1-4, 5

By definition of $\wedge$:

$$(p-p_1) \wedge (p-p_2) \cdot (p-p_3) = \det(p-p_1, p-p_2, p-p_3)$$

Notice that $p = p_1, p_2, p_3$ satisfy the equation since one of the vectors is zero and thus the determinant is zero. Since the det is a linear equation, the equation must be a plane or be a trivial equation (which describes all of $\mathbb{R}^3$). The equation cannot be trivial however since p_1, p_2, p_3 are not colinear

1-4 6

Consdier any point satisfying the equation $p = ut + p_0$ (where $p_0 = (x_0, y_0, z_0)$). Plugging p into either plane equation yields

$$p \cdot v_i - d_i = u \cdot v_i t + p_0 \cdot v_i - d_i = (v_1 \wedge v_2) \cdot v_i + (p_0 \cdot v_i - d_i)$$

We have $(v_1 \wedge v_2) \cdot v_i = 0$ and since p_0 is a point in both planes, p_0 satisfies the plane equation for both planes $p_0 \cdot v_i - d_i = 0$. Thus $p \cdot v_i - d_i = 0$, the line is the intersection of the planes

1-4 10

We have

$$\begin{pmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

Which is easily verified by multiplying out the matricies on the right

The book establishes

$$\begin{vmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{vmatrix} = A^2$$

Thus since determinant is invariant by transposition

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

$$A^2 = \begin{vmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$$

1-4 11a

We have that $|(u \wedge v) \cdot w| = \det(u, v, w)$. It is a commonly known fact that the volume of the parallel pipet spanned by u, v, w is $\det(u, v, w)$. This can be geometrically established by noticing that $|u \wedge v|$ is the area of the quadralateral spanned by u, v and $u \wedge v$ is the perpendicular vector to this quadralateral. We have that the dot product of w with $u \wedge v$ is the length of the projection of w to $u \wedge v$ multiplied by $|u \wedge v|$ which is precisely the area of the parallel pipet

1-4 12

($\Rightarrow$) If there exists u such that $u \wedge v = w$. We have that $w \cdot v = (u \wedge v) \cdot v = \det(u, v, v) = 0$. Thus v and w are perpendicular

($\Leftarrow$) If w, v are perpendicular, define $u = \frac{v \wedge w}{|v|^2}$. Since u, v are perpendicular to w, we know that $w, u \wedge v$ are colinear. Notice that (since u and v are perpendicular) $|u \wedge v| = |u||v| = \frac{|w||v|^2}{|v|^2} = |w|$. Thus $w = \pm(u \wedge v)$. By the right hand rule we have that $w = u \wedge v$ since $u = v \wedge w$. This choice of u was unique

1-4 13

By the product rule for $\wedge$:

$$\frac{d}{dt}(u(t) \wedge v(t)) = u'(t) \wedge v(t) + u(t) \wedge v'(t)$$

By bilinearity of $\wedge$:

$$= a(u(t) \wedge v(t)) + b(v(t) \wedge v(t)) + c(u(t) \wedge u(t)) - a(u(t) \wedge v(t))$$

$v(t) \wedge v(t) = u(t) \wedge u(t) = 0$ yields

$$= a(u(t) \wedge v(t) - u(t) \wedge v(t)) = 0$$

Thus the derivative is zero for all t so $u(t) \wedge v(t)$ is constant

1-5 1

(a) Calculating $|\alpha'(s)|$:

$$|\alpha'(s)| = \sqrt{\frac{a^2}{c^2}\sin^2\frac{s}{c} + \frac{a^2}{c^2}\cos^2\frac{s}{c} + \frac{b^2}{c^2}} = \sqrt{\frac{a^2(\sin^2\frac{s}{c} + \cos^2\frac{s}{c}) + b^2}{c^2}} = \sqrt{\frac{a^2 + b^2}{c^2}} = 1$$

Thus α is parameterized by arclength

(b) Curvature is $|\alpha''(s)|$.

$$\kappa = |\alpha''(s)| = \sqrt{\frac{a^2}{c^4}\cos^2\frac{s}{c} + \frac{a^2}{c^4}\sin^2\frac{s}{c}} = \frac{a}{c^2}$$

Torsion is $\tau = |\frac{d}{ds}(\alpha'(s) \wedge \frac{\alpha''(s)}{\kappa})|$:

We have

$$\frac{\alpha''(s)}{\kappa} = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$$

$$\alpha'(s) = (\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

$$\alpha' \wedge \frac{\alpha''}{\kappa} = (0, 0, \frac{a}{c}\sin^2\frac{s}{c} + \frac{a}{c}\cos^2\frac{s}{c}) = (0, 0, \frac{a}{c})$$

Thus

$$\tau(s) = 0$$

(c) We already found the binormal vector:

$$b(s) = \alpha' \wedge \frac{\alpha''}{\kappa} = (0, 0, \frac{a}{c})$$

We have that the plane is

$$0 = (p - \alpha(s)) \cdot b(s) = \frac{a}{c}(z - \frac{bs}{c})$$

(d) We have the normal vector is

$$n(s) = \frac{\alpha''(s)}{\kappa} = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$$

Since the third component is zero,

$$n(s) \cdot e_3 = 0$$

so $n(s)$ is always perpendicular to the z axis (meets the axis at an angle of $\pi/2$).
we know the line $tn(s) + \alpha(s)$ always intersects the z axis since we can choose t to be a and we get $an(s) + \alpha(s) = (0, 0, \frac{bs}{c})$

