RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2018

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let X be a topological space, let Y be a set, and let $f : X \to Y$ be a function. Construct a topology on Y with the following property: if Z is a topological space and $g : Y \to Z$ is a function, then g is continuous if and only if $g \circ f$ is continuous. Prove that your topology has the required property.

Q.2 Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.

Q.3 Recall that the n-th homotopy group $\pi_n(X, x)$ of a topological space X with a basepoint x is the set of basepoint preserving homotopy classes of maps $S^n \to X$ that send the basepoint of S^n to x. Suppose that $f : X \to Y$ is a covering space. Prove that the map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$, sending $\alpha : S^n \to X$ to $f \circ \alpha$, is a bijection for all $n \geq 2$. (Note that the group structure on $\pi_n(X, x)$ is not relevant to this problem.)

Q.4 Let $a, b \in \mathbb{R}$, and consider the subset S of $\mathbb{R}^3$ defined by the equations

$$xyz = a, \qquad x + y + z = b.$$

(a) Show that if $a \neq 0$ and $b^3 \neq 27a$, then S is a smooth submanifold of $\mathbb{R}^3$.

(b) Suppose that $a = 0$ and $b = 1$. Identify the points of S where S is not a smooth submanifold of $\mathbb{R}^3$.

Q.5 Consider the two vector fields on $\mathbb{R}^3$ with coordinates (x, y, z) given by

$$X = \frac{\partial}{\partial y} + z\frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial z} + y\frac{\partial}{\partial x}.$$

(a) Show that $[X, Y] = 0$.

(b) Compute the flows θ_t of X and ϕ_s of Y, and show directly that for any point $p = (a, b, c) \in \mathbb{R}^3$ and any $s, t \in \mathbb{R}$,

$$\theta_t(\phi_s(p)) = \phi_s(\theta_t(p)).$$

(c) Use part (b) to give a parametrization $(x(s,t), y(s,t), z(s,t))$ for the (unique!) surface passing through the point $p = (1, 0, 0)$ and tangent to the vector fields X and Y at each point. Then give an equation of the form $F(x, y, z) = 0$ that describes this surface.

Q.6 Define a 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\omega = -\left(\frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

(a) Let C be the circle of radius $r > 0$ centered at the origin, oriented counterclockwise. Evaluate the integral $\int_C \omega$ by direct computation.

(b) Calculate $d\omega$.

(c) Let C' be the curve defined implicitly by the equation $x^4 + y^2 = 1$, oriented counterclockwise. Compute the integral $\int_{C'} \omega$.

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1) Let $f: X \to Y$ be any map from a topological space X to a set Y, and define a topology on Y by $V \subseteq Y$ is open if and only if $f^{-1}(V)$ is open in X. We have that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(Y) = X$, $f^{-1}(V \cap U) = f^{-1}(V) \cap f^{-1}(U)$, and $f^{-1}(\bigcup_{\alpha \in A} V_\alpha) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$, so this defines a topology on Y.

Suppose that g is continuous and let $U \subseteq Z$ be open. Then $g^{-1}(U)$ is open in Y and by definition of the topology on Y, we have that $f^{-1}(g^{-1}(U)) = (f \circ g)^{-1}(U)$ is open in X. Hence $f \circ g$ is continuous.

Suppose that $f \circ g$ is continuous, and let $U \subseteq Z$ be open. Then $f^{-1}(g^{-1}(U))$ is open in X with $g^{-1}(U) \subseteq Y$. So by the definition of the topology on Y, we have that $g^{-1}(U)$ is open in Y, so g is continuous.

