Analysis Preliminary Exam Notes

This is a list of most of the definitions, theorems, and propositions contained within *Real Analysis* 2^{nd} *edition* by Gerald B. Folland as well as some extra useful ones. References made in red in this series of notes refer to the actual number of the theorem in the book.

Contents

1. Measures

Definition. Let X be a nonempty set. An ***algebra*** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and compliments. In other words, if $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_1^n E_j \in \mathcal{A}$; and if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$. A ***σ-algebra*** is an algebra that is closed under countable unions.

Definition. If X us any set, $\mathcal{P}(X)$ and $\{\varnothing, X\}$ are σ-algebras. If X is uncountable, then

$$\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$$

is a σ-algebra, called the ***σ-algebra of countable or co-countable sets.***

Definition. If $\mathcal{E}$ is any subset of of $\mathcal{P}(X)$, there is a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$. $\mathcal{M}(\mathcal{E})$ is called the σ-algebra ***generated*** by $\mathcal{E}$.

Lemma 1.1. If $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

Definition. Let X be a metric space. The σ-algebra generated by the family of open sets in X is called the ***Borel σ-algebra*** on X and is denoted by $\mathcal{B}_X$. Its members are called ***Borel sets.***

Notation. There is a standard terminology for the hierarchy of Borel sets. A countable intersection of open sets is called a $\boldsymbol{G_\delta}$ set; a countable union of closed sets is called a $\boldsymbol{F_\sigma}$ set; a countable union of G_δ sets is called a $\boldsymbol{G_{\delta\sigma}}$ set; a countable intersection of F_σ sets is called a $\boldsymbol{F_{\sigma\delta}}$ set; and so forth with δ corresponding to countable intersections and σ corresponding to countable unions.

Proposition 1.2. $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

1. the open intervals: $\mathcal{E}_1 = \{(a, b) \mid a < b\}$,
2. the closed intervals: $\mathcal{E}_2 = \{[a, b] \mid a < b\}$,
3. the half-open intervals: $\mathcal{E}_3 = \{[a, b) \mid a < b\}$ or $\mathcal{E}_4 = \{(a, b] \mid a < b\}$,
4. the open rays: $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$,
5. the closed rays: $\mathcal{E}_7 = \{[a, \infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] \mid a \in \mathbb{R}\}$.

Definition. Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the ***product σ-algebra*** on X is the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha, \ \alpha \in A\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$

Proposition 1.3. If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ-algebra generated by $\left\{\prod_{\alpha \in A} E_\alpha \mid E_\alpha \in \mathcal{M}_\alpha\right\}$.

Proposition 1.4. Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha, \ \alpha \in A$. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha, \ \alpha \in A\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all α, $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \left\{\prod_{\alpha \in A} E_\alpha \mid E_\alpha \in \mathcal{E}_\alpha\right\}$.

Proposition 1.5. Let $X_1, \ldots, X_n$ be metric spaces and let $\prod_1^n X_j$, equipped with the product metric. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset B_X$. If the X_j's are separable, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.

Corollary 1.6. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$.

Definition. An ***elementary family*** is a collection $\mathcal{E}$ of subsets of X such that

- $\varnothing \in \mathcal{E}$,
- if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
- if $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

Proposition 1.7. If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

Definition. Let X be a set equipped with a σ-algebra $\mathcal{M}$. A ***measure*** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

i. $\mu(\varnothing) = 0$,

ii. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigsqcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$.

Property (ii) is called ***countable additivity***. It implies ***finite additivity***:

ii′. if $E_1, \ldots, E_n$ are disjoint sets in $\mathcal{M}$, then $\mu(\bigsqcup_1^n E_j) = \sum_1^n \mu(E_j)$.

A function μ that satisfies (i) and (ii′) but not necessarily (ii) is called a ***finitely additive measure***

Definition. If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a ***measurable space*** and the sets in $\mathcal{M}$ are called ***measurable sets***. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a ***measure space***.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(X) < \infty$, then μ is called ***finite***. If $X = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, μ is called ***σ-finite***. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, μ is called ***semifinite***.

Theorem 1.8. Let $(X, \mathcal{M}, \mu)$ be a measure space

a. **(Monotonicity)** If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

b. **(Subadditivity)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.

c. **(Continuity from below)** If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

d. **(Continuity from above)** If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \supset E_2 \supset \cdots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

Definition. If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a ***null set***

Definition. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true ***almost everywhere*** (abbreviated ***a.e.***), or for ***almost every*** x.

Definition. A measure μ whose domain in $\mathcal{M}$ includes all subsets of null sets is called ***complete***.

Theorem 1.9. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} \mid \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Definition. The measure in Theorem 1.9 is called the ***completion*** of μ, and $\overline{\mathcal{M}}$ is called the ***completion*** of $\mathcal{M}$ with respect to μ.

Definition. An ***outer measure*** of an nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

- $\mu^*(\varnothing) = 0$,
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,
- $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.

Proposition 1.10. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \rho(E_j) \ : \ E_j \in \mathcal{E} \ and \ A \subset \bigcup_1^\infty E_j \right\},$$

then μ^* is an outer measure.

Definition. Let μ^* be an outer measure on a set X. A set $A \subset X$ is called ***μ^*-measurable*** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X.$$

Theorem 1.11. (Carathédory's Theorem) if μ^* is an outer measure on X, the collection $\mathcal{M}^*$ of μ^*-measureable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Definition. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a ***premeasure*** if

- $\mu_0(\varnothing) = 0$,
- if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$.

Proposition 1.13. Let μ_0 be a premeasure on $\mathcal{A}$ and define

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \mu_0(A_j) \ : \ A_j \in \mathcal{A} \ and \ E \subset \bigcup_1^\infty A_j \right\}.$$

Then we have that

- $\mu^*|_{\mathcal{A}} = \mu_0$;
- every set in $\mathcal{A}$ is μ^*-measurable.

Theorem 1.14. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu = \mu^*|_{\mathcal{M}}$ where μ^* is the same as in the previous proposition. If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

Definition. Measures on $\mathbb{R}$ whose domain is the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ are called ***Borel measures*** on $\mathbb{R}$.

Notation. We shall call all subsets of $\mathbb{R}$ of the form $(a, b]$, (a, ∞), or $\varnothing$ where $-\infty \leq a < b < \infty$ ***h-intervals*** (h for "half-open").

Proposition 1.15. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing an right continuous. If $(a_j b_j]$ (j = 1..n) are disjoint h-intervals, let

$$\mu_0\left(\bigcup_1^n (a_j b_j]\right) = \sum_1^n [F(b_j) - F(a_j)],$$

and let $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$ the collection of finite disjoint unions of h-intervals.

Theorem 1.16. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((-x, 0]) & \text{if } x < 0, \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Definition. Let F be a increasing, right continuous function, and let μ_F be its associated Borel measure. Then the completion $\overline{\mu}_F$ (which will often be denoted by μ_F as well) is called the ***Lebesgue-Stieltjes*** measure associated to F.

Notation. For the remainder of this section, we will fix a complete Lebesgue-Stieltjes measure μ on $\mathbb{R}$ associated to the increasing, right continuous function F, and we denote $\mathcal{M}_\mu$ to be the domain of μ.

Lemma 1.17. For any $E \in \mathcal{M}_u$,

$$\mu(E) = \inf\left\{\sum_1^\infty \mu((a_j, b_j)) \ : \ E \subset \bigcup_1^\infty (a_j, b_j)\right\}.$$

Theorem 1.18. If $E \in \mathcal{M}_\mu$, then

$$\begin{aligned} \mu(E) &= \inf\{\mu(U) \mid U \supset E \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(K) \mid K \subset E \text{ and } K \text{ is compact}\}. \end{aligned}$$

Theorem 1.19. If $E \subset \mathbb{R}$, the following are equivalent:

a. $E \in \mathcal{M}_u$

b. $E = V \backslash N_1$ where V is a G_δ set and $\mu(N_1) = 0$

c. $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

Proposition 1.20. If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \varepsilon$.

Definition. Let $F : \mathbb{R} \to \mathbb{R}$ be the identity function. Then the complete measure μ_F associated to F is called the ***Lebesgue measure*** which we will denote by m. The domain of m is called the class of ***Lebesgue measurable*** sets, and we shall denote it by $\mathcal{L}$.

Theorem 1.21. If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

2. Integration

Definition. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-**measurable**, or just **measurable** when $\mathcal{M}$ and $\mathcal{N}$ are understood, if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Proposition 2.1. If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Corollary 2.2. If X and Y are metric (or topological) spaces, every continuous $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Definition. If $(X, \mathcal{M})$ is a measurable space, or real – or complex – valued functions f on X will be called $\mathcal{M}$-**measurable**, or just **measurable**, if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ measurable. In particular, $f : \mathbb{R} \to \mathbb{C}$ is **Lebesgue** (resp. **Borel) measurable** if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$ (resp. $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$) measurable; likewise for $f : \mathbb{R} \to \mathbb{R}$.

