# On Conditions for Intermittent Search in Self-Organizing Neural Networks

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**Abstract.** Self-organizing neural networks (SONN) driven by softmax weight renormalization are capable of finding high quality solutions of difficult assignment optimization problems. The renormalization is shaped by a temperature parameter - as the system cools down the assignment weights become increasingly crisp. It has been recently observed that there exists a critical temperature setting at which SONN is capable of powerful intermittent search through a multitude of high quality solutions represented as meta-stable states of SONN adaptation dynamics. The critical temperature depends on the problem size. It has been hypothesized that the intermittent search by SONN can occur only at temperatures close to the first (symmetry breaking) bifurcation temperature of the autonomous renormalization dynamics. In this paper we provide a rigorous support for the hypothesis by studying stability types of SONN renormalization equilibria.

## 1 Introduction

There have been several successful applications of neural computation techniques in solving difficult combinatorial optimization problems [1, 2]. Self-organizing neural network (SONN) [3] is a neural-based methodology for solving 0-1 assignment problems that has been successfully applied in a wide variety of applications, from assembly line sequencing to frequency assignment in mobile communications.

As usual in self-organizing systems, dynamics of SONN adaptation is driven by a synergy of cooperation and competition. In the competition phase, for each item to be assigned, the best candidate for the assignment is selected and the corresponding assignment weight is increased. In the cooperation phase, the assignment weights of other candidates that were likely to be selected, but were not quite as strong as the selected one, get increased as well, albeit to a lesser degree. The assignment weights need to be positive and sum to 1. Therefore, after each SONN adaptation phase, the assignment weights need to be renormalized back onto the standard simplex e.g. via the softmax function [4, 5]. When endowed with a physics-based Boltzmann distribution interpretation, the softmax function contains a temperature parameter $T > 0$. As the system cools down, the assignments become increasingly crisp. In the original setting SONN is annealed so that a single high quality solution to an assignment problem is found. However, it has been reported recently [6] that there exists a critical temperature $T_*$

at which SONN is capable of powerful intermittent search through a multitude of high quality solutions represented as meta-stable states of SONN adaptation dynamics. It has been hypothesised that the critical temperature may be closely related to the symmetry breaking bifurcation of equilibria in the *autonomous* softmax dynamics.

Unfortunately, at present there is no theory of the SONN adaptation dynamics driven by the softmax renormalization. The first steps towards theoretical underpinning of SONN adaptation driven by softmax renormalization were taken in [7, 8, 6, 9]. For example, [6] suggests to study SONN adaptation dynamics by concentrating on the *autonomous* renormalization process. Indeed, it is this process that underpins the search dynamics in the SONN. Meta-stable states in the SONN intermittent search at the critical temperature $T_*$ are shaped by stable equilibria of the autonomous renormalization dynamics.

In this paper we rigorously show that the only temperature at which stable renormalization equilibria emerge is the first symmetry breaking bifurcation temperature. The stable equilibria appear close to the corners of the assignment weight simplex (0-1 assignment solutions) and act as pulling devices in the intermittent search towards possible assignment solutions. For a rich intermittent search, the stable equilibria should be only weakly attractive, which corresponds to temperatures lower than, but close to the symmetry breaking bifurcation temperature of the autonomous renormalization dynamics. Due to space limitations, we only provide sketches of proofs of the main statements.

The paper has the following organization: After a brief introduction to SONN and autonomous renormalization in section 2, we introduce necessary background related to renormalization equilibria in section 3. We then study stability types of the renormalization equilibria in section 4, focusing on equilibria close to 'one-hot' 0-1 assignment weights in section 4.1. The results are then discussed and summarized in section 5.

