# Geometry/ Topology August 2020

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1. Let $f\colon X \to Y$ be a function between topological spaces.

   (a) Prove that if $f$ is continuous, then whenever a sequence $(x_n)$ converges to $x$ in $X$, then $(f(x_n))$ converges to $f(x)$ in $Y$.

   (b) Prove that the converse of $(a)$ holds if $X$ is first countable.

**Solution:**

(a) Let $f\colon X \to Y$ continuous and suppose $x_n$ converges to $x$. Take a neigbourhood $V$ of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of $x$ by continuity, so there is some $N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(V)$ for all $n \geq N$. Then for $n \geq N$, $f(x_n) \in f(f^{-1}(V)) \in V$. Hence $f(x_n)$ converges to $f(x)$.

(b) First we claim that in a first countable space, $x \in \overline{A}$ iff there is a sequence $(x_n) \subset A$ converging to $x$. For the forward implication, note that if $x \in A$, then just take the constant sequence $x_n = x$. So suppose not: then we have to show that $x$ is a limit point of $A$. Take a nested neighborhood basis $(B_n)$ of $x$, and choose $x_n \in B_n \cap A \neq \emptyset$ (as $x$ is a limit point of $A$). Then any neighborhood $U$ of $x$ contains every $B_n$ for $n \geq$ some $N$. Thus the $x_n$ are a sequence in $A$ converging to $x$. For the backwards implication, we note that convergence of $x_n$ to $x$ means that every nbhd of $x$ contains all but finitely many $x_n$ i.e. every nbhd of $x$ intersects with $A$. Hence either $x \in A$ or $x$ is a limit point of $A$ (for if $x \notin A$, then each punctured neighbourhood of $x$ intersects $A$), so $x \in \overline{A}$ (note that we don't need first countability for this implication).

Now, we recall that an equivalent definition of continuity of $f$ is that $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset X$. Take $y \in f(\overline{A})$. Then $y = f(x)$, where $x \in \overline{A}$. Then by first countability, there is a sequence $x_n$ in $A$ converging to $x$. Then by hypothesis, $f(x_n) \subset f(A)$ converges to $f(x) inf(\overline{A})$. So by the backwards implication in the above paragraph, $f(x) \in \overline{f(A)}$. This show that $f(\overline{A}) \subset \overline{f(A)}$. As $A \subset X$ was arbitrary, we have that $f$ is continuous.

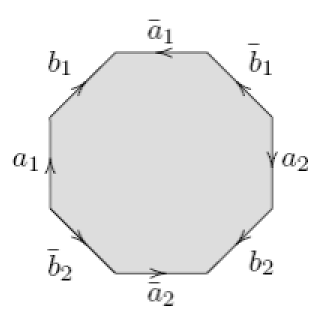
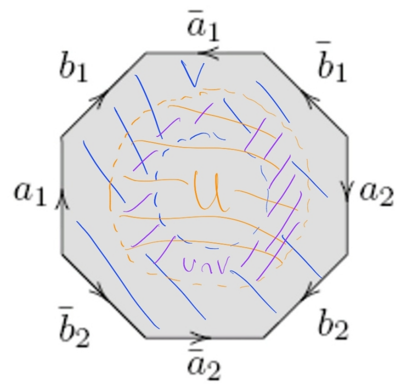



Figure 1: $O \subseteq \mathbb{R}^2$ with gluing data for $\Sigma$

2. Consider the closed octagonal disk $O$ as in the figure, topologized as a subspace of $\mathbb{R}^2$. Let $\Sigma$ be the quotient space obtained by identifying the directed edge $a_1$ with $\overline{a_1}$, $a_2$ with $\overline{a_2}$, $b_1$ with $\overline{b_1}$ and $b_2$ with $\overline{b_2}$. Use the Seifert Van-Kampen Theorem to determine the fundamental group of $\pi_1(\Sigma, y)$ in terms of generators and relations for any base point $y \in \Sigma$.

