ coefficients from the fractional difference filter are a special case of $a_j(d)$. The processes (3) and (4) are well defined because, with $||x||_2$ denoting the $\mathcal{L}_2$-norm, we have from (1) that

$$||x_t||_2 \le ||u_t||_2 + c\sum_{j=1}^{\infty} \ell(j) j^{d-1} ||u_{t-j}||_2 \le c \quad \text{for } d < 0$$

using Karamata's theorem; see Bingham, Goldie, and Teugels (1989, p. 26).

We base our results on the construction of the following two specific processes.

DEFINITION 3. *Let ε_t be i.i.d. with mean zero, variance $\sigma_\varepsilon^2 > 0$, and finite qth moment for some $q \ge 2$ to be chosen later. For such ε_t we define the following two processes.*

(i) $u_{1t} = \varepsilon_t$.
(ii) $u_{2t} = \varepsilon_t + \Delta^{1+d}\varepsilon_t$.

For these two processes we note the following connection with the classes $\mathcal{U}_{LIN}(q)$ and $\mathcal{U}_{NED}(q)$.

LEMMA 1. *For $d \in \left(-\frac{1}{2}, 0\right)$ and for $i = 1, 2$, $u_{it} \in \mathcal{U}_{LIN}(q) \cap \mathcal{U}_{NED}(q)$, and the long-run variance of u_{it} is σ_ε^2.*

Proof. Clearly, $u_{1t} = \varepsilon_t$ is contained in both $\mathcal{U}_{LIN}(q)$ and $\mathcal{U}_{NED}(q)$ and has long-run variance σ_ε^2.

The process $u_{2t} = \varepsilon_t + \Delta^{1+d}\varepsilon_t = \varepsilon_t + \sum_{j=0}^{\infty} b_j(-d-1)\varepsilon_{t-j}$ is a linear process and

$$\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} b_j(-d-1)^2 \le c\sum_{n=1}^{\infty}\sum_{j=n}^{\infty} j^{-2d-4} \le c\sum_{n=1}^{\infty} n^{-2d-3} \le c \quad \text{for } d \in \left(-\frac{1}{2}, 0\right),$$

so that u_{2t} is in $\mathcal{U}_{LIN}(q)$. To see that u_{2t} is in $\mathcal{U}_{NED}(q)$, we calculate

$$\begin{aligned}||u_{2t} - \mathrm{E}(u_{2t}|\varepsilon_{t-m}, \ldots, \varepsilon_{t+m})||_2 &= ||\sum_{n=m+1}^{\infty} b_n(-d-1)\varepsilon_{t-n}||_2 \\ &\le c\left(\sum_{n=m+1}^{\infty} n^{-2d-4}\right)^{1/2} \le cm^{-d-3/2}.\end{aligned}$$

Because $\frac{3}{2} + d > \frac{1}{2}$ for $d \in \left(-\frac{1}{2}, 0\right)$, this shows that u_{2t} is $\mathcal{L}_2$-NED of size $-\frac{1}{2}$ on ε_t, and hence u_{2t} is also in $\mathcal{U}_{NED}(q)$. The generating function for u_{2t} is $f(z) = 1 + (1-z)^{1+d}$, and for $z = 1$ we find because $1 + d > 0$ that $f(1) = 1$. Therefore the long-run variance of u_{2t} is $\lim_{T\to\infty} T^{-1}\mathrm{E}\left(\sum_{t=1}^{T}\left(\varepsilon_t + \Delta^{1+d}\varepsilon_t\right)\right)^2 = f(1)^2\mathrm{Var}(\varepsilon_t) = \sigma_\varepsilon^2$. ■

We next give a general formulation of the FCLT for fractional processes. For this purpose we define the scaled partial sum process

$$X_T(\xi) = \sigma_T^{-1}\sum_{t=1}^{[T\xi]} x_t, \qquad 0 \le \xi \le 1, \tag{5}$$

where $\sigma_T^2 = \mathrm{E}(\sum_{t=1}^{T} x_t)^2$ and $[z]$ is the integer part of the real number z.

