VV285 Recitation Class Week 1

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Some general comments about Part I

This part mainly revolves around the basics of linear algebra discussing the following topics

- Systems Of Linear Equations
- Finite-Dimensional Vector Spaces
- Inner Product Spaces
- Matrices and Linear Maps
- Determinants

It is also important to understand and remember basic manipulations in Mathematica for completing term projects and assignments.

Overview

Definitions

- $\mathbb{F}$ encapsulates the mathematical definition of $\mathbb{R}$ and $\mathbb{C}$. Depending on the context, it can either stand for a real vector space or a complex vector space.
- Homogeneity and Inhomogeneity. A linear system is a homogeneous system if the b_is are all zero. Otherwise, it is inhomogeneous.
- An inhomogeneous system of equations may have either
  - a unique solution or
  - no solution or
  - an infinite number of solutions

  $\diamond$ An inhomogeneous system can not have exact two/three solutions. This is essentially the Fredholm Alternative and will discuss in details in Section 6.
- Underdetermined and Overdetermined. # eqautions $<$ # variables, underdetermined. # eqautions $>$ # variables, overdetermined.

Definitions

- A solution set for a system of equation is the set of n-tuples that solves the linear system of equations. Correspondingly, it either contains:
  - A single point.
  - nothing.
  - An infinite number of elements.
- m equations with n unknowns will have a unique solution if it is diagonizable. i.e, can be transformed into a diagonal form.

Definitions

- Elementary row manipulations.
  - ▶ Swapping(interchanging) two rows
  - ▶ Multiplying each element in a row with a number
  - ▶ Adding a multiple of one row to another row.

  ◇ These manipulations are of great importance in matrix manipulations, say finding the inverse. The details will become clearer.
- Echelon Form.

  ▶The leading entry of each row is 1.

  ▶Each leading entry is in a column that is to the right of the leading entry in the previous row.

  ▶Rows with all zero elements, if any, are below rows having a non-zero element.

  ◇ One of the unknowns acts as a parameter. See for 1.1.6 as an example.

Gauss - Jordan Algorithm: A general recipe

In general, we wish to solve the following

$$\begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array}$$

$$\mathit{Step}1 \quad \begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array} \quad \begin{array}{lll} \cdot(-A) & \cdot(-B) & \\ + & & \cdot(-C) \\ & + & + \end{array} \sim \begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array}$$

$$\mathit{Step}2 \quad \begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \quad \begin{array}{lll} + & + & \\ \cdot(-D) & & + \\ & \cdot(-E) & \cdot(-F) \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array}$$

Fundamental Lemma for homogeneous equation

Lemma.1.18 The homogeneous system

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & 0 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & 0 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & 0
\end{array}$$

has a non trivial solution if n$>$m.

- Double induction.
- Fix m and apply mathematical induction on n.
- If there are more unknowns than equations there are always infinite number of solutions.

Definitions

- Linear Independence. Let V be real or complex vector space. The vectors $v_1, v_2, \ldots, v_n \in$ V, are said to be independent if **for all** λ_1, $\lambda_2, \ldots, \lambda_n \in \mathbb{F}$"only if'

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$$

$\diamond$ When k=2, this is equivalent to discussing if u and v are multiples of each other. If they are not, they are linearly independent.

$$\neg\exists A \cap B \Longleftrightarrow \forall\neg A \cap B \Longleftrightarrow B \Rightarrow \neg A$$

- Independent Set. Given the elements in a finite set is independent M, M is called an independent set.

Definitions

- Linear combination.

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + ... + \lambda_n v_n$$

- Linear Hull/Linear Span. The set that contains all possible linear combination.

$$span\{v_1, v_2, ..., v_n\} = \{y \in V : y = \sum_{k=1}^{n} \lambda_k v_k, \lambda_1, \lambda_2 ..., \lambda_n \in \mathbb{F}\}$$

Definitions

- Span of Subsets.
  If V is a vector space and M is some subset of V, the span of M which is define as the set of all **finite linear combinations** of elements of M.

$$spanM = \{v \in V : \exists_{n\in\mathbb{N}}\exists_{\lambda_1,\ldots,\lambda_n\in\mathbb{F}}, \exists_{m_1,\ldots m_n\in M} : v = \sum_{i=1}^{n} \lambda_i m_i\}$$

$\diamond$ $\exists_{n\in\mathbb{N}}$:finite $\exists_{\lambda_1,\ldots,\lambda_n\in\mathbb{F}}$: linear. $\exists_{m_1,\ldots,m_n} \in M$: combinations.
$\diamond$ M is just a subset in general, but the definition guarantees that spanM is a subspace of V.
$\diamond$ The importance of "finite". (1.2.7)

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

which is simply not in the space of all polynomials in C($\mathbb{R}$).

Definitions

- Basis.
  An n-tuple β=(b_1,...,b_n)$\in$ V_n is a basis if and only if
  i) Uniqueness Statement: the vectors b_1,...b_n are linearly independent.
  ii) Existence statement: V=spanβ
  $\diamond$ Should be useful to prove that an n-tuple is actually a basis.
- The sum.

$$U + W := \{v \in V : \exists_{u\in U}\exists_{w\in W} : v = u + w\}$$

$\diamond$ If U$\cap$W $=\{0\}$, the sum U+W is called **direct**. Notation:U$\oplus$W.

Proofs

- To prove independence in general. Consider
  ▶whether the question can degenerate into proving one can not be the multiple of another or
  ▶by showing the coefficient is all zero if the linear combination is zero.
  ▶ If none of the v_is is contained in the span of others.
- To $(b_1,\ldots,b_n)$is actually a basis:
  ▶ The vectors $b_1,\ldots,b_n$ are linearly independent.
  ▶ V=span β. That is, any vectors in V can be rewritten as the linear combination of the basis.
- Induction can also be an important tool.
- It is general of good practice to take a pencil and paper to try to reformulate the proofs for 1.2.6/1.2.10/1.2.13 and check if you can actually fill in the details yourself.

Proofs

- The sum is direct.
  - ▶ By definition.
  - ▶ For all $x \in U+W$,$x \neq 0$, x has a **unique** representation $x=u+w$.
  - ⋄ Note the relationship between direct and uniqueness.
- . Basis extension theorem.
  Let V be finite-dimensional vector space and $A' \subset V$. There exists a basis of V containing A'.
  - ▶.Independent set A can have at most n elements.
  - ▶ Any independent set A with n elements a basis.

1.2.28

$$dim(U + W) + dim(U \cap W) = dimU + dimW.$$

►The trick is to first define a basis for U∩W(dimension:r) and
►then extend(basis extension theorem) the basis to U(dimension:r**+n**) and W(dimension: r**+m**), using the property(exclusiveness) of U∩W.
►Plug back in the equation is essentially

$$LHS = (r + m + n) + r = (r + m) + (r + n) = RHS$$

Notations

- $\langle \cdot, \cdot \rangle$ Notation:

  ▶ This is essentially a map that takes two elements in the vector space, and maps into the space $\mathbb{F}$.

- The $\cdot$ notation example.

  ▶. (Slide 595)

$$U(p) = -G \int_{B^3} \frac{\rho(\cdot)}{dist(p, \cdot)}$$

  ▶ u($\cdot$, t) is a function/u($\cdot$, t)(ϵ) is a scalar.

- $\mapsto$ and $\rightarrow$

  ▶ $\mapsto$ deals with element in spaces. Eg:(r,ϕ,Φ)$\mapsto$(x,y,z)

  ▶ deals with spaces.(Spereical coordinates) Define: f: (0,∞)$\times$[0,2π)$\times$(0,π)$\rightarrow$ $\mathbb{R}^3 \setminus \{0\}$

Definitions

- The inner product $\langle \cdot, \cdot \rangle$ satisfies
  i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle$ =0 if and only of v=0;
  ii) Additivity: $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
  iii)Homogeneity(Non-symmetric,right entry): $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$
  iv)$\langle u, v \rangle = \overline{\langle v, u \rangle}$
  ▶The examples 1.3.3 contains information on how to define inner product on complex vector space as well as **C([a,b])** can be useful for understanding the slides later.
  ▶ The pair (V, $\langle \cdot, \cdot \rangle$) is called an **inner product space.**
- The induced norm.
  We checked in (1.3.7) that it is actually a norm. The first two properties are easily checked. The triangular inequality requires a bit of more work.

Definition

- **Every inner product space is also a normed vector space and by extension a metric space.**
  $\diamond$: In general, a norm is actually a notion that parallels the concept of distance in the real world. We checked in 1.3.7 for the three properties, triangle inequality/ positive length/Multiplying by a scalar, that the map $\|\cdot\|$:V$\to \mathbb{R}$, v$\mapsto \sqrt{\langle v, v\rangle}$. If we have properly defined the norm, in our case we define the induced norm, in inner product space, it is natural that the inner product space is a normed vector space. A straightforward way to define a metric is by using the norm d(x,y)=$\|x-y\|$. Of course, there are also other ways to define a metric and the norm is just one of them. With the above discussion, it is now clear that **inner product space $\subset$ normed vector space $\subset$ metric space.**

Definitions

- Orthogonality.
  - $\langle u, v\rangle$=0 $\iff$ u$\perp$v.
  - Orthogonal complement. The orthogonal complement of a set M

$$M^{\perp} = \{v \in V : \forall_{m \in M} \langle m, v\rangle = 0\}$$

- Othonormal System.
  Let (V,$\langle\cdot,\cdot\rangle$) be an inner product space. A tuple of vectors ($v_1, ..., v_r$)$\subset$V is called a finite orthonormal system if $\|v_k\|$=1 and $v_j \perp v_k$
- Orthonormal Basis. Given that an n-tuple is a basis, if it is a orthonormal system, then we can show that it is a orthonormal basis.

Vector Space: Revisited

Vector Spaces

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+\colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$ (***associativity***),
   - $u + v = v + u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v + e = v$ for all $v \in V$ (***existence of a unit element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = e$;
3. $\cdot\colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $1 \cdot u = u$ for all $u \in V$,
   - $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

Recap: Subspace

- Let $(V, +, \cdot)$ be real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$, then we say that $(U, +, \cdot)$ is a **subspace** of $(V, +, \cdot)$.
- It is generally recommended if you can not remember the details to go through the details PP326-343 again.
- Consider only a complex or real vector space. Let U be a subspace of V. Is $V \setminus U$ necessary a subspace?
- To check a subspace. Generally, we need to check
  - Closed under addition.
  - Closed under multiplication.
  - Existence of unit element.
- Examples

Subspace: Continue

Lemma

Let U and V be subspaces of W (we only consider the field of real numbers). The intersection of U and V ($U \cap V$) is also a subspace.

Subspace: Continue

Lemma

Let U and V be subspaces of W (we only consider the field of real numbers). The intersection of U and V ($U \cap V$) is also a subspace.

Proof.

We first check the existence of unit element. Since both U and V exists the same unit element $\mathbf{0}$ we are finished.
We then check that the addition and multiplication are closed. We assume u,v$\in$ U $\cap$V. We further argue that αu+v is in U, and αu+v is in V. This gives the fact that αu+v $\in$ U$\cap$V. Note that we checked that the addition is closed and that the scalar multiplication is closed simultaneously. □

Subspace: contd

Lemma

The union $U \cup V$ is only a subspace of W if $U \subseteq V$ or $V \subseteq U$

Subspace: contd

Lemma

The union $U \cup V$ is only a subspace of W if $U \subseteq V$ or $V \subseteq U$

Proof.

We negate the latter statement. That is we show that neither U or V is contained in each other leads to contradiction. Suppose u $\in$U$\setminus$V and vinV$\setminus$U(we examine the part that is solely belong to U/V). Then u+v $\in$ U$\cup$V. That is, u+v must lie either within U or V. If say, u+v lies in U then v=(u+v)+(-1)$\cdot$u, which immediately follows a contradiction. The discussion for V follows the same spirit. $\square$

Theorems and Lemmas

- Pythagoras's Theorem $\|z\|^2=\|x\|^2+\|y\|^2$.
  - y lies in the orthogonal complement of M.
- Basis Representation.(Scalar Term: Fourier Coefficient)

$$v = \sum_{j=1}^{n} \langle e_j, v \rangle e_j$$

- Projection Theorem. **Possibly infinite dimensional**.
  Let (V,$\langle \cdot, \cdot \rangle$) inner product space and $(e_1, ...e_r)$r$\in \mathbb{N}$ be an orthonormal system in V. U = span$\{e_1, ...e_r\}$. Then there **exists** a **unique** representation

$$v = u + w \quad where \quad u \in U \quad and \quad w \in U \perp$$

  and $u = \sum_{i=1}^{r} \langle e_i, v \rangle e_i$, w:=v-u
  ▶The projection theorem essentially states that $\pi_U v$ always exists and is **independent** of the choice for the orthonormal system.

Theorems

- Show existence and uniqueness separately.
- The existence of such decomposition is equivalent to showing that w$\in$U$^\perp$ or u$\perp$v-u.
- Noting that

$$\langle v, \sum_{i=1}^{r} \langle e_i, v \rangle e_i \rangle = \sum_{i=1}^{r} \langle e_i, v \rangle \langle v, e_i \rangle$$

- Corollaries of the projection theorem.
  - Infinite Dimensional: V $=$U$\bigoplus$U$^\perp$
  - Finite dimensional:dim V $=$dim U$+$ dim U$^\perp$
- Bessel's Inequality.

$$\sum_{k=1}^{r} |\langle e_k, v \rangle|^2 \leq \|v\|^2$$

Projection: Continued

You will see in all next week's recitation class that

- P(the projection operator) is actually a linear map.
- It has the property that
  - ►$P^2=P$;
  - ► ran$P\perp$ ker$P$
  - ► $P=P^*$
  - ► $\langle x, Py\rangle=\langle Px, y\rangle$

In general, projection is a concept that is of importance for further discussion in VV286 and VE401.

Gram-Schmidt Orthonormaliztion

Formulas to keep in mind

$$w_1 = \frac{v_1}{\|v_1\|}$$

$$w_k = \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\|}$$

- get some practice to see you actually understand the formalism. Try to do the question on the slide/assignment yourself.

Selected Problems

Note, in the following section, unlike the previous ones, the ▶ notation stands for options instead of facts. And they are all taken from the German textbook "Linear Algebra", you are **HIGHLY RECOMMEND** to take a look at them as an exercise or as some practice for your exam.

- How many subspaces does $\mathbb{R}^2$ have?
  - ▶ two:$\{0\}$ and $\mathbb{R}^2$.
  - ▶ four: $\{0\}$ $\mathbb{R}\times 0$, $0\times\mathbb{R}$ and $\mathbb{R}^2$ itself.
  - ▶ infinitely many.
- Which of the following subsets U$\subset$ $\mathbb{R}^n$ is a vector subspace?
  - ▶ U $=\{x \in \mathbb{R}^n | x_1 = x_2 = ... = x_n\}$
  - ▶ U $=\{x \in \mathbb{R}^n | x_1^2 = x_2^2\}$
  - ▶ U $=\{x \in \mathbb{R}^n | x_1 = 1\}$

Selected Problems

- The vector space V=$\{0\}$ consisting of zero
  - has the basis (0)
  - has the basis $\emptyset$
  - has no basis.

  Hint: The dimension of vector space $\{0\}$ is defined to be zero.
- Is the empty set $\emptyset$ a vector space?
  No. Since there is no unit element.
- If one were to define U_1-U_2=$\{x - y | x \in U_1, y \in U_2\}$ for subspaces U_1, U_2 of V, then one would have
  - U_1-U_1= $\{0\}$
  - $(U_1 - U_2)$+U_2 =U_1
  - U_1-U_2= U_1+U_2

Selected Problems

- One always has
  - $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$
  - $U_1 \cap (U_2 + U_3) = (U_1 + U_2) + (U_1 \cap U_3)$
  - $\mathrm{U}_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$

Mistakes committed in the Recitation Class Clarification

- For the last question, A is still the correct answer. Many thanks to the student who pinpointed that IF you set U_1=span$\{(1,1)\}$, U_2=span$\{(0,1)\}$, U_2=span$\{(1,0)\}$ yields
  B. span$\{(1,1)\}$ and $\mathbb{R}^2$
  C.span$\{(1,1)\}$ and $\mathbb{R}^2$
- I also made a mistake in the fourth question. It should be answer C, using the same argument in the previous item you can find that B is actually false.
- I have to admit using the "basis" and use the union to approach this problem is misleading and absolutely false, since the definition for basis was not standard at all, say (1,0) and (2,0) is absolutely different basis, and set manipulation on that does not yield legitimate result. Instead, I feel like that you can treat the "$+$" notation as some sort of expansion of dimension, and hoping with the counter example you can better appreciate the result.

Mistakes committed in the Recitation Class Clarification

- For the third question, I am not sure if I have answer the questions answered in the class clearly, but the reason why that (0) can not be a basis is that it VIOLATES the uniqueness, that is it has infinite number of representation for 0, ie, $0=\lambda 0$, lambda can take infinite values, and thus 0 can not be a basis and thereby we conclude the dimension is zero.

VV285 Recitation Class Week 2

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Some general comments

For this part, it is important to understand how linear map actually works. After all, matrix is just an "encoding" for the property or the structure of the corresponding linear map. Plus, a golden rule to keep in mind "the columns of the matrix are the images of the standard basis vector." as you have probably have seen from 1.4.4.

As promised, the properties of projection map will be discussed and I hope you can appreciate the mechanism of linear maps and feel more comfortable concepts like dual basis/basis transformation/norm.

Comment on the homework

Exercise 2.3

i) Let V be a real or complex vector space. Show that every norm on V, if it is induced by some inner product, satsfies the *parallelogram rule*:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in V$$

(2 Marks)

ii) Recall that $C([a,b])$ denotes the space of continuous functions on the interval $[a,b] \subset \mathbb{R}$ and that

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$$

defines a norm on this space. Does there exist a scalar product that induces this norm, i.e., can we write $\|f\|_\infty = \sqrt{\langle f, f\rangle}$ for some scalar product? If so, find the scalar product, if not, prove this!
(2 Marks)

- i) is straightforward, manipulate the definition for induced norm and inner product space.
- The infinity norm is not a scalar product. Utilize the proof done in i)

Overview

1. Linear Maps
   - Definitions
   - The operator Norm
   - Theorems
2. Matrices: Part I
3. Exercises

Definitions

1.4.1. Definition. Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be vector spaces that are either both real or both complex. Then a map $L\colon U \to V$ is said to be ***linear*** if it is both ***homogeneous***, i.e.,

$$L(\lambda \odot u) = \lambda \boxdot L(u) \tag{1.4.1a}$$

and ***additive***, i.e.,

$$L(u \oplus u') = L(u) \boxplus L(u'), \tag{1.4.1b}$$

for all $u, u' \in U$ and $\lambda \in \mathbb{F}$. The set of all linear maps $L\colon U \to V$ is denoted by $\mathscr{L}(U, V)$.

- The linear map acting on the 0 element should return the 0 element as well.
- Check if you understand the examples.
- If $\mathbb{C}$ is regarded as a real vector space, the map $z \mapsto \overline{z}$ is linear $\mathbb{C} \to \mathbb{C}$. It is not linear if $\mathbb{C}$ is regarded as a complex vector space.

Definitions

- $$L(\lambda(1+i)) = L(\lambda + \lambda i) = (1+i)\cdot(1-i) = 1-(-1) = 2$$
  $$L((1+i)^2) = L(2i) = -2i$$
  and those two equations yield different result.
- Structure Preserving Map. We endowed multiplication in two vector spaces and check that L is well defined.

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\lambda\odot"'] & V \arrow[d, "\lambda\boxdot"] \\ U & V \arrow[l, "L^{-1}"'] \end{tikzcd}$$

- Linear maps parallels the linear structure of vector space.

Definitions

- The set $\mathcal{L}$(U,V) is again a vector space when endowed with pointwise addition and scalar multiplication.
  $\diamond$Quick Check(Vector Space)–How to add or multiply linear maps.
  - (L+$\tilde{L}$)(u):=Lu+$\tilde{L}$(u)
  - (λL)(u):=λ L(u)

  $\diamond$ Check that(Linearity preserved). For(L+$\tilde{L}$)$\in$ $\mathcal{L}$(U,V)
  - (L+$\tilde{L}$)(λu)=λLu+$\lambda\tilde{L}$u
  - (L+$\tilde{L}$)(u+u')=L(u+u')+$\tilde{L}$(u+u')
- Coordinate Map. V be real/complex vector space and (b_1,...,b_n) a basis.

$$\Phi : V \to \mathbb{F}^n \quad v = \sum_{k=1}^{n} \lambda_k b_k \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Note Φ is linear and bijective(uniqueness of basis)

Definitions

According to their properties, there are several fancy names for linear maps. A homomorphism $L \in \mathscr{L}(U, V)$ is said to be

- an ***isomorphism*** if L is bijective;
- an ***endomorphism*** if $U = V$;
- an ***automorphism*** if $U = V$ and L is bijective;
- ***epimorph*** if L is surjective;
- ***monomorph*** if L is injective.

1.4.10. Remark. If L is an isomorphism, the its inverse, L^{-1} is also linear and hence also an isomorphism.

- To check isomorphic:
  - linear+bijective
  - every basis of $(b_1,..b_n)$ of U the tuple $(\text{L}b_1,..\text{L}b_n)$ is a basis for V
  - dim U= dim V equivalent to (U isomorphic to V).

Definitions

Dual Space

(ii) Let V be a real or complex vector space. Then $\mathscr{L}(V, \mathbb{F})$ is known as the ***dual space*** of V and denoted by V^*. The dual space of V is of course itself a vector space.

Let $\dim V = n < \infty$ and $\mathscr{B} = (b_1, \dots, b_n)$ be a basis of V. Then for every $k = 1, \dots, n$ there exists a unique map

$$b_k^* \colon V \to \mathbb{F}, \qquad b_k^*(b_j) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

It turns out (see exercises) that the tuple of maps $\mathscr{B}^* = (b_1^*, \dots, b_n^*)$ is a basis of $V^* = \mathscr{L}(V, \mathbb{F})$ (called the ***dual basis*** of $\mathscr{B}$) and thus $\dim V^* = \dim V = n$.

Dual Space: contd

- In principle, dual space is special in a way because it the set of linear functionals, i.e. that takes an element in the vector space and returns a scalar(could be complex). Otherwise, mapping a element from one vector space to another is called an linear operator.
- To prove that the tuple of maps $\beta^*=(b_1^*,b_2^*,...,b_n^*)$ is actually a basis, we show by
  - $(b_1^*,b_2^*,...,b_n^*)$ is linearly independent.
  - We show that $V^*=\text{span}\{b_1^*, b_2^*, ..., b_n^*\}$

Proof

- **Linear Independence.** Suppose we have scalar $\lambda_1, \lambda_2, ...\lambda_n \in \mathbb{F}$ such that

$$\lambda_1 b_1^* + \lambda_2 b_2^* + ... + \lambda_n b_n^* = 0$$

It is important to note that at this point 0 is defined as the functional such that every element is mapped to zero. Now, we examine the result acting on basis vectors. Namely,

$$(\lambda_1 b_1^* + \lambda_2 b_2^* + ... + \lambda_n b_n^*)(b_i) = \lambda_1 b_1^*(b_i) + \lambda_2 b_2^*(b_i) + ... + \lambda_n b_n^*(b_i) = \lambda_i$$

The functional on the RHS acting on the basis vectors should return a 0 scalar. We thereby shown the fact that $\lambda_1 = \lambda_2 = ... = \lambda_n$=0

Proof

- **The basis spans V**. Now suppose a more general functional L in the dual space. Generally, we wish to show that(In our case we try to figure out the concrete constructions the lambdas have to follow)

$$L = \lambda_1 b_1^* + \lambda_2 b_2^* + ... + \lambda_n b_n^*$$

Since this is a linear functional, proving this statement is equivalent to showing that **for every** element in the vector space the LHS functional acting on this element is equal to the RHS acting on this element. Now

$$\begin{aligned}(\lambda_1 b_1^* + \lambda_2 b_2^* + ... + \lambda_n b_n^*)(b_i) &= \lambda_1 b_1^*(b_i) + \lambda_2 b_2^*(b_i) + ... + \lambda_n b_n^*(b_i) \\ &= \lambda_i = L(b_i)\end{aligned} \tag{1}$$

where we have construct concrete bi's.

Proof

- For the sake of completeness, we can also conduct the following verification. Namely,
  (note we avoid confusion of linear combination of functional(λ i's) or element in vector space(ai's))

$$\begin{aligned} L(v) &= L(\sum_{i=1}^{n} a_i b_i) = \sum_{i=1}^{n} a_i L(b_i) = \sum_{i=1}^{n} a_i \lambda_i \\ &= \sum_{i=1}^{n} \lambda_i b_i^*(\sum_{j=1}^{n} a_j b_j) = \sum_{i=1}^{n} \lambda_i b_i^*(v) \end{aligned} \tag{2}$$

- The basis statement immediately yields the equality of the dimension argument.
- Note that there is some similar spirit of 1.4.4 in the proof.
- V and V^* are isomorphic.

The double dual space

EX.

Suppose that V is finite dimensional. The map ev: $V \to V^{**}$ defined as

$$ev(v)(f) := f(v)$$

is an isomorphism.

Defintion

- Range and kernel.
  - $ranL := \{v \in V : \exists_{u \in U} v = Lu\}$
  - $kerL := \{u \in U : Lu = 0\}$
- Remark. It is not difficult to see that L$\in\mathcal{L}$(U,V) is injective if and only if kerL=$\{0\}$
  ▶($\Longrightarrow$) If L is injective, then assume an element α in ker L. Then L(α)=0=L(0). The injectiveness triggers the result that α=0 which immediately follows kerL=$\{0\}$
  ▶($\Longleftarrow$) If kerL=0 then we prove by definition. Assuming α_1,$\alpha_2$$\in$V, L$\alpha_1$=L$\alpha_2$.
  By linearity, L(α_1-α_2)=Lα_1-Lα_2=0. The restriction on the space of the kernel guarantees that α_1-α_2=0, completing the proof.

The Operator Norm

- The operator norm is defined by:

$$\|L\| := \sup_{u\in U, u\neq 0} \frac{\|Lu\|_V}{\|u\|_U} = \sup_{u\in U, \|u\|_U=1} \|Lu\|_V$$

  ▶ Applying linearity and put norm of u inside Lu obtaining $\frac{u}{\|u\|}$

- Additional Property. $\|L_1 L_2\| \leq \|L_1\| \cdot \|L_2\|$
  ▶Proof:

$$\begin{aligned} \|L_1 L_2\| &= \max_{x\neq 0} \frac{\|L_1 L_2 x\|}{\|x\|} = \max_{L_2 x\neq 0} \frac{\|L_1 L_2 x\|}{\|L_2 x\|} \cdot \frac{\|L_2 x\|}{\|x\|} \\ &\leq \max_{y\neq 0} \frac{\|L_1 y\|}{\|y\|} \cdot \max_{x\neq 0} \frac{\|L_2 x\|}{\|x\|} \end{aligned} \tag{3}$$

A linear map acting on basis vectors

Theorem

Let U, V be real or complex vector spaces and $(b_1, b_2, ...b_n)$ a basis of U(assuming U finite dimensional). Then for every n-tuple $(v_1, ...v_n) \in V^n$ there exists a unique linear map L: $U \rightarrow V$ such that $Lb_k = v_k$, k=1,2,...n.

- To show uniqueness. Assume there exists a second homomorphism M. We show L=M. Incorporating the linear map and linear structure for vector space.
- Existence. Give concrete reasons/ set forth rules to accurately define the linear map L, and then check that the previously defined map is in fact linear(homogeneity/additivity). In our case the uniqueness of basis argument is crucial for defining L.

In other words, we can examine the properties of(structure) a linear map by simply looking at the image/the result of L acting on the basis vector.

Definition

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n} \tag{4}$$

- You should pretty familiar with the manipulation using a_{ij}, get some practice to make you feel comfortable doing this.

Active and Passive point of view

- Basically the active view shifts the point, while the passive point of view shifts the axis.
- Generally, any transformation can be regarded as first rotate and then stretch.
- See sketch.

Nilpotent

Ex1.

Let N be a square matrix. We say that N is nilpotent if there exists a positive integer r such that $N^r = 0$. where $\mathbf{0}$ is the zero matrix. Prove that if N is nilpotent then id-N is invertible.

Solution

Proof.

To show invertibility, in our case, we prove by finding a concrete example.

$$(id - N) \cdot (id + N + N + ...N^{r-1}) = id - N^r = id$$

Since matrix is simply an encoding of a linear map, it should be reasonbly clear that this also holds for linear maps. $\square$

Discussion

- In etymology, "idem" means the same. The projection map has the property $P^2 = P$, meaning that the final result stays the same even if applied multiple times.($P^n = P$). You could also see(later) that the determinant for idempotent matrices are 0.
- Contrary speaking, "nil" means nothing, In our case, the nilpotent means that applying the same matrix finite times, it will eventually degenerate to 0.($P^r = 0$)

To show invertibility

- Consider a concrete construction and show that

$$A \cdot A^{-1} = id$$

- show that A is bijective.
- Show that A is finite dimensional plus injective plus kerL=$\{0\}$.(Similar to the dual basis example)

Operator Norm / Matrix norm

Exercise 2 a.

