Homework - 02

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Instructions: Completed assignments are due on Monday, January 8, 2018. They are to be uploaded to the Virtual Campus.

Solutions can be submitted by groups of two or three students, which can be different from those in Homework 1. In this case only one copy of each group's work should be uploaded (by any member), clearly stating the names of all contributors.

Format: Exercises should be submitted either as a pair (document + code) or as a Jupyter notebook. Documents should be `.pdf` files *[never as `.doc`/`.docx`/`.odt`]*. Code as ASCII `.r` text files. When more than one file has to be entered, prepare a single compressed file.

Exercise 1 (Estimating a log-odds with a normal prior)**.** *(End-of-chapter Exercise 1 from Jim Albert (2009),* Bayesian computations with R, 2nd ed., *chapter 5.*

Suppose y has a binomial distribution with parameters n and p, and we are interested in the log-odds value $\theta = \log(p/(1-p))$. Our prior for θ is that $\theta \sim \mathrm{N}(\mu, \sigma^2)$. It follows that the posterior density of θ is given, up to a proportionality constant, by

$$g(\theta|y) \propto \frac{exp(y\,\theta)}{(1+exp(\theta))^n}\,\exp\left[-\frac{(\theta-\mu)^2}{2\,\sigma^2}\right].$$

More concretely, suppose we are interested in learning about the probability that a special coin lands heads when tossed. A priori we believe that the coin is fair, so we assign θ a $\mathrm{N}(0, 0.25)$ prior. We toss the coin $n = 5$ times and obtain $y = 5$ heads.

a) Using a normal approximation to the posterior density, compute the probability that the coin is biased toward heads (i.e., that θ is positive).

b) Using the prior density as a proposal density, design a rejection algorithm for sampling from the posterior distribution. Using simulated draws from your algorithm, approximate the probability that the coin is biased toward heads.

c) Using the prior density as a proposal density, simulate values from the posterior distribution using the SIR algorithm. Approximate the probability that the coin is biased toward heads.

Exercise 2 (Bolstad (2010), *Understanding computational Bayesian Statistics,* Chap. 6, Example 8)**.** *The object of this exercise is to build Metropolis-Hastings algorithms with target pdf:*

$$g(\theta|y) \propto 0.8 \times \exp\{-\frac{1}{2}\,\theta^2\} + 0.2 \times \frac{1}{2}\,\exp\{-\frac{1}{2}\,\frac{(\theta-3)^2}{2^2}\},$$

a mixture of a $\mathrm{N}(0,1)$ and a $\mathrm{N}(3,4)$. As a guide, you might want to see the function `normMixMH` from the `Bolstad2` package.

1. *Previous work: g is written as a non-normalized pdf, since that is all we need for MH. However, the normalization constant is known ($1/\sqrt{2\pi}$). Furthermore, we know how to simulate a mixture. Then, plot this pdf, generate a sequence of (independent) random numbers $\sim g$, comparing the histogram with the pdf. Find the sample size needed to obtain a good (or, at least, acceptable) proximity to the target pdf.*

2. *Write a Metropolis-Hastings code with the candidate proposal kernel:*
$$k(\theta'|\theta) = \exp\{-\frac{1}{2}(\theta' - \theta)^2\}$$
The probability of acceptance is:
$$\min\left\{1, \frac{h_x(\theta')\, k(\theta|\theta')}{h_x(\theta)\, k(\theta'|\theta)}\right\} = \min\left\{1, \frac{h_x(\theta')}{h_x(\theta)}\right\}.$$
The symmetry of the candidate generation implies this M-H algorithm reduces to the simple Metropolis algorithm. Use $\theta = 2$ as the start value in the trajectory. Test the algorithm with several values of chain length and burn-in *initial discarded segment.*

3. *Write a Metropolis-Hastings code with the candidate proposal kernel:*
$$k(\theta'|\theta) = q(\theta') = \exp\{-\frac{1}{2}\,(\theta'/3)^2\},$$
a $\mathrm{N}(0, 3^2$ *pdf. M-H algorithm using such a candidate proposal kernel is called* independent candidate M-H algorithm. *With it, the probability of acceptance is:*
$$\min\left\{1, \frac{h_x(\theta')\, k(\theta|\theta')}{h_x(\theta)\, k(\theta'|\theta)}\right\} = \min\left\{1, \frac{h_x(\theta')\cdot q(\theta)}{h_x(\theta)\cdot q(\theta')}\right\}.$$
Use $\theta = 0.4448$ as the start value in the trajectory. Test the algorithm with several values of chain length and burn-in *initial discarded segment. Discuss traceplot, ACF and histogram of the chains tested.*

Exercise 3. *A comparison of the ordinary aceptance/rejection sampling and the M-H algorithm with independent candidate generation: Exercise 6.4 in Robert, Casella (2010),* Introducing Monte Carlo Methods with R, *Chapter 6. See also Robert, Casella (2004)* Monte Carlo Statistical Methods, 2nd ed., *Example 2.19 and Exercise 2.32.* [Remark: I believe there is an erratum in Robert, Casella (2004), pag. 52, paragraph -3, where it reads "the maximum of $b^{-a}(1-b)^{\alpha-a}$" should be *minimum.* Check it].

The problem is to simulate a target $\mathrm{Gamma}(\alpha, \beta)$ *pdf by*

(a) *the ordinary aceptance/rejection sampling method, and*

- *][(b)] with a M-H algorithm with independent candidate generation,*

both with the same candidate generation pdf: a $\mathrm{Gamma}([\alpha], \beta)$ *where $a = [\alpha]$ denotes the integer part of α (the* `floor` *function).*

The reason for choosing this candidate pdf is that a $\mathrm{Gamma}(k, \beta)$ *distribution with an integer k shape parameter is equal to the sum of k independent copies of an* $\mathrm{Exp}(\beta)$, *easy to simulate by the inverse cdf method.*

Package `mcsm` *contains data and functions for the Robert and Casella book. In particular, under* `Code demonstrations` *for Chapter 6 you can find a code excerpt for this exercise.*

Hint: *For the acceptance-rejection method, to derive $b = [\alpha]/\alpha$ as the optimal choice when $\beta = 1$, use the fact that for target f and candidate g, one must find M such that $f/g \leq M$ and that then the acceptance rate is $1/M$, hence one should minimize M to optimize it.*
