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# Geometry/Topology

**Ph.D. Preliminary Exam**
**Department of Mathematics**
**University of Colorado Boulder**

**August, 2014**

*INSTRUCTIONS*:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. Let $Y = X/\sim$ and let $\pi\colon X \to Y$ be the quotient map. Recall that the *quotient topology* on $Y$ is defined as follows: a set $U \subset Y$ is defined to be open if and only if the set $\pi^{-1}(U)$ is open in $X$.

(a) Show that the quotient topology is a topology on $Y$.

(b) Let $X = \mathbb{R}$, and let $\mathbb{Q} \subset \mathbb{R}$ denote the set of rational numbers. Define an equivalence relation on $\mathbb{R}$ by the condition that $x_1 \sim x_2$ if and only if $x_1 - x_2 \in \mathbb{Q}$. Determine the quotient topology on $X/\sim$.

Q.2 Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let $q : \mathbb{R}^2 \to X$ denote the quotient map. Let $\mathbf{x}_0$ denote the image of the point $(0,0)$ in $X$. The fundamental group $\pi_1(X, \mathbf{x}_0)$ is generated by two loops $\alpha, \beta : [0,1] \to X$, defined as follows: let $\xi, \eta : [0,1] \to \mathbb{R}^2$ be the curves

$$\xi(t) = (t, 0), \qquad \eta(t) = (0, t), \qquad 0 \leq t \leq 1,$$

and let

$$\alpha(t) = q(\xi(t)), \qquad \beta(t) = q(\eta(t)).$$

(a) Find a homotopy from $\alpha * \beta$ to $\beta * \alpha$, and conclude that $\pi_1(X, \mathbf{x}_0)$ is abelian.

(b) For integers $m$ and $n$, let $\gamma : [0,1] \to \mathbb{R}^2$ be the curve

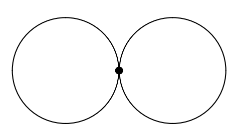
$$\gamma(t) = (mt, nt), \qquad 0 \leq t \leq 1.$$

Show that

$$q \circ \gamma \simeq \alpha^m * \beta^n$$

by constructing an explicit homotopy.

Q.3 Use Van Kampen's theorem to compute the fundamental group of the "figure 8" $X = S^1 \vee S^1$:



Q.4 Let $q : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map. Let $(x, y)$ be the standard coordinates on $\mathbb{R}^2$, and consider the 1-form on $\mathbb{R}^2$ given by

$$\omega = dx + \cos(2\pi y)\, dy.$$

(a) Show that $\omega$ is closed and exact on $\mathbb{R}^2$.

(b) Show that there exists a 1-form $\eta$ on $\mathbb{T}^2$ such that $q^*\eta = \omega$. (Hint: it suffices to show that for any deck transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f^*\omega = \omega$.)

(c) Let $\gamma : [0,1] \to \mathbb{R}^2$ be the path given by $\gamma(a) = (a, 0)$. Compute $\int_\gamma \omega$.

(d) Show that $\eta$ is closed, but *not* exact, on $\mathbb{T}^2$.

Q.5 Let $M_{2\times 2}(\mathbb{R})$ be the space of $2 \times 2$ matrices with real entries, let $S_{2\times 2}(\mathbb{R})$ be the space of *symmetric* $2\times 2$ matrices with real entries, and let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Define a map $f : M_{2\times 2}(\mathbb{R}) \to S_{2\times 2}(\mathbb{R})$ by

$$f(A) = A^T JA.$$

(a) Compute $f$ and the tangent map $Df$ explicitly in terms of coordinates. (Use the standard identifications $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$ and $S_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^3$ to define coordinates on each space, so that $f$ can be regarded as a map from $\mathbb{R}^4$ to $\mathbb{R}^3$.)

(b) Show that the set

$$\{A \in M_{2\times 2}(\mathbb{R}) \mid A^T JA = J\}$$

is a smooth submanifold of $M_{2\times 2}(\mathbb{R})$.

