1

If $\mathcal{X} : \mathcal{I} \to \mathcal{A}$ has limit X, since functors preserve compositions of morphisms $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$, it is the case that functors preserve cones. Thus $F(X) = F(\lim_{i \in \mathcal{I}} A_i)$ is a cone of $F \circ \mathcal{X}$.

Suppose we have the cone Y which yields the diagram

$$\begin{tikzcd} & Y \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & F(X) \arrow[dl, "F(\pi_i)"'] \arrow[dr, "F(\pi_j)"] & \\ F(A_i) \arrow[rr, "F\mathcal{X}(\phi)"] & & F(A_j) \end{tikzcd}$$

Letting G be the left adjoint to F we have that

$$\mathrm{Hom}_{\mathcal{A}}(G(Y), X) \simeq \mathrm{Hom}_{\mathcal{B}}(Y, F(X))$$

We have that $G(Y)$ is a cone of $\mathcal{X}$ and thus there is a unique morphism $G(Y) \to X$ which corresponds to a unique morphism $Y \to F(X)$ making the diagram commute

$$\begin{tikzcd} & Y \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[d, dotted] & \\ & F(X) \arrow[dl, "F(\pi_i)"'] \arrow[dr, "F(\pi_j)"] & \\ F(A_i) \arrow[rr, "F\mathcal{X}(\phi)"] & & F(A_j) \end{tikzcd}$$

And thus $F(X)$ is a limit of $F \circ \mathcal{X} : \mathcal{I} \to \mathcal{B}$

2

For any finite cyclic subgroup $C_n \subset \mathbb{Q}/\mathbb{Z}$ of order n, a generator is $1/n + \mathbb{Z}$ and thus this cyclic group of order n is unique. Thus we have that the cyclic subgroups form a directed set where $\langle 1/n \rangle \leq \langle 1/m \rangle$ iff $n \leq m$ and morphisms

$$f_{ik} : C_i \to C_k$$
$$f_{ik}(1/i) = \frac{k/\gcd(k,i)}{k} = \frac{1}{\gcd(k,i)}$$

For any Cone C of our directed set of cyclic subgroups (over the category of abelian groups)

$$\begin{tikzcd} C_i \arrow[rr, "f_{ik}"] \arrow[dr, "\pi_i"] & & C_k \arrow[dl, "\pi_k"'] \\ & C & \end{tikzcd}$$

We have that we can define a map $\Psi : C \to \mathbb{Q}/\mathbb{Z}$ where for every $x \in C$ if $x = \pi_n(n_x)$ for some $n_x \in C_n$ (here we mean $n_x \cdot g_n$ where g_n generates C_n and $n_x \in \mathbb{Z}$) setting $\Psi(x) = \frac{n_x}{n}$. Let $\Psi(x) = 0$ otherwise.
This map is well defined since if $\pi_n(n_x) = \pi_m(m_x)$ then $\pi_m \circ f_{nm} = \pi_n$ so $n_x = f_{nm}(m_x)$ and

$$\frac{n_x}{n} = \frac{m_x}{\gcd(m,n)n} = \frac{m_x}{m}$$

Notice that composing is just the inclusion map $\Psi \circ \pi_n = i : C_n \to \mathbb{Q}/\mathbb{Z}$ and thus Ψ is $\mathbb{Z}$ linear and establishes $\mathbb{Q}/\mathbb{Z}$ to be the limit of the directed set of cyclic subgroups.

For M and arbitrary $\mathbb{Z}$-module we have the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which corresponds to a long exact sequence (Prop 14 of 17.1 Dummit and Foote)

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}) \to \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to M \otimes \mathbb{Z} \to M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

Since $\mathbb{Q}$ is torsion free it is flat (homework 2 problem 4) and thus $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}) = 0$ so

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \cong \ker(M \otimes \mathbb{Z} \to M \otimes \mathbb{Q})$$

The kernel is precisely $\mathrm{Tor}M$. The reason for this is because $M \cong M \otimes \mathbb{Z}$ from the mapping $M \to M \otimes \mathbb{Z}$ where $x \to x \otimes 1$. $x \in \mathrm{Tor}M$ iff $nx = 0$ for some $n \in \mathbb{Z}$, thus by our mapping $M \to M \otimes \mathbb{Q}$ where $x \to x \otimes 1$ we have $x \otimes 1 = nx \otimes 1/n = 0$ so x is in the kernel iff $x \in \mathrm{Tor}M$

3

We know that $\mathbb{Q}/\mathbb{Z}$ is injective from homework 3 problem 6. If x has order n then as a cyclic group $\langle x \rangle$ embeds into $\mathbb{Q}/\mathbb{Z}$ where $x \to 1/n$. If x has infinite order let $x \to 1/2$. We have the commutative diagram

$$\begin{tikzcd}
 & \mathbb{Q}/\mathbb{Z} \\
\langle x \rangle \arrow[ur, "x \to 1/n"] \arrow[r, hook] & M
\end{tikzcd}$$

Thus we get an induced $\phi : M \to \mathbb{Q}/\mathbb{Z}$ where $\phi(x) = 1/n$ (1/2 in infinite case). For any nonzero x we have such a ϕ so $|\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \backslash \{0\}| \geq 1$ if $|M \setminus \{0\}| \geq 1$. Thus if $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = 0$ then $M = 0$

4

We already know $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an abelian group. For any $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and $r \in A$ since $r \cdot -$ is a group homomorphism $M \to M$ we can define $r\phi(x)$ by composition. The property to check is that $r \to r\phi$ is a ring homomorphism $A \to \mathrm{End}\, M^*$. This is the case since for $a, b \in A$ as linear maps $M \to M$, $a \cdot - + b \cdot - = (a+b) \cdot -$ and $a \cdot - \circ b \cdot - = (ab) \cdot -$

and composition with ϕ will preserve these properties
Thus M^* is an A-module.

