Topology Preliminary Exam Notes

This is a list of most of the definitions, theorems, and propositions contained within *Topology* 2^{nd} *edition* by Munkres as well as some extra useful ones. References made in red in this series of notes refer to the actual number of the theorem in the book.

Contents

2. Topological Spaces and Continuous Functions

- Basis for a Topology -

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

(a) $\varnothing, X \in \mathcal{T}$

(b) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(c) The intersection of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Definition. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies. Then we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$ if $\mathcal{T} \subset \mathcal{T}'$. Alternatively, we could also say that $\mathcal{T}$ is **courser** than $\mathcal{T}'$.

Definition. If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(a) For each $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$.

(b) If x belongs to the intersection of two basis elements B_1 and B_2, there is a basis element B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

Lemma 13.1. Let X be a set, let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Lemma 13.2. Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$.

Lemma 13.3. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(a) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis $B' \in \mathcal{B}'$ such that $x \in \mathbb{B}' \subset B$.

Definition. If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line.

Lemma 13.4. The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology, but are not comparable to each other.

Definition. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all finite intersections of elements of $\mathcal{S}$.

- *The Product Topology on $X \times Y$* -

Definition. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as a basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, there U is an open set of X and V is an open set of Y.

Theorem 15.1. If $\mathcal{B}$ is a basis for the topology of X and C is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Definition. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Theorem 15.2. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_1^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$. [**NB:** We will use this to generalize the product topology later on.]

- *The Subspace Topology* -

Definition. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_y = \{Y \cap U \mid\mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**.

Lemma 16.1. If $\mathcal{B}$ is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Lemma 16.2. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Theorem 16.3. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Theorem 16.4. Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

- Closed Sets and Limit Points -

Theorem 17.1. Let X be a topological space. Then the following conditions hold:

(a) $\varnothing$ and X are closed.

(b) Arbitrary intersections of closed sets are closed.

(c) Finite intersections of closed sets are closed.

Theorem 17.2. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Theorem 17.3. Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Definition. Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

Theorem 17.4. Let Y be a subspace of X; let A be a subset of Y; let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Theorem 17.5. Let A be a subset of the topological space X.

(a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Definition. If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A (at a point other than x itself).

Theorem 17.6. Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'.$$

Corollary 17.7. A subset of a topological space is closed if and only if it contains all its limit points.

Definition. **(Separation Axioms)** Let X be a topological space, then X is called

- $\boldsymbol{T_0}$ if for any pair of distinct points $x, y \in X$ there is an open set U such that $y \notin U$ or there is an open set V such that $y \in V$ and $x \notin V$

- T_1 if for any pair of distinct points $x, y \in X$ there is an open set U such that $y \notin U$ and there is an open set V such that $y \in V$ and $x \notin V$
- T_2 or **Hausdorff** if for any pair of distinct points $x, y \in X$ there are disjoint open sets U, V such that $y \notin U$ and $y \in V$ and $x \notin V$

Proposition (Class). Let $(X, \mathcal{T})$ be a topological space. Then

(a) X is T_0 if and only if for all x, y with $x \neq y$, $\overline{\{x\}} \neq \overline{\{y\}}$.

(b) X is T_1 if and only if for all $x \in X$ $\{x\} = \overline{\{x\}}$.

(c) X is T_2 if and only if $\Delta = \overline{\Delta}$ (where Δ is the diagonal of $X \times X$).

Theorem 17.8. Every finite point set in a Hausdorff space X is closed.

Definition. Let X be T_1 and let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Theorem 17.9. If X is Hausdorff, then a sequence of points of X converges to at most one point of X.

Proof. Suppose that the sequence x_n converges to some point x, and pick any $y \neq x$. Then we know that there are neighborhoods U and V with $x \in U$ and $y \in V$ and $U \cap V = \varnothing$. Since x_n converges to a point in U, then we know that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$, and thus cannot converge to anything in V. Therefore, x_n cannot converge to any $y \neq x$. ♣

Theorem 17.10. Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

- Continuous Functions -

Definition. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Theorem 18.1. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(a) f is continuous.

