2.6 30

We know that any choice set is not measurable, we also know that every set which is countable or finite is measurable. Thus any choice set must be uncountable.

2.7 34

Consider the function Ψ^{-1} which is the inverse of the function $\Psi(x) = \varphi(x) + x$ where φ is the Cantor-Lebesgue function. This inverse function is well defined since Ψ is a bijection. Ψ^{-1} is continuous since it is a strictly increasing surjection (it is strictly increasing since the inverse of a strictly increasing function is strictly increasing). By scaling the function $F(x) = \Psi(2x)$ we have a function defined on $[0, 1]$ which maps a measurable set of positive measure onto the Cantor set

2.7 37 OLD EDITION

It is not true, as illustrated by the function $\Psi^{-1}(x)$ with $E = [0, 2]$ described in 34. We have that the pull back of Ψ^{-1} of a subset of the Cantor set is a nonmeasurable set (the subset of the Cantor set is measurable since the Cantor set is measure 0)

2.7 37

Let B be a measure zero set. For any $\epsilon > 0$ we can cover B with intervals I_k such that $\sum_{k=1}^{\infty} \ell(I_k) < \epsilon/c$. Notice that a implication of lipchitz is

$$m^*(f(I_k)) \leq c\ell(I_k)$$

The reason for this is if $I_k = (a, b)$, letting $x = \frac{a+b}{2}$ we have I_k is a ball of radius $r = \ell(I_k)/2$ around x

$$f(I_k) = f(B_r(x)) \subseteq B_{cr}(f(x))$$

and $B_{cr}(f(x))$ is an interval of length $c\ell(I_k)$
From this we have

$$m^*(f(B)) \leq m^*\left(f\left(\bigcup_{k=1}^{\infty} I_k\right)\right)$$

$$= m^*\left(\bigcup_{k=1}^{\infty} f\left(I_k\right)\right) \leq \sum_{k=1}^{\infty} m^*(f(I_k)) \leq \sum_{k=1}^{\infty} c\ell(I_k) < \epsilon$$

And thus $f(B)$ is measure zero
Let F be a F_σ set. We have that F is a countable union of closed sets

$$F = \bigcup_{i=1}^{\infty} C_i$$

We have

$$f(F) = \bigcup_{i=1}^{\infty} f(C_i)$$

It is the case that every lipchitz function is closed and thus $f(C_i)$ is closed so $f(F)$ is a F_σ set.
To show that Lipshitz $\Rightarrow$ closed:
for an limit point $y \in f(C)$ we have that $f(x_i) \in C$ converges to y. Thus $f(x_i)$ is Cauchy and by the lipchitz condition we get that x_i is Cauchy and thus has a limit $x \in C$. Thus we have that $f(x) = y$ so $y \in f(C)$
Thus since every measurable set $M = F \cup B$ is a union of a F_σ set and a zero set we have that

$$f(M) = f(F) \cup f(B)$$

is also a union of a F_σ set and a zero set and thus measurable

2.7 42 OLD EDITION

Suppose for contradiction we have an enumeration of X :

$$X = \{x_1, x_2, x_3, \dots\}$$

We can construct an element in X that is not enumerated in such a fashion:
Choose any two elements $a_1, a_2 \in X$ and construct the disjoint closed balls B_1, B_2 around a_1, a_2 (whose radius would be $< |a_1 - a_2|/2$). x_1 cannot be in both sets so denote C_1 as the set x_1 is not in. Now choose a_1, a_2 in the interior of C_1 and again construct the disjoint closed balls B_1, B_2 of a_1, a_2 which are also contained in the interior of C_1. Again x_2 cannot be in both sets so denote C_2 as this closed ball.
In general given a closed ball C_n we choose two points $a_1, a_2 \in \text{int } C_n$ and get disjoint balls B_1, B_2 centered around each point. We will denote C_{n+1} as the ball which x_{n+1} is not in
From this we have the nested closed sets $C_1 \subset C_2 \subset C_3 \subset \dots$ which cannot be closed when taking the intersection:

$$\exists x \in \bigcap_{n=1}^{\infty} C_n$$

x was not in the enumeration since for every n, $x \in C_n, x_n \notin C_n$

2.7 42

We know that the inverse image preserves σ-algebra operations:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$f^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} f^{-1}(A_i)$$

$$f^{-1}(A^c) = f^{-1}(A)^c$$

Thus since every borel set B is obtained by various σ-algebra operations of open sets U_i and f^{-1} preserves open sets, we have that $f^{-1}(B)$ is obtained by various σ-algebra operations of open sets $f^{-1}(U_i)$ and thus Borel

3.1 2

This is not the case. Consider the example

$$D = [0.1] \cap \mathbb{Q}, E = [0,1] - \mathbb{Q}$$

D, E are measurable since $m^*(D) = 0$ and $E = [0,1] - D$. Consider the function

$$f(x) = \begin{cases} 1 & x \in D \\ 0 & x \in E \end{cases}$$

f is continuous on D and on E but not continuous on $D \cup E = [0,1]$.

3.1 4

No as illustrated by the following counterexample. We know that Ψ^{-1} described in problem 34 is not a measurable function. It is however one-to-one and thus $(\Psi^{-1})^{-1}(c)$ is just a point and so measurable.

3.1 8

(i) We know that the set of Borel sets are a subset of the set of Lebesgue sets. Thus the definition have direct implications: E is Lebesgue if Borel and $\{x \in E | f(x) > c\}$ is Lebesgue if it is Borel

(ii) We know that the inverse image preserves σ-algebra operations:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$f^{-1}(A^c) = f^{-1}(A)^c$$

We have that the Borel sets are generated by sets of the form (c, ∞) and thus $f^{-1}(B)$ is a set obtained by σ-algebra operations of sets of the form $f^{-1}(c, \infty)$ which are Borel sets since f is Borel measurable. Thus $f^{-1}(B)$ is a Borel set

(iii) We have that $(f \circ g)^{-1}(c, \infty) = g^{-1}(f^{-1}(c, \infty)) = f^{-1}(B)$. Since f is Borel measurable, $f^{-1}(c, \infty) = B$ is a Borel set. From (ii) $g^{-1}(B)$ is Borel and thus $f \circ g$ is Borel measurable

(iv) We have that $(f \circ g)^{-1}(c, \infty) = g^{-1}(f^{-1}(c, \infty)) = g^{-1}(B)$. Since f is Borel measurable, $f^{-1}(c, \infty) = B$ is a Borel set. Notice that the argument for (ii) can be applied for Lebesgue measurable functions as well and so $g^{-1}(B)$ is Lebesgue. Thus $f \circ g$ is Lebesgue measurable

3.1 10

No, consider f is the identity function and g is continuous but not measurable (for instance Ψ^{-1} as described for problem 34 but extended to all of $\mathbb{R}$ by returning $2x$ on $\mathbb{R} \backslash [0,1]$).