Show that the vector-1 norm defined by

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

is actually a norm.

Exercise 2 b.

Show that the vector-1 norm is induced by the operator/matrix norm

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{j=1}^{n} |a_{ij}|$$

Proof a.

We show by three steps,

- $\|x\|_1 \geq 0, \|x\|_1 = 0 \iff$ x=0
- Homogeneity.

$$\|x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \|x\|_1$$

- The triangular inequality.(simply act on each element.)

Proof of b

This proof requires two steps. we first show that the supremum in the definition of the operator norm is bounded above and also exists.(bounded linear maps)

$$\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \leq C$$

and then we show that the supremum has the value that we need to prove. That is,

$$C = \max_{1 \leq j \leq n} \sum_{j=1}^{n} |a_{ij}|$$

- Basically following the recipe in the class, first define bounded linear maps and examine its supremum.
- Do not omit the $x \neq 0$ condition.

Proof contd

Writing out Ax explicitly,

$$\|Ax\|_1 = \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}| \leq C \sum_{j=1}^{n} |x_j|$$

where the first inequality is the triangular inequality and the second equal sign is the interchanging of the summation(you can think of it as summing in for loops, the sequence doesn't really matter) we have shown that it is bounded (above).

Proof contd

Then we also need to show that this bound can be achieved by some vector x_0, (because the maximum argument). Since the maximum must be obtained by one of the j's, let's say, the maximum is obtained by k.

$$max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| = \sum_{i=1}^{k}|a_{ik}|$$

Setting $x_0 = e_k$, where k is the kth standard basis vector, gives

$$\frac{\|Ax_0\|_1}{\|x_0\|_1} = \sum_{i=1}^{n}|a_{ik}| = C$$

which completes the proof.

Transformation of basis

Exercise 3.

A linear map in $\mathbb{R}^3$ with respect to the basis

$$\mu_1 = (a, b, c), \mu_2 = (d, e, f), \mu_3 = (g, h, i)$$

has the matrix representation A$=\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$

Find the matrix representation for A with respect to the basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

The columns of the matrix are the images of the standard basis vector

- . T= $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} A' = TAT^{-1}$

VV285 Recitation Class Week 3

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June 8, 2019

Some general comments

In this part, I hope you appreciate the subtlety of the introduction of cross product. You should be able to appreciate the important properties of determinants det(AB)=detA·detB etc.(det(A+B)=detA+detB ?) On top of that, you should be able to calculate them quite proficiently. If you feel confident about the previous points, I suggest you spend more time understanding the concepts of permutation and groups.
By now, I hope you can appreciate the connection of elementary row manipualtion and elementary matrix manipulation, Fredholm alternative, solutions for linear system.
Try to be familiarize with the concept and "do not be easily confused".

Overview

1. Matrices
   - Invertibility
   - Transformation of Basis
   - Orthogonal Matrices
   - Trace, scalar product, and Pauli Spin matrices
2. Determinants
3. Exercises
   - Calculating the determinant
   - Cramer's Rule

Invertibility

- A matrix A $\in$ Mat(n$\times$n) is invertible if and only if detA$\neq$0
- A matrix is invertible if and only if it the column(row) vectors are linear independent to each other.
- The equation Ax $=$ 0 has only the trivial solution x$=$0
- The kernel of A is trivial, that is, kerA $=\{0\}$
- The equation Ax $=$b has a unique solution for each b. x$=A^{-1}b$.
- A is invertible if and only if there exists an elementary matrix S corresponding to elementary row operations that transform A into the unit matrix SA$=$id.
- A linear map A is invertible if it is bijective(kerA$=\{0\}$ and injective).

Warming-up problem

Exercise

. Calculate the inverse for $A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$

Transformation of basis

Exercise.

A linear map in $\mathbb{R}^3$ with respect to the basis

$$a_1 = (a', b, c), a_2 = (d, e, f), a_3 = (g, h, i)$$

has the matrix representation $\mathrm{T}_a = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$

Find the matrix representation for T with respect to the basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

Solution

$$
\begin{tikzcd}
\vec{u} \arrow[r, mapsto, "T"] \arrow[d, shift right=2, "A^{-1}"'] & \vec{v} \arrow[d, shift left=2, "A^{-1}"] \\
\vec{u}_a \arrow[u, mapsto, shift right=2, "A"'] \arrow[r, mapsto, "T_a"'] & \vec{v}_a \arrow[u, mapsto, shift left=2, "A"]
\end{tikzcd}
$$

We write everything down a bit more formally. Consider the $\overline{u_a}$ and $\overline{v_a}$ under the basis representation(a_1, a_2, a_3), and $\overline{u}$ and $\overline{v}$ represented by standard basis. $\overline{u_a} = u_{a_1} \mathrm{a}_1 + \mathrm{u}_{a_2} \mathrm{a}_2 + \mathrm{u}_{a_3}\ a_3$ and $\overline{u} = u_{e_1} \mathrm{e}_1 + \mathrm{u}_{e_2} \mathrm{e}_2 + \mathrm{u}_{e_3} \mathrm{e}_3$.

Sol contd

For a map from standard basis to nonstandard basis, the rule to remember is that the columns of the matrix are the images of the standard basis vectors. $Ae_i = a_i$, $i \in \{1, 2, 3\}$ Therefore, $A = \begin{pmatrix} a' & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

Now, note that u_a and u are essentially the same thing, and u_{e_i} and u_{a_i} are scalars(So switching A and u is allowed).

$$u_a = u = u_{a_1} Ae_1 + u_{a_2} Ae_2 + u_{a_3} Ae_3 = u_{e_1} e_1 + u_{e_2} e_2 + u_{e_3} e_3$$

$$Au_{a_i} = u_{e_i} \longrightarrow A^{-1} u_{e_i} = u_{a_i}$$

$$A^{-1} : (u_{e_1}, u_{e_2}, u_{e_3})^T \mapsto (u_{a_1}, u_{a_2}, u_{a_3})^T$$

$$\overline{v_a} = T_a \overline{u_a}$$

$$\overline{v_a} = T_a A^{-1} \overline{u}$$

$$A^{-1} \overline{v} = T_a A^{-1} \overline{u}$$

$$\overline{v} = A T_a A^{-1} \overline{u}$$

Remarks and Clarification

- Final answer:

$$T = AT_aA^{-1}$$

- . It is useful to illustrate another example.

Basis Changing Matrix(Explanation for A^{-1})

- A basis for $\mathbb{P}_2$ is T=$\{x+1, x-1, 2x^2\}$. Find the change of basis matrix from B=$\{1, x, x^2\}$ to T. That is to find the linear map A: ▶. Find the matrix representation: 1=$\frac{1}{2}$(x+1)+-$\frac{1}{2}$(x-1),$(\frac{1}{2}, -\frac{1}{2}, 0)$
x=$\frac{1}{2}$(x+1)+$\frac{1}{2}$(x-1),$(\frac{1}{2}, \frac{1}{2}, 0)$
x^2=$\frac{1}{2}2x^2$,$(0,0,\frac{1}{2})$.
- And stacking them column-wise, we get

$$A=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

- Consider f(x)=ax^2+bx+c. Under basis of b, it is $(c,b,a)^T$, while under basis for Under basis for T, it is (Using the method for undetermined coefficient), $(\frac{b+c}{2}, \frac{b-c}{2}, \frac{1}{2}a)^T$. You can check that $A(c,b,a)^T=(\frac{b+c}{2}, \frac{b-c}{2}, \frac{1}{2}a)^T$

Contd

- If you do it in a reverse manner,
  - x+1 = 11+1 x (1,1,0)
  - x-1 = -11+1 x (-1,1,0)
  - $2x^2$ (0,0,2)
- Stacking them up, you will get the inverse
  $$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- Make sure you pay attention to "which maps to which" . In general, you represent the basis that is "being mapped" with the basis that is "mapped into".

Orthogonal Matrices

- A^T $= A^{-1}$, the columns of the orthogonal matrix are orthonormal.
- A$\cdot A^T$=id.
  $\sum_{k=1}^{n} a_{ik} a_{jk} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
- $\|Ax\|^2 = \|x\|^2$
- det A $= \pm 1$

Trace, scalar product, and Pauli Spin matrices

- Consider two vectors x, y in $\mathbb{R}^n$. We have

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i$$

In similar spirit,

$$\langle X, Y \rangle = tr(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

for n by m matrices. Extending it further,to complex numbers

$$\langle X, Y \rangle = tr(X^* Y)$$

Properties: Invariant under TAT^{-1}, $tr(TAT^{-1}) = tr(A)$

Trace, Matrix norm, Orthogonal

Exercise Frobenius Norm Of a matrix.

The Frobenius norm of a matrix A$\in \mathbb{R}^{n\times n}$ is defined as
$\|A\|_F = \sqrt{Tr(A^T A)}$
i) Show that

$$\|A\|_F = (\sum_{i,j} |A_{ij}|^2)^{1/2}$$

Thus the Frobenius norm is simply the Euclidean norm of the matrix when it is considered as an element of $\mathbb{R}^{n^2}$.
ii) Show that if U and V are orthnogonal, then $\|UA\|_F = \|AV\|_F = \|A\|_F$

Solution

Applying the definition, we have

$$\|A\|_F^2 = Tr(A^T A) = \sum_{i=1}^{n}(A^T A)_{ii} = \sum_{i=1}^{n}(\sum_{j=1}^{n} A^T_{ij} A_{ji}) = \sum_{i,j} A^2_{ij}$$

- Remarks.
  ▶ Sometimes you might come across interchanging of variables, if it helps you understand, you can think of it as summing using two for loops, and swapping the iterating variables.
  ▶ Formalism to formulate a formal proof using matrices.

Solution

- We first show that $\|UA\|_F = \|A\|_F$ because

$$\|UA\|_F^2 = Tr((UA)^T UA) = Tr(A^T U^T UA) = Tr(A^T A) = \|A\|_F^2$$

- Similarly. Using the cyclic property of trace.

$$\|AV\|_F^2 = Tr((AV)^T AV) = Tr(AV(AV)^T) = Tr(AVV^T A^T) = tr(AA^T))$$

$$=\|A\|_F^2$$

Calculate the determinant

- Toolkit.
  - Elementary row(matrix) manipulation + Triangular form.

$$detA = det \begin{pmatrix} \lambda_1 & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ & & & & \lambda_n \end{pmatrix} = \prod_{i=1}^{n} \lambda_i$$

  - Laplace Expansion.

$$detA = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} detA_{ij}$$

Calculate the determinant

Exercise 1.

(Credit Chen Xiwen for solution 1 and question.) Calculate the determinant for the matrix A$\in$ Mat(2n$\times$2n,$\mathbb{R}$)

$$A^{(2n)} = \begin{pmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{pmatrix}$$

Solution 1. Using Laplace Expansion

The gist of the proof is basically the following

- Use laplace expansion to eliminate the ith row and jth column.
- obtaining (2n-1)$\times$(2n-1) matrix.
- Obtain a relationship between detA$^{(2n)}$ and detA^{2n-2}.
- Prove by induction.

Sol 1 contd

The determinant can be found by

$$\det A^{(2n)} = a \cdot (-1)^{1+1} \det A^{(2n)}_{11} + b \cdot (-1)^{2n+1} \det A^{(2n)}_{(2n)1} = a \det A^{(2n)}_{11} - b \det A^{(2n)}_{(2n)1}$$

where

$$A^{(2n)}_{11} = \begin{pmatrix} a & \cdots & b & 0 \\ \vdots & & \vdots & \vdots \\ b & \cdots & a & 0 \\ 0 & \cdots & 0 & a \end{pmatrix} \in \mathrm{Mat}((2n-1) \times (2n-1), \mathbb{R})$$

$$A^{(2n)}_{(2n)1} = \begin{pmatrix} 0 & \cdots & 0 & b \\ a & & b & 0 \\ \vdots & & \vdots & \vdots \\ b & \cdots & a & 0 \end{pmatrix} \in \mathrm{Mat}((2n-1) \times (2n-1), \mathbb{R})$$

Sol 1 contd

Then expanding the $(2n-1)$th columns for the two matrices, we obtain

$$\det A = a^2 \det A^{(2n-2)} - b^2 \det A^{(2n-2)}$$

Then we attempt to prove $\det A = (a^2 - b^2)^n$ by induction on $n \geq 1$.

- When $n = 1$, it is clear that

$$\det A^{(2)} = a^2 - b^2$$

- Then based on the hypothesis on $n = k$, from the relation deduced above, we have verified that $\det A^{(2k+2)} = (a^2 - b^2)\det A^{(2k)} = (a^2 - b^2)^n$.

Therefore, the the determinant is given by

$$\det A^{(2n)} = (a^2 - b^2)^n$$

Solution 2. Triangular form.

- Consider the first n rows and the second n rows. Basically we do this for n times.

$$\begin{array}{c} \begin{pmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{pmatrix} \end{array} \begin{array}{l} 1 \quad \cdot(-\frac{b}{a}) \\ 2 \quad \cdot(-\frac{b}{a}) \\ \vdots \\ \text{2n-1} \quad + \\ \text{2n} \quad + \end{array}$$

Therefore we have n a's on the first row(on the diagonal) and n (a-$\frac{b}{a}$).That is

$$(a - \frac{b}{a})^n a^n = (a^2 - b^2)^n$$

Solving linear system's of equation

Exercise 2. Use cramer's rule to solve the following

$$-x + 3y - 2z = 5, 4x - y - 3z = -8, 2x + 2y - 5z = 7$$

Solution

We have Dm=b , where

$$D = \begin{pmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{pmatrix} \quad b = (5, -8, 7)^T$$

We then calculate the determinant for D. Namely, We fix j=1.

$$detD = (-1)^{1+1} \cdot (-1)(5-(-6)) + (-1)^{2+1} \cdot 4(-15+4) + (-1)^{3+1} \cdot 2(-9-2)$$

$$detD = -11 + 44 - 22 = 11$$

To obtain x, y, z we basically replace the first, second, third column with b and we obtain

Solution

$$x = \frac{-110}{11} = -10$$

$$y = \frac{-79}{11}$$

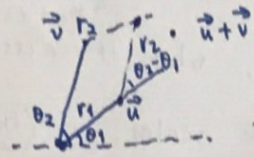
$$z = \frac{91}{11}$$

Exam Checklist(Include but not limited to)

- Trace. Properties. Refer to Exercise 3.7.
- Determinants
- Projection Map. Exercise 3.6/ Notes.
- Operator Norm.
- Transformation of basis.
- Solutions for linear systems.
- Plus the sections we have discussed before: subspaces, linear maps, inner product spaces, norm, linear independence, isomorphism.

↳ P is actually a linear map

→ Polar coordinates



$r_1\cos\theta_1 + r_2\cos\theta_2 \Rightarrow$

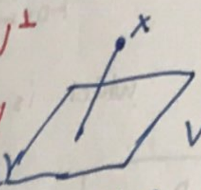
linearity $(r_1\cos\theta_1 + r_2\cos\theta_2,\ r_1\sin\theta_1 + r_2\sin\theta_2)$

P is actually linear

Proof: Take any vectors $x, y \in W$.

$x = P_1 + P_{1\perp}$ $\in V^{\perp}$

$y = P_2 + P_{2\perp}$



$x + y = P_1 + P_2 + (P_{1\perp} + P_{2\perp})$

$P(x+y) = P_1 + P_2 = P(x) + P(y)$

Homogeneity shown similarly.

(Furthermore, ~~supposing~~ assuming scalar λ

$\lambda x = \lambda P_1 + \lambda P_{1\perp}$ (properties of vector space) 3.2D

$P(\lambda x) = \lambda P_1 = \lambda P(x)$

rotate by θ and stretch by $\cos\theta$.

↳ P is idempotent $(P^2 = P)$

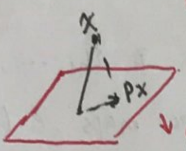
A linear transformation P is called an orthonormal projection if the image of P is V and the kernel is perpendicular to V and $P^2 = P$.

- Intuitive understanding: For any v

$P(Pv) = (Pv)$ (Pv is already on that plane, exerting the linear map again won't hurt at all)

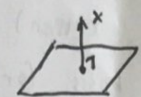
- $\boxed{\operatorname{ran} P \perp \ker P}$: Intuitive understanding

↳ ker P is the set such that $Pv = 0$.



subspace of projection

Px is always in the plane



⇒ Analogy: $V \perp V^{\perp}$

we define the adjoint using matrices.

↳ $P = P^*$

$P = P^* P = \bar{P}^T P$

$Px \perp x - Px$

we've already have $\langle x, A^* y\rangle = \langle Ax, y\rangle$

$0 = \langle x, (P - P^*P)x\rangle = \langle (1 - P)x, Px\rangle$

$P = P^* P$

$P^* = (\bar{P}^T P)^T = (P^T \bar{P})^T = \bar{P}^T (P^T)^T = P^* P$

$A = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$ ↱ Ae_j jth column

$A \in \mathrm{Mat}(m \times n; \mathbb{F})$, $x \in \mathbb{F}^m$, $y \in \mathbb{F}^n$. → WRITE THINGS OUT IF YOU FEEL CONFUSING

$A^* = \begin{pmatrix} \bar{b}_{11} & \cdots & \bar{b}_{m1} \\ \bar{b}_{12} & \ddots & \vdots \\ \vdots & & \\ \bar{b}_{1n} & \cdots & \bar{b}_{mn} \end{pmatrix}$ ↳ $\langle x, Ay\rangle = \langle A^* x, y\rangle$

↳ basis form

Proof: $x = \sum_{i=1}^{m} \lambda_i e_i$ $\quad y = \sum_{j=1}^{n} \mu_j e_j$,

$\langle x, Ay\rangle = \langle \sum_{i=1}^{m} \lambda_i e_i, A \sum_{j=1}^{n} \mu_j e_j\rangle = \sum_{j=1}^{n} \mu_j \sum_{i=1}^{m} \bar{\lambda}_i \langle e_i, Ae_j\rangle$ ith row ⇒ b_{ij}

$A^* e_i$ ↳ ith column

$= \langle A^* x, y\rangle = \langle A^* \sum_{i=1}^{m} \lambda_i e_i, \sum_{j=1}^{n} \mu_j e_j\rangle = \sum_{i=1}^{m} \bar{\lambda}_i \sum_{j=1}^{n} \mu_j \langle A^* e_i, e_j\rangle$ ‖ DEF: complex inner product space

$\bar{b}_{ij} = b_{ij}$

Justification $ev(v)(f) := f(v)$ ← Scalar

Notation: Explained

$f \in \mathcal{L}^*(V, \mathbb{F})$ $ev \in \mathcal{L}(V, \mathcal{L}(V^*, \mathbb{F}))$

$ev(v) \in \mathcal{L}(V^*, \mathbb{F})$

$V^* = \mathcal{L}(V, \mathbb{F})$

$(V^*)^* = \mathcal{L}(V^*, \mathbb{F})$ ← treat as V (unknown vector space)

↳ note that $ev: V \to V^{**}$

Formally speaking, $ev(v)$ ($V^* \to \mathbb{F}$) is a element of V^{**}, when it acts on f, which is an element in the dual space, it returns a scalar. $f(v)$

Proof. : ▷ Linearity. (Stick to $ev: V \to V^{**}$) $(ev(v_1+v_2) = ev(v_1) + ev(v_2))$

$$\underbrace{ev(v_1+v_2)}_{\text{we examine its property}}(f) \overset{\text{by def}}{=} f(v_1+v_2) \underset{\text{in } V^*}{\overset{\text{linearity}}{=}} f(v_1) + f(v_2) \overset{\text{plug back def}}{=} ev(v_1)(f) + ev(v_2)(f)$$

act on f to see clearer

The homogeneity is very similar, $ev(a \cdot v) = a\, ev(v)$ $a \in \mathbb{F}$

▷ Bijective b / $\dim V = \dim V^{**}$ $(\dim V = \dim V^*)$ a + injective

a. ~~Set~~ Is equivalent to showing that ev is injective.

Let v be an element in V $(v \in \ker(ev))$ such that $ev(v)$ is the zero element of V^{**}. That is,

$ev(v)(f) = 0$ for all $f \in V^*$

This is simply $f(v) = 0$ for all $f \in V^*$

In the dual space, the only v that satisfy the property is 0.

Therefore, $\ker(ev) = \{0\}$ ⇒ showing ev is injective

b. (Credit Chen XiWen). Set a basis in V as $\mathcal{B} = \{b_1, \cdots, b_n\}$, then considering the dual basis for V^*, we show that the linear maps $ev(b_i)$ are linearly independent.

an element of V^{**} $(V^* \to \mathbb{F})$

upshot for $ev(v)(f) = f(v)$ where $v = b_i$

$$\sum_{i=1}^{n} \mu_i\, ev(b_i) = 0 \iff \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} y_{ij} b_j^* \right) b_i = 0 \text{ for arbitrary } y_{ij}$$

Namely, $\sum_{i=1}^{n} \mu_i y_{ii} = 0$ for arbitrary y_{ii} ⇒ $\mu_i = 0$

Since $ev(b_i)$ are linearly independent, it set forth concrete rules to define the basis obtained in V^{**} after acting on basis in V.

VV285 Mid 1

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June 9, 2019

Some general comments

This part mainly revolves around the basics

- Systems Of Linear Equations
- Finite-Dimensional Vector Spaces

You do not need to spend extra time on this subject. Just make sure you understand some basics and "how to"s.

Overview

Definitions

- Homogeneity and Inhomogeneity. A linear system is a homogeneous system if the b_is are all zero. Otherwise, it is inhomogeneous.
- An inhomogeneous system of equations may have either
  - a unique solution or
  - no solution or
  - an infinite number of solutions

  ⋄ An inhomogeneous system can not have exact two/three solutions.
- Fredholm alternative. Let A be an n by n matrix
  - either $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$ or
  - $Ax = 0$ has a non-trivial solution.
- There exists solution x for $Ax = b$ if and only if $\text{rank}A = \text{rank}(A|b)$. That is b. is in the span of $\{a_{.1}, ...a_{.n}\}$

Definitions

- A solution set for a system of equation is the set of n-tuples that solves the linear system of equations. Correspondingly, it either contains:
  - A single point.
  - nothing.
  - An infinite number of elements.
- You should be able to make connections between the invertibility of A, the kernel of A, the determinant of A, the columns of A, and the number of solutions of Ax=b.
  - For example, if Ax=0 has only the trivial solution, then kerA=$\{0\}$, then $Ax = b$ has a unique solution, then A is invertible, then detA $\neq 0$, the columns of A are independent.

Definitions

- Elementary row manipulations.
  - Swapping(interchanging) two rows
  - Multiplying each element in a row with a number
  - Adding a multiple of one row to another row.

  $\diamond$ These manipulations correspond to elementary matrix manipulation. Note that the determinant will not change base on elementary matrix manipulation and can be useful to transfer into upper triangular form and find the determinant.

Gauss - Jordan Algorithm: A general recipe

In general, we wish to solve the following

$$\begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array}$$

$$\textit{Step}1 \quad \begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array} \quad \begin{array}{lll} \cdot(-A) & \cdot(-B) & \\ + & & \cdot(-C) \\ & + & + \end{array} \sim \begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array}$$

$$\textit{Step}2 \quad \begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \quad \begin{array}{lll} + & + & \\ \cdot(-D) & & + \\ & \cdot(-E) & \cdot(-F) \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array}$$

Toolkits

- Prove Linear Independence. Let V be real or complex vector space. The vectors v_1,v_2,...,v_n $\in$ V, are said to be independent if **for all** λ_1, λ_2,...,$\lambda_n \in \mathbb{F}$"only if"

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = ... = \lambda_n = 0$$

- Prove V=A$\bigoplus$B
  ▶A$\cap$B= $\{0\}$
  ▶ A+B = V

Definitions

- Basis.
  An n-tuple β=(b_1,...,b_n)$\in$ V_n is a basis if and only if
  i) Uniqueness Statement: the vectors b_1,...b_n are linearly independent.
  ii) Existence statement: V=spanβ
  $\diamond$ Should be useful to prove that an n-tuple is actually a basis.

Proofs

- To prove independence in general. Consider
  ▶whether the question can degenerate into proving one can not be the multiple of another or
  ▶by showing the coefficient is all zero if the linear combination is zero.
  ▶ If none of the v_is is contained in the span of others.
- To $(b_1,...,b_n)$is actually a basis:
  ▶ The vectors $b_1,...,b_n$ are linearly independent.
  ▶ V=span β. That is, any vectors in V can be rewritten as the linear combination of the basis.

Proofs

- The sum is direct.
  - ▶ By definition.
  - ▶ For all $x \in U+W$,$x \neq 0$, x has a **unique** representation $x=u+w$.
  - ⋄ Note the relationship between direct and uniqueness.
- . Basis extension theorem.
  Let V be finite-dimensional vector space and $A' \subset V$. There exists a basis of V containing A'.
  - ▶.Independent set A can have at most n elements.
  - ▶ Any independent set A with n elements a basis.

VV285 Recitation Class Week 5

Shen Yueyang

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June 20, 2019

Some general comments

- Convergence and Continuity
- Functions and Derivatives
- Curves in Vector Spaces.
- Potential Functions
- The Second Derivative
- Extrema Of potential Functions
- Constrained Extrema

Overview

1. Convergence and Continuity
   - Definition
   - Infinite-dimensional Space Example(Exercise)
   - Comments
2. Derivatives
   - Definitions
   - Exercises

Definitions

- Open/ Closed/ Compact.

  ▶ A set U with U$\subset$ V is called open if for every a$\in$U there exists an $\epsilon > 0$ such that B_ϵ(a)$\subset$U.
  - Open set consists of only interior point.
  - In finite dimensional vector space, if a set is open with respect to one norm it is also open for any other norm.
  - Open ball/ Empty set/ Entire Space

  ▶ A set is closed if its complement is open.

  ▶ Compact ⋄ Every sequence has a convergent subsequence. ⋄ The limit is contained in the set.

Theorem of Bolzano Weierstrass

2.1.13. Theorem of Bolzano–Weierstraß in $\mathbb{R}^n$. Let $(x^{(m)})_{m\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^n$, i.e., $x^{(m)} = (x_1^{(m)}, \dots, x_n^{(m)})$. Suppose that there exists a constant $C > 0$ such that $|x_k^{(m)}| < C$ for all $m \in \mathbb{N}$ and each $k = 1, \dots, n$. Then there exists a subsequence $(x^{(m_j)})_{j\in\mathbb{N}}$ that converges to a vector $y \in \mathbb{R}^n$ in the sense that

$$x_k^{(m_j)} \xrightarrow{j\to\infty} y_k \qquad \text{for } k = 1, \dots, n.$$

- Find a convergent subsequence for the first element. In the previous sequence find convergent subsequence for the second element.

Finite Dimension: All norms are Equivalent

Let V be a finite-dimensional vector space, $\|\cdot\|$ be any norm on V and $\{v_1, \dots, v_n\}$ a basis of V. Let $v \in V$ have the representation $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. By the triangle inequality,

$$\|v\| = \|\lambda_1 v_1 + \cdots + \lambda_n v_n\| \le \sum_{i=1}^{n} |\lambda_i| \|v_i\| \le C \sum_{i=1}^{n} |\lambda_i|$$

where $C := \max_{1 \le i \le n} \|v_i\|$ depends only on the basis and not on v. We hence see that for any norm there are constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{i=1}^{n} |\lambda_i| \le \|v\| \le C_2 \sum_{i=1}^{n} |\lambda_i|, \qquad (2.1.7)$$

where the first inequality is just (2.1.5). Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, it follows from their respective inequalities (2.1.7) that (2.1.3) holds. $\square$

The determinant is a continuous function

The proof basically does the following. First, we choose a norm that is to manipulate. Namely, $\|A\|=\max_{i,j}\ |a_{ij}|$, as long as the maximum goes to zero all the other terms smaller than the maximum term goes to zero as well. Assume (A_m) is a sequence converging to A.

$$\|A^{(m)} - A\| = max_{i,j}\|a_{ij}^{m} - a_{ij}\| \to 0$$

We have already argued that the other terms would also go to zero. And we note that det is simply "$+$" and "$\cdot$" operation on the a_{ij}s. Therefore, we have $\det A_m \to \det A$

Comment on the infinite dimensional vector space

In infinite dimensional space,

- It is possible to define non equivalent norms. Not all norms are equivalent.
- Closed and bounded does not necessary means compact.

