# Supersymmetry (SUSY)

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**Class:** Wednesday, Dec 1, Scribe: Yueyang Shen

- Superstring
- Worldsheet supersymmetry
- String Landscape

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We start out with the (gauge fixed for simplicity) Polyakov action

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu \tag{1}$$

1. The $X^\mu$ are worldsheet boson which are background scalars from 2 dimensional point of view.
2. The $\mu$'s carries the 26 dimensional (bosonic) or the spacetime vector index but transform in this 2 dimensional world sheet.

We now think about in supersymmetry (SUSY) in 2 dimensions. In some level, we match (associate) the bosons with fermions.

$$\underbrace{X^\mu}_{\text{boson}} \leftrightarrow \underbrace{\psi^\mu}_{\substack{\text{holomorphic} \\ \text{fermions}}} \quad \underbrace{\tilde{\psi}^\mu}_{\substack{\text{antiholomorphic} \\ \text{fermions}}} \tag{2}$$

1. The worldsheet 2 dimensional fermions are integral of the dirac spinor and weyl spinor etc.
2. In 2D field theory, we can think of starting with 2 dimensional gamma (Pauli-Dirac) matrices, they can be represented as pauli matrices.
3. I have two component spinors, they are arranged in the way that the 2D dirac equation is splitted into holomorphic and antiholomorphic fermions, and by fermions we mean they transform as Grassmans(anticommutator).
4. Introduction of fermionic component reduces the critical dimension from 26 to 10.

The 2 dimensional supersymmetry action is then

$$S = \frac{1}{4\pi}\int d^2z[\frac{2}{\alpha'}\partial X^\mu \overline{\partial} X_\mu + \psi^\mu \overline{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu] \tag{3}$$

- The fermions (spin 1) and bosons transform as vectors(there are 2D fermions but not necessary 10D)

We start by checking the bosonic equation of motion by varying the action.

**EOM:**

$$\partial\overline{\partial}X^\mu = 0 \Longrightarrow X^\mu = X_L^\mu(z) + X_R^\mu(\overline{z}) \tag{4a}$$

$$\overline{\partial}\psi^\mu = 0 \Longrightarrow \psi^\mu = \psi^\mu(z) \tag{4b}$$

$$\partial\widetilde{\psi}^\mu = 0 \Longrightarrow \widetilde{\psi}^\mu = \widetilde{\psi}^\mu(\overline{z}) \tag{4c}$$

**Worldsheet Supersymmetry (Transform like gauge transformation with parameters):**

- Transform bosons into fermions ($\eta$ = Grassman SUSY transformation parameter)

$$\partial(\sqrt{\frac{2}{\alpha'}}X^\mu) = \underbrace{\eta\psi^\mu + \eta^*\widetilde{\psi}^\mu}_{\text{Grassman field}} \tag{5}$$

- Transform fermions back to bosons

$$\delta\psi^\mu = -\eta\partial(\sqrt{\frac{2}{\alpha'}}X^\mu) \tag{6a}$$

$$\delta\widetilde{\psi}^\mu = -\eta^*\partial(\sqrt{\frac{2}{\alpha'}}X^\mu) \tag{6b}$$

**Sanity Check** ($\alpha' = 2$)

$$\begin{aligned}\delta L = {} & \partial X^\mu\overline{\partial}(\eta\psi_\mu + \eta^*\widetilde{\psi_\mu}) + \partial(\eta\psi^\mu + \eta^*\widetilde{\psi^\mu})\overline{\partial}X_\mu + \psi^\mu\overline{\partial}(-\eta\partial X^\mu) + (-\eta\partial X^\mu)\overline{\partial}\psi^\mu \\ & + \widetilde{\psi}^\mu\partial(-\eta^*\overline{\partial}X^\mu) + (-\eta^*\overline{\partial}X^\mu)\overline{\partial}\widetilde{\psi}_\mu = \eta\partial(\psi^\mu\overline{\partial}X_\mu) + \eta^*\overline{\partial}(\widetilde{\psi}^\mu\partial X_\mu)\end{aligned} \tag{7}$$

The Lagrangian being supersymmetric means that the action is invariant under supersymmetry(at least it transforms under a total derivative of the action). For closed string this is easy, there is no boundaries.

