Numerical Analysis II: Homework 8

Peter Shaffery

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1 Problems

Problem 1. Determine the order and region of absolute stability for the s-step Adams-Bashforth methods where $s = 2, 3$.

1. When $s = 2$ then we have $\rho(w) = w^2 - w$ and $\sigma(w) = \frac{3w-1}{2}$. We can Taylor expand $\rho(e^w) - w\sigma(e^w) = \frac{5w^3}{12} + \mathcal{O}(w^4)$, hence the $s = 2$ method is second order. To determine the region of absolute stability we determine for which values of $h\lambda$ the roots of the polynomial in w, $\pi(w; h\lambda) = \rho(w) - h\lambda\sigma(w)$, satisfy the root condition $w^* \leq 1$. This is plotted in Fig. 1a.
2. When $s = 3$ then we have that $\rho(e^w) - w\sigma(e^w) = \frac{3w^4}{8} + \prime(w^5)$; calculating the region of absolute stability as above, the result are shown in Fig 1b.
3. The Adams-Moulton method is order 3 and it's region of stability is given in Fig. 1c.

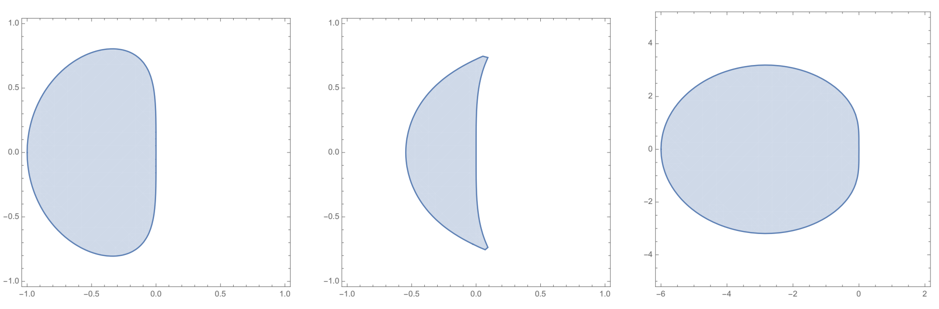
Problem 2. Determine the order and region of absolute stability of the BDF methods of step $s = 2, 3$.

1. Calculated as in problem 1, we have that the BDF method with $s = 2$ is of order 2 with region of stability given in Fig. 2a.
2. Similarly the BDF method with $s = 3$ is of order 3 with region of stability given in Fig. 2b.

Problem 3. Determine the region of absolute stability of the given Runge-Kutta method. Calculate all intersections of this region with the real and imaginary axes.

Proof. For the test problem $y' = \lambda y$ we have that each step of the RK method is equivalent to $y_{n+1} = \frac{1}{24}[24 + 24(h\lambda) + 12(h\lambda)^2 + 4(h\lambda)^3 + (h\lambda)^4]y_n$, hence finding the region of absolute stability requires determining for which values of $z = h\lambda$ is $\frac{1}{24}(25 + 25z + 12z^2 + 4z^3z^4) < 1$. We can plot this region in Mathematica without issue (Fig. 3).

To find the intersection with the real axis we solve $R(z) = \frac{1}{24}(25 + 25z + 12z^2 + 4z^3z^4) = 1$ where $z \in \mathbf{R}$, giving the interval $z \in (-2.79, 0)$. Setting $z = iy$ then we can solve $R(iy)\overline{R(iy)} = 1 - \frac{w^6}{72} + \frac{w^8}{576} = 1$ to determine that the intersection is $y \in (-2\sqrt{2}, 2\sqrt{2})$. □



(a) (b) (c)

Figure 1: Regions of absolute stability in the complex plane for (a) the two-step and (b) the three-step Adams-Bashforth methods and (c) the two-step Adams-Moulton method.

