1.) Elementary algebra

1.1 $$\frac{x^{32}}{x^{9}\cdot x^{2}}\cdot\frac{x^{7}}{x^{2}}=\frac{x^{32}\cdot x^{7}}{x^{9}\cdot x^{2}\cdot x^{2}}=\frac{x^{32+7}}{x^{9+2+2}}=\frac{x^{39}}{x^{13}}=\boxed{x^{26}}$$

$$a^n\cdot a^m=a^{n+m}$$

$$\frac{a^n}{a^m}=a^{n-m}$$

1.2 $$8^2\cdot 4^x\cdot 2^x=8^4 \quad /\div 8^2$$

$$4^x\cdot 2^x=\frac{8^{\cancel{4}2}}{\cancel{8^2}}$$

$$4^x\cdot 2^x=8^2$$

$$(2\cdot 2)^x\cdot 2^x=(2^3)^2$$

$$2^{2x}\cdot 2^x=2^6$$

$$2^{3x}=2^6$$

$$3x=6$$

$$\boxed{x=2}$$

1.3. $$\frac{x}{y}=3 \qquad x^{-4}\cdot y^4=?$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{y}{x}=\frac{1}{\frac{x}{y}}=\frac{1}{3} \qquad x^{-4}=\frac{1}{x^4}$$

$$x^{-4}\cdot y^4=\frac{y^4}{x^4}=\left(\frac{y}{x}\right)^4=\left(\frac{1}{3}\right)^4=\boxed{\frac{1}{81}}$$

1.4. $$\frac{\sqrt{4^{15}}}{\sqrt{16^7}}=\frac{\sqrt{4^{15}}}{\sqrt{(4\cdot4)^7}}=\frac{\sqrt{4^{15}}}{\sqrt{(4^2)^7}}=\frac{\sqrt{4^{15}}}{\sqrt{4^{14}}}=\sqrt{\frac{4^{15}}{4^{14}}}=\sqrt{4}=\boxed{\pm 2}$$

1.5.

(a) $x+(y+z)=(y+x)+z$ **TRUE** (commutativity of addition)

(b) $y(x+z)=xy+zy$ **TRUE** (multiplication is distributive over addition)

(c) $x^{y+z}=x^z+x^y$ **FALSE** $\left(x^{y+z}=x^y \cdot x^z\right)$

(d) $\dfrac{x^z}{x^y}=x^{y-z}$ **FALSE** $\left(\dfrac{x^z}{x^y}=x^{z-y}\right)$

~~B) Functions of one variable~~

1.6.

$\ln(x) \geq e$ $\Rightarrow$ Let's assum it is equal, then:

$\Downarrow$

$\ln(x)=e$

$e^{\ln(x)}=x$

$e^{e}=x$

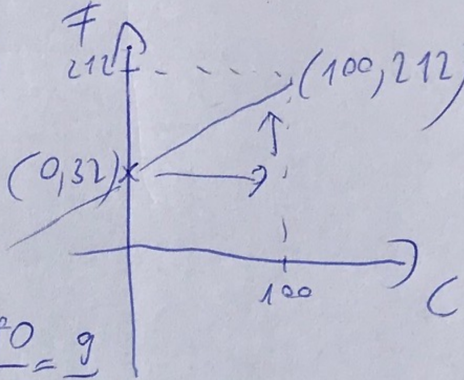
$x \approx 15,15426$

If $\ln(x) \geq e$ $\Rightarrow$ $\boxed{x \geq e^e}$

2.) Functions of one variable

2.1.

1. F (dependent), C (independent)



$y = a + bx$

$°F = 32 + b°C$ where

$b = \text{slope} = \dfrac{212-32}{100-0} = \dfrac{180}{100} = \dfrac{9}{5}$

$$\boxed{°F = 32 + \frac{9}{5}°C}$$

2. C (dependent), F (independent)

$°F = 32 + \dfrac{9}{5}°C$

$\dfrac{9}{5}°C = °F - 32$

$$\boxed{°C = \frac{5}{9}(°F - 32)}$$

2.2

$f(x) = 3x - 12$, find y if $f(y) = 0$

- What is the relation between x and y?
- If there is no relation, then:

$f(x) = 0 = 0 \cdot y$, which is true for $\boxed{\forall y \in \mathbb{R}}$

2.3.

$$9^{x^2-6x+2} = 81$$

$$9^{x^2-6x+2} = 9^2$$

$$x^2-6x+2=2$$

$$x^2-6x=0$$

$$x(x-6)=0$$

$$\boxed{x_1=0} \qquad \boxed{x_2=6}$$

2.4. Let's assume GDP = 1, then:

$$(1+0.03)^x = 3\cdot 1$$

$$(1.03)^x = 3$$

$$\ln(1.03)^x = \ln 3$$

$$x \ln 1.03 = \ln 3$$

$$x = \frac{\ln 3}{\ln 1.03} \approx \boxed{37,167}$$

$\rightarrow$ It takes 37-38 years to triple the GDP.

