RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Recall that a topological space X is said to be *normal* if its points are closed subsets and for every pair of disjoint closed subsets A and B of X there is a pair of disjoint open subsets U and V with $A \subset U$ and $B \subset V$. Let $\pi : X \to Y$ be a surjective continuous and closed map. Show that if X is a normal space, then Y is normal as well.

Problem 2. Let $X = S^1 \times S^1$. Prove that for every $n \geq 2$, every continuous map $S^n \to X$ is homotopic to a constant map.

Problem 3. Let $\mathrm{GL}_n(\mathbb{R})$ be the space of $n \times n$ invertible matrices with entries in $\mathbb{R}$ together with the natural manifold structure inherited from the ambient space of $n \times n$ matrics with entries in $\mathbb{R}$. Prove that the conjugation map

$$\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (g, h) \mapsto ghg^{-1}$$

is differentiable and compute its tangent map. Conclude that the conjugation map is continuous.

Problem 4. (i) Show that $\mathrm{GL}_n(\mathbb{R})$ with the topology as in Problem 3 is disconnected for all positive integers n.

(ii) Show that for even n the matrix $-1_n := \mathrm{diag}(-1, \cdots, -1)$ can be connected by a path in $\mathrm{GL}_n(\mathbb{R})$ with the identity matrix 1_n.

(iii) Let $\mathrm{GL}_n(\mathbb{R})^+$ be the set of matrices in $\mathrm{GL}_n(\mathbb{R})$ with positive determinant. Show that $\mathrm{GL}_n(\mathbb{R})^+$ is path connected. (Hint: Use Problem 3, Problem 4 (ii), and the fact that every matrix is conjugate to an upper triangular matrix to construct a path between a matrix with positive determinant and the identity matrix.)

(iv) Compute $H_0(\mathrm{GL}_n(\mathbb{R}))$ for all positive integers n.

Problem 5. Let M be a compact, oriented n-dimensional manifold (without boundary) and suppose that ω and η are p- and q-forms on M with $p + q = n - 1$. Prove that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

Problem 6. Let N be a compact embedded submanifold of a manifold M. Show that $\Omega^p(M) \to \Omega^p(N)$ is surjective for all p.

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1) Suppose X is Normal and $\pi : X \to Y$ is surjective, continuous, and closed.

Let $C, D \subseteq Y$ be closed, disjoint subsets. Then $A = \pi^{-1}(C)$ and $B = \pi^{-1}(D)$ are closed, disjoint subsets of X. So there exist open, disjoint subsets $A \subseteq U$ and $B \subseteq V$. Consider the sets U^c and V^c which are closed in X with $A \subseteq V^c$ and $B \subseteq U^c$. Well $\pi^{-1}(C) \subseteq V^c$ so $C = \pi(\pi^{-1}(C)) \subseteq \pi(V^c)$ and $\pi(V^c)$ is closed since π is closed. This gives us that $\pi(V^c)^c$ is open with $D \subseteq \pi(V^c)^c$ and similarly $C \subseteq \pi(U^c)^c$. Moreover, $\pi(U^c)^c \cap \pi(V^c)^c = \emptyset$ since

$$\pi(U^c)^c \cap \pi(V^c)^c = Y \setminus \pi(U^c) \cup \pi(V^c)$$

$$\text{and } \pi(U^c) \cup \pi(V^c) = Y.$$

Thus Y is normal.

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2) Let $\varphi : S^n \to X$ be some continuous map, and consider the following diagram:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Consider the universal cover $p : \mathbb{R}^2 \to X = \mathbb{T}^2$. For $n \geq 2$ we know that $\pi_1(S^n) = 1$ so

$$\varphi_*(\pi_1(S^n)) \subseteq \{0\} = p_*(\pi_1(\mathbb{R}^2))$$

and there must exist a lifting $\tilde{\varphi} : S^n \to \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible we know that there exists a homotopy $\tilde{H}$ that sends the image of $\tilde{\varphi}$ to a point, so $\tilde{\varphi}$ is homotopic to a constant map under $\tilde{H}$. Composing p with $\tilde{H}$ we get a homotopy $H = p \circ \tilde{H}$ that sends φ to a constant map.

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3) Define the map

$$F : GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R}) : (g,h) \mapsto ghg^{-1}.$$

First of all, $GL_n(\mathbb{R})$ is a Lie group, so the multiplication and inversion maps are both differentiable and smooth. Let i_1 denote inversion the first coordinate and m denote the multiplication map. Then

$$(g,h) \xmapsto{m \times i_1} (gh, g^{-1}) \xmapsto{m} ghg^{-1}$$

and by the Leibniz rule and composition rules for the differential we get that F is both continuous and differentiable.

