1) G a group of order 385. Show ⓐ G has one Sylow 11-subgroup & that it is $\trianglelefteq G$.
ⓑ G has one Sylow 7-subgroup, that is $\leq Z(G)$.

First, we have $385 = 5 \cdot 7 \cdot 11$.

The # of Sylow 11-subgroups is $\equiv 1 \bmod 11$ and divides $|G|$. As none of 5, 7, 35 are $\equiv 1 \bmod 11$, there is 1 Sylow 11-subgroup P_{11} and as conjugation preserves the order of a subgroup, $gP_{11}g^{-1} = P_{11}$ $\forall g \in G$.

- Similarly, # of Sylow 7-subgroups is $1 \bmod 7$. But none of 5, 11, or 55 are $\equiv 1 \bmod 7$, so G has exactly one Sylow 7-subgroup P_7, necessarily normal in G.

Need to show $P_7 \leq Z(G)$. We have $N_G(P_7) = G$, so $\varphi: G \to \mathrm{Aut}(P_7)$ given by $\varphi(x) = \Psi_x : h \mapsto xhx^{-1}$ is a well defined homomorphism, whose kernel is $C_G(P_7)$. By 1st isom., $G/C_G(P_7) \cong \varphi(G)$ and since $|\mathrm{Aut}(P_7)| = 6$, we must have $|G/C_G(P_7)| = 1$, thus $C_G(P_7) = G$ and $P_7 \in Z(G)$. ∎

2) Let K be a field, $K[[x]]$ the ring of formal power series.

ⓐ Show $\sum_{i=0}^{\infty} a_i x^i$ is a unit iff. $a_0 \neq 0$

ⓑ Show that every nonzero proper ideal is generated by x^k for some k.

If $a_0 = 0$, then the least power of x in $\sum a_i x^i$ is at least 1, and thus for any $\sum b_i x^i$ we have that the least power of x in $(\sum a_i x^i)(\sum b_i x^i)$ is ≥ 1, so the product is not 1

If $a_0 \neq 0$, define the inverse of $\sum a_i x^i$ as follows:

$b_0 = \frac{1}{a_0}$, $b_i = -b_0 \sum_{k=1}^{i} a_k b_{i-k}$

Indeed the coeff. of x^n in $(\sum a_i x^i)(\sum b_i x^i)$ is $a_0 b_n + \sum_{k=1}^{n} a_k b_{n-k}$

and by def $a_0 b_n = -\sum_{k=1}^{n} a_k b_{n-k}$, so $\uparrow = 0$.

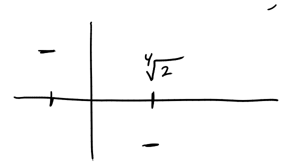
ⓑ If I is a nonzero proper ideal, it has no units.
Thus, all $p(x) \in I$ have $a_0 = 0$ and we may choose a k such that the least power of x in any $p \in I$ is at least k. Then, $I = (x^k)$, since if $p(x) \in I$ is $\sum_{i=0}^{\infty} a_i x^{k+i}$, $p(x) = x^k \sum_{i=0}^{\infty} a_i x^i$. ∎

③ Let $G = \mathrm{Gal}_{\mathbb{Q}}(x^5 - 10x + 5)$ (let $f(x)$). View G as a subgroup of S_5.

ⓐ Show that if an irreducible $g(x) \in \mathbb{Q}[x]$ has prime degree p, then $\mathrm{Gal}_{\mathbb{Q}}(g)$ has an element of order p.

Pf/ As $\mathbb{Q}$ is a perfect field, g has p distinct roots and $\mathrm{Gal}_{\mathbb{Q}}(g) \leq S_p$.
Furthermore, $\mathrm{Gal}_{\mathbb{Q}}(g)$ acts transitively on the roots.
Thus for any root α of g, the orbit of α has size p,
so by orbit-stabilizer theorem, $p = \frac{|\mathrm{Gal}_{\mathbb{Q}}(g)|}{|\mathrm{Stab}_{\mathrm{Gal}_{\mathbb{Q}}(g)}(\alpha)|}$ and $p \mid |\mathrm{Gal}_{\mathbb{Q}}(g)|$.
By Cauchy's theorem, $\mathrm{Gal}_{\mathbb{Q}}(g)$ has an element of order p.

at $\sqrt[4]{2}$
$2\sqrt[4]{2} - 10\sqrt[4]{2} + 5 = -8\sqrt[4]{2} + 5 < 0$
and at $-\sqrt[4]{2}$, $8\sqrt[4]{2} + 5 > 0$



ⓑ Show G has a 5-cycle.

- By ⓐ, G has an element of order 5 if we show the polynomial is irreducible.
As 5 divides -10 and 5, but $25 \nmid 5$ so by Eisenstein, f is irred. over $\mathbb{Q}$ and thus G has a 5-cycle.