   **Solution:** First note that $\Sigma$ is path connected, so $\pi_1(\Sigma, y)$ is independent of the choice of basepoint $y$. We decompose $\Sigma$ as $U \cup V$, where $U$ is an open disc contained inside the octagon $O$, and $V$ is an open set which deformation retracts to the outside of the octogon (see figure below).




   Pick the basepoint $y$ to be contained in $U \cap V$. Then $U$ is path connected and open, and is homeomorphic to an open disc and thus contractible. Thus $\pi_1(U, y) = 1$. $V$ retracts onto the boundary of $O$ which after identifications is just a bouquet of 4 circles $a_1, a_2, b_1$ and $b_2$, which is path connected. Thus $\pi_1(V, y) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, the free group on 4 generators, and thus has presentation $\langle a_1, a_2, b_1, b_2 \rangle$. $U \cap V$ is an annulus, which is path connected, and so SVK can be applied. $U \cap V$ is homotopic to a circle, and hence has fundamental groups $\langle c \rangle \cong \mathbb{Z}$, where $c$ is a single anticlockwise loop inside $U \cap V$. Then SVK says that:

$$\pi_1(\Sigma, y) \cong \pi_1(V, y) *_{\pi_1(U \cap V, y)} \pi_1(U, y)$$

   Where the amalgamation comes from the inclusions of $U \cap V$ into $U$ and $V$. $U$ is simply connected, so the map induced by inclusion into $U$, $i_U^*$ is just the zero map, so $i_U^*(c) = 1$. However, our loop $c$, when included into $V$ as $i_V^*(c)$ can be enlarged via homotopy to be the boundary of $O$, which is $a_1 b_1^{-1} a_1^{-1} b_2 a_2 b_2^{-1} a_2^{-1} b_1$ in $\pi_1(V)$. Hence $i_V^*(c) = a_1 b_1^{-1} a_1^{-1} b_2 a_2 b_2^{-1} a_2^{-1} b_1$. The amalgamated free product means that in $\pi_1(\Sigma, y)$, $i_V^*(c) = i_U^*(c)$. Thus $\pi_1(\Sigma, y)$ has presentation:

$$\pi_1(\Sigma, y) = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1^{-1} a_1^{-1} b_2 a_2 b_2^{-1} a_2^{-1} b_1 = 1 \rangle$$

3. Let $\mathbb{R}P^n$ be the quotient of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by the antipodal map $a \colon S^n \to S^n$, $a(x) = -x$. You may assume that $\mathbb{R}P^n$ is path connected, locally path connected and semilocally simply-connected. You may assume that the product of path connected spaces is path connected and that the product of semilocally simply-connected spaces is semi-locally simply connected.

(a) Prove that the product $X_1 \times X_2$ of locally path connected spaces $X_1$ and $X_2$ is locally path connected.

(b) Determine the number of equivalence classes of path connected covering spaces over $\mathbb{R}P^n \times \mathbb{R}P^n$. Give a representative of each equivalence class. Prove any claim you make about the fundamental groups of $\mathbb{R}P^n$ and $\mathbb{R}P^n \times \mathbb{R}P^n$.

**Solution:**

(a) Recall that $X$ is locally path connected, if given any $x \in X$ and a neighbourhood $U$ of $x$, there exists a path connected open neighbourhood $V$ of $x$ s.t. $V \subset U$. Suppose $X_1$ and $X_2$ are path connected. Pick any point $(x_1, x_2) \in X_1 \times X_2$, and a neighbourhood $N$ of $(x_1, x_2$. As the product topology on $X_1 \times X_2$ has basis $\{U_1 \times U_2 \colon U_1 \in \tau_{X_1}, U_2 \in \tau_{X_2}\}$, there exists $U_1$ open in $X_1$, $U_2$ open in $X_2$ s.t. $U_1 \times U_2 \subset N$. Then by local path connectedness of $X_1$, there contains an open path connected neighborhood $V_1$ of $x_1$ s.t. $V_1 \subset U_1$, and by local path connectedness of $X_2$, there contains an open path connected neighborhood $V_2$ of $x_2$ s.t. $V_2 \subset U_2$. Then $V_1 \times V_2$ is an open path connected neighbourhood of $(x_1, x_2)$ s.t. $V_1 \times V_2 \subset U_1 \times U_2 \subset N$. Hence $X_1 \times X_2$ is locally path connected.