Possible to define non-equivalent norms

Exercise 5.4

Consider the space of continuous functions on $[0, 1]$, $C([0, 1])$. We can define the two norms

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)|\, dx.$$

i) Which of the two inequalities

$$\|f\|_\infty \leq C\|f\|_1 \qquad \text{and} \qquad \|f\|_1 \leq C\|f\|_\infty$$

for some $C > 0$ holds? Prove the one that is valid.
(2 Marks)

ii) Show that the norms are not equivalent by exhibiting a sequence of functions (f_n) with $\|f_n\|_1 \to 0$ but $\|f_n\|_\infty \not\to 0$ as $n \to \infty$.
(2 Marks)

Sol

- The first question is rather straightforward, a common trick is to use the inequality that the integral is smaller than the integral length multiplied by the integration length.
- The second question, you simply disprove the second inequality. Namely(One possibility),

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 - \frac{1}{a^n} \\ a^n(\text{x-}1)+1 & \text{if } 1 - \frac{1}{a^n}\ \text{x} \leq 1 \end{cases}$$

Closed and Bounded Sets are not necessary compact in infinite dimensional space

Exercise 5.5

Consider the vector space of summable complex sequences,

$$l^1 := \left\{(a_n) \colon \mathbb{N} \to \mathbb{C} \colon \sum_{n=0}^{\infty} |a_n| < \infty\right\}.$$

The natural norm is given by

$$\|(a_n)\|_1 := \sum_{n=0}^{\infty} |a_n|.$$

Show that the set

$$\overline{B_1(0)} = \left\{(a_n) \in l^1 \colon \sum_{n=0}^{\infty} |a_n| \leq 1\right\}$$

is closed and bounded, but not compact. *Hint:* Consider the sequence of unit sequences $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the nth position. Show that each $e_n \in \overline{B_1(0)}$ but that the sequence (e_n) does not have a convergent subsequence.

(3 Marks)

Solution

- Continuity. We show the map $\|\cdot\|$: $l^{-1} \to \mathbb{R}$ is continuous.
  $(a_n)^m \to (a_n) \quad \implies \|(a_n)^m\| \to \|(a_n)\|$

$$\left|\|(a_n)^m\| - \|(a_n)\|\right| \leq \|(a_n)^m - (a_n)\| \to 0$$

Solution

Closed. We can't show by showing the fact that the preimage(for the map $\| \cdot \|_1 : (\mathrm{a}_n) \to \mathbb{R}$) of the closed set is also closed. We show by the complement(the preimage of the open set is also open.)
We consider the open set

$$\Omega = \{x \in \mathbb{R} : x > 1\}$$

We have already shown that this map is continuous. By theorem 2.1.28. We immediate the have that the preimage for the map is open. That is to say, the set

$$\| \cdot \|^{-1}(\Omega) = \{(a_n) \in l^1 : \sum_{n=0}^{\infty} |a_n| \geq 1\}$$

Solution

- We construct

$$(a_m)^{(1)} = (1, 0, 0, ...)$$
$$(a_m)^{(2)} = (0, 1, 0, ...)$$
$$.$$
$$.$$
$$.$$
$$\|(a_m)^R - (a_m)^I\| \geq 2$$

for any R and I. Obviously, this is not a Cauchy sequence. Since a set is compact if every sequence in the set has a convergent subsequence, and we can not find a convergent subsequence in this case, we conclude that the set is not compact.

Comments

- The open set is both open and closed. In fact it is the only set that satisfies this property.
- In finite dimensional vector space, closed and bounded is essentially the same as compact, while compact is a stronger statement in infinite dimensional space.
- For a continuous function on a compact set $K \subset V$.
  - $f(K)$ is continuous.
  - f has a maximum on K.
  - f is uniformly continuous on K.

Definitions

2.2.6. Definition. Let $f : X \to V_1$, $g : X \to V_2$ and $x_0 \in X$. We say that

$$f(x) = o(g(x)) \quad \text{as } x \to x_0 \qquad \Leftrightarrow \qquad \lim_{x \to x_0} \frac{\|f(x)\|_{V_1}}{\|g(x)\|_{V_2}} = 0$$

- Compare two functions. Two different normed vector spaces.

2.2.7. Definition. Let X, V be finite-dimensional vector spaces and $\Omega \subset X$ an open set. Then a map $f : \Omega \to V$ is called ***differentiable at*** $x \in \Omega$ if there exists a linear map $L_x \in \mathscr{L}(X, V)$ such that

$$f(x + h) = f(x) + L_x h + o(h) \qquad \text{as } h \to 0. \tag{2.2.1}$$

Derivative

- The Derivative of f **at x**

$$L_x = Df|_x = df|_x$$

- Now h $\in$ X, $L_x \in \mathcal{L}(X, V)$, f (x+h) $\in$ V. For all x in the set Ω, we have a corresponding map defined. Therefore, general speaking, Df is a map

$$Df: \quad \Omega \to \mathcal{L}(X, V)$$

2.2.9. Definition. We define

$$C(\Omega, V) := \{f : \Omega \to V : f \text{ is continuous}\},$$
$$C^1(\Omega, V) := \{f : \Omega \to V : f \text{ is differentiable and } Df \text{ is continuous}\}.$$

We may thus regard the ***derivative*** D as a (linear) map

$$D : C^1(\Omega, V) \to C(\Omega, \mathscr{L}(X, V)), \qquad f \mapsto Df.$$

Linearity

For f: $\mathbb{R} \to \mathbb{R}$, if f is linear, then f(x)=αx, f'(x)=α. However, that does not gives us much useful information about the structure of this map(since we only get a scalar). In fact this α is a linear map that shows α does not depend on x. In this case,

$$Df|_x = \alpha, \quad Df|_x h = \alpha h$$

Back to the trace example. Trace is linear, tr(A+B)=tr(A)+tr(B), so we have DL$|_x$=L. Therefore,

$$Dtr|_A H = trH$$

Calculate the derivative of the inverse.

Exercise 1. Calculate the derivative of the inverse for matrix A.
$(\cdot)^{-1}:\mathsf{A}\rightarrow\mathsf{A}^{-1}$

Solution

We denote the inverse map as f. Then,

$$f(A+H) = (A+H)^{-1} = (A(id + A^{-1}H))^{-1} = (id + A^{-1}H)^{-1}A^{-1}$$

We still need a f(A)=A^{-1} on the right hand side. For that, we apply a mean trick, namely,

$$(id + A^{-1}H)(id - A^{-1}H) = id - A^{-1}H + A^{-1}H - A^{-1}HA^{-1}H$$
$$(id + A^{-1}H)(id - A^{-1}H) = id + o(\|H\|)$$
$$(id + A^{-1}H)^{-1} = id - A^{-1}H + o(\|H\|)$$
$$f(A+H) = A^{-1} - A^{-1}HA^{-1} + o(\|H\|)$$

Writing it out formally, we have

$$D(\cdot)^{-1}|_A H = -A^{-1}HA^{-1}$$

Differentiation

In the class we have verified that the determinant is a continuous function.
In fact, the derivative of the map det : Mat(n×n;$\mathbb{R}$)$\to$ $\mathbb{R}$

$$Ddet|_A H = detA tr(A^{-1}H)$$

VV285 Recitation Class Week 6

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June 27, 2019

Some general comments

For this section, it is recommended that you understand the subtlies of the definition and understand the formalism properly. Theorems like Mean Value Theorem maybe important in terms of problem solving. Plus be familiar with concepts including Jacobian, generalized product and how differentiating under a integral works. For curve integral, basically, you need to calculate concrete examples by hand.

Overview

Jacobian

2.2.18. Theorem. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^m$ such that all partial derivatives $\partial_{x_j} f_i$ exist on Ω.

(i) If all partial derivatives are bounded (there exists a constant $M > 0$ such that $|\partial_{x_j} f_i| \leq M$ on Ω), then f is continuous i.e., $f \in C(\Omega, \mathbb{R}^m)$.

(ii) If all partial derivatives are continuous on Ω, then f is continuously differentiable on Ω, i.e., $f \in C^1(\Omega, \mathbb{R}^m)$. In particular,

$$Df|_x = J_f(x) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}\right|_x$$

for all $x \in \Omega$.

In principle, what is saying here is that, the conditions is either not enough or overwhelming.

partial derivatives bounded $\Longrightarrow$ f is continuous

partial derivatives are continuous $\Longrightarrow$ f is continuously differentiable

In the homework, you have seen that partial derivative is not continuous but f is still differentiable.

The Chain Rule

2.2.23. Chain Rule. Let U, X, V be finite-dimensional vector spaces and $\Omega \subset U$, $\Sigma \subset X$ open sets. Let $g : \Omega \to \Sigma$ and $f : \Sigma \to V$ be differentiable maps. Then the composition $f \circ g : \Omega \to V$ is also differentiable and for all $x \in \Omega$

$$D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x, \tag{2.2.4}$$

where the right-hand side is a composition of linear maps.

Polar coordinate example:(Basically this example does a verification that the chain rule holds)

2.2.24. Example. Consider the polar coordinates $(r, \phi) \in (0, \infty) \times [0, 2\pi)$, defined through the map

$$\Phi(r, \phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}.$$

Product Rule

Rule.jpg

Definition. Let X_1, X_2, V be normed vector spaces. A map $\odot : X_1 \times X_2 \to V$ is called a ***(generalized) product*** if

1. $\odot$ is bilinear and
2. $\|u \odot v\|_V \leq \|u\|_{X_1} \|v\|_{X_2}$ for all $u \in X_1, v \in X_2$.

2.2.22. Product Rule.

- U, X_1, X_2, V are finite-dimensional vector spaces.
- $f : \Omega \to X_1, g : \Omega \to X_2$ are differentiable.
- $\odot : X_1 \times X_2 \to V$ is a generalized product.

Then

- $f \odot g : \Omega \to V$ is differentiable.
- $D(f \odot g) = (Df) \odot g + f \odot (Dg)$.

Example(contd)

Then

$$D\Phi|_{(r,\phi)} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial r} & \frac{\partial \Phi_1}{\partial \phi} \\ \frac{\partial \Phi_2}{\partial r} & \frac{\partial \Phi_2}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}.$$

Next, consider the map $U \colon \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto x_1^2 + x_2^2$. The derivative is

$$DU|_x = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right) = (2x_1, 2x_2)$$

Now $U \circ \Phi = (r\cos\phi)^2 + (r\sin\phi)^2 = r^2$. Clearly, $D(U \circ \Phi)|_{(r,\phi)} = (2r, 0)$. We can also apply the chain rule:

$$\begin{aligned} D(U \circ \Phi)|_{(r,\phi)} &= DU|_{(r\cos\phi, r\sin\phi)} D\Phi|_{(r,\phi)} \\ &= \begin{pmatrix} 2r\cos\phi, & 2r\sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \\ &= (2r\cos^2\phi + 2r\sin^2\phi, -2r^2\cos\phi\sin\phi + 2r^2\sin\phi\cos\phi) \\ &= (2r, 0) \end{aligned}$$

Cylindrical Coordinates

Calculate the derivate of f(x,y,z)=$x^2 + y^2 + z^2$ in cylindrical coordinates

$$\Phi: (0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}, \qquad (r, \phi, \zeta) \mapsto (x, y, z)$$

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = \zeta$$

Cylindrical Coordinates

Calculate the derivate of f(x,y,z)=$x^2 + y^2 + z^2$ in cylindrical coordinates

$$\Phi : (0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}, \qquad (r, \phi, \zeta) \mapsto (x, y, z)$$

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = \zeta$$

$$\mathrm{D}\phi|_{(r,\phi,\zeta)} = \begin{pmatrix} cos\phi & -rsin\phi & 0 \\ sin\phi & rcos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, Df|_{x,y,z} = (2x, 2y, 2z)$$

$$\begin{aligned} D(f \circ \phi)|_{(r,\phi,\zeta)} &= Df|_{(rcos\phi, rsin\phi, \zeta)} D\phi|_{(r,\phi,\zeta)} \\ &= (2rcos\phi, 2rsin\phi, 2\zeta) \begin{pmatrix} cos\phi & -rsin\phi & 0 \\ sin\phi & rcos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (2r, 0, 2\zeta) \end{aligned} \qquad (1)$$

Mean Value Theorem

2.2.30. Mean Value Theorem. Let X, V be finite-dimensional vector spaces, $\Omega \subset X$ open and $f \in C^1(\Omega, V)$. Let $x, y \in \Omega$ and assume that the line segment $x + ty$, $0 \le t \le 1$, is wholly contained in Ω. Then

$$f(x+y) - f(x) = \int_0^1 Df|_{x+ty} y \, dt = \Big(\int_0^1 Df|_{x+ty} \, dt\Big) y. \qquad (2.2.6)$$

The general case(fundamental theorem of calculus)

$$f(x+y) - f(x) = \int_x^{x+y} f'(\xi) \, d\xi.$$

Substituting $t = (\xi - x)/y$ in the integral, we have the equivalent identity

$$f(x+y) - f(x) = \int_0^1 f'(x+yt) y \, dt,$$

Mean Value Therorem

Let f: U $\to$ F be a map and assume that f is differentiable on U. Let A $\subset$ U. Suppose f is all convex in this subset A and that $\|f'\| \leq M$ on A for some M$\geq$0. Then $\|f(x) - f(y)\| \leq M \cdot \|x - y\|$

Solution.

Set h(t)=f((1-t)x+ty). Then h'(t)=f'((1-t)x+ty)(y-x).Then $\|h'(t)\|$$\leq$M$\|x - y\|$. Consider the interval [0,1], we immediately obtain $\|f(x) - f(y)\| = \|h(1) - h(0)\| \leq M\|x - y\|$

6.2

Exercise 6.2

Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$,

$$g(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2)\sin((x_1^2 + x_2^2)^{-\frac{1}{2}}), & (x_1, x_2) \neq (0,0), \\ 0, & (x_1, x_2) = (0,0). \end{cases}$$

i) Prove directly that g is continuous at the origin $(0,0)$.
(1 Mark)

ii) Calculate the partial derivatives $\partial_{x_1} g$ and $\partial_{x_2} g$. Are they continuous at the origin (proof!)?
(2 Marks)

iii) Is g differentiable at the origin? If so, find $Dg|_{(0,0)}$. If not, prove that $Dg|_{(0,0)}$ doesn't exist.
(2 Marks)

6.6

Exercise 6.6

In $\mathbb{R}^2 \setminus \{0\}$, we define polar coordinates by setting

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi.$$

By considering the the map $\Phi\colon (0,\infty)\times[0,2\pi) \to \mathbb{R}^2\setminus\{0\}$, $(r,\varphi)\mapsto(r\cos\varphi, r\sin\varphi)$ and applying the chain rule, show that (in sloppy notation)

$$\Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} =: \Delta_{(r,\phi)},$$

i.e., show

$$\Delta_x f(x_1,x_2)|_{x=(r\cos\varphi, r\sin\varphi)} = \Delta_{(r,\phi)} f(r\cos\varphi, r\sin\varphi).$$

(4 Marks)

Exercises

Let f: $\mathbb{R}^n \to \mathbb{R}$ be the function $f(x)=x^T Ax$ where $x \in \mathbb{R}^n$ and A is a n by n matrix. Find the derivative of f at the point x_0.

Solution

By convention we expand all the terms for $f(x_0+h)$

$$
\begin{aligned}
f(x_0+h) &= (x_0+h)^T A(x_0+h) \\
&= x_0^T Ax_0 + x_0^T Ah + h^T Ax_0 + h^T Ah \\
&= f(x_0) + x_0^T(A+A^T)h + h^T Ah \\
&= f(x_0) + Df|_{x_0} h + h^T Ah
\end{aligned}
\tag{2}
$$

Show

$$\lim_{\|h\|\to 0} \frac{|h^T Ah|}{\|h\|} = 0$$

$$|h^T Ah| = |\langle h, A^T h\rangle| \leq \|h\| \|A^T h\|$$

There are a number of ways to argue that

$$\lim_{\|h\|\to 0} \|A^T h\| = 0$$

Exercises

Let Y $\in$ Mat(m$\times$n, $\mathbb{R}$), X $\in$Mat(p$\times$n,$\mathbb{R}$), Define the following function f:Mat(m$\times$p,$\mathbb{R}$)$\to\mathbb{R}$

$$f(B) = \|Y - BX\|_F^2$$

where $\|\cdot\|_F$ is the Frobenius Norm.

VV285 Midterm 2 review

Shen Yueyang

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July 12, 2019

Exam Checklist(Include but not limit to)

- Infinite dimensional space restriction(closed and bounded not compact)
- Closed, Open, Continuity, Compact
- Norm inequality/ Equivalence of all Norms
- Jacobian

Exam Checklist(Include but not limit to)

- Curve. ▶ Tangent line at a point ▶ Unit tangent vector ▶ Unit normal vector ▶ Open curve length ▶ Curve length to t ▶ Curvature ⋄ A general result and a example in $\mathbb{R}^n$
- Calculate the tangent plane.
- Calculate the derivatives(Directional, manipulating abstract elements,).(cf. Sample exam Exercise 2. Assignment 6.1.)
- Calculate the tangent plane
- Hessian, Extrema, Definiteness
- Find the Extrema
- Implicit Function Theorem
- Bilinear Forms. Multilinear maps.
- Lagrange's Multiplier Rule
- The conclusions on the homework are also important. Don't be overwhelmed by the process of doing it.

Overview

Determinants, Closed, Open

- The set of all invertible matrices(detA$\neq$0) is a open set. To discuss the properties
$$detA = \sum_{\pi \in S_n} sgn\pi \, a_{\pi(1)1} \ldots a_{\pi(n)n}$$
might be more useful.
- The set $\{A : \text{detA}{=}1\}$ is closed.
- The set for all invertible matrices is open.
- The determinant is a continuous function. Further, it is continuously differentiable.

Matrix Calculus

- Derivatives.
  Useful results:
  ▶ $(Ddet)|_A$H =detA $\cdot$tr(A^{-1}H) for invertible matrices, tr(A$^{\#}$H) for non invertible matrices.
  ▶ $D(\cdot)^{-1}|_A$H = -A^{-1}HA^{-1}
  ▶ $Dtr|_A$H =trH. More general if f is a linear map, then Df$|_A$H=f $\circ$ H
  ▶ $D(\cdot)$H$|_A$J = JH
- If at the (n-1) times derivative, the result is independent of the point(or matrix) it acting on. Namely, D^{n-1}f$|_A[H_1, H_2, ..., H_n]$ does not involve A. Then the D^{n}f$|_A$=0(Any element maps into zero).
  ▶The second derivative for a linear map maps any element to zero.
  D^{2}L$|_x$ = 0

Calculating the derivatives

- It is important to utilize the product rule, chain rule in terms of calculating the derivatives.

  ▶ Cross product is a generalized product, T, B, N are orthonormal. $\frac{d(T \times N)}{ds} = \frac{dT}{ds} \times N + \frac{dN}{ds} \times T$

  ▶ When you come cross calculating the second derivative of the determinant.

$$Dtr(A^{-1}H) = Dtr(\cdot)|_{A^{-1}H} \circ D(\cdot)H|_{A^{-1}} \circ D(\cdot)^{-1}|_A$$

Fill in some details

The second derivative for a linear map exists and is equal to zero.

Proof.

Generally, a linear map suffice the following

$$L(x+h) = Lx + Lh.$$

This is essentially saying $\mathrm{DL}|_x\mathrm{h}=\mathrm{Lh}$. Since this derivative is independent of the point/matrix(x), the second derivative is a map that takes any element and always maps into zero.

$$DL|_{x+J}h = Lh$$

Recalling f(x+h) = f(x) +$\mathrm{Df}|_x$h + o(h), in this case

$$DL|_{x+J}h = DL|_x h + (D^2L|_x[J,h]) + o(h)$$

Therefore,$\mathrm{D}^2\mathrm{L}|_x[h, J] = 0$ □

Differentiability

Directional derivative, differentiablility, Partial derivative

- Even though all directional derivatives exists. It is possible that the function is still not differentiable. (cf. Exercise 8.7)
- Note that the directional derivative is a number, in contradistinction to the derivative.
- Even though all the partial derivatives exists, it does not guarantees that the directional derivative exists. If you want to expand the directional derivative as a combination of partial derivatives, you have to check the partial derivatives exists in the first place.(cf. Exercise 6.2)

Differentiability

- If a function is differentiable. Then all directional derivatives exists. Further, the directional derivative can be expressed in terms of $\langle \nabla f, h \rangle$.
- If a function is differentiable then all partial derivatives should also exist
- If f is twice continuously differentiable, the order of differentiation doesn't matter.

Differentiating Under the Integral

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt.$$

Differentiating Under The Integral-

Calculate the integral

$$\int_0^\infty x^n e^{-x}\, dx$$

Let

$$F(t) = \int_0^\infty e^{-tx}\, dx.$$

The integral is easily evaluated, so that $F(t) = \frac{1}{t}$ for all $t > 0$. Differentiating F with respect to t easily leads to the identity

$$F'(t) = -\int_0^\infty x\, e^{-tx}\, dx = -\frac{1}{t^2}.$$

Taking further derivatives yields

$$\int_0^\infty x^n e^{-tx}\, dx = \frac{n!}{t^{n+1}}$$

which immediately implies the formula

$$n! = \int_0^\infty x^n e^{-x}\, dx.$$

The right hand side is the famous gamma function, and does not depend on n being an integer.

Tangent Plane

The basic idea that underlies within calculating tangent plane is y the following:

1) Reduce one variable using implicit equation.

2) Calculate two independent tangent vectors

$$\frac{\partial}{\partial x}(x, y, f(x, y))^T|_{x_0, y_0} = (1, 0, \frac{\partial}{\partial x} f(x_0, y_0))^T$$

$$\frac{\partial}{\partial y}(x, y, f(x, y))^T|_{x_0, y_0} = (0, 1, \frac{\partial}{\partial y} f(x_0, y_0))^T$$

3) Span the vectors and the point we desire

- The normal vector for this plane is basically the cross product of these two independent vectors.

Tangent Plane

Example. The torus is often parametrized by x(θ, Φ) = (R+rsinθ)cosΦ, y(θ, Φ) = (R+rsinθ)sinΦ, z(θ, Φ) = rcosθ, find the tangent plane at the point **r(θ,Φ)**=(x,y,z) = (R,0,r)

Solution

(Based on parametrization)
$\frac{\partial}{\partial\theta}r(\theta,\Phi) = ((R+r\cos\theta)\cos\Phi,\ (R+r\cos\theta)\sin\Phi,\ -r\sin\theta)$
$\frac{\partial}{\partial\Phi}r(\theta,\Phi) = (-(R+r\sin\theta)\sin\Phi,\ (R+r\sin\theta)\cos\Phi,\ 0)$
$\frac{\partial}{\partial\theta}r(0,0) = (0,R,0)$
$\frac{\partial}{\partial\Phi}r(0,0) = (r,0,0)$
The normal vector is $\frac{\partial}{\partial\theta}r(0,0)\times\ \frac{\partial}{\partial\Phi}r(0,0)=(0,0,-Rr)$. *The tangent plane is then* $0\cdot(x-R)+0\cdot(y-0)+(-Rr)(x-r)=0$

A short note on Positive Definiteness

Recall the definition on the slide, that a positive definite matrix A suffice that

$$Q_A(x) = \langle x, Ax \rangle > 0$$

Another common definition for positive definite matrix is such that it can be written in the form of RR^T.
Note the similarity between these two definitions, since

$$\langle x, Ax \rangle = x^T Ax = x^T R^T Rx = \|Rx\|^2 >= 0$$

for any x.

Hessian, Positive definiteness, maxima, minima

Extrema of Real Functions

2.6.12. Theorem. Let $\Omega \subset \mathbb{R}^n$ be open, $f \in C^2(\Omega)$ and $\xi \in \Omega$. Let $\nabla f(\xi) = 0$ (i.e., $Df|_\xi = 0$).

(i) If $\operatorname{Hess} f|_\xi$ is positive definite, f has a strict local minimum at ξ.

(ii) If $\operatorname{Hess} f|_\xi$ is negative definite, f has a strict local maximum at ξ.

(iii) If $\operatorname{Hess} f|_\xi$ is indefinite, f has no extremum at ξ.

Example. F(x,y) $=x^3$-y^3+9xy has DF(x,y)=0 when (x,y)=(0,0) and (x,y) = (3,-3). We calculate the hessian $\begin{pmatrix} 6x & 9 \\ 9 & -6y \end{pmatrix}$

Hessian, Positive definiteness, maxima, minima

At (x,y) =(0,0) then the Hessian is calculated as $D^2F(x)=\begin{pmatrix} 0 & 9 \\ 9 & 0 \end{pmatrix}$

At(x,y) =(3,-3), then the Hessian is calculated as $D^2F(x)=\begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix}$

Recall in the assignment

Let $A \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ be symmetric, i.e.,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Let $\Delta = \det A$. Prove directly that

i) A is positive definite $\Leftrightarrow a > 0$ and $\Delta > 0$

ii) A is negative definite $\Leftrightarrow a < 0$ and $\Delta > 0$

iii) A is indefinite $\Leftrightarrow \Delta < 0$

At (0,0). It is indefinite, so it is neither a maximum or minimum. At(3,-3). It is positive definite, so it is a strict local minimum.

Lagrange's Multiplier Rule

Find the maximum and minimum for the function f (x,y,z) = x^2 - y^2 on the surface of $x^2 + 2y^2 + 3z^2$=1

Solution

$\Phi(x, y, z,\lambda)$ =$x^2 - y^2 + \lambda(x^2 + 2y^2 + 3z^2 - 1)$
1) Calculate the partial derivatives respect to x, y, z and λ, a)2x+2λx = 0, b)-2y+4λy=0, c)6λz=0, d)$x^2 + 2y^2 + 3z^2 - 1 = 0$

A. When $\lambda = 0$

Plug back in the four equations we have two critical points. Namely, (0,0,-$\frac{1}{\sqrt{3}}$), (0,0,$\frac{1}{\sqrt{3}}$).

B. When $\lambda \neq 0$

a) When x$\neq$ 0 Solving the equations, we get two critical points (1,0,0) and (-1,0,0)

Lagrange's Multiplier Rule

Find the maximum and minimum for the function f (x,y,z) = x^2 - y^2 on the surface of $x^2 + 2y^2 + 3z^2$=1

Solution

(cont'd)

b) When x=0 In a similar manner, we can get two critical points $(0, -\frac{1}{\sqrt{2}}, 0)$ *and* $(0, -\frac{1}{\sqrt{2}}, 0)$

Now we plug back in all the critical points we have thus far obtained.

$f(0,0,\frac{1}{\sqrt{3}})=f(0,0,-\frac{1}{\sqrt{3}})=0$

$f(0,\frac{1}{\sqrt{2}},0)=f(0,-\frac{1}{\sqrt{2}},0) = -\frac{1}{2}$

$f(1,0,0) = f(-1,0,0) = 1$

To conclude, (1, 0, 0) and (1, 0, 0) are maxima and $(0, -\frac{1}{\sqrt{2}}, 0)$ *and* $(0, \frac{1}{\sqrt{2}}, 0)$ *are minima. It can be shown that* $(0, 0, \frac{1}{\sqrt{3}})$ *and* $(0, 0, \frac{1}{\sqrt{3}})$ *are saddle points.*

Lagrange's Multiplier Rule

The Lagrange's Multiplier rule itself is not too difficult, what may be tricky is to discuss all possible circumstances carefully and calculate all the possible scenarios.

VV285 RC Week9

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July 25, 2019

Some General Comments

As a convention, I would like to say few words about this part.
In general, part III involves certain amount of calculation. It will be a plus if you understand gain an intuitive perspective toward the theorems like Stokes and Gauss. The contents are developed based on two important concepts circulation and flux, and all the rest are more or less based upon that.
The last part is a fast paced session, but as long as you gain a intuitive understanding the rest is getting some practice then!
As for suggestions, if any, would be practice, practice, and practice!
Lastly, some of the examples in the exercises were borrowed from Chen Xiwen's slides.

Contents in Part III

- Vector Fields and Line Integrals
- Circulation and Flux
- Integration in Practice
- Surfaces and Surface Integrals
- The Theorems of Gauss and Stokes

Overview

Calculate, calculate, and calculate

The line integral for a potential function

3.1.1. Definition. Let $\Omega \subset \mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma\colon I \to \mathcal{C}$. We then define the ***line integral of the potential f along $\mathcal{C}^*$*** by

$$\int_{\mathcal{C}^*} f \, ds := \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)| \, dt$$

The line integral for a vector field

3.1.8. Definition. Let $\Omega \subset \mathbb{R}^n$, $F\colon \Omega \to \mathbb{R}$ be a continuous vector field and $\mathcal{C}^* \subset \Omega$ an oriented open, smooth curve in $\mathbb{R}^n$. We then define the ***line integral of the vector field F along $\mathcal{C}^*$*** by

$$\int_{\mathcal{C}^*} F \, d\vec{s} := \int_{\mathcal{C}^*} \langle F, T \rangle \, ds \qquad (3.1.5)$$

Calculate the mass: A rudimentary example

Example

Suppose a wire is in the shape of a circle, $\mathcal{C}^* : x^2 + y^2 = 1$. The density ρ at a point (x,y) is $\rho(x, y) = 1 + xy$. Calculate the total mass.

Calculate the mass: A rudimentary example

Example

Suppose a wire is in the shape of a circle, $\mathcal{C}^* : x^2 + y^2 = 1$. The density ρ at a point (x,y) is $\rho(x, y) = 1 + xy$. Calculate the total mass.