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2) There are a couple of ways to go about this. The first way is to use the fact that homeomorphic spaces have isomorphic n^{th} homotopy groups. Suppose that $f: \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism and WLOG, assume that $m < n$. This induces a homeomorphism $\hat{f}: \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$. We know that $\mathbb{R}^n \setminus \{p\}$ is homotopic to S^{n-1} and since homotopy groups are invariant under both homotopy and homeomorphism, we get that

$$\pi_k(S^{m-1}) \cong \pi_k(\mathbb{R}^m \setminus \{0\}) \cong \pi_k(\mathbb{R}^n \setminus \{f(0)\}) \cong \pi_k(S^{n-1})$$

for all k. But if $m < n$ then

$$0 \cong \pi_{n-1}(S^{m-1}) \ncong \mathbb{Z} \cong \pi_{n-1}(S^{n-1}).$$

So no such homeomorphism can exist. ■

Alternatively, we can use the Invariance of domain theorem which states if $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous injection of $\mathbb{R}^n$, then f is an open map (so for any $U \subseteq \mathbb{R}^n$, $f(U)$ is open when U is open)

For this, assume again that $m < n$, and consider the inclusion map $\iota: \mathbb{R}^m \to \mathbb{R}^n$, and suppose that there is some homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$. Then $\iota \circ f$ is a continuous, injective map but $\iota \circ f(\mathbb{R}^n) = \{(x^1, \dots, x^n) : x^i = 0 \;\; i > m\}$ is not open. This contradicts the invariance of domain theorem.

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3) Let $f: X \to Y$ be a covering map. Then f induces a map $f_* : \pi_n(X) \to \pi_n(Y)$.

Suppose that $f_*(\alpha) = f_*(\beta)$ for $\alpha, \beta \in \pi_n(X)$, so $f \circ \alpha = f \circ \beta$. Then for every open set containing the image of $f \circ \alpha = f \circ \beta$, we have a collection of corresponding open sets $\{U_i\}_{i \in I} \subseteq \tau_X$ with $U_i \cap U_j = \emptyset$ for $i \neq j$ such that for all i

$$im(\alpha) \cap U_i = im(\beta) \cap U_i$$

since $im(\alpha), im(\beta) \subseteq \bigcup_{i \in I} U_i$. Thus $\alpha = \beta$.

Now let $[\gamma] \in \pi_n(Y, f(x))$. Since $n \geq 2$, $\pi_1(S^n) = \{1\}$ so $\gamma_*(\pi_1(S^n)) = \{1\} \subseteq f_*(\pi_1(X, x))$ so there is a lifting $\tilde{\gamma} : S^n \to X$ with $f \circ \tilde{\gamma} = \gamma = f_*(\tilde{\gamma})$, so f is surjective.

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4) Define a function $F: \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (xyz, x+y+z)$

(a) We have that

$$DF = \begin{pmatrix} yz & xz & xy \\ 1 & 1 & 1 \end{pmatrix}$$

which only drops rank below 2 when

$$yz = xz, \quad yz = xy, \quad \text{and} \quad xz = xy.$$

Suppose $xyz = a \neq 0$, and $b^3 \neq 27a$. Then none of $x, y,$ or z are zero and $z \neq -y$ so DF has full rank in $F^{-1}((a,b))$.

Thus (a,b) is a regular point of F and $S = F^{-1}((a,b))$ is a regular level set of $\mathbb{R}^3$ meaning S is an embedded submanifold of $\mathbb{R}^3$.

(b) If $a = 0$ and $b = 1$ then we have that

$$S = \{(x, y, z) : xyz = 0,\ x+y+z = 1\},$$

so S appears as the union of the three lines

$$z = 1-x, \quad y = 1-x, \quad \text{and} \quad z = 1-y.$$

S cannot then be an embedded submanifold of M since it fails to be locally Euclidean at the points where these lines intersect. Namely $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

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5) (a) $[X,Y] = X(1)\frac{\partial}{\partial z} + X(y)\frac{\partial}{\partial x} - Y(1)\frac{\partial}{\partial y} - Y(z)\frac{\partial}{\partial x}$

$= 0 + 1\frac{\partial}{\partial x} - 0 - 1\frac{\partial}{\partial x}$

$= 0$

(b) We have that the flow of X is given by
$$\Theta_t(x,y,z) = (x+zt,\ y+t,\ z)$$
and that the flow of Y is given by
$$\Psi_s(x,y,z) = (x+ys,\ y,\ z+s).$$

This gives us that
$$\Theta_t \circ \Psi_s(x,y,z) = (x+ys+zt,\ y+t,\ z+s).$$
and
$$\Psi_s \circ \Theta_t(x,y,z) = (x+ys+zt,\ y+t,\ z+s)$$
which are equal.