FRACTIONAL FCLT FOR $\mathcal{U}(q)$. *Let $X_T(\xi)$ be given by (5) and suppose x_t is linear in u_t. We say that the FCLT for fractional Brownian motion holds for a set $\mathcal{U}(q) \subset \mathcal{U}_0(q)$ of processes if, for all $u_t \in \mathcal{U}(q)$, it holds that*

$$X_T(\xi) \xrightarrow{D} X(\xi) \quad \text{in } D[0,1], \tag{6}$$

where $X(\xi)$ is fractional Brownian motion.

Here, $\xrightarrow{D}$ denotes convergence in distribution (weak convergence) in $D[0,1]$ endowed with the metric d_0 in Billingsley (1968, Ch. 3, Sect. 14), which induces the Skorokhod topology and under which $D[0,1]$ is complete.

3. THE NECESSITY RESULTS

Our first main result is the following theorem.

THEOREM 1. *Let $X_T(\xi)$ be defined by (5) for x_t given by the fractional coefficients $b_j(d)$ in (3) with $-\frac{1}{2} < d < 0$, and let $\mathcal{U}(q) \subset \mathcal{U}_0(q)$ be such that $u_{it} \in \mathcal{U}(q)$ for $i = 1, 2$. If the fractional FCLT holds for all u_t in $\mathcal{U}(q)$ for some $q \geq 2$, then $q \geq q_0$.*

Proof. We prove the theorem by assuming that there is a $q_1 \in [2, q_0)$ for which the FCLT holds for $\mathcal{U}(q_1)$ and show that this leads to a contradiction by a careful construction of ε_t and therefore u_{1t} and u_{2t}.

For $u_{it}, i = 1, 2$, we define x_{it} and X_{iT} by (3) and (5). Because u_{it} is in $\mathcal{U}(q_1)$ the fractional FCLT holds by the maintained assumption for u_{it}, and hence $X_{iT}(\xi)$ converges in distribution to fractional Brownian motion.

(a) *The normalizing variance for X_{1T}.* The variance of $\sum_{t=1}^T x_{1t} = \sum_{t=1}^T \Delta^{-d}\, u_{1t} = \sum_{t=1}^T \Delta^{-d}\varepsilon_t$ can be found in Davydov (1970); see also Lemma 3.2 of DDJ:

$$\sigma_{1T}^2 = \mathrm{E}\left(\sum_{t=1}^T x_{1t}\right)^2 \sim \sigma_\varepsilon^2 V_d T^{2d+1}, \tag{7}$$

where

$$V_d = \frac{1}{\Gamma(d+1)^2}\left(\frac{1}{2d+1} + \int_0^\infty ((1+\tau)^d - \tau^d)^2 d\tau\right).$$

(b) *The normalizing variance for X_{2T}.* We write x_{2t} and X_{2T} in terms of x_{1t} and X_{1T}, using (3) and (5):

$$x_{2t} = \Delta^{-d} u_{2t} = x_{1t} + \varepsilon_t - \varepsilon_{t-1}, \tag{8}$$

$$X_{2T}(\xi) = \sigma_{2T}^{-1}\sum_{t=1}^{[T\xi]} x_{2t} = \sigma_{1T}\sigma_{2T}^{-1} X_{1T}(\xi) + \sigma_{2T}^{-1}(\varepsilon_{[T\xi]} - \varepsilon_0). \tag{9}$$