## 2 Self-Organizing Neural Network and Iterative Softmax

First, we briefly introduce Self-Organizing Neural Network (SONN) endowed with weight renormalization for solving assignment optimization problems (see e.g. [6]). Consider a finite set of input elements (neurons) $i \in \mathcal{I} = \{1, 2, ..., M\}$ that need to be assigned to outputs (output neurons) $j \in \mathcal{J} = \{1, 2, ..., N\}$, so that some global cost of an assignment $\mathcal{A} : \mathcal{I} \to \mathcal{J}$ is minimized. Partial cost of assigning $i \in \mathcal{I}$ to $j \in \mathcal{J}$ is denoted by $V(i, j)$. The 'strength' of assigning $i$ to $j$ is represented by the 'assignment weight' $w_{i,j} \in (0, 1)$.

The SONN algorithm can be summarized as follows: The connection weights $w_{i,j}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, are first initialized to small random values. Then, repeatedly, an output item $j_c \in \mathcal{J}$ is chosen and the partial costs $V(i, j_c)$ incurred by assigning all possible input elements $i \in \mathcal{I}$ to $j_c$ are calculated in order to select the 'winner' input element (neuron) $i(j_c) \in \mathcal{I}$ that minimizes $V(i, j_c)$. The 'neighborhood' $\mathcal{B}_L(i(j_c))$ of size $L$ of the winner node $i(j_c)$ consists of $L$ nodes $i \neq i(j_c)$ that yield the smallest partial costs $V(i, j_c)$. Weights from nodes

$i \in \mathcal{B}_L(i(j_c))$ to $j_c$ get strengthened:

$$w_{i,j_c} \leftarrow w_{i,j_c} + \eta(i)(1 - w_{i,j_c}), \quad i \in \mathcal{B}_L(i(j_c)), \tag{1}$$

where $\eta(i)$ is proportional to the quality of assignment $i \to j_c$, as measured by $V(i, j_c)$. Weights[1] $\mathbf{w}_i = (w_{i,1}, w_{i,2}, ..., w_{i,N})'$ for each input node $i \in \mathcal{I}$ are then renormalized using softmax

$$w_{i,j} \leftarrow \frac{\exp(\frac{w_{i,j}}{T})}{\sum_{k=1}^{N} \exp(\frac{w_{i,k}}{T})}, \tag{2}$$

where $T > 0$ is a 'temperature' parameter.

We will refer to SONN for solving an $(M, N)$-assignment problem as $(M, N)$-SONN. Since meta-stable states in the SONN intermittent search are shaped by stable equilibria of the autonomous renormalization dynamics, in this paper we concentrate on conditions for emergence of such equilibria.

The weight vector $\mathbf{w}_i$ of each of $M$ neurons in an $(M, N)$-SONN lives in the *standard* $(N-1)$-*simplex*,

$$S_{N-1} = \{\mathbf{w} = (w_1, w_2, ..., w_N)' \in \mathbb{R}^N \mid w_i \geq 0, i = 1, 2, ..., N, \text{ and } \sum_{i=1}^{N} w_i = 1\}.$$

Given a value of the temperature parameter $T > 0$, the softmax renormalization step in SONN adaptation transforms the weight vector of each unit as follows:

$$\mathbf{w} \mapsto \mathbf{F}(\mathbf{w}; T) = (F_1(\mathbf{w}; T), F_2(\mathbf{w}; T), ..., F_N(\mathbf{w}; T))', \tag{3}$$

where

$$F_i(\mathbf{w}; T) = \frac{\exp(\frac{w_i}{T})}{Z(\mathbf{w}; T)}, \quad i = 1, 2, ..., N, \tag{4}$$

and $Z(\mathbf{w}; T) = \sum_{k=1}^{N} \exp(\frac{w_k}{T})$ is the normalization factor. Formally, $\mathbf{F}$ maps $\mathbb{R}^N$ to $S^0_{N-1}$, the interior of $S_{N-1}$.

Linearization of $\mathbf{F}$ around $\mathbf{w} \in S^0_{N-1}$ is given by the Jacobian $J(\mathbf{w}; T)$:

$$J(\mathbf{w}; T)_{i,j} = \frac{1}{T}[\delta_{i,j} F_i(\mathbf{w}; T) - F_i(\mathbf{w}; T) F_j(\mathbf{w}; T)], \quad i, j = 1, 2, ..., N, \tag{5}$$

where $\delta_{i,j} = 1$ iff $i = j$ and $\delta_{i,j} = 0$ otherwise.

