Exercise 31

(a) We can write U as a disjoint union of intervals through the following iterative method. For any point $x \in U$ we define $U_x = (a_x, b_x)$ where $a_x = \inf\{(a \in \mathbb{R} : (a, x) \subseteq U\}$, $b_x = \sup\{(b \in \mathbb{R} : (x, b) \subseteq U\}$. Now we can construct our union of intervals. Letting $I = U \cap \mathbb{Q}$ we have that

$$U = \bigcup_{q \in I} U_q$$

This is a countable union of intervals since $I \subset \mathbb{Q}$.
To prove this equality we have that $\bigcup_{q\in I} U_q \subseteq U$ since each $U_q \subseteq U$. We know $U_q \subseteq U$ since for any $p \in U_x$ we have that $|p - a|, |p - b| < \epsilon$ for some small $\epsilon > 0$ so $(x, p + \epsilon) \subseteq U$ or $(p - \epsilon, x) \subseteq U$ thus $p \in U$.
We know that $U \subseteq \bigcup_{q\in I} U_q$ since for any $p \in U$ we have either $p \in \mathbb{Q}$ in which case $p \in U_p$ or $p \in \mathbb{R} - \mathbb{Q}$ in which case since U is open there exists $B_r(p) \subseteq U$. Since $\mathbb{Q}$ is dense we know $B_r(p) \cap \mathbb{Q} \neq \emptyset$ so $\exists q \in B_r(p)$ and we have that $B_r(p) \subset U_q$ thus $p \in U_q$.
We can make this union of intervals disjoint using the axiom of choice. We have that if $q \in U_p$, then $U_p = U_q$ since $(a, q) \subset U \Leftrightarrow (a, p) \subset U$ so $a_p = a_q$ and vice versa $b_p = b_q$. Thus we can iterativly choose $q \in I$ then remove $p \in I$ such that $p \in U_q$. Once we have done this for every element in I we have a disjoint union since if $U_q \cap U_p \neq \emptyset$ then $U_q = U_p$ because then there exists $x \in U_q \cap U_p, x \in \mathbb{Q}$ so $U_x = U_p, U_x = U_q$ which means $q = p$.

(b) We have uniqueness since if

$$U = \bigcup U_i = \bigcup V_j$$

yet $\exists U_i = (a, b) \notin \{V_j\}$ then since $U_i \subset \bigcup V_j$ there exists $V = (c, d) \in \{V_j\}$ and $x \in U$ such that $x \in V, x \in U_i$. However we will run into a contradiction: since $V \neq U_i$ we know $a \neq c$ or $b \neq d$, WLOG we assume $a \neq c$ and WLOG we say $a < c$ then we have that $c \in (a, x)$ so $c \in U$ however there is not $V_j \in \{V_j\}$ where $c \in V_j$ which would mean $c \notin U$ which is a contradiction. We know $c \notin V_j \forall j$ since if $c \in V_j$ then is V_j is open so there exist $B_r(c) \subset V_j$. Thus we have $V_j \cap V \neq \emptyset$ since $B_r(c) \cap V = (c, \inf(c + r, d)) \neq \emptyset$ which contradicts the V_js being disjoint.

Exercise 60

(a) If f is not constant, then we have $x, y \in f(M)$, $x < y$. Thus we have that $M = f^{-1}(-\infty, y - 1/2) \cup f^{-1}(y - 1/2, \infty)$. This is a contradiction on M being connected since $f^{-1}(-\infty, y-1/2)$ and $f^{-1}(y-1/2, \infty)$ are clopen and nonempty since both are the preimage of an open set of a continous function and are the complements of each other (so closed).

(b) Again f must be constant. If f is not constant, then we have $x, y \in f(M)$, $x < y$. Since the rational numbers are dense there exists $q \in (x, y) \cap \mathbb{Q}$. Thus we have $M =$

$f^{-1}(-\infty, q) \cup f^{-1}(q, \infty)$. This is a contradiction on M being connected since $f^{-1}(-\infty, q)$ and $f^{-1}(q, \infty)$ are clopen and nonempty since both are the preimage of an open set of a continous function and are the complements of each other (so closed).

Exercise 66

(a) If U is our connected open set in $\mathbb{R}^m$, consider a point $x \in U$ (if U is empty we are trivially done). Consider the set S defined as the set of points $p \in U$ that there exists a path from x to p. We have that S is open since for any $p \in S$ there exists $B_r(p) \subset U$. We have that $B_r(p) \subset S$ as well. The reason for this is as follows. Since we have proven $B_r(p)$ is path connected in lecture we have continous functions $f : [0,1] \to \mathbb{R}^m$ and $g : [0,1] \to \mathbb{R}^m$ with $f(0) = x, f(1) = p$ and $g(0) = p, g(1) = q$ for every $q \in B_r(p)$. Thus we have the continous function $h : [0,1] \to \mathbb{R}^m$ with $h(t) = tg(t) + (1-t)f(t)$. Thus $h(0) = x$ and $h(1) = q$.
We have that S^c is open as well. Thus S is clopen and nonempty so since U is connected, $U = S$ so U is path connected. S^c is open since for any $p \in S^c$ there exists $B_r(p) \subseteq U$ and we have that $B_r(p) \subset S^c$ since if there is any $q \in B_r(p)$ with $q \notin S^c$ then $q \in S$. However we then could construct a path from x to p using the paths from x to q and from q to p the same as before. This contradicts $p \in S^c$ so we must have $B_r(p) \subset S^c$

