Random Priority is not the unique strategyproof, Pareto efficient, and fair allocation mechanism

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Abstract

Random Priority (RP) is a popular mechanism in practice to allocate a set of objects to a set of agents without the use of monetary transfers. Theoretically, RP is appealing because it satisfies desirable efficiency (Pareto efficiency, PE), fairness (equal treatment of equals, ETE), and incentive (strategyproofness, SP) properties. It has been an open question as to whether RP is the *only* mechanism to satisfy these properties. In this note, we show that this is not the case: there exist other PE, SP, and ETE mechanisms besides RP.

1 Introduction

Consider the problem of allocating n indivisible objects to n agents without the use of monetary transfers. Examples of such problems include assigning elementary students to school seats, college students to dormitory rooms, workers to tasks, professors to offices, or time slots on a common machine. A classic and oft-used solution to this problem is the *Random Priority (RP) mechanism*:[1] an ordering of the agents is drawn uniformly at random, and agents are called, one-by-one, to select their favorite object from those that were not selected by earlier agents. The popularity of RP as a solution to this problem can partially be explained theoretically by noting that it satisfies several desirable properties regarding efficiency, fairness, and incentives. Namely, RP is:

- *strategyproof:* in the corresponding direct mechanism, it is always in an agent's best interest to report her preferences over the objects truthfully;[2]
- *Pareto efficient:* for any preferences of the agents, the final allocation will be Pareto efficient;
- *fair:* if two agents have the same preferences, they will receive the same (distribution over) outcomes; in other words, RP satisfies *equal treatment of equals (ETE).*

While it is quite easy to show that RP satisfies the above properties, an open question is whether any *other* mechanism also satisfies these properties, or whether RP is the unique mechanism to do so. In this note, we provide an alternative strategyproof, Pareto efficient, and ETE mechanism that differs from RP, thereby showing that RP is not unique in this class of mechanisms.

This paper contributes to the literature on the analysis and axiomatic characterizations of allocation mechanisms in general, and on RP in particular. Zhou (1990) shows that there is no strategyproof, ETE, and ex-ante efficient mechanism for $n \geq 3$, while Bogomolnaia and Moulin (2001) show that there is no ordinally efficient, strategyproof, and ETE mechanism when $n \geq 4$.[3] Liu and Pycia (2011) study

[1] Random Priority sometimes goes by the name Random Serial Dictatorship (e.g., Abdulkadiroğlu and Sönmez (1998)).

[2] Li (2017) shows that RP satisfies the stronger property of obvious strategyproofness (OSP). See also Pycia and Troyan (2020) for a characterization of RP using OSP.

[3] Ex-ante efficiency is stronger than ordinal efficiency, which in turn is stronger than the Pareto efficiency that we consider. RP is neither ex-ante nor ordinally efficient in finite markets.

the asymptotic equivalence of all ordinally efficient, fair, and strategyproof mechanisms, including RP. Pycia and Troyan (2020) provide the first axiomatic characterization of RP in finite markets using Pareto efficiency, symmetry, and obvious strategyproofness (Li, 2017). Abdulkadiroğlu and Sönmez (1998) and Knuth (1996) show that RP is equivalent to another mechanism called the *core from random endowments*, which works by first randomly assigning the objects to the agents and then allowing the agents to trade according to the top trading cycles (TTC) algorithm of Shapley and Scarf (1974). This equivalence result is extended by Carroll (2014). Pápai (2000) and Pycia and Ünver (2017) generalize TTC to characterize Pareto efficient and and group strategyproof mechanisms.

