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1. Let X be a topological space.

   (a) Prove that if X is compact, then any closed subset of X is also compact.

   (b) Define "X is Hausdorff".

   (c) Prove that if X is compact and Hausdorff, then X is regular. That is, show that given any point $x \in X$ and closed set $V \subset X$ not containing x, there exists disjoint open sets U_1 and U_2 such that $x \subset U_1$ and $V \subset U_2$.

Solution:

(a) If X is compact, then any open cover of X has a finite subcover. Let $A \subset X$ be closed, and $\{U_i\}_{i \in I}$ be an open cover of A. Then $\{U_i\} \cup X \setminus A$ is an open cover of X (as A is closed, $X \setminus A$ is open). By compactness of A, there is a finite subcover $\{U_i\}_{i=1}^n \cup X \setminus A$, and so $\{U_i\}_{i=1}^n$ is a finite subcover of the cover $\{U_i\}_{i \in I}$ of A.

(b) X is Hausdorff if given any two distinct points $x, y \in X$, there exist disjoint open sets U_x, U_y with $x \in U_x$ and $y \in U_y$.

(c) Let X be compact and Hausdorff, and $x \in X$, $V \subset X$ closed with $x \notin V$. Then as X is Hausdorff, there exist two families of open sets $\{U_y\}_{y \in V}, \{V_y\}_{y \in V}$ such that $U_y \cap V_y = \emptyset$, $x \in U_y$ and $y \in V_y$ (note that this works because we never have $y = x$ as $x \notin V$). As X is compact and V is closed in X, by part (b), V is also compact. The $\{V_y\}$ are an open cover of V and so there is a finite subcover $\{V_{y_i}\}_{i=1}^n$. Then $\tilde{V} = \bigcup_{i=1}^n V_{y_i}$ is an open neighborhood of V, as it's a cover of V and an open set as the union of open sets. Moreover, $U_{y_i} \cap V_{y_i} = \emptyset$ for all i, and so $\tilde{V} \cap (\bigcap_{i=1}^n U_{y_i}) = \emptyset$. As the *finite* intersection of open sets, $U = \cap_{i=1}^n U_{y_i}$ is an open set containing x (as x is in each U_{y_i}, it's in their intersection). Hence U and $\tilde{V}$ are the desired disjoint open neighborhoods of $\{x\}$ and V. As $V \subset X$ closed and $x \notin V$ were arbitrary, X is regular.

2. (a) Let X be a topological space and A and B be open subsets of X. Suppose that $A \cap B$ and $A \cup B$ are connected. Prove that A and B are both connected.

   (b) State the Seifert Van Kampen Theorem

   (c) Let X be a connected three dimensional topological manifold. Let x and y be distinct points of X. prove that the inclusion $X \setminus \{x\} \subset X$ induces an isomorphism:

$$\pi_1(X \setminus \{x\}, y) \cong \pi_1(X, y)$$

   (Hint: You may use the fact that if a topological space is locally path connected and connected, then it is path connected.)

Solution:

(a) Note that $A \cap B$ is non-empty, else $A \cup B$ would be a separation of $A \cup B$, which would mean that $A \cup B$ is not connected. Suppose that A is not connected. then there exists a surjective function $f \colon A \to 2$, where 2 is a 2-point discrete space. As $A \cap B$ is connected, f is constant on $A \cap B$, and so we extend f to B by defining it to take the same value on B as it does on $A \cap B$. Then we have a definition of f on A and B, which are open sets, which agree on their overlap $A \cap B$, and so by the pasting lemma, this extension of f is a continuous surjective function from $A \cup B$ to a discrete space. Hence $A \cup B$ is not connected, which is a contradiction. Similarly, by exchanging the roles of A and B, we get that B not connected $\implies A \cup B$ not connected. Hence both A and B are connected.

(b) Let $(X, x_0) = A \cup B$, where A, B and $A \cap B$ are open and path-connected, with $x_0 \in A \cap B$. Then:

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$

Where the amalgamation of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ is given by the maps induced by the inclusions $A \cap B \to A$ and $A \cap B \to B$.

