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Geometry-Topology

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2017

INSTRUCTIONS:

(1) Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
(2) Label each answer sheet with the problem number.
(3) **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. For $j = 0, 1$, let $T_j \subset \mathbb{R}^3$ be the 2-torus obtained by rotating the circle $C_j \subset \mathbb{R}^3$

$$C_j = \left\{ \begin{array}{l} x = 0 \\ (y-3)^2 + (z-3j)^2 = 1 \end{array} \right.$$

about the z-axis; endow T_j with the subspace topology induced from the standard topology on $\mathbb{R}^3$. Parameterize the circle C_j by:

$$c_j(t) = (0, 3 + \cos(t), 3j + \sin(t)),\ \ t \in [0, 2\pi],\ \ j = 0, 1.$$

Let $X := \Big(T_0 \sqcup T_1\Big)/\sim$ be the quotient space of the disjoint union

$$T_0 \sqcup T_1$$

modulo the equivalence relation $\sim$ generated by:

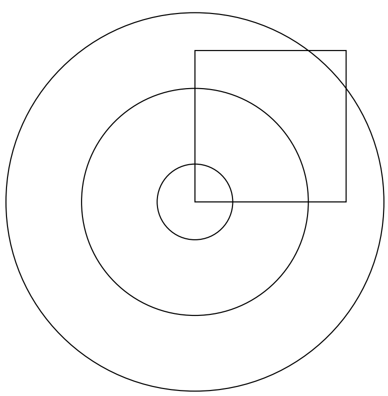
$$\forall t \in [0, 2\pi] :\ \ c_0(t) \sim c_1(t).$$

(1) What is $H_0(X)$? Explain.
(2) Use the Seifert–van Kampen Theorem to compute the fundamental group of X.

Problem 2. Let $(\mathcal{X}, \mathfrak{T})$ be a topological space, and let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of compact subsets of $\mathcal{X}$.

(1) Prove that if Λ is finite, then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is compact.
(2) Give an example of a topological space $(\mathcal{X}, \mathfrak{T})$, an infinite set Λ, and a collection $\{A_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint non-empty compact subsets of $\mathcal{X}$ such that $\bigcup_{\lambda\in\Lambda} A_\lambda$ is compact.

Problem 3. Consider the topological subspace Y of $\mathbb{R}^2$, with the subspace topology induced from the standard topology on $\mathbb{R}^2$, and formed by the lines as in the figure below:



(1) The space Y is homotopic to a bouquet of circles. How many? Explain.
(2) Is the space Y homeomorphic to a bouquet of circles? Explain.

Problem 4. Let n be a positive integer, and let (a_{ij}) be a nonzero symmetric $(n+1) \times (n+1)$ matrix with real entries a_{ij}. Consider the function

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$f(x_0, \ldots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j,$$

and let $Q := f^{-1}(0) \cap (\mathbb{R}^{n+1} - \{0\})$ be the corresponding zero set of f in $\mathbb{R}^{n+1} - \{0\}$.

(1) Use the Implicit Function Theorem to show that Q is a smooth submanifold of $\mathbb{R}^{n+1} - \{0\}$, provided that $\det(a_{ij}) \neq 0$, i.e., the determinant of the matrix (a_{ij}) is nonzero.

(2) Can the converse to part (1) fail? In other words, can Q be a smooth submanifold of $\mathbb{R}^{n+1} - \{0\}$ even if $\det(a_{ij}) = 0$? Explain.

Problem 5. Let M be a smooth manifold. Denote by $\mathfrak{X}(M)$ the vector space of smooth vector fields on M, and for each nonnegative integer k denote by $\Omega^k(M)$ the vector space of smooth k-forms on M.

(1) Let $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. It is a fact that

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}). \end{aligned}$$

Confirm this fact for 1-forms; i.e., for the case $k = 1$. The notation $\hat{X}_i$ indicates that X_i is to be omitted.

