Exercise 1

(a) For pointwise convergence we choose an $x \in M$ and we say $f_n \to f$ if the sequence $f_n(x)$ converges to $f(x)$. For uniform convergence we say $f_n \to f$ if given $\epsilon > 0$ there is an N such that for all $n \geq N$, $x \in M$, $|f_n(x) - f(x)| < \epsilon$

Exercise 2

Given $\epsilon > 0$, since f_n converges uniformly to f, there exists N such that $d(f(m), f_N(m)) < \epsilon/3$ for all $m \in M$. For any point $x \in M$ since f_N is continuous we can choose $r > 0$ such that for all $y \in B_r(x)$, $d(f_N(x), f_N(y)) < \epsilon/3$. From the triangle inequality this yields

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \leq \epsilon$$

For any $y \in B_r(x)$. Thus f is continuous

Exercise 3

(a) Given $\epsilon > 0$, since f_n converges uniformly to f, there exists N such that $d(f(m), f_N(m)) < \epsilon/3$ for all $m \in M$. Since f_N is continuous at x_0 we can choose $r > 0$ such that for all $y \in B_r(x_0)$, $d(f_N(x_0), f_N(y)) < \epsilon/3$. From the triangle inequality this yields

$$d(f(x_0), f(y)) \leq d(f(x_0), f_N(x_0)) + d(f_N(x_0), f_N(y)) + d(f_N(y), f(y)) \leq \epsilon$$

For any $y \in B_r(x_0)$. Thus f is continuous at x

(b) Piecewise continuity is not necessarily true. Consider the function $f : [0, 1] \to \mathbb{R}$ with $f(x) = \frac{1}{n}$ for $x \in [1/(n + 1), 1/n)$ for all $n \in \mathbb{N}$ and $f(0) = 0$. f is not piecewise continuous since it is discontinous at all $1/n$. f is however the uniform limit of the piecewise continuous functions

$$f_n = \chi_{[1/n,1]} \cdot f$$

Exercise 4

(a) This follows from the same argument for continuity:
Given $\epsilon > 0$, since f_n converges uniformly to f, there exists N such that $d(f(m), f_N(m)) < \epsilon/3$ for all $m \in M$. For any point $x \in M$ since f_N is uniformly continuous we can choose $r > 0$ such that for all $x \in \mathbb{R}$ and $y \in B_r(x)$, $d(f_N(x), f_N(y)) < \epsilon/3$. From the triangle inequality this yields

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \leq \epsilon$$

For any $y \in B_r(x)$. Thus f is uniformly continuous

Exercise Additional Problem 1

(a) Consider any convergent sequence

$$f_n \in C_0(\mathbb{R}), f_n \to f$$

For any $\epsilon > 0$ there exists N such that $|f(x) - f_N(x)| < \epsilon/2$ for all $x \in \mathbb{R}$. Since $f_n \in C_0(\mathbb{R})$, there exists $R > 0$ such that $|x| > R \Rightarrow |f_N(x)| < \epsilon$. From the triangle inequality this yields

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \epsilon$$

For all $|x| > R$. Thus $f \in C_0(\mathbb{R})$

(b) Given $\epsilon > 0$ let $R > 0$ be chosen such that $|f(x)| < \epsilon/2$ for all $|x| > R$. $[-R, R]$ is compact and thus f is uniformly continuous on $[-R, R]$: $\exists \delta_0$ such that $\forall x, y \in [-R, R]$, $|x - y| \leq \delta_0 \Rightarrow |f(x) - f(y)| < \epsilon$. We have that for all $x, y \notin [-R, R]$, $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon$. The final case is if $x \in [-R, R]$, $y \notin [-R, R]$. In order for $|x - y| < \delta_0$ it must be the case that $x, y \in [-R - \delta_0, -R + \delta_0] \cup [R - \delta_0, R + \delta_0]$. This is a compact set, and thus there is a δ_1 such that $|x - y| \leq \delta_1 \Rightarrow |f(x) - f(y)| < \epsilon$. Thus by setting $\delta = \min(\delta_0, \delta_1)$, we have uniform continuity: $\forall x, y \in \mathbb{R}, |x - y| \leq \delta \Rightarrow |f(x) - f(y)| < \epsilon$

(c) We have $f(x) = \sin(x^2)$ is not uniformly continuous
The reason for this is because for any $\delta > 0$ we can choose $x, x + \delta$ to get $|f(x) - f(x + \delta)|$ arbitrarily large. By the mean value theorem there exists $x' \in [x, x + \delta]$ where

$$f(x) - f(x + \delta) = \delta f'(x') = \delta x' \sin(x'^2)$$

Which is unbounded for choice of x such that x'^2 is close to $\pi + n\pi$

Exercise Additional Problem 2

For convergence we have for $|x| < |R|$

$$\left| \frac{f^{(k)}(0)}{k!} x^k \right| \leq \left| \frac{Ck!}{R^k k!} x^k \right| = C \left| \frac{x}{R} \right|^k$$

Thus we have the series bounded by the power series

$$\sum_{k=0}^{N} \left| \frac{f^{(k)}(0)}{k!} x^k \right| \leq \sum_{k=0}^{N} C \left| \frac{x}{R} \right|^k$$

Since $\left|\frac{x}{R}\right| < 1$ the power series converges, thus the taylor series converges.
We can show that the series converges to $f(x)$ using the Taylor error bound established in Math 424 (Chapter 3 of Pugh):
Letting $R_r(x) = f(x) - P_r(x)$ (where $P_r(x)$ denotes the rth order taylor series centered at zero) we have for some $\theta \in (0, x)$

$$R_r(x) = \frac{f^{(r+1)}(\theta)}{(r+1)!} x^{r+1}$$

Thus since $|f^{(r+1)}(\theta)| \leq \frac{C(r+1)!}{R^{r+1}}$, and $|x| < R$

$$|R_r(x)| < C\left(\frac{x}{R}\right)^{r+1} \to 0$$

Thus

$$\lim_{r\to\infty} P_r(x) - f(x) = 0 \Rightarrow \lim_{r\to\infty} P_r(x) = f(x)$$