Exercise 91

Given any $\epsilon > 0$, consider the covering of N by $\epsilon/2$- neighborhoods $B = \{B_{\epsilon/2}(q) : q \in N\}$ and the preimage $P = \{f^{-1}(S) : S \in B\}$. Since $\cup_{S\in B} S = N$, we have that $\cup_{U\in P} U = M$. Thus P covers M (and P is a collection open sets since f is continuous and we are taking preimages of open sets) so from the lebesgue number lemma there exists $\lambda > 0$ such that for any $m \in M$ there is a $U \in P$ such that $B_\lambda(m) \subset U$. Thus for any $x, y \in N$ where $d(x, y) < \lambda$ we have that $x, y \in B_\lambda(x)$, thus from what we have shown there is a $m \in M$ such that $B_\lambda(x) \subset f^{-1}(B_{\epsilon/2}(m))$ so $f(x), f(y) \in B_{\epsilon/2}(m)$. Thus from the triangle ineq, $d_M(f(x), f(y)) < d_M(f(x), m) + d_M(f(y), m) \leq \epsilon$. Thus f is uniformly continuous.

Exercise 93

We can consider the complements. Let $\mathcal{U} = \{U = M - C : C \in \mathcal{C}\}$. The finite intersection property translates to for any finite collection $U_1, U_2, \ldots U_n \in \mathcal{U}$, we have that from Demorgans law:

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} M - C_i = M - \bigcap_{i=1}^{n} C_i \neq M$$

Thus $\mathcal{U}$ does not contain a finite subcovering of M. Thus it must be the case that M is not covered by $\mathcal{U}$ or we contradict covering compact. Thus from Demorgans law

$$\bigcup_{U\in\mathcal{U}} U = \bigcup_{C\in\mathcal{C}} M - C = M - \bigcap_{C\in\mathcal{C}} C \neq M$$

which is only the case if $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$

Exercise 94

For any collection of open sets $\mathcal{U}$ which covers M, if the finite intersection property holds, consider the complements $\mathcal{C} = \{C = M - U : U \in \mathcal{U}\}$. Since $\mathcal{U}$ covers M we have

$$M = \bigcup_{U\in\mathcal{U}} U = \bigcup_{C\in\mathcal{C}} M - C = M - \bigcap_{C\in\mathcal{C}} C$$

Thus $\bigcap_{C\in\mathcal{C}} C = \emptyset$. Thus $\mathcal{C}$ must not satisfy the finite intersection property so there exists $C_1, C_2, \ldots C_n$ such that

$$\bigcap_{i=1}^{n} C_i = \emptyset \Rightarrow M = M - \bigcap_{i=1}^{n} C = \bigcup_{i=1}^{n} M - C_i = \bigcup_{i=1}^{n} U_i$$

Thus we have a finite subcover.

Exercise 96

From the definition of dense we have that $B \subset \overline{A}$, thus $\overline{B} \subset \overline{A}$ since $\overline{B}$ is contained in every closed set which contains B. Since B is dense in C we have $C \subset \overline{B} \subset \overline{A}$. Thus A is dense in C

Exercise 1

We have that for all x

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

From what we are givien however we have that

$$\frac{|f(x) - f(t)|}{|x - t|} \leq |x - t|$$

Thus

$$-|x - t| \leq f'(x) = \frac{f(x) - f(t)}{x - t} \leq |x - t|$$

Thus as $t \to x$ we get $0 \leq f'(x) \leq 0 \Rightarrow f'(x) = 0$ for all x. Thus f is constant

Exercise 2

(a) Given any $\varepsilon > 0$, let $\delta = \left(\frac{\varepsilon}{H}\right)^{1/\alpha}$. We have that for any $x, t \in (a, b)$ where $|x - t| \leq \delta$, $|f(x) - f(t)| \leq H|x - t|^\alpha \leq H\delta^\alpha = \varepsilon$, thus f is uniformly continuous.
We can thus extend f by letting $f(a) = \lim_{x \to a} f(x)$, and $f(b) = \lim_{x \to b} f(x)$. This extension of f is also α-Holder since if we fix $u \in (a, b)$ and consider the limits $\lim_{x \to a} |f(x) - f(u)| \leq \lim_{x \to a} H|x - u|^\alpha$, since these functions are continuous at a and we have the inequality $|f(x) - f(u)| \leq H|x - u|^\alpha$ for all $x < a$, the inequality holds at a. This same argument can be applied to b (as well as now $x = a, u = b$ since we showed it holds for $u = a$ and $x \in (a, b)$) by relabeling the terms.

(b) This is the Lipshitz condition.