(b) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(c) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(d) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Definition. We say that a topological space X is **continuous at a point** x if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Definition. Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and f^{-1} are continuous maps, then we say that f is a **homeomorphism**.

Definition. Suppose that $f : X \to Y$ is an injective, continuous map. If the map $g : X \to f(X) : x \mapsto f(x)$ is a homeomorphism, then we say that f is an **imbedding** of X into Y.

Theorem 18.2. (Rules for constructing continuous functions) Let X, Y, and Z be topological spaces.

(a) (Constant function) If $f : X \to Y$ maps all of X into a single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $\iota : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f$ is continuous.

(d) (Restricting the Domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or expanding the Range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Theorem 18.3. (The pasting lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine to make the continuous function $h : X \to Y$ given by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \backslash A. \end{cases}$$

Theorem 18.4. (Maps into products) Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \qquad \text{and} \qquad f_2 : A \to Y$$

are continuous.

- The Product Topology -

Definition. Let J be an index set. Given a set X, we define a ***J*-tuple** of elements to be a function $\mathbf{x} : J \to X$. If α is an element of J, we often denote the value of $\mathbf{x}$ at α by x_α rather than $\mathbf{x}(\alpha)$; we call it the α^{th} **coordinate** of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_\alpha)_{\alpha \in J},$$

which is as close as we can come to a "tuple notation" for an arbitrary index set J. We denote the set of all J-tuples of elements of X by X^J.

Definition. Let $\{A_\alpha\}_{\alpha\in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha\in J} A_\alpha$. The **Cartesian product** of this indexed family, denoted by

$$\prod_{\alpha\in J} A_\alpha,$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha\in J} A_\alpha$$

such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Definition. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis fr a topology on the product space

$$\prod_{\alpha\in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha\in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the **box topology**.

Definition. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\},$$

and let $\mathcal{S}$ denote the union of these collections

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology $\prod_{\alpha\in J} X_\alpha$ is called a **product space**.

Theorem 19.1. The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the for $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Theorem 19.2. Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha\in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as basis for the box topology on $\prod X_\alpha$. The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all remaining indices, will serve as a basis for the product topology on $\prod X_\alpha$.

Theorem 19.3. Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Theorem 19.4. If each space X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff in both the box and the product topology.

Theorem 19.5. Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}.$$

Theorem 19.6. Let $f : A \to \prod X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

- The Metric Topology -

Definition. A **metric** on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

(a) $d(x, y) \geq 0$ for all $x, y \in X$, and equality holds only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition. Let X be a set with a metric d. Define

$$\mathcal{B} = \{B(x_0, \varepsilon) \ : \ x_0 \in X\}.$$

Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}_\mathcal{B}$ on the space X called the **metric topology**.

Definition. If X is a topologcal space. X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X.

- The Quotient Topology -

Definition. Let X and Y be topological spaces. The map $q : X \to Y$ is said to be a **quotient map** if

(a) q is surjective.

(b) If U is open in Y then $q^{-1}(U)$ is open in X.

(c) If U is a subset of Y and $q^{-1}(U)$ is open in X, then U is open in Y.

[**NB:** The last two conditions **do not** mean that q is an open map.]

Definition. We say a subset C of X is **saturated** if C contains every set $p^{-1}(\{y\})$ that it intersects.

Corollary(ish) A map $q : X \to Y$ is a quotient map if and only if q is continuous and q maps saturated open sets in X to open sets in Y.

Definition. A map $f : X \to Y$ is said to be an **open map** if for any open set $U \subset X$, $f(U)$ is open in Y. Similarly, f is a **closed map** if for C closed in X, $f(C)$ is closed in Y.

Definition. If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the **quotient topology** induced by p.

Definition. Let X be a topological space and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

Theorem 22.1. Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.

(a) If A is either open or closed in X, then q is a quotient map.

(b) If p is either an open map or a closed map, then q is a quotient map.

Theorem 22.2. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

Theorem 22.3. Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}.$$

Give X^* the quotient topology.

(a) The map g induces a bijective map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(b) If Z is Hausdorff, so is X^*.