We next find that the variance of $\sum_{t=1}^T x_{2t} = \sum_{t=1}^T \Delta^{-d} u_{2t} = \sum_{t=1}^T (x_{1t} + \varepsilon_t - \varepsilon_{t-1})$ is

$$\begin{aligned}\sigma_{2T}^2 &= \mathrm{E}\left(\sum_{t=1}^T x_{2t}\right)^2 = \mathrm{E}\left(\sum_{t=1}^T (x_{1t} + \varepsilon_t - \varepsilon_{t-1})\right)^2 = \mathrm{E}\left(\varepsilon_T - \varepsilon_0 + \sum_{t=1}^T x_{1t}\right)^2 \\ &= \mathrm{E}(\varepsilon_T - \varepsilon_0)^2 + \mathrm{E}\left(\sum_{t=1}^T x_{1t}\right)^2 + 2\mathrm{E}\left(\sum_{t=1}^T \Delta^{-d}\varepsilon_t(\varepsilon_T - \varepsilon_0)\right).\end{aligned}$$

The first term is constant, the next is σ_{1T}^2, and letting $1_{\{A\}}$ denote the indicator function of the event A, the last term consists of

$$\mathrm{E}\left(\sum_{t=1}^{T}\Delta^{-d}\varepsilon_t\varepsilon_T\right)=\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)\mathrm{E}(\varepsilon_{t-k}\varepsilon_T)$$
$$=\sigma_\varepsilon^2\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)1_{\{k=t-T\}}=\sigma_\varepsilon^2 b_0(d)=\sigma_\varepsilon^2$$

and

$$\mathrm{E}\left(\sum_{t=1}^{T}\Delta^{-d}u_{1t}\varepsilon_0\right)=\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)\mathrm{E}(\varepsilon_{t-k}\varepsilon_0)$$
$$=\sigma_\varepsilon^2\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)1_{\{k=t\}}\leq c\sum_{t=1}^{T}t^{d-1}\leq c\quad \text{for } d<0.$$

Therefore,

$$\sigma_{2T}^2\sim\sigma_{1T}^2+c. \tag{10}$$

(c) *The contradiction*. We now construct the i.i.d. process ε_t so that it has no moment higher than q_1, i.e., $\mathrm{E}|\varepsilon_t|^q=\infty$ for $q>q_1$, by choosing the tail to satisfy

$$P(|\varepsilon_t|^{q_1}\geq c)\sim\frac{1}{c(\log c)^2}\quad \text{as } c\to\infty. \tag{11}$$

In this case we still have $\mathrm{E}|\varepsilon_t|^{q_1}<\infty$. We then find

$$P(\sigma_{1T}^{-1}\max_{1\leq t\leq T}|\varepsilon_t|<c)=P(\sigma_{1T}^{-1}|\varepsilon_1|<c)^T=P(|\varepsilon_1|^{q_1}<c^{q_1}\sigma_{1T}^{q_1})^T$$
$$=(1-P(|\varepsilon_1|^{q_1}\geq c^{q_1}T^{q_1/q_0}))^T$$
$$\sim\left(1-\frac{1}{c^{q_1}T^{q_1/q_0}(q_1(\log c+q_0^{-1}\log T))^2}\right)^T$$
$$\sim\exp\left(-\frac{T^{1-q_1/q_0}}{c^{q_1}(q_1(\log c+q_0^{-1}\log T))^2}\right)\to 0$$

as $T\to\infty$ because $q_1<q_0$. Thus, $\sigma_{1T}^{-1}\max_{1\leq t\leq T}|\varepsilon_t|\xrightarrow{P}\infty$ because the normalizing constant $\sigma_{1T}=\sigma_\varepsilon V_d^{1/2}T^{1/q_0}=\sigma_\varepsilon V_d^{1/2}T^{1/2+d}<\sigma_\varepsilon V_d^{1/2}T^{1/q_1}$ is too small to normalize $\max_{1\leq t\leq T}|\varepsilon_t|$ correctly.

