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(c) Consider the sequence of functions

$$f_n(x) = x^n$$

Notice that $|f_n|_{C^0} = 1$ and $|f_n|_{L^1} = \int_0^1 x^n dx = \frac{1}{n+1}$. This sequence establishes that $|\cdot|_{C^0}$ and $|\cdot|_{L^1}$ are not comparable since

$$|f_n|_{L^1} = \frac{|f_n|_{C^0}}{n+1}$$

becomes an arbitrarily small ratio

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(a) T is linear since we know integration is linear. T is continous since for any convergent sequence f_n under the uniform norm

$$\lim_{n\to\infty} \int_0^x f_n(t)dt = \int_0^x \lim_{n\to\infty} f_n(t)dt$$

and thus T is sequentially continous.
We have that the norm of T is given by

$$|T| = \sup_{f\in C^0} \frac{\left|\int_0^x f(t)dt\right|_{C^0}}{|f|_{C^0}}$$

For arbitrary $f \in C^0$ and letting $M = |f|_{C^0}$ we have

$$\frac{\left|\int_0^x f(t)dt\right|_{C^0}}{|f|_{C^0}} \leq \max_{x\in[0,1]} \frac{\int_0^x M dt}{M} = \max_{x\in[0,1]} x\frac{M}{M} = 1$$

Thus $|T| \leq 1$ notice that when f is constant we get equality and thus $|T| = 1$
(b)

$$T(\cos(nt)) = \int_0^x \cos(nt)\ dt = \frac{\sin(nx)}{n}$$

(c) K is bounded since $|f_n|_{C^0} \leq 1$ for all n.
We will show the only possible Cauchy sequence in K is eventually constant and thus K is closed. This also establishes K is not compact since a nonconstant sequence will not have any Cauchy subsequence.
We can bound the integral for $m \neq n$ from below

$$\int (\cos nt - \cos mt)^2 dt = \int \cos^2 nt - \cos nt \cos mt + \cos^2 mt\ dt$$

We know that if for some $c > 0$ $\int_0^1 |f_n - f_m|^2 dt > c^2$ for all $m \neq n$ then $|f_m - f_n|_{C^0} > c$ and thus we cannot have a Cauchy sequence when $m \neq n$
We have the trig identity

$$\cos mt \cos nt = \frac{\cos(mt+nt) + \cos(mt-nt)}{2}$$

Using this along with the fact $\int \cos^2 x dx = \frac{1}{2}x + \frac{\sin 2x}{4}$

$$\int_0^x \cos^2 nt - \frac{\cos(mt+nt)+\cos(mt-nt)}{2} + \cos^2 mt \ dt$$

$$= x + \frac{\sin 2nx}{4n} + \frac{\sin 2mx}{4m} - \frac{-\sin(mx+nx)}{2(m+n)} + \frac{-\sin(mx-nx)}{2(m-n)}$$

by triangle inequality (we know WLOG $m > n \geq N$ for appropriate N)

$$\left|\frac{\sin 2nx}{4n} + \frac{\sin 2mx}{4m} - \frac{-\sin(mx+nx)}{2(m+n)} + \frac{-\sin(mx-nx)}{2(m-n)}\right| \leq \frac{1}{4n} + \frac{1}{4m} + \frac{1}{2(m+n)} + \frac{1}{2(m-n)} \leq c < 1$$

And thus

$$= x + \frac{\sin 2nx}{4n} + \frac{\sin 2mx}{4m} - \frac{-\sin(mx+nx)}{2(m+n)} \geq x - c$$

and thus evaluating the limit of the integral from 0 to 1:

$$\int_0^1 (\cos nt - \cos mt)^2 dt \geq x - c\Big|_0^1 = 1 - c > 0$$

(d) We have that

$$T(K) = \left\{T(f_n) = \frac{\sin nx}{n}\right\}$$

is compact since any sequence in $T(K)$ either becomes arbitrarily large or is bounded by some N in index. If it is bounded by N then there is some constant f_n which shows up infinitly often in the sequence and is thus a convergent subsequence. Otherwise we can choose a subsequence of indicies increasing to infinity. This is a convergent subsequence since $|f_n|_{C^0} \leq \frac{1}{n}$ yields a Cauchy sequence. The closure is thus compact as well since $\overline{T(K)} = T(K)$

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(a) We have

$$f(q) - f(p) = (\cos 2\pi, \sin 2\pi) - (\cos\pi, \sin\pi) = (1,0) - (-1,0) = (2,0)$$

We have that

$$Df_\theta = (-\sin\theta, \cos\theta)$$

In order to satisfy $Df_\theta(q-p) = f(q) - f(p)$ we have the second coordinate equality

$$\cos\theta = 0$$

which can only happen if $\theta = 3\pi/2$. Plugging in $\theta = 3\pi/2$ does not yield the correct equality however

$$Df_\theta(q-p) = (\pi, 0) \neq (2, 0)$$

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(a) It follows directly from the definition of differentiability. If the limit (letting $h \in \mathbb{R}^m$)

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{|h|}$$

exists (definition of differentiability), then letting $h = tu$ we have the limit

$$\lim_{t\to 0} \frac{f(p+tu) - f(p)}{|tu|} = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}$$

exists

(b) Letting $u = (a, b)$ we have

$$\Delta_{(0,0)} f(u) = \lim_{t\to 0} \frac{(at)^3 bt}{(at)^4 + (bt)^2} = \lim_{t\to 0} \frac{a^3 bt^4}{a^4t^4 + b^2t^2} = \lim_{t\to 0} \frac{a^3 b}{a^4 + b^2 \frac{1}{t^2}} = 0$$

f is not differentiable at $(0,0)$ however since letting $x = y^2$ we have

$$\lim_{x\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^6 y}{y^8 + y^2} = \lim_{y\to 0} \frac{1}{y + \frac{1}{y^5}} \to \infty$$

does not exist

Additional Problem 1

Notice that $\det(A)$ is a continous map. Also notice that a matrix A is invertable if and only if $\det(A) \neq 0$. Thus

$$\mathcal{M} = \det{}^{-1}(\mathbb{R} \setminus \{0\})$$

$\mathbb{R} \setminus \{0\}$ is an open set, thus since the continous preimage of an open set is open, $\mathcal{M}$ is open

$\mathcal{M}$ is dense since if we consider any noninvertable $A \in \mathbb{R}^{n^2} - \mathcal{M}$, we can choose a basis so that A is upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

Since A is singular we know there are some 0s on the diagonal

For any ball of radius r around A we can choose $\epsilon_1, \ldots \epsilon_s$ such that we replace each 0 on a diagonal with an ϵ_i to get a new matrix A'. This new matrix is invertable since it has no zeros on its upper triangular form and the distance from A to A' is less than r by choosing $\epsilon_1, \ldots \epsilon_s$ small enough. Thus any ball centered around a matrix in the complement of $\mathcal{M}$ must intersect $\mathcal{M}$ so $\mathcal{M}$ is dense