Department of Mathematics, University of Colorado Boulder

Topology/Geometry Preliminary Examination

August 2013

The six problems have equal points. Please do all of them.

1. Suppose C_k is a nested sequence of closed subsets of a compact space X; that is, $C_{k+1} \subset C_k$ for each $k \in \mathbb{N}$. Let $C = \cap_{k=1}^{\infty} C_k$. If U is an open set such that $C \subset U$, show that $C_k \subset U$ for some k.

2. Let $\mathbb{T}^2$ denote the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, and let M denote the quotient of $\mathbb{T}^2$ by the relation $(x, y) \equiv (-x, -y)$.
   (a) Is the quotient $Q\colon \mathbb{T}^2 \to M$ a covering map?
   (b) Express M as a quotient of a polygon with sides identified.
   (c) What is the fundamental group of M?

3. State and prove Brouwer's Fixed Point Theorem for the closed unit disc D^2.

4. Let M be a C^∞ manifold.
   (a) Define *orientability* of M.
   (b) Construct coordinate charts for the tangent bundle TM.
   (c) Show that the tangent bundle TM of a smooth manifold M is always orientable, even if M itself is not.

5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^3 + xy + y^3$.
   (a) Show that $f^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^2$.
   (b) Show that $f^{-1}(0)$ is not a smooth submanifold. (Hint: if $(x(t), y(t))$ is a curve in $f^{-1}(0)$ with $x(0) = 0$ and $y(0) = 0$, what is the condition on $x'(0)$ and $y'(0)$?) In fact you can show that $f^{-1}(0)$ is not even a topological submanifold.

6. Let M be a compact, connected, and orientable smooth manifold of dimension 6. Let α and β be two 2-forms on M. Show that there is a point of M where $d\alpha \wedge d\beta = 0$. Hint: integrate.

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1) Suppose that there is no k such that $C_k \subseteq U$. Then since the C_k are nested we have that

$$C_k \setminus U \neq \emptyset \quad \text{for all } k.$$

Define $B_k = C_k \setminus U$ and note that since the C_k are nested and closed we have that $B_{k+1} \subseteq B_k \ \forall k$ and the B_k are all closed. More importantly, since X is compact the B_k are all compact. So since for every finite subcollection $\{B_{k_i}\}_{i=1}^{n}$ $\bigcap_{i=1}^{n} B_{k_i} \neq \emptyset$ then by the finite intersection property we have that there is some b such that

$$b \in \bigcap_{k=1}^{\infty} B_k.$$

But then $b \in \bigcap_{k=1}^{\infty} C_k = C$ and $b \notin U$ while $C \subseteq U$ $\lightning$.

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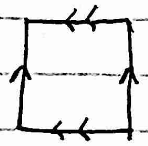
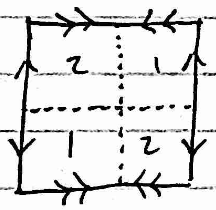
2) (a) No, $Q : \mathbb{T}^2 \to M$ is not a covering map. For Q to be a covering map we need that for any $m \in M$ there is some neighborhood V_m of m such that $Q^{-1}(V_m) = \coprod_{i \in I} U_m^i$ where each U_m^i is homeomorphic to V_m. Take $m = [(0,0)] \in M$. Then for any V_m, $Q^{-1}(V_m)$ will only have one component U_m, ~~and~~ but for each $[(x,y)] \in V_m$ $[(x,y)] \neq (0,0)$ $Q^{-1}([(x,y)]) = \{(x,y), (-x,-y)\}$ so $Q|_{U_m}$ can't be a homeomorphism from $U_m \to V_m$ which means that Q is not a covering map.

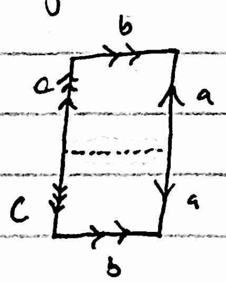
(b) Consider $\mathbb{T}^2$ given by the quotient



we have that the relation $(x,y) \sim (-x,-y)$ divides $\mathbb{T}^2$ as follows



where the items in areas 1 and 2 will be identified with each other. This allows us to rotate the left half of the diagram into the right to obtain the reduced diagram for M



(c) The quotient of the diagram in part (b) gives a sort of "puffed pillow" surface which is easily seen to be homotopic to S^2. Thus

$$\pi_1(M) = 1 .$$

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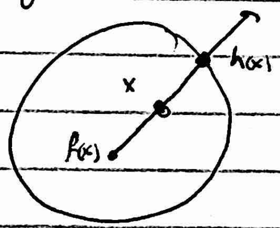
3) Every continuous function $f: D^2 \to D^2$ has at least one fixed point.

Suppose for the sake of contradiction that no such function exists. Well then every $x \in \text{Int}(D^2)$ would be assigned to another distinct point $f(x)$. If we then take the ray from $f(x)$ to x, this ray will intersect with exactly one point on $\partial D^2 = S^1$. We will call the function assigning x to this point on the boundary $h(x)$; and this function actually gives a retraction of D^2 to S^1



Note: $h(s) = s$ for $s \in S^1$

Now consider the inclusion map $\iota: S^1 \to D^2$. This is a continuous map that induces a map on the fundamental groups

$$\pi_1(S^1_*) \xrightarrow{\iota_*} \pi_1(D^2) \xrightarrow{h_*} \pi_1(S^1)$$
$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}.$$

Well then we need $h_* \circ \iota_* = id_{\mathbb{Z}}$ since $h \circ \iota(x) = x$, but this is impossible. $\lightning$.

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4) (a) A smooth manifold M is called orientable if it admits a continuous pointwise orientation. for each point $p \in M$. An orientation for M is just an equivalence class of ordered bases for $T_p M$.

(b) Let (U, φ) be a coordinate chart for M and let $\pi : TM \to M$ denote the standard projection map. Note that $\pi^{-1}(U) \subseteq TM$ is the set of tangent vectors to M at all points in U. Let $(x^1, \dots, x^n)$ denote the coordinate functions of φ and define a map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by:

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right).$$

We claim that for every (U, φ) on M, $(\pi^{-1}(U), \tilde{\varphi})$ is a smooth chart on TM. We only need to check that the transition functions are smooth. Well, given two coordinate charts $(\pi^{-1}(U), \tilde{\varphi})$, $(\pi^{-1}(V), \tilde{\psi})$ we have that

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\, v^i$$

so

$$\tilde{\psi} \circ \tilde{\varphi} = \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x)\, v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j} v^j\right)$$

which is clearly smooth.

(c) We know that a manifold M is orientable if and only if the Jacobian of the transition functions has positive determinant. Computing the Jacobian for the transition functions above, we have that

$$J = \begin{pmatrix} A & * \\ 0 & A \end{pmatrix}$$

where $A = \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)$. So $\det(J) = \det(A)^2 > 0$ and TM is orientable.

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5) (a) We have that

$$Df = (3x^2+y \quad 3y^2+x)$$

which only fails to have full rank when $x=y=0$. However $f(0,0) \neq 1$ so 1 is a regular value of f, so by the Regular Level Set Theorem we have that $f^{-1}(\{1\})$ is a smooth submanifold of $\mathbb{R}^2$.

(b) Consider the Hessian matrix of f given by

$$H = \frac{\partial^2 f}{\partial x^i \partial x^j} = \begin{pmatrix} 6x & 1 \\ 1 & 6y \end{pmatrix}$$

Then

$$H(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ which has eigen values ± 1.

So f has a saddle point at $(0,0) \in f^{-1}(0)$ meaning $f^{-1}(0)$ contains a self-intersecting curve. Then in any neighborhood of this self-intersection we have that $f^{-1}(0)$ fails to be locally Euclidean ($f^{-1}(0)$ is one dimensional so we have that the curve describes f itself), thus $f^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.

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6) First of all, note that $d\alpha \wedge d\beta = d(\alpha \wedge d\beta)$. So by Stokes Theorem we have that

$$\int_M d\alpha \wedge d\beta = \int_{\partial M} \alpha \wedge d\beta = 0 \quad \text{since } \partial M = \emptyset.$$

Suppose that $d\alpha \wedge d\beta$ never vanishes. Then since $d\alpha \wedge d\beta$ is a non-vanishing 6-form on an orientable manifold M, $d\alpha \wedge d\beta$ defines an orientation on M. But if $d\alpha \wedge d\beta$ is an orientation on M then it must be the case that

$$\left| \int_M d\alpha \wedge d\beta \right| > 0 \quad \lightning.$$