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# Geometry/Topology

**Ph.D. Preliminary Exam**
**Department of Mathematics**
**University of Colorado Boulder**

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*INSTRUCTIONS*:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Suppose that $X$ is a topological space, and let

$$\Delta = \{(x,x) \mid x \in X\}.$$

Prove that $X$ is Hausdorff if and only if $\Delta$ is a closed subset of $X \times X$.

Q.2 For each $n$, let $X_n \subset \mathbb{R}^2$ be the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$. Let $X = \bigcup_{n=1}^{\infty} X_n$. Prove that $X$ has no universal cover.

Q.3 Suppose that $E$ is a contractible topological space and $G$ is a group acting freely and properly discontinuously on $E$. Let $e$ be a point of $E$, let $X = E/G$ be the set of $G$-orbits, with the quotient topology, and let $x$ be the image of $e$ in $X$.

Recall that $\mathbb{R}P^2$ is the quotient of the 2-sphere $S^2$ by the equivalence relation $(x,y,z) \sim (-x,-y,-z)$, with the quotient topology. With $X$ as above, prove that there is a continuous map $\mathbb{R}P^2 \to X$ that is not homotopic to a constant if and only if there is an element $g \in G$ such that $g \neq 1$ but $g^2 = 1$. (Hint: What is $\pi_1(X,x)$?)

Q.4 Consider the subset $S \subset \mathbb{R}^3$ defined by the equations

$$x^2 + y^2 = a, \qquad yz = b,$$

where $a, b$ are real numbers with $a > 0$.

(a) Show that if $b \neq 0$, then $S$ is a smooth submanifold of $\mathbb{R}^3$.

(b) Show that $S$ is not a smooth submanifold of $\mathbb{R}^3$ when $b = 0$.

Q.5 Consider the two vector fields on $\mathbb{R}^2$ given by

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \qquad Y = \frac{\partial}{\partial y}.$$

(a) Find the smooth flow on $\mathbb{R}^2$ whose infinitesimal generator is $X$; i.e., find the (unique!) smooth map $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\theta_0 = \mathrm{Id}\,|_{\mathbb{R}^2}$ and

$$\left.\frac{d}{dt}\right|_{t=0} \theta_t(p) = X_p, \qquad \text{for all } \ p \in \mathbb{R}^2.$$

(b) Find $\mathcal{L}_X Y$ using part (a) and the definition of Lie derivative

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})(Y).$$

(c) Compute the Lie bracket $[X,Y]$ directly and check that your answer is the same as your answer to part (b).

Q.6 Let $M$ be a smooth, oriented, $2n$-dimensional manifold. A 2-form $\omega$ on $M$ is called a *symplectic form* on $M$ if $d\omega = 0$ and the $2n$-form $\omega^n = \overbrace{\omega \wedge \cdots \wedge \omega}^{n \text{ times}}$ is a nowhere-vanishing $2n$-form on $M$. (This means that, in terms of any local coordinate chart $(U, (x^1, \ldots, x^{2n}))$ on $M$, we can write $\omega^n = f dx^1 \wedge \cdots \wedge dx^{2n}$ for some nonvanishing function $f : U \to \mathbb{R}$.)

(a) Let $\omega$ be a symplectic form on $M$. Let $(U, \mathbf{x})$ and $(V, \mathbf{y})$ be local coordinate charts on $M$ that are compatible with the orientation of $M$, and suppose that $U \cap V \neq \emptyset$. Show that on $U \cap V$, when we write $\omega^n$ as

$$\omega^n = f\, dx^1 \wedge \cdots \wedge dx^{2n} = g\, dy^1 \wedge \cdots \wedge dy^{2n},$$

the nonvanishing functions $f, g : U \cap V \to \mathbb{R}$ must have the same sign; i.e., they are either both positive-valued or both negative-valued.

(b) Suppose that $M = \mathbb{R}^4$. Show that the 2-form

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$$

is an exact symplectic form on $\mathbb{R}^4$. (Recall that a 2-form $\omega$ on $M$ is *exact* if $\omega = d\alpha$ for some 1-form $\alpha$ on $M$.)

(c) Suppose that $M$ is compact with no boundary. Show that no symplectic form on $M$ is exact.

