Geometry/ Topology January 2022

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Problem 1.

(a) Prove that X is a Hausdorff topological space if and only if the diagonal $\Delta \subset X \times X$

$$\Delta = \{(x, y) \in X \times X \mid x = y\}$$

is a closed subset of $X \times X$ (equipped with the product topology).

(b) Let $f, g \colon X \to Y$ be continuous functions between topological spaces X and Y. Assume that Y is Hausdorff. Show that

$$A = \{x \in X \mid f(x) = g(x)\}$$

is a closed subset of X. (Hint: Construct a new continuous function $F \colon X \to Y \times Y$ and use (a), even if you haven't solved it.)

Solution:

(a) ($\Rightarrow$) Let X be Hausforff. We show that Δ^{C} is open. Let $(x, y) \in \Delta^{\mathsf{C}}$. Then $x \neq y$, where $x, y \in X$. As X is Hausdorff, there exist disjoint open neighborhoods $x \in U, y \in V$. Then $U \times V$ is a neighborhood of (x, y) contained in Δ^{C}: for if there existed $(z, z) \in \Delta \cap U \times V$, then $z \in U \cap V$, which is a contradiction to disjointness of U and V. Hence ever point in Δ^{C} has a neighborhood contained in Δ^{C}, and so Δ^{C} is open and hence Δ is closed.

($\Leftarrow$) Now suppose that Δ is closed. Then Δ^{C} is open. Now pick arbitrary $x \neq y \in X$. Then the point $(x, y) \in \Delta^{\mathsf{C}}$. Thus there exists some open set O with $(x, y) \in O \subset \Delta^{\mathsf{C}}$. Then as $\mathcal{B} = \{U \times V \mid U, V \text{ open in } X\}$ forms a basis for the product topology on X, O is union of basic sets of this type, and thus there is some basic set $U \times V$ with $(x, y) \in U \times V \subset O \subset \Delta^{\mathsf{C}}$. Hence $x \in U$, $y \in V$ with $U \cap V = \emptyset$ (if the intersection were non-empty, then there exists $z \in U \cap V$ so $(z, z) \in U \cap V \subset \Delta^{\mathsf{C}}$, a contradiction). Hence as $x \neq y \in X$ was arbitrary, X is Hausdorff.

(b) Define the function $F \colon X \to Y \times Y$ by $F(x) = (f(x), g(x))$. Then as Y is Hausdorff, $\Delta \subset Y$ is closed. But $F^{-1}(\Delta) = \{x \in X \mid F(x) \in \Delta\} = \{x \in X \mid f(x) = g(x)\} = A$. So as the preimage of a closed set under a continuous function (note that F is continuous as its projections onto each coordinate, f and g are continuous), we have that $F^{-1}(\Delta) = A$ is closed in X.

Problem 2.

Let $B^2 \subseteq \mathbb{R}^2$ be the set of vectors of length less than or equal to one, and $S^1 \subseteq B^2$ be those vectors of length exactly one. In this problem, you may assume without proof that the map $\pi \colon S^1 \times [0, 1] \to B^2$ given by $\pi(x, t) = (1 - t)x$ is a quotient map.

(a) Prove that if $h \colon S^1 \to S^1$ is a continuous map which is nullhomotopic, then there exists a continuous map

$$f \colon B^2 \to S^1$$

such that $f(x) = h(x)$ for all $x \in S^1$.

(b) Prove that a continuous nullhomotopic map $h\colon S^1 \to S^1$ has a fixed point.

Solution:

(a) We use the theorem that if $\pi\colon X \to Y$ is a quotient map and $g\colon X \to Z$ is a map that is constant on the fibers of π, then there exists a map $f\colon Y \to Z$ such that $f \circ \pi = g$. In our case, we have that $h\colon S^1 \to S^1$ is a map which is nullhomotopic, i.e. there exists a map $H\colon S^1 \times I \to S^1$ s.t. $H|_{S^1\times\{0\}} = h$ and $H|_{S^1\times\{1\}} = c$ where c is some fixed element of S^1. Thus we we have the commutative diagram:

$$\begin{tikzcd} S^1 \times I \arrow[d, "\pi"] \arrow[dr, "H"] & \\ B^2 \arrow[r, dotted, "f"] & S^1 \end{tikzcd}$$

