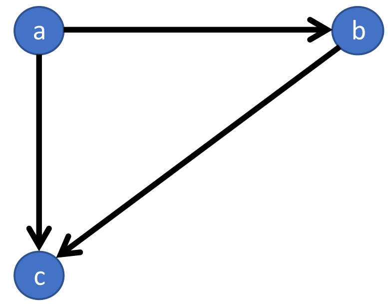
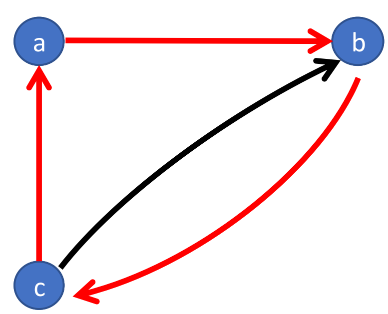
Math 189: Discrete Mathematics

Proof Analysis

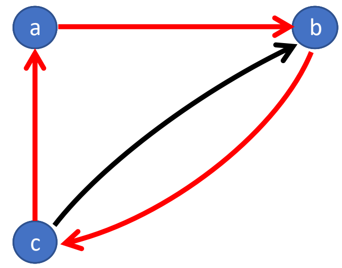
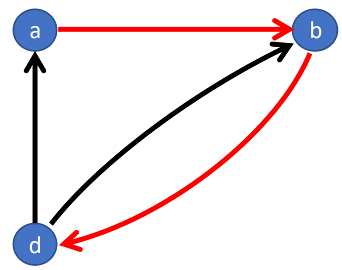
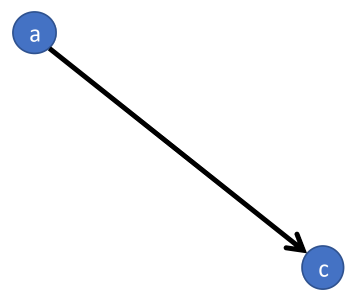
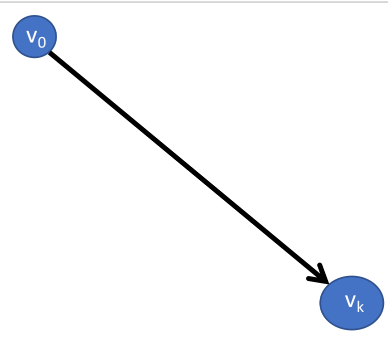
Rob Peterson

Part 1: List of mathematical terms

Mathematical Terms from proof	Definition	Example:
Directed Graph	A directed graph (or digraph, for short) consists of a pair (V, E). V is a set of vertices, and E, a set of directed edges, is a subset of V × V,	
vertex	An element a set, used in a directed graph, and illustrated as dots or points	 a is a vertex
vertices	Plural form of vertex.	 a and b are vertices
cycle	A circuit in which there are no repeated vertices except the first and last vertex.	 The circuit <a,b,c,a> is a cycle

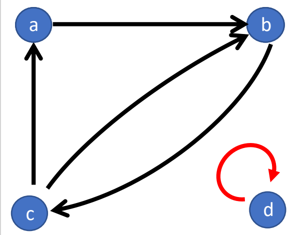
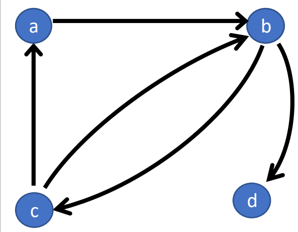
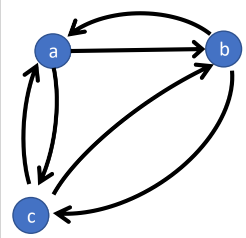
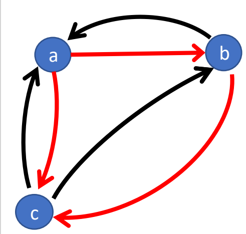
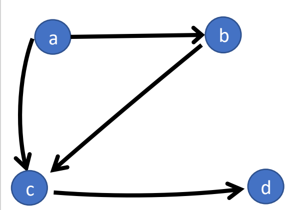
Continued on next page …

Part 1: list of mathematical terms:

Mathematical Terms from proof	Definition	Example:
length	The number of edges in the cycle/circuit/path/walk.	 The circuit <a,b,c,a> has a length of 3
path	A walk in which no vertex is repeated.	 Walk <a,b,c> is a path in this Di-Graph
directed edge	A subset of VxV which is the Cartesian Product of the set of vertices of a graph. These are usually illustrated as arrows. The edge (A,C) would be represented by an arrow whose tail is at vertex A and head is at vertex C.	
edge(v_0,v_k)	An edge with head at labeled v_0 and tail at vertex v_k.	