3. Connectedness and Compactness

- Connected Spaces -

Definition. Let X be a topological space. A **separation** of X is a pair U, V of disjoint, nonempty open sets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Lemma 23.1. If Y is a subspace of X, a separation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Lemma 23.2. If the sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Theorem 23.3. The union of a collection of connected subspaces of X that have a point in common is connected.

Theorem 23.4. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Theorem 23.5. The image of a connected space under continuous map is connected.

Theorem 23.6. A finite Cartesian product of connected spaces is connected.

- Connected Subspaces of the Real Line -

Definition. A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

(a) L has the least upper bound property.

(b) If $x < y$, there exists z such that $x < z < y$.

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.

Corollary 24.2. The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Theorem 24.3. (Intermediate value theorem) Let $f : X \to Y$ be a continuous map where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of x, and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Definition. Given points x and y of the space X, a **path** in X form x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

- Components and Local Connectedness -

Definition. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the **components** (or the "connected components") of X.

Theorem 25.1. The components of X are connected, disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition. We define another equivalence relation on the space X by defining $x \sim y$ if there is a path from x to y in X. The equivalence classes are called the **path components**.

Theorem 25.2. The path components of X are path-connected, disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Definition. A space X is said to be **locally connected at x** if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be **locally connected**. Similarly, a space X is said to be **locally path-connected at x** if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, it is said simply to be **locally path-connected**.

Theorem 25.3. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Theorem 25.4. A space X is locally path-connected if an only if for every open set U of X, each component of U is open in X.

Theorem 25.5. If X is a topological space, each path component of X lies in a component of X. If X is locally path-connected, then the components and the path components of X are the same.

- Compactness -

Definition. A collection $\mathbb{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathbb{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition. **(Open Farci)** A topological space $(X, \mathcal{T})$ is called **compact** if given any open covering $\{U_i\}_{i=1}^{\infty}$, there exists a finite subcover $\{U_i\}_{i=1}^{n}$ where $X = \bigcup_{i=1}^{n} U_i$.m

Definition. **(Closed Farci)** For any given family of closed sets $\mathscr{F}$ with $\bigcap \mathscr{F} = \varnothing$, there exists a finite subfamily $\mathscr{F}'$ such that $\bigcap \mathscr{F}' = \varnothing$.

Lemma 26.1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Theorem 26.2. Every closed subspace of a compact space is compact.

Theorem 26.3. Every compact subspace of a Hausdorff space is closed.

Lemma 26.4. If Y is a compact subspace of the Hausdorff space X, and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Theorem 26.5. If $(X, \mathcal{T})$ and $(Y, \mathcal{W})$ are topological spaces, and $f : X \to Y$ is a continuous map, then $f(X)$ is compact.

Proposition (Class 10/15). Compactness is invariant under homeomorphism.

Theorem 26.6. Let $f : X \to Y$ be a bijective, continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. To show that f is a homeomorphism, we only need to show that f is a closed map. Consider some $C \in X$ which is closed. Then by Theorem 26.5 we have that $f(C)$ is compact. And since Y is Hausdorff, we have that $f(C)$ is closed since it is compact. Thus, f is a closed map, and hence a homeomorphism. ♣

Theorem 26.7. The product of finitely many compact spaces is compact. (The generalized version of this theorem is called Tychonoff's Theorem) (The proof of this involves the use of tubular neighborhoods.)

Proof. (by Farci) Fix $\mathcal{U} = \{U_\alpha \times V\alpha\}_{\alpha \in \alpha}$ and fix some $x \in X$. Consider the space $x \times Y \cong Y$ contained in the product. This space is then compact so there is some finite subcover $\{U_{\alpha_i}^x \times V_{\alpha_i}^x\}_{i=1}^n$. Take $U_x = \bigcap_{j=1}^r U_{\alpha_j}^x$. Then $\{U_x\}_{x \in X}$ is an open cover of X. Since X is compact, there is also a finite subcover $\{U_{x_i}\}_{i=1}^s$.

We know that $U = \{U_\alpha\}$ is an open cover of X and $\{V_\alpha\}$ is an open cover of Y. ♣

Lemma 26.8. (The Tube Lemma) Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood in X.