(2) Given a covariant derivative ∇ on M, there is an induced map

$$D : \mathfrak{X}(M) \times \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\begin{aligned} (D_X\omega)(X_1, \dots, X_k) = &X\left(\omega(X_1, \dots, X_k)\right) \\ &- \sum_{i=1}^{k} \omega\left(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_k\right) \end{aligned}$$

where $X, X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$; here we are denoting $D(X, \omega) = D_X\omega$. It is a fact that for $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, if ∇ is torsion-free, then

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (D_{X_i}\omega)(X_1, \dots, \hat{X}_i, \dots, X_{k+1}).$$

Confirm this fact for 1-forms; i.e., for the case $k = 1$. [Hint: Use part (1), and recall that the torsion T of ∇ is defined by $T(X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2]$.]

(3) Given a covariant derivative ∇ on M, we say that a smooth k-form $\omega \in \Omega^k(M)$ is parallel with respect to ∇ if $D_X\omega = 0$ for all smooth vector fields $X \in \mathfrak{X}(M)$. Use part (2) of this problem (without proof) to show that if a smooth k-form $\omega \in \Omega^k(M)$ is parallel with respect to a torsion-free connection ∇, then ω is d-closed.

Problem 6. Let M be a compact oriented manifold of dimension n without boundary, and let k be an integer such that $0 \le k \le n$. Show that if $\omega \in \Omega^k(M)$ is d-exact and $\eta \in \Omega^{n-k}(M)$ is d-closed, then one has

$$\int_M \omega \wedge \eta = 0.$$

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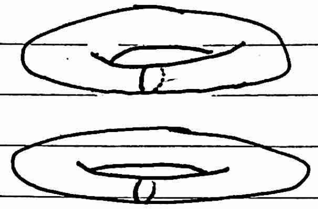
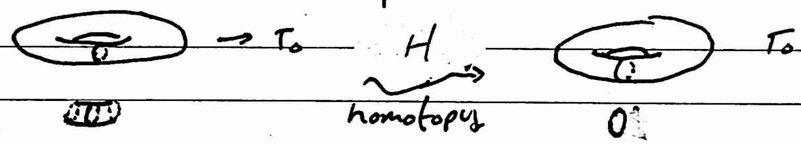
1) This topological space is given by two tori with their u-coordinate circles identified.



(a) We know that $H_0(X)$ is the free abelian group on p generators where p is the number of path components of X. Now, we have that $H_0(T_i) = \mathbb{Z}$ since the Torus is path connected, and $H_0(T_0 \sqcup T_1) = \mathbb{Z} * \mathbb{Z}$. However, since $\sim$ equates a path in T_0 with a path in T_1, both of which are path connected, we get that $T_0 \sqcup T_1/\sim$ is path connect. Hence $H_0(X) = \mathbb{Z}$.

(b) Let $U \subseteq X$ be the set such that

$\pi^{-1}(U) =$ 

and let V be defined similarly but with the Torus T_1 in it's preimage. We can pretty easily see that, since $c_0 \sim c_1$, U and V are homotopic to a torus under the homotopy $\pi \circ H$. Hence $\pi_1(U) = \langle \alpha, \beta \mid \alpha\beta = \beta\alpha \rangle$, $\pi_1(V) = \langle \gamma, \delta \mid \gamma\delta = \delta\gamma \rangle$, and $\pi_1(U \cap V) = \langle z \rangle$. Using SVK, we get that

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \frac{\langle \alpha\beta \mid \alpha\beta = \beta\alpha \rangle * \langle \gamma, \delta \mid \gamma\delta = \delta\gamma \rangle}{\langle i_{1*}(z) = i_{2*}(z) \rangle}$$

$$= \frac{\alpha, \beta \mid \alpha\beta = \beta\alpha \ * \ \gamma, \delta \mid \gamma\delta = \delta\gamma}{\langle \alpha = \delta \rangle}$$

$$= \langle \alpha, \beta, \delta : \alpha\beta = \beta\alpha, \alpha\delta = \delta\alpha \rangle$$

$$= \mathbb{Z} * \mathbb{Z} \oplus \mathbb{Z}.$$

$$\begin{tikzcd} & U \arrow[dr, "j_1"] & \\ U \cap V \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\ & V \arrow[ur, "j_2"'] & \end{tikzcd}$$

