Exersise 7.1

(1) We have that the set $I = \{g \in k[x] : g(T)\}$ is an ideal of $k[x]$. Thus since $k[x]$ is a PID, we know it is generated by one element. This element is m_T since if there was a different generator g of I which is not a unit multiple of m_T then g has degree less than degree of m_T and $g(T) = 0$ which contradicts minimality of m_T. Thus we have that for any $f \in k[x]$, $f(T) = 0 \Leftrightarrow f \in I \Leftrightarrow m_T$ divides f

(2) We know that

$$V \cong k[x]/a_1(x) \oplus k[x]/a_2(x) \cdots \oplus k[x]/a_{n-1}(x) \oplus k[x]/m_T(x)$$

With $a_1|a_2| \ldots a_{n-1}|m_T$. Thus in order for $f \in \mathrm{Ann}_{k[x]}(V)$, it would have to be the case that $a_1|f, a_2|f, \ldots m_T|f$. Which is equivalent to $m_T|f$ since $a_1, a_2 \ldots a_{n-1}|m_T$. Thus $\mathrm{Ann}_{k[x]}(V) = (m_T)$

Exersise 7.2

(1) A is already in rational canonical form so P is just the identity matrix. We have that $\det(xI - A)$ is precisely $(x-1)(x^2 - 3x + 2)$ which is the characteristic polinomial.
(2) We have that the characteristic polinomial splits completely as $(x-1)^2(x-2)$, so the eigenvalues are $1, 1, 2$. Thus the jordan form is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

I solved the equations $Av = v, Aw = 2w$ to get eigenvectors. We get that the eigenvectors are $[0\ -1\ 1], [1\ 0\ 0], [0\ -2\ 1]$ for eigenvalues $2, 1, 1$ respectivly. Thus we know that $S^{-1}JS = A$ where S is the matrix of eigenvectors. So $P^{-1} = S$, a straightforward inverse computation gives us P

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Exersise 7.3

(1) We have that the only possible reduced forms of the $k[x]$ modules are

$$V \cong k[x]/(x) \oplus k[x]/(x(x^2+1)^2)$$

$$V \cong k[x]/(x^2+1) \oplus k[x]/(x^2(x^2+1))$$

$$V \cong k[x]/(x(x^2+1)) \oplus k[x]/(x(x^2+1))$$

Factoring $x(x^2+1)^2 = x^5+2x^3+x$, $x^2(x^2+1) = x^4+x^2, x(x^2+1) = x^3+x$ we have the corresponding rational canonical forms

$$\begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&-1\\0&0&1&0&0&0\\0&0&0&1&0&-2\\0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&-1\\0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}0&0&0&0&0&0\\1&0&-1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix}$$

(2), (3) A has order 4 means that A has a minimal polinomial $M_A(x)$ which divides x^4-1. x^4-1 splits as $(x^2+1)(x+1)(x-1)$ in $\mathbb{Q}$ and splits fully as $(x-1)(x+1)(x-i)(x+i)$ over $\mathbb{C}$. Thus the only possible degree 2 $\mathbb{Q}[x]$ modules are

$$\mathbb{Q}[x]/(x^2-1), \mathbb{Q}[x]/(x^2+1), \mathbb{Q}[x]/(x+1)\oplus\mathbb{Q}[x]/(x+1), \mathbb{Q}[x]/(x-1)\oplus\mathbb{Q}[x]/(x-1)$$

Which leads to the matricies

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

A quick computation yields that the only elements of order 4 are

$$\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

For the complex case, on top of the matricies in $\mathbb{Q}$ we also have the possible $\mathbb{C}[x]$ modules

$$\mathbb{C}[x]/(x^2\pm(1+i)x+i), \mathbb{C}[x]/(x^2\pm(1-i)x-i)$$

$$\mathbb{C}[x]/(x\pm i)\oplus\mathbb{C}[x]/(x\pm i), \mathbb{C}[x]/(x\pm i)\oplus\mathbb{C}[x]/(x\pm 1)$$

This yields the matricies

$$\begin{bmatrix}0&-i\\1&\pm(1+i)\end{bmatrix}\begin{bmatrix}0&i\\-1&\pm(1-i)\end{bmatrix}\begin{bmatrix}\pm i&0\\0&\pm i\end{bmatrix}\begin{bmatrix}\pm 1&0\\0&\pm i\end{bmatrix}\begin{bmatrix}\pm i&0\\0&\pm 1\end{bmatrix}$$

A straightforward computation yields that every one of these matricies is of order 4 (none have order 2)

Exersise 7.4

R is right Noetherian by the following reasoning. For any chain of ideals $0\subset I_1\subset I_2\subset \dots I_n\subset\dots R$, if the chain does not terminate then we can choose elements

$$A_1, A_2, A_3, \dots A_i = \begin{bmatrix}a_i&b_i\\0&c_i\end{bmatrix}\dots$$

where $A_i \in I_i, A_{i+1} \in I_{i+1}$ and $A_{i+1} \notin I_i$. We have that $A_1R + A_2R + \dots A_iR \subset I_i$. We have A_iR is of the form

$$A_iR = \left\{ \begin{bmatrix} a_in & a_ip + b_iq \\ 0 & c_iq \end{bmatrix} |n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\} = \left\{ \begin{bmatrix} a_in & p \\ 0 & q \end{bmatrix} |n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\}$$

