1 If there is a given bipartite graph $G = X \cup Y$ with the given degree sequence. With X, Y the vertix sets that are not connected within eachother. We know that the sum of the degrees of verticies in X must equal the sum of the degrees of verticies in Y. However this cannot be the case, since the sum of degrees of verticies that contains the degree 5 will be 2 modulo 3 since all the other degrees are 0 mod 3, and the sum from the other set will be 0 mod 3. Because the two sums are unequal mod 3, they must be unequal in general. And so no bipartite graph with the given degree sequence can exist.

2 We can induct on d:
The base case is where $d = 1$, in which case there is 1 perfect matching (as proven in class. For the inductive step, given a bipartite graph $G = (X, Y)$ with all verticies having degree $d + 1$. We have proved in class that such a graph has a perfect matching M. If we consider the graph $H = G - M$ where we remove all the edges in M from G, by the definition of a perfect matching, every vertex in G is connected to exactly 1 edge in M, therefore removing M would lower the degree of every vertex by 1. Therefore H is a bitartite graph with all verticies having degree d, which by the inductive hypothesis has d distinct perfect matchings. Since M is does not intersect with the set of edges in H, M cannot intersect with any of the perfect matchings of H. Therefore these matchings of H and M make up $d+1$ distinct perfect matchings of G

3 For any nonempty subset $S \subseteq X$, since every vertex has degree ≥ 4, we know that $|N(S)| \geq 4$ since all verticies in X has 4 neighbors. Therefore the Hall thm conditions is satisified for $|S| < 5$. For $|S| = 5$ we have $S = X$. Since all verticies in Y have degree ≥ 1 we know that $|N(X)| = |Y| > |X|$. Therefore the Hall thm is satisified for all subsets of X which means there exists a perfect matching of X to Y

4 Letting $G = (M, W)$ be the bipartite graph where M are the nodes of men, W the nodes of women, and an edge represents whether the man and woman know each other. If we make a new graph G' by adding two verticies to W and edges between these new two verticies and every node in M, then the conditions in the problem imply that for every $2 \leq k \leq N$, every k nodes in X have at least k neighbors in G' (we add two since every node in X is connected to the two new added nodes in W). Every subset of size 1 of X in G' has a neighbor set of size ≥ 1 as well since every node in X is connected to the added nodes. Therefore the conditions of Halls thm are satisified for G' and so there exists a perfect matching M from X to $W \cup \{v_1, v_2\}$ where v_1, v_2 are the added verticies. When we remove v_1, v_2 and the edges connected to them from M, we also remove two verticies in X from M, and so we are left with a matching M' contained in G with $N - 2$ verticies in X which corresponds to the desired result: a matching of $N - 2$ of the men to the women they know.

5

a. Let $d > 0$ be the smallest degree of the verticies in X. Looking at any nonempty subset $S \subseteq X$, we deduce the following. If we look at all the edges connecting S to $N(S)$, there are at least $d|S|$ edges coming from S since every vertex in S has degree at least d. The number of degrees coming from S must be equal to the number of degrees coming from $N(S)$ to S so the number of edges coming from $N(S)$ to S is at least $d|S|$. Since every vertex in Y has degree at most d, in order for the edges coming from $N(S)$ to be $\geq d|S|$, there must be at least $|S|$ verticies in $N(S)$. And so $|N(S)| \geq |S|$. Therefore the conditions for Hall's thm are satisified, so there is a perfect matching from X into Y

b. We can show that the smallest degree of the verticies in X is $\geq$ the largest degree of the verticies in Y, thus from 5a, it would be concluded that there exists a perfect matching from X into Y. For any $x \in X$ with $x = \{x_1, x_2, \ldots x_k\}$, there are $n-k$ $k+1$ sized subsets that contain x, since it is the count of adding one element $1 \leq x_{k+1} \leq n$ with $x_{k+1} \notin x$ to x to make it a $k+1$ sized subset. Therefore every vertex in X has degree $x - k > n/2$. While the number of k subsets of any element $y \in Y$ is precisely $k+1$ since it is the count of all the ways of removing one element from y. Therefore the degree of y is $k+1 \leq n/2$. And so the degree of every vertex in Y is $<$ the degree of every vertex in X, which means G satisfies the conditions in 5a.

6 We can construct a bipartite graph $G = (A, B)$ on the sets of $A_1, \ldots A_m, B_1, \ldots B_m$ with an edge between A_i and B_j being in G if and only if $A_i \cap B_j \neq \emptyset$. If we look at any nonempty subset $S \subseteq A$, if we look at the size of the union of all the elements in S:

$$\left| \bigcup_{\{A_i \in S\}} A_i \right| = |S|n$$

The reason for this is because all the sets A_i are disjoint and the size of the union of disjoint sets is equal to the sum of the size of the sets. If we look at $N(S)$ we have from the same logic:

$$\left| \bigcup_{\{B_i \in N(S)\}} B_i \right| = |N(S)|n$$

We also know that

$$\bigcup_{\{A_i \in S\}} A_i \subseteq \bigcup_{\{B_i \in N(S)\}} B_i$$

The reason for that is if for any $a \in \bigcup_{\{A_i \in S\}} A_i$, by the definition of edges in G that implies that the set B_i that contains a is in $N(S)$ so $a \in \bigcup_{\{B_i \in N(S)\}} B_i$. Therefore we have

$$\left| \bigcup_{\{A_i \in S\}} A_i \right| \leq \left| \bigcup_{\{B_i \in N(S)\}} B_i \right|$$

And so

$$|S|n \leq |N(S)|n$$

So $|S| \leq |N(S)|$, which means G satisfies the Hall thms criteria, so there exists a perfect matching from A into B. Which means we can reorder the B_i so the i matches with the A_i in this perfect matching.