Proposition 2.3. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathcal{R}$, the following are equivalent:

a. f is $\mathcal{M}$-measurable.
b. $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
c. $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
d. $f^{-1}((\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
e. $f^{-1}((\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proposition 2.4. Let $(X, \mathcal{M})$ and $(Y_\alpha, \mathcal{N}_\alpha)_{\alpha \in A}$ be measurable spaces, $Y = \prod_{\alpha \in A} Y_\alpha,\ N = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha$, and $\pi_\alpha : Y \to Y_\alpha$ the coordinate maps. Then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f_a l = \pi_a l \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all α.

Corollary 2.5. A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\Re(f)$ and $\Im(f)$ are $\mathcal{M}$-measurable.

Proposition 2.6. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f + g$ and fg.

Proposition 2.7. If $\{f_j\}$ is a aequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

$$
\begin{aligned}
g_1(x) &= \sup_j f_j(x), \qquad & g_3 &= \limsup_{j\to\infty} f_j(x) \\
g_1(x) &= \inf_j f_j(x), \qquad & g_3 &= \liminf_{j\to\infty} f_j(x)
\end{aligned}
$$

are all measurable. If $f(x) = \lim_{j\to\infty} f_j(x)$ exists for every $x \in X$, then f is measurable.

Corollary 2.8. If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $\max(f, g)$ and $\min(f, g)$.

Corollary 2.9. If $\{f_j\}$ is a sequence of complex-valued measurable functions and $f(x) = \lim_{j\to\infty} f_j(x)$ exists for all x, then f is measurable.

Definition. If $f : X \to \overline{\mathbb{R}}$, we define the **positive** and **negative parts** of f to be

$$f^+(x) = \max(f(x), 0), \qquad f^-(x) = \max(-f9x), 0).$$

If $f : X \to \mathbb{C}$, we define its **polar decomposition** by

$$f = (\text{sgn}(f))|f|, \quad \text{where} \quad \text{sgn}(z) = \begin{cases} z/|z| & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Definition. Suppose that $(X, \mathcal{M})$ is a measurable space. If $E \subset X$, the **characteristic function** χ_E of E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Definition. A **simple function** of X is a finite, linear combination, with complex coefficients, of characteristic functions of sets in $\mathcal{M}$. Equivalently $f : X \to \mathbb{C}$ is simple if and only if f is measurable and the range of f is a finite subset of $\mathbb{C}$. The **standard representation** of f is given by

$$f = \sum_{j=1}^{n} z_j \chi_{E_j}, \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and range}(f) = \{z_1, \ldots, z_n\}.$$

Theorem 2.10. Let $(X, \mathcal{M})$ be a measurable space.

a. If $f : X \to [0, \infty]$ is measurable, there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$, $\varphi \to f$ pointwise, and $\varphi \to f$ uniformly on any set on which f is bounded.

b. If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f|$, $\varphi \to f$ pointwise, and $\varphi \to f$ uniformly on any set on which f is bounded.

Proposition 2.11. Let $(X, \mathcal{M}, \mu)$ be a measure space. The following statements are true if and only if μ is a complete.

1. If f is measurable and $f(x) = g(x)$ μ-a.e. then $g(x)$ is measurable.
2. If $\{f_n\}$ is a sequence of measureable functions and f is a unction on X such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ μ-a.e., then f is measurable.

Proposition 2.12. If $(X, \mathcal{M}, \mu)$ is a measure space and $\overline{X}, \overline{\mathcal{M}}, \overline{\mu})$ is its completion, then if f is a $(\overline{X}, \overline{\mathcal{M}}, \overline{\mu})$ is a measureable function, there exists a function g that is measureable in $(X, \mathcal{M}, \mu)$ such that $f(x) = g(x)\,\overline{\mu} - a.e.$.

Corollary. If $f : R \to [-\infty, \infty]$ is Lebesgue measurable in $(\mathbb{R}, \mathcal{L}, m)$ then there is a Borel measurable function $g : R \to [-\infty, \infty]$ such that $f(x) = g(x)\,\overline{\mu} - a.e.$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a be a measure space. Let $\varphi : X \to \mathbb{C}$. Recall that if φ is simple if φ is measureable and has finite range. In this case we can write

$$\varphi(x) = \sum_{i=1}^{n} z_i \chi_{E_i}$$

where $\{z_i\}_1^n$ is the range of φ, and $\bigsqcup E_i = X$ with $E_i = \varphi^{-1}(\{z_i\}) \in \mathcal{M}$. Now suppose that φ has non-negative range.

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$$

We define $\int_X \varphi d\mu$ (the integral of φ with respect to μ) by

$$\int_X \varphi = \int_X \sum_{i=1}^{n} a_i \chi_{E_i} d\mu = \sum_{i=1}^{n} a_i \cdot \mu(E_i)$$

with the following arithmatic rules

$$a \cdot \infty = \begin{cases} 0 & a = 0 \\ \infty & a > 0 \end{cases} \qquad a + \infty = \infty$$

We have $L^+ = \{f : X \to [0, \infty] \ : \ f \text{ is measureable}\}$. We will define for $f \in L^+$.

$$\int_X f \, d\mu = \sup \left\{ \int_X \varphi \ : \ \varphi \text{ simple }, \varphi(x) \leq f(x) \right\}$$

Definition. For φ simple $\varphi : X \to [0, \infty)$ and $A \in \mathcal{M}$ define

$$\int_A \varphi \, d\mu = \int_X \varphi \chi_A \, d\mu = \int_X \left(\sum_1^n a_i \chi_{E_i} \right) \chi_A \, d\mu = \int_X \sum_1^n a_i \chi_{E_i \cap A} \, d\mu = \sum_1^n a_i \mu(E_i \cap A).$$

Proposition 2.13. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let φ and ψ be non-negative simple functions on X. Then

(a) If $c \geq 0$, $\int_X c\varphi \, d\mu = c \int_x \varphi \, d\mu$.

(b) $\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$.

(c) If $\psi(x) \leq \phi(x)$ for all $x \in X$ then

$$0 \leq \int_X \psi(x) \, d\mu \leq \int_X \varphi(x) \, d\mu$$

(d) The map $A \mapsto \int_A \varphi \, d\mu$ defines a measure μ_φ on $(X, \mathcal{M})$.

Proposition (Used in Hw). Let $f, g : X \to \overline{\mathbb{R}}$ are measurable. Then $f + g$ is measurable.

Proof. Since f and g are measurable, we know by Theorem 2.10 that there are sequences of simple functions $\{\varphi_n\}$ and $\{\psi_n\}$ such that

$$f = \lim_{n \to \infty} \varphi_n \quad \text{and} \quad g = \lim_{n \to \infty} \psi_n.$$

It then follows that for any $x \in X$,

$$(f + g)(x) = \lim_{n \to \infty} \varphi_n(x) + \psi_n(x)$$

and since the sum of simple functions remains simple, we get that $f + g$ is the (pointwise) limit of simple functions $\{\varphi_n + \psi_n\}$ all of which are measureable. So $f + g$ is the (pointwise) limit of measurable functions, and by Proposition 2.7 $f + g$ is measurable. ♣

Definition. For $(X, \mathcal{M}, \mu)$ a measure space and $f \in L^+$, recall we define

$$\int_X f = \sup\left\{\int_X \varphi \, d\mu : \varphi \in L^+,\ \varphi\text{simple},\ \varphi(x) \leq f(x),\ \forall x \in X\right\}.$$

Let

$$N_f = \left\{\int_X \varphi \, d\mu,\ \varphi \in L^+,\ \varphi \text{ simple},\ \varphi(x) \leq f(x),\ \forall x \in X\right\}$$

Then $\int_x f \, d\mu = \sup N_f \geq 0$. If $\psi \in L^+$ and ψ is simple, then $\inf \varphi$ is nonnegative and simple, and $\varphi(x) \leq \psi(x)$ for all $x \in X$. Then by Proposition 2.13(c) we have that

$$\int_X \varphi \, d\mu \leq \int_x \psi \, d\mu.$$

So

$$\int_X \psi \, d\mu = \sum_{j=1}^{n} b_j \mu(F_j)$$

is a upper bound for N_ψ. So

$$\sup N_\psi \leq \sum_{j=1}^{n} b_j \mu(F_j) = \int_X \phi \, d\mu.$$

Theorem 2.14. (The Monotone Convergence Theorem) If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n (= sup_n f_n)$, then

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n.$$

Theorem 2.15. Let $\{f_j\}_{j\in J}$ be a finite or countably infinite sequence of functions in L^+. Then

$$\int_X \sum f_j \, d\mu = \sum \int_X f_j \, d\mu.$$

Proposition 2.16. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \in L^+$. Then

$$\int_X f \, d\mu = 0 \Leftrightarrow f(x) = 0 \text{ a.e. on } X.$$

Corollary 2.17. Suppose that $\{f_n\}_{n=1}^{\infty} \subset L^+$, $f \in L^+$, and that $f_n(x) \to f(x)$ for almost every x. Then $\int f = \lim \int f_n$.

