RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2012

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Geometry/Topology Prelim **January, 2012**

Q.1 Define a topology on the set $\mathbb{R}$ of real numbers by the condition that $U \subseteq \mathbb{R}$ is open if and only if it is either empty or contains the interval $[0, 1)$. Then

(a) What is the interior of the set $[0, 1]$? And its closure?

(b) Does this topology on $\mathbb{R}$ satisfy the T_0 condition?

(c) Is $\mathbb{R}$ connected in this topology?

(d) Is $\mathbb{R}$ compact in this topology?

Q.2 Consider the 2-torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle.

(a) What is the universal cover of $\mathbb{T}^2$?

(b) Describe the one-point compactification of $\mathbb{T}^2$ minus two distinct points. What is the fundamental group of the one-point compactification of $\mathbb{T}^2$ minus two distinct points?

Q.3 Prove that the singular homology $H_t(X)$ of the space $X = pt$ consisting of a single point is equal to

$$H_t(X) = \begin{cases} \mathbb{Z} & \text{if } \ t = 0 \\ 0 & \text{if } \ t > 0 \end{cases}$$

Q.4 Let M be the subset of Euclidean $\mathbb{R}^3$ defined by the zeros of the function

$$f(x, y, z) = xy - z.$$

(a) Prove that M is a submanifold of $\mathbb{R}^3$.

(b) Define a local coordinate system on M and compute the Riemannian metric induced on M by its embedding into Euclidean $\mathbb{R}^3$ in terms of these local coordinates.

Q.5 Let G be a Lie group. A vector field $\mathbf{v}$ on G is *left-invariant* if, for all $g, h \in G$,

$$(L_g)_*(\mathbf{v}|_h) = \mathbf{v}|_{gh},$$

where L_g denotes left multiplication by g.

(a) Show that the space of left-invariant vector fields on G is isomorphic to the tangent space of G at the identity.

(b) Use part (a) to prove that the tangent bundle of a Lie group is trivial. (A one-sentence description of how to construct a trivialization of the tangent bundle of G is sufficient.)

(c) Show that the vector field

$$\mathbf{v} = x \frac{\partial}{\partial x}$$

on the (abelian) Lie group $(\mathbb{R}_{>0}, \times)$ (i.e., the group of positive real numbers under multiplication) is left-invariant, and compute its flow from an arbitrary point.

Q.6 (a) State Stokes' Theorem.

(b) Let M be a smooth manifold, and $\omega \in \Omega^r(M)$ an r-form on M. Suppose that $\int_\Sigma \omega = 0$ for every r-dimensional submanifold Σ of M which is diffeomorphic to an r-sphere. Prove that $d\omega = 0$.

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1) (a) The interior of a set A is defined to be the largest open set completely contained in A. For $A = [0,1]$ under the given topology, the largest open set contained in A is $[0,1)$, so $\text{int}([0,1]) = [0,1)$. Similarly, the closure of A is the smallest closed set containing A. Under the given ~~top~~ topology we have $cl(A) = \mathbb{R}$ because any other closed set containing $[0,1]$ would have a compliment that was neither empty nor contained $[0,1)$.

(b) The T_0 condition states that for every pair of distinct points x, y there is some open set U around one of them that does not include the other. Our topology is not T_0 since any open set containing 0 or 0.5 contains the whole interval $[0,1)$ and hence contains the other point.

(c) $\mathbb{R}$ is connected in this topology because for all nonempty open sets U, V we have $[0,1) \subseteq U \cap V$.

(d) No, $\mathbb{R}$ is not compact. Consider the open cover

$$\mathcal{U} = \{[0,1) \cup \{r\} : r \in \mathbb{R}\}$$

This clearly has no finite subcover.

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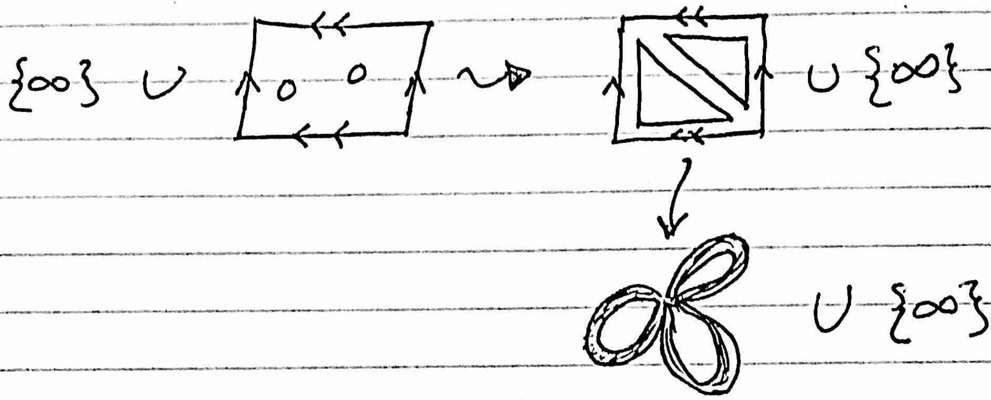
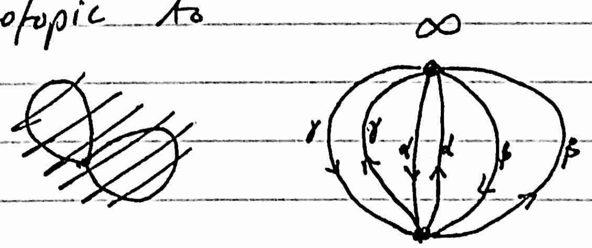
2) (a) The universal cover of $\mathbb{T}^2$ and is given by