(c) We have that for all x

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

From what we are givien however we have that

$$\frac{|f(x) - f(t)|}{|x - t|} \leq |x - t|^{\alpha - 1}$$

Thus

$$-|x - t|^{\alpha - 1} \leq f'(x) = \frac{f(x) - f(t)}{x - t} \leq |x - t|^{\alpha - 1}$$

Thus as $t \to x$ we get $0 \leq f'(x) \leq 0 \Rightarrow f'(x) = 0$ for all x. Thus f is constant

Exercise 3

(a) Suppose for contradiction there exists $x, y \in (a, b)$ with $x < y$ and $f(x) \geq f(y)$. However from the mean value theorem (since f is differentiable on $[x, y] \subset (a, b)$ the mvt holds) we have that $\exists\theta \in (x, y)$ such that

$$f'(\theta) = \frac{f(y) - f(x)}{y - x}$$

However $f(y) - f(x) \leq 0$ and $y - x > 0$ thus $f'(\theta) \leq 0$ which is a contradiction.

(b) f is monotone increasing by similar reasoning:
Suppose for contradiction there exists $x, y \in (a, b)$ with $x < y$ and $f(x) > f(y)$. However from the mean value theorem (since f is differentiable on $[x, y] \subset (a, b)$ the mvt holds) we have that $\exists\theta \in (x, y)$ such that

$$f'(\theta) = \frac{f(y) - f(x)}{y - x}$$

However $f(y) - f(x) < 0$ and $y - x > 0$ thus $f'(\theta) < 0$ which is a contradiction.

Exercise 9

(a) let us define $g(x) = x - f(x)$. Notice that $g'(x) = 1 - f'(x) > 1 - L > 0$ for all x, let $\epsilon = 1 - L$. Notice that for any x, $g(x) = 0 \Leftrightarrow f(x) = x$, thus if we find a unique 0 of $g(x)$ we are done.
For existence:
Suppose $g(0) = A < 0$ (if $A = 0$ we are done). We have from the mvt

$$\frac{g(x) - g(0)}{x} = g'(\theta) > \epsilon$$

for all x and some $\theta \in (0, x)$. Rearranging the inequality yields

$$g(x) > \epsilon x + A$$

Thus if we choose $x > A/\epsilon$ we get $g(x) > 0$ thus we have $g(x) > 0 > g(0)$ and so from the intermedite value theorem there exists $\alpha \in (0, x)$ where $g(\alpha) = 0$
If $g(0) = A > 0$ then now

$$\frac{g(0) - g(x)}{-x} = g'(\theta) > \epsilon$$

and thus rearranging we get

$$-g(x) > -x\epsilon - g(0) \Rightarrow g(x) < x\epsilon + A$$

Thus choosing $x > -A/\epsilon$ we get $g(x) > 0$ and thus from intermedite value theorem we get $\exists\alpha \in (0, x)$ such that $g(\alpha) = 0$.
Uniqueness:
If there exists $\alpha \neq \beta \in \mathbb{R}$ where $g(\alpha) = g(\beta) = 0$ then from the mean value theorem there exists $\theta \in (\alpha, \beta)$ where

$$g'(\theta) = \frac{g(\alpha) - g(\beta)}{\alpha - \beta} = 0$$

But this contradicts $g'(\theta) > \epsilon$. Thus this is not possible

(b) Let $f(x) = x + e^{-x}$. We have that $x \neq x + e^{-x}$ for all x and $f'(x) = 1 - e^{-x} < 1$ for all x

Exercise Additional Problem 1

Given any sequence $x_n \in K$ we can define the chain $A_1 \supset A_2 \supset \dots$ of relatively closed sets in K as $A_n = \overline{B_n} \cap K$ with $B_n = \{x_j : j \geq n\}$. It is clear $A_n \supset A_{n+1}$ since $B_n \supset B_{n+1}$. Thus we have from assumption

$$p \in \bigcap A_n \neq \emptyset$$

We have that p is the limit of some subsequence of x_n (and thus K is compact). We can construct this subsequence inductively as follows (letting $n_k = 1$):
We have that $p \in \overline{B}_n$ for all n, thus for $\epsilon = \frac{1}{k}$ there exists $x_{n_k} \in B_{1+n_{k-1}}$ so that $d(p, x_{n_k}) < \epsilon$. We thus have that $n_k > n_{k-1}$ since $x_{n_k} \in B_{1+n_{k-1}}$ and all the indicies in $B_{1+n_{k-1}}$ are greater than n_{k-1} and thus we have a subsequence. Thus we have the subsequence $(x_{n_k})_k \to p$ since $d(x_{n_k}, p) \to 0$