The softmax map $\mathbf{F}$ induces on $S^0_{N-1}$ a discrete time dynamics known as *Iterative Softmax* (ISM):

$$\mathbf{w}(t+1) = \mathbf{F}(\mathbf{w}(t); T). \tag{6}$$

The renormalization step in an $(M, N)$-SONN adaptation involves $M$ separate renormalizations of weight vectors of all of the $M$ SONN units. For each temperature setting $T$, the structure of equilibria in the $i$-th system, $\mathbf{w}_i(t+1) = \mathbf{F}(\mathbf{w}_i(t); T)$, gets copied in all the other $M-1$ systems. Using this symmetry, it is sufficient to concentrate on a single ISM (6). Note that the weights of different units are coupled by the SONN adaptation step (1). We will study systems for $N \geq 2$.

[1] here $'$ denotes the transpose operator

## 3 Renormalization equilibria

We first introduce basic concepts and notation that will be used throughout the paper. The $n$-dimensional column vectors of 1's and 0's are denoted by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively. The maximum entropy point $N^{-1}\mathbf{1}_N$ of the standard $(N-1)$-simplex $S_{N-1}$ will be denoted by $\overline{\mathbf{w}}$. To simplify the notation we will use $\overline{\mathbf{w}}$ to denote both the maximum entropy point of $S_{N-1}$ and the vector $\overline{\mathbf{w}} - \mathbf{0}_N$.

It is obvious that $\overline{\mathbf{w}}$ is a fixed point of ISM (6) for any temperature setting $T$. We will also use the fact (see [9]) that any other ISM fixed point $\mathbf{w} = (w_1, w_2, ..., w_N)'$ has exactly two different coordinate values: $w_i \in \{\gamma_1, \gamma_2\}$, such that $N^{-1} < \gamma_1 < N_1^{-1}$ and $0 < \gamma_2 < N^{-1}$, where $N_1$ is the number of coordinates $\gamma_1$ larger than $N^{-1}$. The number of coordinates $\gamma_2$ smaller than $N^{-1}$ is then $N_2 = N - N_1$. Of course, since the assignment weights live on the standard simplex $S^0_{N-1}$, we have

$$\gamma_2 = \frac{1 - N_1\gamma_1}{N - N_1}. \tag{7}$$

It is also obvious that if $\mathbf{w} = (\gamma_1\mathbf{1}'_{N_1}, \gamma_2\mathbf{1}'_{N_2})'$ is a fixed point of ISM (6), so are all $\binom{N}{N_1}$ *distinct* permutations of it. We collect $\mathbf{w}$ and its permutations in a set

$$\mathcal{E}_{N,N_1}(\gamma_1) = \left\{\mathbf{v}|\ \mathbf{v} \text{ is a permutation of } \left(\gamma_1\mathbf{1}'_{N_1}, \frac{1 - N_1\gamma_1}{N - N_1}\mathbf{1}'_{N-N_1}\right)'\right\}. \tag{8}$$

Finally, fixed points in $\mathcal{E}_{N,N_1}(\gamma_1)$ exist if and only if the temperature parameter $T$ is set to [9]

$$T_{N,N_1}(\gamma_1) = (N\gamma_1 - 1)\left[-(N - N_1)\cdot\ln\left(1 - \frac{N\gamma_1 - 1}{(N - N_1)\gamma_1}\right)\right]^{-1}. \tag{9}$$

## 4 Stability analysis of renormalization equilibria

We have already mentioned that the maximum entropy point $\overline{\mathbf{w}}$ is a fixed point of ISM (6) for all temperature settings. It is straightforward to show that $\overline{\mathbf{w}}$, regarded as a vector $\overline{\mathbf{w}} - \mathbf{0}_N$, is an eigenvector of the Jacobian $J(\mathbf{w}; T)$ at any $\mathbf{w} \in S^0_{N-1}$, with associated eigenvalue $\lambda = 0$. This simply reflects the fact that ISM renormalization acts on the standard simplex $S_{N-1}$, which is a subset of a $(N-1)$-dimensional hyperplane with normal vector $\mathbf{1}_N$.

