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# Geometry/Topology

**Ph.D. Preliminary Exam**
**Department of Mathematics**
**University of Colorado Boulder**

**August 2015**

*INSTRUCTIONS:*

1. All problems are weighted equally for grading purposes.
2. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
3. Label each answer sheet with the problem number.
4. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

(1) Let $T$ and $T'$ be topologies on a space $X$, with $T \subset T'$. Prove or disprove each of the following statements:

(a) If $X$ is compact in the topology $T$, then $X$ is compact in the topology $T'$.

(b) If $X$ is compact in the topology $T'$, then $X$ is compact in the topology $T$.

(2) (a) State carefully the definitions of:

- Hausdorff topological space;
- Regular topological space.

(b) Prove the following theorem: *Every compact Hausdorff space is regular.* (In proving this, you *may not* use, without proof, the theorem stating that every compact Hausdorff space is normal.)

(3) Let $M$ be a compact connected oriented manifold of dimension $n$ with boundary. Assume that the boundary of $M$ has two connected components $M_0$ and $M_1$, and let $i_k \colon M_k \to M$ be the inclusion maps. Let $\alpha$ be a form of degree $p$ on $M$ and $\beta$ be a form of degree $(n-p-1)$ on $M$. Assume that $i_0^*\alpha = 0$ and $i_1^*\beta = 0$. Prove that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

(4) Let $g = y^\lambda\,(dx^2 + dy^2)$, $\lambda \in \mathbb{R}$, be a Riemannian metric on $M = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. For this metric compute Levi-Civita connection and write (but do not solve) the differential equation for geodesics.

(5) Recall that $\mathbb{R}P^n$ – the real projective space of dimension $n$ – is defined as the quotient of $S^n$ by the equivalence relation $x \sim -x$, $x \in S^n$. Here $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$. Let $n \geq 2$.

(a) Compute the fundamental group of $\mathbb{R}P^n$.

(b) Show that every continuous map $\mathbb{R}P^n \to S^1$ is homotopic to a constant one.

(6) For a matrix $U$ in the ring $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices with real entries, we define the *exponential* $e^U$ of $U$ by

$$e^U = I + U + \frac{U^2}{2!} + \frac{U^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{U^k}{k!}$$

($I$ denotes the identity matrix).

(a) Show that, for any $U \in M_{n\times n}(\mathbb{R})$,

$$U = \frac{d}{dt} e^{tU}\Big|_{t=0}.$$

(Of course, $tU$ denotes the product of the scalar $t$ and the matrix $U$. Also, the differentiation on the right-hand side is element-wise; that is, the derivative of a matrix is the matrix of the derivatives.) You may assume that the derivative on the right-hand side exists, and that differentiation works for infinite sums here just as it would for finite sums.

(b) For a subgroup $N$ of the group $GL(n, \mathbb{R})$ of invertible matrices in $M_{n\times n}(\mathbb{R})$, we define the *Lie algebra* $\mathrm{Lie}(N)$ of $N$ by

$$\mathrm{Lie}(N) = \{U \in M_{n\times n}(\mathbb{R})\colon e^{tU} \in N \text{ for all } t \in \mathbb{R}\}.$$

Show that, if $n = 3$ and $N$ is the *Heisenberg group*:

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subset GL(3, \mathbb{R}),$$

then

$$\mathrm{Lie}(N) = \left\{ \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Hint: part (a) above may be of use.

(c) Any $U \in \mathrm{Lie}(N)$ defines a differential operator on the algebra $C^\infty(N)$ of infinitely differentiable functions on $N$ by the formula

$$Uf(n) = \frac{d}{dt} f\big(n \cdot e^{tU}\big)\Big|_{t=0} \quad (f \in C^\infty(N), n \in N).$$

Let $Z \in \mathrm{Lie}(N)$ be the matrix with a one in the second row, third column, and zeroes elsewhere. Show that, as a differential operator on $C^\infty(N)$,

$$Zf = \frac{\partial f}{\partial b} + a\frac{\partial f}{\partial c}.$$

(Here, we are thinking of $f \in C^\infty(N)$ as a function $f(a, b, c)$ of the given coordinates of $N$.)

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1) (a) This is false. Let $X = \mathbb{R}$ $T = \{\emptyset, \mathbb{R}\}$ ~~not~~ and $T'$ = the standard topology. Then $\mathbb{R}$ is compact in $T$ but not in $T'$.

(b) This is true. Suppose $\{U_i\}_{i \in I}$ is an open cover of $X$ in $T$. Then it is also an open cover of $X$ in $T'$ so there must exist a finite subcover $\{U_{i_k}\}_{k=1}^{n}$ of $X$. Well, each $U_{i_k} \in T$ to begin with, so this is also a finite subcover of $X$ in $T$. Hence, $X$ is compact in $T$.

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2) (a) A topological space $(X, \mathcal{T})$ is called Hausdorff if for every pair of distinct points $a, b \in X$ there exist disjoint open sets $\mathcal{U}, V \in \mathcal{T}$ such that $a \in \mathcal{U}$ and $b \in V$. If $A \subseteq X$ is a closed subset of $X$ and $b \in X$ is such that $b \notin A$, then $X$ is called regular if there exist disjoint open sets $\mathcal{U}, V \in \mathcal{T}$ such that $A \subseteq \mathcal{U}$ and $b \in V$.

