Exersise 7.1

(1) We have that the set $I = \{g \in k[x] : g(T)\}$ is an ideal of $k[x]$. Thus since $k[x]$ is a PID, we know it is generated by one element. This element is m_T since if there was a different generator g of I which is not a unit multiple of m_T then g has degree less than degree of m_T and $g(T) = 0$ which contradicts minimality of m_T. Thus we have that for any $f \in k[x]$, $f(T) = 0 \Leftrightarrow f \in I \Leftrightarrow m_T$ divides f

(2) We know that

$$V \cong k[x]/a_1(x) \oplus k[x]/a_2(x) \cdots \oplus k[x]/a_{n-1}(x) \oplus k[x]/m_T(x)$$

With $a_1|a_2|\ldots a_{n-1}|m_T$. Thus in order for $f \in \mathrm{Ann}_{k[x]}(V)$, it would have to be the case that $a_1|f, a_2|f, \ldots m_T|f$. Which is equivalent to $m_T|f$ since $a_1, a_2 \ldots a_{n-1}|m_T$. Thus $\mathrm{Ann}_{k[x]}(V) = (m_T)$

Exersise 7.2

(1) A is already in rational canonical form so P is just the identity matrix. We have that $\det(xI - A)$ is precisely $(x-1)(x^2-3x+2)$ which is the characteristic polinomial.
(2) We have that the characteristic polinomial splits completely as $(x-1)^2(x-2)$, so the eigenvalues are $1, 1, 2$. Thus the jordan form is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

I solved the equations $Av = v, Aw = 2w$ to get eigenvectors. We get that the eigenvectors are $[0\ -1\ 1], [1\ 0\ 0], [0\ -2\ 1]$ for eigenvalues $2, 1, 1$ respectivly. Thus we know that $S^{-1}JS = A$ where S is the matrix of eigenvectors. So $P^{-1} = S$, a straightforward inverse computation gives us P

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Exersise 7.3

(1) We have that the only possible reduced forms of the $k[x]$ modules are

$$V \cong k[x]/(x) \oplus k[x]/(x(x^2+1)^2)$$

$$V \cong k[x]/(x^2+1) \oplus k[x]/(x^2(x^2+1))$$

$$V \cong k[x]/(x(x^2+1)) \oplus k[x]/(x(x^2+1))$$

Factoring $x(x^2+1)^2 = x^5+2x^3+x$, $x^2(x^2+1) = x^4+x^2, x(x^2+1) = x^3+x$ we have the corresponding rational canonical forms

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(2), (3) A has order 4 means that A has a minimal polinomial $M_A(x)$ which divides x^4-1. x^4-1 splits as $(x^2+1)(x+1)(x-1)$ in $\mathbb{Q}$ and splits fully as $(x-1)(x+1)(x-i)(x+i)$ over $\mathbb{C}$. Thus the only possible degree 2 $\mathbb{Q}[x]$ modules are

$$\mathbb{Q}[x]/(x^2-1), \mathbb{Q}[x]/(x^2+1), \mathbb{Q}[x]/(x+1) \oplus \mathbb{Q}[x]/(x+1), \mathbb{Q}[x]/(x-1) \oplus \mathbb{Q}[x]/(x-1)$$

Which leads to the matricies

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A quick computation yields that the only elements of order 4 are

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For the complex case, on top of the matricies in $\mathbb{Q}$ we also have the possible $\mathbb{C}[x]$ modules

$$\mathbb{C}[x]/(x^2 \pm (1+i)x + i), \mathbb{C}[x]/(x^2 \pm (1-i)x - i)$$

$$\mathbb{C}[x]/(x \pm i) \oplus \mathbb{C}[x]/(x \pm i), \mathbb{C}[x]/(x \pm i) \oplus \mathbb{C}[x]/(x \pm 1)$$

This yields the matricies

$$\begin{bmatrix} 0 & -i \\ 1 & \pm(1+i) \end{bmatrix} \begin{bmatrix} 0 & i \\ -1 & \pm(1-i) \end{bmatrix} \begin{bmatrix} \pm i & 0 \\ 0 & \pm i \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm i \end{bmatrix} \begin{bmatrix} \pm i & 0 \\ 0 & \pm 1 \end{bmatrix}$$

A straightforward computation yields that every one of these matricies is of order 4 (none have order 2)

Exersise 7.4

R is right Noetherian by the following reasoning. For any chain of ideals $0 \subset I_1 \subset I_2 \subset \ldots I_n \subset \ldots R$, if the chain does not terminate then we can choose elements

$$A_1, A_2, A_3, \ldots A_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix} \ldots$$

where $A_i \in I_i, A_{i+1} \in I_{i+1}$ and $A_{i+1} \notin I_i$. We have that $A_1R + A_2R + \dots A_iR \subset I_i$. We have A_iR is of the form

$$A_iR = \left\{ \begin{bmatrix} a_in & a_ip + b_iq \\ 0 & c_iq \end{bmatrix} | n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\} = \left\{ \begin{bmatrix} a_in & p \\ 0 & q \end{bmatrix} | n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\}$$

Thus we have that

$$A_1R + A_2R + \dots A_iR = \left\{ \begin{bmatrix} \gcd(a_1, \dots a_i)n & p \\ 0 & q \end{bmatrix} | n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\}$$

