RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$t.(x_1, \ldots, x_{n+1}) = (tx_1, \ldots, tx_{n+1}).$$

Let $\mathbb{C}\mathrm{P}^n$ be the orbit space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ with the quotient topology. Show that $\mathbb{C}\mathrm{P}^n$ is compact.

Q.2 Prove that there is no homeomorphism between $\mathbb{R}^3$ and $\mathbb{R}^2$.

Q.3 Prove that any continuous map $\mathbb{R}\mathrm{P}^2 \to S^1 \times S^1$ is homotopic to a constant map.

Q.4 Let $M^{2\times 2}$ be the space of 2×2 matrices, and let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Define a map $F : M^{2\times 2} \to M^{2\times 2}$ by

$$F(A) = A^T J A.$$

(a) At which matrices A does the differential $DF(A)$ have maximum rank?

(b) Show that the set

$$\{A \in M^{2\times 2} \mid A^T J A = A\}$$

is a submanifold of $M^{2\times 2}$.

Q.5 (a) Let M be an n-dimensional manifold. Prove that the tangent bundle TM is bundle-isomorphic to $M \times \mathbb{R}^n$ if and only if there exist n vector fields $X_1, \ldots, X_n$ on M such that the vectors $\{X_1(p), \ldots, X_n(p)\}$ are linearly independent for every $p \in M$.

(b) Let $\mathbb{T}^2 = S^1 \times S^1$ be the 2-torus. Use part (a) to show that the tangent bundle of $\mathbb{T}^2$ is isomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$.

Q.6 (a) State Stokes' Theorem.

(b) Let M be the manifold consisting of $\mathbb{R}^3$ minus the z-axis, and consider the 1-form

$$\alpha = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy + (z + 1)\, dz$$

on M.

i. Show that $d\alpha = 0$.

ii. Let C be the unit circle $x^2 + y^2 = 1$ in the xy-plane, and compute $\int_C \alpha$.

iii. Use Stokes' Theorem to show that C is not the boundary of any 2-chain in M.

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1) Consider the projection map $p: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ We want to show $\mathbb{CP}^n$ is compact. Well, for all n S^n is compact, so if we take

$$p(S^{2n+1}) = \mathbb{CP}^n.$$

Since p is a quotient map, we have that p is continuous, and since the continuous image of a compact set is compact ~~so~~ we have that $\mathbb{CP}^n$ is compact.

2) Suppose there is some homeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^2$, then there must me an induced homeomorphism $\tilde{f}: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ and a corresponding isomorphism $\tilde{f}_*: \pi_1(\mathbb{R}^3 \setminus \{0\}) \to \pi_1(\mathbb{R}^2 \setminus \{0\})$ However

$$\pi_1(\mathbb{R}^3 \setminus \{0\}) = 1 \neq \mathbb{Z} = \pi_1(\mathbb{R}^2 \setminus \{0\})$$

so such an $\tilde{f}_*$ cannot exist. Thus $\mathbb{R}^3 \not\cong \mathbb{R}^2$.

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3) Let $\varphi : \mathbb{RP}^2 \to \mathbb{T}^2$ be a continuous map, and let $p : \mathbb{R}^2 \to \mathbb{T}^2$ be the universal cover of $\mathbb{T}^2$:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, dotted, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & \mathbb{T}^2 \end{tikzcd}$$

Now, if $\varphi : \mathbb{RP}^2 \to T^2$ is a continuous map, then it induces a homomorphism $\varphi_* : \pi_1(\mathbb{RP}^2) \to \pi_1(\mathbb{T}^2)$. Well $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, so φ_* must be the trivial homomorphism. This gives us that $\varphi_*(\pi_1(\mathbb{RP}^2)) = \{0\} \subseteq p_*(\pi_1(\mathbb{R}^2))$ so there is a lifting $\tilde{\varphi} : \mathbb{RP}^2 \to \mathbb{R}$ with $\varphi = p \circ \tilde{\varphi}$.
Now, $\mathbb{R}^2$ is contractible the image of $\tilde{\varphi}$ is also contractible meaning im$(\tilde{\varphi})$ is homotopic to a constant map under some homotopy $\tilde{H}$. Well then, under the homotopy described by

$$H = p \circ \tilde{H}$$

we have that φ is homotopic to a constant map in $\mathbb{T}^2$.

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4) THIS QUESTION HAS MULTIPLE TYPOS IN IT. FOR THE CORRECT VERSION SEE QUESTION 5 ON THE AUGUST 2014 EXAM.

5) (a) Suppose M is an n-dimensional manifold with $T_pM \cong M \times \mathbb{R}^n$. Then the standard basis $(e_1, \ldots, e_n)$ produces a smooth global frame for M. which has the desired criterion.

Suppose now that there exist n vector fields $X_1, \ldots, X_n$ such that $T_pM = \text{span}\{X_1(p), \ldots, X_n(p)\}$ Well then in some neighborhood U of any point there is an isomorphism $T_pU \cong U \times \mathbb{R}^n$ and for any $v \in T_pU$ there is a representation of v given by

$$v = v^i \frac{\partial}{\partial x^i} = \tilde{v}^i X_i(p).$$

So in a neighborhood of any point we have an isomorphism φ_p from $\text{span}\{X_i\}_{i=1}^n$ to $\mathbb{R}^n$, and since these vector fields are globally defined these isomorphisms φ_p agree on all overlaps. Thus we have

$$TM \cong M \times \text{span}\{X_1, \ldots, X_n\}$$
$$\cong M \times \mathbb{R}^n.$$

(All we really did here is construct a global coordinate frame).

(b) First of all, S^1 is parallelizable, so $TS^1 = S^1 \times \mathbb{R}^1$. Then we have

$$T\mathbb{T}^2 \cong TS^1 \oplus TS^1$$
$$\cong S^1 \times \mathbb{R}^1 \times S^1 \times \mathbb{R}^1$$
$$\cong S^1 \times S^1 \times \mathbb{R}^2$$
$$\cong \mathbb{T}^2 \times \mathbb{R}^2.$$

If you want to use part (a) use the fact that S^1 is parallelizable to construct ~~of~~ the global coordinate frame $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ for $\mathbb{T}^2$.

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6) (a) Let M be a smooth connected orientable n-dimensional manifold with boundary, and let ω be ~~a~~ a compactly supported $n-1$ form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega.$$

(b) Let

$$\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + (z+1)\,dz.$$

(i) $$d\alpha = \frac{-(x^2+y^2)\,dy\wedge dx - y(2x\,dx + 2y\,dy)\wedge dx + (x^2+y^2)\,dx + x(2x\,dx+2y\,dy)\wedge dy}{(x^2+y^2)^2} + dz\wedge dz$$

$$= 0.$$

(ii) Parameterize the circle $x^2+y^2=1$ by $\gamma(t) = (\cos(t), \sin(t), 0)$ $t \in [0, 2\pi)$. Then

$$\int_\gamma \alpha = \int_{t=0}^{t=2\pi} -\sin(t)(-\sin(t)\,dt) + \cos(t)(\cos(t)\,dt)$$

$$= \int_{t=0}^{t=2\pi} dt = \boxed{2\pi}$$

(iii) Suppose $C = \partial N$ with N a 2-chain in M. Then by Stokes' Theorem we should have

$$\int_C \alpha = \int_{\partial N} \alpha = \int_N d\alpha = 0,$$

but we computed $\int_C \alpha = 2\pi$ ↯.