Exercise 71

(a) If $M \times N = U \sqcup V$ where U, V are clopen. then one of the sets is nonempty. WLOG we have $(x, y) \in U$. If we consider the subspace $M' = M \times \{y\}$. We know that M' is connected since it is homeomorphic to M. We have that $M' = (M' \cap U) \sqcup (M' \cap V)$ where both sets are clopen. Thus since $M' \cap U$ is nonempty, we know $M' \cap U = M'$. We have that $M \times N = U$ and thus $M \times N$ is connected. The reason $M \times N = U$ is as follows. If there existed $p = (w, v) \in V$ then we have that $N' = \{w\} \times N$ is connected since it is homeomorphic to N. We have that $N' = (N' \cap U) \sqcup (N' \cap V)$ and thus since N' is connected and $N' \cap V$ is nonempty we know $N' = N' \cap V$. This is not possible however since $(w, y) \in M' \cap N'$ so $(w, y) \in M' \subset U$ and $(w, y) \in N' \subset V$ which contradicts $U \cap V = \emptyset$

(b) The converse is also true. If we have $M = U \sqcup V$ where U, V clopen in M then we have $M \times N = (U \times N) \sqcup (V \times N)$ is a clopen disjoint union. Thus either U or V must be empty. Switching the labels yields the same result for N.

(c) (a) For any $p = (a, b), q = (x, y) \in M \times N$ since M, N are path connected there exists continous $f : [0,1] \to M, g : [0,1] \to N$ with $f(0) = a, f(1) = x, g(0) = x, g(1) = y$. We can define the continous function $h : [0,1] \to M \times N$ where $h(s) = (f(s), g(s))$. We know h is continous since it is continous in its components. We have that $h(0) = p$ and $h(1) = q$ thus $M \times N$ is path connected.
(b) This is true since we know the projection map $\pi : M \times N \to M$ where $\pi(m, n) = m$ is continous. Thus for any $x, y \in M$ we choose any $n \in N$ so we have the points $(x, n), (y, n) \in M \times N$. There exists a path $f : [0,1] \to M \times N$ with $f(0) = (x, n)$, $f(1) = (y, n)$. We have that $\pi \circ f$ is our path from x to y and thus M is path connected. Relabeling our argument would yield N is path connected as well.

Exercise 124

(a) We have that $\delta S = \bar{S} - \text{int } S$. Thus we have $S \subseteq \bar{S}$ so $S - \delta S = S - (\bar{S} - \text{int } S) = \text{int } S$

(b) $(\overline{S^c})^c \subseteq S$ since $\overline{S^c}$ contains S^c so the complement is contained in S. We have int $S \subseteq (\overline{S^c})^c$ since $(\overline{S^c})^c$ is open and $(\overline{S^c})^c \subseteq S$ (since $S^c \subseteq \overline{S^c}$) and the interior is the largest open set contained in S.

(c) We know that int $U = U$ for any open set U. Thus since int S is open we have int int $S =$ int S

(d) We know int $S \cap$ int $T \subseteq$ int $(S \cap T)$ since int $S \cap$ int T is an open set contained in S and contained in T and thus contained in $S \cap T$ so containment follows from maximality of interior. Conversly int $(S \cap T) \subseteq$ int $S \cap$ int T since for any $p \in$ int $(S \cap T)$ there exists $B_r(p)$ where $B_r(p) \subset S \cap T$ thus $B_r(p) \subset S$, $B_r(p) \subset T$ so $p \in$ int $S \cap$ int T

Exercise 125

(a) By definition of boundary in Ch 2.6 we have $\delta S = \bar{S} - \text{int } S$. It is always the case that int $S \subseteq S \subseteq \bar{S}$. Thus $\delta S = \emptyset \Leftrightarrow$ int $S = \bar{S} = S$. We know that int $S = S$ iff S is open and $\bar{S} = S$ iff S is closed. Thus $\delta S = \emptyset$ iff S is clopen.

(b) This follows directly from the definition of boundary stated in the problem. A point p is in δS iff $\forall r > 0, B_r(p) \cap S \neq \emptyset$ and $B_r(p) \cap S^c \neq \emptyset$. This is equivalent to the condition that $B_r(p) \cap S^c \neq \emptyset$ and $B_r(p) \cap (S^c)^c \neq \emptyset$. Thus $p \in \delta S \Leftrightarrow p \in \delta S^c$