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1) Let $U = p(Q_1)$ $V = p(Q_2)$ (Note: $p^{-1}p(Q_i) = Q_i$ so these are open sets) and $U \cap V = p(Q_1 \cap Q_2)$. Then by SVK we can compute the fundamental group.

$$\begin{tikzcd}
& p(Q_1) \arrow[dr] & \\
p(Q_1 \cap Q_2) \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & Q_1 \cup Q_2/\sim \\
& p(Q_2) \arrow[ur] &
\end{tikzcd}$$

$$\pi_1(p(Q_1 \cap Q_2)) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle.$$

$$\pi_1(p(Q_1)) = \langle a \rangle \qquad \pi_1(p(Q_2)) = \langle b \rangle.$$

$$\iota_{1*}(\alpha) = a \qquad \iota_{2*}(\alpha) = 1$$
$$\iota_{1*}(\beta) = 1 \qquad \iota_{2*}(\beta) = b$$

So

$$\cancel{\pi_1(p(Q_1 \cap Q_2)) =}$$

$$\pi_1(X) = \pi_1(p(Q_1)) \underset{\pi_1(p(Q_1 \cap Q_2))}{*} \pi_1(p(Q_2)) = \frac{\langle a \rangle * \langle b \rangle}{\langle a = 1,\ b = 1 \rangle} = 1$$

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2) (a) Suppose that there is some homeomorphism between $\mathbb{R}$ and $\mathbb{R}^2$ given by f. Then this will induce a homeomorphism from $\mathbb{R}\setminus\{x\}$ to $\mathbb{R}^2\setminus\{f(x)\}$. But $\mathbb{R}\setminus\{x\}$ is not path-connected while $\mathbb{R}^2\setminus\{f(x)\}$ is $\lightning$. So $\mathbb{R}$ and $\mathbb{R}^2$ cannot be homeomorphic.

(b) $\mathbb{R}_{st}$ is not homeomorphic to $\mathbb{R}_{FC}$ because $\mathbb{R}_{st}$ is Hausdorff but $\mathbb{R}_{FC}$ is not.

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3) (a) We only need to show that for any ~~xxxx~~ $(x,y) \in d^{-1}((a,b))$ there is some $r>0$ such that $B_r(x) \times B_r(y) \subseteq d^{-1}((a,b))$. Define $r_1 = \frac{1}{2}(d(x,y)-a)$ and $r_2 = \frac{1}{2}(b-d(x,y))$. We have that for any $(u,v) \in B_{r_1}(x) \times B_{r_2}(y)$

$$d(x,y) \leq d(x,u) + d(u,v) + d(y,v) < r_1 + d(u,v) + r_1$$
$$= d(x,y) - a + d(u,v)$$

So $a < d(u,v)$.
Also for any $(u,v) \in B_{r_2}(x) \times B_{r_2}(y)$

$$d(u,v) \leq d(x,u) + d(x,y) + d(v,y) < b - d(x,y) + d(x,y)$$

So $d(u,v) < b$. If we then take $r = \min\{r_1, r_2\}$, we have that $B_r(x) \times B_r(y) \subseteq d^{-1}((a,b))$, so every point in $d^{-1}((a,b))$ has a basic open set around it contained in $d^{-1}((a,b))$. Thus $d^{-1}((a,b))$ is open and $d: M \times M \to \mathbb{R}$ is continuous.

(b) First of all, since A and B are compact subsets of a Metric Space, they are both closed (Metric Spaces are Hausdorff). Moreover, $A \times B$ is closed and compact in $M \times M$ so any continuous function $f: A \times B \to \mathbb{R}$ attains both a minimum and a maximum on $A \times B$. We showed in part (a) that $d: M \times M \to \mathbb{R}$ is continuous so $d|_{A \times B}$ is continuous and attains a minimum at some point (x_0, y_0). Suppose $d(A,B) = d(x_0, y_0) = 0$. Then $x = y$ and $A \cap B \neq \emptyset$ ↯. So $d(A,B) \neq 0$ and thus must be positive.