Theorem 2.18. (Fatou's Lemma) If $\{f_n\}$ is any sequence in L^+, then

$$\int_X (\liminf f_n) = \liminf \int_X f_n.$$

Example: $(\mathbb{R}, \mathcal{L}, m)$ $f_n(x) = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\lim f_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0$, so

$$\int_{\mathbb{R}} \lim f_n \, dm = \int_{\mathbb{R}} = \, dm = 0 < \infty = \lim \int_{\mathbb{R}} \frac{1}{n} \, dm.$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Let $f \in L^+$. We way that f is integrable if $\int_X f \, d\mu < \infty$.

(ii) Let $F : X \to [-\infty, \infty]$. We say that f is integrable if f^+ and f^- are both integrable (in the sense of (i)), and in this case, we define

$$\int_X f \, d\mu = \int_x f^+ \, d\mu - \int_x f^- \, d\mu.$$

(iii) For $f : X \to \mathbb{C}$ we asy that f is integrable if $\Re(f)$ and $\Im(f)$ are both integrable, and in this case we define

$$\int_X f \, d\mu = \int_X \Re(f) \, d\mu + i \int_X \Im(f) \, d\mu.$$

Note: For (ii) and (iii) it is possible to show that f is integrable if and only if $\int_X |f| \, d\mu < \infty$.

Notation: The set of all integrable function is written $L^1, \; L^1(\mu), \; L^1(X, \mu)$.

Exercise: If $f \in L^1(\mu)$; $\alpha \in \mathbb{C}$, then $\alpha f \in L^1(\mu)$, and

$$\alpha \int_X f \, d\mu = \int_X \alpha f \, d\mu.$$

Exercise: If $f, g \in L^1(\mu)$, then

$$\int_X (f+g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Corollary 2.19. If $\{f_n\} \subset L^+, \; f \in L^+$, and $f_n \to f$ a.e., then $\int f \le \liminf \int f_n$.

Proposition 2.20. If $f \in L^+$ and $\int f < \infty$, then $\{x \; : \; f(x) = \infty\}$ is a null set and $\{x \; : \; f(x) > 0\}$ is σ-finite.

Proposition 2.21. The set of integrable real-valued functions on X is a real vector space, and the integral is a linear functional on it.

Proposition 2.22. If $f \in L^1$, then $|\int f| \le \int |f|$.

Proposition 2.23.

(a) Let $f \in L^1(\mu)$. Then $\{x \in X \; : \; f(x) \neq 0\}$ is σ-finite.

(b) Let $f \in L^1(\mu)$. Then the following are equivalent:

 (i) $\int_E f \, d\mu = \int_E g \, d\mu$ for all $E \in \mathcal{M}$.

(ii) $\int_X |f - g| \, d\mu = 0$.

(iii) $f(x) = g(x)$ μ-a.e.

Proposition (Class). For $f, g \in L^1(\mu)$, we write $f \sim g$ if $f(x) = g(x)$ μ-a.e. By abuse of notation we write $L^1(\mu) = L^1(\mu)/\sim$ and $f = [f]$. Note by Proposition 2.16 $\int_X |f| \, d\mu 00 \Leftrightarrow [f] = 0$. With this identification, we can make $L^1(\mu) = L^1(\mu)/\sim$ a metric space with

$$\rho(f, g) = \int_X |f - g| \, d\mu.$$

Theorem 2.24. (The Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence in L^1 such that $f_n \to f$ a.e. and such that there is some $g \in L^1 \cap L^+$ such that $|f_n| \leq g$ a.e. for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Theorem 2.25. Suppose that $\{f_j\}$ is a sequence in L^1 such that $\sum_1^\infty \int |f_j| < \infty$. Then $\sum_1^\infty f_j$ converges a.e. to a function in L^1, and $\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$.

Theorem 2.26. If $f \in l^1(\mu)$ and $\varepsilon > 0$, there is an integrable simple function $\varphi = \sum a_j \chi_{E_j}$ such that $\int |f - \varphi| \, d\mu < \varepsilon$. If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, the sets E_j in the definition of φ can be taken to be finite unions of open intervals; moreover, there is a continuous function g that vanishes outsided a bounded interval such that $\int |f - g| \, d\mu < \varepsilon$.

Theorem 2.27. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f : X \times [a, b] \to \mathbb{C}$ and suppose that for every $t \in a, b$, $f(\cdot, t) \in L^1(\mu)$, and $\int_X f(x, t)|_{d\mu(x)} < \infty$.

1. Suppose there exists a $g_1 \in L^1 \cap L^+$ such $|f(x, t)| \leq g_1(x)$ for all $(x, t) \in X \times [a, b]$. Define $F(x) = \int_X f(x, t) \, d\mu$. If $\lim_{t\to t_0} f(x, t) = f(x, t_0)$ for all $x \in X$, then $\lim_{t\to t_0} F(t) - F(t_0)$. That is to say
$$\lim_{t\to t_0} \int_X f(x, t) \, d\mu = \int_X \lim t \to t_0 f(x, t) \, d\mu.$$

2. Suppose $\frac{\partial f}{\partial t}$ exists for all $(x, t) \in X \times [a, b]$ and $|\frac{\partial f}{\partial x}| \leq g_2(x)$ for all $(x, t) \in X \times [a, b]$ where $g_2 \in L^1 \cap L^+$. Then $F(x) = \int_x f(x, t) \, d\mu(x)$ is differentiable on $[a, b]$ and $F'(t_0) = \int_x \frac{\partial f}{\partial t}(x, t_0) \, d\mu(x)$ with
$$\lim_{t\to t_0} \frac{F(t) - F(t_0)}{t - t_0} = \lim_{t\to t_0} \int_x \frac{f(x, t) = f(x, t_0)}{t - t_0} \, d\mu(x).$$

Theorem 2.28. Let $f : [a, b] \to \mathbb{R}$ be a bounded function where $[a, b]$ is a closed and bounded interval

1. If f is Riemann integrable over $[a, b]$, then f is Lebesgue measurable and Lebesgue integrable over $[a, b]$ with
$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, dm$$

2. If f is Riemann integrable over $[a, b]$ if and only if
$$m(\{x \in [a, b] \ : \ f \text{ is discontinuous at } x\}) = 0.$$

Example: (of the DCT) Consider

$$\sum_{k=0}^{\infty} x^{2k}$$

on $[0,1]$. Let

$$f_n(x) = \sum_{k=0}^{n} (-1)^k x^{2k} = \frac{1+(-1)^{n+1}x^{2(n+1)}}{1+x^2}.$$

Then $\lim f_n(x)$ exists for all $x \in [0,1)$ and diverges at 1.

$$|f_n(x)| = \left|\frac{1+(-1)^{n+1}x^{2(n+1)}}{1+x^2}\right| \leq \frac{2}{1} = 2$$

on $[0,1]$. Note that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1+(-1)^{n+1}x^{2(n+1)}}{1+x^2} = \frac{1}{1+x^2}.$$

By the DCT, we then get that

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) = \int_{[0,1]} \frac{1}{1+x^2}\, dm = \frac{\pi}{4},$$

and

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) = \lim_{n\to\infty} \int_{[0,1]} \sum_{k=0}^{n} (-1)^k x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

So

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n : X \to \mathbb{C}\}$ all be measurable and let $f : X \to \mathbb{C}$ be measureable

(i) We say that $f_n \to f$ pointwise on X if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$.

(ii) $f_n \to f$ uniformly if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$.

(iii) We say $f_n \to f$ in measure on $(X, \mathcal{M}, \mu)$ if for all $\varepsilon > 0$,

$$\lim_{n\to a} \mu(\{x \in X \ : \ |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(iv) Convergence in L^1, $\{f_n\} \in L^1$, $f \in L^1$, $[f_n] \to [f]$ if

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0.$$

Note: The only implications that we can derive from these types of convergence are that (ii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (iii). THESE WILL BE ON THE EXAM!!!

