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Topology/Geometry

Ph.D. Preliminary Exam

January, 2014

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

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1. Consider the topological spaces $Q_1 = S^1 \times \mathcal{B}^2 \subseteq R^4$ and $Q_2 = \mathcal{B}^2 \times S^1 \subseteq R^4$, where $\mathcal{B}^2$ is the unit disc in $\mathbb{R}^2$ and S^1 is its boundary, the unit circle. Endow Q_j with the topology induced from the standard topology on $\mathbb{R}^4$, $j = 1, 2$. Note in particular that $\partial Q_j = S^1 \times S^1$, $j = 1, 2$. Consider the quotient space X obtained by identifying $(w_1, w_2) \in Q_1$ with $(w_2, w_1) \in Q_2$ whenever w_1 and w_2 are both in the unit circle. Compute the fundamental group of X using the van Kampen theorem.

2. (a) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ (with the standard topologies).
   (b) Is the topological space $\mathbb{R}$ (endowed with the standard topology) homeomorphic to the topological space $\mathbb{R}$ (endowed with the finite complement topology)?

3. Let (M, d) be a metric space.
   (a) Show that the distance function $d : M \times M \to \mathbb{R}$ is continuous. Here $M \times M$ has the product topology.
   (b) If A and B are disjoint compact subsets of M, show that
   $$d(A, B) = \inf_{x \in A, y \in B} d(x, y)$$
   is positive, and that there are points $x_0 \in A$ and $y_0 \in B$ such that $d(A, B) = d(x_0, y_0)$.

4. Suppose M is a smooth orientable compact manifold of dimension $2n$. Suppose ω is a 2-form for which $d\omega = 0$. Let $\mu = \omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ (n times), and suppose that μ is a nowhere-zero $2n$-form on M.
   (a) Show that $d\mu = 0$.
   (b) Show that μ is not $d\beta$ for any $(2n-1)$-form β.
   (c) Conclude that ω is not $d\alpha$ for any 1-form α.

5. Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by
   $$f(u, v) = (u + v, uv, u - v).$$
   (a) If $\omega = y\, dx + x\, dy + y\, dz$, compute $f^*\omega$.
   (b) Is $f^*\omega = d\beta$ for some 1-form β on $\mathbb{R}^2$?
   (c) Calculate $\int_{\partial I^2} f^*\omega$ over the boundary of the unit square I^2 in $\mathbb{R}^2$ defined by the region inside the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$.

6. Consider the map $F \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by
   $$F(u, v, w) = (w^2 - uv, v^2 + u^2).$$
   At which values (a, b) is $F^{-1}(a, b)$ a smooth one-dimensional submanifold of $\mathbb{R}^3$?

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1) Let $U = p(Q_1)$ $V = p(Q_2)$ (Note: $p^{-1}p(Q_i) = Q_i$ so these are open sets) and $U \cap V = p(Q_1 \cap Q_2)$. Then by SVK we can compute the fundamental group.

$$\begin{tikzcd} & p(Q_1) \arrow[dr] & \\ p(Q_1 \cap Q_2) \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & Q_1 \cup Q_2/\sim \\ & p(Q_2) \arrow[ur] & \end{tikzcd}$$

$$\pi_1(p(Q_1 \cap Q_2)) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle.$$

$$\pi_1(p(Q_1)) = \langle a \rangle \qquad \pi_1(p(Q_2)) = \langle b \rangle.$$

$$\iota_{1*}(\alpha) = a \qquad \iota_{2*}(\alpha) = 1$$
$$\iota_{1*}(\beta) = 1 \qquad \iota_{2*}(\beta) = b$$

So

$\cancel{\pi_1(p(Q_1 \cap Q_2)) =}$

$$\pi_1(X) = \pi_1(p(Q_1)) \underset{\pi_1(p(Q_1 \cap Q_2))}{*} \pi_1(p(Q_2)) = \frac{\langle a \rangle * \langle b \rangle}{\langle a = 1, b = 1 \rangle} = 1$$

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2) (a) Suppose that there is some homeomorphism between $\mathbb{R}$ and $\mathbb{R}^2$ given by f. Then this will induce a homeomorphism from $\mathbb{R}\setminus\{x\}$ to $\mathbb{R}^2\setminus\{f(x)\}$. But $\mathbb{R}\setminus\{x\}$ is not path-connected while $\mathbb{R}^2\setminus\{f(x)\}$ is $\lightning$. So $\mathbb{R}$ and $\mathbb{R}^2$ cannot be homeomorphic.

(b) $\mathbb{R}_{st}$ is not homeomorphic to $\mathbb{R}_{FC}$ because $\mathbb{R}_{st}$ is Hausdorff but $\mathbb{R}_{FC}$ is not.

