Geometry/ Topology August 2019

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1. In this problem we consider $\mathbb{R}^n(n \in \mathbb{N}$ with its standard topology, and we let

$$S^n = \left\{x \in \mathbb{R}^{n+1} \colon |x| = 1\right\} \subset \mathbb{R}^{n+1}$$

denote the standard n-sphere with its induced topology. Note that $S^0 = \{\pm 1\} \subset \mathbb{R}$.

(a) Compute the set $[S^1, S^0]$ of homotopy classes of continuous maps $f \colon S^1 \to S^0$.

(b) Prove that the fundamental group $\pi_1(\mathbb{R}^n, 0)$ of $\mathbb{R}^n$ based at the origin $0 \in \mathbb{R}^n$ is trivial.

Solution:

(a) Note that S^0 is a discrete space with 2 points, and S^1 is connected, so any map from S^1 to S^0 is constant (else $f^{-1}(\{-1\})$ and $f^{-1}(\{+1\})$ would be a separation of S^1). So $[S^1, S^0]$ is at most the set $X = \{[c_{-1}], [c_1]\}$, where c_{-1} is the constant map at -1, and c_1 is the constant map to 1. It remains to show that c_1 and c_{-1} are not homotopic. Suppose there exists a homotopy $H \colon S^1 \times I \to S^0$ from c_1 to c_{-1}. Then $H(x, 0) = -1, H(x, 1) = 1$, so H is surjective. But S^1 and I are both connected and hence so is their product. But then H is a surjective continuous map from a connected space to a discrete space with more than one point, which is a contradiction. Hence such a H does not exist, and so c_1 and c_{-1} are not homotopic, and hence $[S^1, S^0] = \{[c_{-1}], [c_1]\}$, where both elements are distinct.

(b) Let $\gamma(x) \colon I \to \mathbb{R}^n$ be any loop based at $0 \in \mathbb{R}^n$. Then the homotopy $H(x, t) = t\gamma(x)$ satisfies $H(x, 0) = \gamma(0) = 0, H(x, 1) = \gamma(x)$. Moreover, $H(x, t) \in \mathbb{R}^n$ for all (x, t) as $\mathbb{R}^n$ is convex. It is continuous as γ is continuous, and so it scaling by t. Hence H is a homotopy from γ to the constant map at 0. Hence there is a single homotopy class of loops in $\mathbb{R}^n$ i.e. $\pi_1(\mathbb{R}^n, 0)$ is trivial.

2. In this problem we again consider $\mathbb{R}^n$ with its standard topology and $S^n \subset \mathbb{R}^{n+1}$, the standard $n-$sphere with its induced topology.

(a) Show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$, for $n > 1$.

(b) For $n \in \mathbb{N}$, denote by $(S^1)^n$ the product of n copies of the unit circle, endowed with the product topology. Prove that $(S^1)^n$ is homeomorphic to $(S^1)^m$ if and only if $n = m$ (where $m, n \in \mathbb{N}$).

Solution:

(a) Suppose there exists a homeomorphism $h \colon \mathbb{R} \to \mathbb{R}^n$. Then this restricts to a homeomorphism $h \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}^n \setminus \{h(0)\}$. But the former is not connected (as it has two components, $(-\infty, 0)$ and $(0, \infty)$), while the latter is (path) connected (by taking paths that are either straight lines or circles). Connectedness is a topological invariant, so no homeomorphism exists.

(b) If $n = m$ then$(S^1)^n$ is homeomorphic to $(S^1)^m$ as they're the same space. If $n \neq m$, then $\pi_1((S^1)^n) \cong \pi_1(S^1) \times \cdots \times \pi_1(S^1) \cong \mathbb{Z}^n$, while $\pi_1((S^1)^m) \cong \mathbb{Z}^m$, where we used that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ plus induction, and ignored basepoints as S^1 is path-connected and hence so are its products. $\mathbb{Z}^n \cong \mathbb{Z}^m$ iff $n = m$, for otherwise these two group have different ranks (n and m respectively). As homeomorphic spaces have isomorphic fundamental groups, we get that $(S^1)^n$ is not homeomorphic to $(S^1)^m$ when $n \neq m$.

3. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X, with $\mathcal{T} \subset \mathcal{T}'$. **TRUE** or **FALSE** (prove or disprove each of the following statements):

(a) If X is limit point compact in the topology $\mathcal{T}$ (i.e. every infinite subset of X has a limit point in X), then X is limit point compact in the topology $\mathcal{T}'$.

(b) If X is limit point compact in the topology $\mathcal{T}'$ then X is limit point compact in the topology $\mathcal{T}$.

