Numerical Analysis II: Homework 3

Peter Shaffery

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1 Problems

Problem 1. Let A be a symmetric positive definite matrix and set $A_0 = A$. Take the Cholesky decomp $A_k = G_k G_k^T$ and then set $A_{k+1} = G_k^T G_k$. Prove that any A_k found this way is also PD and symmetric. Show that, if A is a symmetric, 2×2 matrix with two distinct eigenvalues $\lambda_1 > \lambda_2 \geq 0$ and an ordered diagonal that A_k converges to $\text{diag}(\lambda_1, \lambda_2)$.

Proof that Sequence is Well-Defined. First note that $G_k G_k^T = A_k$ implies $G_k^T = G_k^{-1} A_k$, hence $A_{k+1} = G_k^T G_k = G_k^{-1} A_k G_k$. Since A_{k+1} and A_k are similar then they have the same eigenvalues, so if A_k is PD then so is A_{k+1}. □

Proof of Convergence to Diagonal Matrix. Now consider $A_k = \begin{bmatrix} a_k & b_k \\ b_k & c_k \end{bmatrix} = \begin{bmatrix} l_1 & 0 \\ l_2 & l_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ 0 & l_3 \end{bmatrix}$. Note that $a_k = l_1^2$, $c_k = l_2^2 + l_3^2$, and $b_k = l_1 l_2$. Since $a_k \geq c_k$ then $l_1^2 \geq l_2^2 + l_3^2$. Now consider $A_{k+1} = \begin{bmatrix} l_1 & l_2 \\ 0 & l_3 \end{bmatrix} \begin{bmatrix} l_1 & 0 \\ l_2 & l_3 \end{bmatrix}$. We still have that $a_{k+1} \geq c_{k+1}$, and he off-diagonal elements of A_{k+1} are $b_{k+1} = l_2 l_3$. Assuming $l_2 \neq 0$ (in which case the matrix A_k would already be diagonal), then from $l_1^2 > l_3^2$ we have that $|l_1| > |l_3|$, so $|b_k| > |b_{k+1}|$. Therefore the absolute values of the off-diagonals of the A_k form a strictly decreasing sequence and therefore the $b_k \to 0$.

Since the off-diagonals of A_k approach 0, then for k sufficiently large the eigenvalues of A_k are $\epsilon-$close to the diagonal elements of A_k (this is a direct consequence of the Gershgorin Circle Theorem). Since all A_k have the same, fixed eigenvalues (see previous proof), then this is equivalent to the diagonal elements of the A_k becoming $\epsilon -$ close to λ_1 and λ_2. Therefore the diagonal elements of A_k go to λ_1 and λ_2 and the off-diagonals go to 0, hence the A_k converge in the Frobenius norm, completing the proof. □

Problem 2. Show that the Jacobi eigenvalue algorithm converges quadratically.

Proof. After one rotation with angle parameter θ we have from class that:

$$\frac{a_{pq}}{a_{pp} - a_{qq}} = \frac{\cos(\theta)\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)} = \frac{1}{2}\tan(\theta) \tag{1}$$

Since all $a_{pq} \propto \mathcal{O}(\epsilon)$ then $\theta \propto \arctan(\mathcal{O}(\epsilon))$. From the Taylor expansion of $\arctan(\mathcal{O}(\epsilon))$ we see that $\theta \propto \mathcal{O}(\epsilon)$. We can then show, again from their Taylor expansions, that $\cos(\theta) \propto 1 - \mathcal{O}(\epsilon^2)$ and $\sin(\theta) \propto \mathcal{O}(\epsilon)$. Therefore every rotation matrix in the sweeps is of the form outlined in the problem statment (I really don't want to have to typeset that array).

Since (again from class), $\text{off}^2(B) = \text{off}^2(A) - 2a_{pq}^2$ then $|\text{off}^2(B) - \text{off}^2(A)| \propto |a_{pq}|^2 \propto \mathcal{O}(\epsilon^2)$. Therefore the sum of squared, off-diagonal elements of this sequence decreases quadratically in terms of the individual rotations. □