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1) (a) Suppose $X/\sim$ is T_1, and let $[a],[b]$ be distinct members of $X/\sim$. Then there exists an open subset $V_{[b]}$ such that $[a] \notin V_{[b]}$ but $[b] \in V_{[b]}$. Well then we can determine that

$$\{[a]\}^c = \bigcup_{[a] \neq [b]} V_{[b]}$$

is an open set, so $\{[a]\}$ is closed. Since projection is continuous, we have $\pi^{-1}([a]) \subseteq X$ is closed.

Now suppose $[a] \subseteq X$ is closed for all $a \in X$, and consider $\{[a]\} \subseteq X/\sim$. Well $\pi^{-1}(\{[a]\}) = [a]$ is closed so since π is a quotient map, we get that $\{[a]\}$ must be closed. Therefore, if we have two distinct points $[a]$ and $[b]$ in $X/\sim$ we can take the two open sets that separate them to be $\{[a]\}^c$ and $\{[b]\}^c$. Thus $X/\sim$ is T_1.

(b) Suppose $X/\sim$ is Hausdorff. The set $\sim$ is defined to be $\sim := \{(a,b) \in X \times X : a \sim b\}$. Well, if $X/\sim$ is Hausdorff, we know that the diagonal

$$\Delta_{X/\sim} = \{([a],[b]) : [a]=[b]\} = \{(a,b) : a \sim b\}$$

is closed in $X/\sim \times X/\sim$. We can then take the preimage of the continuous map $\pi \times \pi$ to get

$$(\pi \times \pi)^{-1}(\Delta_{X/\sim}) = \sim$$

is closed in $X \times X$.

(c) Suppose π is an open map. The forward direction was already proved in part (b), so we will deal with the reverse. Since $\sim$ is closed $\sim^c$ is open and

$$\pi \times \pi(\sim^c) = \{(a,b) : a \not\sim b\} = \Delta_{X/\sim}^c$$

is open in $X/\sim \times X/\sim$. Thus the diagonal is closed in $X/\sim \times X/\sim$ and we have that $X/\sim$ is Hausdorff.

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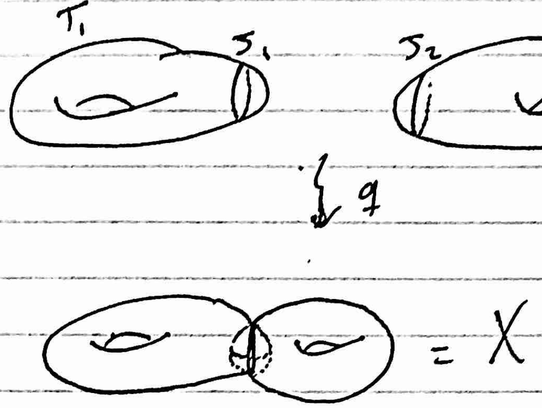
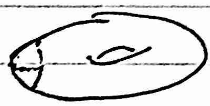
* 2) BE CAREFUL: THIS IS NOT THE 2-HOLED TORUS.

We can represent the quotient defined in this problem as follows



Where the intersection of T_1 and T_2 forms an item homeomorphic to a sphere.

Let

$U =$  $V =$ 

$U \cap V =$ 

Using ~~the~~ deformation retractions, we can see U and V are homotopic to a torus, thus

$$\pi_1(U) = \pi_1(V) = \pi_1(T^2) \quad \text{and} \quad \pi_1(U \cap V) = 1.$$

So by SVK we have

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \pi_1(T^2) *_1 \pi_1(T^2).$$

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3) Let ω be the smooth non-vanishing n-form on Y corresponding to the given orientation on Y. And let η be the smooth non-vanishing orientation form for X. Since X is connected, we know that if α is another non-vanishing orientation form on X then

$$\eta = F\alpha \qquad \text{or} \qquad \eta = -F\alpha$$

for some smooth, positive function F. Let $(E_i|_p)$ be an oriented basis of T_pM. Now, $f^*\omega$ is a non-vanishing orientation form on X, so if $\eta = Ff^*\omega$ then

$$\eta_p(E_1|_p, \ldots, E_n|_p) = F(p)(f^*\omega)(E_1|_p, \ldots, E_n|_p) > 0$$

which means

$$f^*\omega(E_1|_p, \ldots, E_n|_p) = \omega(df_p(E_1|_p), \ldots, df_p(E_n|_p)) > 0$$

so f is orientation preserving. Similarly, if $\eta = -Ff^*\omega$ then f is orientation reversing.

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4) S^1 is parallelizable because it has the non-vanishing global vector field

$$\omega = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

S^2 is not parallelizable by the Hairy Ball Theorem.

S^3 is parallelizable because it is isomorphic to the Lie group $SU(2)$ and all Lie groups are parallelizable.

(We could have also used that $S^1 \cong SO(2)$ which is also a Lie group).

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5) (a) We only need to show ω is exact. Consider

$$\alpha = x + \frac{1}{\pi}\sin(\pi x)\cos(\pi x) + y.$$

Then

$$d\alpha = (1 + \cos(\pi x)^2 - \sin(\pi x)^2)\,dx + dy$$
$$= 2\cos(\pi x)^2\,dx + dy.$$

(b) We have $f(a,b) = (3a+2b,\ a-b)$ $\quad \omega = 2\cos(\pi x)^2 dx + dy$

$$x(a,b) = 3a+2b \qquad y(a,b) = a-b$$
$$dx = 3da + 2db \qquad dy = da - db$$

So

$$f^*\omega = 2\cos[\pi(3a+2b)]^2(3da+2db) + da - db.$$

Along the curve $\gamma(a) = (a,0)$ from $a=0$ to $a=1$ we get

$$\int_\gamma f^*\omega = \int_{a=0}^{a=1} 2\cos(3\pi a)^2 \cdot 3da + da.$$
$$= 4$$

(c) We know that $p^*\eta = \omega$, and

$$d\omega = d(p^*\omega)$$
$$= p^*(d\eta)$$
$$= 0.$$

So for any vector field X $p^*(d\eta)(X) = 0$ which will only be true for every vector field if $d\eta(X) = 0$. Thus η is closed.

(d) $d\bar{f}^*\eta = \bar{f}^* d\eta = 0$ so $\bar{f}^*\eta$ is closed. Now arithmetic in $\mathbb{R}^2/\mathbb{Z}^2$ is done mod 1 (take only the decimal part) so if we were to compute $\bar{f}^*\eta$ with respect to the usual global coordinates (u,v) on $\mathbb{T}^2$, we would get

$$\bar{f}^*\eta = 6\cos(3\pi u + 2\pi v)^2 du + 4\cos(3\pi u + 2\pi v)^2 dv.$$

For $\bar{f}^*\eta$ to be $d\beta$ for some zero form β we would need

$$\beta = 3u + \frac{1}{3\pi}\sin(3\pi u + 2\pi v)\cos(3\pi u + 2\pi v) + 2v + \frac{1}{2\pi}\sin(3\pi u + 2\pi v)\cos(3\pi u + 2\pi v)$$

but these cannot exist in $\mathbb{T}^2$ due to the mod 1 arithmetic.