Solution

It is always important to find out the proper parametrization to facilitate your calculation. In this case, we find $\gamma(\theta) = (cos\theta, sin\theta)$. The total mass is then

$$m = \int_0^{2\pi} (1 + cos\theta \cdot sin\theta) \cdot 1 d\theta = 2\pi$$

Line Integral, Vector Field, Parametrization

Example

In the vector field $\mathbf{F}(x,y) = (x^2y, x\text{-}2y)$and let $\mathcal{C}$ be the curve $y{=}x^2$, running from 0 to 1. Compute the line integral.$\int_{\mathcal{C}} \mathbf{F} dr$

Line Integral, Vector Field, Parametrization

Example

In the vector field $\mathbf{F}$(x,y) $=$ (x^{2}y,x-2y)and let $\mathcal{C}$ be the curve y=x^2, running from (0,0) to (1,1). Compute the line integral.$\int_{\mathcal{C}} \mathbf{F} dr$

Solution

In this case, we choose the parametrization $\gamma(t)$=(t,t^2).$\gamma^{'}(t) = (1, 2t)$
The vector field is then $\mathbf{F}$(x,y)=$(t^4, t - 2t^2)$. Plugging everything in,

$$\int_{\mathcal{C}} \mathbf{F} dr = \int_0^1 (t^4, t - 2t^2) \cdot (1, 2t) dt = -\frac{2}{15}$$

Some Comments

- The line integral is independent on the parametrization of the curve.(You can check so by redoing the previous example using the parametrization $\gamma^{'}(t) = (sint, sin^2(t)))$, where t ranges from 0 to $\frac{\pi}{2}$
- $$\int_{\mathcal{C}*} Fd\bar{s} = \int_{\mathcal{C}*} \langle F, T\rangle ds = \int_{\mathcal{C}*} \langle F, d\bar{s}\rangle$$
- It is equally important to note that d$\bar{s}$=$\gamma^{'}(t)dt$

Definitions

- Flux: The concept that describes the "amount" of vector field that flows (out) through the boundary. Perpendicular

$$\int_{\mathbb{C}^*} \langle F, N \rangle ds$$

- Circulation: The concept that measures the tangential component of all the whole boundary. Parallel

$$\int_{\mathbb{C}^*} \langle F, T \rangle ds$$

Conservative fields, Potential Fields

- A vector field is said to be a potential field if there exists a differentiable potential function U.

$$F(x) = \nabla U(x)$$

  ▶ Every line integral along a closed curve for a potential field yields zero.

- The definition for conservation fields is essentially: The integral along **any open curve** depends only on the initial points and the end points.

  ▶ Every potential field is a conservative field.

  ▶ Every continuous, conservative field on a connected open set is a potential field.

  ▶ Continuous and conservative $\Longrightarrow$ potential field

Potential Fields, Simply Connected Set

- Pathwise Connected.
  For any two points in the set Ω, there exists an open curev that connects these two points

3.1.18. Lemma. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F : \Omega \to \mathbb{R}^n$ is continuously differentiable. Then F is a potential field only if for all $i, j = 1, \ldots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}. \tag{3.1.9}$$

Potential Fields, Conservative fields, Simply connected Set

3.1.21. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a ***simply connected*** open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

then F is a potential field.

We will not have time prove this result here. However, we do need to explain what a "simply connected" set is.

Loosely speaking, a set $\Omega \subset \mathbb{R}^n$ is said to be simply connected if

(i) Ω is pathwise connected and

(ii) every closed curve in Ω can be contracted to a single point within Ω.

Conservative Fields

Example

Denote $\mathbb{H} = (x, y) : y > 0 \subset \mathbb{R}^2$ the upper hals-space of $\mathbb{R}^2$ and consider the two vector fields F,G: $\mathbb{H} \to \mathbb{R}^2$ with (x,y)$\in \mathbb{H}$

$$F(x, y) = (4x^2 + 4y^2, 8xy - lny), \quad G(x, y) = (x + xy, -xy)$$

1. Which of the two fields is(are) conservative?
2. Calculate the potential function for the conservative field.

Solution

.

- We first calculate the partial derivatives.
- check if it is well defined for all (x,y)

Conservative Fields

Example

Denote $\mathbb{H} = (x, y) : y > 0 \subset \mathbb{R}^2$ the upper hals-space of $\mathbb{R}^2$ and consider the two vector fields F,G: $\mathbb{H} \to \mathbb{R}^2$ with (x,y)$\in \mathbb{H}$

$$F(x, y) = (4x^2 + 4y^2, 8xy - lny), \quad G(x, y) = (x + xy, -xy)$$

1. Which of the two fields is(are) conservative?
2. Calculate the potential function for the conservative field.

Solution

. a)

$$\frac{\partial F_1}{\partial y} = 8y, \frac{\partial F_2}{\partial x} = 8y, \frac{\partial G_1}{\partial y} = x, \frac{\partial G_2}{\partial x} = -y$$

G cannot be conservative. Since F is defined on a simply connected set, and all(x,y)'s partial derivatives are defined. Therefore, it is conservative.

Conservative Fields

Example

$$F(x,y) = (4x^2 + 4y^2, 8xy - lny), \quad G(x,y) = (x + xy, -xy)$$

2. Calculate the potential function for the conservative field.

Solution

b) Integrate with respect to x, and y and compare the two results

$$\phi(x,y) = \int F_1(x,y)dx = \frac{4}{3}x^3 + 4y^2x + C_1(y)$$

$$\phi(x,y) = \int F_2(x,y)dy = 4xy^2 - ylny + y + C_2(x)$$

Therefore, the potential is

$$\phi(x,y) = \frac{4}{3}x^3 + 4y^2x - ylny + y$$

A special example on conservative fields

Example

Is $\frac{(-y,x)}{x^2+y^2}$ conservative

A special example on conservative fields

Example

Is $\frac{(-y,x)}{x^2+y^2}$ conservative

Solution

BY convention, we first calculate the partial derivatives. Namely,

$$M_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} = N_x$$

However, this is not properly defined for all (x,y). Therefore, we examine a closed loop in this case. $\mathcal{C}^*$: is the unit circle with the parametrization $\gamma(t) = (cost, sint)$. dx=-sintdt, dy = costdt

$$\int_{\mathcal{C}^*} Fd\bar{r} = \oint_C \frac{-y}{(x^2 + y^2)^2} dx + \frac{x}{(x^2 + y^2)^2} dy = \int_0^{2\pi} dt = 2\pi$$

Definitions

- Divergence. Let $\Omega \subset \mathbb{R}^n$ be open and F: $\Omega \to \mathbb{R}^n$ continuously differentiable. Then,

$$divF := \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + ... + \frac{\partial F_n}{\partial x_n}$$

  - This concept characterize the "source density" at a point x.
- Rotation. Let $\Omega \subset \mathbb{R}^n$ be open and F: $\Omega \to \mathbb{R}^n$. The antisymmetric, bilinear form

$$rot F|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, rotF|_x := \langle DF|_x u, v \rangle - \langle DF|_x v, u \rangle$$

  - Rotation quantifies the circulation density.

Rotation in $\mathbb{R}^2$

See board

Some intuition about divergence theroem and Stokes

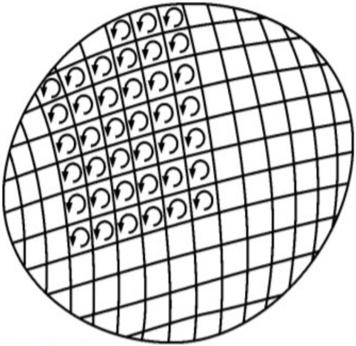
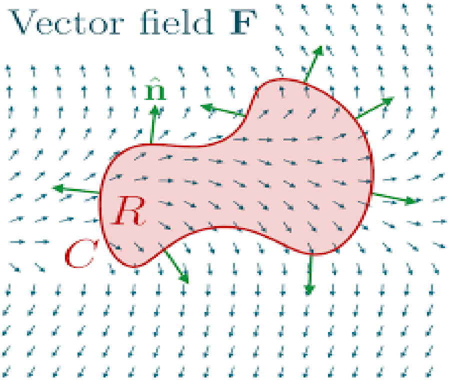


Illustration of Stokes's Theorem. Wikimedia Commons. Wikimedia Foundation. Web. 28 July 2012



VV285 RC Week9

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July 31, 2019

Some General Comments

To help you prepare for the exam, I have prepared some problems at also attached some materials at the end of this slide. In general, you should feel comfortable in terms of interchanging variables(always note how to calculate the jacobian term). Apart from that, polar coordinates, cylindrical coordinates, spherical coordinates are three special scenarios(or methods in terms of changing variables).

At the end of this slide, there are additional materials containing some basic examples and exercises to help you understand the slides. I found them pretty useful when I was preparing for the exams myself. Do not get overwhelmed though, try your best to understand the slides in the first place and then try to apply what you have learnt in the slides to solve the problems will be optimal.

Overview

1. Improper Integrals
   - Exercises
2. Additional Materials

Euler-Poisson Integral

We take a slightly different approach than that in the slides. We discuss the

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Similar to the spirits in the slides,

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2} dx)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

This degenerates into a integration in two dimension. Specifically, we have

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

Euler-Poisson Integral

With the determinant for Jacobian(Cartesian to polar) being r, we thereby have(a bit of transformation on the integration range as well)

$$I^2 = \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drd\theta$$

$$2\pi \int_0^R re^{-r^2} dr = \pi \quad as \quad R \to \infty$$

Therefore,

$$I = \sqrt{\pi}$$

- A useful result follows immediate from interchanging variables.

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Calculate the integral

Example

Calculate

$$\int_0^\infty e^{-ax^2} cosbx \quad dx$$

Calculate the integral

Example

Calculate
$$\int_0^\infty e^{-ax^2} cosbx \quad dx$$

More generally, we consider the function I(b). Building on what we have before, we already see the fact that we have

$$I(0) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Differentiating under the integral, we have

$$I'(b) = \int_0^\infty (e^{-ax^2 cosbx})' dx = \int_0^\infty (-xe^{-ax^2} sinbx) dx$$

$$I'(b) = \frac{1}{2a} e^{-ax^2} sinbx|_0^\infty - \frac{b}{2a}\int_0^\infty e^{-ax^2} cosbxdx = \frac{-b}{2a} I(b)$$

Calculate the Integral

Essentially, we need to solve the following differential equation.

$$I'(b) = \frac{-b}{2a} I(b)$$

The solution being

$$I = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-\frac{b^2}{4a}}$$

Other Examples

See board

44 Show that the spin field **S** does work around every simple closed curve.

45 For $\mathbf{F} = f(x)\mathbf{j}$ and $R =$ unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, integrate both sides of Green's Theorem (1). What formula is required from one-variable calculus?

46 A region R is "*simply connected*" when every closed curve inside R can be squeezed to a point without leaving R. Test these regions:

1. xy plane without $(0, 0)$	2. xyz space without $(0, 0, 0)$
3. sphere $x^2 + y^2 + z^2 = 1$	4. a torus (or doughnut)
5. a sweater	6. a human body
7. the region between two spheres	
8. xyz space with circle removed.	

15.4 Surface Integrals

The double integral in Green's Theorem is over a flat surface R. Now the region moves out of the plane. It becomes a ***curved surface*** S, part of a sphere or cylinder or cone. When the surface has only one z for each (x, y), it is the graph of a function $z(x, y)$. In other cases S can twist and close up—a sphere has an upper z and a lower z. In all cases we want to compute area and flux. This is a necessary step (it is our last step) before moving Green's Theorem to three dimensions.

First a quick review. The basic integrals are $\int dx$ and $\iint dx\,dy$ and $\iiint dx\,dy\,dz$. The one that didn't fit was $\int ds$—the length of a curve. When we go from curves to surfaces, ds becomes dS. ***Area is*** $\iint dS$ ***and flux is*** $\iint \mathbf{F} \cdot \mathbf{n}\, dS$, with double integrals because the surfaces are two-dimensional. The main difficulty is in dS.

All formulas are summarized in a table at the end of the section.

There are two ways to deal with ds (along curves). The same methods apply to dS (on surfaces). The first is in xyz coordinates; the second uses parameters. Before this subject gets complicated, I will explain those two methods.

Method 1 is for the graph of a function: curve $y(x)$ or surface $z(x, y)$.

A small piece of the curve is almost straight. It goes across by dx and up by dy:

$$\text{length } ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\, dx. \tag{1}$$

A small piece of the surface is practically flat. Think of a tiny sloping rectangle. One side goes across by dx and up by $(\partial z/\partial x)dx$. The neighboring side goes along by dy and up by $(\partial z/\partial y)dy$. Computing the area is a linear problem (from Chapter 11), because the flat piece is in a plane.

Two vectors **A** and **B** form a parallelogram. ***The length of their cross product is the area.*** In the present case, the vectors are $\mathbf{A} = \mathbf{i} + (\partial z/\partial x)\mathbf{k}$ and $\mathbf{B} = \mathbf{j} + (\partial z/\partial y)\mathbf{k}$. Then $\mathbf{A}dx$ and $\mathbf{B}dy$ are the sides of the small piece, and we compute $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial z/\partial x \\ 0 & 1 & \partial z/\partial y \end{vmatrix} = -\partial z/\partial x\,\mathbf{i} - \partial z/\partial y\,\mathbf{j} + \mathbf{k}. \tag{2}$$

This is exactly the ***normal vector*** **N** to the tangent plane and the surface, from Chapter 13. Please note: The small flat piece is actually a parallelogram (not always a rectangle). Its area dS is much like ds, but the length of $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ involves two derivatives:

$$\text{area } dS = |\mathbf{A}dx \times \mathbf{B}dy| = |\mathbf{N}|dx\,dy = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,dx\,dy. \tag{3}$$

EXAMPLE 1 Find the area on the plane $z = x + 2y$ above a base area A.

This is the example to visualize. The area down in the xy plane is A. The area up on the sloping plane is greater than A. A roof has more area than the room underneath it. If the roof goes up at a 45° angle, the ratio is $\sqrt{2}$. Formula (3) yields the correct ratio for any surface—including our plane $z = x + 2y$.

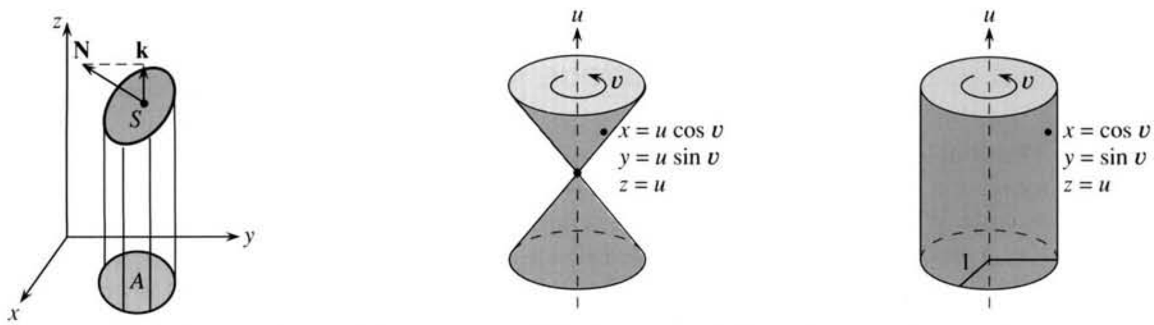


Fig. 15.14 Roof area = base area times $|\mathbf{N}|$. Cone and cylinder with parameters u and v.

The derivatives are $\partial z/\partial x = 1$ and $\partial z/\partial y = 2$. They are constant (planes are easy). The square root in (3) contains $1 + 1^2 + 2^2 = 6$. Therefore $dS = \sqrt{6}\,dx\,dy$. An area in the xy plane is multiplied by $\sqrt{6}$ up in the surface (Figure 15.14a). The vectors $\mathbf{A}$ and $\mathbf{B}$ are no longer needed—their work was done when we reached formula (3)—but here they are:

$$\mathbf{A} = \mathbf{i} + (\partial z/\partial x)\mathbf{k} = \mathbf{i} + \mathbf{k} \quad \mathbf{B} = \mathbf{j} + (\partial z/\partial y)\mathbf{k} = \mathbf{j} + 2\mathbf{k} \quad \mathbf{N} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The length of $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ is $\sqrt{6}$. The angle between $\mathbf{k}$ and $\mathbf{N}$ has $\cos\theta = 1/\sqrt{6}$. ***That is the angle between base plane and sloping plane.*** Therefore the sloping area is $\sqrt{6}$ times the base area. For curved surfaces the idea is the same, except that the square root in $|\mathbf{N}| = 1/\cos\theta$ changes as we move around the surface.

Method 2 is for curves $x(t), y(t)$ ***and surfaces*** $x(u, v), y(u, v), z(u, v)$ ***with parameters.***

A curve has one parameter t. A surface has two parameters u and v (it is two-dimensional). One advantage of parameters is that x, y, z get equal treatment, instead of picking out z as $f(x, y)$. Here are the first two examples:

$$\textbf{\textit{cone}}\ x = u\cos v,\ y = u\sin v,\ z = u \qquad \textbf{\textit{cylinder}}\ x = \cos v,\ y = \sin v,\ z = u. \tag{4}$$

Each choice of u and v gives a point on the surface. By making all choices, we get the complete surface. Notice that a parameter can equal a coordinate, as in $z = u$. Sometimes both parameters are coordinates, as in $x = u$ and $y = v$ and $z = f(u, v)$. That is just $z = f(x, y)$ in disguise—the surface without parameters. In other cases ***we find the*** xyz ***equation by eliminating*** u ***and*** v:

$$\textbf{\textit{cone}} \quad (u\cos v)^2 + (u\sin v)^2 = u^2 \quad \text{or} \quad x^2 + y^2 = z^2 \quad \text{or} \quad z = \sqrt{x^2 + y^2}$$

$$\textbf{\textit{cylinder}} \quad (\cos v)^2 + (\sin v)^2 = 1 \quad \text{or} \quad x^2 + y^2 = 1.$$

The cone is the graph of $f = \sqrt{x^2 + y^2}$. The cylinder is *not* the graph of any function. There is a line of z's through each point on the circle $x^2 + y^2 = 1$. That is what $z = u$ tells us: Give u all values, and you get the whole line. Give u and v all values, and you get the whole cylinder. Parameters allow a surface to close up and even go through itself—which the graph of $f(x, y)$ can never do.

Actually $z = \sqrt{x^2 + y^2}$ gives only the top half of the cone. (A function produces only one z.) The parametric form gives the bottom half also. Similarly $y = \sqrt{1 - x^2}$ gives only the top of a circle, while $x = \cos t$, $y = \sin t$ goes all the way around.

Now we find dS, using parameters. Small movements give a piece of the surface, practically flat. One side comes from the change du, the neighboring side comes from dv. The two sides are given by small vectors $\mathbf{A}du$ and $\mathbf{B}dv$:

$$\mathbf{A} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}. \tag{5}$$

To find the area dS of the parallelogram, start with the cross product $\mathbf{N} = \mathbf{A} \times \mathbf{B}$:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}. \tag{6}$$

Admittedly this looks complicated—actual examples are often fairly simple. The area dS of the small piece of surface is $|\mathbf{N}|\,du\,dv$. The length $|\mathbf{N}|$ is a square root:

$$dS = \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^2}\,du\,dv. \tag{7}$$

EXAMPLE 2 Find $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ and dS for the cone and cylinder.

The cone has $x = u \cos v$, $y = u \sin v$, $z = u$. The u derivatives produce $\mathbf{A} = \partial\mathbf{R}/\partial u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}$. The v derivatives produce the other tangent vector $\mathbf{B} = \partial\mathbf{R}/\partial v = -u \sin v\,\mathbf{i} + u \cos v\,\mathbf{j}$. The normal vector is $\mathbf{A} \times \mathbf{B} = -u \cos v\,\mathbf{i} - u \sin v\,\mathbf{j} + u\,\mathbf{k}$. Its length gives dS:

$$dS = |\mathbf{A} \times \mathbf{B}|\,du\,dv = \sqrt{(u \cos v)^2 + (u \sin v)^2 + u^2}\,du\,dv = \sqrt{2}\,u\,du\,dv.$$

The cylinder is even simpler: $dS = du\,dv$. In these and many other examples, $\mathbf{A}$ is perpendicular to $\mathbf{B}$. ***The small piece is a rectangle.*** Its sides have length $|\mathbf{A}|\,du$ and $|\mathbf{B}|dv$. (The cone has $|\mathbf{A}| = u$ and $|\mathbf{B}| = \sqrt{2}$, the cylinder has $|\mathbf{A}| = |\mathbf{B}| = 1$). The cross product is hardly needed for area, when we can just multiply $|\mathbf{A}|\,du$ times $|\mathbf{B}|dv$.

Remark on the two methods Method 1 also used parameters, but a very special choice—u is x and v is y. The parametric equations are $x = x$, $y = y$, $z = f(x, y)$. If you go through the long square root in (7), changing u to x and v to y, it simplifies to the square root in (3). (The terms $\partial y/\partial x$ and $\partial x/\partial y$ are zero; $\partial x/\partial x$ and $\partial y/\partial y$ are 1.) Still it pays to remember the shorter formula from Method 1.

Don't forget that after computing dS, you have to integrate it. Many times the good way is with polar coordinates. Surfaces are often symmetric around an axis or a point. Those are the *surfaces of revolution*—which we saw in Chapter 8 and will come back to.

Strictly speaking, the integral starts with ΔS (not dS). A flat piece has area $|\mathbf{A} \times \mathbf{B}|\Delta x \Delta y$ or $|\mathbf{A} \times \mathbf{B}|\Delta u \Delta v$. The area of a curved surface is properly defined as a limit. The key step of calculus, from sums of ΔS to the integral of dS, is safe for smooth surfaces. In examples, the hard part is computing the double integral and substituting the limits on x, y or u, v.

EXAMPLE 3 Find the surface area of the cone $z=\sqrt{x^2+y^2}$ up to the height $z=a$.

We use Method 1 (no parameters). The derivatives of z are computed, squared, and added:

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}} \qquad |\mathbf{N}|^2=1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=2.$$

Conclusion: $|\mathbf{N}|=\sqrt{2}$ and $dS=\sqrt{2}\,dx\,dy$. The cone is on a 45° slope, so the area $dx\,dy$ in the base is multiplied by $\sqrt{2}$ in the surface above it (Figure 15.15). The square root in dS accounts for the extra area due to slope. A horizontal surface has $dS=\sqrt{1}\,dx\,dy$, as we have known all year.

Now for a key point. ***The integration is down in the base plane.*** The limits on x and y are given by the "*shadow*" of the cone. To locate that shadow set $z=\sqrt{x^2+y^2}$ equal to $z=a$. The plane cuts the cone at the circle $x^2+y^2=a^2$. We integrate over the inside of that circle (where the shadow is):

$$\text{surface area of cone}=\iint_{\text{shadow}}\sqrt{2}\,dx\,dy=\sqrt{2}\,\pi a^2.$$

EXAMPLE 4 Find the same area using $dS=\sqrt{2}\,u\,du\,dv$ from Example 2.

With parameters, dS looks different and the shadow in the base looks different. The circle $x^2+y^2=a^2$ becomes $u^2\cos^2 v+u^2\sin^2 v=a^2$. In other words $u=a$. (The cone has $z=u$, the plane has $z=a$, they meet when $u=a$.) The angle parameter v goes from 0 to 2π. The effect of these parameters is to switch us "automatically" to polar coordinates, where area is $r\,dr\,d\theta$:

$$\text{surface area of cone}=\iint dS=\int_0^{2\pi}\int_0^a\sqrt{2}\,u\,du\,dv=\sqrt{2}\,\pi a^2.$$

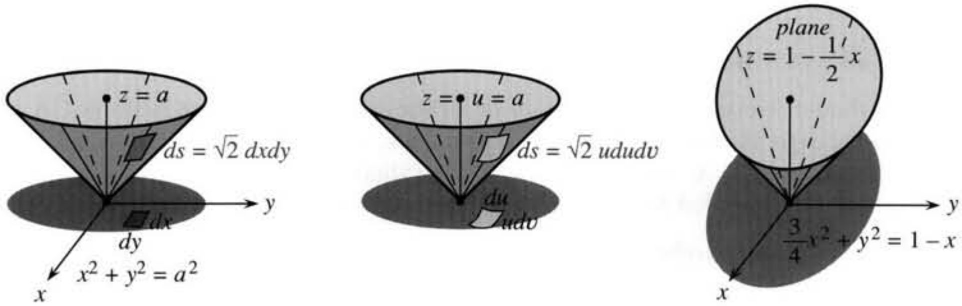


Fig. 15.15 Cone cut by plane leaves shadow in the base. Integrate over the shadow.

EXAMPLE 5 Find the area of the same cone up to the sloping plane $z=1-\frac{1}{2}x$.

Solution The cone still has $dS=\sqrt{2}\,dx\,dy$, but the limits of integration are changed. The plane cuts the cone in an ellipse. Its shadow down in the xy plane is another ellipse (Figure 15.15c). ***To find the edge of the shadow, set*** $z=\sqrt{x^2+y^2}$ ***equal to*** $z=1-\frac{1}{2}x$. We square both sides:

$$x^2+y^2=1-x+\tfrac{1}{4}x^2 \qquad \text{or} \qquad \tfrac{3}{4}(x+\tfrac{2}{3})^2+y^2=\tfrac{4}{3}.$$

This is the ellipse in the base—where height makes no difference and z is gone. The area of an ellipse is πab, when the equation is in the form $(x/a)^2 + (y/b)^2 = 1$. After multiplying by 3/4 we find $a = 4/3$ and $b = \sqrt{4/3}$. Then $\iint \sqrt{2}\, dx\, dy = \sqrt{2}\, \pi ab$ is the surface area of the cone.

The hard part was finding the shadow ellipse (I went quickly). Its area πab came from Example 15.3.2. The new part is $\sqrt{2}$ from the slope.

EXAMPLE 6 Find the surface area of a sphere of radius a (known to be $4\pi a^2$).

This is a good example, because both methods almost work. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$. Method 1 writes $z = \sqrt{a^2 - x^2 - y^2}$. The x and y derivatives are $-x/z$ and $-y/z$:

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{z^2}{z^2} + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{a^2}{z^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

The square root gives $dS = a\, dx\, dy/\sqrt{a^2 - x^2 - y^2}$. Notice that z is gone (as it should be). Now integrate dS over the shadow of the sphere, which is a circle. Instead of $dx\, dy$, switch to polar coordinates and $r\, dr\, d\theta$:

$$\iint_{\text{shadow}} dS = \int_0^{2\pi} \int_0^a \frac{a\, r\, dr\, d\theta}{\sqrt{a^2 - r^2}} = -2\pi a \sqrt{a^2 - r^2}\,\Big]_0^a = 2\pi a^2. \tag{8}$$

This calculation is successful but wrong. $2\pi a^2$ is the area of the *half-sphere* above the xy plane. The lower half takes the negative square root of $z^2 = a^2 - x^2 - y^2$. This shows the danger of Method 1, when the surface is not the graph of a function.

EXAMPLE 7 (same sphere by Method 2: *use parameters*) The natural choice is spherical coordinates. Every point has an angle $u = \phi$ down from the North Pole and an angle $v = \theta$ around the equator. The xyz coordinates from Section 14.4 are $x = a \sin\phi \cos\theta$, $y = a \sin\phi \sin\theta$, $z = a\cos\phi$. The radius $\rho = a$ is fixed (not a parameter). Compute the first term in equation (6), noting $\partial z/\partial\theta = 0$:

$$(\partial y/\partial\phi)(\partial z/\partial\theta) - (\partial z/\partial\phi)(\partial y/\partial\theta) = -(-a\sin\phi)(a\sin\phi\cos\theta) = a^2\sin^2\phi\cos\theta.$$

The other terms in (6) are $a^2\sin^2\phi\sin\theta$ and $a^2\sin\phi\cos\phi$. Then dS in equation (7) squares these three components and adds. We factor out a^4 and simplify:

$$a^4(\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi) = a^4(\sin^4\phi + \sin^2\phi\cos^2\phi) = a^4\sin^2\phi.$$

Conclusion: $dS = a^2 \sin\phi\, d\phi\, d\theta$. A spherical person will recognize this immediately. It is the volume element $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, except $d\rho$ is missing. The small box has area dS and thickness $d\rho$ and volume dV. Here we only want dS:

$$\text{area of sphere} = \iint dS = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi\, d\phi\, d\theta = 4\pi a^2. \tag{9}$$

Figure 15.16a shows a small surface with sides $a\, d\phi$ and $a\sin\phi\, d\theta$. Their product is dS. Figure 15.16b goes back to Method 1, where equation (8) gave $dS = (a/z)\, dx\, dy$.

I doubt that you will like Figure 15.16c—and you don't need it. With parameters ϕ and θ, the shadow of the sphere is a rectangle. The equator is the line down the middle, where $\phi = \pi/2$. The height is $z = a\cos\phi$. The area $d\phi\, d\theta$ in the base is the shadow of $dS = a^2\sin\phi\, d\phi\, d\theta$ up in the sphere. Maybe this figure shows what we don't have to know about parameters.

