Algebra prelim solutions August 2015

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1) Show that if the conjugacy classes of a finite group G have size at most 4, then G is solvable.

Solution: (Thanks to Derek Holt on stackexchange for the hint which enabled me to answer this - see here for a solution which drops the finiteness hypothesis).

We will use the following fact in this proof: Let G be a group with normal subgroup N. Then:

$$G \text{ is solvable} \iff N \text{ is solvable and } G/N \text{ is solvable}$$

Let G be finite and have conjugacy classes of size at most 4. Then this means that:

$$[G\colon C_G(g)] \leq 4 \quad \forall g \in G$$

If all conjugacy classes are of size 1, then G is abelian, so then $[G, G] = \{e\}$ so G is solvable. If not, take some $g \in G$ with conjugacy class of size $n = 2, 3$ or 4. Then G acts transitively on the set of cosets of $C_G(g)$, of which there are n_1 of. So we get a non-trivial homomorphism $\sigma_1\colon G \to S_n$, whence:

$$G/_{\ker \sigma_1} \cong \operatorname{Im} \sigma_1 \leq S_{n_1}$$

So ${}^{G}\!\big/_{\ker\sigma_1}$ is isomorphic to a subgroup of S_4 (as this contains S_3 and S_2 as subgroups), which is non-trivial due to the transitivity of the action. So $\ker\sigma_1$ is a subgroup of G not equal to G. As a subgroup of G, its conjugacy classes are also of size at most 4, because conjugacy classes cannot get bigger within subgroups. We claim that $\ker\sigma_1$ is solvable: if it's abelian, we're done, if not we take some $g \in \ker\sigma_1$ with non-trivial conjugacy class, of size $n_2 \in \{2, 3, 4\}$. Then we get a transitive action from $ker\sigma_1$ to S_{n_2} so then:

$$
{}^{\ker\sigma_1}\!\big/_{\ker\sigma_2} \cong \operatorname{Im}\sigma_2 \leq S_{n_2}
$$

So the quotient $\ker\sigma_1/\ker\sigma_2$ is solvable, and $\ker\sigma_2 \lhd \ker\sigma_1$ again has conjugacy classes of all of size at most 4. So we keep iterating to get:

$$
G \rhd \ker\sigma_1 \rhd \cdots \rhd \ker\sigma_{m-1} \rhd \ker\sigma_m
$$

As these inclusions are proper and G is finite, this set of inclusions does indeed terminate eventually because the order decreases each time: so $\ker\sigma_m$ is abelian for some $m \in \mathbb{N}$ (which precisely means that all of its conjugacy classes are singletons). Then by induction $\ker\sigma_{m-1}$ is solvable as $\ker\sigma_m$ and $\ker\sigma_{m-1}/\ker\sigma_m$ is solvable (subgroup of S_{n_m} where $n_m \leq 4$). So we have $\ker\sigma_m$ solvable $\implies$ $\ker\sigma_{m-1}$ solvable $\implies \cdots \implies$ $\ker\sigma_1$ solvable $\implies$ G solvable .

2) Show that if F is a nontrivial free group, then F has a proper subgroup of finite index.

Solution:
Let $A \neq \emptyset$ be a set, with $F = F(A)$. Then consider:

$$
f\colon F(A) \twoheadrightarrow F(A)^{\mathbf{Ab}} =: G
$$

A surjective homomorphism from $F(A)$ to G, the free *Abelian* group generated by A: Note that this exists by the universal property of free groups: We have the injective set function $\iota\colon A \to F(A)^{\mathbf{Ab}}$, so thus f is the (unique) surjective group homomorphism making the following diagram commute:

$$\begin{tikzcd} F(A) \arrow[r, "f"] & F(A)^{\mathbf{Ab}} \\ A \arrow[u, hook] \arrow[ur, hook, "\iota"] & \end{tikzcd}$$

where the unmarked arrow is the obvious inclusion $A \to F(A)$. Now, G consists of elements of the form:

$$g = \sum_{a \in A} \lambda_a a$$

Where $a \in A$ and $\lambda_a \in \mathbb{Z}$, non-zero for only finitely many $a \in A$. Pick some arbitrary $b \in A$. Then consider the subgroup H of G generated by:

$$\{2b\} \cup \bigcup_{a \in A \setminus \{b\}} a$$

(note this subgroup is automatically normal because G is Abelian). Then considering cosets of H in G we have that in G/H, taking some arbitrary $g \in G$

$$g = \sum_{a \in A} \lambda_a a = \lambda_b b + \sum_{a \in A \setminus \{b\}} a \equiv \lambda_b \equiv \begin{cases} 0 \text{ if } \lambda_b \text{ is even} \\ b \text{ if } \lambda_b \text{ is odd} \end{cases}$$