(b) Assume $n \geq 2$. Then $S^n$ is simply connected, by SVK on $S^n = S^n \setminus N \cup S^n \setminus S$, where $N = \{1, 0, \ldots, 0\}, S = \{-1, 0, \ldots, 0\}$. $S^n \setminus N$ is homeomorphic to $\mathbb{R}^n$ by sterographic projection, and hence has trivial fundamental group, and similarly for $S^n \setminus S$. Both are path connected as $\mathbb{R}^n$ is. Their intersection can be retracted onto the equator $S^{n-1}$, which is path connected for $n \geq 2$. So by SVK, $\pi_1(S^n) = 1$, (where we pick some basepoint on the equator). It is also path connected, which can be seen by using stereographic projection to pullback a path in $\mathbb{R}^n$.

The antipodal map $a$ satisfies $a^2 = id$, so we have a $\mathbb{Z}_2$ action, $\rho$ on $S^n$ given by $\rho(1) = id, \rho(-1) = a$. Then this action acts freely: $ax = x \iff x = -x$, which cannot happen in $S^n$, as the only vector $x$ which satisfies $x = -x$ is the zero vector, which has norm 0 so cannot be in $S^n$. It also acts properly discontinuously: given $x \in S^n$, take an open disc $U$ around $x$ that does not cover more than half of $S^n$. Then $U$ and $-U$ are disjoint. Hence by a theorem in lectures, we have that $\pi_1(\mathbb{R}P^n) = \pi_1(S^n/\rho) = \mathbb{Z}_2$, where we've asssumed that $\mathbb{R}P^n$ is path connected so don't need to specify basepoint. Given our assumptions about $X := \mathbb{R}P^n$, we know that covering spaces of $X$ correspond to conjugacy classes of subgroups of $X$: as $\pi_1(X) = \mathbb{Z}_2$ is abelian, conjugacy classes of subgroups are just subgroups. $\mathbb{Z}_2$ has two subgroups 0 and $\mathbb{Z}_2$. So $X$ has two distinct covering spaces up to isomorphism. These are the identity covering $X \to X$, and the covering given by the quotient map $p \colon S^n \to X$ (this is a covering space, as it's surjective due to being a quotient map, and $[x]$ has a neighborhood evenly covered by $p$ by considering the neigborhoods $U$ of $x$ and $-U$ of $-x$ used in showing that $\mathbb{Z}_2$ acts freely discontinuously). These are non-isomorphic, as $p$ is a two-sheeted cover, while the identity is 1-sheeted.

Now, for path connected spaces, we know that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ via the map $[\gamma_1], [\gamma_2] \to [\gamma_1, \gamma_2] \in \pi_1(X \times Y$. Hence $\pi_1(\mathbb{R}P^n \times \mathbb{R}P^n) \cong \pi_1(\mathbb{R}P^n) \times \pi_1(\mathbb{R}P^n) =$

$\mathbb{Z}_2 \times \mathbb{Z}_2$. This group has the following subgroups: $0$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $0 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times 0$ and $\langle(1,1)\rangle \cong \mathbb{Z}_2$. Hence we are looking for 5 covering spaces. We know that if $(\tilde{X}, p)$ covers $X$ and $(\tilde{Y}, q)$ covers $Y$, then $(\tilde{X} \times \tilde{Y}, p \times q)$ is a covering space for $X \times Y$. Hence we have covers $id \times id \colon \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^n$, $id \times p \colon \mathbb{R}P^n \times S^n \to \mathbb{R}P^n \times \mathbb{R}P^n$, $p \times id \colon S^n \times \mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ and $p \times p \colon S^n \times S^n \to \mathbb{R}P^n \times \mathbb{R}P^n$. The final covering space is given by the covering $S^n \times S^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ given by the quotient map induced by the action $a \times a \colon (x, y) \mapsto (-x, -y)$. This can be shown to be covering map in a similar way to how we showed it for $p$.

