2-6, 1

Assume for contradiction there exists a differentiable field of unit normal vectors

$$N : S \to \mathbb{R}^3$$

Letting $x(u, v), y(s, t)$ be the parametrizations of V_1, V_2, for any $p \in V_1$ we have that with appropriate reordering of u, v

$$N(p) = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$$

on all of V_1 and similarly

$$N(p) = \frac{y_u \wedge y_v}{|y_u \wedge y_v|}$$

on all of V_2. However since the change of coordinate jacobian from x to y is different in sign on W_1 and W_2 and it is the case

$$x_u \wedge x_v = (y_u \wedge y_v)\frac{\partial x}{\partial y}$$

where $\frac{\partial x}{\partial y}$ is the jacobian of the coordinate change, we get the contradiction

$$N(p) = -N(p)$$

for either $p \in W_1$ or $p \in W_2$

2-6 2

Letting $Y = \{Y_i\}$ be a family of coordinate neighborhoods which establish S_2 to be orientable with corresponding parametrizations y_i. For each $p \in S_1$ there is a neighborhood $V_p \subset S_1$ around p such that φ is a diffeomorphism on V_p. Let φ_p be this diffeomorphism. We have that $\varphi_p(p)$ is contained in some Y_i which we will call Y_p with corresponding parametrization y_p. We have the following family of coordinate functions covering S_2

$$\{f_p = \varphi_p^{-1} \circ y_p : y_p^{-1}(Y_p \cap \varphi_p(V_p)) \to S_2\}$$

it is clear this is a covering since each $p \in S_2$ is in the image of $\varphi_p^{-1} \circ y_p$. This covering establishes S_2 to be orientable since the change of coordinate calculation from f_p to f_q

$$f_q^{-1} \circ f_p = y_q^{-1} \circ \varphi \circ \varphi^{-1} \circ y_p = y_q^{-1} \circ y_p$$

is the same as the change of coordinates from y_p to y_q which has positive Jacobian

2-6 4

Let N_α, N_β be the associated normal vector fields of two coordinate neighborhoods $\{U_\alpha\}, \{V_\beta\}$ which satisfy the conditions of Def 1. Notice that

$$F(p) = |N_\alpha(p) - N_\beta(p)| = \begin{cases} 0 & N_\alpha(p) = N_\beta(p) \\ 2 & N_\alpha(p) = -N_\beta(p) \end{cases}$$

Is a continuous function $F : S \to \mathbb{R}$. Thus since S is connected, the image of F is connected so is either entirely 0 or 2. Thus $N_\alpha = N_\beta$ or $N_\alpha = -N_\beta$ on all of S. Thus there is at most two possible orientations since there are at most two possible normal vectors fields (we know that U_α and V_β define the same orientation if and only if they define the same normal vector fields)

2-6 5

(a) Notice that ϕ and ϕ^{-1} are local diffeomorphisms. Thus from problem 2 if S_1 is orientable then S_2 is orientable and similarly if S_2 is orientable then S_1 is orientable

(b) Given a family of coordinate neighborhoods $\{U_\alpha\}$ which establish an orientation on S_1 we have that $\{\varphi(U_\alpha)\}$ is a family of coordinate neighborhood which establishes an orientation on S_2. The reason we know this family establishes an orientation is the same reasoning as in problem 2, we have that the change of basis from $\varphi \circ x_1$ to $\varphi \circ x_2$ is the same as the change of basis from x_1 to x_2 which has positive Jacobian

For the Sphere we will use the stereographic parametrizations

$$\varphi_1 : U_1 \to S - \{(0,0,1)\}, \varphi_2 : U_2 \to S - \{(0,0,-1)\}$$

we have that the normal established by this parametrization has the evaluations

$$(\varphi_{1,u} \wedge \varphi_{1,v})(0,0,0) = N(0,0,1) = (0,0,1)$$

$$(\varphi_{2,u} \wedge \varphi_{2,v})(0,0,0) = N(0,0,-1) = (0,0,-1)$$

that the differential J of the Antipodal map is $-I$ (negative the identity) we have that the normal at $(0,0,1)$ using the parametrizations $A \circ \varphi_1, A \circ \varphi_2$. We have that $(0,0,1) = A(\varphi_2(0,0,0))$ and thus our new normal is

$$N'(0,0,1) = ((A \circ \varphi_2)_u \wedge (A \circ \varphi_2)_v)(0,0,0)$$

by chain rule

$$= ((-I\varphi_{2,u}) \wedge (-I\varphi_{2,v}))(0,0,0)$$

by bilinearity

$$= (-1)^2(\varphi_{2,u} \wedge \varphi_{2,v})(0,0,0) = (0,0,-1) \neq N(0,0,1)$$

Thus we get a different normal and so a different orientation

2-6 6

The notion of orientation is that $\alpha(t), \beta(s)$ have the same orientation if the tangent vectors are the same:

When $\alpha(a) = \beta(b) = p$, $\alpha'(a) = T_\alpha(p) = T_\beta(p) = \beta'(b)$
Each parametrization α induces a continuous tangent vector field $T_\alpha : C \to \mathbb{R}^3$ where $T_\alpha(p) = \alpha'(s)$ (where $\alpha(s) = p$)
Notice that we have a continuous function

$$F(p) = |T_\alpha(p) - T_\beta(p)| : C \to \mathbb{R}$$

where

$$F(p) = \begin{cases} 0 & T_\alpha(p) = T_\beta(p) \\ 2 & T_\alpha(p) = -T_\beta(p) \end{cases}$$

Since C is connected $F(C)$ is connected so $T_\alpha = T_\beta$ or $T_\alpha = -T_\beta$ on all of C.
Thus there is at most two possible orientations since there are at most two possible tangent vector fields defined on C (we know that α and β define the same orientation if and only if they define the same tangent vector fields)