Definition. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if for ever finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9. Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Theorem (Alexander Compactification)[1]. Take a space that is not compact, but is locally compact, and T_2 (e.g. $\mathbb{R}$). Then

$$\widehat{X} = X \sqcup \{\infty\}$$

with topology $\widehat{\mathcal{T}}$ generated by

$$\widehat{\mathcal{B}} = \begin{cases} A \text{ open iff } A \in \mathcal{T} & \text{if } A \subset \widehat{X},\ \infty \notin A, \\ A \text{ open iff } A^c \text{ is compact} & \text{if } A \subset \widehat{X},\ \infty \in A. \end{cases}$$

This fives us a topological space $(\widehat{X}, \widehat{\mathcal{T}})$ which is compact.

Proof. Let $\mathcal{U} = \{\widehat{U}_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $\widehat{X}$. There is some set $\bar{\alpha} \in \Lambda$ such that $\infty \in \widehat{U}_{\bar{\alpha}}$. By construction, this gives us that $\widehat{U}^c_{\bar{\alpha}}$ is compact. Now $\{\widehat{U}_\alpha\}_{\alpha \neq \bar{\alpha}}$ is an open cover of $\widehat{U}^c_{\bar{\alpha}}$ so there is some finite subcover $\{\widehat{U}_{\alpha_k}\}_{k=1}^n$. This then gives us that

$$X = \widehat{U}_{\bar{\alpha}} \cup \bigcup_{k=1}^{n} \widehat{U}_{\alpha_k}$$

which is a finite cover for X. Hence, X is compact. ♣

- Class Notes from 10/19 and 10/22 -

Definition. A topological space is called **second countable** if it has a countable basis.

Theorem 1019.1. (Linelöff) If $(X, \mathcal{T})$ is second countable and $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of X then there is some $\mathcal{U}' \subset \mathcal{U}$ that is a countable subcover of X.

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable basis, and for all $n \in \mathbb{N}$ take $B_n \subset U_{\alpha_n}$ (if such exsts). Then we have that $\mathcal{U}' = \{U_{\alpha_n}\}$ is a countable family that and we claim that $\mathcal{U}'$ covers X. If not, there is some $z \in X$ such that $z \notin \cup U_{\bar{\alpha}}$ for some $\bar{\alpha} \in \Lambda$. Also there is some $B_{\bar{n}} \in \mathcal{B}$ such that $z \in B_{\bar{n}} \subset U_{\bar{\alpha}}$. But then we chose $\mathcal{U}'$ such that $B_{\bar{n}} \subset U_{\alpha_{\bar{n}}}$ which gives us that $z \in U_{\alpha_{\bar{n}}}$. Thus $z \in \bigcup \mathcal{U}'$. ♣

[1] This is also known as the One-Point Compactification Theorem

Theorem 1019.2. For every family of closed sets with empty intersection, there exists a countable subfamily of closed sets with empty intersection.

Theorem 1019.3. Consider $C \subset \mathbb{R}^n$ then TFAE

(a) C is closed and bounded

(b) For every sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$, there exists a convergent subsequence $x_{n_k} \to x_0 \in C$.

(c) Every infinite subset of C has an accumulation point $x_0 \in C$.

(d) C is compact.

Proof. $[(b) \Rightarrow (c)]$ We'll use the definition using families of closed sets. Let $\{F_\alpha\}_{\alpha\in\Lambda}$ with empty intersection. By Lendelöff we have that there is a countable subfamily $\mathscr{F}' = \{F_n\}_{n\in\mathbb{N}}$ with empty intersection. By contradiction, assume that there is no finite subfamily of $\mathscr{F}'$ has empty intersection. So $F_1 \cap F_2 \neq \varnothing$, $F_1 \cap F_2 \cap F_3 \neq \varnothing$,..., $F_1 \cap \cdots \cap F_n \neq \varnothing$. Take $x_1 \in F_1 \cap F_2$,..., $x_n \in F_1 \cap \cdots \cap F_n$. This is a sequence of points in C, so by (b) there is a convergent subsequence $x_n \to x_0 \in C$. But all of the F_k are closed, so we have that the sequence $x_n \to x_0 \in F_k$ for all k, and so $\bigcap_{n\in\mathbb{N}} \neq \varnothing$ $\lightning$.