Proposition 2.29. If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof. Fix some $\varepsilon > 0$. For each $n \in \mathbb{N}$, let $E_n = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$. Well, we know that

$$\int_X |f_n(x) - f(x)|\, d\mu \geq \int_X |f_n(X) - f(X)| \chi_{E_n}(x)\, d\mu = \int_X \varepsilon \chi_{E_n}(x)\, d\mu.$$

This gives us that

$$\mu(E_n) \leq \frac{\int_X |f_n(x) - f(x)|\, d\mu}{\varepsilon}$$

because there is some $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $\int_X |f_n(x) - f(x)|\, d\mu < \varepsilon$. So for $n \geq N$, $\mu(E_n) \leq \frac{\varepsilon^2}{\varepsilon}$ which establishes (iii). ♣

Definition. (Cauchy Sequence for convergence in measure) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measureable functions on $(X, \mathcal{M}, \mu)$. We say that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the sense of convergence in measure if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $m > n \geq N$

$$\mu(\{x \in X : |f_m(x) - f_n(x)| \geq \varepsilon\}) < \varepsilon.$$

Theorem 2.30. Suppose that $\{f_n\}_{n=1}^{\infty}$ are measureable on $(X, \mathcal{M}, \mu)$ and that it is a Cauchy sequence with respect to convergence in measure. Then there is some measurable function f such that $f_n \to f$ in measure. Furthermore, there is a subsequence $\{f_{n_j}\}_{j=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{j \to \infty} f_{n_j}(x) = f(x)$$

pointwise a.e. on $(X, \mathcal{M}, \mu)$. Moreover, if there exists some other measureable function g such that $f_n \to g$ in measure, we have that $f = g$ μ-a.e.

There is an error in the theorem counter in the book.

Corollary 2.32. If $f_n \to f$ in L^1, there is a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ a.e.

Theorem 2.33. (Egoroff) Suppose that $\mu(X) < \infty$, and $f_1, f_2, \ldots$ and f are measureable, complex-valued functions on X such that $f_n \to f$ a.e. The for every $\varepsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Definition. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We define a **measureable rectangle** to be a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

NB: If we take all finite unions of the rectangles, we get an algebra, $\mathcal{A}$ and the σ-algebra generated by this set is

$$\mathcal{A} = \mathcal{M} \otimes \mathcal{N}$$

We get a premeasure on $\mathcal{A}$ if we define $\pi_0(E) := \sum_j \mu(A_j)\nu(B_j)$ for $E = \bigsqcup_j (A_j \times B_j)$. This π behaves as a premeasure, and so it generates an outer measure π^*. This then restricts to a measure

π on $\mathcal{M} \times \mathcal{N}$ which we denote by $\mu \times \nu$. Moreover, if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite, then $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ is σ-finite and is thus unique.

Definition. We define the ***x*-section** E_x and the ***y*-section**, E^y of a product space $X \times Y$ as follows:

$$E_x = \{y \in Y \ : \ (x, y) \in E\}$$
$$E^y = \{x \in X \ : \ (x, y) \in E\}.$$

Also, if f is a function on $X \times Y$, we can define the x-section f_x and the y-section f^y of f by

$$f_x(y) = f^y(x) = f(x, y).$$

Proposition 2.34.

1. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$ for all $x \in X$ and for all $y \in Y$.
2. If f is $\mathcal{M} \otimes \mathcal{N}$-measureable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.

Definition. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. If $\mathcal{C}$ is closed under increasing unions and decreasing intersections, then we call $\mathcal{C}$ is called a **monotone class** of subsets of X.

Definition. Let $\mathcal{E} \subset \mathcal{P}(X)$, we have that the **monotone class generated by $\mathcal{E}$** is the intersection of all monotone classes containing $\mathcal{E}$.

Note: For collection of subsets $\mathcal{E}$, the σ-algebra generated by $\mathcal{E}$ is a monotone class.

Theorem 2.35. (The Monotone Class Lemma) If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by A coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

Theorem 2.36. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measureable on X and Y, respectively, and

$$(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Theorem 2.37. (Fubini-Tonelli) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

1. (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x \, d\nu$ and $h(y) = \int f^y \, d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned}\int f \, d(\mu \times \nu) &= \int \left[\int f(x,y)\, d\nu(y)\right] d\mu(x) \\ &= \int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y)\end{aligned}$$

2. (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$, for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, and the a.e. defined functions $g(x) = \int f_x \, d\nu$ and $h(x) = \int f^y \, d\nu$are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and the equation

Theorem 2.39. (Fubini-Tonelli for Complete Measures) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete, σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in \mathcal{L}^1(\lambda)$, then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y, and in case (b) f_x and f^y are also integrable for a.e. x and y. Moreover, $x \mapsto \int f_x \, d\nu$ and $y \mapsto \int f^y \, d\mu$ are measurable, and in case (b) also integrable, and

$$\int f \, d\lambda = \iint f(x, y) \, d\mu(x) \, d\nu(y) = \iint f(x, y) \, d\nu(y) \, d\mu(x).$$

3. Signed Measures and Differentiation

Definition. Let $(X, \mathcal{M})$ be a measurable space. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : [-\infty, \infty]$ such that

- $\nu(\varnothing) = 0$
- ν assumes at most one of the values $\pm\infty$.
- if $\{E_j\}$ is a sequence of disjoint open sets in $\mathcal{M}$, then $\nu(\bigcup_{i=1}^{\infty} E_j) = \sum_{i=1}^{\infty} \nu(E_j)$ where the latter sum converges absolutely if $\nu(\bigcup_{i=1}^{\infty} E_j)$ is finite.

Proposition 3.1. Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu(\bigcup_1^{\infty} E_j) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a decreasing sequence and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^{\infty} E_j) = \lim_{j\to\infty} \nu(E_j)$

Definition. Let ν be a signed measure on $(X, \mathcal{M})$. Let $E \in \mathcal{M}$. We say that E is a **positive set** for ν if for all $F \in \mathcal{M}$ such that $F \subset E$, $\nu(F) \geq 0$. In particular $\nu(E) \geq 0$. We say that E is a **negative set** for ν if whenever $F \in \mathcal{M}$ and $F \subset E$, $\nu(F) \leq 0$. So in particular $\nu(E) \leq 0$. We say E is a **null set** for ν if whenever $F \subset \mathcal{M}$ and $F \subset E$, $\nu(F) = 0$. In particular, $\nu(E) = 0$.

NB: If we have that $E \in \mathcal{M}$ and $\nu(E) = 0$ *this does not mean that E is a null set.*

Lemma 3.2. Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.

Lemma (Class): Let $(X, \mathcal{M})$ be a measurable space, and let ν be a signed measure on X. Suppose that $A \in \mathcal{M}$ and $0 < \nu A) < \infty$. Then there exists $E \in \mathcal{M}$, $E \subset A$ such that E is positive for ν and such that $0 < \nu(E) < \infty$.

Theorem 3.3. (The Han Decomposition Theorem) If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N such that $P \cup N = X$ and $P \cap N = \varnothing$. If P', N' is another such pair, then $P \bigtriangleup P'$ is null for ν.

Definition. We say that two signed measures μ and ν on $(X, \mathcal{M})$ are **mutually singular** (written $\mu \perp \nu$) if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing,\ E \cup R = X,\ E$ is null for μ, and F is null for ν.

Theorem 3.4. (The Jordan Decomposition Theorem) If ν is a signed measure , there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Definition. The measures ν^+ and ν^- are called the **positive and negative variations** of ν and $\nu = \nu^+ - \nu^-$ is called the **Jordan Decomposition**.

Definition. Integration with respect to a signed measure is defined in the obvious way

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$$

$$\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-.$$

Definition. Suppose taht ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is **absolutely continuous** with respect to μ and write

$$\nu << \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.

Theorem 3.5. Let ν be a finite, signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu << \mu$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\nu(E)| < \delta$ whenever $\mu(E) < \varepsilon$.

Corollary 3.6. If $f \in L^1(\nu)$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_E f\, d\mu| < \varepsilon$ whenever $\mu(E) < \delta$.

Lemma 3.7. Suppose that ν and μ are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exist $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu > \varepsilon\mu$ on E.

Theorem 3.8. (Lebesgue-Radon-Nikodym) Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \rho, \quad \rho << \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f\, d\mu$, and any two such functions are equal μ-a.e.

Definition. We call the function f described in the above theorem the **Radon-Nikodyn derivative** of ν with respect to μ, and we denote it by $d\nu/d\mu$

$$d\nu = \frac{d\nu}{d\mu} d\mu.$$

Proposition 3.9. Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$.

(a) If $g \in L^1(\nu)$, then $g(d\nu/c\mu) \in L^1(\mu)$ and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

(b) We have $\nu \ll \lambda$, and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Corollary 3.10. If $\mu \ll \lambda$ and $\lambda \ll \mu$, then $(d\nu/d\mu)(d\mu/d\lambda) = 1$ a.e.

