Induction Proof

MATH 189: Discrete Mathematics

Section 2

Rob Peterson

# Problem Description

The problem under consideration for this proof consists of a series of steps, described as follows:

1. A stack of 7 disks sits on a surface.
2. One disk is added to the top of the existing stack(s), and a set of disks is added to form a perimeter around the existing stack(s).
3. Step 2 is repeated ad Infinium.

Diagrams showing the steps involved are shown below:

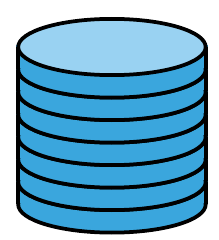


Figure 1: Stage 1

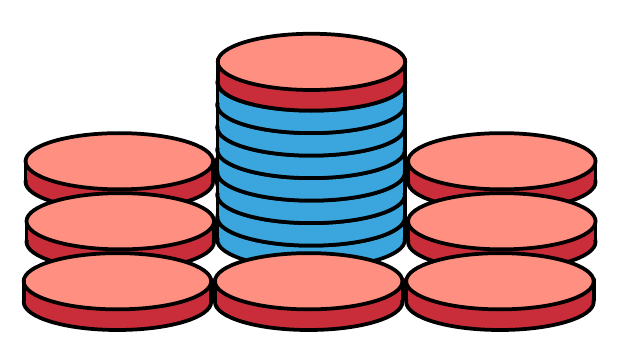


Figure 2: Stage 2

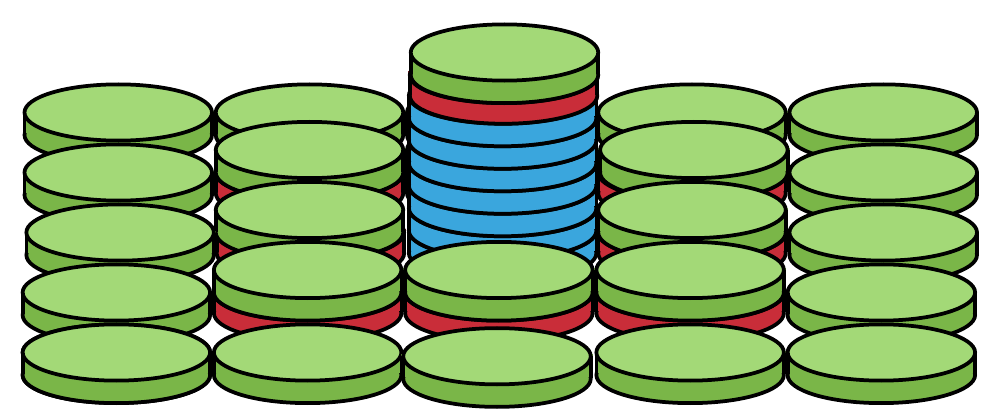


Figure 3: Stage 3

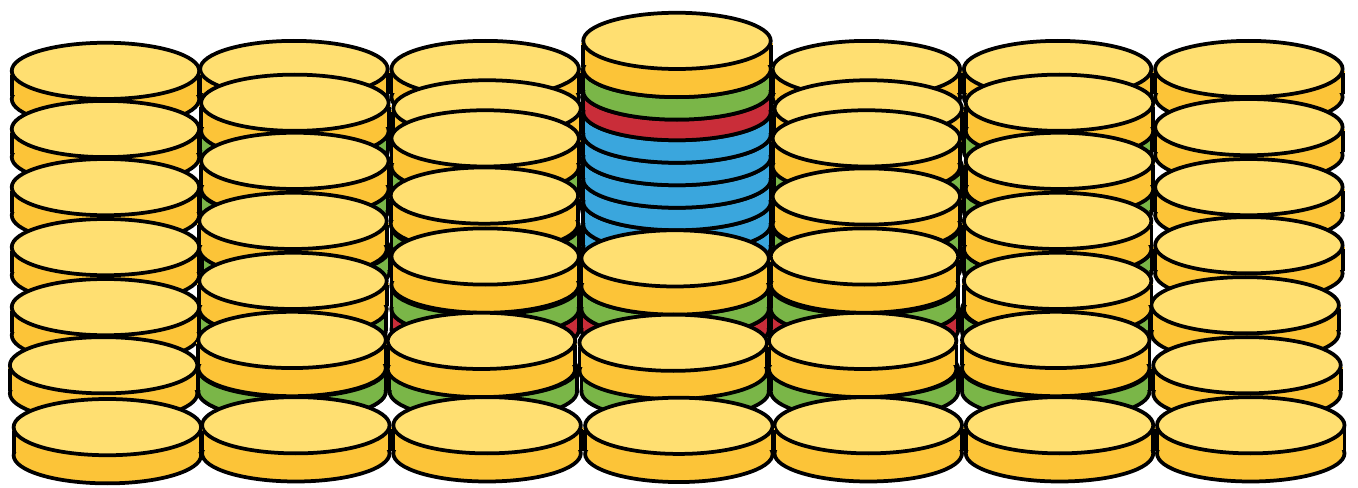


Figure 4: Stage 4

The problem statement regarding these series of stages is to determine a closed for expression for determining the total number of disks depending on what stage the process is in. A good practice in developing a model like this is to develop some data. The following table has been generated from inspection of the problem, and covers the first six stages. It shows the “dimension” of a stage, which denotes the length of a side of the stage, the stages form squares, where is the dimension of a stage. The number of stacks in a stage is also important information to track, as will be shown in the following portions of this proof.