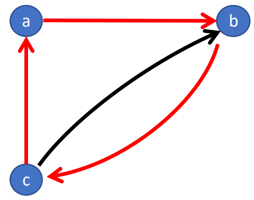
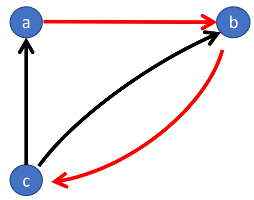
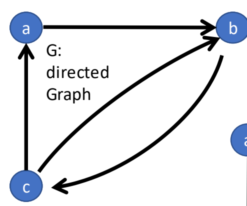
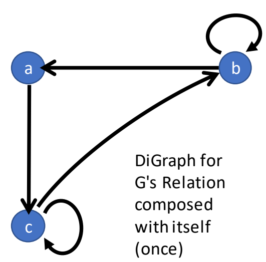
Continued on next page …

Part 1: list of mathematical terms:

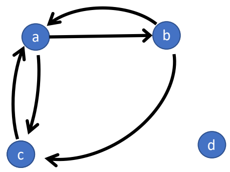
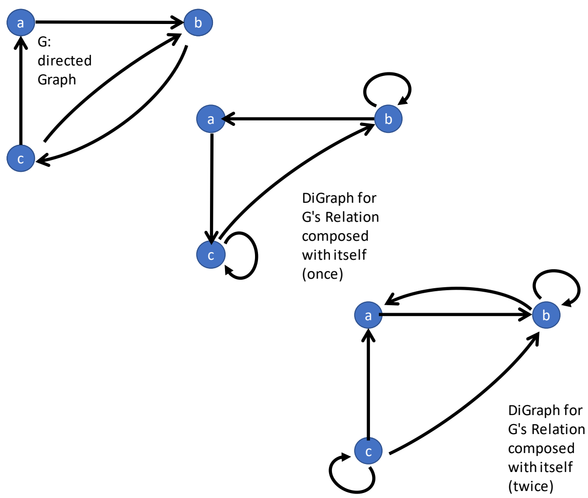
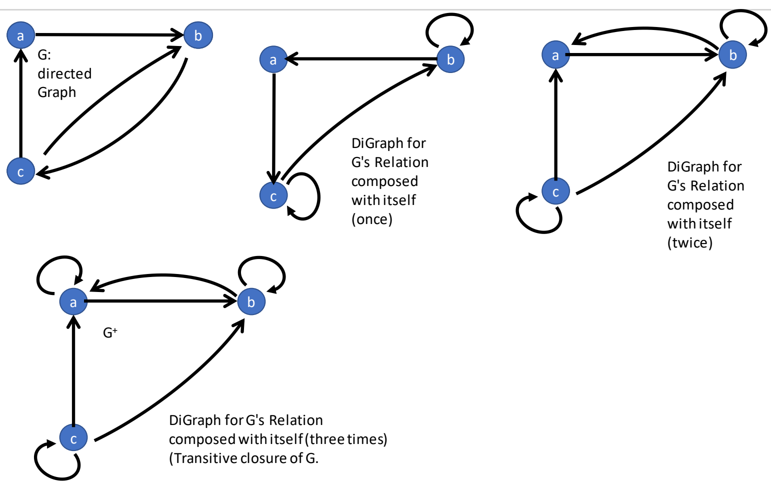
self-loop	An edge that leads from a vertex to itself.	 Vertex d has a self-loop in this Di-Graph
anti-reflexive	The relation R is anti-reflexive if for every x ∈ A, it is not true that xRx	 There are no self-loops in this relation
strict order	A relation R is a strict order if R is transitive and anti-reflexive.	 This is a strict order.
transitive	The relation R is transitive if for every x,y, z ∈ A, xRy and yRz imply that xRz.	 This show transitivity. aRb, bRc, and aRc.
acyclic	A directed Graph is acyclic if it has no positive length cycles.	 There are no cycle lengths > 0.

Continued on next page …

Part 1: List of Mathematical Terms

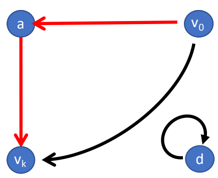
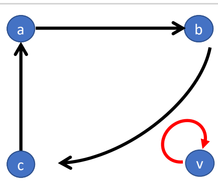
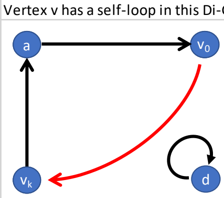
Mathematical Terms from definitions	Definition	Example:
Circuit	A walk in which the fist vertex is the same as the last vetex.	 The circuit <a,b,c,a> is a cycle
Walk	A sequence of alternating vertices and edges that start and end with a vertex.	 <a,b,c> is a walk
Relation composition	The composition of relations R and S on set A is another relation on A, denoted S o R. The pair (a, c) ∈ S o R if and only if there is a b ∈ A such that (a, b) ∈ R and (b, c) ∈ S.	 