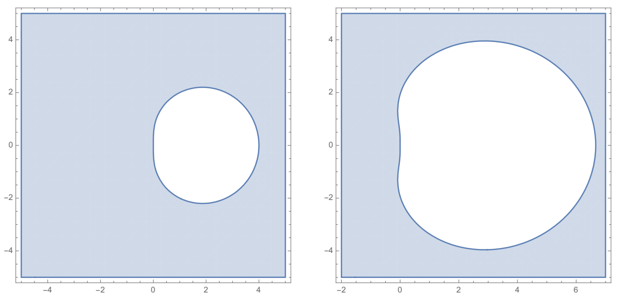

(a) (b)

Figure 2: Regions of absolute stability (shaded blue) in the complex plane for (a) the two-step and (b) the three-step BDF methods.

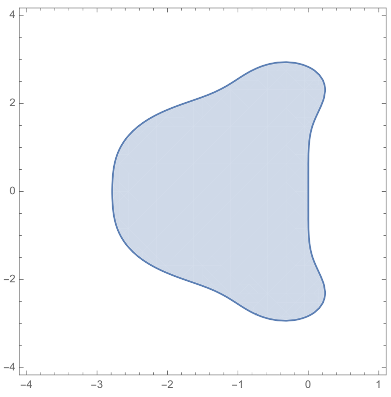


Figure 3: Region of absolute stability for the RK4 method

Problem 4. Given the ODE $\frac{d}{dt}\left[\left(\frac{1}{1+t}\right)x'\right] + \lambda x = 0$ where $x(0) = 0$, we want to find λ such that $x(1) = 0$ using the trap-rule script from a previous assignment.

Setting $y = x'$ we can convert the ODE to system of coupled DEs:

$$\begin{aligned} x' &= y \\ y' &= (1+t)(y - \lambda x) \end{aligned} \tag{1}$$

With IC $x(0) = 0$ and $y(0) = 1$. Let $\phi_x(\lambda) = x(t = 1; \lambda)$ be the flow of this system projected onto x, we want to find λ^* such that $\phi(\lambda^*) = 0$. Qualitatively we note that when $\lambda < \lambda^*$ then $\phi_x(\lambda) > 0$ and when $\lambda > \lambda^*$ then $\phi_x(\lambda) < 0$. Therefore I'm going to just implement a binary search on the interval $[6.6, 6.7]$ (probably due to an error in my implementation of the trap-rule my solutions are all negative for $\lambda \geq 6.7$ so I shifted the given interval).

Doing so gives $\lambda^* = 6.6018303$, see Appendix for code.

Problem 5. Consider the boundary value problem $y'' = f(x, y, y')$ for $a < x < b$ where $y(a) = \gamma_1$ and $y(b) = \gamma_2$.

1. Convert this to an equivalent problem with zero boundary conditions by writing $y(x) = z(x)+w(x)$ with $w(x)$ a straight line statisfying $w(a) = \gamma_1$ and $w(b) = \gamma_2$. Derive a new BVP for $z(x)$.

2. Generalize this procedure to the same problem with BC $a_0y(a)-a_1y'(a) = \gamma_1$ and $b_0y(b)-b_1y'(b) = \gamma_2$. What assumptions, if any, are needed for the coefficients a_0, a_1, b_0, b_1?

1. Well, right off we have that $w(x) = \gamma_1 + \frac{x-a}{b-a}\gamma_2$, hence $z(x) = y(x) - \gamma_1 - \frac{x-a}{b-a}\gamma_2$. Therefore $z''(x) = y''(x) + 0 = f(x, y, y'') = f(x, z(x) + \gamma_1 + \frac{x-a}{b-a}\gamma_2, z'(x) + \frac{\gamma_2}{b-a})$ with BC $z(a) = z(b) = 0$.