$[(a) \Rightarrow (c)]$ Let $S \subset C$ be an infinite subset. By hypothesis, we have that C is bounded. Since S is a subset of C that is infinite, there is an accumulation point of S which we will call x_0 which is also an accumulation point of C. However, C is closed, so it must be the case that $x_0 \in C$.

$[(d) \Rightarrow (c)]$ Assume that C is compact. Assume that there is some infinite subset S of C with no accumulation point. In particular, we have that S is closed since all of the possible accumulation points are already contained in S. This gives us that S^c is open. So for any $x \in S$, x is not an accumulation point of S. Therefore, there is some open neighborhood U_x of x such that $U_x \cap S = \{x\}$. Let us now construct the following family:

$$\mathcal{U} = \{S^c, U_x\}_{x\in S}.$$

This is an open cover of C which is compact, so there is some finite subcover $\{S^c, U_{x_i}\}_{i=1}^n$, which means that S is finite $\lightning$.

$[(d) \Rightarrow (b)]$ Follows from $(d) \Rightarrow (c)$. Consider $S = \{x_n\}_{n\in\mathbb{N}}$. If S is a finite set, then there is some $x_i \in S$ which has countably infinitely many copies. We can then choose this to be the convergent subsequence. ♣

Definition. A topological space $(X, \mathcal{T})$ is first countable if for every $x \in X$ there is a countable basis for the neighborhoods of $x \in X$.

Ex: The real line with the standard topology is first countable. Take any $x \in \mathbb{R}$ then

$$\mathcal{B}_x = \left\{ \left(x - \frac{1}{k}, x + \frac{1}{k} \right) \right\}_{k\in\mathbb{N}}$$

- Compact Subspaces of the Real Line -

Theorem 27.1. Let X be a simply ordered set having the lease upper bound property. In the order topology, each closed interval in X is compact.

Corollary 27.2. Every closed interval in $\mathbb{R}$ is compact.

Theorem 27.3. A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric p.

Theorem 27.4. (Extreme Value Theorem) Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

Definition. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the **distance from x to A** by the equation

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Lemma 27.5. (The Lebesgue number lemma) Let $\mathbb{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathbb{A}$ containing it.

Definition. A function f from the metric space (X, d_x) to the metric space (Y, d_Y) is said to be **uniformly continuous** if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x_0,\ x_1$ of X,

$$d_X(x_0, x_1) < \delta \ \Rightarrow\ d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 27.6. (Uniform Continuity Theorem) Let $f : X \to Y$ be a continuous map from the compact metric space (X, d_x) to the metric space (Y, d_Y). Then f is uniformly continuous.

Definition. If X is a space, a point x of X is said to be an **isolated point** of X if the one point set $\{x\}$ is open in X.

Theorem 27.7. Let X be a nonempty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Corollary 27.8. Every closed interval in $\mathbb{R}$ is uncountable.

- Limit Point Compactness -

Definition. A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 28.1. Compactness implies limit point compactness, but not conversely.

Definition. Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a **subsequence** of the sequence (x_n). The space X is said to be **sequentially compact** if every sequence of points contains a convergent subsequence.

Theorem 28.2. Let X be a metrizable space. Then the following are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

4. Countability and Separation Axioms

- The Countability Axioms -

Definition. A space X is said to have a **countable basis at x** if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Theorem 30.1. Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points in A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition. If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Theorem 30.2. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition. A subset A of a space X is said to be **dense** if $\overline{A} = X$.

Theorem 30.3. Suppose that X has a countable basis. Then:

(a) Every open covering of X contains a countable subcovering.

(b) There exists a countable subset of X that is dense in X.

Definition. A space for which every open covering contains a countable subcovering is called a **Lindelöf space.**

Definition. A space having a countable dense subset is often said to be **separable**.

- The Separation Axioms -

Definition. Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be **normal** if for each pair $A,\ B$ of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Lemma 31.1. Let X be a topological space. Let one-point sets in X be closed.