Example: Why we want σ finite on the Lebesgue-Radon-Nikodym theorem. Consider m the standard Lebesgue measure and ν counting measure on $(R, \mathcal{L})$. So $\nu(E) = card(E)$. Suppose $\exists f : \mathbb{R} \to [0, \infty]$ such that $m(E) = \int_E f(x)\, d\nu$. Fix $a \in \mathbb{R}$. Then $0 = m(a) = \int_{\{a\}} f(x)\, d\nu = f(a) \cdot \nu(\{a\}) = f(a)$. Therefore, $f(a) = 0$ for all $a \in \mathbb{R}$ so $f = 0$ and thus $m = 0$ ↯.

Proposition 3.11. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, there is a measure μ such that $\mu_j \ll \mu$ for all j – namely $\mu = \sum_1^n \mu_j$

Functions of Bounded Variation

Definition. Let $F : \mathbb{R} \to \mathbb{C}$ and fix $x \in \mathbb{R}$. We define $T_F : \mathbb{R} \to [0, \infty]$ by

$$T_F(x) = \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \ : \ n \in \mathbb{N},\ x_0 < x_1 < \cdots < x_n = x \right\}$$

We say that F is of bounded variation on $\mathbb{R}$ if $T_F(x) < \infty$ for all $x \in \mathbb{R}$. If $F : [a, b] \to \mathbb{C}$ we define $T_{F,a}(x)$ the same way but with $x_0 = a$ and $x \leq b$. We say that F is of BV on $[a, b]$ if $T_{F,a}(x) < \infty$ for all $a \leq x \leq b$.

Theorem (Lebesgue 1904) If $F \in BV$, $F \in BV[a, b]$, then $F'(x)$ exists a.e. on $\mathbb{R}$.

Example: Let

$$F(x) = \begin{cases} x \cos\left(\frac{1}{x}\right) & 0 < x \leq \frac{2}{\pi} \\ 0 & o.w. \end{cases}$$

There is no bounded variation of this function. For $k \in \mathbb{N}$ let

$$\mathcal{P}_k = \left\{ 0, \frac{2}{\pi(2k)}, \frac{2}{\pi(2k-1}, \ldots, \frac{2}{\pi 2}.\frac{2}{\pi} \right\}$$

be the partition that we choose of $[0, 2/\pi]$. Then we have that

$$T_{F,0}\left(\frac{2}{\pi}\right) \geq \sum_{j=1}^{2} k|F(x_j) - F(x_{j-1})|$$

Theorem 3.27. Let F be a function

1. $F \in BV$ iff $\Re(F)$ and $\Im(F)$ are BV
2. $F \in BV$ iff $F = F_1 - F_2$ where F_1 and F_2 are bounded and monotone
3. IF $F \in BV$, $F(x+) = \lim_{y \to x^+} F(y)$ and $F(x-)$ exist for all $x \in \mathbb{R}$.
4. If $F \in BV$ the set of points where F is not continuous is countable.
5. If $F \in BV$, set the $G(x) = F(x+)$ for all $x \in \mathbb{R}$. Then $G(x) = F(x)$ a.e. and $G'(x),\ F'(x)$ exist and are equal a.e.

Lemma If $F \in BV[a, b]$ then F is bounded.

Lemma If $F : [a, b] \to \mathbb{R}$, $F \in BV[a, b]$. If $a \leq c < d \leq b$. Then

$$T_{F,a}(c) \leq T_{F_a}(d)$$

Lemma If $F \in BV[a, b]$ and $a \leq c < d \leq b$, then $F \in BV[a, c]$ and $F \in BV[c, d]$ and

$$T_{F,a}(d) = T_{F,a}(c) + T_{F,c}(d).$$

Proposition Let $F \in BV[a,b]$. Then $G = F_1 - F_2$ where F_1 and F_2 are monotone increasing on $[a,b]$.

Definition. Let $E \subset \mathbb{R}$. Let $\mathfrak{I}$ be a collection of intervals in $\mathbb{R}$. We say that $\mathfrak{I}$ is a ***Vitali cover*** for E if for every $x \in E$ and for every $\varepsilon > 0$ there is some $I \in \mathfrak{I}$ such that $x \in I$ and $\ell(I) < \varepsilon$.

Lemma (Vitali Covering Lemma) Let $E \subset \mathbb{R}$ and suppose $m^*(E) < 0$. Suppose $\mathfrak{I}$ is a collection of closed and bounded intervals of $\mathbb{R}$ that form a Vitali cover for E. Then for every $\varepsilon > 0$ there are intervals $I_1, I_2, \ldots, I_n \in \mathfrak{I}$ such that $I_i \cap I_j = \varnothing$ and $m^*\left(E \setminus \bigcup_{k=1}^n I_k\right) < \varepsilon$.

Definition. We define the ***Dini Derivates*** of f at x is given as follows

$$\begin{aligned} D^+ f(x) &= \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}, \\ D^- f(x) &= \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}, \\ D_+ f(x) &= \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}, \\ D_- f(x) &= \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}. \end{aligned}$$

Lemma: If $F \in BV$ is real-valued, then $T_F + F$ and $T_F - F$ are increasing. Moreover

$$F = \frac{1}{2}(T_f + F) + \frac{1}{2}(T_F - F)$$

this is called the ***Jordan decomposition*** of F

Definition. We define the set of ***Normalized Bounded Variation*** functions $NBV \subseteq BV$ by

$$NBV = \{F \in BV \ : \ F \text{ is right continuous and } F(-\infty = 0\}.$$

Theorem: Let $F : [a,b] \to \mathbb{R}$ be a monotone increasing function. Then the derivative F' exists almost everywhere, and $F \in L^1([a,b])$, and

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Definition. We say that a function $F : \mathbb{R} \to \mathbb{C}$ is ***absolutely continuous*** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$,

$$\sum_1^N (b_j - a_j) < \delta \quad \Longrightarrow \quad \sum_1^N |F(b_j) - F(a_j)| < \varepsilon$$

Lemma: Let $F : [a,b] \to \mathbb{R}$ be monotone increasing and AC on $[a,b]$. Then the Borel measure ν_F on $([a,b], \mathbb{B}_{\mathbb{R}})$ defined by

$$\nu_F((c,d]) = F(d) - F(c)$$

is absolutely continuous with respect to m on $[a, b]$.

Corollary: If F is $\nearrow$ and $F \in AC$ on $[a, b]$ then there is an $F \in L^1([a, b]) \cap L^+$ such that

$$F(x) = \int_1^x f(t)\, dt + F(a)$$

for all $x \in [a, b]$.

Corollary: If $F \in AC([a, b])$, then there is a $f \in L^1([a, b])$ such that

$$F(x) - F(a) = \int_a^x f(t)\, dt.$$

Theorem 3.35. Let $F : [a, b] \to \mathbb{R}$. Then TFAE

(a) F is AC on $[a, b]$.

(b) $F(x) - F(a) = \int_a^x f(t)\, dt$ for some $f \in L^1([a, b])$.

(c) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)\, dt$.

Note: go over the proof of $(b) \Rightarrow (c)$ in Folland

Definition. Let $F : a, b \to \mathbb{R}$. We say that F is ***Lipschitz*** if there is some $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

Exercises:

1. If F is Lipschitz then F is $AC([a, b])$.
2. If F is differentiable at all $x \in [a, b]$, and $|F'(x)| \leq M$ then F is Lipschitz $\Rightarrow$ F is in $AC([a, b])$.

Definition. Let $\varphi : (a, b) \to \mathbb{R}$. We say that φ is ***convex*** if for all $x, y \in (a, b)$, and for all $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Proposition. Let $\varphi : (a, b) \to \mathbb{R}$ and suppose $\varphi''(X)$ exists for all $x \in (a, b)$. Then the following are equivalent:

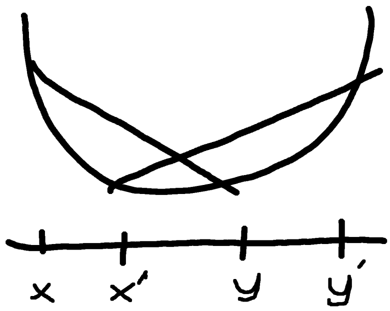
(i) φ is convex on (a, b)

(ii) $\varphi(y) \geq \varphi(x) + \varphi'(x)(y - x)$ for all $x, y \in (a, b)$.

(iii) $\varphi''(x) \geq 0$ for all $x \in (a, b)$

Chordal Slope Lemma: Let $\varphi : (a, b) \to \mathbb{R}$ be a convex function. Suppose $x, x', y, y' \in (a, b)$ with $x \leq x' < y$ and $x < y \leq y'$. Then

$$\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(y') - \varphi(x')}{y' - x'}$$



Corollary, If $x_1 < x_2 \leq x_3 < x_4$ for a convex function φ then

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_4) - \varphi(x_3)}{x_4 - x_3}$$

Theorem. If $\varphi : (a,b) \to \mathbb{R}$ is convex, then φ is AC on every closed sub-interval $[c,d] \subset (a,b)$. Moreover,

$$\lim_{h \to 0^+} = \frac{\varphi(x+h) - \varphi(x)}{h} = \varphi_R'(x+)$$

and

$$\lim_{h \to 0^-} = \frac{\varphi(x+h) - \varphi(x)}{h} = \varphi_L'(x-)$$

exist at every x, and this will imply φ' exists except at a countable number of points in (a,b).