2. Okay so we still want $y(x) = z(x) + w(x)$ where $w(x)$ is some function that will force $a_0z(a) - a_1z'(a) = b_0z(b) - b_1z'(b) = 0$. Since $y'(x) = z'(x) + w'(x)$ then this is equivalent to force

$a_0w(a) - a_1w'(a) = \gamma_1$ and $b_0w(b) - b_1w'(b) = \gamma_2$. Can we find a line that satisfies these conditions? If $w(x) = mx + c$ then:

$$\begin{aligned} a_0ma + a_0c - a_1m &= \gamma_1 \\ b_0mb + b_0c - b_1m &= \gamma_2 \end{aligned} \tag{2}$$

So as long as $a_i \neq b_i$ for $i = 0, 1$ then this gives two equations for two unknowns so there is a unique solution. I find it in the attached Mathematic script, but to shortcut some obnoxious typesetting let's call them m^* and c^*. Then we can just proceed as we did for Part 1 to plug in $w(x) = m^*x+c^*$ into the original BVP.

2 Appendix: Code

```
import numpy as np
import scipy as sp
import scipy.optimize as spot
import numpy.linalg as npla
import matplotlib.pyplot as plt
from tqdm import *

class ode_obliterator_v2(dict):
    def __init__(self, RHS, init_val, init_time=0, h=1e-1, q=2,n=2):
        self.yp = RHS
        self.iv = np.array(init_val)
        self.init_time = init_time
        self.t = init_time
        self.h = h
        self.q = q
        self.n = n
        self.dim = max(self.iv.shape)

        self.sol = self.iv
        self.times = np.array([self.t])

    def sacramento_scramble(self,h,curr,t):
        v = curr + .5*h*np.dot(self.yp(t),curr)
        A = np.eye(self.dim) - .5*h*self.yp(t+h)
        output = npla.solve(A,v)
        return(output)

    def basic_trap(self,end,h):
        hold = self.iv
        times = np.arange(self.init_time,end,h)
        curr = self.iv
        for t in times:
            curr = self.sacramento_scramble(h,curr,t)
            hold = np.vstack([hold,curr])
        return(hold)

    def boston_u_turn(self,init):
        n = len(init)
        table = np.zeros([n,n])
        table[:,0] = init

        for j in range(1,n):
            for i in range(j,n):
                p = 2*j+1
                table[i,j] = table[i,j-1] +(table[i,j-1] - table[i-1,j-1])/float(self.q**p-1)
        update = table[n-1,n-1]

        return(update)

    def boulder_shimmy(self,end_time):
        t_steps = np.arange(self.init_time,end_time,self.h)
        self.multigrid = np.zeros([self.n,len(t_steps),self.dim])
        for k in range(0,self.n):
            h = self.q**(-1*k)*self.h
            curr = self.iv
            for t in tqdm(range(len(t_steps))):
                time = t_steps[t]
                for i in np.arange(self.q**k):
                    curr = self.sacramento_scramble(h,curr,time)
                    time += h

                self.multigrid[k,t,:] = curr

    def houston_plug_n_chug(self,end_time):
        self.boulder_shimmy(end_time)
        hold = np.zeros(self.multigrid.shape[1:3])
        for i in tqdm(range(hold.shape[0])):
            for j in range(hold.shape[1]):
                hold[i,j] = self.boston_u_turn(self.multigrid[:,i,j])

        self.sol = hold
        self.times = np.arange(self.init_time,end_time,self.h)


def ode(t,l):
    A = np.array([[0.,1.],[-l*(1.+t),(1.+t)]])
    return(A)

y0 = np.array([0.,1.])
def test(l):
    trap_solve = ode_obliterator_v2(lambda t: ode(t,l),y0, init_time = 0., h = 1e-5,n=2)
    trap_solve.houston_plug_n_chug(1)
    return(trap_solve.sol[-1,0])

#interval = np.array([6.6,6.7])
interval = np.array([ 6.60182937,6.60184])
for index in range(0,5):
    l = np.mean(interval)
    y1 = test(l)

    if y1 < 0.:
        interval = np.array([interval[0],np.mean(interval)])

    if y1 > 0.:
        interval = np.array([np.mean(interval),interval[1]])

    if y1 == 0.:
        break

#okay it's basically lambda = 6.6018303 (after doing some manual tuning like a freaking caveman)
```