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1) Consider the projection map $p: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ We want to show $\mathbb{CP}^n$ is compact. Well, for all n S^n is compact, so if we take

$$p(S^{2n+1}) = \mathbb{CP}^n.$$

Since p is a quotient map, we have that p is continuous, and since the continuous image of a compact set is compact ~~so~~ we have that $\mathbb{CP}^n$ is compact.

2) Suppose there is some homeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^2$, then there must me an induced homeomorphism $\tilde{f}: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ and a corresponding isomorphism $\tilde{f}_*: \pi_1(\mathbb{R}^3 \setminus \{0\}) \to \pi_1(\mathbb{R}^2 \setminus \{0\})$
However

$$\pi_1(\mathbb{R}^3 \setminus \{0\}) = 1 \neq \mathbb{Z} = \pi_1(\mathbb{R}^2 \setminus \{0\})$$

so such an $\tilde{f}_*$ cannot exist. Thus $\mathbb{R}^3 \not\cong \mathbb{R}^2$.

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3) Let $\varphi : \mathbb{R}P^2 \to \mathbb{T}^2$ be a continuous map, and let $p : \mathbb{R}^2 \to \mathbb{T}^2$ be the universal cover of $\mathbb{T}^2$:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[r, "\varphi"'] \arrow[ur, dotted, "\tilde{\varphi}"] & \mathbb{T}^2 \end{tikzcd}$$

Now, if $\varphi : \mathbb{R}P^2 \to T^2$ is a continuous map, then it induces a homomorphism $\varphi_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{T}^2)$. Well $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, so φ_* must be the trivial homomorphism. This gives us that $\varphi_*(\pi_1(\mathbb{R}P^2)) = \{0\} \subseteq p_*(\pi_1(\mathbb{R}^2))$ so there is a lifting $\tilde{\varphi} : \mathbb{R}P^2 \to \mathbb{R}$ with $\varphi = p \circ \tilde{\varphi}$.
Now, $\mathbb{R}^2$ is contractible the image of $\tilde{\varphi}$ is also contractible meaning $\text{im}(\tilde{\varphi})$ is homotopic to a constant map under some homotopy $\tilde{H}$. Well then, under the homotopy described by

$$H = p \circ \tilde{H}$$

we have that φ is homotopic to a constant map in $\mathbb{T}^2$.

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4) THIS QUESTION HAS MULTIPLE TYPOS IN IT. FOR THE CORRECT VERSION SEE QUESTION 5 ON THE AUGUST 2014 EXAM.

5) (a) Suppose M is an n-dimensional manifold with $T_pM \cong M \times \mathbb{R}^n$. Then the standard basis $(e_1, \dots, e_n)$ produces a smooth global frame for M. which has the desired criterion.

Suppose now that there exist n vector fields $X_1, \dots, X_n$ such that $T_pM = \operatorname{span}\{X_1(p), \dots, X_n(p)\}$ Well then in some neighborhood U of any point there is an isomorphism $T_pU \cong U \times \mathbb{R}^n$ and for any $v \in T_pU$ there is a representation of v given by

$$v = v^i \frac{\partial}{\partial x^i} = \tilde{v}^i X_i(p).$$

So in a neighborhood of any point we have an isomorphism φ_p from $\operatorname{span}\{X_i\}_{i=1}^n$ to $\mathbb{R}^n$, and since these vector fields are globally defined these isomorphisms φ_p agree on all overlaps. Thus we have

$$TM \cong M \times \operatorname{span}\{X_1, \dots, X_n\}$$
$$\cong M \times \mathbb{R}^n.$$

(All we really did here is construct a global coordinate frame).

(b) First of all, S^1 is parallelizable, so $TS^1 = S^1 \times \mathbb{R}^1$. Then we have

$$T\mathbb{T}^2 \cong TS^1 \oplus TS^1$$
$$\cong S^1 \times \mathbb{R}^1 \times S^1 \times \mathbb{R}^1$$
$$\cong S^1 \times S^1 \times \mathbb{R}^2$$
$$\cong \mathbb{T}^2 \times \mathbb{R}^2.$$

If you want to use part (a) use the fact that S^1 is parallelizable to construct ~~of~~ the global coordinate frame $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ for $\mathbb{T}^2$.

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6) (a) Let M be a smooth connected orientable n-dimensional manifold with boundary, and let ω be ~~an~~ a compactly supported $n-1$ form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega.$$

(b) Let

$$\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + (z+1)\,dz.$$

(i) $$d\alpha = \frac{-(x^2+y^2)\,dy\wedge dx - y(2x\,dx + 2y\,dy)\wedge dx + (x^2+y^2)\,dx + x(2x\,dx + 2y\,dy)\wedge dy}{(x^2+y^2)^2} + dz\wedge dz$$

$$= 0.$$

(ii) Parameterize the circle $x^2+y^2=1$ by $\gamma(t) = (\cos(t), \sin(t), 0)$ $t \in [0, 2\pi)$. Then

$$\int_\gamma \alpha = \int_{t=0}^{t=2\pi} -\sin(t)(-\sin(t)\,dt) + \cos(t)(\cos(t)\,dt)$$

$$= \int_{t=0}^{t=2\pi} dt = \boxed{2\pi}$$

(iii) Suppose $C = \partial N$ with N a 2-chain in M. Then by Stokes' Theorem we should have

$$\int_C \alpha = \int_{\partial N} \alpha = \int_N d\alpha = 0,$$

but we computed $\int_C \alpha = 2\pi$ ↯.