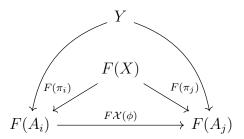
1

If $\mathcal{X}: \mathcal{I} \to \mathcal{A}$ has limit X, since functors preserve compositions of morphisms $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$, it is the case that functors preserve cones. Thus $F(X) = F(\lim_{i \in \mathcal{I}} A_i)$ is a cone of $F \circ \mathcal{X}$.

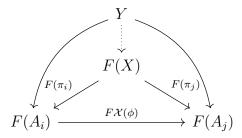
Suppose we have the cone Y which yields the diagram



Letting G be the left adjoint to F we have that

$$\operatorname{Hom}_{\mathcal{A}}(G(Y), X) \simeq \operatorname{Hom}_{\mathcal{B}}(Y, F(X))$$

We have that G(Y) is a cone of \mathcal{X} and thus there is a unique morphism $G(Y) \to X$ which corresponds to a unique morphism $Y \to F(X)$ making the diagram commute



And thus F(X) is a limit of $F \circ \mathcal{X} : \mathcal{I} \to \mathcal{B}$

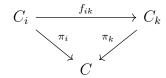
 $\mathbf{2}$

For any finite cyclic subgroup $C_n \subset \mathbb{Q}/\mathbb{Z}$ of order n, a generator is $1/n + \mathbb{Z}$ and thus this cyclic group of order n is unique. Thus we have that the cyclic subgroups form a directed set where $\langle 1/n \rangle \leq \langle 1/m \rangle$ iff $n \leq m$ and morphisms

$$f_{ik}: C_i \to C_k$$

$$f_{ik}(1/i) = \frac{k/\gcd(k,i)}{k} = \frac{1}{\gcd(k,i)}$$

For any Cone C of our directed set of cyclic subgroups (over the category of abelian groups)



We have that we can define a map $\Psi: C \to \mathbb{Q}/\mathbb{Z}$ where for every $x \in C$ if $x = \pi_n(n_x)$ for some $n_x \in C_n$ (here we mean $n_x \cdot g_n$ where g_n generates C_n and $n_x \in \mathbb{Z}$) setting $\Psi(x) = \frac{n_x}{n}$. Let $\Psi(x) = 0$ otherwise.

This map is well defined since if $\pi_n(n_x) = \pi_m(m_x)$ then $\pi_m \circ f_{nm} = \pi_n$ so $n_x = f_{nm}(m_x)$ and

$$\frac{n_x}{n} = \frac{m_x}{\gcd(m, n)n} = \frac{m_x}{m}$$

Notice that composing is just the inclusion map $\Psi \circ \pi_n = i : C_n \to \mathbb{Q}/\mathbb{Z}$ and thus Ψ is \mathbb{Z} linear and establishes \mathbb{Q}/\mathbb{Z} to be the limit of the directed set of cyclic subgroups.

For M and arbitrary \mathbb{Z} -module we have the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which corresponds to a long exact sequence (Prop 14 of 17.1 Dummit and Foote)

$$\cdots \to \operatorname{Tor}_1^{\mathbb{Z}}(M,\mathbb{Q}) \to \operatorname{Tor}_1^{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) \to M \otimes \mathbb{Z} \to M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

Since \mathbb{Q} is torsion free it is flat (homework 2 problem 4) and thus $\operatorname{Tor}_1^{\mathbb{Z}}(M,\mathbb{Q}) = 0$ so

$$\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \cong \ker(M \otimes \mathbb{Z} \to M \otimes \mathbb{Q})$$

The kernel is precisely TorM. The reason for this is because $M \cong M \otimes \mathbb{Z}$ from the mapping $M \to M \otimes \mathbb{Z}$ where $x \to x \otimes 1$. $x \in \text{Tor}M$ iff nx = 0 for some $n \in \mathbb{Z}$, thus by our mapping $M \to M \otimes \mathbb{Q}$ where $x \to x \otimes 1$ we have $x \otimes 1 = nx \otimes 1/n = 0$ so x is in the kernel iff $x \in \text{Tor}M$

