Exercise 26

Consider the contraction

$$\frac{1}{2}x : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

This is a contraction since $|\frac{1}{2}x - \frac{1}{2}y| \leq \frac{1}{2}|x - y| \ \forall x, y$

There is no fixed point since the only possible fixed point is 0

Exercise 27

(a) We have the counterexample

$$x^2 : [0, 1/2] \to \mathbb{R}$$

This is weak contraction since by the mean value theorem for some $\theta \in (x, y) \subset [0, 1/2]$

$$|x^2 - y^2| = 2\theta|x - y| < 2\frac{1}{2}|x - y| = |x - y|$$

This is not a contraction however since for any $0 < L < 1$ we have that for some $\theta \in (L/2, 1/2)$

$$\left|\left(\frac{L}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right| = 2\theta|x - y| > 2\frac{L}{2}|x - y| = L|x - y|$$

(b) Notice the above counterexample was over a compact set

(c) Consider the continuous function

$$g : M \to M$$

$$g(x) = d(x, f(x))$$

since g is over a compact set, it attains its minumum x_0. Thus we have

$$g(x_0) \leq g(f(x_0))$$

x_0 is a fixed point since if it were the case $x_0 \neq f(x_0)$ then since f is a weak contraction we reach the contraction

$$d(x_0, f(x_0)) \leq d(f(x_0), f^2(x_0)) < d(x_0, f(x_0))$$

$$d(x_0, f(x_0)) < d(x_0, f(x_0))$$

The fixed point is unique since if $x \neq y$ are fixed points then we have the contraction

$$d(x, y) = d(f(x), f(y)) < d(x, y)$$

Exercise 34

(a) We have that any function of the form $\gamma(t) : [0, b) \to \mathbb{R}$ where $\gamma(t) = t^2$ solves the ODE since $\gamma'(t) = 2t = 2\sqrt{|\gamma(t)|}$. We also have any function $\beta(t) : (a, c) \to \mathbb{R}$ where $0 \in (a, c)$ and $\beta(t) = 0$ solves the ODE since $\beta'(t) = 0 = 2\sqrt{|\gamma(t)|}$

(b)

(c) This is not a contraction to Picard's theorem since the theorem states uniqueness for each solution whose domain is an open interval (a, b) containing the initial conditions at 0. The nonunique solution t^2 is not defined on the appropriate domain

Exercise Additional Problem 1

Since the polynomials are dense over $C^0([a, b])$, for any $\epsilon > 0$ there is a polynomial $q(x)$ such that $|f(x) - q(x)| < \epsilon$. Thus

$$|f^2(x) - f(x)p(x)| < \epsilon|f(x)|$$

Thus

$$\int_a^b f^2(x)\ dx \leq \int_a^b f(x)p(x) + \epsilon|f(x)|\ dx = \epsilon \int_a^b |f(x)|\ dx$$

Since $\int_a^b |f(x)|\ dx$ is some fixed number > 0 and the inequality holds for all $\epsilon > 0$ it must be the case

$$\int_a^b f^2(x)\ dx = 0$$

This implies that $f(x) = 0$ for all $x \in [a, b]$

Exercise Additional Problem 2

(a) Let $g(x) = f(-\log x) : (0, 1] \to \mathbb{R}$. We have that

$$g(e^{-x}) = f(-\log(e^{-x})) = f(x)$$

(b) Given

$$\lim_{y\to\infty} f(y) = 0$$

Replacing $y = -\log(x)$

$$0 = \lim_{x\to 0} f(-\log(x)) = \lim_{x\to 0} g(x)$$

And thus g can be extended to $C^0([0, 1])$ with $g(0) = 0$

(c) Since the polynomials are dense over $C^0[0,1]$, for any $\epsilon > 0$ there exists $p(x) = \sum_{j=1}^{n} a_j x^j$ such that

$$\sup_{y\in[0,1]} \left| g(y) - \sum_{j=1}^{n} a_j y^j \right| < \epsilon$$

Since e^{-x} is a homeomorphism we can replace $y = e^{-x}$

$$\sup_{x\in[0,\infty)} \left| f(x) - \sum_{j=1}^{n} a_j e^{-jx} \right| < \epsilon$$

Exercise Additional Problem 3

(a) Fix $\epsilon > 0$. Suppose for contradiction that for each f_n there is a x_n such that $f_n(x_n) > \epsilon$. By compactness there is a convergent subsequence x_{n_k} with limit x. We have that $f_{n_k}(x) \geq \epsilon$ for all n_k and thus we contradict $f_n(x)$ converging pointwise to 0. The reason $f_{n_k}(x) \geq \epsilon$ is as follows:
Fixing $N = n_k$, we have that for all $i > k$,

$$f_N(x_{n_i}) \geq f_{n_i}(x_{n_i}) > \epsilon$$

Thus the inequality is preserved in the limit:

$$f_N(x) \geq \epsilon$$

Thus we have proven through contradiction there must exist N such that $f_N(x) < \epsilon$ for all x and since $f_K(x) < f_N(x)$ for all $K > N$ we have uniform convergence

(b) Consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & x < n \\ x - n & x \in (n, n+1) \\ 1 & x > n+1 \end{cases}$$

f_n is a monotonically decreasing sequence of functions which converges pointwise to 0 but does not converge uniformly