The definition (9) implies the evaluation

$$\max_{0\leq\xi\leq1}|\varepsilon_{[T\xi]}|\leq\max_{0\leq\xi\leq1}|\varepsilon_{[T\xi]}-\varepsilon_0|+|\varepsilon_0|$$
$$\leq\max_{0\leq\xi\leq1}|\sigma_{2T}X_{2T}(\xi)|+\max_{0\leq\xi\leq1}|\sigma_{1T}X_{1T}(\xi)|+|\varepsilon_0|$$

such that

$$\sigma_{1T}^{-1} \max_{0\le\xi\le 1} |\varepsilon_{[T\xi]}| \le \max_{0\le\xi\le 1} |\sigma_{1T}^{-1}\sigma_{2T} X_{2T}(\xi)| + \max_{0\le\xi\le 1} |X_{1T}(\xi)| + \sigma_{1T}^{-1}|\varepsilon_0|. \tag{12}$$

We have seen in (7) and (10) that $\sigma_{2T}^2 \sim \sigma_{1T}^2 + c$ and $\sigma_{1T}^2 \sim \sigma_\varepsilon^2 V_d T^{1+2d} \to \infty$ for $d \in \left(-\frac{1}{2}, 0\right)$, so that $\sigma_{1T}\sigma_{2T}^{-1} \to 1$. Therefore, both $\sigma_{1T}^{-1}\sigma_{2T}X_{2T}(\xi)$ and $X_{1T}(\xi)$ converge in distribution by the previous results, and it follows from (12) that $\sigma_{1T}^{-1}\max_{0\le\xi\le 1}|\varepsilon_{[T\xi]}|$ is $O_P(1)$. This contradicts that $\sigma_{1T}^{-1}\max_{1\le t\le T}|\varepsilon_t| \xrightarrow{P} \infty$ and hence completes the proof of Theorem 1. ■

The proof of Theorem 1 implies that the issue is that the rate of convergence, $T^{-(d+1/2)}$, of $\sum_{t=1}^{[T\xi]} \Delta^{-d} u_{1t}$ can be very slow for d close to $-\frac{1}{2}$. Thus, more control on the tail behavior of the u_t sequence is needed when $d \in \left(-\frac{1}{2}, 0\right)$, and this is achieved through the moment condition (1).

We end this section by giving a result that shows when the moment condition $q > q_0$ is necessary instead of $q \ge q_0$. The former is the moment condition applied by, e.g., Taqqu (1975) and Marinucci and Robinson (2000).

THEOREM 2. *Let $X_T(\xi)$ be defined by (5) for x_t given by the general fractional coefficients in (4) with $-\frac{1}{2} < d < 0$, and let $\mathcal{U}(q) \subset \mathcal{U}_0(q)$ be such that $u_{it} \in \mathcal{U}(q)$ for $i = 1, 2$. If the fractional FCLT holds for all slowly varying functions $\ell(\cdot)$ and all u_t in $\mathcal{U}(q)$ for some $q \ge 2$, then $q > q_0$.*

Proof. We assume that there is a $q_1 \in [2, q_0]$ for which the FCLT holds for $\mathcal{U}(q_1)$ and show that this leads to a contradiction. For $u_{it}, i = 1, 2$, we define x_{it} and X_{iT} by (4) and (5) and use the proof of Theorem 1 with the following modifications.

(a) From Karamata's theorem (see Bingham, Goldie, and Teugels, 1989, p. 26), we find that the normalizing variance is $\sigma_{1T}^2 \sim c\ell(T)^2 T^{2d+1} = c\ell(T)^2 T^{1/q_0}$.

(c) We choose the tail of ε_t as in (11) in the proof of Theorem 1 and take $\ell(T) = (\log T)^{-1}$ and find

