2-4, 1

For any curve

$$\alpha : (-\epsilon, \epsilon) \to S$$

with $\alpha(0) = p = (x_0, y_0, z_0)$ we have that

$$f(\alpha(s)) = 0$$

Differentiating both sides and applying the chain rule

$$df(\alpha(0)) \cdot \alpha'(0) = 0$$

Thus we have that

$$df(\alpha(0) = df_p$$

is normal to every tangent vector and thus is the normal vector of the tangent plane. Thus tangent plane has equation

$$df_p \cdot (x - p) = 0$$

2-4 3

We can define the function $g(x, y, z) = z - f(x, y)$

We have that 0 is a regular value of g since g has no critical points

$$dg = (-f_x(x, y), -f_y(x, y), 1) \neq 0$$

Notice that

$$S = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\} = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = f(x, y) - z = 0\}$$

Thus we can apply the result of 2-4 1 to conclude the tangent surface at $p = (x_0, y_0, z_0)$ is given by

$$dg_p \cdot (x - p) = 0$$

which is the same as

$$(-f_x(x_0, y_0), -f_y(x_0, y_0), 1) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$(f_x(x_0, y_0), f_y(x_0, y_0), 1) \cdot (x - x_0, y - y_0, f(x_0, y_0)) = z$$

2-4 4

Letting $g(x,y) = xf(y/x)$ from the previous problem we know the tangent plane for any $p = (x_0, y_0, z_0)$

$$(g_x(x_0,y_0), g_y(x_0,y_0), 1) \cdot (x - x_0, y - y_0, f(x_0,y_0)) = z$$

using the chain rule yields

$$g_x(x_0,y_0) = f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0)$$

$$g_y(x_0,y_0) = f'(y_0/x_0)$$

$$(f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0), f'(y_0/x_0), 1) \cdot (x - x_0, y - y_0, f(x_0,y_0)) = z$$

Plugging in $x = 0, y = 0$ yields

$$(f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0), f'(y_0/x_0), 1) \cdot (-x_0, -y_0, f(x_0,y_0)) = z$$

$$= f'(y_0/x_0)(y_0 - y_0) + f(y_0/x_0)(x_0 - x_0) = 0$$

Thus the point $(0,0,0)$ is on the tangent plane

2-4 8

By definition we know linear transormations are differentiable maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ and thus $L|_S$ is a smooth map
We know that every linear transormation can be represented as a matrix

$$L(x) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that differentiating at any p yields

$$dL = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

And thus

$$L(W) = dL_p(W)$$

For any vector W