Geometry/ Topology January 2020

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1. Suppose that $\{X_i\}_{i\in I}$ is a collection of topological spaces.

   (a) Prove that for each $j \in I$, the projection map $p_j\colon \prod_{i\in I} X_i \to X_j$ defined via $(x_i)_{i\in I} \mapsto x_j$ is continuous with respect to both the product and the box topology. (Recall that the box topology on $\prod_{i\in I} X_i$ is the topology generated by the basis

   $$\beta_{\text{box}} \left\{ \prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ for each } i \in I \right\}.)$$

   (b) Suppose that Y is a topological space and $f\colon Y \to \prod_{i\in I} X_i$ is a function. Prove that f is continuous with respect to the product topology if and only if for each $j \in I, p_j \circ f\colon Y \to X_j$ is continuous.

   (c) Suppose that $I = \mathbb{N}$ and $X_i = \mathbb{R}$ for each $i \in \mathbb{N}$. Prove that the function $f\colon \mathbb{R} \to \prod_{i\in I} X_i$ defined by $t \mapsto (t, t, t, \dots)$ is continuous with respect to the product topology.

   (d) Prove that the function $f\colon \mathbb{R} \to \prod_{i\in I} X_i$ defined the the previous part is **not** continuous with respect to the box topology.

Solution:

(a) First we note that the box topology is finer than the product topology, so if U is open in the product topology, it is open in the box topology. As the domain of p_j has the product/ box topology, this means that we merely need to show that p_j is continuous in the product topology, from which continuity in the box topology immediately follows. Recall that the the product topology has basis:

$$\beta_{\text{prod}} = \left\{ \prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ for each } i \in I, \text{ and } U_i = X_i \text{ for all but finitely } i \right\}$$

Let U_j be open in X_j. Then $p_j^{-1}(U_j) = (U_i)_{i\in I}$, where $U_i = X_i$ for $i \neq j$. This is a basis element in the product topology, so is open in the product topology (and hence open in box as well). As U_j was an arbitrary open set in X_j, this means that p_j is continuous, as the preimage of open sets are open. Hence p_j is continuous (for any $j \in I$, as j was arbitrary).

(b) Suppose f is continuous with respect to the product topology. Then p_j is continuous in the product topology by (a), and hence $p_j \circ f$ is continuous as it's the composition of continuous functions. Now suppose $p_j \circ f$ is continuous for all $j \in I$. We show that f is continuous by showing that the preimage of basis elements are open. Let $\prod_{i\in I} U_i$ be open in the product topology. Notice that $p_j^{-1}(U_j) = (\prod_{i\neq j} X_i) \times U_j$, and $(p_j \circ f)^{-1}(U_j) = f^{-1}(p_j^{-1}(U_j))$. If $\prod U_i$ is any open set in the product topology, where $U_i = X_j$ unless $i \in A$, where A is some finite subset of I, then $\prod U_i = \bigcap_{j\in A} p_j^{-1}(U_j)$, which is open as it's the *finite* intersection of open sets. Then $f^{-1}(\prod U_i) = f^{-1}(\bigcap_{j\in A} p_j^{-1}(U_j)) = \bigcap_{j\in A} f^{-1}(p_j^{-1}(U_j)) = \bigcap_{j\in A}((p_j \circ f)^{-1}(U_j))$, which is open in Y as it's the finite intersection of open sets by continuity of all $p_j \circ f$. Hence f is continuous as the preimages of basis elements in $\prod X_i$ are open in Y.

(c) For all $j \in I$, $p_j \circ f$ is the identity map on $\mathbb{R}$, so f is continuous by part (b).

(d) We show that f is not continuous at 0: $f(0) = (0, 0, \dots)$. Consider the neighborhood $U = \prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ of $f(0)$. If f is continuous at 0, then there exists a neighborhood V of $0 \in \mathbb{R}$ such that $f(V) \subset U$. So suppose V exists. Then V contains some basis element of the topology on $\mathbb{R}$, so we can assume V is an open interval of the form $(-\delta, \varepsilon)$ for some $\delta, \varepsilon > 0$, as if V maps into U then so do any subsets of V. Moreover, by taking the minimum of δ, ε, and replacing δ, ε by their minimum, we can assume that $\delta = \varepsilon$. Then $f(V) = \prod_{n \in \mathbb{N}}(-\varepsilon, \varepsilon)$. By picking $n \geq \frac{1}{\epsilon}$, we have that $(-\varepsilon, \varepsilon) \not\subset \left(-\frac{1}{n}, \frac{1}{n}\right)$, so that $f(V) \not\subset U$. Hence V cannot exist, so f is not continuous in the box topology.

2. Recall that S^n denotes the n-sphere and B^n denotes the closed n-ball. Suppose that to every continuous map $h \colon S^n \to S^n$ we have assigned an integer called its degree (denotes by deg(h)) such that the degree has the following properties:

(a) Homotopic maps have the same degree.