3

We know that \mathbb{Q}/\mathbb{Z} is injective from homework 3 problem 6. If x has order n then as a cyclic group $\langle x \rangle$ embeds into \mathbb{Q}/\mathbb{Z} where $x \to 1/n$. If x has infinite order let $x \to 1/2$. We have the commutative diagram

$$\mathbb{Q}/\mathbb{Z}$$

$$x \to 1/n \longrightarrow M$$

$$\langle x \rangle \longleftrightarrow M$$

Thus we get an induced $\phi: M \to \mathbb{Q}/\mathbb{Z}$ where $\phi(x) = 1/n$ (1/2 in infinite case). For any nonzero x we have such a ϕ so $|\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})\setminus\{0\}| \geq 1$ if $|M\setminus\{0\}| \geq 1$. Thus if $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) = 0$ then M = 0

4

We already know $M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an abelian group. For any $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and $r \in A$ since $r \cdot -$ is a group homomorphism $M \to M$ we can define $r\phi(x)$ by composition. The property to check is that $r \to r\phi$ is a ring homomorphism $A \to \operatorname{End} M^*$. This is the case since for $a, b \in A$ as linear maps $M \to M$, $a \cdot - + b \cdot - = (a+b) \cdot -$ and $a \cdot - \circ b \cdot - = (ab) \cdot -$

and composition with ϕ will preserve these properties Thus M^* is an A-module.

From homework 3 problem 6 we know \mathbb{Q}/\mathbb{Z} is injective and thus $\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})$ is both left and right exact so as \mathbb{Z} modules

$$0 \to N \to M \to L \to 0$$

is exact iff

$$0 \to L^* \to M^* \to N^* \to 0$$

is exact. This is true as A-modules since we can extend our mappings $\phi: B^* \to C^*$ by composing $r\phi(x) = r \cdot - \circ \phi$ which makes each arrow of the commutative diagram A linear while preserving kernels and cokernels

5

 (\Rightarrow) If F is flat then given a short exact sequence of A modules

$$0 \to X \to Y$$

yields from flatness

$$0 \to X \otimes F \to Y \otimes F$$

if we consder any $\phi:X\to F^*$ we have the bilinear mapping $b:X\oplus F\to \mathbb{Q}/\mathbb{Z}$ given by

$$(x,d) \to \phi(x)(d)$$

where $\phi(x): F \to \mathbb{Q}/\mathbb{Z}$. Thus there is a mapping $X \otimes F \to \mathbb{Q}/\mathbb{Z}$ to make the diagram commute.

$$X \otimes F \xrightarrow{x \otimes d} \mathbb{Q}/\mathbb{Z}$$

$$X \oplus F$$

$$X \oplus F$$

Since \mathbb{Q}/\mathbb{Z} is an injective \mathbb{Z} module we can extend our mapping

This mapping $Y \otimes F \to \mathbb{Q}/\mathbb{Z}$ is the same as a mapping $Y \to \operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ which commutes with the mapping ϕ

 (\Leftarrow) If F^* is injective and again we have a short exact sequence

$$0 \to X \to Y$$

we wish to show the induced morphism is injective

$$X \otimes F \to Y \otimes F$$

suppose we have some $x \otimes d \neq 0$ $X \otimes F$. By problem 3 there is a morphism $\phi : X \otimes F \to \mathbb{Q}/\mathbb{Z}$ so that $\phi(x \otimes d) \neq 0$. Since ϕ induces a mapping $X \to F^*$ by sending $x \to \phi(x) : F \to \mathbb{Q}/\mathbb{Z}$ from injectivity we get a mapping $Y \to F^*$ which induces a mapping $Y \otimes F \to \mathbb{Q}/\mathbb{Z}$ which makes the following commute

$$X \otimes F \longrightarrow Y \otimes F$$

$$\downarrow^{\phi} \qquad \qquad \downarrow^{\phi}$$

$$\mathbb{Q}/\mathbb{Z}$$

Thus we have a $x \otimes d$ cannot be in $\ker(X \otimes F \to Y \otimes F)$ so we have injectivity

6

We have that

$$\operatorname{pd}(M) < n \Leftrightarrow \operatorname{Ext}_A^n(M, N) = 0 \ \forall \ A\text{-modules } N \Leftrightarrow \operatorname{id}(M) < n$$

The proof for this is because for any exact sequence

$$0 \to K \to P_{n-2} \to P_{n-3} \to \cdots \to P_0 \to M \to 0$$

where P_i are projective, $\operatorname{Ext}_A^n(M,N) = 0$ for all A-Modules N iff K is projective. It is also the case that $\operatorname{Ext}_A^n(M,N)$ can be computed by an injective resolution and thus for any exact sequence