2 Result

2.1 Definitions

We consider the problem of allocating a set of n indivisible objects X to a set of n agents I. Each agent $i \in I$ has a strict **preference relation** P_i over X, where we write xP_iy to denote that x is strictly preferred to y, and xR_iy if either xP_iy or $x = y$. We use $\mathcal{P}$ to denote the set of all strict preference relations over X. We use $P_I = (P_i)_{i\in I}$ to denote a profile of preferences, one for each agent, and $\mathcal{P}^n$ to denote the set of all preference profiles. A (deterministic) **allocation** $a : I \to X$ is a one-to-one function, where $a(i)$ is the object allocated to agent i. We let $\mathcal{A}$ denote the set of allocations. A **random allocation** $\mu : \mathcal{A} \to [0, 1]$ is a probability distribution over $\mathcal{A}$, where $\sum_{a\in\mathcal{A}} \mu(a) = 1$. We let $\mathcal{M}$ denote the set of random allocations. A **mechanism** $\psi : \mathcal{P}^n \to \mathcal{M}$ is a mapping from preference profiles of the agents to random allocations. Given a mechanism ψ, we write $\psi(P_I)(a)$ to denote the probability that allocation a is implemented when the preferences are P_I. Let $\pi_i^k(\psi(P_I)) = \sum_{a\in\mathcal{A}} \psi(P_I)(a)\mathbb{1}\{a(i) = x_k\}$ be the probability that i receives object x_k at the random allocation $\psi(P_I)$. Finally, we write $\psi_i(P_I) = (\pi_i^k(\psi(P_I)))_{k=1}^n$ to be i's lottery over objects under mechanism ψ at preference profile P_I.

Next, we give formal definitions of the key properties of efficiency, fairness, and incentive that we consider.

- *Efficiency*: A deterministic allocation a is **Pareto efficient** if there is no other allocation a' such that $a(i)R_ia'(i)$ for all $i \in I$ and $a(i)P_ia'(i)$ for some $i \in I$. A mechanism ψ is **Pareto efficient** if, for all $P_I \in \mathcal{P}^n$, every deterministic allocation in the support of $\psi(P_I)$ is Pareto efficient.

- *Fairness:* A mechanism ψ satisfies **equal treatment of equals (ETE)** if for all $P_I \in \mathcal{P}^n$, $P_i = P_j$ implies $\psi_i(P_I) = \psi_j(P_I)$.

- *Incentives:* A mechanism ψ is **strategyproof** if $\psi_i(P_i, P_{-i})$ first-order stochastically dominates $\psi_i(P_i', P_{-i})$ for all $P_i, P_i' \in \mathcal{P}$ and all $P_{-i} \in \mathcal{P}^{n-1}$, where first-order stochastic dominance is defined with respect to i's true preferences P_i.

Finally, we say that two mechanisms ψ and ϕ are **equivalent** if $\psi(P_I) = \phi(P_I)$ for all P_I.

The **Random Priority (RP)** mechanism works as follows. An agent ordering is drawn uniformly at random from the set of all permutations of I. Agents are then assigned objects in this order, with each agent receiving her most preferred object (according to her reported preferences) among the set of objects that have not been assigned to earlier agents. For any preference profile P_I, we define $\psi^{RP}(P_I)$ as the lottery over $\mathcal{A}$ induced by this procedure. It is well-known that ψ^{RP} is strategyproof, Pareto efficient, and satisfies equal treatment of equals.

2.2 Theorem

We are now ready to prove the result.

Theorem 1. *There exists a Pareto efficient, strategyproof, and ETE mechanism ψ that is not equivalent to Random Priority.*

Proof. We prove the theorem using a counterexample with 4 agents, $I = \{1, 2, 3, 4\}$, and 4 objects, $X = \{w, x, y, z\}$. For shorthand, we write $P_i : w, x, y, z$ to denote that i strictly prefers w to x to y to z.

Fix a profile of preferences P_I, and consider the following algorithm.