(b) $\deg(h \circ k) = \deg(\mathrm{h}) \deg(\mathrm{k})$

(c) The identity map has degree one, any constant map has degree zero and the reflection map $h(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1})$ has degree minus one.

Note: You do not need to prove the degree exists, rather you can assume it exists and has the three properties (i)-(iii).

Prove the following:

(a) There is no retraction $f \colon B^{n+1} \to S^n$.

(b) If $h \colon S^n \to S^n$ has degree different from $(-1)^{n+1}$ then h has a fixed point.

(c) If $h \colon S^n \to S^n$ has degree different from one, the there exists $x_0 \in S^n$ such that $h(x_0) = -x_0$.

Solution:

(a) Suppose that there is a retraction $f \colon B^{n+1} \to S^n$. Then by definition, $f \colon i = Id_{S^n}$, where i is the inclusion of S^n into B^{n+1}. Then i is homotopic to the constant map at $0 \in B^{n+1}$, which we will call c_0, via $H(x,t) = xt$. Then $Id_{S^n} = f \circ i \sim f \circ c_0 = c_{f(0)}$, the constant map at $f(0)$. Then using (i), and taking degrees on both sides using (iii), we get that $1 = deg(Id_{S^n}) = deg(c_{f(0)}) = 0$, a contradiction. Hence no retraction f exists.

(b) We use the contrapositive. If h has no fixed point then $h(x) \neq x$ for all $x \in S^n$. Then h is homotopic to the antipodal map $a(x) = -x$ via the homotopy:

$$H(x,t) = \frac{(1-t)h(x) - tx}{|(1-t)h(x) - tx|}$$

The denominator is never zero, for if it were we'd have $tx = (1-t)h(x)$, then taking norms and using the fact that $x, h(x) \in S^n$, we get that $|1-t| = |t|$, so $1 - t = t \implies t = \frac{1}{2}$, where we used that $t \in [0,1]$. Then we get that $\frac{1}{2}x = \frac{1}{2}h(x) \implies x = h(x)$, contrary to assumption. Hence H is continuous, and also $H_0(x) = h(x), H_1(x) = a(x)$, so H

is a homotopy from a to h. Hence by (i), h has the same degree as a. Now note that $a = r_1 \circ \cdots \circ r_{n+1}$, where $r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots x_{n+1})$. Then r_i is homotopic to r_{n+1} via the homotopy $H\colon S^n \times [0, \pi] \to S^n, H(x,t) = (x_1, \ldots, \cos(t)x_i + \sin(t)x_{n+1}, \ldots, -\cos(t)x_{n+1} + \sin(t)x_i)$ (one can check that $|H(x,t)| = 1$ for all (x,t)). (iii) tells us that r_{n+1} has degree -1, so by (i), all r_i have degree -1. a is the composition of all $(n+1)$ r_i, and hence by (ii) has degree $(-1)^{n+1}$. Hence we've proved that a map of degree differing from $(-1)^{n+1}$ has a fixed point.

(c) Again, we use the contrapositive. If $h(x) \neq -x$ for all x, then we have a homotopy from h to the identity map, given by:

$$H(x,t) = \frac{(1-t)h(x) + tx}{|(1-t)h(x) + tx|}$$

(verifying that this is a homotopy is essentially the same as the working in part (b)). Hence by (iii) and (i), h has degree 1. So if h has degree differing from 1, then $h(x_0) = -x_0$ for some $x_0 \in S^n$.

3. Recall that a topological group, G, is a topological space that is also a group with the property that the maps $G \times G \to G$ defined by multiplication and $G \to G$ defined by $g \mapsto g^{-1}$ are continuous functions.

Prove the following theorem:

Suppose that $\tilde{G}$ and G are connected topological groups and the the map $\rho\colon \tilde{G} \to G$ is a both a covering map and a group homomorphism. Then $\tilde{G}$ is abelian if and only if G is abelian.

Solution: The forwards implication is straightforward. ρ is a covering map, and hence is surjective. So suppose $\tilde{G}$ is abelian and take $g, h \in G$. Then $g = \rho(\tilde{g}), h = \rho(\tilde{h})$ for some $\tilde{g}, \tilde{h} \in \tilde{G}$. Then $gh = \rho(\tilde{g})\rho(\tilde{h}) = \rho(\tilde{g}\tilde{h}) = \rho(\tilde{h}\tilde{g}) = \rho(\tilde{h})\rho(\tilde{g}) = hg$, where we used that ρ is a group homomorphism, and that $\tilde{G}$ is abelian. As g, h were arbitrary, G is abelian.

