Math 424, Fall 2017 ***HOMEWORK #1*** ***Peter Gylys-Colwell***

Exersise 7

a. For any even integer n we can write it as the product $n = 2k$ for some $k \in \mathbb{Z}$. Therefore $n^2 = (2k)^2 = 4k^2$ and therefore 4 divides n^2

b. For any even integer n we can write it as the product $n = 4k$ for some $k \in \mathbb{Z}$. Therefore $n^3 = (2k)^3 = 8k^3$ and therefore 8 divides n^3

c. In the prime factorization of twice and odd cube, $2k^3$ where k odd, we know 2 does not divide k and therefore does not divide k^3 and so there is only 2^1 in the prime factorization of $2k^3$. Therefore 8 cannot divide $2k^3$ since $8 = 2^3$ does not divide the powers of 2 in the prime factorization of $2k^3$

d. Suppose for contradiction $\sqrt[3]{2} = \frac{a}{b}$ where a, b are relatively prime. Then we have $2b^3 = a^3$. Since $2b^3$ is even, a^3 is even. The only way it is possible for a^3 to be divisible by 2 is if 2 divides a. Therefore a must be even, which means $a = 2n$ for some $n \in \mathbb{Z}$ so $a^3 = 8n^3 = 2b^3$, So $b^3 = 4n^3$. Therefore b^3 is even which means b must be even

Exersise 10 Let $x = A|B$, by definition we have $-x = C|D$ where $C = \{r \in \mathbb{Q} : \text{ for some } b \in B, \text{ not the smallest element of } B, r = -b\}$ and D is the rest of $\mathbb{Q}$. By definition we have $x + (-x) = E|F$ where $E = \{r \in \mathbb{Q} : \text{ for some } a \in A \text{ and some } c \in C \text{ we have } a + c = r\}$ and F is the rest of $\mathbb{Q}$. Since $0^* = N|M = \{r \in \mathbb{Q} : r < 0\}|\{r \in \mathbb{Q} : r \geq 0\}$, we wish to show $N = E \Rightarrow x + (-x) = 0$. For any $e \in E$ we have $e = a + c$ for some $a \in A$ and $c \in C$. From how C was defined we know $c = -b$ for some $b \in B$. By definition of a cut we know $a < b$, therefore (subtracting b on both sides) we have $a - b < 0$. And so from how N was defined we have that $e = a + (-b) \in N$, and therefore $E \subseteq N$. Now take any element $n \in N$. We know that $n < 0$. Let a be an element of A chosen such that $a + |n/2|$ is not in A. We know such an a exists since if we start with any element of A and iteratively add $|n/2|$ we will get arbitrarily large, since A is bounded from above by some element of B there must be a iteration which is no longer in A, and so the previous iteration is our desired a. Therefore we have $a + |n/2| \in B$ and so (since $n < 0$) we have $x = a \in A$ and $y = a - n \in B$. We have $x + (-y) \in E$ and $x + (-y) = a - (a - n) = n$ so $n \in E$ which means $N \subseteq E$ and thus we have equality of the two sets. Thus $x + (-x) = 0^*$

Exersise 13

a. If there was no $s \in S$ such that $b - \epsilon < s$ then by definition $b - \epsilon$ would be an upper bound of S. However $b - \epsilon < b$ and thus contradicting b being a least upper bound. Therefore there must exist $s \in S$ with $b - \epsilon < s$

b. Yes, as I have proven in part a

c. To show that x is an upper bound:
For any $a \in A$ with $a = A^*|B^*$ and $a \neq x$ we have that $A^* = \{q \in \mathbb{Q} : q < a$. If there was $b \in A^*$ with $b \notin A$ then $b < a$, $b \in B$ but that contradicts every element of B being larger than every element of A, therefore $\forall b \in A^*, b \in A$ and so $A^* \subset A \Rightarrow a < x$.
To show that x is the least upper bound:
If there exists $s < x$ with s an upper bound of A. Let $s^* = C|D$, since $s < x$ we have $C \subset A$. Therefore there exists $a \in A$ with $a \notin C$. Since $a \notin C$, $a \in D$ and so $a > c$ for all $c \in C$. Letting $a^* = E|F$ we have that $E = \{q \in \mathbb{Q} : q < a\}$. Therefore $C \subseteq E$ and so $a \geq s$ since s is an upperbound of A, $a = s$, which contradicts A not containing any upperbounds (condition 3 of cuts). Therefore such an s cannot exist

Exersise 1

a.
$$\{x \in \mathbb{Q} : x^2 = 2\} = \emptyset$$

b. If $x \in \mathbb{Q}$ and $x > 0$ then $\exists\ n \in \mathbb{N}$ such that $\frac{1}{n} < x$

Exersise 2

a. Let $x = A|B$. We know by definition B is nonempty and therefore there exists $y \in B$ with $y \in \mathbb{Q}$. If it is the case that $y = x$ we know that $x + 1 \in \mathbb{Q}$ and we know how ordering works with rational numbers well enough to conclude $x + 1 > x$. Otherwise we have $y = C|D$. For any $a \in A$ we know $y > a$ since $y \in B$. We have by definition $C = \{a \in \mathbb{Q} : a < y\}$ and therefore $a \in C$ so $A \subseteq C$ and so $y > x$

b. Letting $x = A|B$ we know by definition of 0 that $x > 0 \Rightarrow C \subseteq A$ where $C = \{a \in \mathbb{Q} : a < 0\}$ and $\exists y \in A, y \notin C$. Even stronger we can say $\exists z \in A, z > 0$ since if 0 was the only element in A not in C then A would contain an upperbound since 0 would be $\geq a\ \forall a \in A$. Therefore $z > 0$. z is less than x since $z \in A$ so the set E defined by the cut $z = E|F$ is contained in A. This is because $E = \{q \in \mathbb{Q} : q < z\}$ and for any $q < z$ we have $q \in A$ since $q \in B$ would contradict elements of B being larger than elements of A. If $z = x$ then $E = A$, and yet $z \in A = E$ which contradicts how rational cuts are defined. Therefore $0 < z < x$