ⓒ Show that G contains a 2-cycle.

- Identifying $f(x)$ with the function $f(x) \in C^{\infty}(\mathbb{R})$, we see that $f'(x) = 5x^4 - 10 = 5(x^2 - \sqrt{2})(x^2 + \sqrt{2})$ has 2 real zeros, so by analysis, $f(x)$ has at most 3 real roots.

By checking signs at points, we conclude that f has 3 real zeros, and a pair of complex roots $\alpha, \bar{\alpha}$. Thus conjugation swaps $\alpha, \bar{\alpha}$ and is a 2-cycle in $\mathrm{Gal}_{\mathbb{Q}}(f)$.

(d) Use previous results to show that $G \cong S_5$.

- $(12)(12\cdots n)$ generate S_n, so $(12)(12345)$ generate S_5
- $(a\ b)(1\ 2\ \cdots\ n)$ generate S_n if $(b-a, n)=1$.

Since 5 is prime, $(b-a,5)=1$ regardless of b,a.

If $\sigma=(1\,2\,3\,4\,5)$, then $\sigma^{b-a}=(a\ b\ \_\ \_\ \_)$

and we may relabel s.t. $\{(a\ b),(a\ b\ \_\ \_\ \_)\}=\{(12),(12345)\}$ so $G\cong S_5$.

(4a) Prove that there are exactly 2 distinct automorphisms of $\mathbb{F}_5\times\mathbb{F}_{25}$

- One is the identity. The other is the Frobenius map on the second factor.
  - Any automorphism sends $(1,1)\mapsto(1,1)$ and as $\mathbb{F}_5=\langle 1\rangle$, fixes $\mathbb{F}_5$.

    Thus if $\psi\in Aut(\mathbb{F}_5\times\mathbb{F}_{25})$, then $\psi=(Id,\Theta)$ for $\Theta\in Aut(\mathbb{F}_{25})$

    and if $\Theta\neq Id$, then $\Theta=\sigma_5$, the Frobenius map.

(b) Let $f(x)=x^3+3$. Prove $\exists$ exactly 2 isomorphisms $\rho:\mathbb{F}_5[x]/(f(x))\to\mathbb{F}_5\times\mathbb{F}_{25}$.

- Factoring $f(x)$ mod 5: $f(3)=27+3=30\equiv 0 \bmod 5$

  so $f(x)=(x+2)(x^2+3x+4)$

  As x^2+3x+4 has no roots in $\mathbb{F}_5$ and is of deg 2, it is irreducible

  $\mathbb{F}_5[x]/(x+2)\cong\mathbb{F}_5$ and $\mathbb{F}_5[x]/(x^2+3x+4)$ has $\mathbb{F}_5$ as prime subfield but has 25 elements, thus is $\cong\mathbb{F}_{25}$.

  $\mathbb{F}[x]/(f(x))\cong\mathbb{F}_5\times\mathbb{F}_{25}$, and by then applying the automorphisms of $\mathbb{F}_5\times\mathbb{F}_{25}$ we obtain the 2 isomorphisms from $\mathbb{F}[x]/(f(x))\to\mathbb{F}_5\times\mathbb{F}_{25}$.

(5) $G=\mathbb{Z}_{p^2}$, p an odd prime. Classify all semidirect products $G\rtimes G$ up to isomorphism.

$G=\langle 1\rangle$, so any $\varphi: G\to Aut(G)$ is determined by $\varphi(1)$,
and moreover, any automorphism $\rho\in Aut(G)$ is determined by $\rho(1)$.
Note that this shows there are $|Aut(G)|=p(p-1)$, as $\forall\rho\in Aut(G)$, $\rho(1)\neq p,2p,3p,\dots(p-1)p$
If $\varphi(1)$ is the identity map, then $G\rtimes_\varphi G\cong G\times G$.
Note that since $\varphi: G\to Aut(G)$ is a homomorphism, the order of $\varphi(1)$ must divide both p^2 (the order of 1)
and $p(p-1)$ $(|Aut(G)|)$, thus the order of $\varphi(1)$ is p.
Then $G\rtimes_\varphi G$ has order p^4, and the right-factor G is not normal in $G\rtimes_\varphi G$.
Note that all nontrivial $\varphi: G\to Aut(G)$ map to a normal Sylow p-subgroup of $Aut(G)$
so by inner automorphisms of $Aut(G)$ we obtain isomorphic semidirect products.

(6) Consider conjugacy classes of $GL_2(\mathbb{F}_5)$. How many of these conjugacy classes contain matrices whose eigenvalues lie in $\mathbb{F}_5$?

- Look at the Jordan canonical forms.