Q.6 Let

$$M = \mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\})/\sim,$$

where $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^3 \setminus \{0\}$ satisfy $\mathbf{z}_1 \sim \mathbf{z}_2$ if and only if $\mathbf{z}_1 = \lambda \mathbf{z}_2$ for some nonzero $\lambda \in \mathbb{R}$. Denote the equivalence class of a point $(z_0, z_1, z_2) \in \mathbb{R}^3 \setminus \{0\}$ by $[z_0 : z_1 : z_2]$. Define charts $(U_i, \mathbf{x}_i)$ on $\mathbb{RP}^2$ as follows: for $i = 0, 1, 2$, let

$$U_i = \{[z_0 : z_1 : z_2] \in \mathbb{RP}^2 \mid z_i \neq 0\},$$

and define maps $\mathbf{x}_i : U_i \to \mathbb{R}^2$ by

$$\begin{aligned} \mathbf{x}_0([z_0 : z_1 : z_2]) &= \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right), \\ \mathbf{x}_1([z_0 : z_1 : z_2]) &= \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right), \\ \mathbf{x}_2([z_0 : z_1 : z_2]) &= \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right). \end{aligned}$$

(a) Describe the open sets $V_1 = \mathbf{x}_1(U_1 \cap U_2)$ and $V_2 = \mathbf{x}_2(U_1 \cap U_2) \subset \mathbb{R}^2$, and compute the transition function

$$\mathbf{x}_2 \circ (\mathbf{x}_1)^{-1} : V_1 \to V_2$$

in terms of the standard coordinates $(x_1, x_2)$ on $V_1 \subset \mathbb{R}^2$.

(b) Let $(TU_i, T\mathbf{x}_i)$ denote the natural charts on the tangent bundle $T(\mathbb{RP}^2)$. Let

$$\tilde{V}_1 = T\mathbf{x}_1(TU_1 \cap TU_2),\ \tilde{V}_2 = T\mathbf{x}_2(TU_1 \cap TU_2) \subset \mathbb{R}^4,$$

and compute the transition function

$$T\mathbf{x}_2 \circ (T\mathbf{x}_1)^{-1} : \tilde{V}_1 \to \tilde{V}_2$$

in terms of the standard coordinates $(x_1, x_2, v_1, v_2)$ on $\tilde{V}_1 \subset \mathbb{R}^4$.

Top/Geo Prelim Exam August 2014

1) (a) Let $\tau_X$ be the topology on $X$, and $\tau_Y$ be the proposed topology on $Y$. Well

$$\pi^{-1}(\emptyset) = \emptyset, \quad \pi^{-1}(Y) = X \quad \in \tau_X$$

so $\emptyset, Y \in \tau_Y$.

Let $\{U_i\}_{i\in\mathbb{N}} \subseteq \tau_Y$. Then

$$\pi^{-1}\left(\bigcup_{i\in\mathbb{N}} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i) \in \tau_X \quad \text{since } \pi^{-1}(U_i) \in \tau_X,$$

so $\cancel{\pi^{-1}\left(\bigcup_{i\in\mathbb{N}} U_i\right)}$ $\bigcup_{i\in\mathbb{N}} U_i \in \tau_Y$. Finally if $\{K_i\}_{i=1}^{n}$ is s/t $K_i^c \in \tau_Y$ then

$$\pi^{-1}\left(\bigcap_{i=1}^{n} K_i\right) = \bigcap_{i=1}^{n} \pi^{-1}(K_i) \text{ where } \left(\bigcap_{i=1}^{n} \pi^{-1}(K_i)\right)^c \in \tau_X$$

since each $\pi^{-1}(K_i)$ is closed in $X$.

Thus $\tau_Y$ is a topology on $Y$.

(b) We have that $x_1 \sim x_2$ if and only if $x_1 - x_2 \in \mathbb{Q}$. Well, every non-empty basic open set $(a,b) \subseteq \mathbb{R}$ contians a representative of every equivalence class (If $s$ is irrational and $s > b$, there is some rational number $q$ such that $a < s-q < b$ since $q$ is dense). So if $(a,b) \subseteq \pi^{-1}(V)$ for $V \in \tau_Y$ then $\mathbb{R} \subseteq \pi^{-1}(V)$. So the quotient topology must be the indiscrete topology.

