Top /Diff Geo Prelim Exam August 2011.

1) (a) We want to show that X is Hausdorff. Consider the following diagram

$$x \sim y \text{ if } f(x) = f(y)$$

$$\mathbb{R}^2 \xrightarrow{\ \ f\ \ } \mathbb{R}$$
$$q \downarrow$$
$$X = \mathbb{R}^2/_\sim \xrightarrow{\ \ g\ \ } \mathbb{R}$$

Since f is constant on the fibers $q^{-1}([x])$, we know that f induces a continuous map g s.t. $g \circ q = f$. Now consider $[x], [y] \in X$ with $[x] \neq [y]$. Then $g([x]) \neq g([y])$ and since $g([x]), g([y]) \in \mathbb{R}$ which is Hausdorff, $\exists$ open sets U, V in $\mathbb{R}$ with $g([x]) \in U$, $g([y]) \in V$, and $U \cap V = \emptyset$. Well, g is continuous, so $g^{-1}(U)$ and $g^{-1}(V)$ are open, $[x] \in g^{-1}(U)$, $[y] \in g^{-1}(V)$, and $g^{-1}(U) \cap g^{-1}(V) = \emptyset$. Thus, X is Hausdorff.

(b) Suppose that X is not connected so $X = U \sqcup V$ with U and V open. Then $\mathbb{R}^2 = q^{-1}(U) \sqcup q^{-1}(V)$, so $\mathbb{R}^2$ is disconnected $\lightning$.
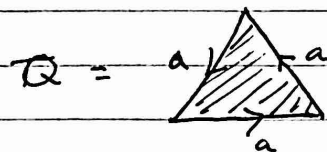
2) Suppose there exists a homeomorphism $f: D^2 \to D^2$ such that $f|_{S^1}$ is not a homeomorphism. Then, there exists some $s \in S^1$ such that $f(s) \notin S^1$. If we then consider the spaces $D^2 \backslash \{x\}$ and $D^2 \backslash \{f(x)\}$, there is an induced homeomorphism $\tilde{f}: D^2 \backslash \{x\} \to D^2 \backslash \{f(x)\}$. Consider some fixed point $y \in D^2$ $x \neq y \neq f(x)$. We know that since $x \in \partial D^2$ and $f(x) \in \text{int}(D^2)$, $D^2 \backslash \{x\}$ is homotopic to D^2 and $D^2 \backslash \{f(x)\}$ is homotopic to S^1, so

$$\pi_1(D^2 \backslash \{x\}, y) = 1, \quad \pi_1(D^2 \backslash \{f(x)\}, y) = \mathbb{Z},$$

but then $\tilde{f}_*: 1 \to \mathbb{Z}$ cannot be an isomorphism and hence $\tilde{f}$ is not a homeomorphism $\rightarrow\leftarrow$. Thus $f(x) \in S^1$ and $f|_{S^1}$ is a homeomorphism.

3)

$$Q = a \triangle a$$
$$\quad\quad a$$

* (a) Q is not a topological manifold because it fails to be locally Euclidean where the corners meet. If Q were to be a topological manifold, it would need to have dimension 2. Therefore, any open neighborhood of the meeting place of the three corners would need to be homeomorphic to $\mathbb{R}^2$. However, with the way that the equivalence relation is defined, any open neighborhood of the point where the corners meet would self-intersect which doesn't happen in $\mathbb{R}^2$ even when embedded in a higher dimensional space.

(b) Let

$$U = \triangle \qquad V = \triangle \qquad \cancel{U \cup V}\; U \cap V = \triangle$$
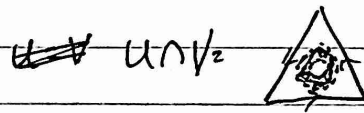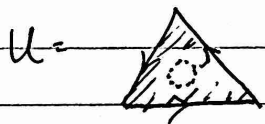
Then we can apply Seifert - Van Kampen (SVK) to get that

$$\pi_1(X) \cong \mathbb{Z}/3\mathbb{Z}.$$

To see this, consider the diagram

$$\langle a \rangle = \pi_1(U \cap V) \xrightarrow[i_{2*}]{i_{1*}} \begin{array}{c} \pi_1(U) \\ \pi_1(V) \end{array} \xrightarrow[j_{2*}]{j_{1*}} \pi_1(X)$$