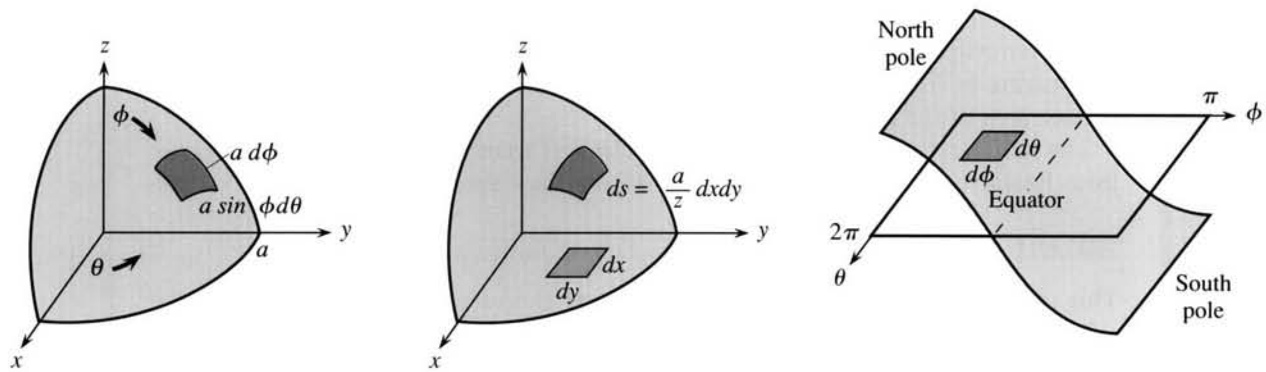


Fig. 15.16 Surface area on a sphere: (a) spherical coordinates (b) xyz coordinates (c) $\phi\theta$ space.

EXAMPLE 8 Rotate $y = x^2$ around the x axis. Find the surface area using parameters.

The first parameter is x (from a to b). The second parameter is the rotation angle θ (from 0 to 2π). The points on the surface in Figure 15.17 are $x = x$, $y = x^2 \cos\theta$, $z = x^2 \sin\theta$. Equation (7) leads after much calculation to $dS = x^2\sqrt{1+4x^2}\,dx\,d\theta$.

Main point: dS agrees with Section 8.3, where the area was $\int 2\pi y\sqrt{1+(dy/dx)^2}\,dx$. The 2π comes from the θ integral and y is x^2. Parameters give this formula automatically.

VECTOR FIELDS AND THE INTEGRAL OF F · n

Formulas for surface area are dominated by square roots. There is a square root in dS, as there was in ds. Areas are like arc lengths, one dimension up. The good point about line integrals $\int \mathbf{F}\cdot\mathbf{n}ds$ is that the square root disappears. It is in the denominator of $\mathbf{n}$, where ds cancels it: $\mathbf{F}\cdot\mathbf{n}ds = M\,dy - N\,dx$. The same good thing will now happen for surface integrals $\iint \mathbf{F}\cdot\mathbf{n}dS$.

> **15I** Through the surface $z = f(x, y)$, the vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ has
>
> $$\textbf{flux} = \iint_{\text{surface}} \mathbf{F}\cdot\mathbf{n}dS = \iint_{\text{shadow}} \left(-M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right)dx\,dy. \qquad (10)$$

This formula tells what to integrate, given the surface and the vector field (f and $\mathbf{F}$). The xy limits come from the shadow. Formula (10) takes the normal vector from Method 1:

$$\mathbf{N} = -\partial f/\partial x\,\mathbf{i} - \partial f/\partial y\,\mathbf{j} + \mathbf{k} \text{ and } |\mathbf{N}| = \sqrt{1+(\partial f/\partial x)^2 + (\partial f/\partial y)^2}.$$

For the *unit* normal vector $\mathbf{n}$, divide $\mathbf{N}$ by its length: $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. The square root is in the denominator, and the same square root is in dS. See equation (3):

$$\mathbf{F}\cdot\mathbf{n}dS = \frac{\mathbf{F}\cdot\mathbf{N}}{\sqrt{}}\sqrt{}\,dx\,dy = \left(-M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right)dx\,dy. \qquad (11)$$

That is formula (10), with cancellation of square roots. The expression $\mathbf{F}\cdot\mathbf{n}dS$ is often written as $\mathbf{F}\cdot d\mathbf{S}$, again relying on boldface to make $d\mathbf{S}$ a vector. Then $d\mathbf{S}$ equals $\mathbf{n}dS$, with direction $\mathbf{n}$ and magnitude dS.

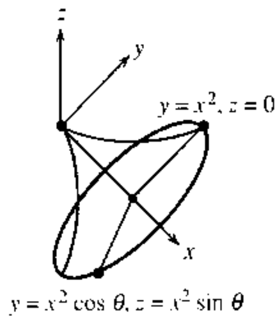


Fig. 15.17 Surface of revolution: parameters x, θ.

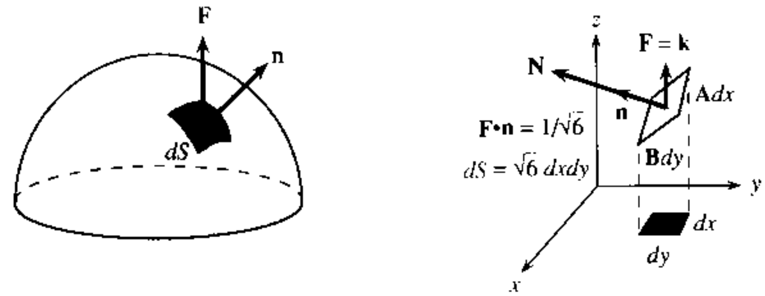


Fig. 15.18 $\mathbf{F}\cdot\mathbf{n}\,dS$ gives flow through dS.

EXAMPLE 9 Find $\mathbf{n}dS$ for the plane $z = x + 2y$. Then find $\mathbf{F}\cdot\mathbf{n}dS$ for $\mathbf{F} = \mathbf{k}$.

This plane produced $\sqrt{6}$ in Example 1 (for area). For flux the $\sqrt{6}$ disappears:

$$\mathbf{n}dS = \frac{\mathbf{N}}{|\mathbf{N}|}dS = \frac{-\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}\sqrt{6}\,dx\,dy = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k})\,dx\,dy.$$

For the flow field $\mathbf{F} = \mathbf{k}$, the dot product $\mathbf{k}\cdot\mathbf{n}dS$ reduces to $1dx\,dy$. The slope of the plane makes no difference! ***The flow through the base also flows through the plane.*** The areas are different, but flux is like rain. Whether it hits a tent or the ground below, it is the same rain (Figure 15.18). In this case $\iint \mathbf{F}\cdot\mathbf{n}dS = \iint dx\,dy =$ shadow area in the base.

EXAMPLE 10 Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the cone $z = \sqrt{x^2 + y^2}$.

Solution $$\mathbf{F}\cdot\mathbf{n}dS = \left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \sqrt{x^2+y^2}\right]dx\,dy = 0.$$

The zero comes as a surprise, but it shouldn't. The cone goes straight out from the origin, and so does $\mathbf{F}$. The vector $\mathbf{n}$ that is perpendicular to the cone is also perpendicular to $\mathbf{F}$. There is no flow *through* the cone, because $\mathbf{F}\cdot\mathbf{n} = 0$. The flow travels out along rays.

$\iint \mathbf{F}\cdot\mathbf{n}dS$ FOR A SURFACE WITH PARAMETERS

In Example 10 the cone was $z = f(x, y) = \sqrt{x^2 + y^2}$. We found dS by Method 1. Parameters were not needed (more exactly, they were x and y). For surfaces that fold and twist, the formulas with u and v look complicated but the actual calculations can be simpler. This was certainly the case for $dS = du\,dv$ on the cylinder.

A small piece of surface has area $dS = |\mathbf{A}\times\mathbf{B}|\,du\,dv$. The vectors along the sides are $\mathbf{A} = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k}$ and $\mathbf{B} = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$. They are tangent to the surface. Now we put their cross product $\mathbf{N} = \mathbf{A}\times\mathbf{B}$ to another use, because $\mathbf{F}\cdot\mathbf{n}dS$ involves not only area but *direction*. We need the unit vector $\mathbf{n}$ to see how much flow goes through.

The direction vector is $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. Equation (7) is $dS = |\mathbf{N}|du\,dv$, so the square root $|\mathbf{N}|$ cancels in $\mathbf{n}dS$. This leaves a nice formula for the "normal component" of flow:

15J Through a surface with parameters u and v, the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ has

$$\text{flux} = \iint \mathbf{F}\cdot\mathbf{n}dS = \iint \mathbf{F}\cdot\mathbf{N}\,du\,dv = \iint \mathbf{F}\cdot(\mathbf{A}\times\mathbf{B})\,du\,dv. \qquad (12)$$

EXAMPLE 11 Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the cylinder $x^2 + y^2 = 1$, $0 \leqslant z \leqslant b$.

Solution The surface of the cylinder is $x = \cos u$, $y = \sin u$, $z = v$. The tangent vectors from (5) are $\mathbf{A} = (-\sin u)\mathbf{i} + (\cos u)\mathbf{j}$ and $\mathbf{B} = \mathbf{k}$. The normal vector in Figure 15.19 goes straight out through the cylinder:

$$\mathbf{N} = \mathbf{A} \times \mathbf{B} = \cos u\, \mathbf{i} + \sin u\, \mathbf{j} \quad (\text{check } \mathbf{A} \cdot \mathbf{N} = 0 \text{ and } \mathbf{B} \cdot \mathbf{N} = 0).$$

To find $\mathbf{F} \cdot \mathbf{N}$, switch $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to the parameters u and v. Then $\mathbf{F} \cdot \mathbf{N} = 1$:

$$\mathbf{F} \cdot \mathbf{N} = (\cos u\, \mathbf{i} + \sin u\, \mathbf{j} + v\, \mathbf{k}) \cdot (\cos u\, \mathbf{i} + \sin u\, \mathbf{j}) = \cos^2 u + \sin^2 u.$$

For the flux, integrate $\mathbf{F} \cdot \mathbf{N} = 1$ and apply the limits on $u = \theta$ and $v = z$:

$$\text{flux} = \int_0^b \int_0^{2\pi} 1\, du\, dv = 2\pi b = \text{surface area of the cylinder.}$$

Note that the top and bottom were not included! We can find those fluxes too. The outward direction is $\mathbf{n} = \mathbf{k}$ at the top and $\mathbf{n} = -\mathbf{k}$ down through the bottom. Then $\mathbf{F} \cdot \mathbf{n}$ is $+z = b$ at the top and $-z = 0$ at the bottom. The bottom flux is zero, the top flux is b times the area (or πb). The total flux is $2\pi b + \pi b = 3\pi b$. Hold that answer for the next section.

Apology: I made u the angle and v the height. Then $\mathbf{N}$ goes outward not inward.

EXAMPLE 12 Find the flux of $\mathbf{F} = \mathbf{k}$ out the top half of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution Use spherical coordinates. Example 7 had $u = \phi$ and $v = \theta$. We found

$$\mathbf{N} = \mathbf{A} \times \mathbf{B} = a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}.$$

The dot product with $\mathbf{F} = \mathbf{k}$ is $\mathbf{F} \cdot \mathbf{N} = a^2 \sin\phi \cos\phi$. The integral goes from the pole to the equator, $\phi = 0$ to $\phi = \pi/2$, and around from $\theta = 0$ to $\theta = 2\pi$:

$$\text{flux} = \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin\phi \cos\phi\, d\phi\, d\theta = 2\pi a^2 \left.\frac{\sin^2\phi}{2}\right]_0^{\pi/2} = \pi a^2.$$

The next section will show that the flux remains at πa^2 through *any surface* (!) that is bounded by the equator. A special case is a flat surface—the disk of radius a at the equator. Figure 15.18 shows $\mathbf{n} = \mathbf{k}$ pointing directly up, so $\mathbf{F} \cdot \mathbf{n} = \mathbf{k} \cdot \mathbf{k} = 1$. The flux is $\iint 1\, dS$ = area of disk $= \pi a^2$. ***All fluid goes past the equator and out through the sphere.***

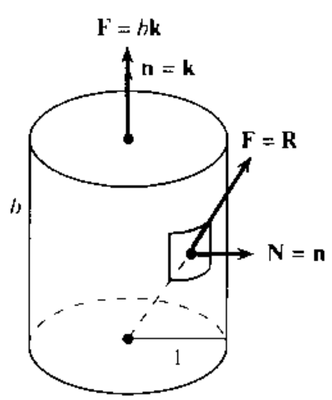


Fig. 15.19 Flow through cylinder.

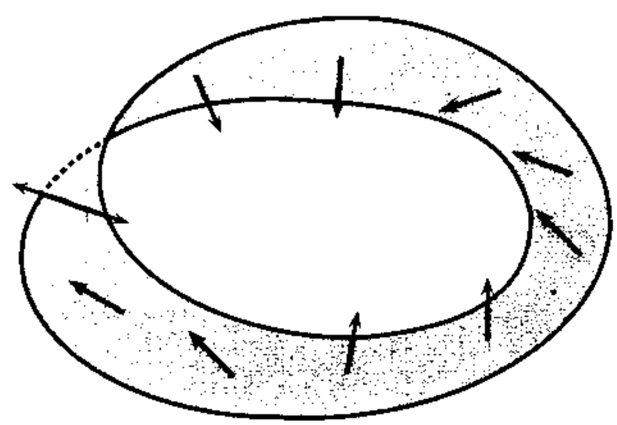


Fig. 15.20 Möbius strip (no way to choose **n**).

I have to mention one more problem. It might not occur to a reasonable person, but sometimes a surface has only *one side*. The famous example is the **Möbius strip**, for which you take a strip of paper, twist it once, and tape the ends together. Its special property appears when you run a pen along the "inside." The pen in Figure 15.20 suddenly goes "outside." After another round trip it goes back "inside." Those words are in quotation marks, because on a Möbius strip they have no meaning.

Suppose the pen represents the normal vector. On a sphere $\mathbf{n}$ points outward. Alternatively $\mathbf{n}$ could point inward; we are free to choose. But the Möbius strip makes the choice impossible. After moving the pen continuously, it comes back in the opposite direction. ***This surface is not orientable.*** We cannot integrate $\mathbf{F}\cdot\mathbf{n}$ to compute the flux, because we cannot decide the direction of $\mathbf{n}$.

A surface is *oriented* when we can and do choose $\mathbf{n}$. This uses the final property of cross products, that they have length and direction and also a *right-hand rule*. We can tell $\mathbf{A}\times\mathbf{B}$ from $\mathbf{B}\times\mathbf{A}$. Those give the two orientations of $\mathbf{n}$. For an open surface (like a wastebasket) you can select either one. For a closed surface (like a sphere) it is conventional for $\mathbf{n}$ to be outward. By making that decision once and for all, the sign of the flux is established: *outward flux is positive.*

FORMULAS FOR SURFACE INTEGRALS

Method 1: Parameters x, y	*Method* 2: Parameters u, v
Coordinates $x, y, z(x, y)$	$x(u,v), y(u,v), z(u,v)$ on surface
$\mathbf{A}=\mathbf{i}+\partial z/\partial x\,\mathbf{k} \qquad \mathbf{N}=\mathbf{A}\times\mathbf{B}$	$\mathbf{A}=\partial x/\partial u\,\mathbf{i}+\partial y/\partial u\,\mathbf{j}+\partial z/\partial u\,\mathbf{k}$
$\mathbf{B}=\mathbf{j}+\partial z/\partial y\,\mathbf{k} \qquad \mathbf{n}=\mathbf{N}/\lvert\mathbf{N}\rvert$	$\mathbf{B}=\partial x/\partial v\,\mathbf{i}+\partial y/\partial v\,\mathbf{j}+\partial z/\partial v\,\mathbf{k}$
$dS=\lvert\mathbf{N}\rvert\,dx\,dy=\sqrt{1+z_x^2+z_y^2}\,dx\,dy$	$dS=\lvert\mathbf{N}\rvert\,du\,dv$
$\mathbf{n}dS=\mathbf{N}\,dx\,dy=(-\partial z/\partial x\,\mathbf{i}-\partial z/\partial y\,\mathbf{j}+\mathbf{k})dx\,dy$	$\mathbf{n}dS=\mathbf{N}\,du\,dv$

15.4 EXERCISES

Read-through questions

A small piece of the surface $z=f(x,y)$ is nearly ___a___. When we go across by dx, we go up by ___b___. That movement is $\mathbf{A}dx$, where the vector $\mathbf{A}$ is $\mathbf{i}+$ ___c___. The other side of the piece is $\mathbf{B}dy$, where $\mathbf{B}=\mathbf{j}+$ ___d___. The cross product $\mathbf{A}\times\mathbf{B}$ is $\mathbf{N}=$ ___e___. The area of the piece is $dS=\lvert\mathbf{N}\rvert\,dx\,dy$. For the surface $z=xy$, the vectors are $\mathbf{A}=$ ___f___ and $\mathbf{B}=$ ___g___ and $\mathbf{N}=$ ___h___. The area integral is $\iint dS=$ ___i___ $dx\,dy$.

With parameters u and v, a typical point on a 45° cone is $x=u\cos v$, $y=$ ___j___, $z=$ ___k___. A change in u moves that point by $\mathbf{A}du=(\cos v\,\mathbf{i}+$ ___l___$)du$. A change in v moves the point by $\mathbf{B}\,dv=$ ___m___. The normal vector is $\mathbf{N}=\mathbf{A}\times\mathbf{B}=$ ___n___. The area is $dS=$ ___o___ $du\,dv$. In this example $\mathbf{A}\cdot\mathbf{B}=$ ___p___ so the small piece is a ___q___ and $dS=\lvert\mathbf{A}\rvert\,\lvert\mathbf{B}\rvert\,du\,dv$.

For flux we need $\mathbf{n}dS$. The ___r___ vector $\mathbf{n}$ is $\mathbf{N}=\mathbf{A}\times\mathbf{B}$ divided by ___s___. For a surface $z=f(x,y)$, the product $\mathbf{n}dS$ is the vector ___t___ (to memorize from table). The particular surface $z=xy$ has $\mathbf{n}dS=$ ___u___ $dx\,dy$. For $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ the flux through $z=xy$ is $\mathbf{F}\cdot\mathbf{n}dS=$ ___v___ $dx\,dy$.

On a 30° cone the points are $x=2u\cos v$, $y=2u\sin v$, $z=u$. The tangent vectors are $\mathbf{A}=$ ___w___ and $\mathbf{B}=$ ___x___. This cone has $\mathbf{n}dS=\mathbf{A}\times\mathbf{B}\,du\,dv=$ ___y___. For $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, the flux element through the cone is $\mathbf{F}\cdot\mathbf{n}dS=$ ___z___. The reason for this answer is ___A___. The reason we don't compute flux through a Möbius strip is ___B___.

In 1–14 find N and $dS=\lvert\mathbf{N}\rvert\,dx\,dy$ and the surface area $\iint dS$. Integrate over the xy shadow which ends where the z's are equal ($x^2+y^2=4$ in Problem 1).

1 Paraboloid $z=x^2+y^2$ below the plane $z=4$.

2 Paraboloid $z=x^2+y^2$ between $z=4$ and $z=8$.

3 Plane $z=x-y$ inside the cylinder $x^2+y^2=1$.

4 Plane $z=3x+4y$ above the square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$.

5 Spherical cap $x^2+y^2+z^2=1$ above $z=1/\sqrt{2}$.

6 Spherical band $x^2+y^2+z^2=1$ between $z=0$ and $1/\sqrt{2}$.

7 Plane $z=7y$ above a triangle of area A.

8 Cone $z^2=x^2+y^2$ between planes $z=a$ and $z=b$.

9 The monkey saddle $z=\frac{1}{3}x^3-xy^2$ inside $x^2+y^2=1$.

10 $z=x+y$ above triangle with vertices $(0,0)$, $(2,2)$, $(0,2)$.

11 Plane $z=1-2x-2y$ inside $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.

12 Cylinder $x^2+z^2=a^2$ inside $x^2+y^2=a^2$. Only set up $\iint dS$.

13 Right circular cone of radius a and height h. Choose $z=f(x,y)$ or parameters u and v.

14 Gutter $z=x^2$ below $z=9$ and between $y=\pm 2$.

In 15–18 compute the surface integrals $\iint g(x,y,z)dS$.

15 $g=xy$ over the triangle $x+y+z=1$, $x,y,z\geqslant 0$.

16 $g=x^2+y^2$ over the top half of $x^2+y^2+z^2=1$ (use ϕ,θ).

17 $g=xyz$ on $x^2+y^2+z^2=1$ above $z^2=x^2+y^2$ (use ϕ,θ).

18 $g=x$ on the cylinder $x^2+y^2=4$ between $z=0$ and $z=3$.

In 19–22 calculate A, B, N, and dS.

19 $x=u$, $y=v+u$, $z=v+2u+1$.

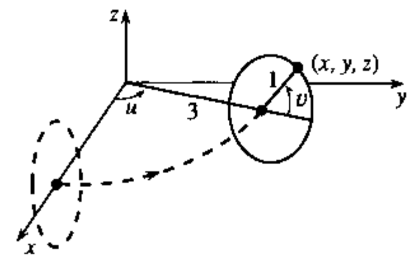
20 $x=uv$, $y=u+v$, $z=u-v$.

21 $x=(3+\cos u)\cos v$, $y=(3+\cos u)\sin v$, $z=\sin u$.

22 $x=u\cos v$, $y=u\sin v$, $z=v$ (not $z=u$).

23–26 In Problems 1–4 respectively find the flux $\iint \mathbf{F}\cdot\mathbf{n}dS$ for $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

27–28 In Problems 19–20 respectively compute $\iint \mathbf{F}\cdot\mathbf{n}dS$ for $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$ through the region $u^2+v^2\leqslant 1$.



29 A unit circle is rotated around the z axis to give a torus (see figure). The center of the circle stays a distance 3 from the z axis. Show that Problem 21 gives a typical point (x,y,z) on the torus and find the surface area $\iint dS=\iint |\mathbf{N}|\,du\,dv$.

30 The surface $x=r\cos\theta$, $y=r\sin\theta$, $z=a^2-r^2$ is bounded by the equator ($r=a$). Find $\mathbf{N}$ and the flux $\iint \mathbf{k}\cdot\mathbf{n}dS$, and compare with Example 12.

31 Make a "double Möbius strip" from a strip of paper by twisting it twice and taping the ends. Does a normal vector (use a pen) have the same direction after a round trip?

32 Make a "triple Möbius strip" with three twists. Is it orientable—does the normal vector come back in the same or opposite direction?

33 If a very wavy surface stays close to a smooth surface, are their areas close?

34 Give the equation of a plane with roof area $dS=3$ times base area $dx\,dy$.

35 The points $(x, f(x)\cos\theta, f(x)\sin\theta)$ are on the surface of revolution: $y=f(x)$ revolved around the x axis, parameters $u=x$ and $v=\theta$. Find $\mathbf{N}$ and compare $dS=|\mathbf{N}|\,dx\,d\theta$ with Example 8 and Section 8.3.

15.5 The Divergence Theorem

This section returns to the fundamental law (***flow out***) − (***flow in***) = (***source***). In two dimensions, the flow was in and out through a closed curve C. The plane region inside was R. In three dimensions, the flow enters and leaves through a closed surface S. The solid region inside is V. Green's Theorem in its normal form (for the flux of a smooth vector field) now becomes the great three-dimensional balance equation—the *Divergence Theorem*:

> **15K** The flux of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ through the boundary surface S equals the integral of the divergence of $\mathbf{F}$ inside V. ***The Divergence Theorem is***
>
> $$\iint \mathbf{F}\cdot\mathbf{n}\,dS = \iiint \operatorname{div}\mathbf{F}\,dV = \iiint\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)dx\,dy\,dz. \qquad (1)$$

In Green's Theorem the divergence was $\partial M/\partial x + \partial N/\partial y$. The new term $\partial P/\partial z$ accounts for upward flow. Notice that a constant upward component P adds nothing to the divergence (its derivative is zero). It also adds nothing to the flux (flow up through the top equals flow up through the bottom). When the whole field $\mathbf{F}$ is constant, the theorem becomes $0 = 0$.

There are other vector fields with $\operatorname{div}\mathbf{F} = 0$. They are of the greatest importance. The Divergence Theorem for those fields is again $0 = 0$, and there is conservation of fluid. ***When*** $\operatorname{div}\mathbf{F} = 0$, ***flow in equals flow out***. We begin with examples of these "*divergence-free*" fields.

EXAMPLE 1 The spin fields $-y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$ and $0\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ have zero divergence.

The first is an old friend, spinning around the z axis. The second is new, spinning around the x axis. Three-dimensional flow has a great variety of spin fields. The separate terms $\partial M/\partial x$, $\partial N/\partial y$, $\partial P/\partial z$ are all zero, so $\operatorname{div}\mathbf{F} = 0$. The flow goes around in circles, and whatever goes out through S comes back in. (We might have put a circle on $\iint_S$ as we did on $\oint_C$, to emphasize that S is closed.)

EXAMPLE 2 The position field $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has $\operatorname{div}\mathbf{R} = 1 + 1 + 1 = 3$.

This is radial flow, straight out from the origin. Mass has to be added ***at every point*** to keep the flow going. On the right side of the divergence theorem is $\iiint 3\,dV$. Therefore the flux is three times the volume.

Example 11 in Section 15.4 found the flux of $\mathbf{R}$ through a cylinder. The answer was $3\pi b$. Now we also get $3\pi b$ from the Divergence Theorem, since the volume is πb. This is one of many cases in which the triple integral is easier than the double integral.

EXAMPLE 3 An ***electrostatic field*** $\mathbf{R}/\rho^3$ or ***gravity field*** $-\mathbf{R}/\rho^3$ almost has $\operatorname{div}\mathbf{F} = 0$.

The vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has length $\sqrt{x^2 + y^2 + z^2} = \rho$. Then $\mathbf{F}$ has length ρ/ρ^3 (inverse square law). Gravity from a point mass pulls *inward* (minus sign). The electric field from a point charge repels *outward*. The three steps almost show that $\operatorname{div}\mathbf{F} = 0$:

Step 1. $\partial\rho/\partial x = x/\rho$, $\partial\rho/\partial y = y/\rho$, $\partial\rho/\partial z = z/\rho$—but do not add those three. $\mathbf{F}$ is not ρ or $1/\rho^2$ (these are scalars). The vector field is $\mathbf{R}/\rho^3$. We need $\partial M/\partial x$, $\partial N/\partial y$, $\partial P/\partial z$.

Step 2. $\partial M/\partial x = \partial/\partial x(x/\rho^3)$ is equal to $1/\rho^3 - (3x\,\partial\rho/\partial x)/\rho^4 = 1/\rho^3 - 3x^2/\rho^5$. For $\partial N/\partial y$ and $\partial P/\partial z$, replace $3x^2$ by $3y^2$ and $3z^2$. Now add those three.

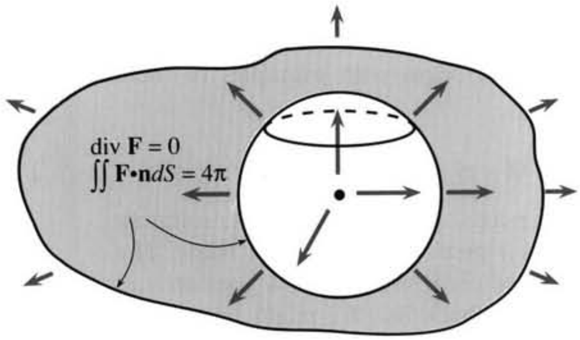
Step 3. $\operatorname{div}\mathbf{F} = 3/\rho^3 - 3(x^2 + y^2 + z^2)/\rho^5 = 3/\rho^3 - 3/\rho^3 = 0$.

The calculation $\operatorname{div}\mathbf{F} = 0$ leaves a puzzle. One side of the Divergence Theorem seems to give $\iiint 0\,dV = 0$. Then the other side should be $\iint \mathbf{F}\cdot\mathbf{n}\,dS = 0$. But the flux is *not* zero when all flow is outward:

> The unit normal vector to the sphere $\rho = \text{constant}$ is $\mathbf{n} = \mathbf{R}/\rho$.
> The outward flow $\mathbf{F}\cdot\mathbf{n} = (\mathbf{R}/\rho^3)\cdot(\mathbf{R}/\rho) = \rho^2/\rho^4$ is always positive.
> Then $\iint \mathbf{F}\cdot\mathbf{n}\,dS = \iint dS/\rho^2 = 4\pi\rho^2/\rho^2 = 4\pi$. ***We have reached*** $4\pi = 0$.

This paradox in three dimensions is the same as for $\mathbf{R}/r^2$ in two dimensions. Section 15.3 reached $2\pi = 0$, and the explanation was a point source at the origin. Same explanation here: M, N, P are infinite when $\rho = 0$. The divergence is a "delta function" times 4π, from the point source. The Divergence Theorem does not apply (unless we allow delta functions). That single point makes all the difference.

Every surface enclosing the origin has flux $= 4\pi$. Our calculation was for a sphere. The surface integral is much harder when S is twisted (Figure 15.21a). But the Divergence Theorem takes care of everything, because div $\mathbf{F} = 0$ in the volume V between these surfaces. Therefore $\iint \mathbf{F}\cdot\mathbf{n}dS = 0$ for the two surfaces together. The flux $\iint \mathbf{F}\cdot\mathbf{n}dS = -4\pi$ into the sphere must be balanced by $\iint \mathbf{F}\cdot\mathbf{n}dS = 4\pi$ out of the twisted surface.