where f is the induced map from the theorem. First we must show that H is constant on fibers of π. Well, $\pi(x,t) = (1-t)x$, which can be rewritten as $\pi(e^{i\theta}, t) = (1-t)e^{i\theta}$. Hence given $0 \neq y = re^{i\theta} \in B^2$, $\pi^{-1}(\{y\}) = \big\{(e^{i\theta}, t)\colon (1-t)e^{i\theta} = re^{i\theta}\big\} = \big\{(e^{i\theta}, t)\colon (1-t-r)e^{i\theta} = 0\big\} = \big\{(e^{i\theta}, 1-r)\big\}$, as $e^{i\theta} \neq 0$. Hence the fibers of non-zero points in B^2 are singletons (because $r \in [0,1]$) is unique and so is $e^{i\theta}$, and $r \in [0,1] \implies 1-r \in [0,1]$), and thus H is constant on these fibers. We also have: $\pi^{-1}(\{0\}) = \{(x,t)\colon (1-t)x = 0\} = S^1 \times \{1\}$, because $x \in S^1$ so $x \neq 0$. As H is a homotopy between h and the constant map at c, we have that $H|_{S^1\times\{1\}} = c$ and thus H is also constant along the fiber of 0. Hence the map $f\colon B^2 \to S^1$ exists, with the property $H = f \circ \pi$, which means that $h(x) = H(x,0) = f(x)$ for all $x \in S^1$ (because the homotopy H starts at the map h).

(b) The map f from part (a) can be considered as a map $f\colon B^2 \to B^2$ by composing with the inclusion $S^1 \hookrightarrow B^2$. Then Brouwer's fixed point theorem implies that f has a fixed point, i.e. $\exists x \in B^2$ s.t. $f(x) = x$. However, the image of f is contained in S^1, so in particular $x \in S^1$. But then by part (a) we get that $h(x) = f(x) = x$, so x is a fixed point of h

Problem 3.

Let $\mathbb{T} = S^1 \times S^1$ (equipped with the product topology) be the torus and $x \in \mathbb{T}$ be any point. Let $\mathbb{R}P^2$ be the real projective plane, which is the quotient of S^2 by the antipodal map. In this problem, you may give the value of the fundamental groups of $\mathbb{R}P^2$ or of S^1 without proof.

(a) Use the method of your choice to compute $\pi_1(\mathbb{T}, x)$.

(b) Prove that any continuous map $f\colon \mathbb{R}P^2 \to \mathbb{T}$ is nullhomotopic.

Solution:

(a) There are several methods to do this which I will list now:

- Use without proof that $\pi_1(S^1) \cong \mathbb{Z}$, and that for path-connected spaces, the fundamental group functor respects products. Hence

$$\pi_1(\mathbb{T}, x) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$$

- Use Seifert van-Kampen on the torus as the quotient of a square, with U being a neighborhood of the boundary of the square and V being a disc in the middle of the square (such that U and V cover all of $\mathbb{T}$, and pick $x \in U \cap V$). Then U is homotopic to the wedge of two circles, which has fundamental group $\mathbb{Z} \star \mathbb{Z}$, the free group generated by two symbols (say a and b), and V is contractible. The circle generating $\pi_1(U \cap V)$ embeds into U as the boundary of the square, which in $\pi_1(U)$ is represented by $aba^{-1}b^{-1} = [a, b]$, while the inclusion into V maps this element of $\pi_1(U \cap V)$ to the identity element, as $\pi_1(V)$ is the trivial group. Hence SVK tells us that

$$\pi_1(\mathbb{T}, x) = \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

  where we used that $\langle a, b \mid [a, b] = 1 \rangle$ is the abelianization of $\mathbb{Z} \star \mathbb{Z}$ (the free group on two generators), which gives $\mathbb{Z} \times \mathbb{Z}$ (the free *Abelian* group on two generators).

- Finally, we could also use that $\mathbb{R}^2$ is a simply connected space that covers $S^1 \times S^1$ by the map $(x, y) \to (e^{2\pi x}, e^{2\pi y})$ and so this is the universal covering map. This map has kernel $\mathbb{Z} \times \mathbb{Z}$, so this is the fundamental group of $S^1 \times S^1 = \mathbb{T}$.

(b) Without proof: $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. If $f\colon \mathbb{R}P^2 \to \mathbb{T}$, then we have the induced map $f_*\colon \mathbb{Z}_2 \to \mathbb{Z}^2$ between fundamental groups. f_* must send 0 to 0 as it's a homomorphism, and it must also send $1 \in \mathbb{Z}_2$ to a torsion element of $\mathbb{Z}^2$, because 1 is torsion in $\mathbb{Z}_2$. However, $\mathbb{Z}^2$ is torsion-free, so its only torsion element is $(0, 0)$. So f_* must send 1 to $(0, 0)$ as well, and so f_* is the zero-map, i.e. $f_*(\mathbb{Z}_2) = 0$.
Now consider the universal covering map $p\colon \mathbb{R}^2 \to \mathbb{T}$. We have that $f_*(\mathbb{Z}_2) = 0 \subset 0 = p_*(\mathbb{R}^2)$, so the map lifting lemma can be applied to give a lift $\tilde{f}\colon \mathbb{R}P^2 \to \mathbb{R}^2$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{T} \end{tikzcd}$$

Now, as $\mathbb{R}^2$ is contractible, there exists a homotopy $H\colon \mathbb{R}P^2 \times I \to \mathbb{R}^2$ between $\tilde{f}$ and the constant map at some $c \in \mathbb{R}^2$. Then $p \circ H\colon \mathbb{R}P^2 \times I \to \mathbb{T}$ gives a homotopy between $\tilde{f} \circ p = f$ and the constant map at $p(c)$. Hence f is nullhomotopic.

