RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry-Topology

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2015

INSTRUCTIONS:

(1) Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
(2) Label each answer sheet with the problem number.
(3) **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Let $(X, \mathfrak{T})$ be a topological space, and let $\sim$ be an equivalence relation defined on X. Denote by $(X/\sim, \mathfrak{Q})$ the set of equivalence classes with the quotient topology $\mathfrak{Q}$. Prove that if $(X/\sim, \mathfrak{Q})$ is Hausdorff, then every equivalence class C in X is a closed set of $(X, \mathfrak{T})$.

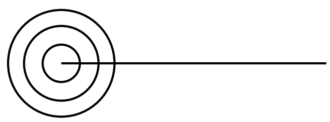
Problem 2. Let $Q = [0,1] \times [0,1]$ be the unit square with the topology induced from the standard topology in $\mathbb{R}^2$. Define the following equivalence relation $\sim$ on Q:

- every point is equivalent to itself;
- $(0, x)$ is equivalent to $(1, x)$ $\forall x \in [0,1]$;
- $(x, 0)$ is equivalent to $(1-x, 1)$ $\forall x \in [0,1]$.

The Klein bottle $\mathbb{KB}$ is defined to be the topological space $(Q/\sim, \mathfrak{Q})$ where $\mathfrak{Q}$ is the quotient topology.

(1) Show (pictorially if you like) that the 2-torus $\mathbb{T}^2$ is a double cover of the Klein bottle $\mathbb{KB}$.
(2) Can a closed orientable surface that is not homeomorphic to the 2-sphere or the 2-torus be a cover of the 2-torus $\mathbb{T}^2$? Explain.

Problem 3. Let $(\mathbb{R}^2, \mathcal{ST}_2)$ be $\mathbb{R}^2$ with the standard topology $\mathcal{ST}_2$. Let C be the subset of $\mathbb{R}^2$ defined below, and let $(C, \mathcal{ST}_2|_C)$ be C with the induced topology from $\mathcal{ST}_2$. Use the Seifert–Van Kampen Theorem to find the fundamental group of $(C, \mathcal{ST}_2|_C)$.



C

$$C := \left(S_1 \bigcup S_2 \bigcup S_3 \bigcup \{(x,0) | 0 \leq x \leq 5, x \in \mathbb{R}\}\right) \subset \mathbb{R}^2,$$

where

$$S_i = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = i\},\ i = 1,2,3.$$

Problem 4. Consider the vector fields on $\mathbb{R}^3$:

$$X = x(2y + \cos y)\frac{\partial}{\partial x} - \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(1) Write down a submanifold N of $\mathbb{R}^3$ containing the point $(0,1,0)$ and having the property that there is an open neighborhood $U \subseteq N$ of $(0,1,0)$ such that for each point $n \in U$ the tangent space $T_nN \subseteq T_n\mathbb{R}^3$ is equal to the span of $X(n)$ and $Y(n)$.
(2) Compute the Lie bracket $[X, Y]$.
(3) Let $\mathcal{D}$ be the set of vector fields that can be written in the form $fX + gY$ for some functions $f, g \in C^\infty(\mathbb{R}^3)$. Is $[X, Y]$ in $\mathcal{D}$? Explain.
(4) Can you write down a submanifold N of $\mathbb{R}^3$ containing the point $(1,0,0)$ and having the property that there is an open neighborhood $U \subseteq N$ of $(1,0,0)$ such that for each point $n \in U$ the tangent space $T_nN \subseteq T_n\mathbb{R}^3$ is equal to the span of $X(n)$ and $Y(n)$? Explain.

Problem 5. Let M be a smooth manifold.

(1) Let $p \in M$. Show there is an open neighborhood U of p such that for all $q \in U$, there is a diffeomorphism $f : M \to M$ such that $f(p) = q$. [Hint: you may want to consider flows of appropriately chosen vector fields, or, alternatively maps $\mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto x + \eta(x)y$ with $y \in \mathbb{R}^n$ and $\eta(x)$ an appropriately chosen function.]

(2) Deduce from part (1) that if M is connected, then the action of the diffeomorphism group of M on M is transitive; i.e., show that given any two points $p, q \in M$ there is a diffeomorphism $f : M \to M$ with $f(p) = q$.

Problem 6. Let (M, g) be a Riemannian manifold with Riemannian curvature tensor R. Consider the $(0, 2)$-tensor ω, defined by the trace

$$\omega(u, v) := \mathrm{tr}(w \mapsto R(u, v)(w)),$$

where u, v, w are vector fields on M. Show that $\omega = 0$.

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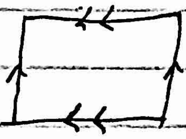
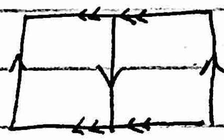
1) Suppose that $(X/\sim, Q)$ is Hausdorff. Then $\{[x]\} \subseteq X/\sim$ is closed for all $[x] \in X/\sim$. So $\{[x]\}^c$ is open and $\pi^{-1}(\{[x]\})^c = \{y \in X : y \not\sim x\}$ is open in X. Thus

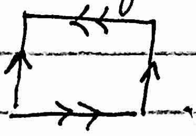
$$\pi^{-1}(\{[x]\}^c)^c = \{y \in X : y \sim x\} = \pi^{-1}(\{[x]\})$$

is closed, so every equivalence class of X is closed in (X, τ).