Part 2: List of mathematical Symbols

Symbols	Definition	Example:
G	Label for a directed graph	 This is a directed graph.
G^k	Label for the directed graph who's relation has been composed with itself k times.	
G^+	The transitive closure of G. It is the graph that shows the relation that is the smallest relation that is both transitive, and shows all the pairs of G's relation.	

Continued on next page ...

Part 2: List of Mathematical Symbols:

Symbols	Definition	Example:
X	Cartesian Product of two sets. The set of all ordered pairs in which the first entry comes from the first set, and the second entry comes from the second set.	$A = \{a, b, c\}$ $B = \{x, y, z\}$ $A \times B = \begin{matrix} \{(a,x), & (b,x), & (c,x) \\ (a,y), & (b,y), & (c,y) \\ (a,z), & (b,z), & (c,z)\} \end{matrix}$
∈	"element of"	A=$\{a, b, c, x\}$ x∈ A
$\langle v_0, \ldots, v_k \rangle$	Denotes a walk from vertex v_0 to vertex v_k	 A walk from v_0 to v_k
$v_0 = v_k$	Indicates that v_0 and v_k are the same vertex.	N/A
k	length of a path	N/A
(v,v)	An edge from a vertex to itself (a self-loop).	 Vertex v has a self-loop in this Di-Graph
(v_0, v_k)	An edge from vertex v_0 to vertex v_k.	 Vertex v_0 shares an edge with vertex v_k

Part 3: Proof StoryBoard

The following is a storyboard that illustrates the steps of the proof in Zybooks Discrete mathematics text theorem 6.8.1:

<u>If G has a positive length cycle, then there is a path <v_0,...,v_k> in G where $v_0 = v_k$ and $k \geq 1$.</u>