Jensen's Inequality. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable on $[0,1]$ so that

$$\int_{[0,1]} |f| \, dm < \infty.$$

Suppose also that $\varphi \circ f : [0,1] \to \mathbb{R}$ is Lebesgue integrable over $[0,1]$. Then

$$\varphi\left(\int_{[0,1]} f(x) \, dx\right) \leq \int_{[0,1]} (\varphi \circ f)(x) \, dx.$$

4. Point Set Topology

Definition. Let $(X, \mathcal{T})$ be a topological space. We say that X is locally compact Hausdorff if X is Hausdorff and if every $X \in X$ has a basis compact neighborhoods

Theorem 4.31. If $(X, \mathcal{T})$ is locally compact Hausdorff, and if $K \subset U \subset X$ where K is compact and U is open, then there exists an open set V with $\overline{V}$ compact. $K \subset V$ and $\overline{V} \subset U$.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{W})$ be topological spaces. Let $f : X \to I$.

F is continuous at $x_0 \in X$ if for all open sets V with...

$f : X \to Y$ is continuous if for all open sets $V \subset Y$, $f^{-1}(V)$ is open in X.

Definition. A ***directed set*** is a nonempty set D on which there is a defined a partial order $\preceq$ such that

(i) $\alpha \preceq \alpha$ for all $\alpha \in D$

(ii) for every $\alpha, \beta \in D$ there is some $\gamma \in D$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Definition. Let $\mathcal{C}, \mathcal{D}$ be two directed sets. We say that a function $h : \mathcal{C} \to \mathcal{D}$ is ***order-preserving*** if whenever $\alpha\beta \in \mathcal{C}$ with $\alpha \preceq \beta$ then $h(\alpha) \preceq h(\beta)$.

Definition. Let $h : \mathcal{C} \to \mathcal{D}$ be an order preserving map of directed set. We say that h is ***cofinal*** if for all $\gamma \in \mathcal{D}$ there is a $\beta \in \mathcal{C}$ such that $\gamma \preceq h(\beta)$.

Definition. Let $(X, \mathcal{T})$ be a topological space. Then a ***net*** on X is a function $f : \mathcal{D} \to X$ where $\mathcal{D}$ is some directed set. We usually write $\{x_\alpha = f(\alpha) \mid \alpha \in \mathcal{D}\}$ or $\{x_\alpha\}_{\alpha \in \mathcal{D}}$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ be a net on X. Then we say that $\{x_\alpha\}$ converges to $x_o \in X$ and write $\lim x_\alpha = x_0$ if for every neighborhood U of x_0 there is a $\alpha' \in \mathcal{D}$ such that for all $\alpha \in \mathcal{D}$ such that $\alpha' \preceq \alpha$ we have $x_\alpha \in U$.

Definition. Let $\mathcal{D}$ be a directed set and let $f : \mathcal{D} \to X$ be a net in the topological space $(X, \mathcal{T})$. We say that $\{y = g(\beta)\}_{\beta \in \mathcal{C}}$ is a ***subnet*** of the original net if there is an order-preserving, cofinal function $h : \mathcal{C} \to \mathcal{D}$ such that

$$\{y_\beta = g(\beta) = f \circ h(\beta)\}_{\beta \in \mathcal{C}}$$

Proposition 4.19. Let $(X, \mathcal{T})$ and $(Y, \mathcal{W})$ be topological spaces with $f : X \to Y$ then f is continuous at $x_0 \in X$ if and only if for every net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ converging to $x_0 \in X$, $\{f(x_\alpha)\}_{\alpha \in \mathcal{D}}$ is a net in Y converging to $f(x_0)$.

Definition. We say that a point $x_0 \in X$ is a ***cluster point*** of the net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ if for every neighborhood U of x_0, and for all $\alpha' \in \mathcal{D}$, there exits a $\beta \in \mathcal{D}$ with $\alpha' \preceq \beta$ and $x_\beta \in U$.

Proposition 4.20. The point $x_0 \in X$ is a cluster point of the net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ if and only if $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ has a subnet $\{y_\beta\}_{\beta \in \mathcal{C}}$ with $\lim y_\beta = x_0$.

Bolzano Weirstrauss Theorem for Nets. (Prop 4.29) Let $(X, \mathcal{T})$ be a topological space. Then TFAE

1. $(X, \mathcal{T})$ is compact.
2. Every net in $(X, \mathcal{T})$ has a cluster point in X.
3. Every net in $(X, \mathcal{T})$ has a convergent subnet.

Urysohon's Lemma. (Compact version) Let X be a compact normal space. If A and B are disjoint sets in X then ther eexists $f \in C(X, [0,1])$ such that

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Theorem 4.32. (Urysohon) (Locally compact Hausdorff version) let $(X, \mathcal{T})$ be a LCH space. Suppose $K \subset U \subset X$ where K is compact and U is open. Then there is an $f \in C(X, [0.1])$ such that

$$f(x) = \begin{cases} 1 & x \in K \\ 0 & x \in X \backslash V \end{cases}$$

where $K \subset V \subset \overline{V} \subset U$ and V is open and $\overline{V}$ is compact. (**Note:** This means that F is compactly supported since $support f(x) = \{x \in X \mid f(x) \neq 0\} \subset \overline{V}$.

Definition. Let $(X, \mathcal{T})$ be a LCH space

$$C_c(X) := \{F : X \to \mathbb{R}(\text{or } \mathbb{C}) \mid f \text{ is cont's on } X \text{ and } supp(F) \text{ is in a compact subset of } X\}$$

NB: If $f \in C_c(X)$ then f is bdd since $K + supp(f)$ gives us that $f(K)$ is compact in $\mathbb{C}$ so $f(K)$ is bdd.

Definition. For X LCH define

$$C_0(X) = \{f \in C(X) \mid f \text{ "vanishes at infinity" }\}$$

Proposition 4.35. Let $(X, \mathcal{T})$ be LCH. Fix $f \in C_0(X)$. Then there is a sequence $\{F_n\}_{n=1}^{\infty} \subset C_c(X)$ such that $f_n \to f$ uniformly X.

Tychonoff's Theorem. If $\{X_\alpha\}_{\alpha \in A}$ is any family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact.

Definition. If X is a topological space that $\mathscr{F} \subset C(X)$, $\mathscr{F}$ is called ***equicontinuous at*** $x \in X$ if for every $\varepsilon > 0$ there is a neighborhood U of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U$ and all $f \in \mathscr{F}$. And $\mathscr{F}$ is called **equicontinuous** if it is equicontinuous at each point $x \in X$. Also, $\mathscr{F}$ is said to be ***pointwise bounded*** if $\{f(x) \ : \ f \in \mathscr{F}\}$ is a bounded subset of $\mathbb{C}$ for each $x \in X$.

Arzelà-Ascoli Theorem I. Let X be a compact Hausdorff space. If $\mathscr{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathscr{F}$ is totally bounded in the uniform metric, and the closure of $\mathscr{F}$ in $C(X)$ is compact.

Arzelà-Ascoli Theorem II. Let X be a σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, there exists $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.

Corollary. If $\mathscr{F}$ is as in the statement of Arzelà-Ascoli, then $\mathscr{F}$ is uniformly bounded.

The Stone-Weierstrass Theorem. Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ that separates points, then either $A = C(X, \mathbb{R})$ or $A = \{f \in C(X, \mathbb{R}) \ : \ f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds iff $\mathcal{A}$ contains the constant functions.

5. Elements of Functional Analysis

Definition. Let K denote either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{X}$ be a vecotor space over K. A ***seminorm*** on $\mathcal{X}$ is a function $x \mapsto \|x\|$ from $\mathcal{X}$ to $[0, \infty)$ such that

- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$
- $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in \mathcal{X}$ and $\lambda \in K$

A seminorm such that $\|x\| = 0$ iff $x = 0$ is called a ***norm*** and a vector space equipped with a norm is called a ***normed vector space***.

Definition. A ***Banach Space*** is a complete, normed vector space.

Theorem 5.1. A normed vector space $\mathcal{X}$ is complete iff every AC convergent series in $\mathcal{X}$ converges.

Theorem. Let $(\mathcal{X}, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$. Then $\mathcal{X}$ is a Banach Space if and only if every absolutely convergent summable in $\mathcal{X}$ converges to an element in X.

Corollary. $L^1(X, \mu)$ is a Banach space with norm

$$\|f\| = \int_X |f| \, d\mu.$$

Notation. We denote the normed vector space of bounded sequences by ℓ^∞.