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2) (a) Suppose $\{A_\lambda\}_{\lambda\in\Lambda}$ is a collection of compact subsets with Λ finite. Define $\tilde{A} = \bigcup_{\lambda\in\Lambda} A_\lambda$ and consider any open cover $\{U_i\}_{i\in I}$ of $\tilde{A}$. For any $A_\lambda \subseteq \tilde{A}$, this is also an open cover of A_λ, so there must exist some finite subcover $\{U_j^\lambda\}_{j=1}^{n_\lambda}$. Then the collection

$$\bigcup_{\lambda\in\Lambda} \{U_j^\lambda : j = 1, \dots, n_\lambda\} \subseteq \{U_i\}_{i\in I}$$

is a finite subcover of A since it is a finite union of sets with finitely many elements.

(b) Consider $X = \mathbb{R}$ and $\tau = \{\emptyset, \mathbb{R}\}$. Then every singleton set is compact in (X, τ), and $\mathbb{R}$ is also compact. So $\{\{x\} : x\in\mathbb{R}\}$ is an infinite collection of disjoint compact sets whose union is also compact.

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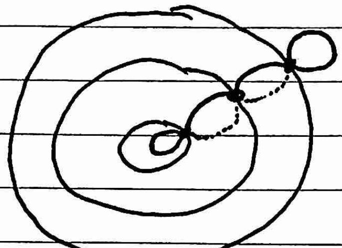
3) (a) Yes. Homotopy allows us to contract the space Y along the arcs of the circles contained within the square to produce the space



Contracting again along the dotted lines given above, we obtain a boquet of 7 circles. So Y is homotopic to a boquet of 7 circles.

(b) No. Homeomorphic spaces have isomorphic fundamental groups and Homotopic spaces have isomorphic fundamental groups. Now the wedge of n circles has fundamental group isomorphic to the free group on n generators, and we just saw in part (a) that Y is homotopic to a wedge of 7 circles, so if Y is to be homeomorphic to some boquet of circles, it would have to be the boquet of 7 circles. Suppose that there was some homeomorphism $\varphi : \bigvee_{i=1}^{7} S^1 \to Y$, and let p be the point of intersection of the 7 circles in $\bigvee_{i=1}^{7} S^1$. Then φ induces a homeomorphism $\psi : \bigvee_{i=1}^{7} S^1 \setminus \{p\} \to Y \setminus \{\varphi(p)\}$, but this is impossible since $\bigvee_{i=1}^{7} S^1 \setminus \{p\}$ has 7 path-components while $Y \setminus \{\varphi(p)\}$ will still only have 1. Thus Y is not homeomorphic to a boquet of circles.

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4) a) Suppose $\det(a_{ij}) \neq 0$, and let $U \subseteq \mathbb{R}^n \times \mathbb{R}$ contain Q with $0 \notin U$. We have that

$$\frac{\partial f}{\partial x^n} = \sum_{i=0}^{n-1} (a_{n,i} x_i + a_{i,n} x_i) + 2a_{n,n} x_n$$

$$= 2 \sum_{i=0}^{n} a_{n,i} x_i .$$

Since $\det(a_{ij}) \neq 0$ we have that no column or row of (a_{ij}) is zero. Then $\forall (\hat{x}, \hat{y}) \in Q$ $[(\hat{x},\hat{y}) \in \mathbb{R}^n \times \mathbb{R}]$, $(\hat{x},\hat{y}) \in Q$, $f(\hat{x},\hat{y}) = 0$, and

