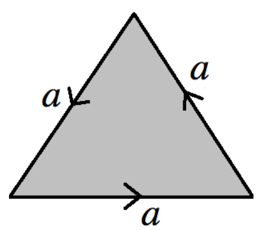
GEOMETRY/TOPOLOGY PRELIMINARY EXAM
AUGUST 2011

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Define an equivalence relation on $\mathbb{R}^2$ by $x \sim y$ if and only if $f(x) = f(y)$. Let X be the quotient space.
   (a) Show that X is always Hausdorff.
   (b) Must X be connected?
2. Let D^2 denote the unit disc in $\mathbb{R}^2$ with the unit circle S^1 its boundary. If $f\colon D^2 \to D^2$ is a homeomorphism, show that the restriction $f|_{S^1}$ is a homeomorphism onto S^1. (Hint: one way to do this is to assume it is not and obtain a contradiction by considering fundamental groups.)
3. Consider the quotient space Q formed by identifying the sides of a triangle T as in the diagram.



   (a) Is Q a topological manifold? (An intuitive explanation is sufficient.)
   (b) Use the Seifert-van Kampen theorem to compute the fundamental group of Q.
4. A *contact form* on a three-dimensional manifold M is a C^∞ 1-form on M such that $\alpha \wedge d\alpha$ is nowhere zero. A *Reeb field* for a contact form is a C^∞ vector field X on M such that $\alpha(X) = 1$ everywhere and $d\alpha(X, Y) = 0$ everywhere for every C^∞ vector field Y on M.
   (a) Prove that $\alpha = (\cos z)\, dx + (\sin z)\, dy$ is a contact form on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.
   (b) Show that there is a unique Reeb field for this contact form, and compute it.

(c) Describe the flow of this Reeb field. Is it periodic?

5. Let a and b be real numbers with $a > 0$. Consider the set M_{ab} of 2×2 matrices $A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ satisfying $w^2+x^2+y^2+z^2 = a$ and $wz - xy = b$. Show that if $a \neq 2|b|$, then M_{ab} is a smooth submanifold of $\mathbb{R}^4$.

6. Suppose M is an annulus $[a, b] \times S^1$, for numbers $b > a > 0$, with C^∞ Riemannian metric given in polar coordinates $r \in [a, b]$ and $\theta \in S^1$ by $ds^2 = dr^2 + \varphi(r)^2 \, d\theta^2$ for some function φ. Let ∇ denote the usual Levi-Civita covariant derivative, and let R denote the radial vector field $R = r \frac{\partial}{\partial r}$.

   (a) Find all vector fields of the form $V = f(r) \frac{\partial}{\partial r} + g(r) \frac{\partial}{\partial \theta}$ satisfying $\nabla_R V = 0$ everywhere on M.

   (b) How does your answer change if M is a disc rather than an annulus, with the same metric? (Hint: what does $\varphi(r)$ look like near the origin $r = 0$?)

Top / Diff Geo Prelim Exam August 2011.

1) (a) We want to show that X is Hausdorff. Consider the following diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "q"'] \arrow[dr, "f"] & \\ X = \mathbb{R}^2/\sim \arrow[r, dashed, "g"'] & \mathbb{R} \end{tikzcd}$$

$x \sim y$ if $f(x) = f(y)$

Since f is constant on the fibers $q^{-1}([x])$, we know that f induces a continuous map g s.t. $g \circ q = f$. Now consider $[x], [y] \in X$ with $[x] \neq [y]$. Then $g([x]) \neq g([y])$ and since $g([x]), g([y]) \in \mathbb{R}$ which is Hausdorff, $\exists$ open sets U, V in $\mathbb{R}$ with $g([x]) \in U$, $g([y]) \in V$, and $U \cap V = \emptyset$. Well, g is continuous, so $g^{-1}(U)$ and $g^{-1}(V)$ are open, $[x] \in g^{-1}(U)$, $[y] \in g^{-1}(V)$, and $g^{-1}(U) \cap g^{-1}(V) = \emptyset$. Thus, X is Hausdorff.

(b) Suppose that X is not connected so $X = U \sqcup V$ with U and V open. Then $\mathbb{R}^2 = q^{-1}(U) \sqcup q^{-1}(V)$, so $\mathbb{R}^2$ is disconnected $\lightning$.

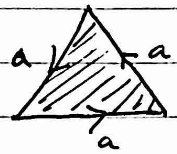
Top / Diff Geo Prelim Exam August 2011

2) Suppose there exists a homeomorphism $f: D^2 \to D^2$ such that $f|_{S^1}$ is not a homeomorphism. Then, there exists some $s \in S^1$ such that $f(s) \notin S^1$. If we then consider the spaces $D^2 \setminus \{x\}$ and $D^2 \setminus \{f(x)\}$, there is an induced homeomorphism $\tilde{f}: D^2 \setminus \{x\} \to D^2 \setminus \{f(x)\}$. Consider some fixed point $y \in D^2$ $x \neq y \neq f(x)$. We know that since $x \in \partial D^2$ and $f(x) \in \mathrm{int}(D^2)$, $D^2 \setminus \{x\}$ is homotopic to D^2 and $D^2 \setminus \{f(x)\}$ is homotopic to S^1, so

$$\pi_1(D^2 \setminus \{x\}, y) = 1, \quad \pi_1(D^2 \setminus \{f(x)\}, y) = \mathbb{Z},$$

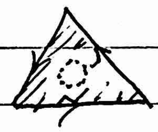
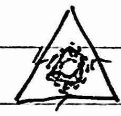
but then $\tilde{f}_*: 1 \to \mathbb{Z}$ cannot be an isomorphism and hence $\tilde{f}$ is not a homeomorphism $\Rightarrow\Leftarrow$. Thus $f(x) \in S^1$ and $f|_{S^1}$ is a homeomorphism.