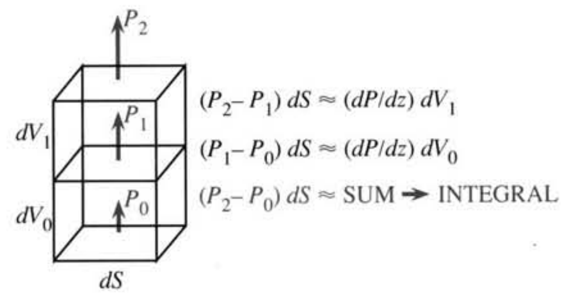


Fig. 15.21 Point source: flux 4π through all enclosing surfaces. Net flux upward $= \iiint (\partial P/\partial z) dV$.

Instead of a paradox $4\pi = 0$, this example leads to Gauss's Law. A mass M at the origin produces a gravity field $\mathbf{F} = -GM\mathbf{R}/\rho^3$. A charge q at the origin produces an electric field $\mathbf{E} = (q/4\pi\varepsilon_0)\mathbf{R}/\rho^3$. The physical constants are G and ε_0, the mathematical constant is the relation between divergence and flux. Equation (1) yields equation (2), in which the mass densities $M(x, y, z)$ and charge densities $q(x, y, z)$ need not be concentrated at the origin:

15L Gauss's law in differential form: div $\mathbf{F} = -4\pi GM$ and div $\mathbf{E} = q/\varepsilon_0$.
Gauss's law in integral form: Flux is proportional to total mass or charge:

$$\iint \mathbf{F}\cdot\mathbf{n}dS = -\iiint 4\pi GM dV \text{ and } \iint \mathbf{E}\cdot\mathbf{n}dS = \iiint q\, dV/\varepsilon_0. \qquad (2)$$

THE REASONING BEHIND THE DIVERGENCE THEOREM

The general principle is clear: Flow out minus flow in equals source. Our goal is to see why ***the divergence of* F *measures the source***. In a small box around each point, we show that div $\mathbf{F}\, dV$ balances $\mathbf{F}\cdot\mathbf{n}dS$ through the six sides.

So consider a small box. Its center is at (x, y, z). Its edges have length $\Delta x, \Delta y, \Delta z$. Out of the top and bottom, the normal vectors are $\mathbf{k}$ and $-\mathbf{k}$. The dot product with $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is $+P$ or $-P$. The area ΔS is $\Delta x\Delta y$. So the two fluxes are close to $P(x, y, z + \frac{1}{2}\Delta z)\Delta x\Delta y$ and $-P(x, y, z - \frac{1}{2}\Delta z)\Delta x\Delta y$. When the top is combined with the bottom, the difference of those P's is ΔP:

$$\textit{net flux upward} \approx \Delta P\Delta x\Delta y = (\Delta P/\Delta z)\Delta x\Delta y\Delta z \approx (\partial P/\partial z)\Delta V. \qquad (3)$$

Similarly, the combined flux on two side faces is approximately $(\partial N/\partial y)\Delta V$. On the front and back it is $(\partial M/\partial x)\Delta V$. Adding the six faces, we reach the key point:

$$\textit{flux out of the box} \approx (\partial M/\partial x + \partial N/\partial y + \partial P/\partial z)\Delta V. \tag{4}$$

This is (div $\mathbf{F})\Delta V$. For a constant field both sides are zero—the flow goes straight through. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, a little more goes out tban comes in. The divergence is 3, so $3\Delta V$ is created inside the box. By the balance equation the flux is also $3\Delta V$.

The approximation symbol $\approx$ means that the leading term is correct (probably not the next term). The ratio $\Delta P/\Delta z$ is not exactly $\partial P/\partial z$. The difference is of order Δz, so the error in (3) is of higher order $\Delta V \Delta z$. Added over many boxes (about $1/\Delta V$ boxes), this error disappears as $\Delta z \to 0$.

The sum of (div $\mathbf{F})\Delta V$ over all the boxes approaches $\iiint$(div $\mathbf{F})dV$. On the other side of the equation is a sum of fluxes. There is $\mathbf{F}\cdot\mathbf{n}\Delta S$ out of the top of one box, plus $\mathbf{F}\cdot\mathbf{n}\Delta S$ out of the bottom of the box above. The first has $\mathbf{n} = \mathbf{k}$ and the second has $\mathbf{n} = -\mathbf{k}$. ***They cancel each other—the flow goes from box to box.*** This happens every time two boxes meet. The only fluxes that survive (because nothing cancels them) are at the outer surface S. The final step, as $\Delta x, \Delta y, \Delta z \to 0$, is that those outside terms approach $\iint \mathbf{F}\cdot\mathbf{n}dS$. Then the local divergence theorem (4) becomes the global Divergence Theorem (1).

Remark on the proof That "final step" is not easy, because the box surfaces don't line up with the outer surface S. A formal proof of the Divergence Theorem would imitate the proof of Green's Theorem. On a very simple region $\iiint(\partial P/\partial z)dx\,dy\,dz$ equals $\iint P\,dx\,dy$ over the top minus $\iint P\,dx\,dy$ over the bottom. After checking the orientation this is $\iint P\mathbf{k}\cdot\mathbf{n}dS$. Similarly the volume integrals of $\partial M/\partial x$ and $\partial N/\partial y$ are the surface integrals $\iint M\mathbf{i}\cdot\mathbf{n}dS$ and $\iint N\mathbf{j}\cdot\mathbf{n}dS$. Adding the three integrals gives the Divergence Theorem. Since Green's Theorem was already proved in this way, the reasoning behind (4) is more helpful than repeating a detailed proof.

The discoverer of the Divergence Theorem was probably Gauss. His notebooks only contain the outline of a proof—but after all, this is Gauss. Green and Ostrogradsky both published proofs in 1828, one in England and the other in St. Petersburg (now Leningrad). As the theorem was studied, the requirements came to light (smoothness of F and S, avoidance of one-sided Möbius strips).

New applications are discovered all the time—***when a scientist writes down a balance equation in a small box.*** The source is known. The equation is div $\mathbf{F}$ = source. After Example 5 we explain $\mathbf{F}$.

EXAMPLE 4 If the temperature inside the sun is $T = \ln 1/\rho$, find the heat flow $\mathbf{F} = -\text{grad } T$ and the source div F and the flux $\iint \mathbf{F}\cdot\mathbf{n}dS$. The sun is a ball of radius ρ_0.

Solution $\mathbf{F}$ is $-\text{grad} \ln 1/\rho = +\text{grad} \ln \rho$. Derivatives of $\ln \rho$ bring division by ρ:

$$\mathbf{F} = (\partial\rho/\partial x\,\mathbf{i} + \partial\rho/\partial y\,\mathbf{j} + \partial\rho/\partial z\,\mathbf{k})/\rho = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho^2.$$

This flow is radially outward, of magnitude $1/\rho$. The normal vector $\mathbf{n}$ is also radially outward, of magnitude 1. The dot product on the sun's surface is $1/\rho_0$:

$$\iint \mathbf{F}\cdot\mathbf{n}dS = \iint dS/\rho_0 = (\text{surface area})/\rho_0 = 4\pi\rho_0^2/\rho_0 = 4\pi\rho_0. \tag{5}$$

Check that answer by the Divergence Theorem. Example 5 will find div $\mathbf{F} = 1/\rho^2$. Integrate over the sun. In spherical coordinates we integrate $d\rho$, $\sin\phi\, d\phi$, and $d\theta$:

$$\iiint_{\text{sun}} \text{div } \mathbf{F}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\rho_0} \rho^2 \sin\phi\; d\rho\, d\phi\, d\theta/\rho^2 = (\rho_0)(2)(2\pi) \text{ as in (5).}$$

This example illustrates ***the basic framework of equilibrium.*** The pattern appears everywhere in applied mathematics—electromagnetism, heat flow, elasticity, even relativity. There is usually a constant c that depends on the material (the example has $c = 1$). The names change, but we always take ***the divergence of the gradient***:

potential $f \rightarrow$ ***force field*** $-c$ grad f.	Then div$(-c\,\text{grad}\,f) =$ electric charge
temperature $T \rightarrow$ ***flow field*** $-c$ grad T.	Then div$(-c\,\text{grad}\,T) =$ heat source
displacement $\mathbf{u} \rightarrow$ ***stress field*** $+c$ grad $\mathbf{u}$.	Then div$(-c\,\text{grad}\,\mathbf{u}) =$ outside force.

You are studying calculus, not physics or thermodynamics or elasticity. But please notice the main point. The equation to solve is div$(-c\,\text{grad}\,f) =$ known source. The divergence and gradient are exactly what the applications need. Calculus teaches the right things.

This framework is developed in many books, including my own text *Introduction to Applied Mathematics* (Wellesley-Cambridge Press). It governs equilibrium, in matrix equations and differential equations.

PRODUCT RULE FOR VECTORS: INTEGRATION BY PARTS

May I go back to basic facts about the divergence? First the definition:

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \text{ has div } \mathbf{F} = \nabla \cdot \mathbf{F} = \partial M/\partial x + \partial N/\partial y + \partial P/\partial z.$$

The divergence is a scalar (not a vector). At each point div $\mathbf{F}$ is a number. In fluid flow, it is the rate at which mass leaves—the "flux per unit volume" or "flux density."

The symbol ∇ stands for a vector whose components are *operations not numbers*:

$$\nabla = \text{"}\mathbf{del}\text{"} = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z. \tag{6}$$

This vector is illegal but very useful. First, apply it to an ordinary function $f(x, y, z)$:

$$\nabla f = \text{"}\mathbf{del}\,f\text{"} = \mathbf{i}\,\partial f/\partial x + \mathbf{j}\,\partial f/\partial y + \mathbf{k}\,\partial f/\partial z = \textbf{\textit{gradient of }} f. \tag{7}$$

Second, take the dot product $\nabla \cdot \mathbf{F}$ with a vector function $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$:

$$\nabla \cdot \mathbf{F} = \text{"}\mathbf{del\ dot\ F}\text{"} = \partial M/\partial x + \partial N/\partial y + \partial P/\partial z = \textbf{\textit{divergence of }} \mathbf{F}. \tag{8}$$

Third, take the cross product $\nabla \times \mathbf{F}$. This produces the vector curl $\mathbf{F}$ (next section):

$$\nabla \times \mathbf{F} = \text{"}\mathbf{del\ cross\ F}\text{"} = \ldots \text{(to be defined)} \ldots = \textbf{\textit{curl of }} \mathbf{F}. \tag{9}$$

The gradient and divergence and curl are ∇ and $\nabla \cdot$ and $\nabla \times$. The three great operations of vector calculus use a single notation! You are free to write ∇ or not—to make equations shorter or to help the memory. Notice that Laplace's equation shrinks to

$$\nabla \cdot \nabla f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 0. \tag{10}$$

Equation (10) gives the potential when the source is zero (very common). $\mathbf{F} = \text{grad}\,f$ combines with div $\mathbf{F} = 0$ into Laplace's equation div grad $f = 0$. This equation is so important that it shrinks further to $\nabla^2 f = 0$ and even to $\Delta f = 0$. Of course $\Delta f = f_{xx} + f_{yy} + f_{zz}$ has nothing to do with $\Delta f = f(x + \Delta x) - f(x)$. Above all, remember that f is a scalar and $\mathbf{F}$ is a vector: ***gradient of scalar is vector*** and ***divergence of vector is scalar.***

Underlying this chapter is the idea of extending calculus to vectors. So far we have emphasized the Fundamental Theorem. The integral of df/dx is now the integral of div $\mathbf{F}$. Instead of endpoints a and b, we have a curve C or surface S. But it is the *rules* for derivatives and integrals that make calculus work, and we need them now for vectors. Remember the derivative of u times v and the integral (by parts) of $u\,dv/dx$:

15M Scalar functions $u(x, y, z)$ and vector fields $\mathbf{V}(x, y, z)$ obey the ***product rule***:

$$\operatorname{div}(u\mathbf{V}) = u \operatorname{div} \mathbf{V} + \mathbf{V} \cdot (\operatorname{grad} u). \qquad (11)$$

The reverse of the product rule is integration by parts (***Gauss's Formula***):

$$\iiint u \operatorname{div} \mathbf{V}\, dx\,dy\,dz = -\iiint \mathbf{V} \cdot (\operatorname{grad} u)\, dx\,dy\,dz + \iint u\, \mathbf{V} \cdot \mathbf{n}\, dS. \qquad (12)$$

For a plane field tbis is ***Green's Formula*** (and $u = 1$ gives Green's Theorem):

$$\iint u\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = -\iint \left(M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y}\right) dx\,dy + \int u(M\mathbf{i} + N\mathbf{j}) \cdot \mathbf{n}\, ds. \qquad (13)$$

Those look like heavy formulas. They are too much to memorize, unless you use them often. The important point is to connect vector calculus with "scalar calculus," which is not heavy. Every product rule yields two terms:

$$(uM)_x = u\,\partial M/\partial x + M\,\partial u/\partial x \quad (uN)_y = u\,\partial N/\partial y + N\,\partial u/\partial y \quad (uP)_z = u\,\partial P/\partial z + P\,\partial u/\partial z.$$

Add those ordinary rules and you have the vector rule (11) for the divergence of $u\mathbf{V}$.

Integrating the two parts of div($u\mathbf{V}$) gives $\iint u\mathbf{V} \cdot \mathbf{n}dS$ by the Divergence Theorem. Then one part moves to the other side, producing the minus signs in (12) and (13). ***Integration by parts leaves a boundary term***, in three and two dimensions as it did in one dimension: $\int uv'dx = -\int u'v dx + [uv]_a^b$.

EXAMPLE 5 Find the divergence of $\mathbf{F} = \mathbf{R}/\rho^2$, starting from grad $\rho = \mathbf{R}/\rho$.

Solution Take $\mathbf{V} = \mathbf{R}$ and $u = 1/\rho^2$ in the product rule (11). Then div $\mathbf{F} = (\operatorname{div} \mathbf{R})/\rho^2 + \mathbf{R} \cdot (\operatorname{grad} 1/\rho^2)$. The divergence of $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is 3. For grad $1/\rho^2$ apply the chain rule:

$$\mathbf{R} \cdot (\operatorname{grad} 1/\rho^2) = -2\mathbf{R} \cdot (\operatorname{grad} \rho)/\rho^3 = -2\mathbf{R} \cdot \mathbf{R}/\rho^4 = -2/\rho^2.$$

The two parts of div $\mathbf{F}$ combine into $3/\rho^2 - 2/\rho^2 = 1/\rho^2$—as claimed in Example 4.

EXAMPLE 6 Find the balance equation for flow with velocity $\mathbf{V}$ and fluid density ρ.

$\mathbf{V}$ is the rate of movement of fluid, while $\rho\mathbf{V}$ is the rate of movement of *mass*. Comparing the ocean to the atmosphere shows the difference. Air has a greater velocity than water, but a much lower density. So normally $\mathbf{F} = \rho\mathbf{V}$ is larger for the ocean. (Don't confuse the density ρ with the radial distance ρ. The Greeks only used 24 letters.)

There is another difference between water and air. Water is virtually incompressible (meaning ρ = constant). Air is certainly compressible (its density varies). The balance equation is a fundamental law—the conservation of mass or the "*continuity equation*" for fluids. This is a mathematical statement about a physical flow without sources or sinks:

$$\textbf{\textit{Continuity Equation}}\text{: } \operatorname{div}(\rho\mathbf{V}) + \partial\rho/\partial t = 0. \qquad (14)$$

Explanation: The mass in a region is $\iiint \rho\, dV$. Its rate of decrease is $-\iiint \partial\rho/\partial t\, dV$. The decrease comes from flow out through the surface (normal vector $\mathbf{n}$). The dot product $\mathbf{F}\cdot\mathbf{n} = \rho\mathbf{V}\cdot\mathbf{n}$ is the rate of mass flow through the surface. So the integral $\iint \mathbf{F}\cdot\mathbf{n}dS$ is the total rate that mass goes out. By the Divergence Theorem this is $\iiint \operatorname{div} \mathbf{F}\, dV$.

To balance $-\iiint \partial\rho/\partial t\, dV$ in every region, div $\mathbf{F}$ must equal $-\partial\rho/\partial t$ at every point. The figure shows this continuity equation (14) for flow in the x direction.

$$\underset{\rho\mathbf{V}\,dS\,dt}{\text{mass in}} \rightarrow \boxed{\text{mass } \rho dS\, dx} \rightarrow \underset{d(\rho\mathbf{V})\,dS\,dt}{\text{extra mass out}} = \underset{-d\rho\,dS\,dx}{\text{mass loss}}$$

Fig. 15.22 Conservation of mass during time dt: $d(\rho\mathbf{V})/dx + d\rho/dt = 0$.

15.5 EXERCISES

Read-through questions

In words, the basic balance law is __a__. The flux of $\mathbf{F}$ through a closed surface S is the double integral __b__. The divergence of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is __c__, and it measures __d__. The total source is the triple integral __e__. That equals the flux by the __f__ Theorem.

For $\mathbf{F} = 5z\mathbf{k}$ the divergence is __g__. If V is a cube of side a then the triple integral equals __h__. The top surface where $z = a$ has $\mathbf{n} =$ __i__ and $\mathbf{F}\cdot\mathbf{n} =$ __j__. The bottom and sides have $\mathbf{F}\cdot\mathbf{n} =$ __k__. The integral $\iint \mathbf{F}\cdot\mathbf{n}dS$ equals __l__.

The field $\mathbf{F} = \mathbf{R}/\rho^3$ has div $\mathbf{F} = 0$ except __m__. $\iint \mathbf{F}\cdot\mathbf{n}dS$ equals __n__ over any surface around the origin. This illustrates Gauss's Law __o__. The field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$ has div $\mathbf{F} =$ __p__ and $\iint \mathbf{F}\cdot\mathbf{n}dS =$ __q__. For this $\mathbf{F}$, the flux out through a pyramid and in through its base are __r__.

The symbol ∇ stands for __s__. In this notation div $\mathbf{F}$ is __t__. The gradient of f is __u__. The divergence of grad f is __v__. The equation div grad $f = 0$ is __w__'s equation.

The divergence of a product is div$(u\mathbf{V})$ = __x__. Integration by parts is $\iiint u \operatorname{div} \mathbf{V}\, dx\, dy\, dz =$ __y__ + __z__. In two dimensions this becomes __A__. In one dimension it becomes __B__. For steady fluid flow the continuity equation is div $\rho\mathbf{V} =$ __C__.

In 1–10 compute the flux $\iint \mathbf{F}\cdot\mathbf{n}dS$ by the Divergence Theorem.

1 $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, S: unit sphere $x^2 + y^2 + z^2 = 1$.

2 $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, V: unit cube $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$.

3 $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, S: unit sphere

4 $\mathbf{F} = x^2\mathbf{i} + 8y^2\mathbf{j} + z^2\mathbf{k}$, V: unit cube.

5 $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j}$, S: sides $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

6 $\mathbf{F} = \mathbf{u}_r = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho$, S: sphere $\rho = a$.

7 $\mathbf{F} = \rho(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, S: sphere $\rho = a$

8 $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, S: sphere $\rho = a$.

9 $\mathbf{F} = z^2\mathbf{k}$, V: upper half of ball $\rho \leqslant a$.

10 $\mathbf{F} = \operatorname{grad}(xe^y \sin z)$, S: sphere $\rho = a$.

11 Find $\iiint \operatorname{div}(x^2\mathbf{i} + y\mathbf{j} + 2\mathbf{k})dV$ in the cube $0 \leqslant x, y, z \leqslant a$. Also compute $\mathbf{n}$ and $\iint \mathbf{F}\cdot\mathbf{n}dS$ for all six faces and add.

12 When a is small in problem 11, the answer is close to ca^3. Find the number c. At what point does div $\mathbf{F} = c$?

13 (a) Integrate the divergence of $\mathbf{F} = \rho\mathbf{i}$ in the ball $\rho \leqslant a$.
(b) Compute $\iint \mathbf{F}\cdot\mathbf{n}dS$ over the spherical surface $\rho = a$.

14 Integrate $\iint \mathbf{R}\cdot\mathbf{n}dS$ over the faces of the box $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3$ and check by the Divergence Theorem.

15 Evaluate $\iint \mathbf{F}\cdot\mathbf{n}dS$ when $\mathbf{F} = x\mathbf{i} + z^2\mathbf{j} + y^2\mathbf{k}$ and:
(a) S is the cone $z^2 = x^2 + y^2$ bounded above by the plane $z = 1$.
(b) S is the pyramid with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

16 Compute all integrals in the Divergence Theorem when $\mathbf{F} = x(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and V is the unit cube $0 \leqslant x, y, z \leqslant 1$.

17 Following Example 5, compute the divergence of $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho^7$.

18 (grad f)$\cdot\mathbf{n}$ is the ______ derivative of f in the direction ______. It is also written $\partial f/\partial n$. If $f_{xx} + f_{yy} + f_{zz} = 0$ in V, derive $\iint \partial f/\partial n\, dS = 0$ from the Divergence Theorem.

19 Describe the closed surface S and outward normal $\mathbf{n}$:
(a) V = hollow ball $1 \leqslant x^2 + y^2 + z^2 \leqslant 9$.
(b) V = solid cylinder $x^2 + y^2 \leqslant 1$, $|z| \leqslant 7$.
(c) V = pyramid $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, $x + 2y + 3z \leqslant 1$.
(d) V = solid cone $x^2 + y^2 \leqslant z^2 \leqslant 1$.

20 Give an example where $\iint \mathbf{F}\cdot\mathbf{n}dS$ is easier than $\iiint \operatorname{div} \mathbf{F}\, dV$.

21 Suppose $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, R is a region in the xy plane, and (x, y, z) is in V if (x, y) is in R and $|z| \leqslant 1$.

(a) Describe V and reduce $\iiint \text{div}\,\mathbf{F}\,dV$ to a double integral.

(b) Reduce $\iint \mathbf{F}\cdot\mathbf{n}dS$ to a line integral (check top, bottom, side).

(c) Whose theorem says that the double integral equals the line integral?

22 Is it possible to have $\mathbf{F}\cdot\mathbf{n} = 0$ at all points of S and also $\text{div}\,\mathbf{F} = 0$ at all points in V? $\mathbf{F} = \mathbf{0}$ is not allowed.

23 Inside a solid ball (radius a, density 1, mass $M = 4\pi a^3/3$) the gravity field is $\mathbf{F} = -GM\mathbf{R}/a^3$.

(a) Check $\text{div}\,\mathbf{F} = -4\pi G$ in Gauss's Law.

(b) The force at the surface is the same as if the whole mass M were ________.

(c) Find a gradient field with $\text{div}\,\mathbf{F} = 6$ in the ball $\rho \leqslant a$ and $\text{div}\,\mathbf{F} = 0$ outside.

24 The outward field $\mathbf{F} = \mathbf{R}/\rho^3$ has magnitude $|\mathbf{F}| = 1/\rho^2$. Through an area A on a sphere of radius ρ, the flux is ________. A spherical box has faces at ρ_1 and ρ_2 with $A = \rho_1^2 \sin\phi d\phi d\theta$ and $A = \rho_2^2 \sin\phi d\phi d\theta$. Deduce that the flux out of the box is zero, which confirms $\text{div}\,\mathbf{F} = 0$.

25 In Gauss's Law, what charge distribution $q(x, y, z)$ gives the unit field $\mathbf{E} = \mathbf{u}_r$? What is the flux through the unit sphere?

26 If a fluid with velocity $\mathbf{V}$ is incompressible (constant density ρ), then its continuity equation reduces to ________. If it is irrotational then $\mathbf{F} = \text{grad}\,f$. If it is both then f satisfies ________ equation.

27 ***True or false***, with a good reason.

(a) If $\iint \mathbf{F}\cdot\mathbf{n}dS = 0$ for every closed surface, $\mathbf{F}$ is constant.

(b) If $\mathbf{F} = \text{grad}\,f$ then $\text{div}\,\mathbf{F} = 0$.

(c) If $|\mathbf{F}| \leqslant 1$ at all points then $\iiint \text{div}\,\mathbf{F}\,dV \leqslant$ area of the surface S.

(d) If $|\mathbf{F}| \leqslant 1$ at all points then $|\text{div}\,\mathbf{F}| \leqslant 1$ at all points.

28 Write down statements **E–F–G–H** for source-free fields $\mathbf{F}(x, y, z)$ in three dimensions. In statement **F**, paths sharing the same endpoint become surfaces sharing the same boundary curve. In **G**, the stream function becomes a *vector field* such that $\mathbf{F} = \text{curl}\,\mathbf{g}$.

29 Describe two different surfaces bounded by the circle $x^2 + y^2 = 1, z = 0$. The field $\mathbf{F}$ automatically has the same flux through both if ________.

30 The boundary of a bounded region R has no boundary. Draw a plane region and explain what that means. What does it mean for a solid ball?

15.6 Stokes' Theorem and the Curl of F

For the Divergence Theorem, the surface was closed. S was the boundary of V. Now the surface is not closed and S has its own boundary—a curve called C. We are back near the original setting for Green's Theorem (region bounded by curve, double integral equal to work integral). But Stokes' Theorem, also called Stokes's Theorem, is in three-dimensional space. There is a ***curved surface*** S bounded by a ***space curve*** C. This is our first integral around a space curve.

The move to three dimensions brings a change in the vector field. The plane field $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ becomes a space field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. The work $Mdx + Ndy$ now includes Pdz. The critical quantity in the double integral (it was $\partial N/\partial x - \partial M/\partial y$) must change too. We called this scalar quantity "curl **F**," but in reality it is only the third component of a vector. Stokes' Theorem needs all three components of that vector—which is curl **F**.

DEFINITION The curl of a vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector field

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}. \qquad (1)$$

The symbol $\nabla \times \mathbf{F}$ stands for a determinant that yields those six derivatives:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}. \tag{2}$$

The three products $\mathbf{i}\, \partial/\partial y\, P$ and $\mathbf{j}\, \partial/\partial z\, M$ and $\mathbf{k}\, \partial/\partial x\, N$ have plus signs. The three products like $\mathbf{k}\, \partial/\partial y\, M$, down to the left, have minus signs. There is a cyclic symmetry. This determinant helps the memory, even if it looks and is illegal. A determinant is not supposed to have a row of vectors, a row of operators, and a row of functions.

EXAMPLE 1 The plane field $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ has $P = 0$ and $\partial M/\partial z = 0$ and $\partial N/\partial z = 0$. Only two terms survive: curl $\mathbf{F} = (\partial N/\partial x - \partial M/\partial y)\mathbf{k}$. Back to Green.

EXAMPLE 2 The cross product $\mathbf{a} \times \mathbf{R}$ is a ***spin field*** $\mathbf{S}$. Its axis is the fixed vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. The flow in Figure 15.23 turns around $\mathbf{a}$, and its components are

$$\mathbf{S} = \mathbf{a} \times \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\mathbf{i} + (a_3 x - a_1 z)\mathbf{j} + (a_1 y - a_2 x)\mathbf{k}. \tag{3}$$

Our favorite spin field $-y\mathbf{i} + x\mathbf{j}$ has $(a_1, a_2, a_3) = (0, 0, 1)$ and its axis is $\mathbf{a} = \mathbf{k}$.

The divergence of a spin field is $M_x + N_y + P_z = 0 + 0 + 0$. Note how the divergence uses M_x while the curl uses N_x and P_x. ***The curl of* S *is the vector*** $2\mathbf{a}$:

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 2a_1\mathbf{i} + 2a_2\mathbf{j} + 2a_3\mathbf{k} = 2\mathbf{a}.$$

This example begins to reveal the meaning of the curl. It measures the spin! The direction of curl $\mathbf{F}$ is the ***axis of rotation***—in this case along $\mathbf{a}$. The magnitude of curl $\mathbf{F}$ is ***twice the speed of rotation***. In this case $|\operatorname{curl} \mathbf{F}| = 2|\mathbf{a}|$ and the angular velocity is $|\mathbf{a}|$.

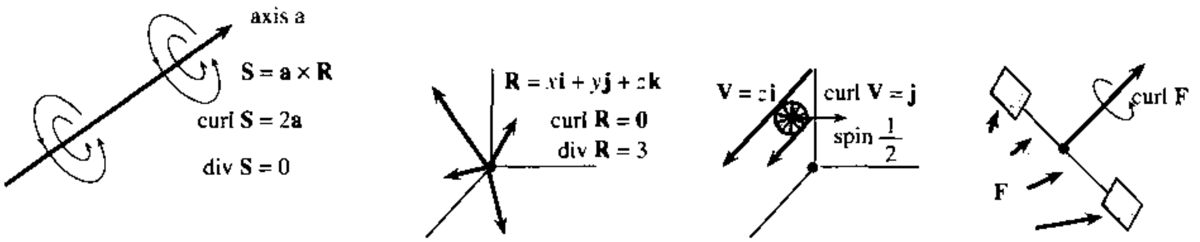


Fig. 15.23 Spin field $\mathbf{S} = \mathbf{a} \times \mathbf{R}$, position field $\mathbf{R}$, velocity field (shear field) $\mathbf{V} = z\mathbf{i}$, any field $\mathbf{F}$.