(e) We know the tangent line direction vector is

$$t(s) = \alpha'(s) = (\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

We have that

$$t(s) \cdot e_3 = \frac{b}{c}$$

is constant, thus the angle between the two vectors is constant

1-5 2

From lecture and in the book we have the following identities

$$\alpha'(s) = t(s)$$

$$\alpha''(s) = \kappa(s)n(s)$$

$$\alpha'''(s) = \kappa'(s)n(s) + \kappa(s)n'(s)$$

$$n'(s) = -\tau(s)b(s) - \kappa(s)t(s)$$

So we have

$$-\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2} = -\frac{\kappa(s)t(s) \wedge n(s) \cdot (\kappa'(s)n(s) + \kappa(s)n'(s))}{|\kappa(s)|^2}$$

$t(s) \wedge n(s) \cdot n(s) = 0$ so we have

$$= -\frac{\kappa(s)t(s) \wedge n(s) \cdot \kappa(s)n'(s)}{|\kappa(s)|^2} = -\frac{\kappa^2(s)(t(s) \wedge n(s)) \cdot (-\tau(s)b(s) - \kappa(s)t(s))}{|\kappa(s)|^2}$$

$t(s) \wedge n(s) \cdot t(s) = 0$ so

$$= \tau(s)t(s) \wedge n(s) \cdot b(s)$$

Since $b(s)$ is defined as $t(s) \wedge n(s)$ and the norm is 1, this yields

$$= \tau(s)b(s) \cdot b(s) = \tau(s)$$

1-5 4

Letting $\alpha(s)$ be the curve parameterized by arc length and p the point, there exists a scalar function $c(s)$ where $c(s)n(s) + \alpha(s) = p$. Differentiating both sides yields,

$$c'(s)n(s) + c(s)n'(s) + \alpha'(s) = 0$$

We have the identity $n' = -\kappa t - \tau b$

$$c'(s)n(s) + c(s)(-\kappa(s)t(s) - \tau(s)b(s)) + t(s) = 0$$

$$c'(s)n(s) + (-\kappa(s)c(s) + 1)t(s) - c(s)\tau(s)b(s) = 0$$

Notice that this is a sum of the linear independent vectors n, t, b equal to zero. Thus it must be the case that $c'(s) = -\kappa(s)c(s) + 1 = c(s)\tau(s) = 0$. This shows that $\tau(s) = 0$ which means α is contained in a plane (as we established in lecture). We have that $c'(s) = 0$ so $c(s)$ is constant and so since $-\kappa(s)c(s) + 1$, $\kappa(s)$ is constant, which we know means α is contained in a circle

1-5 5

(a) Letting p be the point, there exists a scalar function $c(s)$ where $c(s)t(s) + \alpha(s) = p$. Differentiating yields

$$c'(s)t(s) + c(s)t'(s) + \alpha'(s) = 0$$

$$(c'(s) + 1)t(s) + c(s)\kappa(s)n(s) = 0$$

Since t, n are linearly independent, $c'(s) + 1 = \kappa(s)c(s) = 0$. Since $c'(s) \neq 0$, it must be the case that $c(s) \neq 0$ so $\kappa(s) = 0$. Thus $\alpha''(s) = 0$, solving this differential equation yields $\alpha(s) = c_1 t + c_0$ for some constants c_0, c_1. Thus α is a line

(b) Not necessarily, consider the curve

$$\alpha(t) = (|t|, t, 0)$$

This curve is not contained in a line, yet the tangent line (when defined) crosses the orgin

1-5 6

(a) By definition of $|\cdot|$ and orthogonal transformations: $|p(u)| = \sqrt{p(u) \cdot p(u)} = \sqrt{u \cdot u} = |u|$. We have $\cos(\theta_p)|p(u)||p(v)| = |p(u) \cdot p(v)| = |u \cdot v| = |u||v|\cos(\theta)$. Since norms are invariant we get $\cos(\theta_p) = \cos(\theta)$ since $0 \leq \theta \leq \pi$ this implies $\theta = \theta_p$

(b) Since p is linear, it suffices to show it is true for the basis e_1, e_2, e_3 since we have $p(v) = v_1 p(e_1) + v_2 p(e_2) + v_3 p(e_3)$. So

$$p(v) \wedge p(w) = \sum_{i,j \in [3]} v_i w_k p(e_i) \wedge p(e_k)$$

We have that since $p(e_j) \cdot p(e_k) = e_j \cdot e_k$, We have that $p(e_1), p(e_2), p(e_3)$ are still all orthogonal to eachother. Thus the vector product of two must be the other. (Not the negative of the other since the transformation is orientation preserving). Thus $p(e_i) \wedge p(e_j) = p(e_k) = p(e_i \wedge e_j)$

(c) Since each of the described quatities is obtained by taking derivatives and norms of a parameterized curve, it suffices to show differentiation is invariant.
By Linearity we have

$$p(\frac{d}{ds}\alpha(s)) = p(\alpha_1'(s)e_1 + \alpha_2'(s)e_2 + \alpha_3'(s)e_3)$$

$$= p(\alpha_1'(s)e_1) + p(\alpha_2'(s)e_2) + p(\alpha_3'(s)e_3) = \alpha_1'(t)p(e_1) + \alpha_2'(s)p(e_2) + \alpha_3'(s)p(e_3) = \frac{d}{ds}p(\alpha(s))$$