Now suppose G is abelian. Consider the map $[,]\colon \tilde{G} \times \tilde{G} \to \tilde{G}$ given by $[\tilde{g}, \tilde{h}] = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$, the commutator map. Then the commutator is continuous as it's a composition of inversion and multiplication. Connectedness of $\tilde{G}$ implies that $\tilde{G} \times \tilde{G}$ is connected, and so $[\tilde{G}, \tilde{G}]$ is connected in $\tilde{G}$ as it's the continuous image of a connected space. Now note that for any $\tilde{g}, \tilde{h}$, $\rho([\tilde{g}, \tilde{h}]) = \rho(\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}) = \rho(\tilde{g})\rho(\tilde{h})\rho(\tilde{g})^{-1}\rho(\tilde{h})^{-1} = e_G$, as G is abelian and ρ is a group homomorphism. Hence $[\tilde{G}, \tilde{G}] \subset \rho^{-1}(e_G)$. As ρ is a covering map, $\rho^{-1}(e_G)$ is a discrete set of points in $\tilde{G}$. Hence $[\tilde{G}, \tilde{G}]$ is a single point in $\rho^{-1}(e_G)$, as it's connected. But $[e_{\tilde{G}}, e_{\tilde{G}}] = e_{\tilde{G}}$, so this point is $e_{\tilde{G}}$, the identity in $\tilde{G}$. Hence $[\tilde{G}, \tilde{G}] = e_{\tilde{G}}$ i.e. $\tilde{G}$ is Abelian.

4. Let $M = \mathbb{R}^2$, and consider the vector fields:

$$X = (x+1)\frac{\partial}{\partial x} - (y+1)\frac{\partial}{\partial y}, \quad Y = (x+1)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}$$

on M.

(a) Show that there exist local coordinates (s,t) in some neighborhood U of the point $(1,0)$ such that the restriction of X and Y to U are given by $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

(b) Find such coordinates explicitly, and verify directly that they satisfy the conditions $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

Solution:

(a) First we show that X and Y are commuting vector fields: indeed, using the formula:

$$[X,Y] = (XY^j - YX^j)\frac{\partial}{\partial x_j}$$

we compute that:

$$[X,Y] = ((x+1)-(x+1))\frac{\partial}{\partial x} + (-(y+1)-(-(y+1)))\frac{\partial}{\partial y} = 0$$

Also, letting $p = (1,0)$, we find that $X_p = 2\frac{\partial}{\partial x}|_p - \frac{\partial}{\partial y}|_p$, and $Y_p = 2\frac{\partial}{\partial x}|_p + \frac{\partial}{\partial y}|_p$, which are linearly independent as the vectors $(2,-1)$ and $(2,1)$ are linearly independent in any basis, and $\left\{\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p\right\}$ form a basis for T_pM.
Hence by the theorem for canonical form of commuting vector fields, there exists local coordinates (s,t) in a neighborhood U of p s.t. $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

(b) We find these coordinates explicitly by acting with the flow θ of X and the flow ψ of Y on $p = (1,0)$. First we find the flows by computing integral curves: Suppose $\gamma = (f(t), g(t))$ is an integral curve for X. Then:

$$f'(t)\frac{\partial}{\partial x} + g'(t)\frac{\partial}{\partial y} = X_{\gamma(t)} = (f(t)+1)\frac{\partial}{\partial x} - (g(t)+1)\frac{\partial}{\partial y}$$

So we need to solve $f'(t) = f(t)+1$ and $g'(t) = -g(t)-1$. Solutions are given by $f(t) = ae^t - 1$ and $g(t) = be^{-t} - 1$. The flow $\theta_t(x,y)$ is given by $\theta_{(x,y)}(t)$ being an integral curve for X starting at (x,y). Hence

$$\theta_{(x,y)}(t) = ((x+1)e^t - 1, (y+1)e^{-t} - 1)$$

, as $\theta_{(x,y)}(t)$ is an integral curve for X starting at (x,y). Similarly, the flow of Y is given by:

$$\psi_{(x,y)}(t) = ((x+1)e^t - 1, (y+1)e^t - 1)$$

Now, to find the coordinates (s,t) in terms of (x,y) we calculate:

$$\begin{aligned}(x,y) &= \theta_s \circ \psi_t(1,0) = \theta_s(2e^t - 1, e^t - 1)\\ &= (2e^t e^s - 1, e^t e^{-s} - 1)\\ &= (2e^{t+s} - 1, e^{t-s} - 1)\end{aligned}$$

which tells us that:

$$\begin{aligned}t + s &= \ln\left(\frac{x+1}{2}\right)\\ t - s &= \ln(y+1)\end{aligned}$$