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

- Normal Spaces -

Theorem 32.1. Every regular space with a countable basis is normal.

Theorem 32.2. Every metrizable space is normal.

Theorem 32.3. Every compact Hausdorff space is normal.

Theorem 32.4. Every well-ordered set X is normal in the order topology.

- The Urysohn Lemma -

Theorem 33.1. (Urysohn's Lemma) Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [a, b]$$

such that $f(x) = a$ for every $x \in A$, and $f(y) = b$ for all $y \in B$.

9. The Fundamental Group

Note: We will consider all maps to be continuous from this point forward unless otherwise stated.

- Homotopy of Paths -

Definition. If f and f' are maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ $(I = [0, 1])$such that

$$F(x, 0) = f(x) \qquad F(x, 1) = f'(x)$$

for each x. The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$, and f' is a constant map, we say that f is **nullhomotopic**.

Definition. Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be **path homotopic** if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s, 0) = f(s) \qquad &\text{and} \qquad F(s, 1) = f'(s), \\ F(0, t) = x_0 \qquad &\text{and} \qquad F(1, t) = x_1, \end{aligned}$$

for each $x \in I$ and for each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

Lemma 51.1. The relations $\simeq$ and $\simeq_p$ are equivalence relations

Example: Take $n \geq 1$ and consider $[S^n, S^0]$ = set of equivalence classes of $Maps(S^n, S^0)/h.e.$ where $h.e.$ is the homotopic equivalence relation.

We have that $S^0 = \{-1, 1\}$. This gives us that $[S^n, S^0] = \{*_{-1}, *_1\}$ where $*_{-1} = [f_{-1} : S^n \to -1 \in S^0]$ and $*_1 = [f_1 : S^n \to 1 \in S^1]$.

Definition. If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the **product** $f * g$ of f and g to be the path h given by the equations

$$h(x) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}], \\ g(2s - 1) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

Theorem 51.2. The operation $*$ on paths in X behaves as a group operation.

- The Fundamental Group -

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{W})$ be topological, let $A \subset X$ be closed, and let $f, g : X \to Y$ be maps that agree on A. Then we say $f \underset{\text{rel. A}}{\simeq} g$ (**homotopic relative to A**) if

$$H|_{X \times \{0\}} = f \quad H|_{X \times \{1\}} = g \quad \text{and} \quad H_{A \times \{t\}} = f|_A = g|_A$$

where H is a homotopy.

Note: The set A can, an often will, just be a point.

Definition. A **loop** is a map $f : O \to Y$ such that $f(0) = f(1)$.

Definition. Let $\mathscr{F} = \{\text{all loops centered at } x_0\}$. Then define, $\underset{\text{rel. }\{0,1\}}{\sim}$ as an equivalence relation on $\mathscr{F}$. We say that the **fundamental group** of X relative to the point x_0 is $\pi_1(X, x_0) = X/\underset{\text{rel. }\{0,1\}}{\sim}$.

Definition. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\alpha^{-1}] * [f] * [\alpha].$$

Theorem 52.1. The map $\hat{\alpha}$ is a group isomorphism.

Theorem (Class): Let $(X, \mathcal{T})$ $(Y, \mathcal{W})$ be topological spaces. Then fix $x_0 \in X$, $y_0 \in Y$ and consider $\pi_1(X, x_0)$, and $\pi_1(Y, y_0)$ (Recall that the group operation on these groups is concatenation of loops denoted $\alpha * \beta$ standing for $[\alpha] * [\beta] = [\alpha * \beta]$). If $f : X \to Y : x_0 \mapsto y_0$, then f induces a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which is a homeomorphism.

Theorem (Class) If $(X, \mathcal{T})$ is path connected, then for all $x_0, x_1 \in X$, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Definition. A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma 52.2. In a simply connected space X, any two paths having the same initial and final points are homotopic.

Definition. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the **homomorphism induced by $\boldsymbol{h}$**, relative to the base point x_0.

Theorem 52.3. If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $id : (X, x_0) \to (X, x_0)$ is the identity map, then id_* is the identity homomorphism.