We will now show that if $\mathbf{w}$ is a fixed point of ISM, then $\overline{\mathbf{w}} - \mathbf{w}$ is an eigenvector of the ISM Jacobian at $\mathbf{w}$.

**Theorem 1.** *Let $\mathbf{w} \in \mathcal{E}_{N,N_1}(\gamma_1)$ be a fixed point of ISM (6). Then, $\mathbf{w}_* = \overline{\mathbf{w}} - \mathbf{w}$ is an eigenvector of the Jacobian $J(\mathbf{w}; T_{N,N_1}(\gamma_1))$ with the corresponding eigenvalue $\lambda_*$, where*

1. *if $\lceil N/2 \rceil \le N_1 \le N - 1$, then $0 < \lambda_* < 1$,*

2. *if* $1 \leq N_1 < \lceil N/2 \rceil$,
   (a) *and* $N^{-1} < \gamma_1 < (2N_1)^{-1}$, *then* $\lambda_* > 1$.
   (b) *then there exists* $\bar{\gamma}_1 \in ((2N_1)^{-1}, N_1^{-1})$, *such that for all ISM fixed points* $\mathbf{w} \in \mathcal{E}_{N,N_1}(\gamma_1)$ *with* $\gamma_1 \in (\bar{\gamma}_1, N_1^{-1})$, $0 < \lambda_* < 1$.

Sketch of the Proof:
To simplify the notation, we will denote the Jacobian of ISM at $\mathbf{w} = (w_1, w_2, ..., w_N)'$ by $J$ and the temperature at which $\mathbf{w}$ exists by $T$. From

$$J\mathbf{w}_* = J(\overline{\mathbf{w}} - \mathbf{w}) = J\overline{\mathbf{w}} - J\mathbf{w} = -J\mathbf{w},$$

and using (5), we have for the $i$-th element of $J\mathbf{w}_*$:

$$\frac{w_i}{T}(\mathbf{w}'\mathbf{w} - w_i\mathbf{e}_i'\mathbf{w}) = \frac{w_i}{T}(\|\mathbf{w}\|^2 - w_i).$$

But

$$J\mathbf{w}_* = \lambda_*\overline{\mathbf{w}} - \lambda_*\mathbf{w},$$

and so the $i$-th element of $J\mathbf{w}_*$ must also be equal to $\lambda_* N^{-1} - \lambda_* w_i$. In other words,

$$\frac{w_i}{T}(\|\mathbf{w}\|^2 - w_i) = \lambda_* N^{-1} - \lambda_* w_i \tag{10}$$

should hold for all $i = 1, 2, ..., N$. Since $w_i \in \{\gamma_1, \gamma_2\}$, $\gamma_2 = (1 - N_1\gamma_1)/(N - N_1)$, we have that

$$\frac{\gamma_1 \cdot (\gamma_1 - \|\mathbf{w}\|^2)}{\gamma_2 \cdot (\gamma_2 - \|\mathbf{w}\|^2)} = \frac{\gamma_1 - N^{-1}}{\gamma_2 - N^{-1}} \tag{11}$$

would need to be true. This indeed can be verified after some manipulation using $|\mathbf{w}\|^2 = N_1\gamma_1^2 + N_2\gamma_2^2$.

From (10), by plugging in $\gamma_1$ for $w_i$, we obtain

$$\lambda_* = \frac{\gamma_1 \cdot (\gamma_1 - \|\mathbf{w}\|^2)}{T \cdot (\gamma_1 - N^{-1})}. \tag{12}$$

Now, it can be shown that $\gamma_1 > \|\mathbf{w}\|^2$, and since $\gamma_1 > N^{-1} > 0$, $T > 0$, we have that $\lambda_*$ is positive.