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1) ($\Rightarrow$) Suppose that $X$ is Hausdorff, and consider $(x,y) \in \Delta^c$. Then $x \neq y$, so there exist basic open sets $U_x$ and $V_y$ with $x \in U_x$ and $y \in V_y$ such that $U_x \cap V_y = \emptyset$. Since $U_x$ and $V_y$ are basic open sets $U_x \times V_y$ is open in $X \times X$. More importantly $(x,y) \in U_x \times V_y$ for an arbitrary $(x,y) \in \Delta^c$. Hence, $\Delta^c$ is open and $\Delta$ is closed.

($\Leftarrow$) Suppose $\Delta$ is closed in $X \times X$ Then for every point $(x,y) \in \Delta^c$ there is some open set $U \times V \subseteq \Delta^c$ where $U$ and $V$ are basic open sets since $\Delta^c$ is open. Well, $U$ and $V$ are open sets of $X$ with $U \cap V = \emptyset$, $x \in U$, and $y \in V$ since if $U$ and $V$ shared a point $z$ then $(z,z) \in U \times V$, but this is impossible since $U \times V \in \Delta^c$.

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2) Suppose that some universal cover $\tilde{X}$ exists, and let $p = (0,0)$. Then if $U$ is an evenly covered neighborhood of $p$, it lifts to a neighborhood $V$ of $q$ in the fiber of $p$ with $V$ homeomorphic to $U$. Now consider the following diagram

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d, "f"', "\wr"] & \tilde{X} \arrow[d, "\pi"] \\ U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

where $i$ and $j$ are the standard inclusion maps, and $f$ is any homeomorphism. Under this identification we have that the induced maps on the fundamental groups are related as follows:

$$i_* \circ f_* = \pi_* \circ j_*$$

but since $\tilde{X}$ is simply connected, both of these must be the trivial map, and so $i_*$ must also be the trivial map. This means that $U$ is some neighborhood of $p$ such that $U$ is homotopic to a point. This can't be the case, however, since every neighborhood of $p$ in the subspace topology contains some circle.

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3) First of all, consider the following diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, dotted, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X = E/G \end{tikzcd}$$

Since $G$ is a covering space action (free and properly discontinuous action) on $E$, we have that $\pi_1(X,x) \cong G$. Moreover, if $\varphi$ is a continuous map then it induces a homeomorphism $\varphi_* : \pi_1(\mathbb{RP}^2) \to G$. Well, $\pi_1(\mathbb{RP}^2) = C_2$ the cyclic group of two elements, so for $\varphi_* : C_2 \to G$ to be a homeomorphism, we need for either $\varphi_*$ to be trivial or for $G$ to have a cyclic subgroup of order 2, e.g., there needs to be a $g \in G$ with $g \neq 1$ but $g^2 = 1$.

Suppose that no such $g$ exists. Then $\varphi_*(\pi_1(\mathbb{RP}^2)) = \{0\} \subseteq p_*(\pi_1(E))$ so there exists a lifting $\varphi$ of $\varphi$. Since $E$ is contractible, we then know that there is a homotopy $\tilde{H}$ of $\tilde{\varphi}$ to a constant map. Well then, if we take $H = p \circ \tilde{H}$, we will have that $H$ is a homotopy from $\varphi$ to a constant map.

For the other direction suppose there is some $g \in G$ with $g \neq 1$ and $g^2 = 1$, and let $\psi : C_2 \to \langle g \rangle$ be a homomorphism from $C_2 \to G$ that maps into the given subgroup $(0 \mapsto e \;\; 1 \mapsto g)$. Since $\mathbb{RP}^2$ is a connected CW complex and $E$ is a contractible cover of $X = E/G$, we have that every homomorphism $\pi_1(\mathbb{RP}^2) \to \pi_1(X)$ is induced by a continuous map $f : \mathbb{RP}^2 \to X$. Specifically, this gives us that for some $\varphi : \mathbb{RP}^2 \to X$, $\varphi_* = \psi$. Moreover, $\varphi$ cannot be homotopic to a constant map since $\psi(C_2) = \langle g \rangle$.
[See Hatcher Proposition 1B.9]

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4) Consider the map
$$F: \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x^2+y^2, yz)$$

(a) We have that
$$DF = \begin{pmatrix} 2x & 2y & 0 \\ 0 & z & y \end{pmatrix}$$

which only drops rank when $2xz = 0$, $2xy = 0$, and $2y^2 = 0$. Then for any point $(a,b)$ with $b \neq 0$ we will have $y \neq 0$ so $DF$ will have full rank. Therefore, every such $(a,b)$ will be a regular value of $F$ and we have that $S = F^{-1}((a,b))$ is a smooth submanifold of $\mathbb{R}^3$ by the Regular Level Set Theorem.