Corollary 52.4. If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0$ with $\pi_1(Y, y_0)$.

- Covering Spaces -

Standing Hypotheses: All topological spaces are Hausdorff, path connected, and locally path connected.

Definition. Let $p : E \to B$ be a continuous, surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into **slices**.

Definition. (Farsi) Let X and $\widetilde{X}$ be topological spaces, and consider the map $\pi : \widetilde{X} \to X$. We say that π is a cover if:

(a) π is surjective

(b) These is an open cover of $X, \mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ such that

$$\pi^{-1}(U_\alpha) = \bigsqcup_{z \in Z} \widetilde{U}_{\alpha,z}$$

where the $\widetilde{U}_{\alpha,z}$ are open in $\widetilde{X}$ and $\pi|_{\widetilde{U}_{\alpha,z}} \to U_\alpha$ is a homeomorphism.

2.′ For all $x \in X$, there exists U_x open such that

$$\pi^{-1}(U_x) = \bigsqcup_{z \in Z} \widetilde{U}_{x,z}$$

which restricts to a homeomorphism.

Note: $\pi^{-1}(x)$ is discrete since

$$\pi^{-1}(U_x) = \bigsqcup_{z \in Z} \widetilde{U}_{x,z}$$

Class Example: Take $\psi : S^1 \to S^1 : z \mapsto z^3$. This is a 3-fold cover.

Definition. Let $p : E \to B$ be continuous and surjective. If every point $b \in B$ has a neighborhood U that is evenly covered by p, then p is called a **covering map**, and E is said to be a **covering space** of B.

Example: Consider $\pi : \mathbb{R} \to S^1 : t \mapsto e^{2\pi i t}$. Fix $x = 1$ and take $U_1 = S^1 \backslash \{-1\}$. Then we have that

$$\pi^{-1}(U_k) = \left(-\frac{1}{2}, \frac{1}{2}\right) + k \qquad k \in \mathbb{Z}.$$

This gives us that $\pi|_{\widetilde{U}_k} : \widetilde{U}_k \to U_k$ is a homeomorphism by calculus using ln.

- The Fundamental Group of the Circle -

Definition. Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a **lifting** of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 54.1. (Path Lifting Lemma) Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Path lifting (Farci) Let $\pi : \widetilde{X} \to X$ be a covering map. Then let $\alpha : I \to X$ be a path in X, and fix $\tilde{e} \in \widetilde{X}$, $\tilde{e} \in \pi^{-1}(\alpha(0))$. Then there exists a unique lift $\tilde{\alpha}$, $\tilde{\alpha} : I \to \widetilde{X}$ such that $\tilde{\alpha}(0) = \tilde{e}$.

Corollary (Farsi) Given $pi : \tilde{X} \to X$ a covering map, then the Path Lifting Lemma gives a function from a loop α to $\pi^{-1}(x_0)$ where $x_0 = \alpha(0)$ where $\alpha \mapsto \tilde{\alpha}(1)$.

Lemma 54.2. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting F to a continuous map

$$\widetilde{F} : I \times I \to E$$

such that $\widetilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Homotopy Lifting (Farci) Let $p : \widetilde{X} \to X : \tilde{x}_0 \mapsto x_0$ be a covering map. Define a map $F : I \times I \to X$ with $x_0 = \alpha(0) = \beta(0)$ and

$$F(x,0) = \alpha(s) \qquad \text{and} \qquad F(s,1) = \beta(s).$$

Then there exists a unique homotopy of paths $\widetilde{F} : I \times I \to \widetilde{X}$ such that

$$\widetilde{F}(0,0) = \tilde{\alpha}(0) = \tilde{x}_0.$$

Theorem (Farci) $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Proof. Let $S : \{\text{Loops in } S^1 \text{ at } 1\} \to \mathbb{Z} : \alpha \mapsto \tilde{\alpha}(1)$. We can then define $\widehat{S} : \pi_1(S^1, 1) \to \mathbb{Z}$ using the homotopy lifting lemma...... ♣

12. The Seifert-van Kampen Theorem

Definition. Push-out diagrams and amalgamated products. Let $A_1,\ A_2,\ B$ be groups with

$$
\begin{tikzcd}
B \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\
A_2 \arrow[r, "g_2"'] \arrow[drr, bend right, "h_2"'] & C \arrow[dr, dashed, "\exists!\varphi"] & \\
& & D
\end{tikzcd}
$$

where (C, g_1, g_2) is a unique solution that satisfies the universal property.