$n = 1$: $\mathbb{R}P^1 \cong S^1$, so this has countably many covers from $S^1 \to S^1$ given by $z \mapsto z^n$, $n \in \mathbb{N}$, and the map $\epsilon \colon \mathbb{R} \to S^1$, $\epsilon(t) = e^{2\pi i t}$, corresponding to the subgroups $n\mathbb{Z} \subset \mathbb{Z}$ and $0 \subset \mathbb{Z}$ respectively.

4. (a) Give a careful definiton of the tangent bundle of a manifold $M$. Define the manifold structure on $TM$.

   (b) Show that the tangent bundle of the sphere $S^2$ is not bundle isomorphic to $S^2 \times \mathbb{R}^2$

**Solution:**

(a) (see Lee pg65 for all of the details). Let $M$ be an $n-$manifold. The tangent bundle $TM$ of $(M, U_\alpha, \varphi_\alpha)$ is defined as $\bigsqcup_{p \in M} T_pM$, where $T_pM$ is the tangent space at the point $p \in M$. Charts on $TM$ are given by the following: for each smooth chart $(U, \phi_\alpha)$ on $M$, a smooth chart on $\bigsqcup_{p \in U} T_pM$ is given by $\tilde{\varphi}_\alpha \colon \bigsqcup_{p \in U} T_pM \to \mathbb{R}^n \times \mathbb{R}^n$ by:

$$\tilde{\varphi}_\alpha \left( p, v_1 \frac{\partial}{\partial x_1}\bigg|_p, \ldots, v_n \frac{\partial}{\partial x_n}\bigg|_p \right) = (\varphi(p), v_1, \ldots v_n)$$

(b) If $TS^2$ were bundle isomorphic to $S^2 \times \mathbb{R}^2$ then this would be the same as $S^2$ being parallelizable, i.e. that there exists a non-vanishing vector field on $S^2$. But the hairy ball theorem says that this is not the case. Thus $TS^2$ is *not* bundle isomorphic to $S^2 \times \mathbb{R}^2$.

5. Consider the following two vector fields on $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \text{ and } Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

Show that there is no nonempty smooth surface $S \subset \mathbb{R}^3$ that is tangent to both vector fields at each of its points.

**Solution:** Suppose such a surface $S$ exists. Then if $X, Y \in \mathfrak{X}(S)$, then so is $[X, Y]$. We use the formula:

$$[X, Y] = (XY^j - YX^j)\frac{\partial}{\partial x_j}$$

so that

$$\begin{aligned}[X,Y] &= \left((x\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(1)-(\frac{\partial}{\partial x}-\frac{\partial}{\partial z})(x)\right)\frac{\partial}{\partial x}+\left((x\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(0)-(\frac{\partial}{\partial x}-\frac{\partial}{\partial z})(1)\right)\frac{\partial}{\partial y}\\ &\quad+\left((x\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(-1)-(\frac{\partial}{\partial x}-\frac{\partial}{\partial z})(0)\right)\frac{\partial}{\partial z}\\ &= -\frac{\partial}{\partial x}\end{aligned}$$

Then in the basis $(\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p, \frac{\partial}{\partial z}|_p)$, the value of these vector fields at $p \in S$ can be written as $(x,1,0),(1,0,-1),(-1,0,0)$. Then we have:

$$\begin{vmatrix} x & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{vmatrix} = 1 \neq 0$$

Proving that for any $p \in \mathbb{R}^3$, $X(p), Y(p)$ and $[X,Y](p)$ are linearly independent elements of $T_p\mathbb{R}^3$. If a non-empty surface $S$ that is tangent to both $X$ and $Y$ existed, it would also be tangent to $[X,Y]$, and hence $dim(T_pS) \geq 3$ for all $p \in S$, as $T_pS$ is a vector space so would have to contain the span of $X(p), Y(p)$ and $[X,Y](p)$, which are all linearly independent. But a surface is a $2-$manifold, so has tangent spaces of dimension 2, which is a contradiction. Hence no non-empty surface $S$ can exist.