2.5.

$$\log_{\pi}\left(\frac{1}{\pi^5}\right) = x \Rightarrow \pi^x = \frac{1}{\pi^5} \Rightarrow \pi^x = (\pi)^{-5}$$

$$\boxed{x=-5}$$

3, Calculus

(a) (b)

3.1 $$\sum_{i=0}^{\infty}\left(\frac{1}{5^i}+0.3^i\right) = \sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^i + \sum_{i=0}^{\infty}\left(\frac{3}{10}\right)^i$$

if $a_n = c b^n$ & $0 < b < 1$ $\sum_{i=1}^{\infty} a_i = \frac{cb}{1-b}$

(a) $$\sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^i = \left(\frac{1}{5}\right)^0 + \sum_{i=1}^{\infty}\left(\frac{1}{5}\right)^i = \left(\frac{1}{5}\right)^0 + \frac{1\cdot\frac{1}{5}}{1-\frac{1}{5}} = \left(\frac{1}{5}\right)^0 + \frac{\frac{1}{5}}{\frac{4}{5}} =$$

$$= \left(\frac{1}{5}\right)^0 + \frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{4} = 1 + \frac{1}{4}$$

(b) $$\sum_{i=0}^{\infty}\left(\frac{3}{10}\right)^i = \left(\frac{3}{10}\right)^0 + \sum_{i=1}^{\infty}\left(\frac{3}{10}\right) = \left(\frac{3}{10}\right)^0 + \frac{1\cdot\frac{3}{10}}{1-\frac{3}{10}} = \left(\frac{3}{10}\right)^0 + \frac{\frac{3}{10}}{\frac{7}{10}} =$$

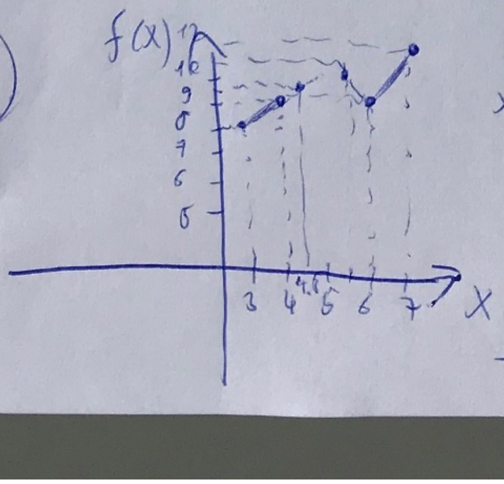
$$= \left(\frac{3}{10}\right)^0 + \frac{3}{\cancel{10}}\cdot\frac{\cancel{10}}{7} = 1 + \frac{3}{7} = \frac{10}{7}$$

$$(a) + (b) = 1 + \frac{1}{4} + \frac{10}{7} = \frac{5}{4} + \frac{10}{7} = \frac{35}{28} + \frac{40}{28} = \boxed{\frac{75}{28}}$$

3.3 $f(x) = x^3 - 4 \quad f'(x) = 3x^2$

$$f'(2) = 3(-2)^2 = 3\cdot 4 = \boxed{12}$$

3.2



$$\lim_{x\to 5}\frac{x^2-25}{x-5} = \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = x+5 = \boxed{10}$$

3.4. $f(x)=\frac{x^5+3}{x^2-1} \Rightarrow g(x)=x^5+3, \quad h(x)=x^2-1$

$$f'(x)=\frac{g'(x)\cdot h(x)-h'(x)\,g(x)}{(h(x))^2}=\frac{5x^4(x^2-1)-(2x)(x^5+3)}{(x^2-1)^2}=$$

$$=\boxed{\frac{5x^6-5x^4-2x^6-6x}{(x^2-1)^2}}$$

3.5.