(c) If X is connected in the topology $\mathcal{T}$, then X is connected in the topology $\mathcal{T}'$.

(d) If X is connected in the topology $\mathcal{T}'$, then X is connected the the topology $\mathcal{T}$.

Solution:

(a) **FALSE**: Let $X = \mathbb{Q}$ with the indiscrete topology, and take any infinite subset $A \subset \mathbb{Q}$. Then if we take any $x \in X$, the only nbhd of x is X itself, so $(X \setminus \{x\}) \cap A = A \neq \emptyset$ or $(X \setminus \{x\}) \cap A = A \setminus \{x\} \neq \emptyset$. Hence x is a limit point of A, and so $(X, \tau_{\text{indiscrete}})$ is limit point compact. However, now consider X with the discrete topology, and let A again be any infinite subset. Given a point $x \in X$, $\{x\}$ is a nbhd of x, and $(\{x\} \setminus \{x\}) \cap A = \emptyset$, so x is not a limit point of A. As x was arbitrary, A has no limit point. Hence $(X, \tau_{\text{discrete}})$ is *not* limit point compact. We have that $\tau_{\text{indiscrete}} \subset \tau_{\text{discrete}}$, which proves that the statement is false.

(b) **TRUE**: Suppose $\mathcal{T} \subset \mathcal{T}'$ and $(X, \mathcal{T}')$ is limit compact. Let A be any infinite subset of X. Then A has a limit point a in the topology $\mathcal{T}'$. Then a is also a limit point of A in the topology $\mathcal{T}$: Let $N(a)$ be any nhbd of a in $(X, \mathcal{T})$. Then as $\mathcal{T} \subset \mathcal{T}'$, $N(a)$ is also a nhbd of a in $(X, \mathcal{T}')$. Hence as $(X, \mathcal{T}')$ is limit point compact, $(N(a) \setminus \{a\}) \cap A \neq \emptyset$, so a is a limit point of A in $(X, \mathcal{T})$, and so $(X, \mathcal{T})$ is limit point compact.

(c) **FALSE**: Let $X = \mathbb{R}$ with $\mathcal{T}$ being the standard topology and $\mathcal{T}'$ being the discrete topology. Then $(X, \mathcal{T})$ is connected, but $(X, \mathcal{T}')$ is not (a separation by open sets is given by $\mathbb{R} = \{0\} \cup \mathbb{R} \setminus \{0\}$).

(d) **TRUE**: By the contrapositive, suppose that $(X, \mathcal{T})$ is *not* connected. Then there is a set $A \in \mathcal{T}, A \neq X, \emptyset$ that is clopen. Then as $\mathcal{T} \subset \mathcal{T}'$, $A \in \mathcal{T}'$ is a clopen set in $(X, \mathcal{T}')$ that is neither X nor $\emptyset$. Hence $(X, \mathcal{T}')$ is *not* connected.

4. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $f \colon M \to \mathbb{R}^n$ for any $n > 0$.

Solution: If f is a smooth submersion, then f is an open map. Hence $f(M)$ is open in $\mathbb{R}^n$. As $f(M)$ is the continuous image of a compact space, it is compact, and hence also is closed as compact subsets of Hausdorff spaces are closed. Hence $f(M)$ is clopen, and thus as $\mathbb{R}^n$ is connected, $f(M) = \mathbb{R}^n$ (can't have $f(M) = \emptyset$ as f has nonempty domain and codomain). But then this implies that $\mathbb{R}^n$ is compact, a contradiction.

5. Let M be a smooth manifold and let $U \subset M$ be an open set. Let $\mathfrak{X}(U)$ denote the vector space of vector fields on U, let $\mathfrak{D}(U) \subset \mathfrak{X}(U)$ denote a vector subspace, and for each nonnegative integer k let $\Omega^k_M(U)$ denote the vector space of smooth k-forms on U, with exterior derivative d.

   (a) Denote by $\mathcal{I}^k_{\mathfrak{D}}(U) \subset \Omega^k_M(U)$ the set of k-forms that annihilate $\mathfrak{D}(U)$; i.e., the set of $\omega \in \Omega^k_M(U)$ such that $\omega(X_1, \dots, X_K) = 0$ whenever $X_1, \dots, X_k \in \mathfrak{D}(U)$. For all nonnegative integers k, l, show that if $\omega \in \mathcal{I}^k_{\mathfrak{D}}(U)$ and $\eta \in \Omega^l_M(U)$, then $\omega \wedge \eta \in \mathcal{I}^{k+l}_{\mathfrak{D}}(U)$

   (b) We say that $\mathfrak{D}(U)$ is involutive if it is preserved by the Lie bracket; i.e., we have that $X_1, X_2 \in \mathfrak{D}(U)$ implies that $[X_1, X_2] \in \mathfrak{D}(U)$. Show that is $\mathfrak{D}(U)$ is involutive, then $\omega \in \mathcal{I}^1_{\mathfrak{D}}(U)$ implies that $d\omega \in \mathcal{I}^2_{\mathfrak{D}}(U)$. [Hint: you may want to use Cartan's formula for the exterior derivative.]

