Advection equation

A skew-symmetric weak form is given by: find u_h such that, for each fix $t \in (0, t_f]$, $u_h \in V_h^p$ and

$$m(v, \dot{u}_h) + a(v, u_h) = L(v), \qquad \forall v \in V_h^p, \tag{1a}$$

where

$$m(v, \dot{u}_h) = (v, \dot{u}_h), \tag{1b}$$

$$a(v, u_h) = (v, \alpha\beta \cdot \nabla u_h) - (\nabla v, (1-\alpha)\beta u_h) - (v, \alpha(\beta \cdot n)u_h)_{\Gamma} + (v, (\beta \cdot n)u_h)_{\Gamma^{out}}, \tag{1c}$$

$$L(v, u_h) = -(v, (\beta \cdot n)u^{+})_{\Gamma^{in}}. \tag{1d}$$

Advection equation (cont.)

Stabilized form:

$$M(v, \dot{u}_h) + A(v, u_h) = L(v), \tag{2a}$$

with

$$M(v, \dot{u}_h) = m(v, \dot{u}_h) + \gamma_M j(v, \dot{u}_h), \tag{2b}$$

$$A(v, u_h) = a(v, u_h) + \gamma_A h^{-1} j(v, u_h), \tag{2c}$$

where $\gamma_M, \gamma_A > 0$ are penalty parameters. We use the following stabilization term:

$$j(v, u_h) = \sum_{F \in \mathcal{F}_\Gamma} \sum_{k=1}^{p} h^{2k+1} \left\langle \left[\partial_n^k v\right], \left[\partial_n^k u_h\right] \right\rangle. \tag{3}$$

Other equations

- Poisson equation $\rightarrow$ *Sticko, 2022*

$$a(v, u_h) = L(v) \tag{4}$$

- heat equation $\rightarrow$ *Ludvigsson, et al., 2018*

$$m(v, \dot{u}_h) + a(v, u_h) = L(v) \tag{5}$$

- wave equation $\rightarrow$ *Sticko, Kreiss, 2019*

$$m(v, \ddot{u}_h) + a(v, u_h) = L(v) \tag{6}$$

with

$$a(v, u_h) = (\nabla v, \nabla u_h) - \left\langle v, \frac{\partial u_h}{\partial n} \right\rangle_{\Gamma_D} - \left\langle \frac{\partial v_h}{\partial n}, u_h \right\rangle_{\Gamma_D} + \frac{\gamma_D}{h} \langle v, u_h \rangle_{\Gamma_D} \tag{7a}$$

$$L(v) = (v, f) + \left\langle \frac{\gamma_D}{h} v - \frac{\partial v}{\partial n}, g_D \right\rangle_{\Gamma_D} + \langle v, g_N \rangle_{\Gamma_N} \tag{7b}$$

... *Nitsche's method*