RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Let X be a topological space, $\sim$ an equivalence relation on X, and $\pi : X \to X/\sim$ the canonical projection. Prove the following claims:

(a) $X/\sim$ is a T_1-space, if and only if each equivalence class is closed in X.

(b) If $X/\sim$ is Hausdorff, then $\sim$ is closed in $X \times X$.

(c) If the canonical projection is open, then $X/\sim$ is Hausdorff, if and only if $\sim$ is closed in $X \times X$.

Problem 2. Let T_1 and T_2 be tori and J_1 and J_2 be homotopically trivial simple closed curves on T_1 and T_2 respectively. Let X be the quotient space obtained by identifying J_1 and J_2 by a homeomorphism. Use the Seifert–van Kampen Theorem to compute the fundamental group of X.

Problem 3. Let $f : X \to Y$ be a local diffeomorphism between connected, oriented manifolds, with X compact. Prove that f either preserves orientation at every $x \in X$ or reverses orientation at every $x \in X$.

Problem 4. Recall that a manifold is called *parallelizable*, if its tangent bundle is trivial. Determine for which $n \in \{1, 2, 3\}$ the sphere S^n is parallelizable. Prove your claim.

Problem 5. Let $p : \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map. Let x and y be the standard coordinates on $\mathbb{R}^2$ and consider the 1-form

$$\omega = 2\cos^2(\pi x)dx + dy$$

on $\mathbb{R}^2$. Then ω descends to a 1-form η on T^2; i.e. there exists a 1-form η on T^2 such that $p^*\eta = \omega$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $f(a, b) = (3a + 2b, a - b)$. Then f descends to a map $\bar{f} : T^2 \to T^2$; i.e. there is a commutative diagram:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{R}^2 \arrow[d, "p"'] \\ T^2 \arrow[r, "\bar{f}"] & T^2 \end{tikzcd}$$

(a) Show that ω is closed and exact.

(b) Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the path given by $\gamma(a) = (a, 0)$. Compute $\int_\gamma f^*\omega$.

(c) Show that η is closed on T^2.

(d) Show that $\bar{f}^*\eta$ is closed, but *not* exact on T^2.

Problem 6. Let X be a C^∞ surface. Suppose that X is covered by two open sets U and V with corresponding charts

$$\varphi_U : U \to \mathbb{R}^2 \quad \text{and} \quad \varphi_V : V \to \mathbb{R}^2,$$

which are surjective. Assume further that the transition function

$$\tau_{VU} : \phi_U(U \cap V) \to \phi_V(U \cap V)$$

is given by

$$\tau_{VU}(a_1, a_2) = \left(\frac{1}{a_1}, \frac{1}{a_2} \right).$$

(a) Let x_1, x_2 be the coordinate functions on $\mathbb{R}^2$. The tensor

$$\frac{dx_1 \otimes dx_1}{(1 + x_1^2)^2} + \frac{dx_2 \otimes dx_2}{(1 + x_2^2)^2}$$

on $\mathbb{R}^2$ determines a Riemannian metric on V (via φ_V). Show there is a Riemannian metric g on X extending this metric on V.

(b) Let ∇ be the Levi-Civita connection on X with respect to g. The vector fields $\partial/\partial x_1$ and $\partial/\partial x_2$ provide a frame for the tangent bundle on U. Compute ∇ explicitly in terms of this frame (i.e. compute $\nabla_{\partial/\partial x_i}(\partial/\partial x_j)$ for $1 \leq i, j \leq 2$).

(c) Compute the curvature tensor R associated to ∇ explicitly on U in terms of the frame $\partial/\partial x_1$ and $\partial/\partial x_2$.
Hint: For given vector fields χ_1, χ_2, χ_3 express the vector field $R(\chi_1, \chi_2)\chi_3$ in terms of $\chi_1, \chi_2, \chi_3, \nabla$ and the Lie bracket $[\ ,\ ]$.

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1) (a) Suppose $X/\sim$ is T_1, and let $[a],[b]$ be distinct members of $X/\sim$. Then there exists an open subset $V_{[b]}$ such that $[a] \notin V_{[b]}$ but $[b] \in V_{[b]}$. Well then we can determine that

$$\{[a]\}^c = \bigcup_{[a]\neq[b]} V_{[b]}$$

is an open set, so $\{[a]\}$ is closed. Since projection is continuous, we have $\pi^{-1}([a]) \subseteq X$ is closed.

Now suppose $[a] \subseteq X$ is closed for all $a \in X$, and consider $\{[a]\} \subseteq X/\sim$. Well $\pi^{-1}(\{[a]\}) = [a]$ is closed so since π is a quotient map, we get that $\{[a]\}$ must be closed. Therefore, if we have two distinct points $[a]$ and $[b]$ in $X/\sim$ we can take the two open sets that separate them to be $\{[a]\}^c$ and $\{[b]\}^c$. Thus $X/\sim$ is T_1.

(b) Suppose $X/\sim$ is Hausdorff: The set $\sim$ is defined to be $\sim := \{(a,b) \in X \times X : a \sim b\}$. Well, if $X/\sim$ is Hausdorff, we know that the diagonal

$$\Delta_{X/\sim} = \{([a],[b]) : [a]=[b]\} = \{(a,b) : a\sim b\}$$

is closed in $X/\sim \times X/\sim$. We can then take the preimage of the continuous map $\pi \times \pi$ to get

$$(\pi\times\pi)^{-1}(\Delta_{X/\sim}) = \sim$$

is closed in $X \times X$.

