Gamma and Beta Distribution
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Gamma Distribution

The Gamma Distribution is used in distributions of data that are skewed while containing all random variables x that are nonnegative, as Gamma Distribution is defined for $x \geq 0$. In this distribution most of the density is located near the left side of the distribution or the origin and gradually becomes less dense as y the input to your probability density function $f(y)$ increases. Before looking at the formula for Gamma Distribution, it is important to introduce the gamma function represented by $\Gamma(\alpha)$, this function is defined for any positive real number α as,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \qquad for\ \alpha > 0.$$

The function $\Gamma(\alpha)$ in gamma distribution, is to ensure that the integral of the Probability Density Function over all possible values of x equals 1. Or in simpler terms, it normalizes the Probability Density Function, thus making it a valid probability distribution. In the gamma distribution β influences that scale of the distribution. Larger β values will make the scale bigger and smaller β values will make the distribution smaller respectively. α is another important variable in gamma distribution α will influence the shape if α is a higher value it will result in a more peaked shape while if α is lower the distribution will be more evenly shaped.

PDF

The Probability Density Function or PDF describes the likelihood of a continuous random variable taking on a specific value within a certain range in its distribution. The gamma distribution itself is provided by its PDF. The PDF for Gamma Distribution is given by,

$$f(y) = \frac{y^{\alpha-1} e^{\frac{-y}{\beta}}}{\beta^\alpha \Gamma(\alpha)}\ for\ 0 \leq y < \infty,$$

CDF

The Cumulative Distribution Function or CDF represents the probability that a random variable with a gamma distribution with shape parameter α and scale parameter β is less than or equal to y. The CDF is often expressed in terms of the incomplete gamma function as given by,

$$F(y, \alpha; \beta) = \frac{\gamma(\alpha,\ \beta y)}{\Gamma(\alpha)}$$

Where $\gamma(\alpha,\ \beta y)$ is the lower incomplete gamma function.

Mean and Variance

The mean represents the average value of the distribution, and the variance provides the spread of the distribution. Given by,

$$\mu = E(Y) = \alpha\beta \quad and \quad \sigma^2 = V(Y) = \alpha\beta^2$$

Use Cases

The gamma distribution is applicable in various fields due to its ability to model continuous, nonnegative random variables. One common use case is in reliability engineering, where it is employed to model the time until a system or component fails, such as aircraft engines as mentioned in the text. The distribution's ability to capture skewed data and its sensitivity to shape parameter adjustments can be attributed to the α variable, which make it particularly suitable for scenarios involving varying degrees of wear and tear. In the realm of finance, the gamma distribution can be used to model the distribution of stock prices and returns, which could potentially provide insights into market movements. Overall, the gamma distribution is very adaptable and can be used in diverse situations and fields that require skewed statistical modeling.

Beta Probability Distribution

The Beta Probability function focuses on three main components the density function, that is defined over a closed interval $0 \leq y \leq 1$, and the two parameters α & β. The two parameters, α and β, play a crucial role in shaping the distribution to fit different scenarios. The α parameter represents the number of successes, while β represents the number of failures, providing a formula for capturing a wide range of distribution shapes. These allow Beta Probability Distribution to be used for analyzing and interpreting proportions with many different applications.

PDF

The Beta Probability Distribution itself is characterized by its Probability Density Function, defined over the closed interval $0 \leq y \leq 1,$ given by,

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$$

Where,

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The Probability Density Function or PDF of a beta distribution returns the probability density at a given point y within the interval [0, 1]. From α *and* β parameters that determine the shape of the distribution.

CDF

The Cumulative Distribution Function or CDF of a beta distribution gives the probability that a random variable following that distribution is less than or equal to a specific value y.

$$\boldsymbol{F(y)} = \int_{\mathbf{0}}^{y} \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}dt = I_y(\alpha,\beta).$$

Mean and Variance

The mean represents the average value of the distribution, and the variance provides the spread of the distribution. These are calculated with the given α *and* β parameters that determine the shape of the distribution. Given by,

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \; and \; \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

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Use Cases

The Beta Probability Distribution is used in many different applications and fields due to its ability to model proportions effectively. One use case, mentioned in the textbook, involves determining impurity levels in chemical products. The math works because the distribution's parameters can capture various impurity distribution patterns, providing a realistic representation of uncertainty. Another use case is in manufacturing for quality control, the distribution's capacity to represent the proportion of defective items is very important to keep track of quality. The math works here by integrating prior knowledge with observed data, estimating proportions in situations with binary outcomes and uncertainty.

References

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Beta Distribution

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