Exersise 3.1

The algebraic closure of $\mathbb{F}_p$ is the infinite vectorspace over $\mathbb{F}_p$

$$K = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$$

We know that the algebraic closure must contain the splitting field of $x^{p^n} - x$ and thus each $\mathbb{F}_{p^n}$ is contained in the closure. Therefore the algebraic closure necessarily contains K. Conversly every algebraic extension of $\mathbb{F}_p(\alpha)/\mathbb{F}_p$ is a finite vectorspace over $\mathbb{F}_p$ and thus letting $n = [\mathbb{F}_p(\alpha) : \mathbb{F}_p]$ it is the case that $\mathbb{F}(\alpha) \cong \mathbb{F}_{p^n}$ and thus is a subfield of K. Thus K contains all algebraic extensions of $\mathbb{F}_p$ which means K contains the algebraic closure.

Exersise 3.2

We have that $[\mathbb{F}_p(\sqrt{\alpha}) : \mathbb{F}_p] = [\mathbb{F}_p(\sqrt{\beta}) : \mathbb{F}_p] = 2$. Therefore $|\mathbb{F}_p(\sqrt{\alpha})| = |\mathbb{F}_p(\sqrt{\beta})| = p^2$. As we have established in lecture it is necessarily the case that they are the splitting field of $x^{p^2} - x$ over $\mathbb{F}_p$ and are thus isomorphic to $\mathbb{F}_{p^2}$.

Exersise 3.3

Exersise 3.4

As we have established in lecture, every finite field is of the form $\mathbb{F}_{p^n}$ which is the splitting field for the seperable polynomial $x^{p^n} - x$ over $\mathbb{F}_p$. Thus since $x^{p^n} - x$ is seperable, $\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois $|\mathrm{Aut}(\mathbb{F}_{p^n}/\mathbb{F}_p)| = n$. Since the orbit of $F \in \mathrm{Aut}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is of size n, it must be the case $\mathrm{Aut}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathrm{orb}(F)$

Exersise 3.5

Exersise 3.6