Now, we know that

$$dm_{(e,e)}(X,Y) = X + Y \quad \text{and} \quad di_e(X) = -X$$

This gives us that

$$dF_{(e,e)} = g + h - g = h.$$

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4) (i) Let $U = \{A \in GL_n(\mathbb{R}) : \det(A) > 0\}$ and $V = \{A \in GL_n(\mathbb{R}) : \det(A) < 0\}$. We can see that $U \cap V = \emptyset$ and $GL_n(\mathbb{R}) = U \cup V$. We have that U and V are open when $GL_n(\mathbb{R})$ as viewed as the metric space $(GL_n(\mathbb{R}), \det)$.

(ii) If n is even we have that $\det(-I_n) = 1$. Well, the determinant function $\det_+ : U \to (0, \infty)$ where U is the set given in part (i). ~~So~~ This map is continuous and surjective so $\det_+^{-1}((0,\infty)) = U$ must be path connected. Hence, there is a path between $-I_n$ and I_n.

(iii) This was actually done already in (ii). Another way to do this, however, is to use the exponential map.

(iv) $H_0(X) = \mathbb{Z}^n$ where n is the number of connected components of X, so we have that

$$H_0(GL_n(\mathbb{R})) = \mathbb{Z} \times \mathbb{Z}.$$

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5) First of all, since $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$, we have that.

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

Moreover, by Stoke's Theorem we get that

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^p \omega \wedge d\eta = 0$$

since $\partial M = \emptyset$. Rearranging this we get that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

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6) Let $\iota : N \to M$ be the inclusion map and pick a point $q \in N$. Let $(U_q, (x^i))$ be an open chart centered at $\iota(q) = q$ such that $U_q \cap N = V_q$ is a local k-slice of U_q. Then we know that there is an open chart $(\iota^{-1}(V_q), (y^i)) = (V_q, (y^i))$ of N such that $y^i(a) = x^i(a)$ for $a \in N$ and $i \in \{1, \dots, k\}$. Then for any $\omega \in \Omega^p(V_q)$, we can express ω in terms of these local coordinates as a sum over increasing multi-indices I by

$$\omega = \sum_I{}' \omega_I \, dy^{i_1} \wedge \dots \wedge dy^{i_p}$$

where each ω_I is a smooth coordinate function defined on V_q. Well, if we keep the same indexing set and define $\eta \in \Omega^p(U_q)$ by

$$\eta = \sum_I{}' \eta_I \, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with $\eta_I(a) = \omega_I(a)$, then

$$\begin{aligned} \iota^* \eta &= \sum_I{}' (\eta_I \circ \iota) \, d(x^{i_1} \circ \iota) \wedge \dots d(x^{i_p} \circ \iota) \\ &= \sum_I{}' \omega_I \, dy^{i_1} \wedge \dots \wedge dy^{i_p} = \omega; \end{aligned}$$

hence we have $\Omega^p(U_q) \to \Omega^p(V_q)$ is surjective for all p.

Now, we know that $\{U_q\}_{q \in N}$ forms an open cover of N and admits a smooth partition of unity $\{\varphi_q\}_{q \in N}$ subordinate to this open cover. Define $\psi_q = \varphi_q|_N$. Then the collection $\{\psi_q\}_{q \in N}$ will be a smooth partition of unity of N subordinate to the open cover $\{V_q\}_{q \in N}$. Let $\tilde{\omega} \in \Omega^p(N)$ be any smooth p-form on N. Since this form is smooth, we know that there is an indexed collection of p-forms $\{\omega^q\}_{q \in N}$ such that

$$\tilde{\omega} = \sum_{q \in N} \psi_q \omega^q.$$

Now define the p-form $\tilde{\eta} \in \Omega^p(M)$ by

$$\tilde{\eta} = \sum_{q \in N} \varphi_q \eta^q$$

where each η^q is such that $\iota^* \eta^q = \omega^q$ as given in the first part of this proof, and $\tilde{\eta}|_y = 0$ for any $y \notin \bigcup_{q \in N} U_q$. This will be defined and smooth on all of M since $\operatorname{supp}(\varphi_q \eta^q) \subseteq U_q$. Furthermore, since this partition of unity is locally finite, for any $x \in N$, we have that $\tilde{\omega}|_x = \iota^* \tilde{\eta}|_x$, and so $\tilde{\omega} = \iota^* \eta$. Hence $\Omega^p(M) \to \Omega^p(N)$ is surjective for all p.