Problem 4.

In this problem, $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ equipped with its canonical real manifold structure. Define $f\colon \mathbb{C}^2 \to \mathbb{C}$ by $f(w, z) = w^2 - z^3$, and let $V = f^{-1}(0)$.

(a) Show that f is a submersion at each $(w, z) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and conclude from that that $V \setminus \{(0, 0)\}$ is a submanifold of $\mathbb{C}^2$ of real dimension 2.

(b) Show that V intersects the unit sphere $S^2 \subset \mathbb{C}^2$ transversally which means that

$$T_pV + T_pS^3 = \mathbb{C}^2$$

for all $p \in S^3 \cap V$
(Hint: Consider the path $\gamma\colon \mathbb{R}_{>0} \to V$ given by $\gamma(t) = (t^3 w, t^2 z)$.)

(c) Conclude from (b) that the intersection $S^3 \cap V$ is a 1-manifold K.

Solution:

(a) We have that $Df_{(w,z)} = (2w, -3z^2)$ which has full rank (1) unless $2w = 0 = -3z^2$, which can only occur when $w = 0 = z$. Hence $Df\colon T_p\mathbb{C}^2 \to T_{f}(p)\mathbb{C}$ is surjective at all points $p \in \mathbb{C}^2 \setminus \{0\}$. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ we get that $Df\colon T_p\mathbb{R}^4 \to T_p\mathbb{R}^2$ is surjective as well. Note that we can restrict f to a submersion $\tilde{f}\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$, because have restricted to an open submanifold so the behaviour of the differential of f locally does not change. Then 0 is a regular value of $\tilde{f}$, and the regular level set theorem implies that $\tilde{f}^{-1}(0) = V \setminus \{(0,0)\}$ is an embedded submanifold of $\mathbb{C}^2$ of real codimension $\dim_{\mathbb{R}} \mathbb{C} = 2$, i.e. of real dimension $\dim_{\mathbb{R}} \mathbb{C}^2 - 2 = 4 - 2 = 2$.

(b) Note that we can replace V with $V \setminus \{(0,0)\}$ if needed, as $S^3 \cap V = S^3 \cap (V \setminus \{(0,0)\})$, because $0 \notin S^3$. We consider the path given in the question, with $p = (w,z) \in S^3 \cap V$. Then $f(\gamma(t)) = t^6(w^2 - z^3) = t^6(0) = 0$, so that γ is a path in V. Note that $\gamma(1) = p$, so that $\gamma'(1)$ is an element of T_pV. Now recall that S^3 has real dimension 3, as $S^3 = g^{-1}(\{1\})$, where $g\colon \mathbb{R}^4 \to \mathbb{R}$, $g(x) = \|x\|^2$. So in order to show that $T_pV + T_pS^3 = \mathbb{C}^2$, we only need to show that T_pV is not contained inside T_pS^3, as $\mathbb{C}^2$ has real dimension 4, which is only one more than that of T_pS^3. Also recall that as the preimage of a regular value, we have that $T_pS^3 = \ker Dg_p$. So we calculate that

$$Dg(\gamma'(1)) = (g \circ \gamma)'(1) = (t^6|w|^2 + t^4|z|^2)'(1) = (6t^5|w|^2 + 4t^3|z|^2)|_{t=1} = 6|w|^2 + 4|z|^2 \neq 0$$

where we used that $|w|$ and $|z|$ are not both zero because $(0,0) \notin S^3$ and so $p \neq (0,0)$. Hence $\gamma'(1)$ is an element of T_pV which is not an element of T_pS^3. Hence $T_pV + T_pS^3 = \mathbb{C}^2$ by comparing dimensions. As $p \in S^3 \cap V$ was arbitrary, we are done.

(c) By theorem 6.30 (b) of Lee, we have that $S^3 \cap (V \setminus \{(0,0)\}) = S^3 \cap V$ is an embedded submanifold of $\mathbb{C}^2$ with codimension equal to the sum of the codimensions of S^3 and $V \setminus \{(0,0)\}$ i.e. of codimension 2+1 = 3, and so $S^3 \cap V$ has real dimension $\dim_{\mathbb{R}} \mathbb{C}^2 - 3 = 4 - 3 = 1$.

