16.1 This is a special case of thm 16.1 ii:
We have
$$(-1)a + a = (-1 + 1)a = 0 \cdot a = 0$$
and so subtracting a on both sides yields
$$(-1)a = -a$$

16.7 Since F is a field we know there is $a^{-1} \in F$ such that $aa^{-1} = 1$. Therefore if we let $x = a^{-1}(-b)$ we satisfy the equation:
$$a(a^{-1}(-b)) + b = (aa^{-1})(-b) + b = -b + b = 0$$
We get that first equality since $\cdot$ is associative

16.11

a. The only unit is $(1, 1)$ since for any $a, b \in \mathbb{Z}$, $ab = 1 \Leftrightarrow a = 1, b = 1$. The only zero-divisor is $(0, 0)$ since for any $a, b \in \mathbb{Z}$, $ab = 0 \Leftrightarrow a = 0$ and/or $b = 0$. Since the set of nilpotents elements is a subset of zero-divisors, it follows that the only nilpotent is also $(0, 0)$.

b. From previous knowledge of groups we know every element in $\mathbb{Z}_3$ has an inverse under the group operation of multiplication modulo 3, therefore we know for any $(a, b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ there is a $(a^{-1}, b^{-1}) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ such that $(a, b)(a^{-1}, b^{-1}) = (1, 1)$ and so every element in $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is a unit. Since 3 is prime there is no two numbers that can multiply together to be a multiple of 3 unless one of the two numbers is already a multiple of 3, only $(0, 0)$ is a zero-divisor and from that it follows (since the set of nilpotents is a subset of zero-divisors) that $(0, 0)$ is the only nilpotent

c. The units are $(1, 1), (1, 5), (3, 1), (3, 5)$ with respective inverses $(1, 1), (1, 5), (3, 1), (3, 5)$. The zero-divisors are all the rest of the elements: $(0, 2), (0, 3), (0, 4), (2, 2), (2, 3), (2, 4)$. The nilpotents are $(0, 0), (2, 0)$.

16.13

a. If there were two multiplicative identities: $1 \neq 1'$ we would have by definition of the multiplicative identity
$$1 = 1 \cdot 1' = 1'$$
and so $1 = 1'$

b. If there were two multiplicative inverses, let β and α be multiplicative inverses of a. We have

$$\beta = \beta(a\alpha) = (\beta a)\alpha = \alpha$$

And so $\beta = \alpha$

A From the definition we know that the center is abelian and from the definition of a division ring we know every element is a unit. Now all we need to show is that the center is closed under multiplication and addition. Given any $a, b \in$ the center of R we have for any $x \in R$

$$(a+b)x = ax + bx = xa + xb = x(a+b)$$

and so $a + b$ is in the center. We also have

$$(ab)x = axb = x(ab)$$

and so ab is in the center. Therefore the center is a field.

B $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain. Consider any $a, b \in \mathbb{Z}/\{0\}$

$$(a, 0) \cdot (0, b) = (0, 0)$$

and so $(a, 0)$ and $(0, b)$ are non-zero zero-divisors.

C $\mathbb{Z}_{10}$ is not an integral domain. Consider

$$2 \cdot 5 = 0$$

and so 2 and 5 are non-zero zero-divisors. Observing that S is the set of all even integers in R we know that S is closed under addition and multiplication since multiplying or adding to even numbers yields an even number. Addition is still commutative in S. Therefore S is a subring of R.
S is an integral domain since for any $s \in S$ in order for $s \cdot a = 0$, 10 must divide sa and so $2 \cdot 5$ must divide sa. However since s is even if it also has a factor of 5 then it is a multiple of 10 since it has a factor of both 5 and 2. If $s \neq 0 \mod 10$ then, a must have a factor of 5 and so if $a \in S$ then a is 0 since a would have a factor of 5 and a factor of 2. Therefore there is no non-zero term $a \in S$ such that $sa = 0$.
S is a field since S is commutative (since R is commutative) and each term is a unit with 6 the multiplicative identity: $2 \cdot 8 = 6$, $4 \cdot 4 = 6$, $6 \cdot 6 = 6$, $6 \cdot 2 = 2$, $6 \cdot 4 = 4$, and $6 \cdot 8 = 8$.

D We in order for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be in the center we have for any $w, x, y, z \in \mathbb{R}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} wa+xc & wb+xd \\ ya+zc & yb+zd \end{bmatrix}$$

Equating the top left and bottom right corners gives us $by = cx$. The only way for those quantities be equal for any x, y is if $b = c = 0$. From there, equating the top right and bottom left corners gives us $ax = xd$ and $dy = ya$. Dividing by x for the first equation or y for the second equation yields $a = d$. Therefore the center consisits of all matricies of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

With $a \in \mathbb{R}$

E We already know the addition operation is commutative and the operations have the distributive property, so we just have to show S is closed under addition and multiplication. For any

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \in S$$

$$A + B = \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix} \in S$$

$$AB = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & -bd+ac \end{bmatrix} \in S$$

Therefore S is a subring of $M_2(\mathbb{R})$. To show $S \cong \mathbb{C}$ we define the bijection $\varphi :$

$$\varphi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + bi$$

To show that its a homomorphism we have using the multiplaction equalities above

$$\varphi(A)\varphi(B) = (a+ib)(c+id) = ac - bd + (ad+bc)i = \varphi(AB)$$

As well as

$$\varphi(A) + \varphi(B) = a + ib + c + id = (a+c) + (b+d)i = \varphi(A+B)$$

And finally we know that φ is bijective since for any $a + ib \in \mathbb{C}$ the matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ will map to that number, and if any two matricies map to the same number then the two numbers that determine their entries must be the same so they must be the same matrix