Advanced Numerical Methods: Assignment 1: Maxwell's Equations

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1 Model

We consider the following 1D model of Maxwell's equations,

$$Cu_t = Au_x, \tag{1}$$

where

$$u = \begin{pmatrix} E \\ H \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}. \tag{2}$$

We remark that

$$A = A^*,\ C = C^*. \tag{3}$$

2 Boundary Conditions

Now we apply the energy method. Using both (1) and (3) we get

$$\begin{aligned} (u, Cu_t) &= (u, Au_x) = u^*Au|_{x_l}^{x_r} - (u_x, Au) \\ (u_t, Cu) &= (Cu_t, u) = (Au_x, u) = (u_x, Au) \end{aligned}$$

where

$$(u, v) := u^*v.$$

If we add the results and define

$$\|u\|_C^2 := u^*Cu \tag{4}$$

we obtain

$$\frac{d}{dt}\|u\|_C^2 = (u, Cu_t) + (u_t, Cu) = u^*Au|_{x_l}^{x_r}$$

For the Dirichlet BC we calculate the latter with (2).

$$\begin{aligned} u^*Au|_{x_l}^{x_r} &= HE + EH|_{x_l}^{x_r} = 2EH|_{x_l}^{x_r} \\ &= 2EH|^{x_r} - 2EH|^{x_l} \end{aligned}$$

We can take one of the following Dirichlet BC:

$$E = g_r, \quad x = x_r, \quad E = g_l, \quad x = x_l \quad \text{or} \tag{5}$$
$$H = g_r, \quad x = x_r, \quad H = g_l, \quad x = x_l, \tag{6}$$

either involving the electric or magnetic component. For example if $H|^{x_r}$ we could choose the BC $E^{r_l} = g_r <= 0$ and analogously for the rest.

For the characteristic BC we use the partition $A = A^+ + A^-$ and obtain

$$\frac{d}{dt}\|u\|_C^2 = u^*A^+u|_{x_l}^{x_r} + u^*A^-u|_{x_l}^{x_r} = u^*A^+u|^{x_r} + u^*A^-u|^{x_r} - u^*A^+u|^{x_l} - u^*A^-u|^{x_l}.$$

A Matlab calculation leads further to

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = S^{-1} = \begin{pmatrix} -\sqrt{0.5} & \sqrt{0.5} \\ \sqrt{0.5} & \sqrt{0.5} \end{pmatrix},$$
$$A^+ = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \quad A^- = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}.$$

with

$$S^{-1}DS = A = A^+ + A^-.$$

Since the set of eigenvalues $\rho(A) = \{1, -1\}$ we need one boundary condition at $x = x_l$ and one BC at $x = x_r$. In addition to that,

$$u^*A^+u \geq 0 \quad \text{and} \quad u^*A^-u \leq 0 \tag{7}$$

due to definition of A^+ and A^-. That's why we define the following BCs:

$$u^*A^+u|^{x_r} = g_r(\leq 0) \tag{8}$$
$$u^*A^-u|^{x_l} = g_l(\geq 0). \tag{9}$$

Inserting this in $\frac{d}{dt}\|u\|_C^2$ yields

$$\frac{d}{dt}\|u\|_C^2 \leq 0,$$

as desired.

3 SBP-SAT

Let

$$D_1 = H^{-1}\left(Q + \frac{1}{2}B\right) \tag{10}$$

be an SBP-Operator approximating the first derivative, i.e.

$$H = H^T > 0,\ Q + Q^T = 0,\ B = e_m e_m^T - e_1 e_1^T \tag{11}$$

An SBP-SAT approximation v of (1) is then given by

$$Cv_t = (A \otimes D_1)v - (A^+ \otimes H^{-1}e_m)(v_r - g_r) + (A^- \otimes H^{-1}e_1)(v_l - g_l).$$

We use the energy method in order to proof stability. If we multiply v_t with $v^*(I_k \otimes H)$ we get

$$\begin{aligned}
&v^*(I_k \otimes H)Cv_t \\
&\quad = v^*(I_k \otimes H)(A \otimes D_1)v - v^*(I_k \otimes H)(A^+ \otimes H^{-1}e_m)(v_r - g_r) + v^*(I_k \otimes H)(A^- \otimes H^{-1}e_1)(v_l - g_l) \\
&\quad = v^*(A \otimes HD_1)v - v^*(A^+ \otimes e_m)(v_r - g_r) + v^*(A^- \otimes e_1)(v_l - g_l) \\
&\quad = v^*\left(A \otimes Q + \frac{1}{2}B\right)v - v_r^* A^+(v_r - g_r) + v_l^* A^-(v_l - g_l)
\end{aligned}$$

The conjugate transpose can be written as

$$v_t^* C^*(I_k \otimes H)v = v^*\left(A \otimes Q^* + \frac{1}{2}B\right)v - (v_r - g_r)^* A^+ v_r + (v_l - g_l)^* A^- v_l$$

We used (3) and (11). Adding those equalities leads with the properties of Q and the definition of B as proposed in (3) to

$$\begin{aligned}
\frac{d}{dt}\|v\|_H^2 &= v^*(A \otimes B)v - v_r^* A^+(v_r - g_r) + v_l^* A^-(v_l - g_l) - (v_r - g_r)^* A^+ v_r + (v_l - g_l)^* A^- v_l \\
&= v_r^* A v_r - v_l^* A v_l - v_r^* A^+(v_r - g_r) + v_l^* A^-(v_l - g_l) - (v_r - g_r)^* A^+ v_r + (v_l - g_l)^* A^- v_l \\
&= v_r^* A^- v_r - v_l^* A^+ v_l + g_r^* A^+ g_r - g_l^* A^- g_l \underbrace{-(v_r - g_r)^* A^+(v_r - g_r) + (v_l - g_l)^* A^-(v_l - g_l)}_{\text{Additional damping}}
\end{aligned}$$

For the last equality we inserted $A = A^+ + A^-$. Now we can estimate the latter with (7), (8) and (9) and conclude

$$\begin{aligned}
&\frac{d}{dt}\|v\|_h^2 \\
&\quad = \underbrace{v_r^* A^- v_r}_{\leq 0,(7)} - \underbrace{v_l^* A^+ v_l}_{>=0,(7)} + \underbrace{g_r^* A^+ g_r}_{\leq 0\,(8)} \underbrace{-g_l^* A^- g_l}_{\geq 0,(9)} \underbrace{-(v_r - g_r)^* A^+(v_r - g_r) + (v_l - g_l)^* A^-(v_l - g_l)}_{\text{Additional damping}} \\
&\quad \leq 0
\end{aligned}$$

with additional damping and therefore stability is proven for the CBCs.