Definition. $L^\infty(\mathbb{R}, m)$. We say that a vunction $f : \mathbb{R} \to \mathbb{F}$ is ***essentially bounded*** if

1. f is lebesgue measurable and
2. $\exists Z \subseteq \mathbb{R}$ with $m(Z) = 0$
$$\sup_{x \in \mathbb{R} \setminus Z} |f(x)| < \infty.$$

If $M \geq 0$ and there is some $Z_M \subseteq \mathbb{R}$ with

$$\sup_{x \in \mathbb{R} \setminus Z} |f(x)| < M$$

we say that M is an ***essential upper bound***. For f essentially bounded, we write

$$\|f\|_\infty = \inf\{M \geq 0 \ : \ M \text{ an essential upper bound for } f\}$$

Proposition 5.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces over $\mathbb{F}$. Let $T : \mathcal{X} \to \mathcal{Y}$ then TFAE:

1. T is bounded.
2. T is uniformly continuous as a function from $\mathcal{X}$ to $\mathcal{Y}$.
3. T is continuous at some point $x_0 \in \mathcal{X}$.

Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces over $\mathbb{F}$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the set of all bounded linear maps form $\mathcal{X}$ to $\mathcal{Y}$.

Proposition. If $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces over some field $\mathbb{F}$ then $L(\mathcal{X}, \mathcal{Y})$ is a normed linear space over $\mathbb{F}$.

Proposition. $\mathcal{X}^* = L(\mathcal{X}, \mathbb{F})$ is a Banach space.

Definition. A ***sublinear functional*** on $\mathcal{X}$ is a map $p : \mathcal{X} \to \mathbb{R}$ such that

$$p(x+y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in \mathcal{X},\ \lambda \geq 0.$$

Hahn-Banach Theorem Let $\mathcal{X}$ be a normed linear space over $\mathbb{R}$, and let $p : \mathcal{X} \to \mathbb{R}$ be a sublinear functional on $\mathcal{X}$. Let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Let f be a linear functional on $\mathcal{M}$ st $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $F : \mathcal{X} \to \mathbb{R}$ with $F(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $F(x) = f(x)$ for all $x \in \mathcal{M}$.

Theorem. Let $\mathcal{X}$ be a normed linear space over $\mathbb{R}$, and let $\mathcal{M}$ be a subspace of $\mathcal{X}$ and let $f : \mathcal{M} \to \mathbb{R}$ be a bounded linear functional on $\mathcal{M}$. Then there exists a bounded linear functional $F : \mathcal{X} \to \mathbb{R}$ such that

1. $F(x) = f(x)$ for all $x \in \mathcal{M}$
2. $\|F\|_{\mathcal{X}^*} = \|f\|_{\mathcal{M}^*}$

Corollary. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$. Let $\mathcal{M}$ be a *closed* linear subspace of $\mathcal{X}$ ($\mathcal{M}$ a proper linear subset). Let $x_0 \in \mathcal{M}^c$ then there exists a $f \in \mathcal{X}^*$ such that $f(x_0) \neq 0$ but $f|_{\mathcal{M}} = 0$. Indeed, if

$$\delta = \inf_{y \in \mathcal{M}} \|x_0 - y\| > 0$$

we can choose F so that $\|F\| = 1$ and $F(x_0) = \delta$.

Complex Hahn-Banach Theorem Let $\mathcal{X}$ be a normed linear space over $\mathbb{R}$, and let $p : \mathcal{X} \to \mathbb{R}$ be a sublinear functional on $\mathcal{X}$. Let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Let f be a linear functional on $\mathcal{M}$ st $|f(x)| \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $F : \mathcal{X} \to \mathbb{R}$ with $|F(x)| \leq p(x)$ for all $x \in \mathcal{X}$ and $F(x) = f(x)$ for all $x \in \mathcal{M}$.

Theorem 5.8. Let $\mathcal{X}$ be a normed vector space.

a. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \backslash \mathcal{M}$, there exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $f|_{\mathcal{M}} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x_0 - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.
b. If $x \neq 0 \in \mathcal{X}$, there exits $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$
c. The bounded linear functionals on $\mathcal{X}$ separate points.
d. If $x \in \mathcal{X}$, define $\widehat{x} : \mathcal{X}^* \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from $\mathcal{X}$ to $\mathcal{X}^{**}$.

Definition. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. We say that $(X, \|\cdot\|)$ is ***reflexive*** if the map from $\mathcal{X}$ into $\mathcal{X}^{**}$ given by $x \mapsto \widehat{x}$ is an isometric isometry.

Definition. We define the space $L^p([0,1]) = \{f \; : \; \int_0^1 |f|^p \, dx < \infty\}$. This space has a norm defined by

$$\|f\|_p = \left[\int_0^1 |f|^p \, dx\right]^{\frac{1}{p}}.$$

Note: L^p is reflexive.

Definition. Let (X,d) be a metric space. We say $F \subset X$ is a ***meager*** set (or a ***set of the first category***) if we can write

$$F = \bigcup_{n \in \mathbb{N}} A_n$$

where $\{A_n\}$ is a countable collection of nowhere dense sets. We say that $S \subset X$ is a ***set of the second category*** if S is not a set of the first category.

Baire Category Theorem. Let (X,d) be a complete metric space.

1. If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.
2. X is not a countable union of nowhere dense sets (Recall: A set is nowhere dense if its closure has empty interior).

Corollary. Let (X,d) be a complete metric space and suppose that each $x \in X$ is an accumulation point for X. Then X is uncountable.

Definition. Let $\mathcal{H}$ be a complex vector space. An ***inner product*** on $\mathcal{H}$ is a map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

satisfying

1. $\langle x + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
2. $\langle y, x \rangle = \overline{\langle x, y \rangle}$
3. $\langle x, x \rangle \geq 0$ and is 0 iff $x = 0$.

Given an inner product $\langle \cdot, \cdot \rangle$, we can define a norm by

$$\| \cdot \| = (\langle \cdot, \cdot \rangle)^{\frac{1}{2}}.$$

A compact vector $\mathcal{H}$ space equipped with an inner product is called a ***pre-Hilbert space***. If $\mathcal{H}$ is complete with respect to the norm defined above, then it is called a ***Hilbert Space***.

The Cauchy-Schwarz Inequality. Let $\mathcal{H}$ be an inner product space over $\mathbb{C}$. then for all $x, y \in \mathcal{H}$

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|.$$

Proposition 5.21. If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

The Parallelogram Law. For all $x, y \in \mathcal{H}$,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Definition. If $x, y \in \mathcal{X}$, we say that x is ***orthogonal*** to y and write $x \perp y$ fi $\langle x, y\rangle = 0$. If $E \subset \mathcal{H}$, we define

$$E^\perp = \{x \in \mathcal{H} \ : \ \langle x, y\rangle = 0 \text{ for all } y \in E\}.$$

The Pythagorean Theorem. If $x_1, \dots, x_n \in \mathcal{H}$ and $x_j \perp x_k$ for $j \neq k$,

$$\left\| \sum_1^n x_j \right\|^2 = \sum_1^n \|x_j\|^2.$$

Theorem 5.24. If $\mathcal{M}$ is a closed subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Moreover, if $x \in \mathcal{H}$ $x = m + n$ with $m \in M, n \in M^\perp$ and

1. m is the nearest point in $\mathcal{M}$ to x
2. n is the nearest point in $\mathcal{M}^\perp$ to x
3. $\|x\|^2 = \|m\|^2 + \|n\|^2$

Theorem 5.25. Let $\mathcal{H}$ be a Hilbert space. Then if $f \in \mathcal{H}^*$ there exist a unique $y_0 \in \mathcal{H}$ such that $f(x) = \langle x, y_0\rangle$ for all $x \in \mathcal{H}$. Moreover,

$$\|f\| = \|y_0\|.$$

Bessel's Inequality. If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$,

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2.$$

In particular, $\{\alpha \ : \ \langle x, u_\alpha\rangle \neq 0\}$ is countable.

Definition. Let $\{u_n\}$ be a countable orthonormal set in $\mathcal{H}$ a Hilbert space. Let $\mathcal{M}$ be the closed linear subspace containing all finite linear combinations of $\{u_n\}$. Then the projection $P_\mathcal{M} : \mathcal{H} \to \mathcal{M}$ is given by

$$P_\mathcal{M}(x) = \sum \widehat{x}(i) u_i = \sum \langle x, u_i\rangle u_i.$$

Definition. We say that a set of orthonormal vectors in $\mathcal{H}$ is ***complete*** if whenever $x \in \mathcal{H}$ is such that $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.

Theorem 5.27. If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, the following are equivalent:

1. **(Completeness)** If $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.
2. **(Parseval's Identity)** $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all $x \in \mathcal{H}$.