$$\begin{aligned}
&P(\sigma_{1T}^{-1} \max_{1\le t\le T} |\varepsilon_t| < c)\\
&\quad = P(\sigma_{1T}^{-1}|\varepsilon_1| < c)^T = P(|\varepsilon_1|^{q_1} < c^{q_1}\sigma_{1T}^{q_1})^T\\
&\quad = (1 - P(|\varepsilon_1|^{q_1} \ge c^{q_1} T^{q_1/q_0}\ell(T)^{q_1}))^T\\
&\quad \sim \left(1 - \frac{1}{c^{q_1} T^{q_1/q_0}\ell(T)^{q_1}(q_1(\log c + q_0^{-1}\log T + \log\ell(T)))^2}\right)^T\\
&\quad \sim \exp\left(-\frac{T^{1-q_1/q_0}\ell(T)^{-q_1}}{c^{q_1}(q_1(\log c + q_0^{-1}\log T + \log\ell(T)))^2}\right) \to 0
\end{aligned}$$

as $T \to \infty$ because $q_1 \leq q_0$. Note that even with $q_1 = q_0$ (and $q_0 > 2$ because $d < 0$) we have the factor $\exp(-c(\log T)^{q_1-2}) \to 0$, which ensures the convergence to zero. The contradiction follows exactly as in the proof of Theorem 1. ■

4. DISCUSSION

In this section we present two corollaries that demonstrate how our results apply to the processes in Marinucci and Robinson (2000) and to those in DDJ, respectively.

We first discuss the implications of Theorem 1 for the results of DDJ, who state (in our notation) the following claim (given as Theorem 3.1 in DDJ).

DDJ CLAIM. *If $X_T(\xi)$ is defined by (3) and (5), where $|d| < \frac{1}{2}$ and $u_t \in \mathcal{U}_{NED}(q)$ for $q > 2$, then $X_T(\xi) \xrightarrow{D} X(\xi)$ in $D[0,1]$, where $X(\xi)$ is fractional Brownian motion.*

It is noteworthy that $\mathcal{U}_{NED}(q)$ allows u_t to have a very general dependence structure through the NED assumption, but in particular that DDJ assume only that $\sup_t \mathrm{E}|u_t|^q < \infty$ for $q > 2$, which is weaker than $q \geq q_0$ if $d < 0$. The following corollary to Theorem 1 shows how our result disproves Theorem 3.1 in DDJ.

COROLLARY 1. *Let $X_T(\xi)$ be defined by (3) and (5) with $-\frac{1}{2} < d < 0$. If the fractional FCLT holds for all u_t in $\mathcal{U}_{NED}(q)$ then $q \geq q_0$.*

Proof. From Lemma 1 we know that u_{1t} and u_{2t} are in $\mathcal{U}_{NED}(q)$, which by Theorem 1 proves the corollary. ■

It follows from Corollary 1 that Theorem 3.1 of DDJ (and their subsequent results relying on Theorem 3.1) does not hold under their Assumption 1. We finish with an application of our results to the processes in Marinucci and Robinson (2000).

COROLLARY 2. *Let $X_T(\xi)$ be defined by (4) and (5) with $-\frac{1}{2} < d < 0$. If the fractional FCLT holds for all slowly varying functions $\ell(\cdot)$ and all u_t in $\mathcal{U}_{LIN}(q)$ then $q > q_0$. Thus, the moment condition (1) with $q > q_0$ is both necessary and sufficient for Theorem 1 of Marinucci and Robinson (2000).*

Proof. The necessity statement follows from Theorem 2 because u_{1t} and u_{2t} are in $\mathcal{U}_{LIN}(q)$ by Lemma 1. The sufficiency statement follows because the univariate version of Assumption A of Marinucci and Robinson (2000) (translated to type I processes) was in fact used to define the class $\mathcal{U}_{LIN}(q)$ and the coefficients $a_j(d)$. ■

It follows from Corollary 2 that $q > q_0$ is both necessary and sufficient for the fractional FCLT when using the coefficients $a_j(d)$ to define a general fractional process and u_t is a linear process of the type in $\mathcal{U}_{LIN}(q)$. However, it does not follow from our results that $q \geq q_0$ is necessary for the FCLT when u_t is an i.i.d. or autoregressive moving average (ARMA) process because the process u_{2t} needed in the construction is neither i.i.d. nor ARMA.

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