EXAMPLE 3 (!!) ***Every gradient field*** $\mathbf{F} = \partial f/\partial x\, \mathbf{i} + \partial f/\partial y\, \mathbf{j} + \partial f/\partial z\, \mathbf{k}$ ***has*** curl $\mathbf{F} = \mathbf{0}$:

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\mathbf{k} = \mathbf{0}. \tag{4}$$

Always f_{yz} equals f_{zy}. They cancel. Also $f_{xz} = f_{zx}$ and $f_{yx} = f_{xy}$. So curl grad $f = \mathbf{0}$.

EXAMPLE 4 (twin of Example 3) The divergence of curl **F** is also automatically zero:

$$\text{div curl } \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 0. \quad (5)$$

Again the mixed derivatives give $P_{xy} = P_{yx}$ and $N_{xz} = N_{zx}$ and $M_{zy} = M_{yz}$. The terms cancel in pairs. In "curl grad" and "div curl", everything is arranged to give zero.

> **15N** The curl of the gradient of every $f(x, y, z)$ is curl grad $f = \nabla \times \nabla f = \mathbf{0}$.
> The divergence of the curl of every $\mathbf{F}(x, y, z)$ is div curl $\mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$.

The spin field **S** has no divergence. The position field **R** has no curl. **R** is the gradient of $f = \frac{1}{2}(x^2 + y^2 + z^2)$. **S** is the curl of a suitable **F**. Then div **S** = div curl **F** and curl **R** = curl grad f are automatically zero.

You correctly believe that curl **F** measures the "spin" of the field. You may expect that curl (**F** + **G**) is curl **F** + curl **G**. Also correct. Finally you may think that a field of parallel vectors has no spin. That is wrong. Example 5 has parallel vectors, but their different lengths produce spin.

EXAMPLE 5 The field $\mathbf{V} = z\mathbf{i}$ in the x direction has curl $\mathbf{V} = \mathbf{j}$ in the y direction.

If you put a wheel in the xz plane, ***this field will turn it.*** The velocity $z\mathbf{i}$ at the top of the wheel is greater than $z\mathbf{i}$ at the bottom (Figure 15.23c). So the top goes faster and the wheel rotates. The axis of rotation is curl $\mathbf{V} = \mathbf{j}$. The turning speed is $\frac{1}{2}$, because this curl has magnitude 1.

Another velocity field $\mathbf{v} = -x\mathbf{k}$ produces the same spin: curl $\mathbf{v} = \mathbf{j}$. The flow is in the z direction, it varies in the x direction, and the spin is in the y direction. Also interesting is $\mathbf{V} + \mathbf{v}$. The two "shear fields" add to a perfect spin field $\mathbf{S} = z\mathbf{i} - x\mathbf{k}$, whose curl is $2\mathbf{j}$.

THE MEANING OF CURL F

Example 5 put a paddlewheel into the flow. This is possible for any vector field **F**, and it gives insight into curl **F**. The turning of the wheel (if it turns) depends on its location (x, y, z). The turning also depends on the *orientation* of the wheel. We could put it into a spin field, and if the wheel axis **n** is perpendicular to the spin axis **a**, the wheel won't turn! The general rule for turning speed is this: ***the angular velocity of the wheel is*** $\frac{1}{2}(\text{curl } \mathbf{F}) \cdot \mathbf{n}$. This is the "*directional spin*," just as $(\text{grad} f) \cdot \mathbf{u}$ was the "*directional derivative*"—and **n** is a unit vector like **u**.

There is no spin anywhere in a gradient field. It is ***irrotational***: curl grad $f = \mathbf{0}$.

The pure spin field $\mathbf{a} \times \mathbf{R}$ has curl $\mathbf{F} = 2\mathbf{a}$. The angular velocity is $\mathbf{a} \cdot \mathbf{n}$ (note that $\frac{1}{2}$ cancels 2). This turning is everywhere, ***not just at the origin.*** If you put a penny on a compact disk, it turns once when the disk rotates once. That spin is "*around itself*," and it is the same whether the penny is at the center or not.

The turning speed is greatest when the wheel axis **n** lines up with the spin axis **a**. Then $\mathbf{a} \cdot \mathbf{n}$ is the full length $|\mathbf{a}|$. The gradient gives the direction of fastest growth, and the curl gives the direction of fastest turning:

maximum growth rate of f is $|\text{grad} f|$ in the direction of grad f

maximum rotation rate of **F** is $\frac{1}{2}|\text{curl } \mathbf{F}|$ in the direction of curl **F**.

STOKES' THEOREM

Finally we come to the big theorem. It will be like Green's Theorem—a line integral equals a surface integral. The line integral is still the work $\oint \mathbf{F} \cdot d\mathbf{R}$ around a curve. The surface integral in Green's Theorem is $\iint (N_x - M_y)\, dx\, dy$. The surface is flat (in the xy plane). Its normal direction is $\mathbf{k}$, and we now recognize $N_x - M_y$ as the $\mathbf{k}$ component of the curl. Green's Theorem uses only this component because the normal direction is always $\mathbf{k}$. For Stokes' Theorem on a curved surface, we need all three components of curl $\mathbf{F}$.

Figure 15.24 shows a hat-shaped surface S and its boundary C (a closed curve). Walking in the positive direction around C, with your head pointing in the direction of $\mathbf{n}$, the surface is *on your left*. You may be standing straight up ($\mathbf{n} = \mathbf{k}$ in Green's Theorem). You may even be upside down ($\mathbf{n} = -\mathbf{k}$ is allowed). In that case you must go the other way around C, to keep the two sides of equation (6) equal. The surface is still on the left. A Möbius strip is not allowed, because its normal direction cannot be established. The unit vector $\mathbf{n}$ determines the "counterclockwise direction" along C.

15O (***Stokes' Theorem***) $$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS. \qquad (6)$$

The right side adds up small spins in the surface. The left side is the total circulation (or work) around C. That is not easy to visualize—this may be the hardest theorem in the book—but notice one simple conclusion. ***If*** curl $\mathbf{F} = \mathbf{0}$ ***then*** $\oint \mathbf{F} \cdot d\mathbf{R} = 0$. ***This applies above all to gradient fields***—as we know.

A gradient field has no curl, by (4). A gradient field does no work, by (6). In three dimensions as in two dimensions, ***gradient fields are conservative fields***. They will be the focus of this section, after we outline a proof (or two proofs) of Stokes' Theorem.

The first proof shows *why* the theorem is true. The second proof shows that it really is true (and how to compute). You may prefer the first.

First proof Figure 15.24 has a triangle ABC attached to a triangle ACD. Later there can be more triangles. S will be ***piecewise flat***, close to a curved surface. Two triangles are enough to make the point. In the plane of each triangle (they have different $\mathbf{n}$'s) Green's Theorem is known:

$$\oint_{AB+BC+CA} \mathbf{F} \cdot d\mathbf{R} = \iint_{ABC} \text{curl } \mathbf{F} \cdot \mathbf{n} dS \qquad \oint_{AC+CD+DA} \mathbf{F} \cdot d\mathbf{R} = \iint_{ACD} \text{curl } \mathbf{F} \cdot \mathbf{n} dS.$$

Now add. The right sides give $\iint$ curl $\mathbf{F} \cdot \mathbf{n} dS$ over the two triangles. On the left, ***the integral over CA cancels the integral over AC***. The "crosscut" disappears. That leaves $AB + BC + CD + DA$. This line integral goes around the outer boundary C—which is the left side of Stokes' Theorem.

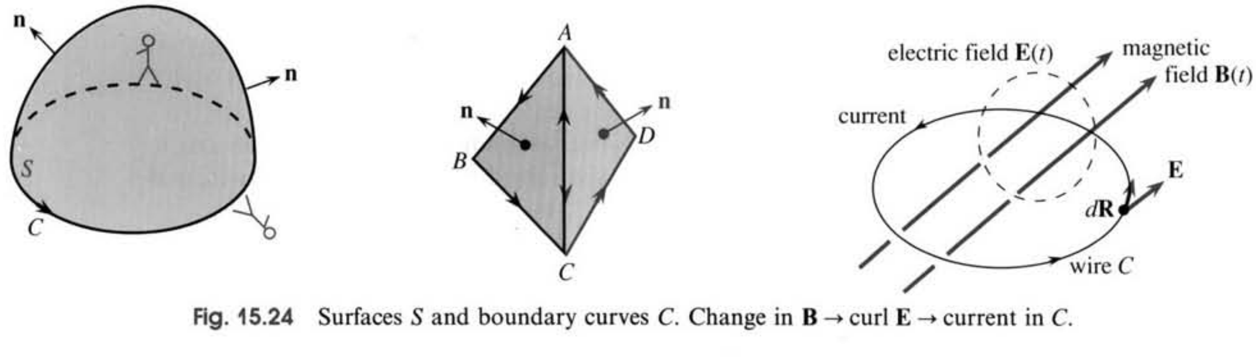


Fig. 15.24 Surfaces S and boundary curves C. Change in $\mathbf{B} \rightarrow$ curl $\mathbf{E} \rightarrow$ current in C.

Second proof Now the surface can be curved. A new proof may seem excessive, but it brings formulas you could compute with. From $z = f(x, y)$ we have

$$dz = \partial f/\partial x\, dx + \partial f/\partial y\, dy \qquad \text{and} \qquad \mathbf{n}dS = (-\partial f/\partial x\, \mathbf{i} - \partial f/\partial y\, \mathbf{j} + \mathbf{k})\, dx\, dy.$$

For $\mathbf{n}dS$, see equation 15.4.11. With this dz, the line integral in Stokes' Theorem is

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \oint_{\text{shadow of } C} M\, dx + N\, dy + P(\partial f/\partial x\, dx + \partial f/\partial y\, dy). \tag{7}$$

The dot product of curl $\mathbf{F}$ and $\mathbf{n}dS$ gives the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}dS$:

$$\iint_{\text{shadow of } S} [(P_y - N_z)(-\partial f/\partial x) + (M_z - P_x)(-\partial f/\partial y) + (N_x - M_y)]\, dx\, dy. \tag{8}$$

To prove (7) = (8), change M in Green's Theorem to $M + P\partial f/\partial x$. Also change N to $N + P\partial f/\partial y$. Then (7) = (8) is Green's Theorem down on the shadow (Problem 47). This proves Stokes' Theorem up on S. Notice how Green's Theorem (flat surface) was the key to both proofs of Stokes' Theorem (curved surface).

EXAMPLE 6 Stokes' Theorem in electricity and magnetism yields Faraday's Law.

Stokes' Theorem is not heavily used for calculations—equation (8) shows why. But the spin or curl or ***vorticity*** of a flow is absolutely basic in fluid mechanics. The other important application, coming now, is to electric fields. Faraday's Law is to Gauss's Law as Stokes' Theorem is to the Divergence Theorem.

Suppose the curve C is an actual wire. We can produce current along C by varying the magnetic field $\mathbf{B}(t)$. The flux $\varphi = \iint \mathbf{B} \cdot \mathbf{n}dS$, passing within C and changing in time, creates an electric field $\mathbf{E}$ that does work:

$$\textit{Faraday's Law} \text{ (integral form): work} = \oint_C \mathbf{E} \cdot d\mathbf{R} = -d\varphi/dt.$$

That is physics. It may be true, it may be an approximation. Now comes mathematics (surely true), which turns this integral form into a differential equation. Information at points is more convenient than information around curves. Stokes converts the line integral of $\mathbf{E}$ into the surface integral of curl $\mathbf{E}$:

$$\oint_C \mathbf{E} \cdot d\mathbf{R} = \iint_S \operatorname{curl} \mathbf{E} \cdot \mathbf{n}dS \text{ and also } -\partial\varphi/\partial t = \iint_S -(\partial \mathbf{B}/\partial t) \cdot \mathbf{n}dS.$$

These are equal for any curve C, however small. So the right sides are equal for any surface S. We squeeze to a point. The right hand sides give one of Maxwell's equations:

$$\textit{Faraday's Law} \text{ (differential form): } \operatorname{curl} \mathbf{E} = -\partial \mathbf{B}/\partial t.$$

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

The chapter ends with our constant and important question: Which fields do no work around closed curves? Remember test **D** for plane curves and plane vector fields:

$$\text{if } \partial M/\partial y = \partial N/\partial x \text{ then } \mathbf{F} \text{ is conservative and } \mathbf{F} = \operatorname{grad} f \text{ and } \oint \mathbf{F} \cdot d\mathbf{R} = 0.$$

Now allow a three-dimensional field like $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 + z)\mathbf{j} + y\mathbf{k}$. Does it do work around a space curve? *Or is it a gradient field*? That will require $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2 + z$ and $\partial f/\partial z = y$. ***We have three equations for one function*** $f(x, y, z)$. Normally they can't be solved. When test **D** is passed (now it is the three-dimensional test: curl $\mathbf{F} = \mathbf{0}$) they *can* be solved. This example passes test **D**, and f is $x^2y + yz$.

15P $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a conservative field if it has these properties:
A. The work $\oint \mathbf{F} \cdot d\mathbf{R}$ around every closed path in space is zero.
B. The work $\int_P^Q \mathbf{F} \cdot d\mathbf{R}$ depends only on P and Q, not on the path in space.
C. $\mathbf{F}$ is a ***gradient field***: $M = \partial f/\partial x$ and $N = \partial f/\partial y$ and $P = \partial f/\partial z$.
D. The components satisfy $M_y = N_x$, $M_z = P_x$, and $N_z = P_y$ (curl $\mathbf{F}$ ***is zero***).
A field with one of these properties has them all. **D** is the quick test.

A detailed proof of $\mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \mathbf{C} \Rightarrow \mathbf{D} \Rightarrow \mathbf{A}$ is not needed. Only notice how $\mathbf{C} \Rightarrow \mathbf{D}$: curl grad $\mathbf{F}$ is always zero. The newest part is $\mathbf{D} \Rightarrow \mathbf{A}$. ***If*** curl $\mathbf{F} = \mathbf{0}$ ***then*** $\oint \mathbf{F} \cdot d\mathbf{R} = 0$. But that is not news. It is Stokes' Theorem.

The interesting problem is to solve the three equations for f, when test **D** is passed. The example above had

$$\partial f/\partial x = 2xy \Rightarrow f = \int 2xy\,dx = x^2y \text{ plus any function } C(y, z)$$

$$\partial f/\partial y = x^2 + z = x^2 + \partial C/\partial y \Rightarrow C = yz \text{ plus any function } c(z)$$

$$\partial f/\partial z = y = y + dc/dz \Rightarrow c(z) \text{ can be zero.}$$

The first step leaves an arbitrary $C(y, z)$ to fix the second step. The second step leaves an arbitrary $c(z)$ to fix the third step (not needed here). Assembling the three steps, $f = x^2y + C = x^2y + yz + c = x^2y + yz$. Please recognize that the "fix-up" is only possible when curl $\mathbf{F} = \mathbf{0}$. Test **D** must be passed.

EXAMPLE 7 Is $\mathbf{F} = (z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k}$ the gradient of any f?

Test **D** says *no*. This $\mathbf{F}$ is a spin field $\mathbf{a} \times \mathbf{R}$. Its curl is $2\mathbf{a} = (2, 2, 2)$, which is not zero. A search for f is bound to fail, but we can try. To match $\partial f/\partial x = z - y$, we must have $f = zx - yx + C(y, z)$. The y derivative is $-x + \partial C/\partial y$. That never matches $N = x - z$, so f can't exist.

EXAMPLE 8 What choice of P makes $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + P\mathbf{k}$ conservative? Find f.

Solution We need curl $\mathbf{F} = 0$, by test **D**. First check $\partial M/\partial y = z^2 = \partial N/\partial x$. Also

$$\partial P/\partial x = \partial M/\partial z = 2yz \qquad \text{and} \qquad \partial P/\partial y = \partial N/\partial z = 2xz.$$

$P = 2xyz$ passes all tests. To find f we can solve the three equations, or notice that $f = xyz^2$ is successful. Its gradient is $\mathbf{F}$.

A third method defines $f(x, y, z)$ as ***the work to reach*** (x, y, z) ***from*** $(0, 0, 0)$. The path doesn't matter. For practice we integrate $\mathbf{F} \cdot d\mathbf{R} = M\,dx + N\,dy + P\,dz$ along the straight line (xt, yt, zt):

$$f(x, y, z) = \int_0^1 (yt)(zt)^2(x\,dt) + (xt)(zt)^2(y\,dt) + 2(xt)(yt)(zt)(z\,dt) = xyz^2.$$

EXAMPLE 9 Why is div curl grad f automatically zero (in two ways)?

Solution First, curl grad f is zero (always). Second, div curl $\mathbf{F}$ is zero (always). Those are the key identities of vector calculus. We end with a review.

Green's Theorem:
$$\oint \mathbf{F} \cdot d\mathbf{R} = \iint (\partial N/\partial x - \partial M/\partial y)\,dx\,dy$$

$$\oint \mathbf{F} \cdot \mathbf{n}\,ds = \iint (\partial M/\partial x + \partial N/\partial y)\,dx\,dy$$

$$\textit{Divergence Theorem}: \quad \iint \mathbf{F}\cdot\mathbf{n}\,dS = \iiint (\partial M/\partial x + \partial N/\partial y + \partial P/\partial z)\,dx\,dy\,dz$$

$$\textit{Stokes' Theorem}: \quad \oint \mathbf{F}\cdot d\mathbf{R} = \iint \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS.$$

The first form of Green's Theorem leads to Stokes' Theorem. The second form becomes the Divergence Theorem. You may ask, ***why not go to three dimensions in the first place***? The last two theorems contain the first two (take $P=0$ and a flat surface). We could have reduced this chapter to two theorems, not four. I admit that, but a fundamental principle is involved: "It is easier to generalize than to specialize." For the same reason df/dx came before partial derivatives and the gradient.

15.6 EXERCISES

Read-through questions

The curl of $M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is the vector __a__. It equals the 3 by 3 determinant __b__. The curl of $x^2\mathbf{i}+z^2\mathbf{k}$ is __c__. For $\mathbf{S}=y\mathbf{i}-(x+z)\mathbf{j}+y\mathbf{k}$ the curl is __d__. This $\mathbf{S}$ is a __e__ field $\mathbf{a}\times\mathbf{R}=\frac{1}{2}(\text{curl }\mathbf{F})\times\mathbf{R}$, with axis vector $\mathbf{a}=$ __f__. For any gradient field $f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}$ the curl is __g__. That is the important identity curl grad $f=$ __h__. It is based on $f_{xy}=f_{yx}$ and __i__ and __j__. The twin identity is __k__.

The curl measures the __l__ of a vector field. A paddlewheel in the field with its axis along $\mathbf{n}$ has turning speed __m__. The spin is greatest when $\mathbf{n}$ is in the direction of __n__. Then the angular velocity is __o__.

Stokes' Theorem is __p__ = __q__. The curve C is the __r__ of the __s__ S. This is __t__ Theorem extended to __u__ dimensions. Both sides are zero when $\mathbf{F}$ is a gradient field because __v__.

The four properties of a conservative field are $\mathbf{A}=$ __w__, $\mathbf{B}=$ __x__, $\mathbf{C}=$ __y__, $\mathbf{D}=$ __z__. The field $y^2z^2\mathbf{i}+2xy^2z\mathbf{k}$ (passes)(fails) test $\mathbf{D}$. This field is the gradient of $f=$ __A__. The work $\int\mathbf{F}\cdot d\mathbf{R}$ from $(0,0,0)$ to $(1,1,1)$ is __B__ (on which path?). For every field $\mathbf{F}$, $\iint$ curl $\mathbf{F}\cdot\mathbf{n}\,dS$ is the same out through a pyramid and up through its base because __C__.

In Problems 1–6 find curl F.

1 $\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$

2 $\mathbf{F}=\text{grad}(xe^y\sin z)$

3 $\mathbf{F}=(x+y+z)(\mathbf{i}+\mathbf{j}+\mathbf{k})$

4 $\mathbf{F}=(x+y)\mathbf{i}-(x+y)\mathbf{k}$

5 $\mathbf{F}=\rho^n(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

6 $\mathbf{F}=(\mathbf{i}+\mathbf{j})\times\mathbf{R}$

7 Find a potential f for the field in Problem 3.

8 Find a potential f for the field in Problem 5.

9 When do the fields $x^m\mathbf{i}$ and $x^n\mathbf{j}$ have zero curl?

10 When does $(a_1x+a_2y+a_3z)\mathbf{k}$ have zero curl?

In 11–14, compute curl F and find $\oint_C\mathbf{F}\cdot d\mathbf{R}$ by Stokes' Theorem.

11 $\mathbf{F}=x^2\mathbf{i}+y^2\mathbf{k}$, $C=$ circle $x^2+z^2=1$, $y=0$.

12 $\mathbf{F}=\mathbf{i}\times\mathbf{R}$, $C=$ circle $x^2+z^2=1$, $y=0$.

13 $\mathbf{F}=(\mathbf{i}+\mathbf{j})\times\mathbf{R}$, $C=$ circle $y^2+z^2=1$, $x=0$.

14 $\mathbf{F}=(y\mathbf{i}-x\mathbf{j})\times(x\mathbf{i}+y\mathbf{j})$, $C=$ circle $x^2+y^2=1$, $z=0$.

15 (important) Suppose two surfaces S and T have the same boundary C, and the direction around C is the same.
(a) Prove $\iint_S$ curl $\mathbf{F}\cdot\mathbf{n}\,dS=\iint_T$ curl $\mathbf{F}\cdot\mathbf{n}\,dS$.
(b) Second proof: The difference between those integrals is $\iiint$ div(curl $\mathbf{F}$)dV. By what Theorem? What region is V? Why is this integral zero?

16 In 15, suppose S is the top half of the earth ($\mathbf{n}$ goes out) and T is the bottom half ($\mathbf{n}$ comes in). What are C and V? Show by example that $\iint_S\mathbf{F}\cdot\mathbf{n}\,dS=\iint_T\mathbf{F}\cdot\mathbf{n}\,dS$ is *not* generally true.

17 Explain why $\iint$ curl $\mathbf{F}\cdot\mathbf{n}\,dS=0$ over the closed boundary of any solid V.

18 Suppose curl $\mathbf{F}=\mathbf{0}$ and div $\mathbf{F}=0$. (a) Why is $\mathbf{F}$ the gradient of a potential? (b) Why does the potential satisfy Laplace's equation $f_{xx}+f_{yy}+f_{zz}=0$?

In 19–22, find a potential f if it exists.

19 $\mathbf{F}=z\mathbf{i}+\mathbf{j}+x\mathbf{k}$.

20 $\mathbf{F}=2xyz\mathbf{i}+x^2z\mathbf{j}+x^2y\mathbf{k}$

21 $\mathbf{F}=e^{x-z}\mathbf{i}-e^{x-z}\mathbf{k}$

22 $\mathbf{F}=yz\mathbf{i}+xz\mathbf{j}+(xy+z^2)\mathbf{k}$

23 Find a field with curl $\mathbf{F}=(1,0,0)$.

24 Find all fields with curl $\mathbf{F}=(1,0,0)$.

25 $\mathbf{S}=\mathbf{a}\times\mathbf{R}$ is a spin field. Compute $\mathbf{F}=\mathbf{b}\times\mathbf{S}$ (constant vector $\mathbf{b}$) and find its curl.

26 How fast is a paddlewheel turned by the field $\mathbf{F} = y\mathbf{i} - x\mathbf{k}$ (a) if its axis direction is $\mathbf{n} = \mathbf{j}$? (b) if its axis is lined up with curl $\mathbf{F}$? (c) if its axis is perpendicular to curl $\mathbf{F}$?

27 How is curl $\mathbf{F}$ related to the angular velocity ω in the spin field $\mathbf{F} = \omega(-y\mathbf{i} + x\mathbf{j})$? How fast does a wheel spin, if it is in the plane $x + y + z = 1$?

28 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S} = y\mathbf{i} - x\mathbf{j}$.

29 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S} = \mathbf{a} \times \mathbf{R}$.

30 ***True or false***: when two vector fields have the same curl at all points: (a) their difference is a constant field (b) their difference is a gradient field (c) they have the same divergence.

In 31–34, compute $\iint$ curl $\mathbf{F} \cdot \mathbf{n}dS$ over the top half of the sphere $x^2 + y^2 + z^2 = 1$ and (separately) $\oint \mathbf{F} \cdot d\mathbf{R}$ around the equator.

31 $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ **32** $\mathbf{F} = \mathbf{R}/\rho^2$

33 $\mathbf{F} = \mathbf{a} \times \mathbf{R}$ **34** $\mathbf{F} = (\mathbf{a} \times \mathbf{R}) \times \mathbf{R}$

35 The circle C in the plane $x + y + z = 6$ has radius r and center at $(1, 2, 3)$. The field $\mathbf{F}$ is $3z\mathbf{j} + 2y\mathbf{k}$. Compute $\oint \mathbf{F} \cdot d\mathbf{R}$ around C.

36 S is the top half of the unit sphere and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$. Find $\iint$ curl $\mathbf{F} \cdot \mathbf{n}dS$.

37 Find $g(x, y)$ so that curl $g\mathbf{k} = y\mathbf{i} + x^2\mathbf{j}$. What is the name for g in Section 15.3? It exists because $y\mathbf{i} + x^2\mathbf{j}$ has zero ________.

38 Construct $\mathbf{F}$ so that curl $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} - 5z\mathbf{k}$ (which has zero divergence).

39 Split the field $\mathbf{F} = xy\mathbf{i}$ into $\mathbf{V} + \mathbf{W}$ with curl $\mathbf{V} = 0$ and div $\mathbf{W} = 0$.

40 Ampère's law for a steady magnetic field $\mathbf{B}$ is curl $\mathbf{B} = \mu\mathbf{J}$ ($\mathbf{J}$ = current density, μ = constant). Find the work done by $\mathbf{B}$ around a space curve C from the current passing through it. Maxwell allows varying currents which brings in the electric field.

41 For $\mathbf{F} = (x^2 + y^2)\mathbf{i}$, compute curl (curl $\mathbf{F}$) and grad (div $\mathbf{F}$) and $\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz}$.

42 For $\mathbf{F} = v(x, y, z)\mathbf{i}$, prove these useful identities:

(a) curl(curl $\mathbf{F}$) = grad (div $\mathbf{F}$) $- (\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz})$.

(b) curl$(f\mathbf{F}) = f$ curl $\mathbf{F}$ + (grad f) $\times \mathbf{F}$.

43 If $\mathbf{B} = \mathbf{a} \cos t$ (constant direction $\mathbf{a}$), find curl $\mathbf{E}$ from Faraday's Law. Then find the alternating spin field $\mathbf{E}$.

44 With $\mathbf{G}(x, y, z) = m\mathbf{i} + n\mathbf{j} + p\mathbf{k}$, write out $\mathbf{F} \times \mathbf{G}$ and take its divergence. Match the answer with $\mathbf{G} \cdot$ curl $\mathbf{F} - \mathbf{F} \cdot$ curl $\mathbf{G}$.

45 Write down Green's Theorem in the xz plane from Stokes' Theorem.

46 ***True or false***: $\nabla \times \mathbf{F}$ is perpendicular to $\mathbf{F}$.

47 (a) The second proof of Stokes' Theorem took $M^* = M(x, y, f(x, y)) + P(x, y, f(x, y))\partial f/\partial x$ as the M in Green's Theorem. Compute $\partial M^*/\partial y$ from the chain rule and product rule (there are five terms).

(b) Similarly $N^* = N(x, y, f) + P(x, y, f)\partial f/\partial y$ has the x derivative $N_x + N_z f_x + P_x f_y + P_z f_x f_y + P f_{yx}$. Check that $N_x^* - M_y^*$ matches the right side of equation (8), as needed in the proof.

48 "The shadow of the boundary is the boundary of the shadow." This fact was used in the second proof of Stokes' Theorem, going to Green's Theorem on the shadow. Give two examples of S and C and their shadows.

49 Which integrals are equal when C = boundary of S or S = boundary of V?

$\oint \mathbf{F} \cdot d\mathbf{R}$ $\quad \oint (\text{curl } \mathbf{F}) \cdot d\mathbf{R}$ $\quad \oint (\text{curl } \mathbf{F}) \cdot \mathbf{n}ds$ $\quad \iint \mathbf{F} \cdot \mathbf{n}dS$

$\iint \text{div } \mathbf{F} dS$ $\quad \iint (\text{curl } \mathbf{F}) \cdot \mathbf{n}dS$ $\quad \iint (\text{grad div } \mathbf{F}) \cdot \mathbf{n}dS$ $\quad \iiint \text{div } \mathbf{F} dV$

50 Draw the field $\mathbf{V} = -x\mathbf{k}$ spinning a wheel in the xz plane. What wheels would *not* spin?

Contiune on 16.7 Triple Integrals

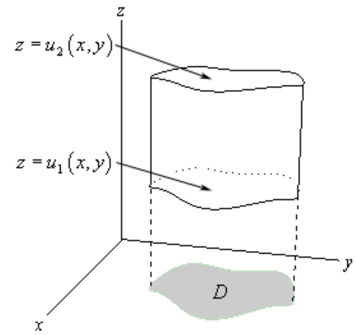


Figure 1:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

Applications of Triple Integrals Let E be a solid region with a density function $\rho(x,y,z)$.