We can make the supersymmetric parameters local in case I have local supersymmetry(it is vaguely super gravity- it has a 2-dimensional worldsheet gravitino, and if I want to quantize the theory I can still get into some nice gauge where the gravitino becomes very trivial, but when I gauge fix the local supersymmetry transformations, I introduce additional superconformal ghosts), so quantization becomes really complicated.

We will ignore the ghost sectors and just do matter fields. We promote the normal conformal algebra to superconformal algebra. As a theory of local diffeomorphism and local gravity, you have the stress energy tensor and from the 2D CFT point of view, the stress energy tensor plays a key role in CFT.

The energy momentum tensor being

$$T_{B(osonic)} = -\frac{1}{\alpha'}\partial X^{\mu}\partial X_{\mu} - \frac{1}{2}\psi^{\mu}\partial\psi_{\mu} \quad \text{Energy Momentum Tensor of Weight 2} \tag{8a}$$

$$T_F = i\sqrt{\frac{2}{\alpha'}}\psi^{\mu}\partial X_{\mu} \quad \text{Supercurrent of weight 3/2} \tag{8b}$$

This is taking the supersymmetric Polyakov action and use a Noether's procedure and obtain a conserved energy momentum tensor and a conserved supercurrent. By looking at the algebras (the generic OPE of two stress tensors)

$$T_B(z)T_B(0) \sim \frac{c/2}{z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \tag{9a}$$

$$T_B(0) \sim \underbrace{\frac{3/2}{z^2}T_F(0)}_{chiral \quad parameter} + \frac{1}{z}\partial T_F(0) \tag{9b}$$

$$T_F(z)T_F(0) \sim \frac{2c}{3z^3} + \frac{2}{z}T_B(0) \tag{9c}$$

For the bosonic string and polyakov action, I have c=D. Here for each dimension, I have 1 boson and 1 fermion and the central charge,

$$c = (\underbrace{1}_{boson} + \overbrace{\frac{1}{2}}^{fermion})\underbrace{D}_{\text{spacetime dimension}} = \frac{3}{2}D \tag{9d}$$

"$\mathcal{N} = 1$ supersymmetry (one supercurrent associate with the holomorphic factor)" are $T_F$ supercurrent

left & right mover $\longrightarrow \mathcal{N} = (1,1)$ SUSY(can be referred as 2D CFTs or theories in 6D)

For the mode expansions,

We start with a closed string,

$$X^{\mu}(\sigma + 2\pi) = X^{\mu}(\sigma) \quad \psi^{\mu}(\sigma + 2\pi) = -\psi^{\mu}(\sigma) \tag{10}$$

There are two possibilities,

$$\text{Neumann-Schwarz Sector} \quad \psi^{\mu}(\sigma + 2\pi) = -\psi^{\mu} \tag{11a}$$

$$\text{Ramond Sector} \quad \psi^{\mu}(\sigma + 2\pi) = \psi^{\mu}(\sigma) \tag{11b}$$

Consider the mode expansions,

$$X^{\mu}(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n}\frac{\alpha_n^{\mu}}{z^{n+1}} \quad n \in \mathcal{Z} \tag{12}$$

The +1 in the denominator comes from the conformal dimension. Splitting cylinder into plane uses $Z = e^{-iw} = e^{-i\sigma}e^{\tau}$

The general solution that accounts for both sectors is the following:

$$\psi^\mu(z) = \sum_r \frac{\psi_r^\mu}{z^{r+1/2}} \tag{13}$$

- $r \in \mathcal{Z} + \frac{1}{2}$ Is NS and half integer quantized
- $r \in \mathcal{Z}$ Is R sector and integer quantized; becomes weird because of branchcuts.