Note: If B is trivial, then C is the free product $A_1 * A_2$ of A_1 and A_2, i.e. C is the free group generated by A_1 and A_2.

Seifert-van Kampen Theorem (Baby Case) Assume that $X = A \cup B$ with $A \cap B$ path-connected, locally path-connected, and $\pi_1(A \cap B) = 1$. Then we have that $\pi_1(X, x_0) \cong \pi(A, x_0) * \pi(B, x_0)$.

Definition. Will denote the join of two spaces $X \vee Y$ where the join glues together the two spaces at exactly one point.

Seifert-van Kampen Theorem (General Case) Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let

$$\varphi_1 : \pi_1(U, x_0) \to H \qquad \text{and} \qquad \varphi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by inclusion.

$$
\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\varphi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
& \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}
$$

If $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \varphi_1$ and $\Phi \circ j_2 = \varphi_2$.

99. Extra Things

Lemma 99.1. If (U, V) is a separation of X and if Y is a connected subset of X, then $Y \subseteq U$ or $Y \subseteq V$.

Proof. Let (U, V) be a separation of X and let $Y \subseteq X$ be connected. Suppose for the sake of contradiction that $Y \nsubseteq U$ and $Y \nsubseteq Y$. Then there exists $a, b \in Y$ such that $a \in U$ and $b \in Y$. We know that (U, V) is a separation of X, so if we consider the sets $U \cap Y$ and $V \cap Y$ we know that these sets are disjoint, nonempty, and $(U \cap Y) \sqcup (V \cap Y) = Y$ which contradicts the assumption that Y was connected. Therefore either $Y \subseteq U$ or $Y \subseteq V$. ♣

Theorem 99.2. Suppose $\{A_\alpha\}_{\alpha \in J}$ is a collection of connected subsets of X, each of which intersects a connected set B. Then $B \cup (\cup_{\alpha \in J} A_\alpha)$ is connected.

Proof. Let $S = B \cup (\cup_{\alpha \in J} A_\alpha)$ be as described above. We want to show that S is connected. Suppose that it isn't. Then there is a separation (U, V) of S. B is a connected subset of S so we have that B is contained in either U or V by the previous lemma. WLOG assume that $B \subseteq U$. Now, we want to show that $B \cup (\cup_{\alpha \in J} A_\alpha) = \cup_{\alpha \in J}(B \cup A_\alpha) \subseteq U$, and, hence, that $V = \varnothing$, so if we can show that any arbitrary set $B \cup A_\alpha \subseteq U$ then we are done. To do this, we can simply show that $B \cup A_\alpha$ is connected (if $B \cup A_\alpha$ is a connected subset of S then $B \cup A_\alpha \subseteq U$ or $B \cup A_\alpha \subseteq V$ and if $B \subseteq U$ then we must have $B \cup A_\alpha \subseteq U$).

First, suppose for the sake of contradiction that $B \cup A_\alpha$ is disconnected. Well then, there exists a separation (W, Z) of $B \cup A_\alpha$, and, since both B and A_α are connected subsets we have that they are either contained in W or contained in Z by the previous lemma. We also know that there is some $p \in B \cup A_\alpha$. Now $W \cup Z$ covers $B \cup A_\alpha$, so we must have that either $p \in W$ or $p \in Z$. WLOG assume that $p \in W$. Well then, since p is in both B and A_α we know that $B, A_\alpha \subseteq W$, and we get that $W = B \cup A_\alpha$ and $Z = \varnothing$ ↯. So every $B \cup A_\alpha$ is connected and we have that every $B \cup A_\alpha \subseteq U$ which means that S is connected.

♣

100. More on Covering Spaces

This is coming soon.