Top / Diff Geo Prelim Exam August 2014

2) (a) First of all, we will note that if $H: I \times I \to X$ is a homotopy from $\varphi_1: I \to X$ to $\varphi_2 : I \to X$ then given a continuous map $f: X \to Y$, $f \circ H : I \times I \to Y$ is also a homotopy.
Consider the paths in $\mathbb{R}^2$

$$\varphi_1(t) = \begin{cases} \xi(2t) & t \in [0, \frac{1}{2}] \\ (1,0) + \eta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

$$\varphi_2(t) = \begin{cases} \eta(2t) & t \in [0, \frac{1}{2}] \\ (0,1) + \xi(2t-1) & t \in [\frac{1}{2}, 0] \end{cases}$$

Then since $\mathbb{R}^2$ is contractible, $\varphi_1(0) = \varphi_2(0)$, and $\varphi_1(1) = \varphi_2(1)$, we have that $\varphi_1$ and $\varphi_2$ are homotopic. More importantly $q \circ \varphi_1 = \alpha * \beta$ and $q \circ \varphi_2 = \beta * \alpha$ are homotopic since $q$ is continuous.

(b) Define the path in $\mathbb{R}^2$

$$\delta(t) = \begin{cases} \xi(2mt) & t \in [0, \frac{1}{2}] \\ (m,0) + \eta(2nt - n) & t \in [\frac{1}{2}, 0] \end{cases}$$

Then $\delta(0) = \gamma(0)$ and $\delta(1) = \gamma(1)$. Since $\mathbb{R}^2$ is contractible, we have that $\delta$ and $\gamma$ are homotopic via the straight-line homotopy given by

$$H(s,t) = t\,\gamma(s) + (1-t)\,\delta(s).$$

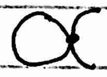
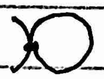
Now, since $q \circ \delta = \alpha^m \circ \beta^n$, we then get that the homotopy

$$q \circ H(s,t) = q(t\,\gamma(s) + (1-t)\,\delta(s))$$

is a path homotopy from $q \circ \gamma$ to $\alpha^m * \beta^n$. Thus $q \circ \gamma \simeq \alpha^m * \beta^n$.

Top / Diff Geo Prelim Exam August 2014

3) Take

$\mathcal{U} =$  , $V =$  , and $\mathcal{U} \cap V =$  .

Then

$$\pi_1(\mathcal{U}) = \pi_1(V) = \mathbb{Z} \quad \text{and} \quad \pi_1(\mathcal{U} \cap V) = 1$$

So, by SVK we have that

$$\pi_1(X) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V) = \mathbb{Z} *_1 \mathbb{Z} = \mathbb{Z} * \mathbb{Z}.$$

Top/Diff Geo Prelim Exam August 2014

1) (a) We only need to show that $\omega$ is exact since $d^2=0$. Consider the 0-form given by

$$\alpha = x + \frac{1}{2\pi}\sin(2\pi y)$$

then $d\alpha = \omega$.

(b) Consider $\mathbb{T}^2$ with its standard coordinates $(u,v)$. Then if we take

$$\eta = du + \cos(2\pi v)\, dv$$

then $q^*\eta = \omega$.

(c) We have

$$\int_\gamma \omega = \int_{a=0}^{a=1} da = 1 \qquad \begin{array}{ll} x(a)=a & dx = da \\ y(a)=0 & dy = 0. \end{array}$$

(d) We have that

$$0 = d\omega = dp^*\eta = p^* d\eta$$

so $d\eta = 0$ and $\eta$ is closed.

We also know that every exact covector field is conservative, so if we can show that

$$\int_c \eta \neq 0$$

for $c =$ u-axis of $\mathbb{T}^2$ then we are done. Well

$$\int_c \eta = \int_\gamma p^*\eta = \int_\gamma \omega = 1 \neq 0$$

where $\gamma(a) = (a,0)$ $a \in [0,1]$ so $\eta$ is not exact.

