Financial Econometrics A
Assignment #1: The DAR Model

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Please hand in an answer to:

- **Problems 1, 3 and 4 are mandatory.**
- **Either of: Problem 2 or Problem 5**

  Problem 2: if you are interested in theory for stationarity conditions.

  Problem 5: if you want to try to model "bubbles" in financial data.

Submission deadline: Friday, October 27, 2022 at 3 PM. Late hand-ins will be rejected.
Please upload your answer to Absalon **and hand in a hard-copy version to Jacob**

You are allowed to hand in your answer in groups of up to **3 students**. Please **include name, student ID, and class number** of all group members on the front page of your submission.

1 Double Autoregressive (DAR) model

Consider the so-called double autoregressive (DAR) model given by

$$\begin{aligned}\Delta x_t &= \pi x_{t-1} + \varepsilon_t, \ \varepsilon_t = \sigma_t z_t, \\ \sigma_t^2 &= \omega + \alpha x_{t-1}^2,\end{aligned}$$

where z_t is an iid $N(0,1)$ distributed sequence, and $t = 1, ..., T$ with x_0 fixed in the statistical analysis.

The parameters are given by $\pi \in \mathbb{R}$, $\omega > 0$, and $\alpha \geq 0$.
Note that the model reduces to the well-known ARCH(1) model if $\pi = -1$, and to the well-known AR(1) model if $\alpha = 0$.

Problem 1: The drift criterion

1. Find $E(x_t|x_{t-1})$ and $\mathrm{Var}(x_t|x_{t-1})$. Be precise about what results you use for the derivations.

2. Argue that the process x_t is a Markov chain, with a conditional density of x_t, i.e. $f(x_t|x_{t-1})$, that satisfies Assumption I.1 for the drift criterion.

3. Consider the drift function $\delta(x) = 1 + x^2$, and show that x_t satisfies the drift criterion in this case if $(1+\pi)^2 + \alpha < 1$.

4. Explain why $E(x_t^2) < \infty$ and $E|x_t| < \infty$ for all parameter values which satisfy $(1+\pi_0)^2 + \alpha_0 < 1$. Explain how it may be possible for parameter values which satisfy $(1+\pi_0)^2 + \alpha_0 < 1$, that $E(x_t^2) < \infty$ and $E(x_t^4) = \infty$.

Problem 2: Strict stationarity and Drift Criterion

In the following we want to show that if the DAR process satisfies

$$E[\log(|1 + \pi + \sqrt{\alpha} z_t|)] < 0$$

the drift criterion with drift function $\delta(x) = 1 + |x|^s$ for some arbitrary small $s > 0$. That is, we want to establish that Assumption I.2 is satisfied for the drift function $\delta(x) = 1 + |x|^\kappa$ for small $\kappa > 0$ (rather than $\kappa = 2$ as in Problem 1).

It will be useful to use that the DAR proces x_t may equivalently be written on the so-called random coefficient autoregressive representation:

$$x_t = \phi_t x_{t-1} + \eta_t, \tag{1}$$

where ϕ_t and η_t are independent, ϕ_t is i.i.d. $\mathrm{N}(1+\pi, \alpha)$ and η_t i.i.d. $\mathrm{N}(0, \omega)$.

1. With $\kappa \in (0,1)$, use repeatedly that $|x+y|^a \leq |x|^a + |y|^a$ for any $0 \leq a \leq 1$ in order to show that
$$E[\delta(x_t)|x_{t-1}] \leq 1 + E|\eta_t|^\kappa + E|\phi_t|^\kappa |x_{t-1}|^\kappa. \tag{2}$$

2. Note that we can always write $\phi_t = 1 + \pi + \sqrt{\alpha} z_t$, with z_t iid $N(0,1)$. Use (2) to argue that we need
$$E|\phi_t|^\kappa = E[|1+\pi+\sqrt{\alpha}z_t|^\kappa] < 1$$
for the drift criterion to hold.