- Draw an ordering of the agents uniformly from the set of all permutations of I. Denote this ordering as $\sigma : \sigma_1, \sigma_2, \sigma_3, \sigma_4$.
- If $P_1 = P_2 = w, x, y, z$, $\sigma_1 = 1$, and $\sigma_2 = 2$, then assign agent 1 to w, agent 2 to x, agent 3 to her top choice among $\{y, z\}$ (according to P_3), and agent 4 to the remaining unassigned object.
- If $P_1 = P_2 = w, x, y, z$, $\sigma_1 = 2$, and $\sigma_2 = 1$, then assign agent 2 to w, agent 1 to x, agent 4 to her top choice among $\{y, z\}$ (according to P_4), and agent 3 to the remaining unassigned object.
- In all other cases, assign agents in the order $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to their favorite object among those that were not selected by earlier agents (the entire set X for agent σ_1).

Define ψ as the mechanism that results from applying the above algorithm to any preference profile P_I. Note that this is very similar to standard Random Priority, except in two special cases described in the second and third bullet points. In these specific instances, the third agent to select is not chosen randomly from the remaining agents.

It is trivial to see that this mechanism is Pareto efficient, as the algorithm always results in a Pareto efficient deterministic allocation. It is also easy to see that the mechanism is strategyproof: agents cannot affect their place in the selection order, and at their turn, it is optimal to have reported their true preferences.

For equal treatment of equals, note first that it is well-known that RP satisfies ETE. Further, on any preference profile P_I in which either $P_1 \neq w, x, y, z$ or $P_2 \neq w, x, y, z$, $\psi(P_I)$ produces the same lottery over deterministic allocations as RP (and therefore immediately satisfies ETE on all such profiles). Thus, consider any P_I where $P_1 = P_2 = w, x, y, z$. For all σ such that $\{\sigma_1, \sigma_2\} \neq \{i_1, i_2\}$, ψ once again leads to the same deterministic allocation as the corresponding case under RP.

Thus, there are 4 cases left, $\sigma : 1, 2, 3, 4$, $\sigma' : 1, 2, 4, 3$, $\sigma'' : 2, 1, 3, 4$, and $\sigma''' : 2, 1, 4, 3$. It is obvious that 1 and 2 receive the same allocations under each of these as under RP, and so $\psi_1(P_I)$ and $\psi_2(P_I)$ are the same as 1 and 2's lotteries under RP. Finally, consider agents 3 and 4, and note that since 1 and 2 will take w and x, only the relative rankings of y and z matter. If P_3 and P_4 rank y and z differently, then they each receive their favorite from the set $\{y, z\}$, which once again is the same as under RP. So, consider the case that both P_3 and P_4 prefer y to z. Note that under both RP and ψ, 3 receives y and 4 receives z for exactly 2 of $\{\sigma, \sigma', \sigma'', \sigma'''\}$, and the allocation is reversed for the other 2 selections: under RP, 3 receives y and 4 receives z under σ and σ'', and vice-versa under σ' and σ''', while under ψ, 3 receives y and 4 receives z under σ and σ', while 4 receives y and 3 receives z under σ'' and σ'''. The remaining case where both 3 and 4 prefer z to y is analogous, and so, summing up, we conclude that both $\psi_3(P_I)$ and $\psi_4(P_I)$ are equivalent to the respective lotteries under RP. Therefore, ψ also satisfies ETE.

Finally, we argue that ψ is not equivalent to RP. To see this, consider the preference profile $P_i = w, x, y, z$ for all $i \in I$, and the following allocation:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ w & x & z & y \end{pmatrix}.$$

Note that $\psi(P_I)(a) = 0$, while $\psi^{RP}(P_I)(a) > 0$. Therefore, the two mechanisms are not equivalent. ■

Note that while the proof constructs a mechanism ψ that produces a different distribution over global allocations (full assignments of all individuals to objects) than ψ^{RP}, from the perspective of an individual agent i, the marginal distributions of i's assignment over individual objects are the same for both mechanisms. Whether all strategyproof, Pareto efficient, and ETE mechanisms are equivalent to RP under a weaker such notion of marginal equivalence remains an open question.

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