|  |  |  |  |
| --- | --- | --- | --- |
| ***Stage (n)*** | ***Dimension*** | ***# stacks*** | ***# disks*** |
| 1 | 1 | 1 | 7 |
| 2 | 3 | 9 | 16 |
| 3 | 5 | 25 | 41 |
| 4 | 7 | 49 | 90 |
| 5 | 9 | 81 | 171 |
| 6 | 11 | 121 | 292 |

Table 1: First six stages of problem

If the reader examines the previous figures showing the disks, the dimension of the square (seen from the top down) can be derived from the current step number:

[LaTeX 1: Expression showing the summation of different parts of the Dn definition]

The following portion of this document will derive expressions for the above expressions.

Part of this expression depends on defining the number of stacks in the previous step based on the current step number. The dimension of a square for the present step is simply an increase of the dimension of the previous step by two. It is also related to the current step number according to the following equation:

[LaTeX 2: Expression showing the dimension derivation [2n-3]

The top and bottom row lengths which will be added to this problem are an increase by two of the top and bottom row lengths from the previous step. It can also be derived from the current step number as shown:

[LaTeX 3: Expression showing the top/bottom length expression based on the current step number (n)

2(2(n)-1)]

The lengths of the sides that are to be added with each new step are an increment from the lengths of the sides from the previous step. The following expression shows how the length of the left/right sides depends on the current step number:

[LaTeX 4: Expression showing the right/left length expression based on the current step number (n)

2(2(n)-3)]

Using these expressions, we can form a recursive formula to define the number of disks in a current step:

[LaTeX 5: Equation showing the recursive formula (D(n)= D(n-1)+(2n-3)2+2(2n-1)+2(2n-3)) (for n>1) ]

To validate the closed form formula for this problem, we use n=3 and n=4, plugging those values (step numbers) into the recursive formula and verifying that the recursive formula produces the correct output value.

[LaTeX 6: D(3)= D3-1+(2(3)-3)2+2(2(3)-1)+2(2(3)-3)) = 41 ]

[LaTeX 7: D(4)= D4-1+(2(4)-3)2+2(2(4)-1)+2(2(4)-3)) = 90]

Next, we need to derive a closed form formula for the number of disks based on the current step number. A good method for deriving formulas of this kind is to perform an analysis of the input/output relationships and how they change from step to step.

We can take the difference of step from step . If the differences are consistent from step n to step n+1, and step n+2, …, then the closed form expression is a linear combination of degree 1. If the differences are not consistent, then we can take the difference of the previous differences, and see if those are consistent from n,n+1,n+2, and so on. This process is repeated until we have a consistent difference from n to n+1. This initial process is shown below using real outputs from this problem statement:

[LaTeX 8: Show this:

1| 7

2| 16 9

3| 41 25 16

4| 90 49 24 8

5| 171 81 32 8

6| 292 121 40 8

7| 461 169 48 8

]

As shown above, the process produces consistent outputs after the 4th column. This means that the closed form expression we are looking for is a 3rd order polynomial of the following form:

[LaTeX 9: An3+Bn2+Cn+D=# disks in current step]

The next step is to use some of the known outputs to produce a system of equations based on the above equation. Using data from the first four steps shown in table 1, we have the following system of equations:

[LaTeX 10: System of equations for the known table outputs]

Using matrix format makes these equations easier to solve.

The following steps solve these equations:

[LaTeX 11: Solving the system of equations]

The closed form expression for the number of disks based on a particular step is shown:

[LaTeX 12: D(n)=(4/3)n3+0n2-(1/3)n-6]

To show that the closed formula expression is valid for a particular n, choose n=3 and n=4, since we already have data for those steps:

[LaTex 13: D(3)=(1/3)(3)3-(1/3)(3)+6=90]

[LaTex 14: D(4)=(1/3)(4)3-(1/3)(4)+6=171]

To prove that equation 12 is a correct predictor for the number of disks present in a particular step in this problem we will proceed with an induction proof.

The base step is to prove that equation 12 is valid for the base step. In the base step n=1, and there is a single stack of 7 disks. Using equation 12, we see that the closed form expression is validated in the base case:

[LaTeX 15: D(1)=(1/3)(1)3-(1/3)(1)+6=7 (check mark)]

Next we assume that equation 12 has been shown to be true for some n=k, where k>=1. According to equation 12, the number of disks in this step would be D(k)=(4/3)k3-(1/3)k+6.

Based on the recursive formula shown in equation 5, we see that the number of disks in the k+1 step is expressed as follows:

[LaTeX 16: D(k+1)= D(k)+(2(k+1)-1)2+2(2(k+1)-1)+2(2(k+1)-3)) ]

We have an expression for D(k), plugging it in to equation 16:

[LaTeX 17: D(k+1)= (4/3)k3-(1/3)k+6+(2(k+1)-1)2+2(2(k+1)-1)+2(2(k+1)-3)) ]

((4)k3-(1)k+18+3\*(2(k+1)-1)2+3\*2(2(k+1)-1)+3\*2(2(k+1)-3)))/3

Induction proof to follow …