RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2017

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Compute the fundamental group of $\mathbb{R}^3 - C$ where

$$C = \{(x, y, z) \mid x = 0, y^2 + z^2 = 2\}.$$

(Hint: Consider a tube around C whose inner hole has been filled by a disk.)

Problem 2. Let $p : X \to Y$ be a continuous closed surjection.

(a) Let $U \subset X$ be an open set which contains $p^{-1}(\{y\})$. Prove that there is an open neighborhood W_y of y such that $p^{-1}(W_y) \subset U$. (Hint: Consider $X \backslash U$.)

(b) Recall that X is normal if the one point sets in X are closed and, for every pair disjoint of closed sets A and B, there exists disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Show that if X is normal, then so is Y.

Problem 3. Let $p : \widetilde{X} \to X$ be the universal cover of a connected and locally-path connected space X and let $A \subset X$ be a connected and locally path-connected subspace. Let $\widetilde{A}$ be a path component of $p^{-1}(A)$.

(a) Show that $\widetilde{A} \to A$ is a covering space.

(b) Prove that the image of

$$\pi_1\big(\widetilde{A}, \widetilde{a}_0\big) \to \pi_1(A, a_0)$$

coincides with the kernel of $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$, where $\widetilde{a}_0 \in \widetilde{A}$ is any basepoint, $a_0 = p\big(\widetilde{a}_0\big)$, and $\iota : A \hookrightarrow X$ is the canonical embedding.

Problem 4. (a) Show that there is no immersion $S^1 \to \mathbb{R}$.

(b) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x^3 z, xy + z)$. At which point is f a submersion? Determine the regular values of f.

Problem 5. Define $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$ as a 2-form on $\mathbb{R}$. Show that no diffeomorphism $\varphi : \mathbb{R}^6 \to \mathbb{R}^6$ satisfying $\varphi^*\omega = \omega$ can map the unit sphere S^5 to a sphere of radius $r \neq 1$.
Hint: consider $\omega \wedge \omega \wedge \omega$.

Problem 6. Recall that an n-dimensional manifold M is called *parallelizable* if its tangent bundle is trivial. Which of the following manifolds are parallelizable? Provide a short justification of your answer in a sentence.

(i) The n-torus $\left(S^1\right)^n = \mathbb{R}^n/\mathbb{Z}^n$ (where $\mathbb{Z}^n$ is the subgroup of the additive group of $\mathbb{R}^n$ consisting of points whose coordinates are all integers);

(ii) the sphere S^2;

(iii) the real projective plane $\mathbb{RP}^2$.

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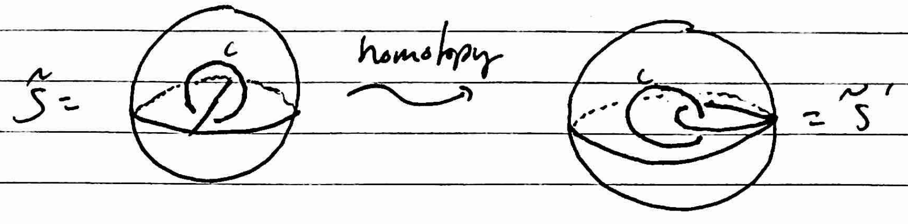
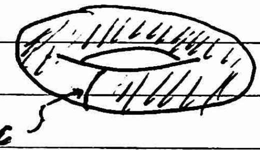
1) There are a couple of ways to go about this. The first way is to note that we can contract $\mathbb{R}^3 - C$ to the following sphere $\tilde{S}$ which can then be homotoped to $\tilde{S}'$



So the space $\mathbb{R}^3 - C$ is homotopic to $S^1 \wedge S^2$. Thus $\pi_1(\mathbb{R}^3 - C) = \pi_1(S^1 \wedge S^2) = \mathbb{Z}$. ∎