Top/Diff Geo Prelim Exam August 2014

5) We will use the canonical isomorphisms between $M_{2\times 2}$ and $\mathbb{R}^4$ and $S_{2\times 2}$ and $\mathbb{R}^3$ given by

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \longmapsto (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$$

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \longmapsto (s_1, s_2, s_3) \in \mathbb{R}^3.$$

(a) Under the above identification we have

$$A^t J A = \begin{pmatrix} a_1^2 - a_3^2 & a_1a_2 - a_3a_4 \\ a_1a_2 - a_3a_4 & a_2^2 - a_4^2 \end{pmatrix}$$

so

$$f(a_1, a_2, a_3, a_4) = (a_1^2 - a_3^2,\ a_1a_2 - a_3a_4,\ a_2^2 - a_4^2).$$

This gives us

$$Df = \begin{pmatrix} 2a_1 & 0 & -2a_3 & 0 \\ a_2 & a_1 & -a_4 & -a_3 \\ 0 & 2a_2 & 0 & -2a_4 \end{pmatrix}.$$

(b) The matrix $Df$ ~~drops~~ drops rank only when $a_1 = -a_2$ and $a_3 = -a_4$ or when $a_1 = a_3$ and $a_2 = a_4$. In either of these cases, we would have $\det(A) = 0$. However, if $A^t J A = J$ then we would have

$$\det(A^t J A) = \det(A^t)\det(J)\det(A)$$
$$= \det(A)^2$$
$$= 1 = \det(J)$$

So $J$ is a regular value of $f$. This means

$$\{A \in M_{2\times 2}(\mathbb{R}) : A^t J A = J\} = f^{-1}(J)$$

is a smooth submanifold of $M_{2\times 2}(\mathbb{R})$.

# Top/Diff Geo Prelim Exam August 2014

6) (a) We can describe $U_i := \{[z_0 : z_1 : z_2] \in \mathbb{RP}^2 \mid z_i \neq 0\}$, so

$$V_1 = x_1(U_1 \cap U_2) = \{(\tfrac{z_0}{z_1}, \tfrac{z_2}{z_1}) \mid [z_0 : z_1 : z_2] \in \mathbb{RP}^2 \text{ and } z_1, z_2 \neq 0\}$$
$$= \{(x_1, x_2) : x_2 \neq 0\} \text{ and}$$
$$V_2 = x_2(U_1 \cap U_2) = \{(x_1, x_2) : x_1 \neq 0\}.$$

Also,

$$x_2 \circ (x_1)^{-1}(x_1, x_2) = x_2([1 : x_1 : x_2])$$
$$= \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right).$$

(b) Let $(y_1, y_2, w_1, w_2)$ represent the standard coordinates on $V_2$. We have that

$$\sum w_i \frac{\partial}{\partial y_i}\Big|_p = \sum v_i \frac{\partial}{\partial x_i}$$

So

$$v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} = w_1 \frac{\partial}{\partial y_1} + w_2 \frac{\partial}{\partial y_2}$$
$$v_1\left(\frac{-1}{(x_1)^2}\frac{\partial}{\partial y_1} + \frac{-x_2}{(x_1)^2}\frac{\partial}{\partial y_2}\right) + v_2\left(\frac{1}{x_1}\frac{\partial}{\partial y_2}\right)$$

which gives us

$$-v_1 \frac{1}{x_1^2}\frac{\partial}{\partial y_1} + \left(v_2 \frac{1}{x_1} - v_1 \frac{x_2}{x_1^2}\right)\frac{\partial}{\partial y_2} = w_1 \frac{\partial}{\partial y_1} + w_2 \frac{\partial}{\partial y_2}.$$

Thus

$$Tx_2 \circ (Tx_1)^{-1}(x_1, x_2, v_1, v_2) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{-v_1}{x_1^2}, \frac{v_2}{x_1} - \frac{x_2 v_2}{x_1^2}\right).$$