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4) (a) We have that

$$d\mu = d\omega \wedge \omega^{n-1} + \omega \wedge d\omega \wedge \omega^{n-2} + \cdots + \omega^{n-1} \wedge d\omega$$
$$= 0 + 0 + \cdots + 0 = 0.$$

(b) We have that μ is a nonvanishing $2n$-form on a $2n$-dimensional orientable manifold, μ defines an orientation on M. This gives us that

$$\left| \int_M \mu \right| > 0.$$

Suppose now that $\mu = d\beta$ for some $(2n-1)$-form β. Then we have that by Stokes' Theorem

$$\int_M \mu = \int_M d\beta = \int_{\partial M} \beta = 0 \quad \lightning.$$

(c) Suppose that $\omega = d\alpha$ for some 1-form α. Then we have that

$$\mu = d\alpha \wedge (d\alpha)^{n-1}$$
$$= d(\alpha \wedge (d\alpha)^{n-1})$$

so $\mu = d\beta$ for $\beta = \alpha \wedge (d\alpha)^{n-1}$ which contradicts part (b).

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5) Let $f: \mathbb{R}^2 \to \mathbb{R}^3 : (u,v) \mapsto (u+v,\ uv,\ u-v)$.

(a) Suppose $\omega = y\,dx + x\,dy + y\,dz$. Well

$$x(u,v) = u+v \qquad dx = du + dv$$
$$y(u,v) = uv \qquad dy = v\,du + u\,dv$$
$$z(u,v) = u-v \qquad dz = du - dv$$

So

$$f^*\omega = (2uv + v^2)\,du + (u^2 + uv)\,dv.$$

(b) No. If $f^*\omega = d\beta$ for some 0-form β, we would need

$$\beta = u^2 v + v^2 u + F(v) = u^2 v + \tfrac{1}{2}v^2 u + G(u)$$

which is impossible since the coefficients on the $v^2 u$ term cannot agree. Alternatively, you can compute

$$d(f^*\omega) = (2u - 2v - 1)\,du \wedge dv \neq 0$$

so $f^*\omega$ cannot be exact since it is not closed.

(c) Let

$$\gamma_1(t) = (t,0) \quad \gamma_2(t) = (1,t) \quad \gamma_3(t) = ((1-t),1) \quad \gamma_4(t) = (0,1-t) \quad t \in [0,1]$$

Then

$$\int_{\partial I^2} f^*\omega = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f^*\omega = \int_{\gamma_1} f^*\omega + \int_{\gamma_2} f^*\omega + \int_{\gamma_3} f^*\omega + \int_{\gamma_4} f^*\omega$$

$$= 0 + \int_0^1 1+t\,dt + \int_0^1 2(1-t)+1\,dt + 0$$

$$= \int_0^1 4-t\,dt = \boxed{3.5}$$

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6) Let $F : \mathbb{R}^3 \to \mathbb{R}^2 : (u,v,w) \mapsto (w^2 - uv, v^2 + u^2)$

Then we have that

$$DF = \begin{pmatrix} -v & -u & 2w \\ 2u & 2v & 0 \end{pmatrix}$$

which has full rank except when $u+v=0, w=0$; $u-v=0, w=0$, or $u=v=0$. So the critical points of F are the points of the form $(u,-u,0)$, $(u,u,0)$, $(0,0,w)$.

Plugging these critical points in, we get that our set of critical values of F is given by

$$S = \{(x,y) : 0 \le x,\ y = 2x\} \cup \{(x,y) : 0 \ge x,\ y = -2x\} \cup \{(x,y) : 0 \le x,\ y = 0\}$$

and $F^{-1}((a,b))$ is a smooth one-dimensional submanifold whenever $(a,b) \notin S$.