Thus we have that

$$A_1R + A_2R + \dots A_iR = \left\{ \begin{bmatrix} \gcd(a_1, \dots a_i)n & p \\ 0 & q \end{bmatrix} |n \in \mathbb{Z}, p, q \in \mathbb{Q} \right\}$$

Since $A_{i+1}R \not\subseteq A_1R + A_2R + \dots A_iR$, we know that $\gcd(a_1, \dots a_i) \not| a_{i+1}$ and thus we have that $\gcd(a_1, \dots a_i) < \gcd(a_1, \dots a_{i+1})$. So in a finite amount of iterations, there is an n such that $\gcd(a_1, \dots a_n) = 1$ which means

$$A_1R + A_2R + \dots A_nR = R \Rightarrow I_n = R$$

R is not left Noetherian as illustrated in this chain

$$R \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \subset R \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \subset R \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \subset \dots \subset R \begin{bmatrix} 1 & \frac{1}{2^n} \\ 0 & 0 \end{bmatrix} \subset \dots$$

Elements of each Ideal are of the form

$$\begin{bmatrix} z & z \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} z & \frac{z}{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} z & \frac{z}{4} \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 1 & \frac{z}{2^n} \\ 0 & 0 \end{bmatrix} \dots$$

For $z \in \mathbb{Z}$ and are thus each proper ideals.

Exersise 7.5

(1) Let $g_1 \dots g_n$ be a basis for R over k. We can consider the number of these generators in an ideal I which we will denote by $d(I)$. For any ascending chain

$$0 \subset I_1 \subset I_2 \dots I_k \subset \dots \subset R$$

Since $I_i \subset I_{i+1}$ we know that $d(I_i) \leq d(I_{i+1})$. Also notice that if $d(I_i) = d(I_{i+1})$ then $I_i = I_{i+1}$. This is because the set of generators in I_i and I_{i+1} must be the same since $I_i \subseteq I_{i+1}$. Thus for any $a \in I_{i+1}$ we can write it as a linear combination of those generators in I_{i+1} and thus a is in I_i as well. Thus we have a monotonic bounded sequence in the integers.

$$0 \leq d(I_1) \leq d(I_2) \leq \dots d(I_k) \dots \leq n$$

Thus it must be constant after some N. So $I_N = I_{N+1} = I_{N+2} \dots$ the chain terminates. The same reasoning shows R is artinian. For any descending chain

$$R \supseteq I_1 \supseteq I_2 \dots I_k \supseteq \dots \supset 0$$

We have a monotonic bounded sequence

$$n \geq d(I_1) \geq \dots d(I_k) \geq \dots 0$$

And thus past some N $d(I_k)$ is constant so $I_N = I_{N+1} = I_{N+2} \dots$ the chain terminates.

(2) R/I is also a PID and thus Noetherian. We know PIDs are Noetherian since if we consider an infinite chain $0 \subseteq I_1 \subseteq I_2 \subseteq \dots I_k \subseteq \dots R/I$ the union of all these ideals is an ideal (and principle) $J = \bigcup_{k \in \mathbb{N}} I_k = a(R/I)$. Thus a must be in one of the I_k and then the chain is constant past that ideal.
We can do a similar argument for Artinian. If we have a descending chain $R/I \supseteq I_1/I \supseteq I_2/I \supseteq \dots I_k/I \supseteq \dots \supseteq I$ then the intersection of all these ideals is an ideal $J = \bigcap_{k \in \mathbb{N}} I_k = (a)$. We have that $I \subseteq J$ and therefore letting $I = (b) \neq 0$ we have that $a|b$ so $a \neq 0$. Thus since PIDs are UFDs, a has a factorization

$$a = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

For each $I_i = (a_i)$ of our chain, we have that $a_i | a$ and $a_i | a_{i+1}$. Thus if we consider the number of prime factors of a present in a_i which we will denote as $d(a_i)$, we have a bounded monotonic sequence in $\mathbb{N}$

$$d(a_1) \leq d(a_2) \leq \dots d(a_k) \leq \dots d(a)$$

Thus it converges. So for some $N \in \mathbb{N}$, $d(a_N) = d(a_{N+1}) = d(a_{N+2}) \dots$ which means $a_N = a_{N+1} \dots \Rightarrow I_N = I_{N+1} = I_{N+2} \dots$

Exersise 7.6

(1) If we consider any nonzero ideal I, then there is a unit $G \in I$. The reason for this is because if $B \in I$ with $B \neq 0$ then we can perform row operations (which works the same as in the vector space case) to multiply every column and row except for the row and column of some nonzero entry of B by 0. This new matrix B' is still in I since it is the product of row operation matricies with an element in I and it has only one entry b that is nonzero. Thus by performing column and row swaps we get matricies $B_1, B_2, B_3 \dots B_n \in I$ where B_i has the (i,i) entry be the nonzero entry b while every other entry is 0. Thus $G = \sum_{i=1}^{n} B_i$ is the identity matrix multiplied by b. Since D is a division ring G is a unit since b^{-1} multiplied by the identity matrix is the inverse of G. Thus $I = R$

(2) It is clear that I_k is closed under addition since we add component-wise so the columns that are not the kth column will stay zero.
I_k is a left ideal since for any $A \in R, B \in I_k$,

$$(AB)_{m,l} = \sum_{i=0}^{n} A_{m,i} B_{i,l}$$

Thus for every $l \neq k$ we get that $B_{i,l} = 0$ so $(AB)_{m,l} = 0$ so AB has zero columns for every column that is not the kth column. Thus $AB \in I_k$. So I_k is a left ideal.