Numerical Analysis II: Homework 3

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February 15, 2017

1 Problems

Problem 1a. Prove that, if all the singular values of $A \in \mathbb{C}^{n\times n}$ are equal then $A = \gamma U$, where γ is a constant and U is unitary.

Proof. This follows almost immediately from the SVD of A. Since $A = V\Sigma W^*$, where V and W are unitary and Σ is a diagonal matrix of the singular values of A, then if all singular values are equal then $\Sigma = \gamma I$, where I is the identity matrix. Hence $A = \gamma V W^*$, and since V and W are unitary clearly $U = VW^*$ is unitary as well. □

Problem 1b. Prove that, if A is nonsingular and λ is an eigenvalue of A, then $\|A^{-1}\|_2^{-1} \leq |\lambda| \leq \|A\|_2$.

Proof. By the definition of $\|A\|_2 := \sup\limits_x \frac{\|Ax\|}{\|x\|}$ we have that $|\lambda| \leq \|A\|_2$. This can be seen by letting v be the eigenvector of λ then $\|Av\| = |\lambda|\|v\|$. Now note that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, so we have that $\frac{1}{|\lambda|} \leq \|A^{-1}\|$, therefore $|\lambda| \geq \|A^{-1}\|^{-1}$, and the result is proven. □

Problem 1c. Prove that A may be represented in the form $A = SU$, where S is positive semidefinite and U is unitary. Furthermore prove that if A is invertible then this representation is unique.

Proof. Couldn't make headway with this one □

Problem 2. Here we want to prove some eigenvector perturbation results. Consider the following perturbed eigenproblem:

$$(A + \epsilon B)u_k(\epsilon) = \lambda_k(\epsilon)u_k(\epsilon)$$

We would like to show that:

$$u_k(\epsilon) = u_k + \epsilon(a_k u_k + \sum_{j\neq k} \frac{v_j^* B u_k}{(\lambda_k - \lambda_j)s_j} u_j) + \mathcal{O}(\epsilon^2)$$

Where $A^* v_k = \bar{\lambda}_k v_k$ and $s_k = v_k^* u_k$.

Proof. To start let's assume that $\lambda_k(\epsilon)$ and $u_k(\epsilon)$ are continuous functions in ϵ, so we can write:

$$u_k(\epsilon) = u_k(0) + \epsilon u_k'(0) + \mathcal{O}(\epsilon^2)$$

Take the derivative of both sides of the perturbed eigenproblem and set $\epsilon = 0$ to get $Bu_k + Au_k'(0) = \lambda_k'(0)u_k + \lambda_k u_k'(0)$, so:

$$(A - \lambda_k)u_k'(0) = (B - \lambda_k'(0))u_k \tag{1}$$

Note that we can expaned $u_k'(0)$ can be expanded in terms of the basis $(u_1, ..., u_n)$, ie. $u_k'(0) = \sum\limits_{i=1}^n a_i u_i$ which can be inserted into (1) to get:

$$(B - \lambda_k'(0))u_k = (A - \lambda_k)\sum_{i=1}^n a_i u_i = \sum_{i\neq k} a_i(\lambda_i - \lambda_k)u_i \tag{2}$$

From the hint we have that $\lambda_k'(0) = \frac{v_k^* B u_k}{s_k}$, so this means that:

$$(B - \frac{v_k^* B u_k}{s_k})u_k = \sum_{i\neq k} a_i(\lambda_i - \lambda_k)u_i \tag{3}$$

Now we can pick out an individual term from the sum of the RHS of (3) by multiplying on both sides by v_i^* (since $v_i^* u_j = \delta_{ij} s_i$, where δ_{ij} is the Kronecker delta), which gives:

$$v_i^* B u_k - \frac{v_k^* B u_k}{s_k} v_i^* u_k = a_i(\lambda_i - \lambda_k) s_i$$

And since $i \neq k$ (the k term dropped out of the sum a few steps back), then the second term in the LHS is always 0 and we can solve for a_i to obtain:

$$a_i = \frac{v_i^* B u_k}{(\lambda_i - \lambda_k) s_i} \tag{4}$$

Plugging (4) back into our expansion of $u_k'(0)$ and plugging *that* into the expansion of $u_k(\epsilon)$ gives the desired result. $\square$