Numerical Analysis II: Homework 6

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1 Problems

Problem 1. The linear system $y' = Ay$, $y(0) = y_0$ is solved by Euler's method. Let $e_n = y_n - y(nh)$ and show that:

$$\|e_n\| \leq \|y_0\| \max_{\lambda \in \sigma(A)} |(1 + h\lambda)^n - e^{nh\lambda}|$$

Where the norm above is the 2-norm and $\sigma(A)$ denotes the set of eigenvalues of matrix A.

Proof. Some useful results:

1. Since A is symmetric it is also diagonalizable and hence we can show that $y(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} u_i$ where λ_i are the eigenvalues of A (with repetition), u_i are the eigenvectors, and $c_i = \langle c_i, y_0 \rangle$. More usefully, $\vec{c} = [c_1, ..., c_n]^T = P^T y_0$ where P is the matrix whose columns are the eigenvectors of A.

2. Note that $y_n = (1 + hA)y_{n-1}$ hence by induction $y_n = (1 + hA)^n y_0$.

Let $\Lambda = diag(\lambda_1, ..., \lambda_n)$, so then $y(t) = Pe^{\Lambda t}P^T y_0$. From this we have that $e_n = y(nh) - y_n = Pe^{\Lambda t}P^T y_0 - (1 + hA)^n y_0 = (Pe^{\Lambda t}P^T - (1 + hA)^n)y_0$. Since norms are sub-multiplicative we then have that $\|e_n\| \leq \|Pe^{\Lambda t}P^T - (1 + hA)^n\| \|y_0\|$ where the matrix norm in the left factor on the right of the inequality is just the induced 2-norm for operators.

Now we note that $Pe^{\Lambda t}P^T - (1 + hA)^n$ is still a symmetric matrix, hence (assuming that it's real) its 2-norm is just its spectral radius. Use the fact that $\sigma(Pe^{\Lambda t}P^T) = e^{\sigma(A)*t}$, and furtheremore that the eigenvalue $e^{\lambda_i t}$ of $Pe^{\Lambda t}P^T$ has eigenvector u_i (this can be seen from the series definition of matrix exponentials and the fact that the matrix is similar to $e^{\Lambda t}$). We can similarly show that $\sigma((1 + hA)^n) = (1 + h\sigma(A))^n$ and again, the eigenvalue $(1 + h\lambda_i)^n$ has eigenvector u_i.

Therefore $(Pe^{\Lambda t}P^T - (1+hA)^n)u_i = (e^{\lambda_i t} - (1+h\lambda_i)^n)u_i$, so the spectrum $\sigma(Pe^{\Lambda t}P^T - (1+hA)^n) = e^{\sigma(A)t} - (1 + h\sigma(A))^n$. From this we have that $\|Pe^{\Lambda t}P^T - (1 + hA)^n\| = \rho(Pe^{\Lambda t}P^T - (1 + hA)^n) = \max_{\lambda \in \sigma(A)} |e^{\lambda t} - (1 + h\lambda)^n|$, and then replace the t by nh because I just now realized that I forgot to do that earlier. This completes the proof. $\square$

Problem 2. The IVP $y' = \sqrt{y}$, $y(0) = 0$ clearly has a non-trivial solution $y(t) = \frac{t^2}{4}$, but Euler's method just returns the trivial solution $y(t) = 0$. Explain this paradox.

Euler's method basically uses iterated, first order Taylor approximations to estimate the function $y(t)$, but since $y(t)$ is a quadratic polynomial it's Taylor coefficients for the non-quadratic terms are all 0. Therefore the first order Taylor approximation to $y(t)$ at any point is a constant, and since $y(0) = 0$ Euler just returns a constant $y(t) = 0$.

More directly, each step of Euler pushes the approximation forward by $hf(y_{n-1})$, but since $f(y_0) = 0$ then $y_1 = 0$ as well. If $f(y_{n-1}) = 0$ and $y_{n-1} = 0$ then it can be quickly verified that $y_n = 0$ for all steps n.