$$\left(\frac{\partial f^i}{\partial y^j} (\hat{x},\hat{y}) \right)$$

is nonsingular, so by the Implicit Function Theorem, there exist $V^{\hat{x}} \subseteq \mathbb{R}^n$, $W^{\hat{y}} \subseteq \mathbb{R}$ with $V^{\hat{x}} \times W^{\hat{y}} \subseteq U$ and a function $F_{(\hat{x},\hat{y})} : V^{\hat{x}} \to W^{\hat{y}}$ such that $f^{-1}(0) \cap V^{\hat{x}} \times W^{\hat{y}}$ is the graph of $F_{(\hat{x},\hat{y})}$. Taking $V_0 \times W_0 = \bigcup_{(\hat{x},\hat{y}) \in Q} V^{\hat{x}} \times W^{\hat{y}}$ we can then

define a function $F : V_0 \to W_0$ such that $F|_{V^{\hat{x}} \times W^{\hat{y}}} = F_{(\hat{x},\hat{y})}$

and $f^{-1}(0) \cap V_0 \times W_0 = f^{-1}(0) \cap (\mathbb{R}^{n+1} \setminus \{0\})$ is the graph of F. Thus Q appears as the graph of a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$ and is therefore a smooth submanifold of $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.

(b) The converse can fail. If $a_{ij} = 0$ whenever $i \neq n \neq j$ but $a_{ij} = 1$ if $i = n$ or $j = n$, then $\det(a_{ij}) = 0$ for $n > 2$ but

$$\left(\frac{\partial f^i}{\partial y^j} (\hat{x},\hat{y}) \right)$$

is still nonsingular for $(\hat{x},\hat{y}) \in Q$, so we can still apply the IFT to get the result.

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5) (a) If $\omega \in \Omega^1(M)$ then ω is expressable as $u\,dv = \omega$ for some smooth functions u and v. Then we have

$$d(u\,dv)(X_1, X_2) = du \wedge dv\,(X_1, X_2)$$
$$= X_1 u X_2 v - X_2 u X$$

and

$$\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_n)) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], \dots \hat{X}_i, \dots, \hat{X}_j, \dots X_{k+1})$$

$$= X_1(u\,dv(X_2)) - X_2(u\,dv(X_1)) - u\,dv([X_1, X_2])$$
$$= (X_1 u X_2 v + u X_1 X_2 v) - (X_2 u X_1 v + u X_2 X_1 v) - u[X_1, X_2] v$$
$$= X_1 u X_2 v - X_2 u X_1 v.$$

So the formula holds when ω is a 1-form.

(b) We have that

$$\sum_{i=1}^{k+1} (-1)^{i+1} D_{X_i} \omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) = D_{X_1}\omega(X_2) - D_{X_2}\omega(X_1)$$
$$= X_1(\omega(X_2)) - \omega(\nabla_{X_1} X_2) - X_2(\omega(X_1)) + \omega(\nabla_{X_2} X_1)$$

$$= X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega(\nabla_{X_1} X_2 - \nabla_{X_2} X_1)$$

Since ∇ is torsion-free, we have that $T(X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2] = 0$
so

$$X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) = (\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])$$
$$= d\omega \text{ by part (a).}$$

So the formula holds for 1-forms.

(c) Suppose ω is parallel with respect to ∇; then by part (b) we have

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} D_{X_i} \omega(X_1, \dots, \hat{X}_i, \dots, X_n)$$

$$= \sum_{i=1}^{k+1} 0$$

$$= 0$$

So ω is d-closed.

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(6) Suppose $\omega \in \Omega^k(M)$ is d-exact. Then there is some $\alpha \in \Omega^{k-1}(M)$ with $d\alpha = \omega$. Suppose in addition that $\eta \in \Omega^{n-k}(M)$ is d-closed so $d\eta = 0$. Then we have that

$$\begin{aligned} d(\alpha \wedge \eta) &= d\alpha \wedge \eta + (-1)^{k-1} \alpha \wedge d\eta \\ &= d\alpha \wedge \eta \\ &= \omega \wedge \eta \end{aligned}$$

Then if we apply Stokes' Theorem, we get that

$$\int_M \omega \wedge \eta = \int_M d(\alpha \wedge \eta) = \int_{\partial M} \alpha \wedge \eta = \int_{\emptyset} \alpha \wedge \eta = 0.$$