It can be established that the values of coordinates $\gamma_1, \gamma_2$ of each ISM fixed point $\mathbf{w}$ must satisfy

$$\gamma_1 = \frac{1}{N}\left(1 + \tau\frac{N_2}{N1}\right) \tag{13}$$

$$\gamma_2 = \frac{1}{N}(1 - \tau), \tag{14}$$

for some $\tau \in [0, 1)$.

In order to prove 1, we use parametrization (13-14) to obtain (after some manipulation)

$$\lambda_* = \frac{\gamma_1}{T}(1 - \tau). \tag{15}$$

Assume $\lambda_* \geq 1$. That means (using (9), (14), (15), and after some manipulations)

$$\frac{T}{\gamma_1} = \frac{1-\frac{\gamma_2}{\gamma_1}}{-\ln\frac{\gamma_2}{\gamma_1}} \leq 1-\tau = \gamma_1 N \frac{\gamma_2}{\gamma_1},$$

which can be written as

$$\ln a \leq -\rho\frac{(1-a)^2}{a} + a - 1 = f(a;\rho), \tag{16}$$

where

$$0 < \rho = \frac{N_1}{N} < 1 \tag{17}$$

and

$$0 < a = \frac{\gamma_2}{\gamma_1} < 1. \tag{18}$$

On $a > 0$, both $\ln a$ and $f(a)$ are continuous concave functions with $\ln 1 = f(1) = 0$ and $\ln'(1) = f'(1) = 1$. So the function values, as well as the slopes of $\ln a$ and $f(a)$ coincide at $a = 1$. Since it can be shown that for $N_1 \geq \lceil N/2 \rceil$ (hence $\rho \geq 1/2$), the slope of $f(a)$ exceeds that of $\ln a$ on the whole interval $(0,1)$, it must be that $f(a) < \ln a$. This is a contradiction to (16), and so $\lambda_* < 1$.

The proof of 2a proceeds analogously to the proof of 1, this time we are interested in conditions on $\gamma_1$ in the neighborhood of $a = 1$, such that, given $\rho \in (0, 1/2)$, we have $f'(a) < \ln'(a)$, and hence $f(a) > \ln a$.

Finally, for $\rho \in (0, 1/2)$, we have $f'(a) > \ln'(a)$ for $a > 0$ in the neighborhood of 0. Now, $\ln a$ and $f(a)$ are continuous concave functions with $\ln 1 = f(1) = 0$, $\ln'(1) = f'(1) = 1$, and for $a \in (0,1)$ in the neighborhood of 1, we have $f'(a) < \ln'(a)$, implying $\ln a < f(a)$. Since $\lim_{a\to 0^+} f(a) = \lim_{a\to 0^+} \ln a = \infty$ and $\lim_{a\to 0^+} \frac{\ln a}{f(a)} = 0$, there exists $\bar{a} \in (0,1)$, such that on $a \in (0,\bar{a})$, $f(a) < \ln a$. It can be shown that $a \in (0,\bar{a})$ corresponds to $\gamma_1 \in (\bar{\gamma}_1, N_1^{-1})$ with $\bar{\gamma}_1 \in ((2N_1)^{-1}, N_1^{-1})$. This proves 2b.

*Q.E.D.*

We have established that for an ISM equilibrium $\mathbf{w}$, both $\overline{\mathbf{w}}$ and $\mathbf{w}_* = \overline{\mathbf{w}} - \mathbf{w}$ are eigenvectors of the ISM Jacobian at $\mathbf{w}$. Stability types of the remaining $N-2$ eigendirections are characterized in the next two theorems.