(c) Let X be a topological 3-manifold. Then each point $x \in X$ has a neighborhood U_x homeomorphic to an open 3-ball. Hence any neighborhood N of $x \in X$ contains a path connected neighborhood, as this is true in $\mathbb{R}^3$ (either N contains a nbhd homeomorphic to $\mathbb{R}^3$ which is path connected, or N is inside this nbhd, so we can apply the homeomorphism $\varphi \colon U_x \to \mathbb{R}^3$ to find a path connected nbhd V of $\varphi(x)$ contained in $\varphi(N)$, as $\mathbb{R}^3$ is locally path-connected. Then $\varphi^{-1}(V)$ is a path connected nbhd of x contained in N) . Hence X is locally path-connected and connected, and hence path-connected. Thus the fundamental group of X is independent of basepoint.

We now apply the Seifert Van Kampen Theorem to compute $\pi_1(X, y)$. We decompose X as $X = U \cup V$, where $U = X \setminus \{x\}$ and V is an open nhbd of x homeomorphic to $\mathbb{R}^3$. Note that U is open as it's the complement of a closed set: all manifolds are Hausdorff, and so singleton $\{x\}$ are closed $T_2 \implies T_1$, and V is open by definition. $U \cap V$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ which is path connected. $U \cap V = X$ is connected by assumption. Hence by (a), both U and V are connected. As open subsets of manifolds, U and V are submanifolds, and so are also locally path-connected and hence path-connected. Note that $y \neq x$ so we can choose $y \in U \cap V$. $U \cap V$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$, which is homotopic to S^2, which is simply connected (by SVK on $S^2 \setminus \{N\}\, S^2 \setminus \{S\}$). Hence $\pi_1(U \cap V, y) = 0$. V is homoemorphic to $\mathbb{R}^3$, which is simply connected as it's contractible by straight line homotopy. Hence SVK tells us that:

$$\pi_1(X, x_0) \cong \pi_1(U, y) *_0 0 = \pi_1(X \setminus \{x\}, y)$$

Morever, SVK tells us that this isomorphism is induced by the inclusion of U into X, and so we're done.

3. Let $n \geq 2$. Consider the real projective n-space $\mathbb{R}P^n$ obtained from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by identifying the points x and $-x$ for every $x \in S^n$.

(a) Compute the fundamental group of $\mathbb{R}P^n$.

(b) Classify the path connected covering spaces of $\mathbb{R}P^n$ up to isomorphism.

(c) Prove that every continuous map $f: \mathbb{R}P^2 \to S^1$ is nullhomotopic.

Solution:

(a) Consider the $\mathbb{Z}_2$ action on S^n given by $1(x) = x, -1(x) = -x$. First we observe that S^n is simply connected for $n \geq 2$. Then we note that this action is free: for $x = gx \iff g = 1$, for $x = -x$ is only true in $\mathbb{R}^n$ is $x = 0$, but $0 \notin S^n$. It also acts properly discontinuously: Given any $x \in S^n$, pick an open neighborhood U_x of x small enough so that it covers less than half of S^n. Then $U_x \cap -U_x = \emptyset$. Hence by the theorem linking fundamental groups and group actions, we have that:

$$\pi_1(S^n/\mathbb{Z}_2) = \mathbb{Z}_2$$

But the orbit space $S^n/\mathbb{Z}_2$ is precisely $\mathbb{R}P^n$, for the orbits of $\mathbb{Z}_2$ in S^2 are $\{x, -x\} = [x]$, where $[x]$ is the equivalence class of x considered as any element of the quotient space $\mathbb{R}P^n$.

(Note that $\mathbb{R}P^n$ is path connected as it's the quotient (and hence the image under a continuous map) of the path connected space S^n. Hence we could ignore basepoints).

Alternatively: Show that the quotient map is a covering map and hence a universal covering map as S^n is simply connected, and then use the fact that fundamental group of base space is in bijection with fibres of the universal covering map (in this case the fibres are $\{x, -x\}$. Then we can use that $\mathbb{Z}^2$ is the only 2-element group.