The other way to do this is to let D represent the disk bounded by the circle C. Let T be the filled torus with outer meridian given by C. Then define the sets

$$U = \text{int}(T)$$
$$V = \mathbb{R}^3 - D$$
$$U \cap V = \text{int}(D).$$



We can see that U contracts to a circle and V contracts to a sphere. This gives us that $\pi_1(U) = \mathbb{Z}$, $\pi_1(V) = 1$, and $\pi_1(U \cap V) = 1$. So by SVK we have that

$$\pi_1(\mathbb{R}^3 - C) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \mathbb{Z} *_1 1 = \mathbb{Z}.$$

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2) (a) Since $\mathcal{U}$ is open, then $X \setminus \mathcal{U}$ is closed, and since $p: X \to Y$ is a surjective, closed map we have that $p(X \setminus \mathcal{U})$ is closed. Take $W_y = Y \setminus p(X \setminus \mathcal{U})$. We obviously have that $y \in W_y$ since $p^{-1}(\{y\}) \subseteq \mathcal{U}$. Now, if we take any $x \in p^{-1}(W_y)$ then $p(x) \in Y \setminus p(X \setminus \mathcal{U})$, so $x \notin X \setminus \mathcal{U}$. Thus $x \in \mathcal{U}$ and $p^{-1}(W_y) \subseteq \mathcal{U}$

(b) Since p is closed and surjective, we get that the one point sets are closed in Y for free. Now suppose $A, B \subseteq Y$ are disjoint closed sets. Since p is continuous, we then have that $p^{-1}(A)$ and $p^{-1}(B)$ are closed in X. So there exist open sets $\mathcal{U}, V \subseteq X$, $\mathcal{U} \cap V = \emptyset$ $p^{-1}(A) \subseteq \mathcal{U}$ and $p^{-1}(B) \subseteq V$. Now, for each $a \in A$, $p^{-1}(\{a\}) \subseteq \mathcal{U}$ so by part (a) there is some $W_a \subseteq Y$ such that $p^{-1}(W_a) \subseteq \mathcal{U}$. Define the sets $W_A := \bigcup_{a \in A} W_a$, $W_B = \bigcup_{b \in B} W_b$. Then we have that $A \subseteq W_A$, $B \subseteq W_B$ and that U_A and W_B are open. with $p^{-1}(W_A) \subseteq \mathcal{U}$ and $p^{-1}(W_B) \subseteq V$. Well then $p^{-1}(W_A \cap W_B) \subseteq p^{-1}(W_A) \cap p^{-1}(W_B) \subseteq \mathcal{U} \cap V = \emptyset$ so $W_A \cap W_B = \emptyset$ and we have that Y is normal.

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3) (a) Let $\tilde{A}$ be a path component of $p^{-1}(A)$ and consider the restriction $p|_{\tilde{A}} : \tilde{A} \to A$ of the covering map p. For any $x \in A$, there is some evenly covered neighborhood U of x. Since A is locally path connected, we can take some smaller path connected neighborhood of x relative to the topology on A. Call this set V. Then V is evenly covered and the path components of $p^{-1}(V)$ are mapped homeomorphically onto V. Since $\tilde{A}$ is a path component of $p^{-1}(A)$, the path components of $p|_{\tilde{A}}^{-1}(V)$ are path components of $p^{-1}(A)$ and are hence mapped homeomorphically onto V. Thus $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering map and $\tilde{A} \to A$ is a covering space.

* (b) Consider the following diagram.

$$1 \to \pi_1(\tilde{A}, \tilde{a}_0) \xrightarrow{q_*} \pi_1(A, a_0) \xrightarrow{\iota_*} \pi_1(X, a_0) \to 1$$

we want to show that this is a short exact sequence ($im(q_*) = ker(\iota_*)$)
Let $[\alpha] \in im(q_*)$. Then there exists $[\beta] \in \pi_1(\tilde{A})$ such that $q_*[\beta] = [\alpha]$.
But then we have that

$$\iota_*[\alpha] = \iota_*(q_*[\beta]) = (\iota \circ q)_*[\beta] = (p \circ j)_*[\beta] = p_*[j \circ \beta]$$
$$= p_*[c_{\tilde{a}_0}] = [c_{a_0}] \quad \text{since } \tilde{X} \text{ is simply connected.}$$

So $im(q_*) \subseteq ker(\iota_*)$.

