Math 380a, Fall 2016 ***HOMEWORK #5*** ***Peter Gylys-Colwell***

1. We will prove this by contradiction, consider

$$z = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$$

for some n with z being an integer.
We will now define a value M as the number with the prime factorization of $n!$, except that the power of 2 in M's prime factorization is equal to $i-1$, where i is the value such that

$$2^i \leq n < 2^{i+1}$$

The following property holds for M:

$$k \in \{1, 2, 3, \dots n\}/\{2^i\} \Rightarrow k|M$$

The reason is that the largest possible power of 2 that divides a number $\leq n$ that is not 2^i would be $i-1$, that is because the smallest possible number that has a factor of 2^i besides 2^i would be 2^{i+1} which is larger than n. All the other prime factors for each number $\leq n$ must be present in M since M has the same prime factorization as $n!$ besides the powers of 2.
Therefore we have by multiplying by M on both sides of the original assumption:

$$Mz = M + \frac{M}{2} + \frac{M}{3} + \dots \frac{M}{2^i} + \dots \frac{M}{n}$$

However we reach a contradiction here, since the left hand side of the equality is an integer, and each term of the right hand side of the equality is an integer except for the term

$$\frac{M}{2^i}$$

Which means the right hand side cannot be an integer. Therefore the equality above cannot hold

2. We can prove this by contradiction.
Let $\frac{x}{y}$ be a root of p such that $\frac{x}{y}$ is the most reduced form, so $\gcd(x, y) = 1$.
That means

$$-a_0 = a_1\frac{x}{y} + a_2\frac{x^2}{y^2} + a_3\frac{x^3}{y^3} + \dots a_r\frac{x^r}{y^r}$$

$$-y^r a_0 = a_1 x y^{r-1} + a_2 x^2 y^{r-2} \dots a_r x^r$$

Now we work through the following cases:
If y is even then since y and x are relatively prime, $2 \not| x$.
Therefore we have

$$2| - y^r a_0$$

However we have, $2|a_i x^i y^{r-i}$ for $i \neq r$, and since $2 \not| x$ and $2 \not| a_r$, we have $2 \not| x^r a_r$. Which means

$$a_1 x y^{r-1} + a_2 x^2 y^{r-2} \ldots a_r x^r \equiv 0 + 0 + 0 \cdots + 1 \pmod 2$$

And so

$$2 \not| a_1 x y^{r-1} + a_2 x^2 y^{r-2} \ldots a_r x^r$$

Which means

$$-y^r a_0 \neq a_1 x y^{r-1} + a_2 x^2 y^{r-2} \ldots a_r x^r$$

which is a contradiction.
If y is odd then we know $-y^r a_0$ must be odd since a_0 is also odd.
However if x is even, we know

$$a_1 x y^{r-1} + a_2 x^2 y^{r-2} \ldots a_r x^r$$

Is even which would mean

$$-y^r a_0 \neq a_1 x y^{r-1} + a_2 x^2 y^{r-2} \ldots a_r x^r$$

which would be a contradiction.
And if x is odd, we have the following: Since $p(1)$ is odd, we know

$$a_0 + a_1 + a_2 + \ldots a_r \equiv 1 \mod 2$$

And since a_0, a_r are odd

$$a_1 + a_2 + \ldots a_{r-1} \equiv 1 \mod 2$$

which means an odd amount of the terms above must be odd.
Therefore since $x \equiv 1 \mod 2$ and $y \equiv 1 \mod 2$, we have

$$a_i x^i y^{r-i} \equiv a_i \mod 2$$

And so

$$x y^{r-1} a_1 + x^2 y^{r-2} a_2 + \ldots x^{r-1} y a_{r-1} \equiv 1 \mod 2$$

And we have $x^r a_r \equiv 1 \mod 2$

$$x y^{r-1} a_1 + x^2 y^{r-2} a_2 + \ldots x^{r-1} y a_{r-1} + x^r a_r \equiv 0 \mod 2$$