Since $A_{i+1}R \not\subseteq A_1R + A_2R + \dots A_iR$, we know that $\gcd(a_1, \dots a_i) \not| a_{i+1}$ and thus we have that $\gcd(a_1, \dots a_i) < \gcd(a_1, \dots a_{i+1})$. So in a finite amount of iterations, there is an n such that $\gcd(a_1, \dots a_n) = 1$ which means

$$A_1R + A_2R + \dots A_nR = R \Rightarrow I_n = R$$

R is not left Noetherian as illustrated in this chain

$$R \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \subset R \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \subset R \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \subset \dots \subset R \begin{bmatrix} 1 & \frac{1}{2^n} \\ 0 & 0 \end{bmatrix} \subset \dots$$

Elements of each Ideal are of the form

$$\begin{bmatrix} z & z \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} z & \frac{z}{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} z & \frac{z}{4} \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 1 & \frac{z}{2^n} \\ 0 & 0 \end{bmatrix} \dots$$

For $z \in \mathbb{Z}$ and are thus each proper ideals.

Exersise 7.5

(1) Let $g_1 \dots g_n$ be a basis for R over k. We can consider the number of these generators in an ideal I which we will denote by $d(I)$. For any ascending chain

$$0 \subset I_1 \subset I_2 \dots I_k \subset \dots \subset R$$

Since $I_i \subset I_{i+1}$ we know that $d(I_i) \leq d(I_{i+1})$. Also notice that if $d(I_i) = d(I_{i+1})$ then $I_i = I_{i+1}$. This is because the set of generators in I_i and I_{i+1} must be the same since $I_i \subseteq I_{i+1}$. Thus for any $a \in I_{i+1}$ we can write it as a linear combination of those generators in I_{i+1} and thus a is in I_i as well. Thus we have a monotonic bounded sequence in the integers.

$$0 \leq d(I_1) \leq d(I_2) \leq \dots d(I_k) \dots \leq n$$

Thus it must be constant after some N. So $I_N = I_{N+1} = I_{N+2} \dots$ the chain terminates. The same reasoning shows R is artinian. For any descending chain

$$R \supseteq I_1 \supseteq I_2 \dots I_k \supseteq \dots \supset 0$$

We have a monotonic bounded sequence

$$n \geq d(I_1) \geq \dots d(I_k) \geq \dots 0$$

And thus past some N $d(I_k)$ is constant so $I_N = I_{N+1} = I_{N+2} \dots$ the chain terminates.

(2) R/I is also a PID and thus Noetherian. We know PIDs are Noetherian since if we consider an infinite chain $0 \subseteq I_1 \subseteq I_2 \subseteq \dots I_k \subseteq \dots R/I$ the union of all these ideals is an ideal (and principle) $J = \bigcup_{k\in\mathbb{N}} I_k = a(R/I)$. Thus a must be in one of the I_k and then the chain is constant past that ideal.
We can do a similar argument for Artinian. If we have a descending chain $R/I \supseteq I_1/I \supseteq I_2/I \supseteq \dots I_k/I \supseteq \dots \supseteq I$ then the intersection of all these ideals is an ideal $J = \bigcap_{k\in\mathbb{N}} I_k = (a)$. We have that $I \subseteq J$ and therefore letting $I = (b) \neq 0$ we have that $a|b$ so $a \neq 0$. Thus since PIDs are UFDs, a has a factorization

$$a = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

For each $I_i = (a_i)$ of our chain, we have that $a_i|a$ and $a_i|a_{i+1}$. Thus if we consider the number of prime factors of a present in a_i which we will denote as $d(a_i)$, we have a bounded monotonic sequence in $\mathbb{N}$

$$d(a_1) \leq d(a_2) \leq \dots d(a_k) \leq \dots d(a)$$

Thus it converges. So for some $N \in \mathbb{N}$, $d(a_N) = d(a_{N+1}) = d(a_{N+2}) \dots$ which means $a_N = a_{N+1} \dots \Rightarrow I_N = I_{N+1} = I_{N+2} \dots$

Exersise 7.6

(1) If we consider any nonzero ideal I, then there is an $A \in I$ with $\det A \neq 0$. The reason for this is because if $B \in I$ with $\det B = 0$ then we can perform row operations (which works the same as in the vector space case) to get B in diagonalized form. This diagonalized form B' is still in I since it is the product of row operation matricies with an element in I.

$$B' = \begin{bmatrix} b_1 & 0 & \dots & \dots & 0 \\ 0 & b_2 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \dots & \vdots \\ 0 & \dots & 0 & b_m & \ddots \\ 0 & \dots & 0 & 0 \ddots & \vdots \end{bmatrix}$$

we now have that $AR \subseteq I$ and $RA \subseteq I$. However A is a unit and thus $R = AR = RA = I$. The reason A is a unit is because we can perform row operations in D the same as over a vector space until we get the identity matrix since A has invertable determinant.

(2) It is clear that I_k is closed under addition since we add component-wise so the columns

that are not the kth column will stay zero.
I_k is a left ideal since for any $A \in R, B \in I_k$,

$$(AB)_{m,l} = \sum_{i=0}^{n} A_{m,i} B_{i,l}$$

Thus for every $l \neq k$ we get that $B_{i,l} = 0$ so $(AB)_{m,l} = 0$ so AB has zero columns for every column that is not the kth column. Thus $AB \in I_k$. So I_k is a left ideal.