(c) Suppose π is an open map. The forward direction was already proved in part (b), so we will deal with the reverse. Since $\sim$ is closed $\sim^c$ is open and

$$\pi\times\pi(\sim^c) = \{(a,b) : a \not\sim b\} = \Delta_{X/\sim}^c$$

is open in $X/\sim \times X/\sim$. Thus the diagonal is closed in $X/\sim \times X/\sim$ and we have that $X/\sim$ is Hausdorff.

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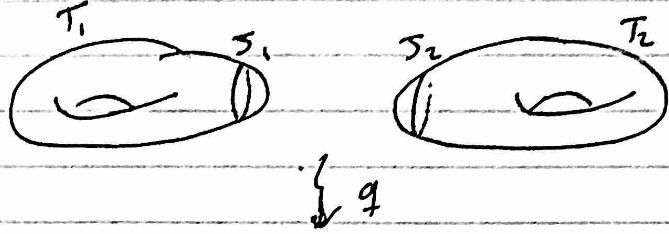
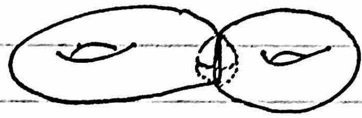
* 2) BE CAREFUL: THIS IS NOT THE 2-HOLED TORUS.

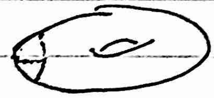
We can represent the quotient defined in this problem as follows



 $= X$

Where the intersection of T_1 and T_2 forms an item homeomorphic to a sphere.

Let

$U =$  $V =$ 

$U \cap V =$ 

Using ~~the~~ deformation retractions, we can see U and V are homotopic to a torus, thus

$$\pi_1(U) = \pi_1(V) = \pi_1(T^2) \quad \text{and} \quad \pi_1(U \cap V) = 1.$$

So by SVK we have

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \pi_1(T^2) *_1 \pi_1(T^2).$$

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3) Let ω be the smooth non-vanishing n-form on Y corresponding to the given orientation on Y. And let η be the smooth non-vanishing orientation form for X. Since X is connected, we know that if α is another non-vanishing orientation form on X then

$$\eta = F\alpha \quad \text{or} \quad \eta = -F\alpha$$

for some smooth, positive function F. Let $(E_i|_p)$ be an oriented basis of T_pM. Now, $f^*\omega$ is a non-vanishing orientation form on X, so if $\eta = F f^*\omega$ then

$$\eta_p(E_1|_p, \dots, E_n|_p) = F(p)(f^*\omega)(E_1|_p, \dots, E_n|_p) > 0$$

which means

$$f^*\omega(E_1|_p, \dots, E_n|_p) = \omega(df_p(E_1|_p), \dots, df_p(E_n|_p)) > 0$$

so f is orientation preserving. Similarly, if $\eta = -F f^*\omega$ then f is orientation reversing.

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4) S^1 is parallelizable because it has the non-vanishing global vector field

$$\omega = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

S^2 is not parallelizable by the Hairy Ball Theorem.

S^3 is parallelizable because it is isomorphic to the Lie group $SU(2)$ and all Lie groups are parallelizable.

(We could have also used that $S^1 \cong SO(2)$ which is also a Lie group).

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5) (a) We only need to show ω is exact. Consider

$$\alpha = x + \frac{1}{\pi}\sin(\pi x)\cos(\pi x) + y.$$

Then

$$d\alpha = (1 + \cos(\pi x)^2 - \sin(\pi x)^2)\,dx + dy$$
$$= 2\cos(\pi x)^2\,dx + dy.$$

(b) We have $f(a,b) = (3a+2b,\ a-b)$ $\quad \omega = 2\cos(\pi x)^2 dx + dy$

$$x(a,b) = 3a+2b \qquad y(a,b) = a-b$$
$$dx = 3da + 2db \qquad dy = da - db$$

So

$$f^*\omega = 2\cos[\pi(3a+2b)]^2(3da+2db) + da - db.$$

Along the curve $\gamma(a) = (a,0)$ from $a=0$ to $a=1$ we get

$$\int_\gamma f^*\omega = \int_{a=0}^{a=1} 2\cos(3\pi a)^2 \cdot 3da + da.$$
$$= 4$$

(c) We know that $p^*\eta = \omega$, and

$$d\omega = d(p^*\omega)$$
$$= p^*(d\eta)$$
$$= 0.$$

So for any vector field X $p^*(d\eta)(X) = 0$ which will only be true for every vector field if $d\eta(X) = 0$. Thus η is closed.

(d) $d\bar{f}^*\eta = \bar{f}^* d\eta = 0$ so $\bar{f}^*\eta$ is closed. Now arithmetic in $\mathbb{R}^2/\mathbb{Z}^2$ is done mod 1 (take only the decimal part) so if we were to compute $\bar{f}^*\eta$ with respect to the usual global coordinates (u,v) on $\mathbb{T}^2$, we would get

$$\bar{f}^*\eta = 6\cos(3\pi u + 2\pi v)^2\,du + 4\cos(3\pi u + 2\pi v)^2\,dv.$$

For $\bar{f}^*\eta$ to be $d\beta$ for some zero form β we would need

$$\beta = 3u + \frac{1}{3\pi}\sin(3\pi u + 2\pi v)\cos(3\pi u + 2\pi v) + 2v + \frac{1}{2\pi}\sin(3\pi u + 2\pi v)\cos(3\pi u + 2\pi v)$$

but these cannot exist in $\mathbb{T}^2$ due to the mod 1 arithmetic.