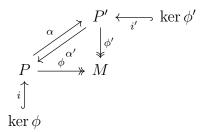
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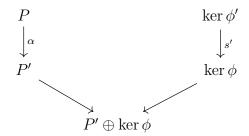
We have the following commutative diagram



We get the mappings α and α' from projectivity since $\phi: P \to M$ surjects $\exists \alpha': P' \to P$ such that $\phi = \phi' \circ \alpha'$ and $\exists \alpha: P \to P'$ such that $\phi' = \phi \circ \alpha$.

Since $\phi \circ \alpha' \circ i' = \phi' \circ i' = 0$, $\exists !s' : \ker \phi' \to \ker \phi$ with $\alpha' \circ i' = i \circ s'$

Thus we have the mappings



which induces a unique mapping $f: P \oplus \ker \phi' \to P' \oplus \ker \phi$. Swapping the labeling of the above argument would also yield a unique mapping $g: P' \oplus \ker \phi \to P \oplus \ker \phi'$. I am stuck on where to go from here

An example where the condition fails is with the \mathbb{Z} modules $P = \mathbb{Z}, P' = \mathbb{Z}/(4)$ and $M = \mathbb{Z}/(2)$. We then have the canonical mappings with kernels

$$\phi:P\to M, \phi':P'\to M$$

$$\ker \phi = 2\mathbb{Z}, \ker \phi' = 2\mathbb{Z}/(4)$$

Then

$$\ker \phi' \oplus P = 2\mathbb{Z}/(4) \oplus \mathbb{Z} \not\cong 2\mathbb{Z} \oplus \mathbb{Z}/(4) = \ker \phi \oplus P'$$

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Since P is free of infinite rank and P' is finitly generated we can surject $\phi: P \to P'$. Thus letting M = P' we have the maps $\phi: P \to M$, id: $P' \to M$ which yields

$$P \simeq P' \oplus \ker \phi$$

We have that any module that is the direct summand of a free module is projective. This is because if $F = M \oplus N$ is free the functor

$$\operatorname{Hom}(F,-) = \operatorname{Hom}(M,-) \oplus \operatorname{Hom}(N,-)$$

is exact which means the summands must be exact. We did not need the hypothesis to apply for every M, just the case when P' = M

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From Maschke's Theorem we know that any $\mathbb{C}[G]$ module is semisimple, thus for any injection

$$\phi: I \to M$$

we have that $M \simeq I \oplus \operatorname{coker} \phi$ and thus ϕ has the left inverse defined by

$$\phi^{-1} \oplus 0 : I \oplus \operatorname{coker} \phi \to I$$

Where ϕ^{-1} denotes the inverse of ϕ over im ϕ . A well known equivalent definition of an injective module is that every injection has a right inverse, thus we have shown I to be injective.

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For any embedding $N \subseteq M$ of R-modules and R-module homomorphism $\alpha: N \to I$ we can consider the collection of pairs

$$\mathcal{C} = (D, \beta)$$

Where $D \subseteq M$ are submodules containing N and $\beta: D \to I$ are mappings which restricted to N yield the equality $\beta|_N = \alpha$

We have that C forms a partially ordered set where $(D_1, \beta_1) < (D_2, \beta_2)$ iff $D_1 \subset D_2$ and $\beta_1 = \beta_2|_{D_2}$. We have that any ascending chain

$$(D_1, \beta_1) \le (D_2, \beta_2) \le (D_3, \beta_3) \le \dots$$

has the upper bound

$$\left(\bigcup_{i=1}^{\infty} D_i, \bigcup_{i=1}^{\infty} \beta_i\right)$$

Thus by Zorns lemma there is a maximal element (M', α') We have that M' = M and thus α' is a mapping $M \to I$ where $\alpha'|_{N} = \alpha$ thus establishing I to be injective. We have M' = M as follows

Suppose for contradiction there is $m \in M \setminus M'$, then we have the ideal

$$J=\{r\in R|rm\in M'\}$$

We can restrict α' to $\alpha'|_{Jm}: Jm \to I$. Thus this mapping extends to a mapping $\alpha''|_{Rm}: Rm \to I$. Thus we have a new pair $(M' \cup Rm, \alpha'')$ which is strictly larger than (M', α') , contradicting maximality.

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 (\Rightarrow) : If an abelian group G is divisible, then as a $\mathbb Z$ module, we have that G satisfies Baer's criterion and thus is injective.

For any ideal

$$J = \langle n \rangle \subset \mathbb{Z}$$

with a morphism $\phi: J \to G$ by divisiblity there exists y such that $ny = \phi(n)$ and thus we have the extension

$$\psi: \mathbb{Z} \to G$$

defined by $\psi(1) = y$

 (\Leftarrow) : Suppose G was injective. Given any element $y \in G$ and $n \in \mathbb{N}$ consider the subgroup $\langle y \rangle = Y \subseteq G$ which has canonical inclusion map $i: Y \to G$. We will define the group

$$H = \mathbb{Z}/(n \cdot |Y|)$$

(if $|Y| = \infty$ then $H = \mathbb{Z}$). We have the injective map

$$f: Y \to H$$

where f(a) = n, f(ka) = kn thus we have the diagram

$$Y \xrightarrow{f} H$$

thus from injectivity there is an induced map $h: H \to I$ with $h \circ f = i$. Thus $n \cdot h(1) = h(n) = h(f(a)) = i(a) = a$ so h(1) is a solution to ng = y so G is divisible