(b) Let $X$ be a compact Hausdorff space, $A \subseteq X$ closed and $b \in X \setminus A$. Since $A$ is a closed subset of a compact space, we know that $A$ is compact. Since $X$ is Hausdorff, we know that for every $a \in A$ there are open subsets $\mathcal{U}_a$, $V_a$ such that $\mathcal{U}_a \cap V_a = \emptyset$, $a \in \mathcal{U}_a$, and $b \in V_a$. Well, $\{\mathcal{U}_a\}_{a \in A}$ forms an open cover of $A$, so there must exist some finite subcover $\{\mathcal{U}_{a_i}\}_{i=1}^{n}$ of $A$. Then take the sets

$$V = \bigcap_{i=1}^{n} V_{a_i} \qquad \mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}_{a_i}$$

This will give us that $\mathcal{U}$ and $V$ are open, disjoint sets with $A \subseteq \mathcal{U}$ and $b \in V$. as desired.

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3) First of all, we know that since $\alpha \in \Omega^p(M)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Moreover, by Stokes' Theorem, we know that

$$\int_M d(\alpha \wedge \beta) = \int_{\partial M} \alpha \wedge \beta.$$

Now, the boundary of $M$ has 2 connected components $M_0$ and $M_1$ so we can decompose this integral into

$$\int_{\partial M} \alpha \wedge \beta = \int_{M_0} \alpha \wedge \beta + \int_{M_1} \alpha \wedge \beta.$$

However, since $i_0^* \alpha = 0$ we have $i_0^*(\alpha \wedge \beta) = 0$ on $M_0$ and since $i_1^* \beta = 0$ we have $i_1^*(\alpha \wedge \beta) = 0$ on $M_1$. So

$$\int_{M_0} \alpha \wedge \beta + \int_{M_1} \alpha \wedge \beta = 0.$$

This gives us that

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

So, rearranging this equation we get that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta.$$

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5) (a) First of all, we know that for $n \geq 2$, $S^n$ is simply connected. We also know that $p: S^n \to \mathbb{RP}^n$ is a two sheeted cover of $\mathbb{RP}^n$. Since $S^n$ is simply connected and $q$ is a covering map, we have that each fiber of $q$ has the same cardinality as the fundamental group of $\mathbb{RP}^n$. Well $|q^{-1}([x])| = 2$, so $|\pi_1(\mathbb{RP}^n)| = 2$. This gives us that

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}.$$

(b) Suppose $\varphi: \mathbb{RP}^n \to S^1$ is a continuous map and consider the standard covering map $p: \mathbb{R} \to S^1$. This gives us the following diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & S^1 \end{tikzcd}$$

Now $\varphi$ induces a homomorphism $\varphi_*: \pi_1(\mathbb{RP}^n) \to \pi_1(S^1)$, so $\varphi_*: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ which means that $\varphi_*$ is the trivial homomorphism. Thus $\varphi_*(\pi_1(\mathbb{RP}^n)) = \{0\} \subseteq p_*(\pi_1(\mathbb{R}))$ so there exists a lifting $\tilde{\varphi}: \mathbb{RP}^n \to \mathbb{R}$. Since $\mathbb{R}$ is contractable, we know that there is a homotopy $\tilde{H}$ that sends $\tilde{\varphi}$ to a constant map. Well then, the map $p \circ \tilde{H}$ is a homotopy that sends $\varphi$ to a constant map.

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6) (a) We have that

$$\frac{d}{dt} e^{tU}\Big|_{t=0} = \left[U + tU^2 + \frac{t^2U^3}{2!} + \frac{t^3U^4}{3!} + \dots\right]_{t=0}$$
$$= U.$$

(b) Suppose first that

$$U = \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow U^2 = \begin{pmatrix} 0 & 0 & uv \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow U^n = 0 \quad n \geq 3$$

Then

$$e^{tU} = \begin{pmatrix} 1 & ut & wt + tuvt^2 \\ 0 & 1 & vt \\ 0 & 0 & 1 \end{pmatrix} \in N$$

So $U \in Lie(N)$.

Alternatively if $U \in Lie(N)$, then

$$e^{tU} = I + tU + \frac{tU^2}{2!} + \dots = \begin{pmatrix} 1 & a(t) & b(t) \\ 0 & 1 & c(t) \\ 0 & 0 & 1 \end{pmatrix}$$

where $a(t)$, $b(t)$, and $c(t)$ are infinite "polynomials" in $t$.

Well, we know that $U = \frac{d}{dt} e^{tU}|_{t=0}$, so we have that

$$U = \begin{pmatrix} 0 & a'(0) & b'(0) \\ 0 & 0 & c'(0) \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$Lie(N) = \left\{ \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} : u, v, w \in \mathbb{R} \right\}.$$

(c) We have that $Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $e^{tZ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$. If we then take $n = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, we get that

$$Zf(n) = \frac{d}{dt} f(n \cdot e^{tZ})\Big|_{t=0}$$
$$= \frac{d}{dt} f\begin{pmatrix} 1 & a & at + c \\ 0 & 1 & t + b \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \frac{d}{dt} f(a, t+b, at+c)$$
$$= \frac{df_a}{dt}\frac{\partial f}{\partial a} + \frac{df_b}{dt}\frac{\partial f}{\partial b} + \frac{df_c}{dt}\frac{\partial f}{\partial c}$$
$$= 0\frac{\partial f}{\partial a} + 1\frac{\partial f}{\partial b} + a\frac{\partial f}{\partial c}$$
$$= \frac{\partial f}{\partial b} + a\frac{\partial f}{\partial c}.$$