**Exersise §1, 9**

For any $s \in S = A - (B \cup C)$ we have $s \in A$ and $s \notin B$, as well as $s \in A$ and $s \notin B$. Therefore $s \in R = (A - B) \cap (A - C)$ and so $S \subseteq R$. For any $r \in R$ we have $r \in A - B$ as well as $r \in A - C$ so $r$ must be in $A$. Also since $r \in A - B$, $r \notin B$ and similarly since $r \in A - C$, $r \notin C$. Therefore $R \subseteq S$, and so $R = S$

For the other law we have for any $s \in S = A - (B \cap C)$ we have $s \in A$ and $s$ is not in both $B$ and $C$. Therefore $s$ must not be in either $B$ or $C$ so $s \in A - B$ or $s \in A - C$ whcih means $s \in R = (A - B) \cup (A - C)$. Therefore $S \subseteq R$. We also have for any $r \in R$, $r$ is in $A - B$ or $A - C$ which means $r \in A$ and $r$ is not in both $B$ and $C$ which means $r \in S$. Therefore $R \subseteq S$ and so $R = S$

**Exersise §2, 1**

a. For any $a \in A_0$, by definition we have $f(a) \in f(A_0)$ and therefore

$$a \in f^{-1}(f(A_0))$$

which means $A_0 \subseteq f^{-1}(f(A_0))$. If $f$ is injective then if there exists $b \notin A_0$ with $b \in A_0 - f^{-1}(f(A_0))$ then $f(b) \in f(A_0)$ which means there exists $a \in A_0$ such that $f(b) = f(a)$ which contradicts injectivity. Therefore $A_0 - f^{-1}(f(A_0)) = \emptyset$ and so $A_0 = f^{-1}(f(A_0))$

b. For any $b \in B_0$ we have by definition $f(f^{-1}(b)) \subseteq B_0$ and so $f(f^{-1}(B_0)) \subseteq B_0$. If $f$ is surjective then for any $b \in B_0$ there is a $a \in A$ such that $f(a) = b$ and therefore $a \in f^{-1}(b)$ and so $b \in f(f^{-1}(b)) \subseteq f(f^{-1}(B_0))$ and therefore $B_0 \subseteq f(f^{-1}(B_0))$. This means that $B_0 = f(f^{-1}(B_0))$

**Exersise §2, 2**

a. Given any $b \in B_0$, since $B_0 \subseteq B_1$ we know $b \in B_1$. By the definition of $f^{-1}(B_1)$ we have that $f^{-1}(b) \subseteq f^{-1}(B_1)$ since $b \in B_1$. And since $f^{-1}(B_0)$ is a unioun of these preimages which are contained in $B_1$, we know $f^{-1}(B_0) \subseteq f^{-1}(B_1)$

b. Given any $a \in A$ with $a \in f^{-1}(B_0 \cup B_1)$ or equivalently $f(a) \in B_0 \cup B_1$ we know that $f(a)$ must be in either $B_0$ or $B_1$ and so $a$ is in either $f^{-1}(B_0)$ or $f^{-1}(B_1)$. Therefore $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$ and so $f^{-1}(B_0 \cup B_1) \subseteq f^{-1}(B_0) \cup f^{-1}(B_1)$. Conversly if $f(a)$ is in $B_0$ or in $B_1$ then $f(a) \in B_0 \cup B_1$ and so $f^{-1}(B_0) \cup f^{-1}(B_1) \subseteq f^{-1}(B_0 \cup B_1)$. Therefore we have equality

c. Given any $a \in A$ with $a \in f^{-1}(B_0 \cap B_1)$ or equivalently $f(a) \in B_0 \cap B_1$ we know that $f(a)$ must be in both $B_0$ and $B_1$ and so $a$ is in $f^{-1}(B_0)$ and $f^{-1}(B_1)$. Therefore $a \in f^{-1}(B_0) \cap f^{-1}(B_1)$ and so $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$. Conversly if $f(a)$ is in $B_0$ and in $B_1$ then $f(a) \in B_0 \cap B_1$ and so $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$. Therefore we have equality

d. Given any $a \in A$ with $a \in f^{-1}(B_0 - B_1)$ or equivalently $f(a) \in B_0 - B_1$ we know that $f(a)$ must be in $B_0$ and not $B_1$ and so $a$ is in $f^{-1}(B_0)$ and $f^{-1}(B_1)$. Therefore $a \in f^{-1}(B_0) - f^{-1}(B_1)$ and so $f^{-1}(B_0 - B_1) \subseteq f^{-1}(B_0) - f^{-1}(B_1)$. Conversly if $f(a)$ is in $B_0$ and not in $B_1$ then $f(a) \in B_0 - B_1$ and so $f^{-1}(B_0) - f^{-1}(B_1) \subseteq f^{-1}(B_0 - B_1)$. Therefore we have equality