Volume: $V(E) = \iiint_E 1dV$

Mass: $m = \iiint_E \rho(x,y,z)dV$

Moments about the coordinate planes:

$$M_{xy} = \iiint_E z\rho(x,y,z)dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)dV$$

$$M_{yz} = \iiint_E x\rho(x,y,z)dV$$

Center of mass: $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = M_{yz}/m \quad , \quad \bar{y} = M_{xz}/m \quad , \quad \bar{z} = M_{xy}/m \quad .$$

Remark: The center of mass is just the weighted average of the coordinate functions over the solid region. If $\rho(x,y,z) = 1$, the mass of the solid equals its volume and the center of mass is also called the **centroid** of the solid.

Example Find the volume of the solid region E between $y = 4 - x^2 - z^2$ and $y = x^2 + z^2$.

Soln: E is described by $x^2+z^2 \le y \le 4-x^2-z^2$ over a disk D in the xz-plane whose radius is given by the intersection of the two surfaces: $y=4-x^2-z^2$ and $y=x^2+z^2$.

$4-x^2-z^2 = x^2+z^2 \Rightarrow x^2+z^2=2$. So the radius is $\sqrt{2}$.

Therefore

$$V(E)=\iiint_E 1dV=\iint_D\left[\int_{x^2+z^2}^{4-x^2-z^2} 1dy\right]dA=\iint_D 4-2(x^2+z^2)dA$$
$$=\int_0^{2\pi}\int_0^{\sqrt{2}}(4-2r^2)rdrd\theta=\int_0^{2\pi}\left[2r^2-\frac{1}{2}r^4\right]_0^{\sqrt{2}}=4\pi$$

Example Find the mass of the solid region bounded by the sheet $z=1-x^2$ and the planes $z=0, y=-1, y=1$ with a density function $\rho(x,y,z)=z(y+2)$.

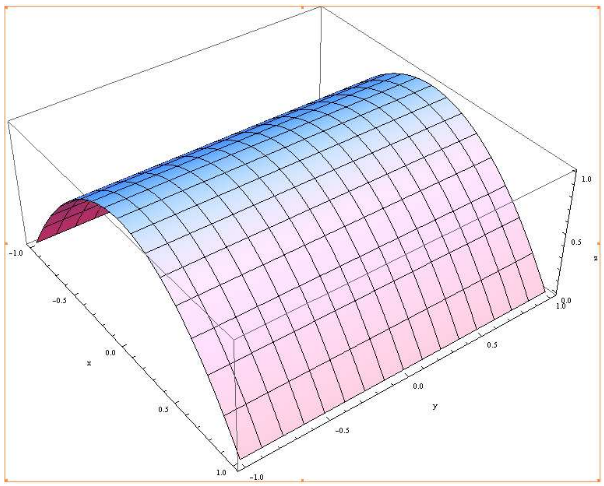


Figure 2:

Soln: The top surface of the solid is $z=1-x^2$ and the bottom surface is $z=0$ over the region D in the xy-plane which is bounded by the other equations in the xy-plane and the intersection of the top and bottom surfaces.

The intersection gives $1-x^2=0 \Rightarrow x=\pm 1$. Therefore D is a square $[-1,1]\times[-1,1]$.

$$m=\iiint_E \rho(x,y,z)dV=\iiint_E z(y+2)dV=\iint_D\left[\int_0^{1-x^2} z(y+2)dz\right]dA$$
$$=\int_{-1}^1\int_{-1}^1\int_0^{1-x^2} z(y+2)dzdxdy=\frac{1}{2}\int_{-1}^1\int_{-1}^1(1-x^2)^2(y+2)dxdy=$$
$$\frac{8}{15}\int_{-1}^1(y+2)dy=32/15$$

Example Find the centroid of the solid above the paraboloid $z=x^2+y^2$ and below the plane $z=4$.

Soln: The top surface of the solid is $z=4$ and the bottom surface is $z=x^2+y^2$ over the region D defined in the xy-plane by the intersection of the top and bottom surfaces.

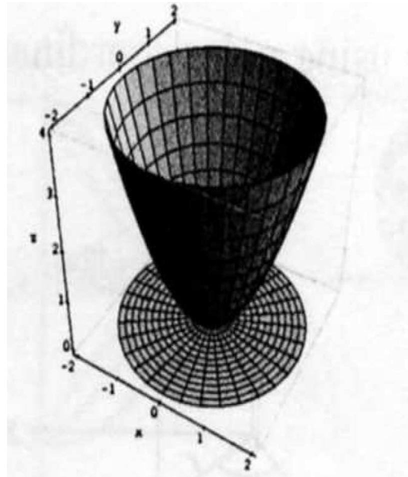


Figure 3:

The intersection gives $4 = x^2 + y^2$. Therefore D is a disk of radius 2.

By the symmetry principle, $\bar{x} = \bar{y} = 0$. We only compute $\bar{z}$:

$$m = \iiint_E 1dV = \iint_D \left[\int_{x^2+y^2}^4 1dz \right] dA = \iint_D 4 - (x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 8\pi$$

$$M_{xy} = \iiint_E z dV = \iint_D \left[\int_{x^2+y^2}^4 z dz \right] dA = \iint_D 8 - \frac{1}{2}(x^2 + y^2)^2 dA =$$

$$\int_0^{2\pi} \int_0^2 (8 - \frac{1}{2} r^4) r dr d\theta = \int_0^{2\pi} [4r^2 - \frac{1}{12} r^6]_0^2 d\theta = 64\pi/3.$$

Therefore $\bar{z} = M_{xy}/m = 8/3$ and the centroid is $(0, 0, 8/3)$.

16.8 Triple Integrals in Cylindrical and Spherical Coordinates

1. Triple Integrals in Cylindrical Coordinates

A point in space can be located by using polar coordinates r, θ in the xy-plane and z in the vertical direction.

Some equations in cylindrical coordinates (plug in $x = r\cos(\theta), y = r\sin(\theta)$):

Cylinder: $x^2 + y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$;
Sphere: $x^2 + y^2 + z^2 = a^2 \Rightarrow r^2 + z^2 = a^2$;
Cone: $z^2 = a^2(x^2 + y^2) \Rightarrow z = ar$;
Paraboloid: $z = a(x^2 + y^2) \Rightarrow z = ar^2$.

The formula for triple integration in cylindrical coordinates:

If a solid E is the region between $z = u_2(x, y)$ and $z = u_1(x, y)$ over a domain D in the xy-plane, which is described in polar coordinates by $\alpha \leq \theta \leq \beta$, $h_1(\theta) \leq r \leq h_2(\theta)$, we plug

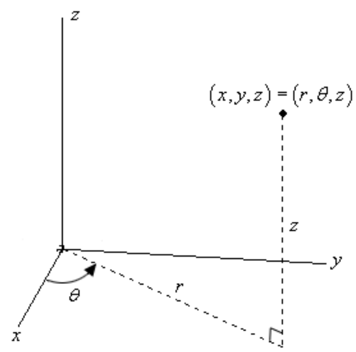


Figure 4:

in $x = r\cos(\theta), y = r\sin(\theta)$

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA =$$
$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

Note: $dV \to r dz dr d\theta$

Example Evaluate $\iiint_E z dV$ where E is the portion of the solid sphere $x^2+y^2+z^2 \le 9$ that is inside the cylinder $x^2+y^2=1$ and above the cone $x^2+y^2=z^2$.

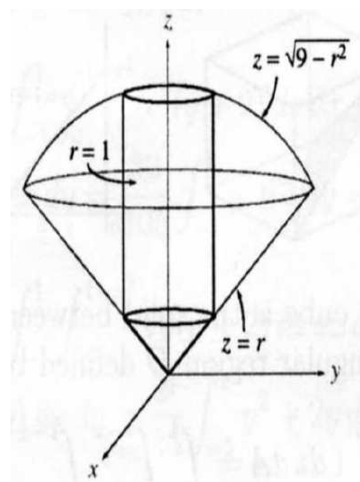


Figure 5:

Soln: The top surface is $z = u_2(x,y) = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$ and the bottom surface is $z = u_1(x,y) = \sqrt{x^2+y^2} = r$ over the region D defined by the intersection of the top (or bottom) and the cylinder which is a disk $x^2 + y^2 \leq 1$ or $0 \leq r \leq 1$ in the xy-plane.

$$\iiint_E zdV = \iint_D \left[\int_r^{\sqrt{9-r^2}} zdz\right] dA = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{9-r^2}} zrdzdrd\theta =$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2}[9-2r^2]rdrd\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}[9r-2r^3]drd\theta = \int_0^{2\pi}[9/4-1/4]d\theta = 4\pi$$

Example Find the volume of the portion of the sphere $x^2+y^2+z^2 = 4$ inside the cylinder $(y-1)^2 + x^2 = 1$.

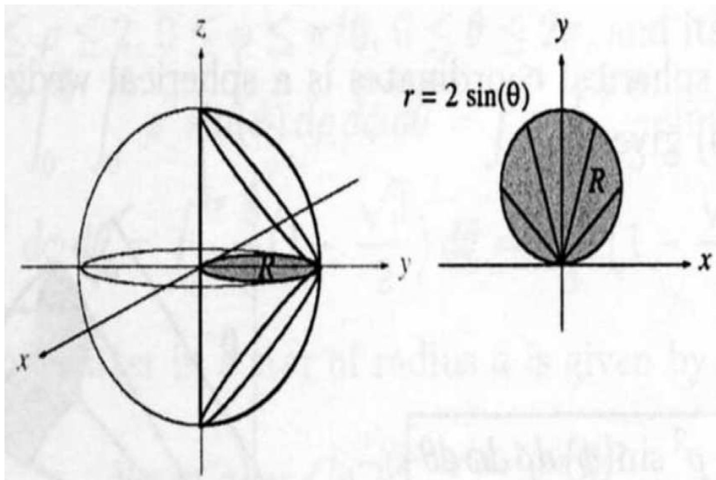


Figure 6:

Soln: The top surface is $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$ and the bottom is $z = -\sqrt{4-x^2-y^2} = -\sqrt{4-r^2}$ over the region D defined by the cylinder equation in the xy-plane. So rewrite the cylinder equation $x^2 + (y-1)^2 = 1$ as $x^2+y^2-2y+1 = 1 \Rightarrow r^2 = 2r\sin(\theta) \Rightarrow r = 2\sin(\theta)$.

$$V(E) = \iiint_E 1dV = \iint_D \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1dzdA = \int_0^{\pi}\int_0^{2\sin(\theta)}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1rdzdrd\theta =$$

$$\int_0^{\pi}\int_0^{2\sin(\theta)} 2r\sqrt{4-r^2}drd\theta \text{ (by substitution u} = 4-\text{r}^2\text{)} =$$

$$\int_0^{\pi} -\frac{2}{3}[(4-4\sin^2(\theta))^{3/2} - (4)^{3/2}]d\theta \text{ (use identity } 1 = \cos^2(\theta)+\sin^2(\theta)\text{)} =$$

$$\int_0^{\pi}\frac{16}{3}[1-|\cos(\theta)|^3]d\theta = \int_0^{\pi/2}\frac{16}{3}[1-\cos^3(\theta)]d\theta + \int_{\pi/2}^{\pi}\frac{16}{3}[1+\cos^3(\theta)]d\theta =$$

$$\int_0^{\pi/2}\frac{16}{3}[1-(1-\sin^2\theta)\cos\theta]d\theta + \int_{\pi/2}^{\pi}\frac{16}{3}[1+(1-\sin^2\theta)\cos\theta]d\theta =$$

$$16/3[(\theta-\sin\theta+\sin^3\theta/3)|_0^{\pi/2} + (\theta+\sin\theta-\sin^3\theta/3)|_{\pi/2}^{\pi}] = 16\pi/3 - 64/9$$

2. Triple Integrals in Spherical Coordinates

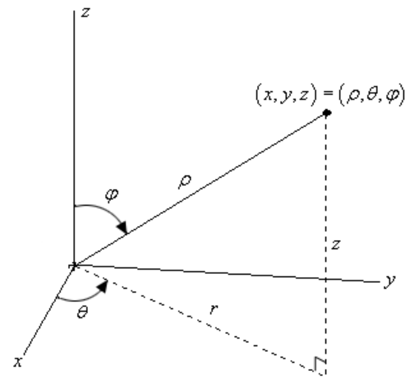


Figure 7:

In spherical coordinates, a point is located in space by longitude, latitude, and radial distance.

Longitude: $0 \leq \theta \leq 2\pi$;

Latitude: $0 \leq \phi \leq \pi$;

Radial distance: $\rho = \sqrt{x^2 + y^2 + z^2}$.

From $r = \rho \sin(\phi)$

$$x = r\cos(\theta) = \rho \sin(\phi)\cos(\theta)$$

$$y = r\sin(\theta) = \rho \sin(\phi)\sin(\theta)$$

$$z = \rho \cos(\phi)$$

Some equations in spherical coordinates:

Sphere: $x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = a$

Cone: $z^2 = a^2(x^2 + y^2) \Rightarrow \cos^2(\phi) = a^2 \sin^2(\phi)$

Cylinder: $x^2 + y^2 = a^2 \Rightarrow r = a$ or $\rho \sin(\phi) = a$

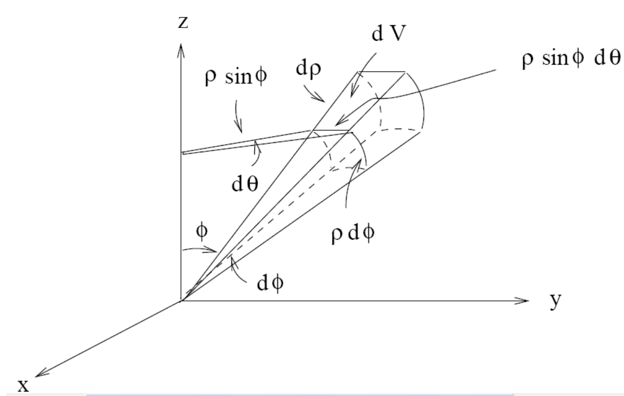


Figure 8: Spherical wedge element

The volume element in spherical coordinates is a spherical wedge with sides $d\rho, \rho d\phi, rd\theta$. Replacing r with $\rho\sin(\phi)$ gives:

$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$$

For our integrals we are going to restrict E down to a spherical wedge. This will mean $a \le \rho \le b$, $\alpha \le \theta \le \beta$, $c \le \phi \le d$,

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_c^d \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)d\rho d\phi d\theta$$

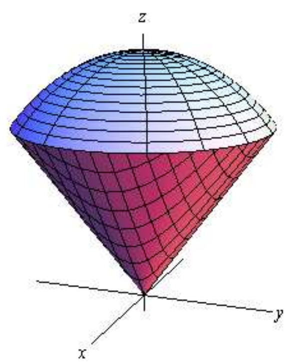


Figure 9: One example of the sphere wedge, the lower limit for both ρ and ϕ are 0

The more general formula for triple integration in spherical coordinates:
If a solid E is the region between $g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)$, $\alpha \le \theta \le \beta$, $c \le \phi \le d$, then

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)d\rho d\phi d\theta$$

Example Find the volume of the solid region above the cone $z^2 = 3(x^2+y^2)$ $(z \ge 0)$ and below the sphere $x^2+y^2+z^2 = 4$.

Soln: The sphere $x^2+y^2+z^2 = 4$ in spherical coordinates is $\rho = 2$. The cone $z^2 = 3(x^2+y^2)$ $(z \ge 0)$ in spherical coordinates is $z = \sqrt{3(x^2+y^2)} = \sqrt{3}r \Rightarrow \rho\cos(\phi) = \sqrt{3}\rho\sin(\phi) \Rightarrow \tan(\phi) = 1/\sqrt{3} \Rightarrow \phi = \pi/6$.

Thus E is defined by $0 \le \rho \le 2$, $0 \le \phi \le \pi/6$, $0 \le \theta \le 2\pi$.

$$V(E) = \iiint_E 1dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin(\phi)d\rho d\phi d\theta =$$
$$\int_0^{2\pi}\int_0^{\pi/6} \frac{8}{3}\sin(\phi)d\phi d\theta = \frac{16\pi}{3}(1-\frac{\sqrt{3}}{2})$$

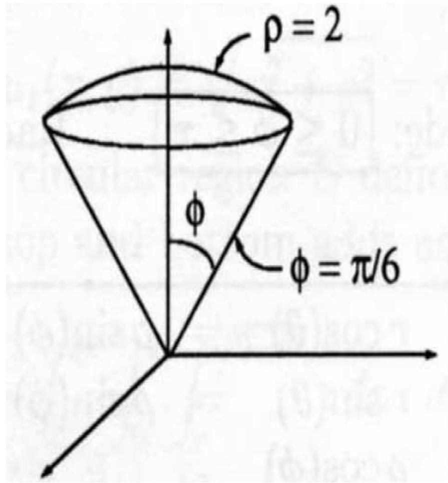


Figure 10:

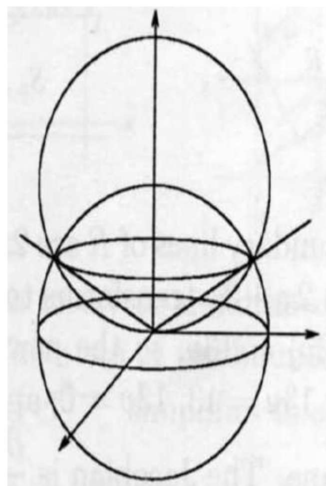


Figure 11:

Example Find the centroid of the solid region E lying inside the sphere $x^2+y^2+z^2 = 2z$ and outside the sphere $x^2 + y^2 + z^2 = 1$ *Soln:* By the symmetry principle, the centroid lies on the z axis. Thus we only need to compute $\bar{z}$

The top surface is $x^2 + y^2 + z^2 = 2z \Rightarrow \rho^2 = 2\rho\cos(\phi)$ or $\rho = 2\cos(\phi)$. The bottom surface is $x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$. They intersect at $2\cos(\phi) = 1 \Rightarrow \phi = \pi/3$.

$$m = \iiint_E 1dV = \int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos(\phi)} \rho^2\sin(\phi)d\rho d\phi d\theta =$$
$$\int_0^{2\pi}\int_0^{\pi/3}\frac{8}{3}\cos^3(\phi)\sin(\phi)d\phi d\theta - \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin(\phi)d\phi d\theta = \frac{11\pi}{12}$$

$$\bar{z} = M_{xy}/m = \frac{12}{11\pi}\iiint_E z dV = \frac{12}{11\pi}\int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos(\phi)} \rho\cos(\phi)\rho^2\sin(\phi)d\rho d\phi d\theta =$$

$$\frac{12}{11\pi}\left[\int_0^{2\pi}\int_0^{\pi/3} 4\cos^5(\phi)\sin(\phi)d\phi d\theta - \int_0^{2\pi}\int_0^{\pi/3} 1/4\cos(\phi)\sin(\phi)d\phi d\theta\right] =$$

$$\frac{12}{11\pi}\left[\frac{-4}{6}\cos^6(\phi)|_0^{\pi/3} - \frac{1}{4}\frac{\sin^2(\phi)}{2}|_0^{\pi/3}\right]$$

$$\frac{12}{11\pi}[9\pi/8] \simeq 1.2$$

Example Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 dzdxdy$ into spherical coordinates.

Soln: We first write down the limits of the variables:

$$0 \leq y \leq 3$$

$$0 \leq x \leq \sqrt{9-y^2}$$

$$\sqrt{x^2+y^2} \leq z \leq \sqrt{18-x^2-y^2}$$

.

The range for x tells us that we have a portion of the right half of a disk of radius 3 centered at the origin. Since $y \geq 0$, we will have the quarter disk in the first quadrant. Therefore since D is in the first quadrant the region, E, must be in the first octant and this in turn tells us that we have the following range for θ

$$0 \leq \theta \leq \pi/2$$

.

Now, lets see what the range for z tells us. The lower bound, $z = \sqrt{x^2+y^2}$ is the upper half of a cone $z^2 = x^2 + y^2$. The upper bound, $z = \sqrt{18-x^2-y^2}$ is the upper half of the sphere $x^2 + y^2 + z^2 = 18$. So the range for ρ

$$0 \leq \rho \leq \sqrt{18}$$

Now we try to find the range for ϕ. We can get it from the equation of the cone. In spherical coordinates, the equation of the cone is $1 = \tan(\phi)$, which gives $\phi = \pi/4$. We have the range for ϕ

$$0 \leq \phi \leq \pi/4$$

.

Thus

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 dzdxdy = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{18}} \rho^4\sin(\phi)d\rho d\theta d\phi$$

Final Review

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August 6, 2019

Overview

1. Green's Identities
2. Exercises
3. Substitution Rule
   - Green's Theorem(Similar to Slide 608)

Green's Identities

- The upshot of Green's identities is that it can shown that the Neumann and Dirichlet problems are unique. In the sample exam you have shown that neuman problem is a constant and in fact if you plug in similar details and argue that the square of an integral is zero then it must be zero almost everywhere. You can see that the Dirichet condition also yield similar result.

Boundary operator

$$Bu := \alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$$

where $\alpha, \beta : \partial\Omega \to \mathbb{R}$ with

$$\alpha(x) \geq 0, \qquad \beta(x) \geq 0, \qquad \alpha(x) + \beta(x) > 0 \qquad \text{on } \partial\Omega.$$

Special cases: boundary conditions

- of the first kind (Dirichlet): $\beta(x) = 0$ for all x
- of the second kind (Neumann): $\alpha(x) = 0$ for all x
- of the third kind (Robin): $\alpha(x), \beta(x) \neq 0$ for all x

Green's Identities

in the theory of differential equations. Green's identities are your friend here.

Exercise 8. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set and $u\colon \Omega \to \mathbb{R}$ a twice differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0.$$

i) Prove that u is constant, i.e., there exists some $c \in \mathbb{R}$ such that $u(x) = c$ for all $x \in \Omega$.

ii) Interpret the statement of this exercise physically in terms of fluid flow (potential flow) in a container.

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega u \nabla^2 u dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds. \tag{5}$$

If $\nabla^2 u = 0$ in Ω then for $u \in C^2(\bar{\Omega})$, it follows that

$$\begin{aligned} & \int_\Omega |\nabla u|^2 dx = 0 \\ \Longrightarrow \quad & \nabla u = 0 \\ \Longrightarrow \quad & u = \text{constant}. \end{aligned}$$

This observation leads to uniqueness theorems for the Dirichlet problem and the Neumann problem.

Results

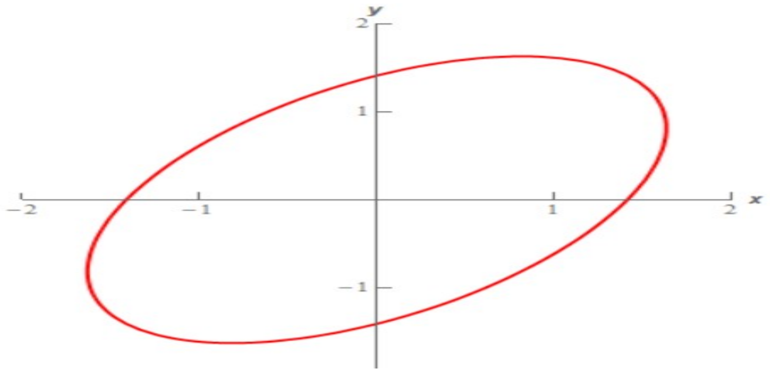
- In polar coordinates dA=rdrdθ
  (r,θ)$\mapsto$(x,y) x=g_1(r,θ)=rcosθ y=g_2(r,θ)=rsinθ
  $$\det J_g(r,\theta)=\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}=r$$
  $$dxdy = dA = rdrd\theta$$
- In spherical coordinates the factor is $R^2 sin\theta$. That is, dV=r^2sinϕ drdθdϕ, where ϕ is the angle formed with respect to the xy plane
- In cylindrical coordinates the factor is R^2

A more sophisticated example

Example

Evaluate $\iint_R x^2 - xy + y^2 dA$ where R is the ellipse given by x^2-xy+$y^2 \leq 2$.

Consider the parametization such that x=au-bv and y=au+bv(To transform a eclipse to a circle). And use undetermined coefficients to sort out a and b.



A more sophisticated Example

Solution

To start with,

$$a = \sqrt{2}, b = \sqrt{\frac{2}{3}}$$

And the transformation yields

$$u^2 + v^2 \leq 1$$

which is a lot friendlier to deal with. Now the Jacobian,

$$\det J_g(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{pmatrix} = \frac{4}{\sqrt{3}}$$

The integration region is essentially a circle where the area is π

$$\iint_R x^2 - xy + y^2 dA = \frac{4}{\sqrt{3}}\pi$$

Comments

- As a general rule of thumb, you might find parametrize the following objects in cylindrical coordinates more convenient

$$\begin{aligned}
&\text{Cylinder: } x^2+y^2=a^2 \Rightarrow r^2=a^2 \Rightarrow r=a;\\
&\text{Sphere: } x^2+y^2+z^2=a^2 \Rightarrow r^2+z^2=a^2;\\
&\text{Cone: } z^2=a^2(x^2+y^2) \Rightarrow z=ar;\\
&\text{Paraboloid: } z=a(x^2+y^2) \Rightarrow z=ar^2.
\end{aligned}$$

- While in Spherical coordinates you might find the following useful

$$\begin{aligned}
&\text{Sphere: } x^2+y^2+z^2=a^2 \Rightarrow \rho=a\\
&\text{Cone: } z^2=a^2(x^2+y^2) \Rightarrow \cos^2(\phi)=a^2\sin^2(\phi)\\
&\text{Cylinder: } x^2+y^2=a^2 \Rightarrow r=a \text{ or } \rho\sin(\phi)=a
\end{aligned}$$

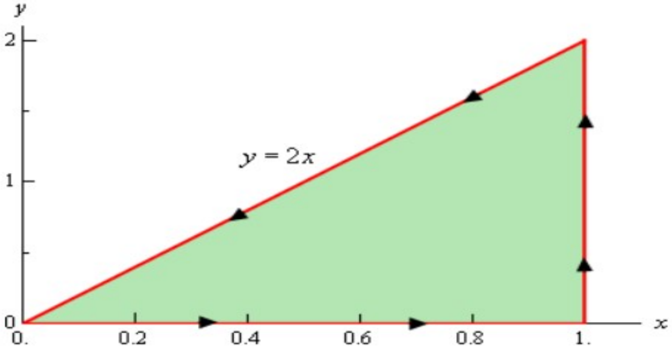
Some other integration formula

- Mass:
$$m = \iiint_E \rho(x, y, z)dV$$
where ρ is the mass density at (x,y,z)
- Volume:
$$V = \iiint_E 1 dV$$
- Moments(i th, usually three dimension):
$$m_k(E) = \iiint_E x_i \rho dV$$

Green's theorem

Example

Use Green's Theorem to evaluate $\oint_C xydx + x^2y^3dy$ where C is the triangle with vertices (0,0), (1,0),(1,2)



Green's Theorem

Apply the theorem To start with F(x,y)=(xy,x^2y^3). Therefore,

$$\oint_C xydx + x^2y^3dy = \iint_D -\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} dA = \iint_D 2xy^3 - xdA$$

Parametrize the region We put down the region R:

$$R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], 0 \leq y \leq 2x\}$$

Integrate

$$\iint_D 2xy^3 - xdA = \int_0^1 \int_0^{2x} 2xy^3 - xdydx = (\frac{4}{3}x^6 - \frac{2}{3}x^3)|_0^1 = \frac{2}{3}$$

A common yet important example

EXAMPLE 3 An ***electrostatic field*** $\mathbf{R}/\rho^3$ or ***gravity field*** $-\mathbf{R}/\rho^3$ almost has div $\mathbf{F}=0$.

The vector $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ has length $\sqrt{x^2+y^2+z^2}=\rho$. Then $\mathbf{F}$ has length ρ/ρ^3 (inverse square law). Gravity from a point mass pulls *inward* (minus sign). The electric field from a point charge repels *outward*. The three steps almost show that div $\mathbf{F}=0$:

Step 1. $\partial\rho/\partial x = x/\rho$, $\partial\rho/\partial y = y/\rho$, $\partial\rho/\partial z = z/\rho$—but do not add those three. $\mathbf{F}$ is not ρ or $1/\rho^2$ (these are scalars). The vector field is $\mathbf{R}/\rho^3$. We need $\partial M/\partial x$, $\partial N/\partial y$, $\partial P/\partial z$.

Step 2. $\partial M/\partial x = \partial/\partial x(x/\rho^3)$ is equal to $1/\rho^3-(3x\,\partial\rho/\partial x)/\rho^4 = 1/\rho^3-3x^2/\rho^5$. For $\partial N/\partial y$ and $\partial P/\partial z$, replace $3x^2$ by $3y^2$ and $3z^2$. Now add those three.

Step 3. div $\mathbf{F} = 3/\rho^3 - 3(x^2+y^2+z^2)/\rho^5 = 3/\rho^3-3/\rho^3 = 0$.