3. For each $x \in \mathcal{H}$, $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$, where the sum on the right has only countably many nonzero terms and converges in the norm topology no mater how these terms are ordered.

Proposition 5.28. Every Hilbert space has an orthonormal basis.

Proposition 5.29. A Hilbert space $\mathcal{H}$ is separable iff it has a countable orthonormal basis, in which case every orthonormal basis for $\mathcal{H}$ is countable.

Proposition 5.30. Let $\{u_\alpha\}$ be a compelte orthonormal set for $\mathcal{H}$ a Hilbert space. Then the map

$$\Phi : X \to \{(\widehat{x}(\alpha))_{\alpha \in I} \ : \ \widehat{x}(\alpha) = \langle x, u_\alpha \rangle\}$$

is a Hilbert space isomorphism from $\mathcal{H}$ to $\ell^2(I)$.

6. L^p Spaces

Young's Inequality. For $1 < p < \infty$, with $q = \frac{p}{p-1}$ and $A, B \geq 0$

$$[A, B] \leq \frac{A^p}{p} + \frac{B^q}{q}$$

where $[A, B]$ denotes a general product.

Corollary. If $a, b \geq 0$ and $p > 1$ then

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Definition. Let $p \in (0, \infty)$ and letr $(X, \mathcal{M}, \mu)$ ve a measure space. We say that a function $f : X \to \mathbb{C}$ is in $L^p(X)$ if

$$\int_X |f|^p \, d\mu < \infty.$$

In this case, we write that

$$||f||_p = \left[\int_X |f|^p \, d\mu\right].$$

Proposition. $L^p(X, \mu)$ is a vector space over $\mathbb{C}$.

Definition. If $p \in (1, \infty)$ we define the ***conjugate exponent*** q to p by $q = \frac{p}{p-1}$. And we have that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Also, if $p = 1$, we define the conjugate exponent q by $q = \infty$.

Hölder's Inequality. Let $p \in (1, \infty)$ and q be its conjugate exponent in $(X, \mathcal{M}, \mu)$ a measure space. Then if f and g are measureable functions, then

$$||fg||_1 \leq ||f||_p ||g||_q.$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds iff $\alpha|f|^p = \beta|g|^q$ a.e. for some constants α, β with $\alpha\beta \neq 0$.

Minkowski's Inequality. If $1 \leq p < \infty$ and $f, g \in L^p$, then

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Theorem 6.6. For $1 \leq p < \infty$, L^p is a Banach space.

Proposition 6.7. For $1 \leq p < \infty$, the set of simple functions is dense in L^p.

Definition. Let $f : X \to \mathbb{C}$ be a $\mathcal{M}$-measurable function. We say that f is ***essentially bounded*** with respect to $\mathcal{M}$ if there exits an N, $0 \leq N < \infty$ such that

$$|f(x)| \leq N \quad \mu - a.e.$$

i.e.

$$\mu(\{x \in X \ : \ |f(x)| > N\}) = 0.$$

Definition. If f is essentially bounded with respect to an X, we write

$$||f||_\infty = \inf\{N \geq 0 \ : \ N \text{ is an essential upper bound for } |f|\}.$$

Proposition. If $(X, \mathcal{M}, \mu)$ is a measure space and if $f : X \to \mathbb{C}$ is essentially bounded with respect to $\mathcal{M}$ then $||f||_\infty$ is an essential upper bound for $|f|$.

Theorem 6.8. Let $f, g : X \to \mathbb{C}$ be measurable. Then

1. $||fg||_1 \leq ||f||_1 ||g||_\infty$. If $f \in L^1$ and $g \in L^\infty$, $||fg||_1 = ||f||_1||g||_\infty$ if and only if $|g(x)| = ||g||_\infty$ a.e. on the set where $f(x) \neq 0$.
2. $||\cdot||_\infty$ is a norm on L^∞.
3. $||f_n - f||_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.
4. L^∞ is a Banach space.
5. The simple functions are dense in L^∞.

Proposition 6.9. If $0 < p < q < r \leq \infty$, then $L^q \subset L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.

Proposition 6.10. If $0 < p < q < r \leq \infty$, then $L^q \cap L^r \subset L^p$ and $||f||_q \leq ||f||_p^\lambda ||f||_r^{1-\lambda}$, where $\lambda \in (0, 1)$ is defined by

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}, \text{ that is, } \lambda = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}}.$$

Proposition 6.11. If A is any set and $0 < p < q \leq \infty$, then $\ell^p(A) \subset \ell^q(A)$ and $||f||_q \leq ||f||_p$.

Proposition 6.12. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q(\mu) \subset L^p(\mu)$ and $||f||_p \leq ||f||_q \, \mu(X)^{\frac{1}{p} - \frac{1}{q}}$.

Proposition 6.13. Let $p \in [1, \infty]$, let $(X, \mathcal{M}, \mu)$ be a measure space, let q be the conjugate exponent to p. Then, each $g \in L^q$ defines a continuous bounded linear functional T_g on L^q by

$$T_g(f) = \int_X fg \, dx.$$

and

$$||g||_q = ||T_g|| = \sup\left\{\left|\int fg\right| \ : \ ||f||_p = 1\right\}$$

if $q < \infty$. If $(X, \mathcal{M}, \mu)$ is semifinite, then equality holds for all q.

Proposition 6.14. Let p and q be conjugate exponents. Suppose that g is a measurable functions on X such that $fg \in L^1$ for all f in tghe space Σ of simple functions that vanish outside a set of finite measure, and the quantity

$$M_q(g) = \sup \left\{ \left| \int fg \right| \ : \ f \in \Sigma, \ ||f||_p = 1 \right\}$$

is finite. Also, suppose that either $S_g = \{x \ : \ g(x) \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q$ and $M_q(g) = ||g||_q$.

Theorem 6.15. Let p and q be conjugaet exponents. If $1 < p < \infty$, for each $\varphi \in (L^P)^*$ there exists $g \in L^q$ such that $\varphi(f) = \int fg$ for all $f \in L^p$, and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ provided μ is σ-finite.

Corollary 6.16. If $1 < p < \infty$, L^p is reflexive.

Chebyshev's Inequality. If $f \in L^p$ $(0 < p < \infty)$, the for any $\alpha > 0$,

$$\mu(\{x \ : \ |f(x)| > \alpha\}) \leq \left[\frac{||f||_p}{\alpha} \right]^p.$$

Theorem 6.18. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let K be an $(\mathcal{M} \otimes \mathcal{N})$-measureable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x,y) \, d\mu(x) \leq C$ for a.e. $y \in Y$ and $\int |K(x,y)| \, d\nu(y) \leq C$ for a.e. $x \in X$, and that $1 \leq p \leq \infty$. If $f \in L^p(\nu)$, the integral

$$Tf(x) = \int K(x,y) f(y) \, d\nu(y)$$

converges absolutely for a.e. $x \in X$, the function Tf thus defined is in $L^p(\mu)$, and $||Tf||_p \leq C||f||_p$.

Minkowski's Inequality for Integrals. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measureable function on $X \times Y$.

1. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int \left(\int f(x,y) \, d\nu(y) \right)^p d\mu(x) \right]^{\frac{1}{p}} \leq \int \left[\int f(x,y)^p \, d\mu(x) \right]^{\frac{1}{p}} d\nu(y).$$

2. If $1 \leq p \leq \infty$, $f(\cdot, y) \in L^p(\mu)$ for a.e. y, and the function $y \mapsto ||f(\cdot, y)||_p$ is in $L^1(\nu)$, then $f(x, \cdot) \in L^1(\nu)$ for a.e. x, the function $x \mapsto \int f(x,y) \, d\nu(y)$ is in $L^p(\mu)$, and

$$\left\| \int f(\cdot, y) \, d\nu(y) \right\|_p \leq \int ||f(\cdot, y)||_p \, d\nu(y).$$

Definition. If $f : X \to \mathbb{C}$ is a measurable function on $(X, \mathcal{M}, \mu)$, we define its ***distribution function*** $\lambda_f : (0, \infty) \to [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x \ : \ |f(x)| > \alpha\}).$$

Proposition 6.22.

1. λ_f is decreasing and right continuous.
2. If $|f| \leq |g|$, then $\lambda_f \leq \lambda_g$.
3. If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.
4. If $g = g + h$, then $\lambda_f(\alpha) \leq \lambda_g(\frac{1}{2}\alpha) + \lambda_h(\frac{1}{2}\alpha)$.

Proposition 6.23. If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and φ is a nonnegative Borel measurable function on $(0, \infty)$, then

$$\int_X \varphi \circ |f| \, d\mu = -\int_0^\infty \varphi(\alpha) \, d\lambda_f(\alpha).$$

Proposition 6.24. If $0 < p < \infty$, then

$$\int |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$