(c) Let S be the codimension 1 submanifold of $\mathbb{R}^3$ corresponding to the x-axis. Then at $p=(1,0,0)$, we have that the tangent space to S, T_pS, is complimentary to span$\{X|_p, Y|_p\}$, and we can construct a smooth function
$$\Psi_r(s,t) = \Theta_t \circ \Psi_s(r,0,0) = (r+st,\ t,\ s) = (x(s,t), y(s,t), z(s,t))$$
for any $r \in \mathbb{R}$ that parameterizes a surface passing through $(r,0,0)$ with
$$\frac{\partial}{\partial t}\Psi_r(t,s) = s\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = z(s,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X$$
and
$$\frac{\partial}{\partial s}\Psi_r(t,s) = t\frac{\partial}{\partial x} + \frac{\partial}{\partial z} = y(s,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial z} = Y.$$

So the parameterization of the unique surface passing through $(1,0,0)$ and tangent to X and Y is given by
$$\Psi_1(t,s) = (1+st,\ t,\ s).$$

If we want to represent this as $F(x,y,z)=0$, then we solve this system to get $F(x,y,z) = x - yz - 1$.

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6) (a) Parameterize the circle of radius r by

$$\gamma_r(t) = (r\cos(t),\ r\sin(t)).$$

Then we have that

$$\int_C \omega = \int_{\gamma_r} \omega = \int_{t=0}^{2\pi} \frac{-r\sin(t)}{r^2}(-r\sin(t))\,dt + \frac{r\cos t}{r^2}(r\cos(t))\,dt$$

$$= \int_{t=0}^{2\pi} 1\,dt$$

$$= \boxed{2\pi}$$

(b) $$d\omega = d\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$

$$= \frac{1}{(x^2+y^2)^2}\left[\left(-(x^2+y^2)dy + y(2x\,dx + 2y\,dy)\right)\wedge dx + \left((x^2+y^2)dx - x(2x\,dx + 2y\,dy)\right)\wedge dy\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[(-x^2-y^2+2y^2)dy\wedge dx + (x^2+y^2-2x^2)dx\wedge dy\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[(-x^2+y^2+x^2-y^2)dy\wedge dx\right] = 0.$$

(c) If we take the 0-form $\alpha = \arctan\left(\frac{y}{x}\right)$ then

$$d\alpha = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \omega$$

so ω is exact. The integral of an exact form is path-independent, and since C and C' share the point $(1,0)$, we have that

$$\int_C \omega = \int_{C'} \omega = \boxed{2\pi}$$

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$$= \int_{t=0}^{2\pi} 1\, dt$$
$$= \boxed{2\pi}$$

(b) $d\omega = d\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$
$$= \frac{1}{(x^2+y^2)^2}\left[\left(-(x^2+y^2)dy + y(2x\,dx + 2y\,dy)\right)\wedge dx + \left((x^2+y^2)dx - x(2x\,dx + 2y\,dy)\right)\wedge dy\right]$$
$$= \frac{1}{(x^2+y^2)^2}\left[(-x^2-y^2+2y^2)dy\wedge dx + (x^2+y^2-2x^2)dx\wedge dy\right]$$
$$= \frac{1}{(x^2+y^2)^2}\left[(-x^2+y^2+x^2-y^2)dy\wedge dx\right] = 0.$$

(c) From part (b) we have that ω is closed. Then since ω is a 1-form and in $\mathbb{R}^2\setminus\{0\}$ we have that C is homotopic to C', we get that
$$\int_C \omega = \int_{C'} \omega = 2\pi$$