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-2} \to K \to 0$$

with I_i injective, $\operatorname{Ext}_A^n(M,N)=0$ for all A-Modules N iff K is injective

Thus it must be the case that pd(M) = id(M) since if pd(M) < id(M) then we'd have the contradiction id(M) < id(M) or vise versa. Hence

$$\sup\{\operatorname{pd}(M)|M\text{ is an A-Module}\}=\sup\{\operatorname{id}(M)|M\text{ is an A-Module}\}$$

7

- (\Rightarrow) If R is semisimple then every R module M is semisimple and is thus injective. Therefore $\mathrm{id}(M)=0$
- (\Leftarrow) We have that as an R module $\mathrm{id}(R)=0$ so R is injective. Thus R is semisimple

Let $g_1, g_2, \ldots g_n$ be a minimal set of generators for the A module M (where M is finitely generated, projective, and A is a commutative local ring)

Letting A^n be the free module generated by $g_1 \dots g_n$ we have the exact sequence

$$0 \longrightarrow K \longrightarrow A^n \xrightarrow{g_i \to g_i} M \longrightarrow 0$$

where $K = \ker(g_i \to g_i)$.

From projectivity we have splitting

$$A^n = M \oplus K$$

We can apply the functor $A/m \otimes -$ where m is the maximal ideal of A

$$A/m \otimes A^n = (A/m \otimes M) \oplus (A/m \otimes K)$$

We have that $A/m \otimes D$ is an A/m vector space for any A-module D. We can compare dimensions of vector spaces, $n = \dim A/m \otimes M = \dim A/m \otimes A^n$ and thus $\dim A/m \otimes K = 0$. Therefore mK = K. From Nakayama's Lemma Corollary 2 (section 2 of Reid) this implies K = 0 and thus $M = A^n$

9

We can proceed by induction, if $\operatorname{pd}_{A/xA}(M/xM) = 0$ then M/xM is a projective A/xA module and thus $M/xM \oplus K = F$ where F is a free A/xA module. Tensoring with A yields as A modules

$$M/xM \oplus (K \otimes_{A/xA} A) = F/xF$$

where F now denotes a free A module

Thus we have the projective resolution

$$0 \to xF \oplus (K \otimes_{A/xA} A) \to F/xF \to M/xM$$

Thus $\operatorname{pd}_A(M/xM) \leq 1$ and it cannot be 0 since M/xM is not torsion free and so not projective.

We of course have $\operatorname{pd}_A(M) \geq 0$ as well.

Let P be any projective A module projecting onto M we have the exact sequence

$$0 \to K \to P \to M \to 0$$

by inductive hypothesis $\operatorname{pd}_{A/xA}(K/xK) = \operatorname{pd}_A(K/xK) - 1 \leq \operatorname{pd}_A(K)$. Notice that $\operatorname{pd}_A(K) \leq \operatorname{pd}_A(M)$ since any projective resolution of K will show up above as a projective resolution to $P \to M$. When tensoring with $\otimes_A A/xA$ we get the exact sequence

$$\cdots \to \operatorname{Tor}_A^1(M, A/xA) \to K/xK \to P/xP \to M/xM \to 0$$

We have that $\operatorname{Tor}_A^1(M, A/xA) = 0$. This is calculated with the projective resolution

$$0 \to xA \to A \to A/xA \to 0$$

and thus $\operatorname{Tor}_1^A(M,A/xA)=\ker(xA\to A)$ which is 0 since x is not a zero divisor. Thus we have

$$0 \to K/xK \to P/xP \to M/xM \to 0$$

This was the case for any K so letting K/xK be the kernel which realizes the smallest projective resolution of M/xM we have the desired result (by either viewing the mapping $K/xK \to M/xM$ as A linear or A/xA linear)

$$\operatorname{pd}_A(M/xM) - 1 = \operatorname{pd}_{A/xA}(M/xM) = \operatorname{pd}_{A/xA}(K/xK) + 1$$

And since $\operatorname{pd}_{A/xA}(K/xK) + 1 \leq \operatorname{pd}_A(K) \leq \operatorname{pd}_A(M)$ we get the final result

$$\operatorname{pd}_A(M/xM) - 1 = \operatorname{pd}_{A/xA}(M/xM) \le \operatorname{pd}_A(M)$$