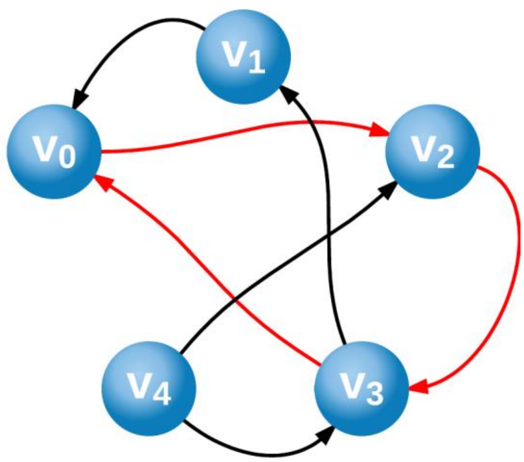


Figure 1: Directed Graph G having positive length cycle <v_0, v_2, v_3, v_0>

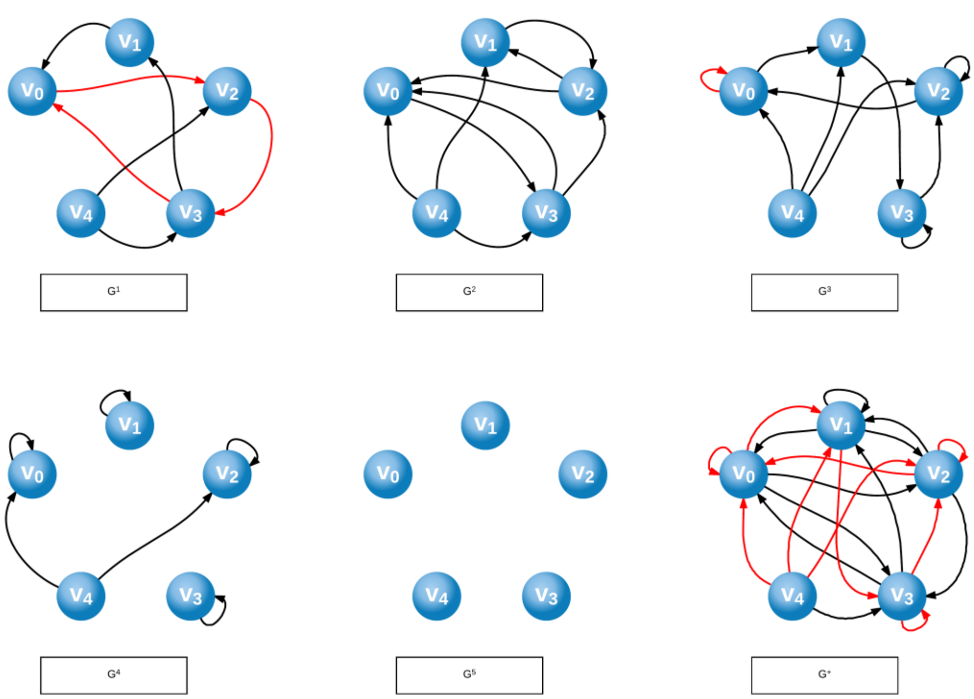


Figure 2: The 5 graph powers and the transitive closure are pictured for reference.

The existence of the path of length k implies that there is an edge (v_0, v_k) in G^k.

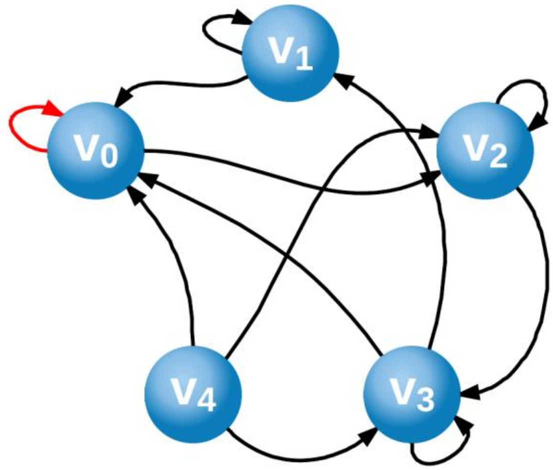


Figure 3:G^3, depicting an edge <v_0,v_0>

All the edges in G^k are included in G^+.

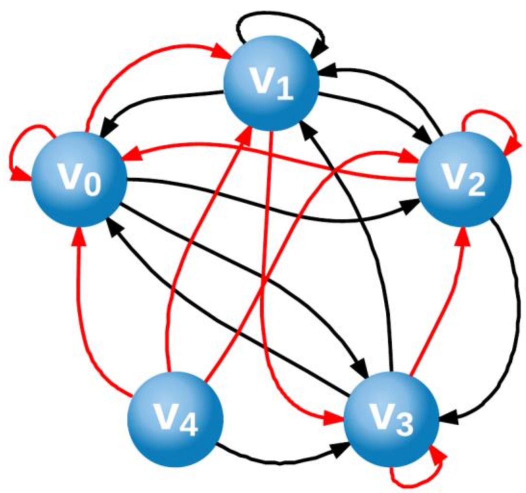


Figure 4: G^+ (Transitive closure of G), with all the edges from G^3 highlighted in red

Since $v_0 = v_k$, the edge is a self-loop, implying that G^+ cannot be anti-reflexive. Therefore G^+ is not a strict order.

Definition of Strict Order:

The relation is transitive and anti-reflexive.

G^+ is by definition transitive, so in order for it not to be a strict order, it must not be anti-reflexive which means that it has a self-loop.

Suppose that G^+ has a self loop at vertex v.

See figure 3, vertex v_0.

The edge (v, v) must be in G^k for some $k \geq 1$.

Figure 2 illustrates G^3 containing a self-loop at V_0.

The existence of the edge (v, v) in G^k means that there is a path of length k in G that begins and ends at vertex v.

Figure 1 shows a path of length 3: $<v_0, v_2, v_3, v_0>$

Thus, there is a cycle of length k in G and G is not acyclic.

Part 4: Additional Questions

The first paragraph is a proof of the statement "If G^+ is a strict order, then G has no positive length cycles." What type of proof structure is used in the first paragraph of the proof?

The first paragraph of the proof negates the conclusion (*G has no positive length cycles*), and shows that the hypothesis cannot be true as a result. This is a proof by contrapositive.

What is the value of k in your illustration of the first paragraph?

The value of *k* in my illustration of the first paragraph is 3.

Why does a self-loop imply that G^+ cannot be anti-reflexive?

The definition if anti-reflexive is give in Zybook as:

"… for every x ∈ A, it is not true that xRx." If the graph G of the relation R contains a self-loopat vertex x, then xRx. Therefore the existence of self-loops dictates that the relation R cannot be anti-reflexive.

What justifies the author's statement that "Therefore $G^+$is not a strict order"?

A strict order is transitive and anti-reflexive. The author has shown that the transitive closure under discussion is not anti-reflexive, therefore it cannot be a strict order.