3. Consider the function $h(\kappa) = E[|1+\pi+\sqrt{\alpha}z_t|^\kappa]$. Note that $h(0) = 1$. It can be shown that the derivative of h at zero is given by
$$h'(0) = E[\log(|1+\pi+\sqrt{\alpha}z_t|)].$$
Argue that $E[|1+\pi+\sqrt{\alpha}z_t|^\kappa] < 1$ for some small $\kappa > 0$, if indeed
$$E[\log(|1+\pi+\sqrt{\alpha}z_t|)] < 0.$$
Hint: By definition,
$$h'(0) = \lim_{\kappa \to 0} \frac{h(\kappa) - h(0)}{\kappa} = \lim_{\kappa \to 0} \frac{E[|1+\pi+\sqrt{\alpha}z_t|^\kappa] - 1}{\kappa}$$

Problem 3: Strict stationarity

As argued in (*the optional)* Problem 2, it can be shown that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + |x|^\kappa$ for some small $\kappa > 0$, if
$$E[\log(|1+\pi+\sqrt{\alpha}z_t|)] < 0.$$
This condition is also known as the strict stationarity condition.

Figure 1 illustrates the regions for $(\phi, \sqrt{\alpha})$ with $\phi = \pi + 1$ in order to ensure $\phi^2 + \alpha < 1$ and $E[\log(|\phi + \sqrt{\alpha}z_t|)] < 0$.

1. Based on the figure, it appears that $E[\log(|\phi+\sqrt{\alpha}z_t|)] < 0$ for $\phi = \alpha = 1$. Verify this using Monte Carlo integration, that is:
Draw N independent realizations of $\log(|1+z_i|)$, $i = 1, ..., N$, with N large (e.g. 1,000), and report $N^{-1}\sum_{i=1}^{N} \log(|\phi + \sqrt{\alpha}z_i|)$. Comment.

2. Simulate two realizations of the DAR process with $\omega_0 = 1$, $T = 500$ and starting value $x_0 = 0$:

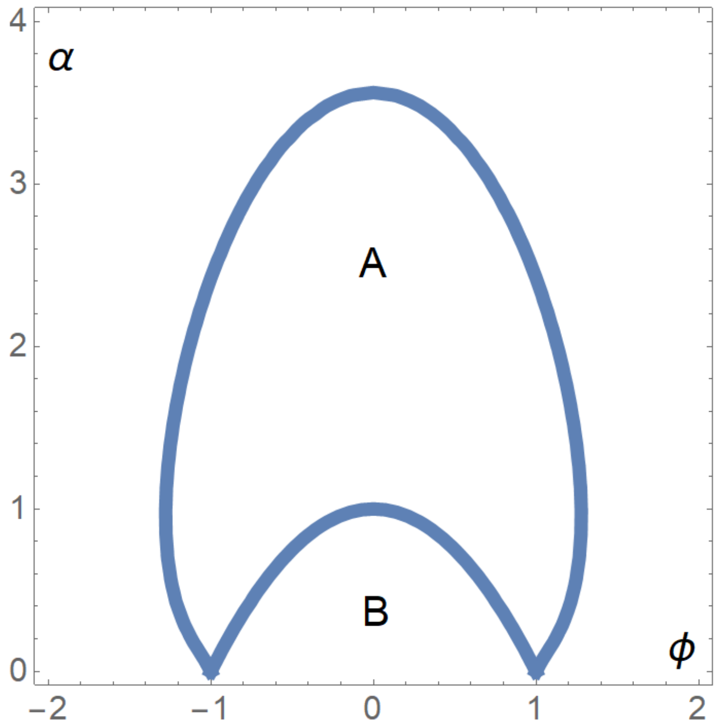

Figure 1: Stationarity region for DAR process.

(a) For the first realization, choose $\pi_0 = 0$ and $\alpha_0 = 0$.

(b) For the second realization, choose $\pi_0 = 0$ and $\alpha_0 = 1$.

(c) Make a plot of both series. Briefly comment on how the two series differ.

Recall that the AR(1) process is non-stationary when the autoregressive coefficient is equal to one ($\pi = 0$). The fact that $\alpha > 0$ ensures that the DAR process may be stationary even though $\pi = 0$, and hence that the process is a "unit-root"-type process. This property is known as *volatility-induced stationarity*.

