Gamma and Beta Distribution
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Normal Distribution

The most commonly used probability distribution is the normal distribution. This gives us the classic bell curve. This shape is formed by the nature of normal distribution, it implies that most values are centered around the mean rather than being on either extreme end. One of the remarkable features of the normal distribution is its ability to transform various data distributions into a Gaussian-like form through proper mathematical transformations.

The normal distribution's mathematical symmetry extends its applicability to various fields. It serves as a fundamental part of statistical analyses, enabling the utilization of parametric statistical methods, as these require the distribution to be Gaussian. Additionally, the normal distribution's characteristic bell curve simplifies the representation and analysis of data, making it a powerful tool in fields such as finance, biology, and quality control. The versatility of the normal distribution solidifies its status as one of the most important distributions in statistical modeling and data analysis. The normal distribution is applicable if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$, the formula for normal distribution is seen below,

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}, \quad -\infty < y < \infty,$$

Use Cases

The normal distribution, or Gaussian distribution, finds widespread application across various disciplines owing to its versatile properties. This distribution is commonly found in many everyday applications One prominent use case is in grades. A standard distribution can be used to curve grades. The normal distribution serves as the foundation for grading systems. The concept of curving grades, a practice employed daily, and is an application of the normal distribution. This ensures that grades are representative of the overall distribution of student performance, aligning with the bell curve's principle of concentration around the mean.

The normal distribution also has many uses outside the classroom, the normal distribution is applicable in the distribution various human attributes. Characteristics such as age, height, and IQ scores often exhibit a normal distribution pattern in populations. This not only simplifies the representation of these traits but also enables statistical analyses that aid in understanding the distribution and variation within a given population. In the realm of probability, the normal distribution is also present in the summation of the rolls of two dice. The distribution of possible outcomes follows the characteristic bell curve, with the most probable sum being in the middle and the extremes becoming increasingly less likely. This application extends to various scenarios involving random variables and probabilities, showcasing the normal distribution's major role in probability theory.

The normal distribution's use cases are the most common, shaping the way we interpret and analyze data in real-world scenarios, from educational assessments to understanding human attributes and even predicting outcomes in probability-driven situations. Its omnipresence signifies its importance as a fundamental concept in diverse fields. Even if you do not start with a Gaussian-like distribution if you apply the proper transform it allows for standard distribution to be applied to the data which allows for greater statistical analysis.

Gamma Distribution

The Gamma Distribution is used in distributions of data that are skewed while containing all random variables x that are nonnegative, as Gamma Distribution is defined for $x \geq 0$. In this distribution most of the density is located near the left side of the distribution or the origin and gradually becomes less dense as y the input to your probability density function $f(y)$ increases. Before looking at the formula for Gamma Distribution, it is important to introduce the gamma function represented by $\Gamma(\alpha)$, this function is defined for any positive real number α as,

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \qquad for\ \alpha > 0.$$

The function $\Gamma(\alpha)$ in gamma distribution, is to ensure that the integral of the Probability Density Function over all possible values of x equals 1. Or in simpler terms, it normalizes the Probability Density Function, thus making it a valid probability distribution. In the gamma distribution β influences that scale of the distribution. Larger β values will make the scale bigger and smaller β values will make the distribution smaller respectively. α is another important variable in gamma distribution α will influence the shape if α is a higher value it will result in a more peaked shape while if α is lower the distribution will be more evenly shaped.

PDF

The Probability Density Function or PDF describes the likelihood of a continuous random variable taking on a specific value within a certain range in its distribution. The gamma distribution itself is provided by its PDF. The PDF for Gamma Distribution is given by,

$$f(y) = \frac{y^{\alpha-1} e^{\frac{-y}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}\ for\ 0 \leq y < \infty,$$

CDF

The Cumulative Distribution Function or CDF represents the probability that a random variable with a gamma distribution with shape parameter α and scale parameter β is less than or equal to y. The CDF is often expressed in terms of the incomplete gamma function as given by,

$$F(y, \alpha; \beta) = \frac{\gamma(\alpha,\ \beta y)}{\Gamma(\alpha)}$$

Where $\gamma(\alpha,\ \beta y)$ is the lower incomplete gamma function.

Mean and Variance

The mean represents the average value of the distribution, and the variance provides the spread of the distribution. Given by,

$$\mu = E(Y) = \alpha\beta \quad and \quad \sigma^2 = V(Y) = \alpha\beta^2$$

Use Cases

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The gamma distribution is applicable in various fields due to its ability to model continuous, nonnegative random variables. One common use case is in reliability engineering, where it is employed to model the time until a system or component fails, such as aircraft engines as mentioned in the text. The distribution's ability to capture skewed data and its sensitivity to shape parameter adjustments can be attributed to the α variable, which make it particularly suitable for scenarios involving varying degrees of wear and tear. In the realm of finance, the gamma distribution can be used to model the distribution of stock prices and returns, which could potentially provide insights into market movements. Overall, the gamma distribution is very adaptable and can be used in diverse situations and fields that require skewed statistical modeling.

Beta Probability Distribution

The Beta Probability function focuses on three main components the density function, that is defined over a closed interval $0 \leq y \leq 1$, and the two parameters α & β. The two parameters, α and β, play a crucial role in shaping the distribution to fit different scenarios. The α parameter represents the number of successes, while β represents the number of failures, providing a formula for capturing a wide range of distribution shapes. These allow Beta Probability Distribution to be used for analyzing and interpreting proportions with many different applications.

PDF

The Beta Probability Distribution itself is characterized by its Probability Density Function, defined over the closed interval $0 \leq y \leq 1$, given by,

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$$

Where,

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The Probability Density Function or PDF of a beta distribution returns the probability density at a given point y within the interval [0, 1]. From α *and* β parameters that determine the shape of the distribution.

CDF

The Cumulative Distribution Function or CDF of a beta distribution gives the probability that a random variable following that distribution is less than or equal to a specific value y.

$$\boldsymbol{F(y)} = \int_{\mathbf{0}}^{\boldsymbol{y}} \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}dt = I_y(\alpha,\beta).$$

Mean and Variance

The mean represents the average value of the distribution, and the variance provides the spread of the distribution. These are calculated with the given α *and* β parameters that determine the shape of the distribution. Given by,

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \ \ and \ \ \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Use Cases

The Beta Probability Distribution is used in many different applications and fields due to its ability to model proportions effectively. One use case, mentioned in the textbook, involves determining impurity levels in chemical products. The math works because the distribution's parameters can capture various impurity distribution patterns, providing a realistic representation of uncertainty. Another use case is in manufacturing for quality control, the distribution's capacity to represent the proportion of defective items is very important to keep track of quality. The math works here by integrating prior knowledge with observed data, estimating proportions in situations with binary outcomes and uncertainty.

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