Top / Diff Geo Prelim Exam August 2011

3) $Q =$ 

* (a) Q is not a topological manifold because it fails to be locally Euclidean where the corners meet. If Q were to be a topological manifold, it would need to have dimension 2. Therefore, any open neighborhood of the meeting place of the three corners would need to be homeomorphic to $\mathbb{R}^2$. However, with the way that the equivalence relation is defined, any open neighborhood of the point where the corners meet would self-intersect which doesn't happen in $\mathbb{R}^2$ even when embedded in a higher dimensional space.

(b) Let

$U =$  $V =$  $U \cap V =$ 

Then we can apply Seifert-Van Kampen (SVK) to get that

$$\pi_1(X) = \mathbb{Z}/3\mathbb{Z}.$$

To see this, consider the diagram

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "j_{1*}"] & \\ \langle \alpha \rangle = \pi_1(U \cap V) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(X) \\ & \pi_1(V) \arrow[ur, "j_{2*}"'] & \end{tikzcd}$$

Since $\pi_1(U) = \langle a \rangle$ $\pi_1(V) = 1$ $i_{1*}(\alpha) = a^3$ $i_{2*}(\alpha) = 1$
$\pi_1(U \cap V) = \langle \alpha \rangle$

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \frac{\langle a \rangle * 1}{\langle a^3 = 1 \rangle} = \langle a \mid a^3 = 1 \rangle = \mathbb{Z}/3\mathbb{Z}.$$

Top / Diff Geo Prelim Exam August 2011

4) (a) $\alpha = \cos(z)\,dz + \sin(z)\,dy$

$d\alpha = -\sin(z)\, dz \wedge dx + \cos(z)\, dz \wedge dy$

$$\alpha \wedge d\alpha = \cos(z)^2\, dx \wedge dz \wedge dy - \sin(z)^2\, dy \wedge dz \wedge dx$$
$$= \cos(z)^2 + \sin(z)^2\, dx \wedge dz \wedge dy$$
$$= dx \wedge dz \wedge dy \neq 0. \ \forall p \in T^3$$

So α is a contact form.

(b) Suppose that there were two Reeb fields for α A, B. Well if $\alpha(A) = \alpha(B)$ then we have that for every point (x, y, z)

$$\cos(z)\, dx(A) + \sin(z)\, dy(B) = \cos(z)\, dx(B) + \sin(z)\, dy(B).$$

If we then pick the points $(0,0,0)$ and $(0,0,\pi/2)$ we get

$$dx(A) = dx(B) \text{ and } dy(A) = dy(B).$$

Similarly, we would have

$$d\alpha(A,B) = -\sin(z)\, dz \wedge dx\,(A,B) + \cos(z)\, dz \wedge dy\,(A,B)$$

and by using the relation $dx(B) = dx(A)$ derived previously and evaluating at $(0,0,\pi)$ we have

$$(dz(A) - dz(B))\, dx(A) = 0.$$

We can also use $dy(B) = dy(A)$ and $(0,0,0)$

$$(dz(A) - dz(B))\, dy(A) = 0,$$

and putting this all together, we get that

$$dz(A) = dz\, B$$

so $A \equiv B$.

(cont.) 4) (b) For X to be the Reeb field of α we need $\alpha(X)=1$ and $d\alpha(X)=0$, so solving the set of equations that this gives us we have that the Reeb field of α is

$$X = \cos(z)\frac{\partial}{\partial x} + \sin(z)\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}.$$

(c) Taking X as above, we can compute that the flow of X is given by

$$\Theta_t(x,y,z) = (x+\cos(z)t,\ y+\sin(z)t,\ z)$$

and this is not periodic because if $(x_0,y_0,z_0) \neq (x_1,y_1,z_1)$

$$(x_0+\cos(z_0)t,\ y_0+\sin(z_0)t,\ z_0) \neq (x_1+\cos(z_1)t,\ y_1+\sin(z_1)t,\ z_1)$$

since any ~~coordinates~~ values z_0, z_1 that could result in equality of the first two coordinates will result in inequality of the third.

Top / Diff Geo Prelim Exam August 2011

5) Consider the canonical isomorphism $\mathcal{M}_{2\times 2} \to \mathbb{R}^4 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,b,c,d)$ and define the map F by
$$F(w,x,y,z) = (w^2+x^2+y^2+z^2,\ wz - xy).$$

Then we have that

$$DF = \begin{pmatrix} 2w & 2x & 2y & 2z \\ z & -y & -x & w \end{pmatrix}$$

and this matrix has rank <2 only when all of the following are zero:

$$-2wy - 2xz,\ -2wx - 2yz,\ 2w^2 - 2z^2,$$
$$-2x^2 + 2y^2,\ 2xw + 2xz,\ 2wy + 2xz$$
(Determinants of all 2×2 minors)

This never happens so long as $a \neq 2|b|$ so each point in the set
$$S = \{(a,b) : a \neq 2|b|\}$$
is a regular value of F so $\forall (a,b) \in S$

$$M_{ab} = F^{-1}(\{(a,b)\})$$

is a smooth submanifold of $\mathbb{R}^4$ by the Regular Level Set Theorem.