(b) We use that there is a bijection between isomorphism classes of covering spaces of $\mathbb{R}P^n$ and subgroups of $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. The only subgroups of $\mathbb{Z}_2$ are 0 and $\mathbb{Z}_2$ itself, and hence there are only two isomorphism classes. $\mathbb{Z}_2$ corresponds to the cover $S^n \to \mathbb{R}P^2$ given in the definition of $\mathbb{R}P^n$. 0 corresponds to the identity covering $id: \mathbb{R}P^n \to \mathbb{R}P^n$.

(c) Suppose that $f: \mathbb{R}P^2 \to S^1$. Then $f_*: \pi_1(\mathbb{R}P^2) \to \pi_1(S^1)$, i.e. $f_*: \mathbb{Z}_2 \to \mathbb{Z}$, so f_* is the zero map, as $f_*(1)$ must be a torsion element of $\mathbb{Z}$, which is torsion-free. Now if we consider $p: \mathbb{R} \to S_1$, we have that $0 f_*(\mathbb{R}P^2) \subset p_*(\pi_1(\mathbb{R})) = 0$, so by the map lifting lemma, f lifts to a map $\tilde{f}: \mathbb{R}P^2 \to \mathbb{R}$. Then as $\mathbb{R}$ is contractible, $\tilde{f}$ is homotopic to the constant map at some $c \in \mathbb{R}$ via a homotopy H, which implies that $f = p \circ \tilde{f}$ is homotopic to the constant map at $p(c)$ via the homotopy $p \circ H$.

4. Let G be a Lie group.

(a) Describe a linear isomorphism between the vector space of left invariant vector fields on G and the tangent space at the identity. Prove that your map is a linear isomorphism.

(b) Prove that the tangent bundle of a Lie group is trivial.

(c) Is there a group structure on S^2 so that the 2-sphere becomes a Lie group?

Solution:

(a) Give a left-invariant vector field V, the given isomorphism is given by $V \to V(e)$, where e is the identity element of G. It is linear, as $V + \lambda W \to (V + \lambda W)(e) = V(e) + \lambda W(e)$. The inverse is given by $\phi\colon T_eG \to \mathrm{Lie}(G)$, $\phi(v)|_g = (dL_g)_e(v)$. This field is left invariant, as:

$$d(L_{gg'}(\phi(v)|_{g'})) = (dL_g)_{g'}(dL_{g'})_e(v) = (dL_{gg'})_e(v) = \phi(v)|_{gg'}$$

It is an inverse to the evaluation at e map, as $\phi(v)|_e = (dL_e)_e(v) = id_{T_eG}(v) = v$, and if V is left invariant, then $V_g = (dl_g)_e V_e = \phi(V_e)$. The inverse is smooth as its action on any smooth function is smooth: Given $f \in C^\infty(G)$, and $\gamma\colon I \to G$ s.t. $\gamma(0) = e, \gamma'(0) = v$, we calculate:

$$(\phi(v)f)_g = \phi(v)_g f = (dL_g)_e v f = (dL_g)_e \gamma'(0) f = \gamma'(0)(f \circ L_g) = \frac{d}{dt}\Big|_{t=0} (f \circ L_g \circ \gamma)(t)$$

This shows that for $g \in G$, $\phi(v)f_g = \partial\psi/\partial t(0, g)$, where $\psi(t, g) = f \circ L_g \circ \gamma(t)$, which is the composition of smooth functions, and hence $\partial\psi/\partial t(0, g)$ depends smoothly on g, i.e. $\phi(v)f$ is smooth. Hence ϕ is a *smooth* inverse to evaluation at e, and hence is an isomorphism between T_eG and *smooth* left-invariant vector fields on G.

(b) If G is a Lie group, then any basis for T_eG gives a global frame on G, by the correspondence in (a). Local frames on an open subset U of a manifold give rise to smooth local trivializations $\pi^{-1}(U) \to U \times \mathbb{R}^n$, where $\pi\colon TM \to M$ is the canonical projection. hence this *global* frame gives us a smooth trivialization $\pi^{-1}(G) = TG \to G \times \mathbb{R}^n$. This means that TG is trivial, as it admits a smooth global trivialization.