Then H obviously has only 2 cosets. So we have a surjection $\pi \colon G \to \mathbb{Z}_2$. But then $\pi \circ f$ is a surjection from $F(A)$ to $\mathbb{Z}_2$, so by the first isomorphism theorem:

$$F(A) \big/ \ker(\pi \circ f) \cong \mathbb{Z}_2$$

i.e. $\ker(\pi \circ f)$ is an index 2 subgroup of $F(A)$, so $F(A)$ has a proper subgroup of finite index (proper as the above isomorphism isn't with the trivial group, so $\ker(\pi \circ f)$ can't be all of $F(A)$. As A was an arbitrary non-singleton set (only non-singletons generate non-trivial free groups) we have the result.

3) Show that if R is PID and S is an integral domain containing no subfield, then any homomorphism $\varphi\colon R \to S$ is injective.

Solution:
By the first isomorphism theorem for rings:

$$R/\ker(\varphi) \cong \operatorname{Im}\varphi$$

Note that $\operatorname{Im}\varphi$ is a subring of S (and so is also an integral domain). Then we combine the facts that for any commutative ring R with non-trivial ideal I:

$$R/I \text{ is a field} \iff I \text{ is a maximal ideal of } R$$

And that PID $\implies$ commutative ring, and that in a PID an ideal I is maximal iff it is prime. We also have for arbitrary commutative ring R:

$$R/I \text{ is an integral domain} \iff I \text{ is a prime ideal of } R$$

For any proper ideal I. So S being an integral domain implies that $\operatorname{Im}\varphi$ is an integral domain (as it's a subring of S), which implies that $\ker(\varphi)$ is a prime ideal, which in turn implies that it's maximal as R is a PID, which then means that $\operatorname{Im}(\varphi) \cong R/\ker(\varphi)$ is a field. So we must have that $\ker(\varphi)$ is *not* a proper ideal of R: if it's R then this is impossible as this implies that 1_R is in $\ker(\varphi)$, but ring homomorphisms send 1 to 1, so S is the zero ring, which is *not* an integral domain. So the only other option is that $\ker(\varphi) = \{0\}$ i.e. φ is injective.

4) Let A be an $n \times n$ matrix over a field K. Show that if A has exactly one invariant factor, then any matrix B that commutes with A must be a polynomial in A (That is, show that if $BA = AB$, then $B = p(A)$ for some $p(x) \in K[x]$.)

Solution:

5) Suppose that $f \in \mathbb{Z}[x]$ is a monic irreducible polynomial of degree 4. Sup-

pose further that there is a complex number α such that both α and α^2 are roots of f. What is f?

Solution:
Consider $\alpha = \zeta_5$, primitive 5th root of unity in $\mathbb{C}$. Then:

$$x^5 - 1 = (x - \zeta_5)(x - \zeta_5^2)(x - \zeta_5^3)(x - \zeta_5^4)(x - 1)$$

so

$$f(x) = (x - \zeta_5)(x - \zeta_5^2)(x - \zeta_5^3)(x - \zeta_5^4) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Z}[x]$$

Has α and α^2 as roots and is of degree 4. It is irreducible as:

$$f(x + 1) = \frac{(x + 1)^5 - 1}{x + 1 - 1}$$

is irreducible by Eisenstein's criterion with $p = 5$. So f satisfies the hypotheses of the question.

6) Let ζ be a primitive 8th root of unity and let $K = \mathbb{Q}[\zeta]$. Determine $\mathrm{Gal}(K/\mathbb{Q})$ and all the intermediate fields between $\mathbb{Q}$ and K.

Solution: