# CU Boulder: *Algebra* Prelim
# *August 2008*

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These are my solutions to the questions on the CU Boulder *Algebra* preliminary exam from *August 2008* found here. I worked on these solutions over the summer of 2019 in preparation for the preliminary exam in the Fall 2019. Please send any questions, comments, or corrections to juan.moreno-1@boulder.edu.

**Problem 1.** *(17 pts) Show that a group $G$ of order $2^3 \cdot 5 \cdot 13$ cannot be simple.*

*Proof.* By Sylow's Theorem, the number of Sylow 13-subgroups of $G$ must satisfy $n_{13} \equiv 1(\mathrm{mod} 13)$ and $n_{13} \big| 2^3 \cdot 5$. The only possibilities are $n_{13} = 1$ or 40. If $n_{13} = 40$, then since the order of these Sylow 13-subgroups is prime, these subgroups must all have trivial intersection. So we count $12 \cdot 40 = 480$ distinct elements of order 13 in $G$. Similarly the number of Sylow 5-subgroups of $G$ must satisfy $n_5 \equiv 1(\mathrm{mod} 5)$ and $n_5 \big| 104$. The only possibilities are $n_5 = 1$ or 26. If $n_5 = 26$ then by the same reasoning as above we count $4 \cdot 26 = 104$ distinct elements of order 5. If both $n_{13} = 40$ and $n_5 = 26$ then we would have $480 + 104 = 584 > |G|$ distinct elements, a contradiction. We then have that at least one of these must be 1, implying $G$ has a unique Sylow subgoup which, by Sylow's Theorem must be normal. Thus $G$ cannot be simple. □

**Problem 2.** *(17 pts) Let $G$ be a finite group which acts on a set $S$ on both the left and the right. For an element $s \in S$, let $Gs$ and $sG$ denote the orbit of $s$ under these respective actions.These actions can be combined into a single (left) action of$G \times G$ on $S$ via $(g,h)s = gsh^{-1}$. The corresponding orbit of $s$ under this action is denoted $GsG$. There are two independent questions one wants to answer about such orbits: what is their size, and how many of them are there?*
*(a) (12 pts) Show that for $s \in S$ the size of $GsG$ is*

$$|GsG| = \frac{|Gs||sG|}{|Gs \bigcap sG|}.$$

*Proof.* □