M189 Discrete Mathematics
Mathematical Induction Proof

Claim: The nth triangular number can be computed using the formula $T(n) = \frac{n(n+1)}{2}$.

Proof: The proof will be by mathematical induction on n.

Base Step: When $n = 1$ there is just one ball, so the first triangular number is 1. When $n = 1$ the formula yields $\frac{1(1+1)}{2} = \frac{2}{2} = 1$. Therefore, the claim is true when $n = 1$.

Induction Step: Assume that the claim has been shown to be true for some $n = k$ with $k \geq 1$. That means the kth triangular number is $T(k) = \frac{k(k+1)}{2}$.

Now, we need to show that the claim is true for $n = k + 1$. From the illustration we see that the $k + 1^{\text{st}}$ triangular number is the previous triangular number $T(k)$ plus $k + 1$.

Here is the story that is inferred from the illustrations.

$$
\begin{aligned}
T(k+1) &= T(k) + (k+1) && \textit{From the illustrations shown above.} \\
&= \frac{k(k+1)}{2} + (k+1) && \textit{Using the assumption that } T(k) = \frac{k(k+1)}{2} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} && \textit{Write } (k+1) \textit{ as an equivalent expression with a denominator of 2.} \\
&= \frac{k(k+1) + 2(k+1)}{2} && \textit{Add fractions.} \\
&= \frac{(k+1)(k+2)}{2} && \textit{Factor } (k+1) \textit{ out of each term.} \\
&= \frac{(k+1)((k+1)+1)}{2} && \textit{Write } (k+2) \textit{ as } ((k+1)+1).
\end{aligned}
$$

This last line is the closed form formula when $n = k + 1$.

By the principle of mathematical induction, we can conclude that the claim is true for all natural numbers.■