Which would lead to the same contradiction,

$$x y^{r-1} a_1 + x^2 y^{r-2} a_2 + \ldots x^{r-1} y a_{r-1} + x^r a_r \neq -a_0 y^r$$

Therefore there can be no rational roots for p

3. To prove this, we choose a set of n primes:

$$P = \{p_1^2, p_2^2, p_3^2, \ldots p_n^2\}$$

We know any two elements of P are relatively prime since each element is a square of distinct primes

And now consider the set

$$S = \{0, -1, -2, \dots, -n+2, -n+1\}$$

Then by the chinese remainder thm, we know there exits some A such that

$$A \equiv 0 \mod p_1^2, A+1 \equiv 0 \mod p_2^2, \quad \dots \quad A+n-1 \equiv 0 \mod p_n^2$$

Which is the desired result

4. For any prime p we know the number of times p divides the numerator and denominator of $a_{m,n}$ is given by the following:

number of times p divides the numerator:

$$\lfloor \frac{2m}{p} \rfloor + \lfloor \frac{2m}{p^2} \rfloor + \lfloor \frac{2m}{p^3} \rfloor \dots + \lfloor \frac{2m}{p^k} \rfloor + \lfloor \frac{2n}{p} \rfloor + \lfloor \frac{2n}{p^2} \rfloor + \lfloor \frac{2n}{p^3} \rfloor \dots + \lfloor \frac{2n}{p^k} \rfloor$$

number of times p divides the denominator:

$$\lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor + \lfloor \frac{m}{p^3} \rfloor \dots + \lfloor \frac{m}{p^k} \rfloor + \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor \dots + \lfloor \frac{m+n}{p} \rfloor + \lfloor \frac{m+n}{p^2} \rfloor + \lfloor \frac{m+n}{p^3} \rfloor \dots + \lfloor \frac{m+n}{p^k} \rfloor$$

Comparing term for term these counts we will conclude

$$\lfloor \frac{2m}{p^i} \rfloor + \lfloor \frac{2n}{p^i} \rfloor \geq \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{m+n}{p^i} \rfloor$$

For any $i \in \{1, 2, \dots n\}$

Which would imply that the count of times p divides the numerator is $\geq$ the count of times p divides the denominator. This would mean that the denominator divides the numerator since any prime that divides the denominator, divides the numerator at least the same number of times. And so we can conclude that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer.

To prove

$$\lfloor \frac{2m}{p^i} \rfloor + \lfloor \frac{2n}{p^i} \rfloor \geq \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{m+n}{p^i} \rfloor$$

We will go through the following cases, if

$$\lfloor \frac{2m}{p^i} \rfloor + \lfloor \frac{2n}{p^i} \rfloor = \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor$$

Then that would mean the rounded off term for $\frac{m}{p^i}$ and $\frac{n}{p^i}$ must both be $< \frac{1}{2}$, and so

$$\lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{m+n}{p^i} \rfloor = \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{m}{p^i} \rfloor + \lfloor \frac{n}{p^i} \rfloor$$

Similarly if

$$\lfloor\frac{2m}{p^i}\rfloor+\lfloor\frac{2n}{p^i}\rfloor=\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+1$$

Then the rounded off term for either $\frac{m}{p^i}$ or $\frac{n}{p^i}$ is $<\frac{1}{2}$ and the other rounded off term is $\geq\frac{1}{2}$, which would mean

$$\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m+n}{p^i}\rfloor\leq\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+1$$

And lastly if

$$\lfloor\frac{2m}{p^i}\rfloor+\lfloor\frac{2n}{p^i}\rfloor=\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+2$$

Then the rounded off term for both $\frac{m}{p^i}$ and $\frac{n}{p^i}$ are $>\frac{1}{2}$, which would mean

$$\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m+n}{p^i}\rfloor=\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+\lfloor\frac{m}{p^i}\rfloor+\lfloor\frac{n}{p^i}\rfloor+1$$

Therefore the inequality holds for each of the cases.