**Theorem 2.** *Consider a ISM fixed point* $\mathbf{w} = (\gamma_1 \mathbf{1}'_{N_1}, \gamma_2 \mathbf{1}'_{N_2})'$ *for some* $1 \leq N_1 < N$ *and* $N^{-1} < \gamma_1 < N_1^{-1}$. *Let* $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{N-N_1-1}\}$ *be a set of* $(N-N_1)$*-dimensional unit vectors, such that* $\mathcal{B}$*, together with* $\mathbf{1}_{N-N_1}/\|\mathbf{1}_{N-N_1}\|$*, form an orthonormal basis of* $\mathbb{R}^{N-N_1}$*. Then, there are* $N - N_1 - 1$ *eigenvectors of the ISM Jacobian at* $\mathbf{w}$ *of the form:*

$$\mathbf{v}_-^i = (\mathbf{0}'_{N_1}, \mathbf{u}'_i)', \quad i = 1, 2, ..., N - N_1 - 1. \tag{19}$$

*All eigenvectors* $\mathbf{v}_-^1, \mathbf{v}_-^2, ..., \mathbf{v}_-^{N-N_1-1}$ *have the same associated eigenvalue*

$$0 < \lambda_- = \frac{1 - N_1\gamma_1}{(N-N_1)T_{N,N_1}(\gamma_1)} = \frac{\gamma_2}{T_{N,N_1}(\gamma_1)} < 1. \tag{20}$$

<u>Sketch of the Proof:</u>
The Jacobian can be written as

$$J = \frac{-1}{T}\begin{bmatrix} G_1 & G_{12} \\ G_{12}' & G_2 \end{bmatrix}, \tag{21}$$

where

$$G_1 = \gamma_1(\gamma_1 \mathbf{1}_{N_1}\mathbf{1}_{N_1}' - I_{N_1}), \tag{22}$$

$$G_2 = \gamma_2(\gamma_2 \mathbf{1}_{N_2}\mathbf{1}_{N_2}' - I_{N_2}), \tag{23}$$

and

$$G_{12} = \gamma_1\gamma_2 \mathbf{1}_{N_1}\mathbf{1}_{N_2}'. \tag{24}$$

Since all $\mathbf{u} \in \mathcal{B}$ are orthogonal to $\mathbf{1}_{N_2}$, we have

$$\begin{bmatrix} G_{12} \\ G_2 \end{bmatrix}\mathbf{u} = -\gamma_2 \begin{bmatrix} \mathbf{0}_{N_1} \\ \mathbf{u} \end{bmatrix}.$$

and so for all $i = 1, 2, ..., N - N_1 - 1$, it holds

$$J\mathbf{v}_-^i = \frac{\gamma_2}{T}\mathbf{v}_-^i.$$

Since both $\gamma_2$ and $T$ are positive, $\lambda_- = \gamma_2/T > 0$.
It can be shown that

$$T = \frac{\gamma_1 - \gamma_2}{\ln\gamma_1 - \ln\gamma_2}. \tag{25}$$

Then,

$$\lambda_- = \frac{\ln\frac{\gamma_1}{\gamma_2}}{\frac{\gamma_1}{\gamma_2} - 1} = \frac{\ln b}{b-1}, \tag{26}$$

where $b = \gamma_1/\gamma_2 > 1$. But $0 < \ln b < b - 1$ on $b \in (1, \infty)$, and so $\lambda_- < 1$.
*Q.E.D.*

Note that even though theorem 2 is formulated for $\mathbf{w} = (\gamma_1\mathbf{1}_{N_1}', \gamma_2\mathbf{1}_{N_2}')'$, by the symmetry of ISM, the result translates to all permutations of $\mathbf{w}$ in a straightforward manner. The same applies to the next theorem.

**Theorem 3.** *Consider a ISM fixed point* $\mathbf{w} = (\gamma_1\mathbf{1}_{N_1}', \gamma_2\mathbf{1}_{N_2}')'$ *for some* $1 \le N_1 < N$ *and* $N^{-1} < \gamma_1 < N_1^{-1}$. *Let* $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{N_1-1}\}$ *be a set of* $N_1$*-dimensional unit vectors, such that* $\mathcal{B}$*, together with* $\mathbf{1}_{N_1}/\|\mathbf{1}_{N_1}\|$*, form an* *orthonormal basis of $\mathbb{R}^{N_1}$. Then, there are $N_1 - 1$ eigenvectors of the ISM Jacobian at $\mathbf{w}$ of the form:*

$$\mathbf{v}_+^i = (\mathbf{u}_i', \mathbf{0}_{N-N_1}')', \quad i = 1, 2, ..., N_1 - 1. \tag{27}$$