e. Given any $b \in f(A_0)$ we know there exists some $a \in A$ with $f(a) = b$, since $a \in A_0 \subseteq A_1$ we have that $a \in A_1$ and so $f(a) \in f(A_1)$. Therefore $f(A_0) \subseteq f(A_1)$

f. Given any $b \in f(A_0 \cup A_1)$ we know there exists $a \in A_0 \cup A_1$ with $f(a) = b$ and so $a$ is either in $A_0$ or $A_1$ so $f(a) \in f(A_0) \cup f(A_1)$ and so $f(A_0 \cup A_1) \subseteq f(A_1) \cup f(A_0)$. Conversly for any $f(a) \in f(A_0) \cup f(A_1)$ we know $f(a)$ is in either $f(A_0)$ or $f(A_1)$ and so $a \in A_0$ or $a \in A_1$ therefore $a \in A_0 \cup A_1$ and therefore $f(a) \in f(A_0 \cup A_1)$. Therefore we have equality

g. Given any $b \in f(A_0 \cap A_1)$ we know there exists $a \in A_0 \cap A_1$ with $f(a) = b$ and therefore since $a$ is in $A_0$ and $A_1$, $f(a) \in f(A_0)$ and $f(a) \in f(A_1)$ so $f(a) f(A_0) \cap f(A_1)$. Therefore $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$. If $f$ is injective, for any $b \in f(A_0) \cap f(A_1)$ we know $b$ is in both $f(A_0)$ and in $f(A_1)$. Therefore there exists elements $a_0 \in A_0, a_1 \in A_1$ such that $f(a_0) = b \in f(A_0)$ and $f(a_1) = b \in f(A_1)$. Since $f$ is injective however $a_0 = a_1$ and we know $a = a_0 = a_1$ is in both $A_0$ and $A_1$. Therefore $f(a) \in f(A_0 \cap A_1)$, and thus $f(A_0) \cap f(A_1) \subseteq f(A_0 \cap A_1)$ and we have equality of the sets

h. For any $b \in f(A_0) - f(A_1)$ we know $b$ is in $f(A_0)$ and not $f(A_1)$. Since $b \in f(A_0)$ we know there exists $a \in A_0$ such that $f(a) = b$. Since $f(a) \notin f(A_1)$, we know $a \notin A_1$ and so $a \in A_0 - A_1$, thus $f(a) \in f(A_0 - A_1)$ and thus $f(A_0) - f(A_1) \subseteq f(A_0 - A_1)$. If $f$ is injective, given any $b \in f(A_0 - A_1)$ we know there exists $a \in A_0 - A_1$ with $f(a) = b$ and therefore since $a$ is in $A_0$ and not $A_1$, $f(a) \in f(A_0)$. We also have $f(a) \notin f(A_1)$ since if there exists $a_1 \in A_1$ with $f(a_1) = f(a)$ then $a = a_1$ since $f$ is injective, but this is not possible since $a \notin A_1$ while $a_1 \in A_1$. Thus $f(a) \in f(A_0) - f(A_1)$. Therefore $f(A_0 - A_1) \subseteq f(A_0) - f(A_1)$. Thus we have equality of the sets

**Exersise §2, 4**

a. For any $a \in (g \circ f)^{-1}(C_0)$ there must exist a $c \in C_0$ such that $g \circ f(a) = c$. Therefore $f(a) \in g^{-1}(c)$ and thus $a \in f^{-1}(g^{-1}(c))$. Therefore $(g \circ f)^{-1}(C_0) \subseteq f^{-1}(g^{-1}(C_0))$. Conversly if $a \in f^{-1}(g^{-1}(C_0))$ then there exists a $b \in g^{-1}(C_0)$ such that $f(a) = b$, and then there must exist a $c \in C_0$ such that $g(b) = c$. Therefore $g(f(a)) = c$. Therefore