$f(x)=x^9+3$

$f'(x)=9x^8$

$\boxed{f''(x)=72x^7}$

3.6 $f(x)=\frac{1}{x} \Rightarrow$ Not continuous at $x=0$ because $\frac{1}{x}=\frac{1}{0}$ not defined

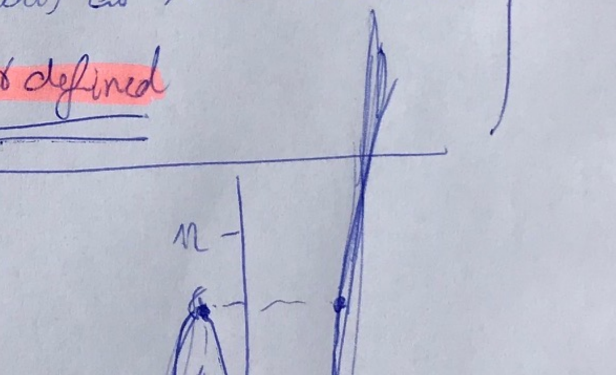
3.7 $f(x)=4x^3-12x$

$f'(x)=12x^2-12=0$

$12x^2=12$

$x^2=1$

$\boxed{x=\pm 1}$



$f''(x)=24x$

$x=$	-1	0	1
$f(x)=$	8	0	-8
$f''(x)=$	-24	0	24
	concave	infl. point	convex

$\Rightarrow$ (−1) local maximum

(−1) local minimum

(0) inflexion point

3.8 $f(x,y) = x^3 - y^2$

$f(2,3) = 2^3 - 3^2 = 8 - 9 = \boxed{-1}$

3.9 $f(x,y) = \ln(x - 3y)$

It is defined if $x - 3y > 0$, because $\ln(\cancel{x})$ cannot be ≤ 0

$\Downarrow$

$x - 3y > 0$

$\boxed{x > 3y}$, where x and $y \in \mathbb{R}$

3.10 $\frac{\partial}{\partial x}\left(x^5 y^7 + \frac{x^2}{y^3}\right) = \boxed{y^7 \cdot 5x^4 + \frac{1}{y^3} \cdot 2x}$ (y considered as constant)

3.11

$f(x,y) = \sqrt{xy} - x - y$

$\Downarrow$

$\left.\begin{array}{l} g(x) = \sqrt{x} \\ h(x) = xy \end{array}\right]$ $g(h(x))' = g'(h(x)) \cdot h'(x) = \frac{1}{2\sqrt{xy}} \cdot y$

$\frac{\partial f(x,y)}{\partial x} = \frac{1}{2\sqrt{xy}} \cdot y - 1$ $\frac{\partial f(x,y)}{\partial y} = \frac{1}{2\sqrt{xy}} x - 1$

$\frac{y}{2\sqrt{xy}} - 1 = 0$ $\frac{x}{2\sqrt{xy}} - 1 = 0$ $\frac{y}{2\sqrt{xy}} - \cancel{1} = \frac{x}{2\sqrt{xy}} - \cancel{1}$

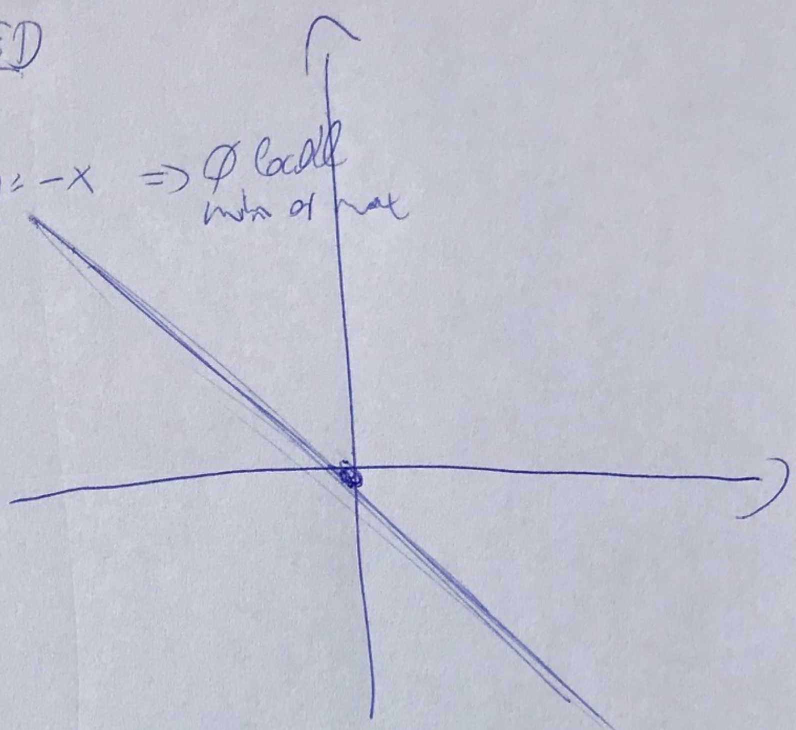
$y = x \longrightarrow \frac{x}{2\sqrt{x^2}} - 1 = 0$