Problem 4: Maximum likelihood estimation

For a realization of the DAR-process $(x_t : t = 0, 1, ..., T)$, the log-likelihood function is given by,

$$\begin{aligned} L_T(\pi, \omega, \alpha) &= \sum_{t=1}^{T} l_t(\pi, \omega, \alpha), \quad l_t(\pi, \omega, \alpha) = -\frac{1}{2} \log[\sigma_t^2(\omega, \alpha)] - \frac{1}{2} \frac{(\Delta x_t - \pi x_{t-1})^2}{\sigma_t^2(\omega, \alpha)}, \\ \sigma_t^2(\omega, \alpha) &= \omega + \alpha x_{t-1}^2. \end{aligned}$$

The maximum likelihood estimator of (π, ω, α) is obtained by maximizing $L_T(\pi, \omega, \alpha)$ with respect to (π, ω, α).

Let $\theta = (\pi, \omega, \alpha)'$ such that $l_t(\theta) = l_t(\pi, \omega, \alpha)$. Moreover, let $\theta_0 = (\pi_0, \omega_0, \alpha_0)'$ denote the vector of true parameter values.

1. Show that

$$\frac{\partial l_t(\theta)}{\partial \alpha} = \frac{1}{2} \frac{x_{t-1}^2}{\omega + \alpha x_{t-1}^2} \left(\frac{(\Delta x_t - \pi x_{t-1})^2}{\omega + \alpha x_{t-1}^2} - 1 \right).$$

2. Use that for the true value $\theta_0 = (\pi_0, \omega_0, \alpha_0)'$,

$$\frac{(\Delta x_t - \pi_0 x_{t-1})^2}{\omega_0 + \alpha_0 x_{t-1}^2} = \frac{\varepsilon_t^2}{\omega_0 + \alpha_0 x_{t-1}^2} = z_t^2.$$

to show that,

$$\frac{\partial l_t(\theta_0)}{\partial \alpha} := \left. \frac{\partial l_t(\theta)}{\partial \theta} \right|_{\theta = \theta_0} = \frac{1}{2} \frac{x_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2} \left(z_t^2 - 1 \right).$$

Assume that $\alpha_0 > 0$. State conditions such that $\{\partial l_t(\theta_0)/\partial \alpha : t = 1, ..., T\}$ satisfies a CLT from the lecture notes, i.e.

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial l_t(\theta_0)}{\partial \alpha} \xrightarrow{D} N(0, \Omega).$$

Problem 5: "Bubbles" in the Bitcoin/USD exchange rate

During the recent years, the Bitcoin cryptocurrency has gained a lot of attention.

1. The data set `xbtusd` contains the Bitcoin/USD exchange rate from February 20 - July 19, 2013.[1] The second column of the data set contains the variable `xbtusd`, which is the raw data series. The third column contains the variable `xbtusd_detrend`, which is a de-trended version of the exchange rate. We will here focus on the latter series. Make a graph of the series `xbtusd_detrend`. Comment briefly.

2. A leading paper on Bicoin modeling argues that: "*The trajectory of the Bitcoin/USD exchange rate displays repetitive episodes of upward trends, followed by instantaneous drops, which are called bubbles. In general, a bubble has two phases: (1) a phase of fast upward (or downward) departure from the stationary path that resembles an explosive pattern and displays an exponential rate of growth, followed by (2) a phase of sudden almost vertical drop (or upspring) back to the underlying fundamental path.*"
   Based on the simulations from Problem 3.2, we may argue that a *stationary* DAR model should also be able to capture such features.

   (a) Run one of the codes in R, Ox or Python which estimate a DAR model based on the `xbtusd_detrend` series.

   (b) Based on the point estimates of (π, ω, α) simulate a realization of a DAR process with the same starting value and sample length as the `xbtusd_detrend` series. Compare briefly the simulated series with the true series.

   (c) Based on the estimation output, does the `xbtusd_detrend` appear to be stationary? Explain briefly.

[1] Data were downloaded from a Bloomberg terminal. Ticker: "XBTUSD Curncy".