SCUDEM Model: Quality of AI Content

1 The Core Differential Equation System

The dynamics of content diversity, $Q(\alpha)$, in the presence of AI-generated content, α, are described by the following system of coupled ordinary differential equations (ODEs).

The Model Equations

$$\frac{dQ}{d\alpha} = -k\,Q(\alpha) + \lambda\,P_H(\alpha) \qquad \text{(Equation 1: Quality Dynamics)}$$
$$P_H(\alpha) = 1 - \alpha \qquad \text{(Equation 2: Proportion Constraint)}$$

Variable and Parameter Definitions

- $Q(\alpha)$: The **Content Diversity** (originality) of the content pool.
- $P_H(\alpha)$: The **Proportion of Human-Generated Content** in the pool.
- k: **Rate of Elimination** constant (strength of model collapse effect).
- λ: **Rate of Injection** constant (human content input).
- α: **Amount** of the AI content proportion.

2 Analytical Solution of the ODE

The Governing Linear ODE

The derived governing equation for $Q(\alpha)$ is:

$$\frac{dQ}{d\alpha} + kQ(\alpha) = \lambda(1-\alpha) \quad \text{(Equation 1)} \tag{1}$$

Solution by Integrating Factor

Equation (1) is a linear ODE of the form $\frac{dQ}{d\alpha} + P(\alpha)Q = F(\alpha)$, where $P(\alpha) = k$ (a constant). The solution proceeds via the integrating factor method.

2.0.1 1. Integrating Factor

The integrating factor, $\mu(\alpha)$, is:

$$\mu(\alpha) = e^{\int k d\alpha} = e^{k\alpha}$$

2.0.2 2. General Solution Derivation

Multiplying Equation (1) by the integrating factor and rewriting the left side as the derivative of a product:

$$\frac{d}{d\alpha}\left[e^{k\alpha}Q(\alpha)\right] = \lambda e^{k\alpha}(1-\alpha)$$

Integrating both sides with respect to α:

$$e^{k\alpha}Q(\alpha) = \lambda \int e^{k\alpha}(1-\alpha)d\alpha + C$$

We split the integral on the right:

$$e^{k\alpha}Q(\alpha) = \lambda\left(\int e^{k\alpha}d\alpha - \int \alpha e^{k\alpha}d\alpha\right) + C$$

The first integral is straightforward: $\int e^{k\alpha}d\alpha = \frac{1}{k}e^{k\alpha}$.

2.0.3 3. Integration by Parts

The second integral, $\int \alpha e^{k\alpha}d\alpha$, requires integration by parts ($\int u\,dv = uv - \int v\,du$). We choose $u = \alpha$ (so $du = d\alpha$) and $dv = e^{k\alpha}d\alpha$ (so $v = \frac{1}{k}e^{k\alpha}$).

$$\begin{aligned}
\int \alpha e^{k\alpha}d\alpha &= \alpha\left(\frac{1}{k}e^{k\alpha}\right) - \int\left(\frac{1}{k}e^{k\alpha}\right)d\alpha \\
&= \frac{\alpha}{k}e^{k\alpha} - \frac{1}{k}\left(\frac{1}{k}e^{k\alpha}\right) \\
&= \frac{\alpha}{k}e^{k\alpha} - \frac{1}{k^2}e^{k\alpha}
\end{aligned}$$

2.0.4 4. Final General Solution

Substituting the integrated terms back into the general solution form:

$$\begin{aligned}
e^{k\alpha}Q(\alpha) &= \lambda\left[\frac{1}{k}e^{k\alpha} - \left(\frac{\alpha}{k}e^{k\alpha} - \frac{1}{k^2}e^{k\alpha}\right)\right] + C \\
e^{k\alpha}Q(\alpha) &= \frac{\lambda}{k}e^{k\alpha} - \frac{\lambda\alpha}{k}e^{k\alpha} + \frac{\lambda}{k^2}e^{k\alpha} + C \\
e^{k\alpha}Q(\alpha) &= \frac{\lambda e^{k\alpha}}{k}\left(1 - \alpha + \frac{1}{k}\right) + C
\end{aligned}$$

Dividing by the integrating factor, $e^{k\alpha}$:

$$Q(\alpha) = \frac{\lambda}{k}\left(1 - \alpha + \frac{1}{k}\right) + Ce^{-k\alpha} \quad \text{(General Solution)}$$

Applying Initial Conditions

We apply the initial condition: at the start, when there is no AI content, $\alpha = 0$, and the Content Diversity is $Q(0) = Q_0$.

Setting $\alpha = 0$ in the general solution:

$$Q_0 = \frac{\lambda}{k}\left(1 - 0 + \frac{1}{k}\right) + Ce^0$$

Solving for the integration constant C:

$$Q_0 = \frac{\lambda}{k}\left(1 + \frac{1}{k}\right) + C$$

$$C = Q_0 - \frac{\lambda}{k}\left(1 + \frac{1}{k}\right)$$

The Complete Solution

Substituting the constant C back into the general solution yields the complete, particular solution for the Content Diversity $Q(\alpha)$:

$$Q(\alpha) = \frac{\lambda}{k}\left(1 - \alpha + \frac{1}{k}\right) + \left(Q_0 - \frac{\lambda}{k}\left(1 + \frac{1}{k}\right)\right) e^{-k\alpha}$$

This equation defines the Content Diversity as a function of the AI content proportion, where Q_0 is the initial content diversity.