$a \in f^{-1}(g^{-1}(c))$. Thus we have $f^{-1}(g^{-1}(C_0)) \subseteq (g \circ f)^{-1}(C_0)$. Thus we have equality of the sets

b. For any $a, \bar{a} \in A$ with $a \neq \bar{a}$ then $f(a) \neq f(\bar{a})$ since $f$ is injective. Since $f(a) \neq f(\bar{a})$ we have $g(f(a)) \neq g(f(\bar{a}))$ since $g$ is injective. Therefore $g \circ f(a) \neq g \circ f(\bar{a})$ and so $g \circ f$ is injective

c. If $g \circ f$ is injective then $f$ must be injective and $g$ must be injective on $f(A)$. If there exists $a, \bar{a} \in A$ with $a \neq \bar{a}$ and $f(a) = f(\bar{a})$ then we would have $g(f(a)) = g(f(\bar{a}))$ which would contradict injectivity of $g \circ f$. If there exists $b, \bar{b} \in f(A)$ with $b \neq \bar{b}$ and $g(b) = g(\bar{b})$ then since $b, \bar{b} \in f(A)$ there exists $a, \bar{a} \in A$ with $f(a) = b, f(\bar{a}) = \bar{b}$ and thus $g(f(a)) = g(f(\bar{a}))$ which contradicts injectivity of $g \circ f$

d. If $f, g$ are surjective then $f(A) = B$, $g(B) = C$. Therefore we have $g(f(A)) = g(B) = C$ and thus $g \circ f$ is surjective.

e. $g$ must be surjective on $f(A)$. This is because $g(f(A)) = C$. Not much can be said about surjectivity of $f$ for instance we could have $A = \mathbb{Z}, B = \mathbb{Z}, C = \{1\}$ and have $f = 1$, $g(x) = x$.

f. **Theorem:** $g \circ f$ is bijective if and only if $g$ is bijective on $f(A)$ and $f$ is injective

**Exersise §2, 5**

a. For any $a, \bar{a} \in A$ with $a \neq \bar{a}$, if $f$ has a left inverse $g$ and $f(a) = f(\bar{a})$ then we would have $g(f(a)) = g(f(\bar{a}))$ but $g(f(a)) = a \neq \bar{a} = g(f(\bar{a}))$ which is not possible, therefore $f(a) \neq f(\bar{a})$ and so $f$ is injective.
For any $b \in B$, if $f$ has a right inverse $h$ then $f(h(b)) = b$ which means $b \in f(A)$ therefore $B \subseteq f(A)$ and since $f(A) \subseteq B$ we have $f(A) = B$, so $f$ is surjective.

b. let $A = \{0, 1\}$ and $B = \mathbb{Z}$, and $f(x) = x$. We have the left inverse

$$g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 0 \end{cases}$$

but $f$ is not surjective so cannot have a right inverse

c. let $A = \mathbb{Z}$ and $B = \{1\}$, and $f(x) = 1$. We have the right inverse $h(x) = 1$, but $f$ is not injective so cannot have a left inverse

d. There can be multiple left inverses and right inverses. From my answer to problem $b$ there is another possible left inverse:

$$g(x) = \begin{cases} 0 & x \in 2\mathbb{Z} \\ 1 & x \notin 2\mathbb{Z} \end{cases}$$

This $g$ has the right inverse $f$ from problem $b$ as well as the right inverse

$$f(x) = x + 2$$

e. We already know that if $f$ has a left inverse then it is injective, and if it has a right then it is surjective. Therefore $f$ is bijective. To show $g = h$, for any $b \in B$, since $f$ is bijective we know there exists one and only one $a \in A$ with $f(a) = b$. Therefore $a = f^{-1}(b)$. By definition we have $g(f(a)) = g(b) = a = f^{-1}(b)$. Similarly $f(h(b)) = b$ and so $h(b) \in f^{-1}(b) = \{a\}$ so $h(b) = a$. Therefore $g = h = f^1$

**Exersise §3, 4**

a. Checking properties:
Reflexivity: $f(x) = f(x)$ so $x \sim x$
Symmetry: $f(x) = f(y)$ then $f(y) = f(x)$ so $x \sim y$ means $y \sim x$
Transitivity: $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$, so $x \sim y$ and $y \sim z$ then $x \sim z$

b. We can define a mapping $f^* : A^* \to B$ with $f^*(r) = f(a)$ for any $a \in r$. This mapping is well defined since $f^*(r) = f(a) = f(b)$ for any $a, b \in r$. This mapping is surjective since for any $b \in B$, since $f$ is surjective there is a $a \in A$ with $f(a) = b$, therefore letting $a^*$ be the equivalence class of $a$ we have $f^*(a^*) = b$. If $f^*(a^*) = f^*(b^*)$ for some $a^*, b^* \in A^*$ then for any $a \in a^*$ and $b \in b^*$ we have $f(a) = f(b)$ and therefore $a$ and $b$ must be in the same equivalently class: $a^* = b^*$ so $f^*$ is injective. Therefore $f^*$ is a bijection