(c) Suppose S^2 were a Lie group. Then a basis of T_eS^2 would give rise to a global frame on S^2. Pick one element of this frame, call it V. Then as V_p is part of a basis of T_pS^2 for any $p \in S^2$, $V_p \neq 0$ for any $p \in S^2$. Hence V is a non-vanishing vector field on S^2. But the Hairy Ball Theorem forbids this. Hence there is no Lie group structure on S^2.

5. (a) Define differential k-forms and what it means for a form to be *closed* and to be *exact*

(b) Prove that any closed 1-form on $\mathbb{R}^2$ is exact.

(c) Give an example of a 1-form on the two-dimensional torus $S^1 \times S^1$ that is closed but not exact.

Solution:

(a) A differential k-form on M is a smooth section of the bundle of alternating vector fields on M, $\Lambda^k T^*M$. A $k-$form ω is *closed* if $d\omega = 0$, and ω is *exact* if $\omega = d\eta$ for some $(k-1)$ form η.

(b) By De Rham's Theorem, $H^1_{dR}(\mathbb{R}^2) \cong H^1_{\text{sing}}(\mathbb{R}^2, \mathbb{R})$. The latter is trivial, for it is equivalent to $\mathrm{Hom}_{\mathrm{Grp}}(H_1(\mathbb{R}^2), \mathbb{R}))$. By homotopy invariance, $H_1(\mathbb{R}^2) \cong H_1(\{*\})$, the first homology group of the one-point space. This is trivial, as there's only one p-simplex in $\{*\}$ for any $p \geq 0$, namely the constant map. Then the boundary maps alternate between isomorphisms and zero maps, giving 0 in all degrees but zero. Hence $0 \cong \mathrm{Hom}_{\mathrm{Grp}}(0, \mathbb{R}) = \mathrm{Hom}_{\mathrm{Grp}}(H_1(\mathbb{R}^2), \mathbb{R})) \cong H^1_{dR}(\mathbb{R}^2)$. Hence the 1st de-Rham homology group of $\mathbb{R}^2$ is trivial, which says that all closed forms on $\mathbb{R}^2$ are exact. One could also use homotopy invariance of de-Rham cohomology, and observe that the only $1-$forms on a single point are zero, as a point is a 0-dimensional manifold, so has 0 as its (co)tangent space (so a 1-form is a linear functional on 0, and thus is the zero map).

(c) There is a 1-form on S^1 which we shall call $d\theta$, which is given by the restriction of $\frac{xdy-ydx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ to S^1. This can easily be seen to be closed. There are also two smooth projection maps $p_1 \colon S^1 \times S^1 \to S^1, \quad p_1(a,b) = a$, and $p_2 \colon S^1 \times S^1 \to S^1, \quad p_2(a,b) = b$. Then $p_1^*(d\theta)$ is a smooth 1-form on $S^1 \times S^1$. Moreover, as d commutes with pullbacks, $d(p_1^*(d\theta)) = p_1^*(d(d\theta)) = p_1^*(0) = 0$, so that $p_1^*(d\theta)$ is closed. If $p_1^*(d\theta)$ is exact, then stokes says that its integral must be 0 around any smooth closed path in $S^1 \times S^1$, as $\partial(S^1 \times S^1) = 0$. But if we consider the path $\gamma(t) = ((\cos(t), \sin(t)), (0,0))$ for $t \in [0, 2\pi]$, a quick calculation shows that:

$$\int_\gamma p_1^*(d\theta) = \int_{p_1 \circ \gamma} d\theta = \int_0^{2\pi} \cos(\theta) d(\sin\theta) - \sin(\theta) d(\cos\theta) d\theta$$
$$= \int_0^{2\pi} \left(\cos^2\theta + \sin^2\theta\right) d\theta = \int_0^{2\pi} 1 d\theta = 2\pi \neq 0$$

And hence $p_1^*(d\theta)$ is *not* exact.