6. (a) Let $a\colon S^n \to S^n$ be the antipodal map. Suppose that $\omega$ is a smooth form on $S^n$ such that $a^*\omega = \omega$. Prove that if $\omega$ is exact, then there is a smooth form $\eta$ with $\omega = d\eta$ and $a^*\eta = \eta$

   (b) Use $(6a)$ to deduce that on the projective space $\mathbb{R}P^n$ ever closed $k-$form with $0 < k < n$ is exact.

   **Hint:** You may use that for $0 < k < n$ every closed $k-$form on $S^n$ is exact.

**Solution:**

(a) Suppose that $\omega$ is an exact smooth $k-$form on $S^n$. Then by definition of exactness, $\omega = d\eta$ for some smooth $(k-1)-$form $\eta$. Then consider the form $\tilde{\eta} := \frac{1}{2}(\eta + a^*\eta)$. Then as $d$ is linear and commutes with pullbacks, we have:

$$d\tilde{\eta} = \frac{1}{2}(d\eta + d(a^*\eta)) = \frac{1}{2}(d\eta + a^*(d\eta)) = \frac{1}{2}(\omega + a^*\omega) = \frac{1}{2}(\omega+\omega) = \omega.$$

Moreover, we also have that:

$$a^*\tilde{\eta} = \frac{1}{2}(a^*\eta + a^*a^*\eta) = \frac{1}{2}(a^*\eta + (a\circ a)^*\eta) = \frac{1}{2}(a^*\eta + id^*\eta) = \frac{1}{2}(a^*\eta+\eta) = \tilde{\eta}$$

where we used that $a(a(x)) = a(-x) = (-(-x)) = x$, and hence $a \circ a = id$. Hence $\tilde{\eta}$ is the desired form.

(b) (see Lemma 17.33 on page 456 of Lee for a practically identical argument). Take a closed $k$-form $\omega$ on $\mathbb{R}P^n$ where $0 < k < n$. Then we have the quotient/ covering map $\pi\colon S^n \to \mathbb{R}P^n$, induced by the equivalence relation $x \sim a(x)$. Consider the form $\pi^*\omega$. Then $d(\pi^*\omega) = \pi^*(d\omega) = \pi^*(0) = 0$, so $\pi^*\omega$ is a closed $k$-form on $S^n$. Moreover,

$\pi^*\omega = a^*(\pi^*\omega)$, as $\pi$ identifies $x$ with $a(x)$ (so $\pi \circ a = \pi$). By the hint, $\pi^*\omega$ is exact, and hence by $(6a)$, there exists a $(k-1)$ from $\eta$ on $S^n$ s.t. $d\eta = \pi^*\omega$ and $a^*\eta = \eta$.

Let $U \subset \mathbb{R}P^n$ be an open subset evenly covered by $\pi$. Then there are two local sections $\sigma_1, \sigma_2 \colon U \to S^n$ (as $\pi$ is a 2-sheeted covering), which are related by $\sigma_2 = a \circ \sigma_1$, as $a$ is a covering transformation of $\pi \colon S^n \to \mathbb{R}P^n$ (and the only non-trivial one). Thus:

$$\sigma_2^*\eta = (a \circ \sigma_1)^*\eta = \sigma_1^* a^* \eta = \sigma_1^*\eta$$

So we can define a smooth global $(k-1)$ form $\beta$ on $\mathbb{R}P^n$ by setting $\beta|_U = \sigma^*\eta$ for any smooth local section $\sigma \colon U \to S^n$. By the above, this definition is independent of the smooth section chosen, so the definitions agree where they overlap. Finally, given $p \in \mathbb{R}P^n$, choose a local section $\sigma \colon U \to S^n$ where $U$ is a neighbourhood of $p$. Then:

$$d\beta = d\sigma^*\eta = \sigma^* d\eta = \sigma^*\pi^*\omega = (\pi \circ \sigma)^*\omega = \omega$$

($\pi \circ \sigma = id_U$, as this is the definition of $\sigma$ being a loca section of $\pi$). Hence $\beta$ a smooth form on $\mathbb{R}P^n$ s.t. $d\beta = \omega$, and so $\omega$ is exact.