Top / Diff Geo Prelim Exam January 2012

1) (a) The interior of a set A is defined to be the largest open set completely contained in A. For $A = [0,1]$ under the given topology, the largest open set contained in A is $[0,1)$, so $\text{int}([0,1]) = [0,1)$. Similarly, the closure of A is the smallest closed set containing A. Under the given ~~tots~~ topology we have $cl(A) = \mathbb{R}$ because any other closed set containing $[0,1]$ would have a compliment that was neither empty nor contained $[0,1)$.

(b) The T_0 condition states that for every pair of distinct points x, y there is some open set U around one of them that does not include the other. Our topology is not T_0 since any open set containing 0 or 0.5 contains the whole interval $[0,1)$ and hence contains the other point.

(c) $\mathbb{R}$ is connected in this topology because for all nonempty open sets U, V we have $[0,1) \subseteq U \cap V$.

(d) No, $\mathbb{R}$ is not compact. Consider the open cover

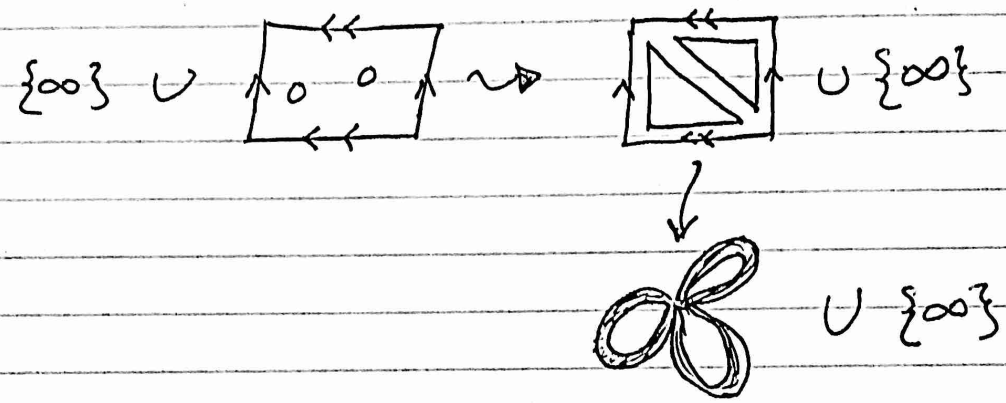
$$\mathcal{U} = \{[0,1) \cup \{r\} : r \in \mathbb{R}\}$$

This clearly has no finite subcover.

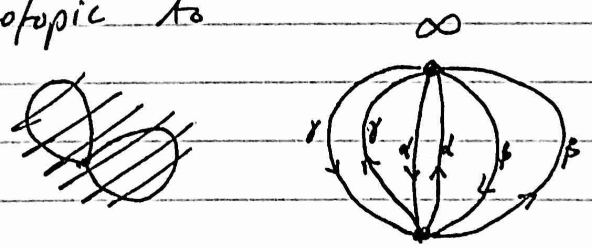
Top / Diff Geo Prelim Exam January 2012

2) (a) The universal cover of $\mathbb{T}^2$ and is given by
$$p : \mathbb{R}^2 \to \mathbb{T}^2 : (x,y) \mapsto (x,y) \bmod 1.$$

* (b) Fundamentally, we can describe the one point compactification of $\mathbb{T}^2$ minus two distinct points can be given in the following series of deformations



Using a deformation retraction we can then see that this space is Homotopic to



This is a graph with the following fundamental group:

$$\pi_1(X) = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = \alpha\beta = \beta\alpha = \alpha\gamma = \gamma\alpha = \beta\gamma = \gamma\beta = 1 \rangle.$$

Top / Diff Geo Prelim Exam January 2012

3) Let $C_n(X)$ denote the free abelian group with basis the sets of singular n-simplices, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be defined by

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma|_{(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)}$$

where v_i are the 0-simplices of σ. We know that

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Note that for $X = pt$ we have $C_n(X) = 0$ for $n \geq 1$, so

$$H_t(X) = 0 \quad \text{for} \quad t > 0.$$

And when $t = 0$, we have

$$\ker(\partial_0) = \langle pt \rangle$$
$$\operatorname{im}(\partial_1) = 0$$

So $H_0(X) = {}^{\langle pt \rangle}/_{0} = \langle pt \rangle \cong \mathbb{Z}$.

Top / Diff Geo Prelim Exam January 2012

4) (a) $\mathcal{M} = \{(x,y,z) : xy - z = 0\}$. We have that

$$df = \begin{pmatrix} y & x & -1 \end{pmatrix}$$

which always has full rank, so every point $p \in \mathbb{R}$ is a regular point. Thus $\mathcal{M} = f^{-1}(0)$ is a smooth submanifold.

(b) Not done. This question concerns Riemannian Geometry.

Top / Diff Geo Prelim Exam January 2012

5) (a) The space of left-invariant vector fields of G is called the Lie Group of G, which we will denote by $Lie(G)$. We want to show $Lie(G) \cong T_e G$. Consider the map

$$ev : Lie(G) \to T_e G : X \mapsto X_e;$$

all we need to show is that this is an isomorphism of vector spaces.

It is fairly clear that ev is linear over $\mathbb{R}$, so we now want to show ev is surjective. Let $\vec{v} \in T_e G$ be an arbitrary vector and define a vector field v^L by

$$v^L = d(L_g)_e(v)$$

Then if there was such a vector field $v^L \in Lie(G)$, it would have to be given by this formula. Thus we just need for v^L to be a smooth vector field. Well, for any $g \in G$

$$(v^L f)(g) = v^L|_g f = d(L_g)_e(v)(f) = v(f\,dg)$$
$$= v'(0)(f \circ L_g)$$
$$= \frac{d}{dt}\Big|_{t=0} (f \circ L_g \circ \gamma)(t).$$

(b) We have that $Lie(G) \cong T_e G$ is finite dimensional, so we can describe a basis for $T_e G$ and use this to trivialize $Lie(G)$ under the previosly given isomorphism ev.

(c) We have that

$$(L_g)_*(\vec{V}|_h) = (g x \tfrac{\partial}{\partial x})(h) = gh \tfrac{\partial}{\partial x} = \vec{V}|_{gh},$$

so X is left-invariant. Now if $p \in \mathbb{R}_{>0}$ we have that the flow of X from p is given by

$$\theta_t(p) = p e^t.$$

Top / Diff Geo Prelim Exam January 2012

6) (a) Let M be a smooth oriented n-manifold and let ω be a compactly supported $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega .$$

* (b) Suppose that M is smooth manifold, $\omega \in \Omega^r(M)$, and that for every r-dimensional submanifold $\Sigma \subseteq M$ we have

$$\int_\Sigma \omega = 0.$$

We know that Σ is diffeomorphic to a surface in $\mathbb{R}^r$ that satisfies the equation

$$x_1^2 + \cdots + x_r^2 = k^2 \qquad k \in \mathbb{R},$$

so it appears as the boundary of some manifold S ~~to~~ diffeomorphic to the $r+1$ dimensional dish

$$x_1^2 + \cdots + x_r^2 \leq k^2$$

which itself is diffeomorphic to $\mathbb{R}^{r+1}$. Applying Stokes's Theorem we get that

$$\int_S d\omega = \int_{\partial S} \omega = \int_\Sigma \omega = 0$$

so $d\omega = 0$.