This gives (super)Virasoro algebra

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \quad \{\psi_r^\mu, \psi_s^\nu\} = \delta_{r+s,0}\eta^{\mu\nu} \tag{14}$$

$$T_B(z) = \sum_n \frac{L_n}{z^{n+2}} \quad T_F(z) = \sum_r \frac{G_r}{z^{r+3/2}} \tag{15}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{16}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r} \tag{17}$$

$$[G_r, G_s] = 2L_{r+2} + \frac{c}{12}(4r^2-1)\delta_{r+s,0} \tag{18}$$

We want a complete theory with no anomaly. In these theories, the critical dimension is

$$c_{tot} = c(ghosts) + c(matter) = 0 \tag{19}$$

Bosonic string

$$c_{tot} = -\underbrace{2G}_{\text{bc ghost}} + \overbrace{1}^{\text{1 boson per dimension}} \cdot \underbrace{D}_{\text{matter}} \Longrightarrow D = 26 \tag{20}$$

Superstring

$$c_{tot} = \underbrace{-26}_{\text{bc ghosts}} + \underbrace{11}_{\beta\gamma\text{ghosts}} + (\underbrace{1}_{\text{X boson}} + \underbrace{\frac{1}{2}}_{\text{fermion}})D \Longrightarrow 3/2D = 15; D = 10 \tag{21}$$

- bc ghost: (Diff/ Weyl) anticommuting boson
- $\beta\gamma$ ghost: superconformal ghost; local worldsheet supercurrent/supersymmetry; commuting fermions; fixating fermionic supercurrent

There is a bosonic zero mode:

1. Bosonic zero mode (corresponds to momentum operator that generates translations of the string worldsheet; natural in the Minkowski spacetime assumption) :

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu \quad \text{open string} \tag{22}$$

2. Fermions (either (half) /integer quantized) in the Ramond sector has a zero mode $\psi_0^\mu$

   - Fermions also have creation(negative modes) and annihilation(positive modes) operators(we can split into positive and negative modes). We stuck at $\psi_0$

$$\text{Grassman/fermion “square” should not equal to 1}: \quad \{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \tag{23}$$

This looks like a Dirac Matrix(Clifford) algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{24a}$$

$$\text{More common convention in D dimension context: } \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{24b}$$

$$\psi_0^\mu \sim \frac{1}{\sqrt{2}}\Gamma^\mu \tag{24c}$$

- Gamma matrix is linear operator and should act on spinors $\longrightarrow$ The Radmond ground state should transform as spinors

Ramond ground state should transform as a 10 dimensional (Clifford algebra) spinor

$$NS: \text{Superconformal ghost issue} \quad |0;k\rangle \quad R: \text{10 dimensional spinor} \quad |R;k\rangle \tag{25}$$

Transform as spacetime fermion (although we don't transform it that way).
We start with 2 dimensional worldsheet supersymmetry; 2 dimensional fermions.
Everything 10 dimensional in the Polyakov action is basically a vector, but when we quantize we realize that the radmond vaccum should be designed to transform as spacetime spinor (how we get spacetime fermions/ but we don't start that way)

Dirac Equation (Spacetime fermions) The dirac equation gives massless representation for higher spin states. (For a bosonic string we have the mass shell equation EOM $\Box + m^2$=0; for old canonical quantization, we look at the oscillator excitation numbers-modes of the stress tensor)

$$L_n|\text{state}\rangle = 0; \quad (L_0 - \underbrace{A}_{\substack{\text{normal ordering constant}\\ \text{bosonic/fermionic}}})|\text{state}\rangle = 0; \quad G_r|\text{state}\rangle = 0 \tag{26}$$

The last equation of $G_r$ specifies the modes of the supercurrent. ($r \geq 0$). Note for the ramond sector we also have $G_0|\text{state}\rangle = 0$ - the zero mode of the super current(only shows up the radmond sector).

$$T_F \sim \psi^\mu \partial X_\mu \tag{27a}$$

$$G_0 \sim \underbrace{\psi_0^\mu \alpha_{0\mu}}_{(22)\longrightarrow p_\mu \psi_0^\mu} + \underbrace{\psi_1^\mu \alpha_{-1\mu} + \psi_{-1}^\mu \alpha_{1\mu} + ...}_{\text{cancel due to } L_n \text{ condition; sufficient number of lowering operators}} \tag{27b}$$