* Since $\pi_1(U) = \langle a \rangle \quad \pi_1(V) = 1 \quad i_{1*}(a) = a^3 \quad i_{2*}(a) = 1$
$$\pi_1(U \cap V) = \langle a \rangle$$

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \frac{\langle a \rangle * 1}{\langle a^3 = 1 \rangle} = \langle a \mid a^3 = 1 \rangle = \mathbb{Z}/3\mathbb{Z}$$

4) (a) $\alpha = \cos(z)\,dx + \sin(z)\,dy$

$d\alpha = -\sin(z)\,dz \wedge dx + \cos(z)\,dz \wedge dy$

$\alpha \wedge d\alpha = \cos(z)^2\,dx \wedge dz \wedge dy - \sin(z)^2\,dy \wedge dz \wedge dx$

$= \cos(z)^2 + \sin(z)^2\,dx \wedge dz \wedge dy$

$= dx \wedge dz \wedge dy \neq 0. \quad \forall\, p \in T^3$

So α is a contact form.

(b) Suppose that there were two Reeb fields for α A, B. Well if $\alpha'(A) = \alpha(B)$ then we have that for every point (x, y, z)

$\cos(z)\,dx(A) + \sin(z)\,dy(B) = \cos(z)\,dx(B) + \sin(z)\,dy(B)$.

If we then pick the points $(0,0,0)$ and $(0,0,\pi/2)$ we get

$dx(A) = dx(B)$ and $dy(A) = dy(B)$.

Similarly, we would have

$d\alpha(A, B) = -\sin(z)\,dz \wedge dx\,(A, B) + \cos(z)\,dz \wedge dy\,(A, B)$

and by using the relation $dx(B) = dx(A)$ derived previously and evaluating at $(0, 0, \pi)$ we have

$\big(dz(A) - dz(B)\big)\,dx(A) = 0.$

We can also use $dy(B) = dy(A)$ and $(0,0,0)$

$\big(dz(A) - dz(B)\big)\,dy(A) = 0,$

and putting this all together, we get that

$dz(A) = dz B$

So $A \equiv B$.

(cont.) 4) (b) For X to be the Reeb field of α we need $\alpha(X) = 1$ and $d\alpha(X) = 0$, so solving the set of equations that this gives us we have that the Reeb field of α is

$$X = \cos(z)\frac{\partial}{\partial x} + \sin(z)\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}.$$

(c) Taking X as above, we can compute that the flow of X is given by

$$\Theta_t(x,y,z) = (x + \cos(z)t, \; y + \sin(z)t, \; z)$$

and this is not periodic because if $(x_0, y_0, z_0) \neq (x_1, y_1, z_1)$

$$(x_0 + \cos(z_0)t, \; y_0 + \sin(z_0)t, \; z_0) \neq (x_1 + \cos(z_1)t, \; y_1 + \sin(z_1)t, \; z_1)$$

since any ~~coordinates~~ values z_0, z_1 that could result in equality of the first two coordinates will result in inequality of the third.

5) Consider the canonical isomorphism $M_{2\times 2} \to \mathbb{R}^4 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$ and define the map F by
$$F(w, x, y, z) = (w^2 + x^2 + y^2 + z^2, \ wz - xy).$$

Then we have that

$$DF = \begin{pmatrix} 2w & 2x & 2y & 2z \\ z & -y & -x & w \end{pmatrix}$$

and this matrix has rank < 2 only when all of the following are zero:

$$-2wy - 2xz, \quad -2wx - 2yz, \quad 2w^2 - 2z^2, \qquad \text{(Determinants of all}$$
$$-2x^2 + 2y^2, \quad 2xw + 2xz, \quad 2wy + 2xz \qquad 2\times 2 \text{ minors)}$$

This never happens so long as $a \neq 2|b|$ so each point in the set
$$S = \{(a, b) : a \neq 2|b|\}$$
is a regular value of F so $\forall (a, b) \in S$

$$M_{ab} = F^{-1}(\{(a, b)\})$$

is a smooth submanifold of $\mathbb{R}^4$ by the Regular Level Set Theorem.