$$p : \mathbb{R}^2 \to \mathbb{T}^2 : (x,y) \mapsto (x,y) \mod 1.$$

* (b) Fundamentally, we can describe the one point compactification of $\mathbb{T}^2$ minus two distinct points can be given in the following series of deformations



Using a deformation retraction we can then see that this space is Homotopic to



This is a graph with the following fundamental group:

$$\pi_1(X) = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = \alpha\beta = \beta\alpha = \alpha\gamma = \gamma\alpha = \beta\gamma = \gamma\beta = 1 \rangle.$$

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3) Let $C_n(X)$ denote the free abelian group with basis the sets of singular n-simplices, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be defined by

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma\big|_{(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)}$$

where v_i are the 0-simplices of σ. We know that

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Note that for $X = pt$ we have $C_n(X) = 0$ for $n \geq 1$, so

$$H_t(X) = 0 \quad \text{for } t > 0.$$

And when $t = 0$, we have

$$\ker(\partial_0) = \langle pt \rangle$$
$$\operatorname{im}(\partial_1) = 0$$

So $H_0(X) = \langle pt \rangle / 0 = \langle pt \rangle \cong \mathbb{Z}$.

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4) (a) $\mathcal{M} = \{(x, y, z) : xy - z = 0\}$. We have that

$$df = \begin{pmatrix} y & x & -1 \end{pmatrix}$$

which always has full rank, so every point $p \in \mathbb{R}$ is a regular point. Thus $\mathcal{M} = f^{-1}(0)$ is a smooth submanifold.

(b) Not done. This question Concerns Riemannian Geometry.

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5) (a) The space of left-invariant vector fields of G is called the Lie Group of G. which we will denote by $Lie(G)$. We want to show $Lie(G) \cong T_e G$. Consider the map

$$ev : Lie(G) \to T_e G : X \mapsto X_e;$$

all we need to show is that this is an isomorphism of vector spaces.

It is fairly clear that ev is linear over $\mathbb{R}$, so we now want to show ev is surjective. Let $\vec{v} \in T_e G$ be an arbitrary vector and define a vector field v^L by

$$v^L = d(L_g)_e(v)$$

Then if there was such a vector field $v^L \in Lie(G)$, it would have to be given by this formula. Thus we just need for v^L to be a smooth vector field. Well, for any $g \in G$

$$(v^L f)(g) = v^L|_g f = d(L_g)_e(v)(f) = v(f\, dg)$$
$$= v'(0)(f \circ L_g)$$
$$= \frac{d}{dt}\Big|_{t=0} (f \circ L_g \circ \gamma)(t).$$

(b) We have that $Lie(G) \cong T_e G$ is finite dimensional, so we can describe a basis for $T_e G$ and use this to trivialize $Lie(G)$ under the previosly given isomorphism ev.

(c) We have that

$$(L_g)_*(\vec{V}|_h) = (g x \tfrac{\partial}{\partial x})(h) = gh \tfrac{\partial}{\partial x} = \vec{V}|_{gh},$$

so X is left-invariant. Now if $p \in \mathbb{R}_{>0}$ we have that the flow of X from p is given by

$$\theta_t(p) = p e^t.$$

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6) (a) Let M be a smooth oriented n-manifold and let ω be a compactly supported $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega .$$

* (b) Suppose that M is smooth manifold, $\omega \in \Omega^r(M)$, and that for every r-dimensional submanifold $\Sigma \subseteq M$ we have

$$\int_\Sigma \omega = 0.$$

We know that Σ is diffeomorphic to a surface in $\mathbb{R}^r$ that satisfies the equation

$$x_1^2 + \cdots + x_r^2 = k^2 \qquad k \in \mathbb{R},$$

so it appears as the boundary of some manifold S ~~to~~ diffeomorphic to the $r+1$ dimensional disk

$$x_1^2 + \cdots + x_r^2 \leq k^2$$

which itself is diffeomorphic to $\mathbb{R}^{r+1}$. Applying Stokes's Theorem we get that

$$\int_S d\omega = \int_{\partial S} \omega = \int_\Sigma \omega = 0$$

so $d\omega = 0$.