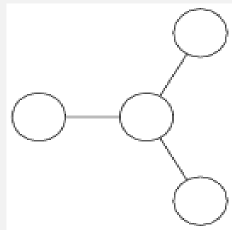
$$\underbrace{p_\mu \Gamma^\mu |Ramond\rangle}_{\text{Dirac equation in momentum space}} = 0 \tag{27c}$$

What we find in Ramond sectors are spacetime fermions(and they satisfy dirac (like-for higher spin states) equation). The "physical" Ramond ground state for the spacetime lorentz group SO(1,9) has a 32 component spinor $|R;k\rangle$(The first order relation relates the top half to the bottom half of the spinor,Due to dirac equation removes half of these components, so 16 physical states survive the OCQ quantization-to look at this we look at the little group).

Massless representation of Poincare can be classified under the little group symmetry(take the massless condition and move in 1 or 9 dimension 9-1=8). The little group SO(8) (transverse) representation

- Spinors: $8_s$ $|R+,k\rangle$ left-handed
- Conjugate spinors: $8_c$ $|R-,k\rangle$ right-handed

Small aside: SO(8) D4 algebra



There is a triality between these three nodes for the Dynkin diagram for D4. The irreducible representations are:

- Comes in triples: $8_v$(vector, singlet; component vector);$8_s$; $8_c$
  Rmk: $8_v$ means if I have massless photon in 10D, I have 8 transverse direction(little group SO(8)).
- Antisymmetric $B_{\mu\nu}$ 28 dimensional
- $56_v$,$56_s$,$56_c$

Now consider the open string spectrum (1/2 superstring spectrum; instead of doing left and right movers at the same time). Fermions can have integer quantized modes (Ramond sector) and half integer quantized modes (NS sector).
NS Sector (easy): modes $\alpha_n^\mu, \psi_r^\mu$; "creation" operators $\alpha_{-n}^\mu, \psi_{-r}^\mu$; Modes numbers: n, r(number of modes excitations; see from $L_0$)

| State | $\alpha' m^2$ | $(-1)^F$ |
|---|---|---|
| Tachyon: $\|0;k\rangle$ | $-\frac{1}{2}$ | - (ghost issue; superconformal ghosts) |
| (Next level half integer fermions): $8_v$ $\psi^\mu_{-\frac{1}{2}}\|0;k\rangle$ | 0 (massless vector) | $+$ |
| $\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}$ or $\alpha^\mu_{-1}\|0;k\rangle$ | $\frac{1}{2}$ | $-$ |
| $\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}\psi^\rho_{-\frac{1}{2}}$ or $\psi^\mu_{-\frac{3}{2}}\alpha^\nu_{-1}$ or $\alpha^\mu_{-\frac{3}{2}}\|0;k\rangle$ | 1 | $+$ |

The fermion number operator $F$ is introduced to quantify the split more systematically (any single application of half integer fermion will flip between the integer pattern to half integer pattern).

$$F = \text{worldsheet fermion operator} = \#\text{ of }\psi^\mu\text{ excitations} \tag{28}$$

- There are no fermions in Tachyons, so it should be zero but
- The states with an odd (even) number of world-sheet fermions are even (odd) under the action of $(-1)^F$

- GSO projection: In 1976, Gliozzi-Scherk-Olive proposed to truncate the spectrum (tachyons are eliminated) by insisting $(-1)^F = 1$

Integer Quantized Ramond Sector: $\alpha^\mu_{-n}\psi^\mu_{-r(=n;integer)}\underbrace{(\psi^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu)}_{\text{hiding}}$

$$\text{Ramond} \Longrightarrow \underbrace{R_+ \quad R_-}_{\text{Ramond ground state}} \tag{29}$$

| State | $\alpha' m^2$ | $(-1)^F$ |
|---|---|---|
| $8_s$ $\|R_+; k\rangle$ | 0 | + |
| $8_c$ $\|R_-; k\rangle$ | 0 | − |
| $\alpha^\mu_{-1}\|R_+; k\rangle$ (bosons won't change) | 1 | + |
| $\alpha^\mu_{-1}\|R_-; k\rangle$ (bosons won't change) | 1 | − |
| $\psi^\mu_{-1}\|R_+; k\rangle$ (fermions flips) | 1 | − |
| $\psi^\mu_{-1}\|R_-; k\rangle$ (fermions flip) | 1 | + |