Conversely, let $[\alpha] \in ker(\iota_*)$. Then $\iota_* = [\iota \circ \alpha] = [c_{a_0}]$. But $[c_{a_0}] \in p_*(\pi_1(\tilde{x}, \tilde{a}_0))$, so there is a loop $\tilde{\alpha}$ in $\tilde{X}$ at $\tilde{a}_0$ such that $p_*[\tilde{\alpha}] = [c_{a_0}]$. Since $\tilde{A}$ is a path component and $\tilde{a}_0 \in \tilde{A}$, we must have $[\tilde{\alpha}] \in \pi_1(\tilde{A}, \tilde{a}_0)$. Therefore, $q_*[\tilde{\alpha}] = [\alpha]$ and $[\alpha] \in im(q_*)$. and $ker(\iota_*) \subseteq im(q_*)$.

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4) (a) Consider some 0-form $f : S^1 \to \mathbb{R}$. We know that S^1 is a compact, orientable manifold without boundary, so

$$\int_{S^1} df = \int_{\partial S^1} f = 0$$

by Stokes' Theorem. Well, this is only true if df vanishes at some point, so df is not injective and hence f cannot be an immersion.

(b) Given $f : \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x^3 z, xy+z)$ we have

$$Df = \begin{pmatrix} 3x^2 z & 0 & x^3 \\ y & x & 1 \end{pmatrix}$$

which only fails to have full rank when

$$3x^3 z = 0, \quad 3x^2 z - x^3 y = 0, \quad \text{and} \quad -x^4 = 0.$$

This only happens when $x = 0$, so f is a submersion at all points (x,y,z) with $x \neq 0$. This gives us that the Regular values of f are (a,b) with $a \neq 0 \neq b$.

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5) Suppose $\varphi : \mathbb{R}^6 \to \mathbb{R}^6$ is a diffeomorphism that maps S^5 to some other sphere $\tilde{S}$. Let $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6$. Then we have that

$$\frac{1}{6}\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6$$

which is the standard volume form on $\mathbb{R}^6$. Since φ is a diffeomorphism, we then get that when $\varphi^*\omega = \omega$

$$\begin{aligned}
\mathrm{vol}(\tilde{S}) &= \frac{1}{6}\int_{\tilde{S}} \omega \wedge \omega \wedge \omega \\
&= \frac{1}{6}\int_{S^5} \varphi^*(\omega \wedge \omega \wedge \omega) \\
&= \frac{1}{6}\int_{S^5} \varphi^*\omega \wedge \varphi^*\omega \wedge \varphi^*\omega \\
&= \frac{1}{6}\int_{S^5} \omega \wedge \omega \wedge \omega \\
&= \mathrm{vol}(S^5).
\end{aligned}$$

So $\tilde{S}$ must be a sphere with the same radius as S^5. Hence, the radius of $\tilde{S}$ is 1.

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6) (a) We know that for the product space of any finite set of smooth manifolds

$$T\left(\bigoplus_i^n M_i\right) = \bigoplus_i^n TM_i$$

S^1 is a parallelizable smooth manifold (it is a Lie group) so we have $T(S^1) \cong S^1 \oplus \mathbb{R}$. This gives us that

$$T((S^1)^n) = T\left(\bigoplus_i^n S^1\right)$$
$$= \bigoplus_i^n T(S^1)$$
$$= \bigoplus_i^n (S^1 \oplus \mathbb{R})$$
$$= (S^1)^n \oplus \mathbb{R}^n$$

so $(S^1)^n$ is parallelizable.

(b) Every parallelizable manifold admits a smooth global frame. However, by the Hairy Ball Theorem, we know that no such frame can exist on S^2. Therefore, S^2 is not parallelizable.

(c) Every parallelizable manifold is orientable. $\mathbb{R}P^2$ is not orientable, so it cannot be parallelizable.