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3) (a) We only need to show that for any $(x,y) \in d^{-1}((a,b))$ there is some $r>0$ such that $B_r(x) \times B_r(y) \subseteq d^{-1}((a,b))$. Define $r_1 = \frac{1}{2}(d(x,y)-a)$ and $r_2 = \frac{1}{2}(b-d(x,y))$. We have that for any $(u,v) \in B_{r_1}(x) \times B_{r_2}(y)$

$$d(x,y) \leq d(x,u) + d(u,v) + d(y,v) < r_1 + d(u,v) + r_1$$
$$= d(x,y) - a + d(u,v)$$

So $a < d(u,v)$.
Also for any $(u,v) \in B_{r_2}(x) \times B_{r_2}(y)$

$$d(u,v) \leq d(x,u) + d(x,y) + d(v,y) < b - d(x,y) + d(x,y)$$

So $d(u,v) < b$. If we then take $r = \min\{r_1, r_2\}$, we have that $B_r(x) \times B_r(y) \subseteq d^{-1}((a,b))$, so every point in $d^{-1}((a,b))$ has a basic open set around it contained in $d^{-1}((a,b))$. Thus $d^{-1}((a,b))$ is open and $d: M \times M \to \mathbb{R}$ is continuous.

(b) First of all, since A and B are compact subsets of a Metric Space, they are both closed (Metric Spaces are Hausdorff). Moreover, $A \times B$ is closed and compact in $M \times M$ so any continuous function $f: A \times B \to \mathbb{R}$ attains both a minimum and a maximum on $A \times B$. We showed in part (a) that $d: M \times M \to \mathbb{R}$ is continuous so $d|_{A \times B}$ is continuous and attains a minimum at some point (x_0, y_0). Suppose $d(A,B) = d(x_0, y_0) = 0$. Then $x = y$ and $A \cap B \neq \emptyset$ ↯. So $d(A,B) \neq 0$ and thus must be positive.

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4) (a) We have that

$$d\mu = d\omega \wedge \omega^{n-1} + \omega \wedge d\omega \wedge \omega^{n-2} + \cdots + \omega^{n-1} \wedge d\omega$$
$$= 0 + 0 + \cdots + 0 = 0.$$

(b) We have that μ is a nonvanishing $2n$-form on a $2n$-dimensional orientable manifold, μ defines an orientation on M. This gives us that

$$\left| \int_M \mu \right| > 0.$$

Suppose now that $\mu = d\beta$ for some $(2n-1)$-form β. Then we have that by Stokes' Theorem

$$\int_M \mu = \int_M d\beta = \int_{\partial M} \beta = 0 \quad \lightning.$$

(c) Suppose that $\omega = d\alpha$ for some 1-form α. Then we have that

$$\mu = d\alpha \wedge (d\alpha)^{n-1}$$
$$= d(\alpha \wedge (d\alpha)^{n-1})$$

so $\mu = d\beta$ for $\beta = \alpha \wedge (d\alpha)^{n-1}$ which contradicts part (b).

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5) Let $f:\mathbb{R}^2 \to \mathbb{R}^3 : (u,v) \mapsto (u+v,\ uv,\ u-v)$.

(a) Suppose $\omega = y\,dx + x\,dy + y\,dz$. Well

$$\begin{aligned} x(u,v) &= u+v & dx &= du+dv \\ y(u,v) &= uv & dy &= v\,du + u\,dv \\ z(u,v) &= u-v & dz &= du-dv \end{aligned}$$

So

$$f^*\omega = (2uv+v^2)\,du + (u^2+uv)\,dv.$$

(b) No. If $f^*\omega = d\beta$ for some 0-form β, we would need

$$\beta = u^2v + v^2u + F(v) = u^2v + \tfrac{1}{2}v^2u + G(u)$$

which is impossible since the coefficients on the v^2u term cannot agree. Alternatively, you can compute

$$d(f^*\omega) = (2u-2v-1)\,du\wedge dv \neq 0$$

so $f^*\omega$ cannot be exact since it is not closed.

(c) Let

$$\gamma_1(t) = (t,0) \quad \gamma_2(t) = (1,t) \quad \gamma_3(t) = ((1-t),1) \quad \gamma_4(t) = (0,1-t) \quad t\in[0,1]$$

Then

$$\int_{\partial I^2} f^*\omega = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f^*\omega = \int_{\gamma_1} f^*\omega + \int_{\gamma_2} f^*\omega + \int_{\gamma_3} f^*\omega + \int_{\gamma_4} f^*\omega$$

$$= 0 + \int_0^1 1+t\,dt + \int_0^1 2(1-t)+1\,dt + 0$$

$$= \int_0^1 4-t\,dt = \boxed{3.5}$$

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6) Let $F : \mathbb{R}^3 \to \mathbb{R}^2 : (u, v, w) \mapsto (w^2 - uv, v^2 + u^2)$

Then we have that

$$DF = \begin{pmatrix} -v & -u & 2w \\ 2u & 2v & 0 \end{pmatrix}$$

which has full rank except when $u+v=0, w=0$; $u-v=0, w=0$, or $u=v=0$. So the critical points of F are the points of the form $(u,-u,0)$, $(u,u,0)$, $(0,0,w)$.

Plugging these critical points in, we get that our set of critical values of F is given by

$$S = \{(x,y) : 0 \leq x, y = 2x\} \cup \{(x,y) : 0 \geq x, y = -2x\} \cup \{(x,y) : 0 \leq x, y = 0\}$$

and $F^{-1}((a,b))$ is a smooth one-dimensional submanifold whenever $(a,b) \notin S$.