- If I think about spacetime supersymmetry, I have 8 bosonic components, and 8 fermionic on shell components. In Ramond we have 16 massless individual components. ▶ Single gamma matrix flips the left/right chiralities. Every action of $\psi_0$ which essentially is the gamma matrix flips chiralities, which is the number of fermions. So I declare some highest weight representation of the clifford algebra. The high dimensional spinors are a bit tricky(For example, spin ↑↑↑ .. should be given a fermion number). Each action of the clifford algebra moves around between the positive and negative chirality spinors of these fermion numbers.

- GSO Projection choice (we have too many fermions in the massless states $16 > 8$ to be supersymmetric to the bosons; either on $8_s$ or $8_c$):
▶ Not so obvious. Two choices. 1) $(-1)^F$=1; using left hand spinors $8_s$. 2)$(-1)^F = -1$ using right hand spinors. If I can flip between these spinors, these are related by spacetime parity. This does not give physically distinct theory

Open String

Massless states

$$\underbrace{8_v}_{A_\mu \text{ gauge bosons (photon)}} + \underbrace{\overbrace{8_s}^{(-1)^F=1 \text{ GSO choice}}}_{X \text{ gaugino}} \tag{30}$$

- On shell supersymmetry is really nice. A bosnic equals a fermionic degrees of freedom.
- Spacetime point of view: $D = 10, \quad N = 1$ supermaxwell; super yang mills (Why super YM come in? Chan Paton fatcors)

Right + Left Movers

$$(NS + R) \otimes (NS + R) = \underbrace{NSNS}_{\text{boson}} + \underbrace{RR}_{\substack{\text{bispinor-bilinear spinor} \\ \text{transform like bosons}}} + \underbrace{NSR + RNS}_{\text{spacetime fermions}} \tag{31}$$

- Chiral Type II-B (Left-Handed+Left-Handed)

$$\underbrace{(8_v + 8_s)}_{\text{left}} \otimes \underbrace{(8_v + 8_s)}_{\text{right}} \tag{32a}$$

$$\begin{aligned} F_{p+1} = dC_p &= 8_v \times 8_v + 8_s \times 8_s + 8_v \times 8_s + 8_s \times 8_v \\ &= \overbrace{\underbrace{1}_{\phi} + \underbrace{28}_{B_{\mu\nu}} + \underbrace{35_v}_{g_{\mu\nu}}}^{\text{rank two tensor } 8_v \times 8_v} + \overbrace{\underbrace{1}_{c_0} + \underbrace{28}_{c_2} + \underbrace{35_s}_{c^{(+)}_{-1}}}^{\text{even RR potential } 8_s \times 8_s : F_1 F_3 F_5^\dagger} + \underbrace{8_c + 8_c}_{\text{dilatino}} + \underbrace{56_c + 56_c}_{\text{gravitino}} \end{aligned} \tag{32b}$$

- Non Chiral Type II-A ▶ Flipping left/right movers still give the same $(8_v + 8_s) \otimes (8_v + 8_c)$. Thus, nonchiral.

$$(8_v + 8_s) \otimes (8_v + 8_c) \tag{33a}$$

$$\begin{aligned} F_{p+1} = dC_p &= 8_v \times 8_v + 8_s \times 8_c + 8_v \times 8_c + 8_s \times 8_v \\ &= \overbrace{\underbrace{1}_{\phi} + \underbrace{28}_{B_{\mu\nu}} + \underbrace{35_v}_{g_{\mu\nu}}}^{\text{rank two tensor } 8_v \times 8_v} + \overbrace{\underbrace{8}_{c_1} + \underbrace{56}_{c_3}}^{\text{odd RR potential } 8_s \times 8_c : F_2 F_4} + \underbrace{8_s + 8_c}_{\text{dilatino}} + \underbrace{56_s + 56_c}_{\text{gravitino}} \end{aligned} \tag{33b}$$