6. In $\mathbb{R}^n$ consider the Euler vector field

$$V \colon \mathbb{R}^n \to T\mathbb{R}^n, \quad x \mapsto V(x) = x^1\partial_1 + \cdots + x^n\partial_n$$

(a) Let $c \in \mathbb{R}$ and $f \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a c-homogeneous smooth function, i.e.

$$f(\lambda x) = \lambda^c f(x), \quad \text{for all } \lambda > 0, x \in \mathbb{R}^n \setminus \{0\}$$

Show that $V(f) = cf$

(b) Find the flow of the Euler vector field.

(c) Prove that the only 1-form on $\mathbb{R}^n$ invariant under the flow of V is the trivial 1-form $\alpha = 0$.

Solution:

(a) We calculate:

$$\begin{aligned}
f(\lambda x_1, \lambda x_2, \dots \lambda x_n) &= \lambda^c f(\lambda x_1, \dots \lambda x_n) \\
\frac{\partial}{\partial \lambda} f(\lambda x_1, \dots \lambda x_n) &= \frac{\partial}{\partial \lambda}(\lambda^c f(\lambda x_1, \dots \lambda x_n)) \\
\left.\frac{\partial f}{\partial x_i}\right|_{(\lambda x_1, \dots, \lambda x_n)} \frac{\partial \lambda x_i}{\partial \lambda} &= c\lambda^{c-1} f(\lambda x_1, \dots \lambda x_n) \\
x_i \left.\frac{\partial f}{\partial x_i}\right|_{(\lambda x_1, \dots, \lambda x_n)} &= c\lambda^{c-1} f(\lambda x_1, \dots \lambda x_n)
\end{aligned}$$

(summation convention is in effect). Setting $\lambda = 1$, this gives:

$$x_i \frac{\partial}{\partial x_i} f(x_1, \dots, x_n) = cf(x_1, \dots, x_n)$$

But notice that we can write any $x \in \mathbb{R}^n \setminus \{0\}$ as $x = (x_1, \dots, x_n)$, and that:

$$(Vf)(x) = x_i \frac{\partial}{\partial x_i} f(x) = x_i \frac{\partial}{\partial x_i} f(x_1, \dots, x_n) = cf(x_1, \dots, x_n) = cf(x)$$

This holds for all x, and thus $Vf = cf$.

(b) We calculate that $\theta_t(x) = x + tx$ satisfies:

$$\theta_0(x) = x, \quad \theta'_x(0) = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} = V(x)$$

and:

$$\theta_t \circ \theta_s(x) = \theta_t(x + sx) = (x + sx + tx) = x + (t+s)x = \theta_{t+s}(x)$$

And so θ is the flow of the Euler vector field.

(c) Let α be a 1-form on $\mathbb{R}^n$. Then α is a covector field. If α is invariant under θ iff $\mathcal{L}_V \alpha = 0$. Then by Cartan's magic formula:

$$0 = \mathcal{L}_V \alpha = i_V d\alpha + d(i_V \alpha) = d(i_V \alpha)$$

Where we used that all 1-forms on $\mathbb{R}^n$ are exact, as $\mathbb{R}^n$ is contractible. Hence $i_V \alpha$ is a closed smooth function on $\mathbb{R}^n$. As $\mathbb{R}^n$ is connected, this means that $i_V \alpha$ is constant. Say this constant is c. This tells us that, if we write $\alpha = \alpha_j dx^j$ (this is possible as the dx^i are *global* coframes on $\mathbb{R}^n$):

$$c = i_V(\alpha)(e_i) = \alpha(V(e_i)) = \alpha\left(\frac{\partial}{\partial x_i}\right) = \alpha\left(\frac{\partial}{\partial x^i}\right) = a_j dx^j \frac{\partial}{\partial x_i} = a_j \delta^j_i = \alpha_i$$

However, we also have:

$$c = i_V(\alpha)(2e_i) = \alpha(V(2e_i)) = \alpha\left(2\frac{\partial}{\partial x_i}\right) = 2\alpha\left(\frac{\partial}{\partial x^i}\right) = 2a_j dx^j \frac{\partial}{\partial x_i} = 2a_j \delta^j_i = 2\alpha_i$$

So $\alpha_i = 2\alpha_i$ for all i, and so $\alpha_i = 0$ for all i, i.e. α is the trivial 1-form.