(b) If $b = 0$ we have
$$S = \{(x,y,z) : x^2+y^2 = a,\ z=0\} \cup \{(x,y,z) : x^2 = a,\ y=0\}.$$

If we then remove the point $p = (\sqrt{a}, 0, 0)$ from $S$, we can see that $S \setminus \{p\}$ has 3 connected components, so $S$ cannot be locally Euclidean under the subspace topology. Hence, when $b = 0$, $S$ is not a smooth submanifold of $\mathbb{R}^3$.

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5) (a) We need to find some map

$$\theta_t(x,y) = (x(t), y(t)).$$

such that

$$\dot{x}(t) = x(t) \quad \text{and} \quad \dot{y}(t) = -2y(t)$$
$$x(0) = x \qquad\qquad y(0) = y.$$

These are simple exponential differential equations, so we have

$$\theta_t(x,y) = (xe^{t}, ye^{-2t}).$$

(b) $$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})Y$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \frac{\partial\theta^1_{-t}}{\partial x} & \frac{\partial\theta^1_{-t}}{\partial y} \\ \frac{\partial\theta^2_{-t}}{\partial x} & \frac{\partial\theta^2_{-t}}{\partial y} \end{pmatrix} \begin{pmatrix} 0\frac{\partial}{\partial x} \\ 1\frac{\partial}{\partial y} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0\frac{\partial}{\partial x} \\ 1\frac{\partial}{\partial y} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} e^{2t}\frac{\partial}{\partial y} = 2\frac{\partial}{\partial y}$$

(c) $$[X,Y] = XY - YX$$

$$= X(1)\frac{\partial}{\partial y} - Y(x)\frac{\partial}{\partial x} - Y(-2y)\frac{\partial}{\partial y}$$

$$= 0\frac{\partial}{\partial y} - 0\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} = 2\frac{\partial}{\partial y}$$

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6) (a) Since $\omega^n$ is nonvanishing on an orientable manifold, we have that for any open subset $U$, $U \neq \emptyset$,

$$\left| \int_U \omega^n \right| > 0.$$

On the open set $U \cap V$ we have that $\omega = f\, dx^1 \wedge \cdots \wedge dx^{2n} = g\, dy^1 \wedge \cdots \wedge dy^{2n}$. Suppose $f$ and $g$ have different sign, then we have that

$$\int_{U \cap V} \omega^n = \int_{U \cap V} f\, dx^1 \wedge \cdots \wedge dy^{2n}$$

$$\int_{U \cap V} \omega^n = \int_{U \cap V} g\, dy^1 \wedge \cdots \wedge dy^{2n}$$

but $\int_{U \cap V} f\, dx^1 \wedge \cdots \wedge dx^n = -\int_{U \cap V} g\, dy^1 \wedge \cdots \wedge dy^{2n}$ which implies that

$\int_{U \cap V} \omega^n = -\int_{U \cap V} \omega^n = 0$. ↯. So $f$ and $g$ must have the same sign.

(b) Consider $\alpha = x^1 d\ \ + x^3 dx^4$. Then $d\alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ so $\omega$ is exact. To see that $\omega^2$ is nowhere vanishing we compute

$$\begin{aligned}
\omega^2 &= (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\
&= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 \\
&= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
&= 2\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,
\end{aligned}$$

which obviously vanishes nowhere. Since $\omega$ is exact $d\omega = 0$, so we have $\omega$ is symplectic.

(c) Suppose $\omega = d\alpha$ is an exact symplectic form on $M$. Then by Stokes' Theorem we have

$$\int_M \omega^n = \int_M d(\alpha \wedge (d\alpha)^{n-1}) = \int_{\partial M} \alpha \wedge (d\alpha)^{n-1} = \int_{\emptyset} \alpha \wedge (d\alpha)^{n-1} = 0$$

and this contradicts the fact that we need $\omega^n$ to be nonvanishing.