From which we add and subtract these two equations to get that $(s,t) = (\frac{1}{2}(\ln(\frac{x+1}{2}) - \ln(y+1)), \frac{1}{2}(\ln(\frac{x+1}{2}) + \ln(y+1)))$. To verify that these are the correct coordinates, we use $\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y}$, which gives:

$$\begin{aligned}
\frac{\partial}{\partial s} &= 2e^t e^s \frac{\partial}{\partial x} + -e^{t-s}\frac{\partial}{\partial y} = (x+1)\frac{\partial}{\partial x} - (y+1)\frac{\partial}{\partial y} = X \\
\frac{\partial}{\partial t} &= 2e^{t+s}\frac{\partial}{\partial x} + e^{t-s}\frac{\partial}{\partial y} = (x+1)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y} = Y
\end{aligned}$$

5. Let $a, b \in \mathbb{R}$, and consider the subset S of $\mathbb{R}^3$ defined by the equations

$$x^2 - z^2 = a^2, \quad x^2 + y^2 + z^2 + b^2$$

(a) Show that if $a, b \neq 0$ and $a^2 \neq b^2$, then S is a regular manifold of $\mathbb{R}^3$.

(b) Describe the set S when $a = b = 1$. Is it a regular submanifold of $\mathbb{R}^3$?

Solution:

(a) (Note that regular submanifold is another name for embedded submanifold).

We show that S is an embedded submanifold by exhibiting it as the regular level set of a smooth function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^2$, defined by $\Phi(x,y,z) = (x^2 - z^2, x^2 + y^2 + z^2)$. Then $S = \Phi^{-1}(a^2, b^2)$, and Φ is smooth because its coordinate functions are polynomials. It remains to be shown that (a^2, b^2) is a regular value of Φ when $a, b \neq 0$ and $a^2 \neq b^2$.

We compute:

$$d\Phi_{(x,y,z)} = \begin{pmatrix} 2x & 0 & -2z \\ 2x & 2y & 2z \end{pmatrix}$$

We perform row reduction to reduce this matrix to:

$$\begin{pmatrix} 2x & 0 & -2z \\ 0 & 2y & 4z \end{pmatrix}$$

Which has rank 2 if no more than one of $x, y, z = 0$. So if $x = y = 0$ we have $-z^2 = a^2$ so $z = a = 0$, which is not allowed. If $x = z = 0$ we have again that $a = 0$, contrary to assumption. If $y = z = 0$ then $x^2 = a^2$ and $x^2 = b^2$ i.e. $a^2 = b^2$, again contrary to assumption. So the only points where $d\Phi$ does not have full rank are points *not* in $S = \Phi^{-1}(a^2, b^2)$. Hence S is a regular level set of Φ, and so is a properly embedded submanifold of $\mathbb{R}^3$.

(b) Suppose $a = b = 1$. Then $x^2 - z^2 = 1 = x^2 + y^2 + z^2$. Hence $z^2 = x^2 - 1$ and so $2x^2 + y^2 = 2$ i.e. $y^2 = 2(1 - x^2)$. But of course $x^2, y^2, z^2 > 0$, so the only possible values of x^2 in this solution set satisfy $x^2 \geq 1$ and $x^2 \leq 1$, i.e. $x^2 = 1$, so $x = \pm 1$, which in turn implies that $y^2 = z^2 = 0$, so $S = \{(\pm 1, 0, 0)\}$. This is an embedded 0-dimensional submanifold of $\mathbb{R}^3$, as it's the disjoint union of two points, so the inclusion of S into $\mathbb{R}^3$ is an immersion as it's an inclusion map so it smooth, and the differential is trivially injective as 0-dimensional manifolds have the zero space as their tangent spaces. It is also embedded as the subspace topology is the discrete topology, which gives a manifold structure.

6. Let M be a compact, oriented n-dimensional manfiold without boundary. A *volume form* on M is a nowhere, vanishing n-form Ω on M with the property that $\int_M \Omega > 0$.

(a) Show that every volume form Ω on M is closed.

(b) Show that a volume form Ω on M cannot be exact.

Solution:

(a) If Ω is an n-form on M, then $d\Omega$ is an $(n+1)$-form. But as M is a dimension n manifold, any $(n+1)$ form is 0, as at each point $p \in M$, an $(n+1)$-form at that point is an alternating $(n+1)$-linear map on T_pM, which has dimension n, i.e. $d\Omega_p(v_1, \ldots v_{n+1}) = 0$ for any choice of v_i, as if $\{v_1, \ldots v_{n+1}\} \in T_pM$, then the v_i are not linearly independent, and so $d\Omega$ evaluated on this $(n+1)$ tuple is 0 (as $(d\Omega)_p$ is alternating). Hence $d\Omega = 0$ i.e. Ω closed.

(b) Suppose Ω is a volume form that is exact. Then $\Omega = d\eta$ for some $(n-1)$-form η. As M is compact, η is compactly supported, and as M is also orientable, Stokes' theorem applies and tells us that:

$$\int_M \Omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0$$

contradicting $\int_M \Omega > 0$. Hence a volume form Ω cannot be exact.