*All eigenvectors $\mathbf{v}_+^1, \mathbf{v}_+^2, ..., \mathbf{v}_+^{N_1-1}$ have the same associated eigenvalue*

$$\lambda_+ = \frac{\gamma_1}{T_{N,N_1}(\gamma_1)} > 1. \tag{28}$$

Sketch of the Proof:
The proof proceeds analogously to the proof of theorem 2. *Q.E.D.*

### 4.1 Equilibria near vertices of the standard simplex

We have proved that components of *stable* equilibria of the SONN renormalization can only be found when $N_1 = 1$, which corresponds to equilibria near vertices of the simplex $S_{N-1}$. Stable manifold of the linearized ISM system at such an equilibrium $\mathbf{w} \in S_{N-1}^0$ is given by the span of $\mathbf{w}_* = \overline{\mathbf{w}} - \mathbf{w}$ (by theorem 1 the corresponding eigenvalue $\lambda_*$ is in $(0, 1)$ if $\mathbf{w}$ lies sufficiently close to a vertex of $S_{N-1}$) and $N - 2$ vectors orthogonal to both $\overline{\mathbf{w}}$ and $\mathbf{w}_*$ (theorem 2). Since $N_1 = 1$, there is no invariant manifold of the linearized ISM with expansion rate $\lambda_+ > 1$ (theorem 3). Also, no dynamics takes place outside interior of $S_{N-1}$ (zero eigenvalue corresponding to the eigenvector $\overline{\mathbf{w}}$ of the linearized ISM). It is important to note that by theorem 3, *all* ISM fixed points with $N_1 \geq 2$ are unstable. From now on we will concentrate on ISM equilibria with $N_1 = 1$.

The temperature $T_{N,1}(\gamma_1)$ (see eq. (9)) at which fixed points $\mathbf{w} \in \mathcal{E}_{N,1}(\gamma_1)$ exist is a concave function attaining a unique maximum at some $\gamma_1^0 \in (N^{-1}, 1)$ [9]. Denote the corresponding temperature $T_{N,1}(\gamma_1^0)$ by $T_E(N, 1)$. Then, no fixed point $\mathbf{w} \in \mathcal{E}_{N,1}(\gamma_1)$ for any $\gamma_1 \in (N^{-1}, 1)$ can exist for temperatures $T > T_E(N, 1)$. For temperatures $T$ slightly below $T_E(N, 1)$ there are two fixed points $\mathbf{w}_-(T)$ and $\mathbf{w}_+(T)$, corresponding to the increasing and decreasing branches of the concave temperature curve $T_{N,1}(\gamma_1)$, respectively. Temperature $T_E(N, 1)$ is the first symmetry breaking bifurcation temperature of the ISM when new fixed points other than $\overline{\mathbf{w}}$ emerge as the system cools down.

In this section we show that one of the fixed points, $\mathbf{w}_+(T)$, is a stable equilibrium with increasingly strong attraction rate as the system cools down ($\mathbf{w}_+(T)$ moves towards a vertex of $S_{N-1}$), while the other one, $\mathbf{w}_-(T)$, is a saddle ISM fixed point.

By (12) (and some manipulation), $\overline{\mathbf{w}} - \mathbf{w}$ is an eigenvector of the ISM Jacobian at $\mathbf{w}$ with the corresponding positive eigenvalue

$$\lambda_*(\gamma_1) = \frac{N\gamma_1(1 - \gamma_1)}{N\gamma_1 - 1} \ln \frac{(N - 1)\gamma_1}{1 - \gamma_1}. \tag{29}$$