Problem 5.

(a) Let $\gamma\colon (a,b) \to M$ with $a < b$ be an integral curve to a smooth vector field on a manifold M. Suppose that $\gamma'(t) = 0$ for some t. Prove that then γ is a constant map.

(b) Find the integral curves of the following vector field X on $\mathbb{R}^2$

$$X(x,y) = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

where x, y are coordinates of $\mathbb{R}^2$.

Solution:

(a) Suppose that $\gamma'(t) = 0$ and let $p = \gamma(t)$. Then as γ is the integral curve to a smooth vector field, call it V, we have that $V_p = \gamma'(t) = 0$. But now observe that $\tilde{\gamma}\colon \mathbb{R} \to M$, $\tilde{\gamma}(t) = p$ is also an integral curve to V which goes through the point p. By existence and uniqueness of integral curves, γ and $\tilde{\gamma}$ must agree on the overlap of their domains, which is (a,b), and so $\gamma(t) = \tilde{\gamma}(t) = p$ for all $t \in (a,b)$.

(b) For an integral curve to X, we want to find a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that $\gamma_1'(t) = \gamma_1^2(t),\ \gamma_2'(t) = \gamma_1(t)\gamma_2(t)$. First, observe that:

$$\gamma_1(t) = (a-t)^{-1}$$

solves the first equation, with $\gamma_1(0) = 1/a$, and

$$\gamma_2(t) = b(a-t)^{-1}$$

solves the second equation.
Now fix $(x, y) \in \mathbb{R}^2$. We want to find the integral curves that go through (x, y) at $t = 0$. Note that if $x = 0$ then $X(x, y) = 0$ and so $\gamma(t) = (x, y)$ a constant curve (by part (a)). So now we only need to find integral curves starting at points where $x \neq 0$.

Choose $a = x^{-1}$ and $b = y/x$ (which makes sense as $x \neq 0$) to get:

$$\gamma(t) = \left(\left(\frac{1}{x} - t\right)^{-1}, \frac{y}{x}\left(\frac{1}{x} - t\right)^{-1}\right)$$

which satisfies $\gamma'(t) = \gamma_1(t)^2 \frac{\partial}{\partial x} + \gamma_1(t)\gamma_2(t)\frac{\partial}{\partial y}$, and $\gamma(0) = (x, y)$

Problem 6.

Suppose that η is a closed k-form on an n-manifold M and let N be a closed, oriented k-manifold. Assume that $f_0 \colon N \to M$ and $f_1 \colon N \to M$ are smooth maps. Prove that if f_0 is smoothly homotopic to f_1 then

$$\int_N f_0^* \eta = \int_N f_1^* \eta$$

(Hint: Recall that *smooth homotopy from* f_0 *to* f_1 is a smooth map $F \colon N \times [0,1] \to M$ such that $F \circ i_t = f_t$ for $t = 0, 1$, where $i_t \colon N \to N \times [0,1], p \mapsto (p, t)$.)

Solution: As N and I are compact, oriented manifolds, we have that $N \times I$ is also a compact, oriented manifold, with boundary $N \times \{0\} \cup N \times \{1\}$, where $N \times \{0\}$ will have the opposite orientation to the orientation to that of N. We have a smooth homotopy $F \colon N \times I \to M$. As η is a closed k-form, so if $F^*\eta$, as d commutes with pullbacks. $N \times I$ is a $(k+1)$ manifold, because N is a k manifold, and I is a 1-manifold. Hence we can apply Stokes's theorem to $F^*\eta$ on $N \times I$ to get:

$$0 = \int_{N \times I} F^* 0 = \int_{N \times I} F^* d\eta = \int_{N \times I} dF^*\eta = \int_{\partial(N \times I)} F^*\eta = \int_{N \times \{0\}} F^*\eta + \int_{N \times \{1\}} F^*\eta = -\int_N f_0^*\eta + \int_N f_1^*\eta$$

Where we used that F restricted to $N \times \{0\}$ is f_0, and similarly for f_1 (implicitly we've used that i_0 is an orientation-reversing diffeomorphism from N to $N \times \{0\}$ so that $\int_{N \times \{0\}} F^*\eta = -\int_N i_0^* F^*\eta = -\int_N (F \circ i_0)^*\eta = -\int_N f_0^*\eta$, and a similar argument for f_1, but now i_1 is orientation-preserving so we get a + sign).
Rearranging gives:

$$\int_N f_0^*\eta = \int_N f_1^*\eta$$