No local max/min $\Longleftarrow \boxed{N/A} \Longleftarrow \frac{x}{2x} = 1$

3.11. CONTINUED

if $x=y$

$f(x,y)=\sqrt{xy}-x-y=-x \Rightarrow \emptyset$ locall min or max



3.12 $\max x^2y^2$ s.t. $2x+y=9$

$$\begin{cases} g(x,y)=2x+y-9=0 \\ f(x,y)=x^2y^2 \end{cases}$$

$\mathcal{L}=f(x,y)-\lambda\cdot g(x,y)$

$\mathcal{L}=x^2y^2-\lambda(2x+y-9)$

$\frac{\partial\mathcal{L}}{\partial x}=y^2 2x-2\lambda=0 \Rightarrow 2\lambda=y^2 2x \Rightarrow \lambda=y^2x$

$\frac{\partial\mathcal{L}}{\partial y}=x^2\cdot 2y-\lambda=0 \Rightarrow x^2\cdot 2y-y^2x=0$

$x^2 2y=y^2x$

$x2=y$

$\frac{\partial\mathcal{L}}{\partial\lambda}=2x+y-9=0 \Rightarrow 2y-9=0$

$\boxed{y=4.5}$

$x=\frac{y}{2}=\boxed{2.25}$

$\boxed{f(x,y)=102.51}$

what if not?

e.g. $x=3$, $y=3$ $\rightarrow f(3,3)=81$ e.g. $x=2$, $y=5$ $\rightarrow f(2,5)=100$

4.) Linear algebra

4.1

$$A = \begin{bmatrix} 2 & 5 \\ 2 & 1 \\ 7 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 9 & 1 & 5 \end{bmatrix}$$

$B \cdot A$

$$\begin{array}{ccc|cc} & & & 2 & 5 \\ & & & 2 & 1 \\ & & & 7 & 6 \\ \hline 1 & 0 & 1 & 9 & 11 \\ 9 & 1 & 5 & 55 & 76 \end{array}$$

4.2.

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 4 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

$A \cdot B$

$$\begin{array}{cc|ccc} & & 8 & 4 & 0 \\ & & 2 & 1 & 2 \\ \hline 5 & 3 & 46 & 23 & 6 \\ 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 12 & 6 & 4 \end{array}$$

4.3. ~~Transpose~~

$$A = \begin{bmatrix} e & 93 & 4.7 \\ 2 & 6.1 & 4.22 \\ 4 & \pi & 0 \end{bmatrix}$$

$$\boxed{A^T = \begin{bmatrix} e & 2 & 4 \\ 93 & 6.1 & \pi \\ 4.7 & 4.22 & 0 \end{bmatrix}}$$

4.4 $A = \begin{bmatrix} 2 & 6 \\ 2 & 8 \end{bmatrix}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = ad - bc = 2 \cdot 8 - 6 \cdot 2 = 16 - 12 = \boxed{\underline{\underline{4}}}$

5. Probability theory

5.1

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$\# \Omega = 6^2 = \boxed{\underline{\underline{36}}}$

5.2

	Drug user	Other population
# pop	0.1%	99.9%
Test positive	98%	0.3% (= 1 − 99.7%)

$\Downarrow$

$\mathbb{P}(B_1) = 0.1\%$ $\mathbb{P}(B_2) = 99.9\%$

$\mathbb{P}(A / B_1) = 98\%$ $\mathbb{P}(A | B_2) = 0.3\%$

Apply Bayes' rule

$$\mathbb{P}(B_i | A) = \frac{\mathbb{P}(A | B_i) \cdot \mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(B_j) \cdot \mathbb{P}(A | B_j)} = \frac{98\% \times 0.1\%}{98\% \times 0.1\% + 0.3\% \times 99.9\%} = \boxed{\underline{24.64189\%}}$$

5.3 $p = \frac{1}{6}$ - probability in each throw that we throw "5"

$n = 20$ - we throw n-times

x-times we throw "5" $\Rightarrow$ follows a binomial distribution

$$\mathbb{P}(x) = p^x (1-p)^{n-x} \binom{n}{x} =$$

$$E(x) = n \cdot p = 20 \cdot \frac{1}{6} = \boxed{\frac{20}{6}}$$