It is straightforward to show that on $\gamma_1 \in (N^{-1}, 1)$, $\lambda_*(\gamma_1)$ is a concave function of positive slope at[2] $\gamma_1 = N^{-1}$ with

$$\lim_{\gamma_1 \to N^{-1}} \lambda_*(\gamma_1) = 1$$

and

$$\lim_{\gamma_1 \to 1} \lambda_*(\gamma_1) = 0.$$

Hence there exists a unique $\gamma_1^* \in (N^{-1}, 1)$, such that $\lambda_*(\gamma_1^*) = 1$ and for $\gamma_1 \in (N^{-1}, \gamma_1^*)$ we have $\lambda_*(\gamma_1) > 1$, whereas for $\gamma_1 \in (\gamma_1^*, 1)$, it holds $0 < \lambda_*(\gamma_1) < 1$.

From (29) and $\lambda_*(\gamma_1^*) = 1$, we have

$$\ln \frac{(N-1)\gamma_1^*}{1-\gamma_1^*} = \frac{N\gamma_1^* - 1}{N\gamma_1^*(1-\gamma_1^*)}. \tag{30}$$

It is not difficult to show that $\gamma_1^*$ is actually equal to $\gamma_1^0$, the value of $\gamma_1$ at which $T_{N,1}(\gamma_1)$ attains maximum. As the system gets annealed, the eigenvalue $\lambda_*(\gamma_1)$ decreases below 1 and increases above 1 for the two fixed points $\mathbf{w}_+(T)$ and $\mathbf{w}_-(T)$, respectively. The weakest contraction is at equilibria $\mathbf{w}_+(T)$ existing at temperatures close to $T_E(N, 1)$.

It can be easily shown that the other contraction rate (theorem 2, (20)),

$$\lambda_-(\gamma_1) = \frac{1-\gamma_1}{(N-1)T_{N,1}(\gamma_1)}, \tag{31}$$

is a decreasing function of $\gamma_1$ as well. Hence, $\mathbf{w}_+(T)$ are *stable* equilibria with weakest contraction for temperatures close to $T_E(N, 1)$. As the system cools down the contraction gets increasingly strong.

## 5 Discussion - SONN adaptation dynamics

In the intermittent search regime by SONN, the search is driven by pulling promising solutions temporarily to the vicinity of the 0-1 'one-hot' assignment values - vertices of $S_{N-1}$ [6]. The critical temperature for intermittent search should correspond to the case where the attractive forces already exist in the form of *attractive* equilibria near the 'one-hot' assignment suggestions (vertices of $S_{N-1}$), but the convergence rates towards such equilibria should be sufficiently weak so that the intermittent character of the search is not destroyed. In this paper, we have rigorously shown that this occurs at temperatures lower than, but close to the first bifurcation temperature $T_E(N, 1)$.

The hypothesis that there is a strong link between the critical temperature for intermittent search by SONN and bifurcation temperatures of the autonomous ISM has been formulated in [6]. Both [9] and [6] hypothesize that even though there are many potential ISM equilibria, the critical bifurcation points are related only to equilibria near the vertices of $S_{N-1}$, as only those can be ultimately

[2] $\lambda_*(\gamma_1)$ can be continuously extended to $\gamma_1 = 1/N$

responsible for 0-1 assignment solution suggestions in the course of intermittent search by SONN. In this study, we have rigorously shown that the stable equilibria can in fact exist *only* near the vertices of $S_{N-1}$. Only when $N_1 = 1$, there are no expansive eigendirections of the local Jacobian with $\lambda_+ > 1$.

Even though the present study helps to shed more light on the prominent role of the first (symmetry breaking) bifurcation temperature $T_E(N, 1)$ in obtaining the SONN intermittent search regime, many interesting open questions remain. For example, no theory as yet exists of the role of abstract neighborhood $\mathcal{B}_L(i(j_c))$ of the winner node $i(j_c)$ in the cooperative phase of SONN adaptation. [6] report a rather strong pattern of increasing neighborhood size with increasing assignment problem size (tested on N-queens), but this issue deserves a more detailed study.

We conclude by noting that it may be possible to apply the theory of ISM in other assignment optimization systems that incorporate the softmax assignment weight renormalization e.g. [10, 11].

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