   (c) Suppose that M is m-dimensional, and assume there exist vector fields $X_1, \dots, X_m \in \mathfrak{X}(U)$ with $X_1(p), \dots X_m(p)$ linearly independent for every $p \in U$, and such that for some integer r, any $X \in \mathfrak{D}(U)$ can be written in the form $X = \sum_{i=1}^r f_i X$, for some $f_1, \dots, f_r \in C^\infty(U)$. Show that $\mathfrak{D}(U)$ is involutive if $\omega \in \mathcal{I}^1_{\mathfrak{D}}(U)$ implies that $d\omega \in \mathcal{I}^2_{\mathfrak{D}}(U)$.

Solution:

(a) Using the fact that locally we can write ω as $\omega^1 \wedge \cdots \wedge \omega^k$ and η as $\omega^{k+1} \wedge \cdots \wedge \omega^{k+l}$ for covector fields ω^j, and the fact that $(\omega \wedge \eta)(X_1, \dots, X_{k+l}) = \det \omega^i(X_j)$ by definition of the wedge product, and combining this with the permutation definition of the determinant: $\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{k+l} a_{i,\sigma(i)}$, it follows that:

$$(\omega \wedge \eta)(X_1, \dots, X_{k+l}) = \sum_{\sigma \in S_n} sgn(\sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \dots X_{\sigma(k+1)})$$

So if $X_1, \dots, X_{k+l} \in \mathfrak{D}(U)$, then in particular $X_{\sigma(1)}, \dots, X_{\sigma(k)} \in \mathfrak{D}(U)$ for any permutation , and hence as $\omega \in \mathcal{I}^k_{\mathfrak{D}}(U)$, $\omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) = 0$ and so every term in the sum is zero, i.e. $\omega \wedge \eta(X_1, \dots, X_{k+l})$ for any choice of $(X_1, \dots, X_{k+1}) \in \mathfrak{D}(U)$, and so $\omega \wedge \eta \in \mathcal{I}^{k+l}_{\mathfrak{D}}(U)$.

(b) Suppose $X, Y \in \mathfrak{D}(U)$, and $\omega \in \mathcal{I}^1_{\mathfrak{D}}(U)$. Then applying the formula:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

gives:

$$d\omega(X, Y) = X0 - Y0 - 0 = 0$$

(as $\omega \in \mathcal{I}^1_{\mathfrak{D}}(U)$ and $X, Y, [X, Y] \in \mathfrak{D}(U)$).

(c) Not sure how to do this one: let me know if you have any ideas!

6. Let G be a Lie group and let T_eG be the tangent space at the origin. A Riemannian metric g on G is said to be left-invariant if it is invariant under all left translations; i.e. $L_x^* g = g$ for all $x \in G$, where $L_x : G \to G$ is left multiplication by x.

   (a) Show that the restriction map $g \mapsto g|_{T_eG}$ gives a bijection between left-invariant Riemannian metrics on G and inner products on T_eG.

   (b) Show that every left-invariant Riemannian metric on a torus $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n (n \in \mathbb{N})$ is flat.

Solution:

(a) Idea: Same as correspondence between left-invariant vector fields and elements of T_eG: Given an inner product $\langle \ , \ \rangle$ on T_eG, we can define a left-invariant metric by $\langle v, w \rangle_x = \langle dL_{x^{-1}}(v), dL_{x^{-1}}(w) \rangle_e$. This choice of inner product at each point is smooth by smoothness of L_x. Then left-invariance of g means that this is an inverse to $g \mapsto g|_{T_eG}$.

(b) Idea for $n = 2$: If a metric is left-invariant, then the curvature tensor is the same at each point. So if the curvature is positive at one point, it is positive everywhere, similarly if it's negative at one point then it's negative everywhere. In either of these cases, the integral of curvature across $\mathbb{T}$ will be positive or negative respectively. But the Gauss-Bonnet Theorem says that said integral is 0, hence by left-invariance the curvature tensor must be 0 everywhere, i.e. the metric is flat.