2) (a) A Torus can be given by the quotient on the polygon.

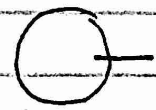


Dividing in half, we can get  which provides an obvious double cover of the Klein bottle represented by



* (b) No. The universal cover of T^2 is $\mathbb{R}^2$. We have that the other covers of $\mathbb{T}^2$ are derived from how subgroups of $\mathbb{Z}^2$ act on $\mathbb{R}^2$. So the only covers for $\mathbb{T}^2$ are homeomorphic to $\mathbb{R}^2/\mathbb{Z}$, $\mathbb{R}^2/\mathbb{Z}^2$, or $\mathbb{R}/\{0\}$.

3) Take

$U =$  $V =$  $U \cap V = -$

Then by SVK we have

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = (\mathbb{Z} * \mathbb{Z}) *_1 \mathbb{Z} = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

($\pi_1(U) = \mathbb{Z} * \mathbb{Z}$ can be seen by homotoping the intersecting line down to a point producing a boquet of two Circles).

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4) (a) At $p=(0,1,0)$, we have that $X_p = -\frac{\partial}{\partial z}$ and $Y_p = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, so $\text{span}\{X_p, Y_p\} = \text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, so we can simply take the yz plane since at every point of this plane $X_p = -\frac{\partial}{\partial z}$ and $Y_p = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

(b) $[X,Y] = X(1)\frac{\partial}{\partial y} + X(1)\frac{\partial}{\partial z} - Y(x(z_y + \cos(y)))\frac{\partial}{\partial x} + Y(1)\frac{\partial}{\partial z}$
$= (x\sin(y) - 2x)\frac{\partial}{\partial x}$.

(c) No, $[X,Y] \neq fX + gY$ for $f,g \in C^\infty(\mathbb{R}^3)$. For any combination $fX + gY$ we have that the $\frac{\partial}{\partial y}$ component will have g as it's only component function. If this is the case then for $fX + gY = [X,Y]$ we would need $g=0$, but then for the $\frac{\partial}{\partial z}$ component to disappear, we would need $f=0$, so $fX + gY = 0 \neq [X,Y]$.

* (d) No. If we compute the flow of X, we get that
$$\theta_t(x,y,z) = (xe^{2yt + t\cos(y)}, y, z-t),$$
and the flow of Y is
$$\psi_s(x,y,z) = (x, y+s, z+s).$$
If we then take the cross-product we will get that the normal vector for the desired submanifold at a given point along one of the flow lines is determined by
$$\theta_t(1,0,0) \times \psi_s(1,0,0) = (st, -se^t - t, se^t).$$
In particular, at the start of the flows when $s=t=0$ we have that the normal vector would be
$$\theta_0(1,0,0) \times \psi_0(1,0,0) = (0,0,0)$$
Which isn't possible since this would imply that $\dim(N)=3$ but $\dim(T_pN) = \dim(\text{span}\{X_p, Y_p\}) = 2$.

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5) (a) Fix some $y \in M$ and define

$$S_y = \{x \in M : \exists f \in \mathrm{Diff}(M),\ f(x) = y\}.$$

S_y is nonempty since $y \in S_y$.

Consider any $p \in M$ and a chart (U, φ) centered at p where $\varphi : U \to B^n(0,1)$. Then for $z \in U$, denote by $\hat{c} = \varphi(z)$ the coordinates of z and choose some r, with $\|\hat{c}\| < r < 1$. Then the constant vector field $X = c^i \frac{\partial}{\partial x^i}$ is well defined on U, and by multiplying X by a bump function, we can extend X to a globally defined compactly supported vector field $\tilde{X}$ with $\tilde{X}|_{\varphi^{-1}(B^n(0,1))} = X$. Since $\tilde{X}$ has compact support, it generates a globally defined flow θ_t. The curve $\gamma(t) = f^{-1}(t\hat{c})$ is an integral curve of $\tilde{X}$ satisfying $\gamma(0) = x$ and $\gamma(1) = z$. Since all global flows of global vector fields are diffeomorphisms, we have the desired result.

(b) From the above argument, we have that $x \in S_z$ and $z \in S_x$. Let $x \in S_y$ and choose U as above. Then for all $z \in U$, $z \in S$ and in turn $z \in S_y$. Thus S_y is open.

Next if $\{x_n\} \in S_y$ and $x_n \to x$, then by choosing U around x as above, we have both $x_n \in S_y$ for all n and for all $n \geq N \in \mathbb{N}$ $x_n \in U$, so $x \in S_y$ and S_y is closed. Since M is connected, it has no nontrivial clopen sets which gives us that $S_y = M$.