From homework 3 problem 6 we know $\mathbb{Q}/\mathbb{Z}$ is injective and thus $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is both left and right exact so as $\mathbb{Z}$ modules

$$0 \to N \to M \to L \to 0$$

is exact iff

$$0 \to L^* \to M^* \to N^* \to 0$$

is exact. This is true as A-modules since we can extend our mappings $\phi : B^* \to C^*$ by composing $r\phi(x) = r \cdot - \circ \phi$ which makes each arrow of the commutative diagram A linear while preserving kernels and cokernels

5

($\Rightarrow$) If F is flat then given a short exact sequence of A modules

$$0 \to X \to Y$$

yields from flatness

$$0 \to X \otimes F \to Y \otimes F$$

if we consder any $\phi : X \to F^*$ we have the bilinear mapping $b : X \oplus F \to \mathbb{Q}/\mathbb{Z}$ given by

$$(x, d) \to \phi(x)(d)$$

where $\phi(x) : F \to \mathbb{Q}/\mathbb{Z}$. Thus there is a mapping $X \otimes F \to \mathbb{Q}/\mathbb{Z}$ to make the diagram commute.

$$\begin{tikzcd}
X \otimes F \arrow[r, dotted] & \mathbb{Q}/\mathbb{Z} \\
X \oplus F \arrow[u, "x \otimes d"] \arrow[ur, "\phi(x)(d)"'] &
\end{tikzcd}$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$ module we can extend our mapping

$$\begin{tikzcd}
Y \otimes F \arrow[r, dotted] & \mathbb{Q}/\mathbb{Z} \\
X \otimes F \arrow[u] \arrow[ur] & X \oplus F \arrow[l, "x \otimes d"] \arrow[u, "\phi(x)(d)"'] \\
0 \arrow[u] &
\end{tikzcd}$$

This mapping $Y \otimes F \to \mathbb{Q}/\mathbb{Z}$ is the same as a mapping $Y \to \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ which commutes with the mapping ϕ

($\Leftarrow$) If F^* is injective and again we have a short exact sequence

$$0 \to X \to Y$$

we wish to show the induced morphism is injective

$$X \otimes F \to Y \otimes F$$

suppose we have some $x \otimes d \neq 0$ $X \otimes F$. By problem 3 there is a morphism $\phi : X \otimes F \to \mathbb{Q}/\mathbb{Z}$ so that $\phi(x \otimes d) \neq 0$. Since ϕ induces a mapping $X \to F^*$ by sending $x \to \phi(x) : F \to \mathbb{Q}/\mathbb{Z}$ from injectivity we get a mapping $Y \to F^*$ which induces a mapping $Y \otimes F \to \mathbb{Q}/\mathbb{Z}$ which makes the following commute

$$\begin{tikzcd}
X \otimes F \arrow[r] \arrow[dr, "\phi"] & Y \otimes F \arrow[d, dotted] \\
 & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Thus we have a $x \otimes d$ cannot be in $\ker(X \otimes F \to Y \otimes F)$ so we have injectivity

6

We have that

$$\mathrm{pd}(M) < n \Leftrightarrow \mathrm{Ext}^n_A(M, N) = 0 \ \forall \ A\text{-modules } N \Leftrightarrow \mathrm{id}(M) < n$$

The proof for this is because for any exact sequence

$$0 \to K \to P_{n-2} \to P_{n-3} \to \cdots \to P_0 \to M \to 0$$

where P_i are projective, $\mathrm{Ext}^n_A(M, N) = 0$ for all A-Modules N iff K is projective. It is also the case that $\mathrm{Ext}^n_A(M, N)$ can be computed by an injective resolution and thus for any exact sequence

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-2} \to K \to 0$$

with I_i injective, $\mathrm{Ext}^n_A(M, N) = 0$ for all A-Modules N iff K is injective

Thus it must be the case that $\mathrm{pd}(M) = \mathrm{id}(M)$ since if $\mathrm{pd}(M) < \mathrm{id}(M)$ then we'd have the contradiction $\mathrm{id}(M) < \mathrm{id}(M)$ or vise versa. Hence

$$\sup\{\mathrm{pd}(M) | M \text{ is an A-Module}\} = \sup\{\mathrm{id}(M) | M \text{ is an A-Module}\}$$

7

($\Rightarrow$) If R is semisimple then every R module M is semisimple and is thus injective. Therefore $\mathrm{id}(M) = 0$

($\Leftarrow$) We have that as an R module $\mathrm{id}(R) = 0$ so R is injective. Thus R is semisimple

8

Let $g_1, g_2, \dots g_n$ be a minimal set of generators for the A module M (where M is finitely generated, projective, and A is a commutative local ring)
Letting A^n be the free module generated by $g_1 \dots g_n$ we have the exact sequence

$$0 \longrightarrow K \longrightarrow A^n \xrightarrow{g_i \to g_i} M \longrightarrow 0$$

where $K = \ker(g_i \to g_i)$.
From projectivity we have splitting

$$A^n = M \oplus K$$

We can apply the functor $A/m \otimes -$ where m is the maximal ideal of A

$$A/m \otimes A^n = (A/m \otimes M) \oplus (A/m \otimes K)$$

We have that $A/m \otimes D$ is an A/m vector space for any $A-$module D. We can compare dimensions of vector spaces, $n = \dim A/m \otimes M = \dim A/m \otimes A^n$ and thus $\dim A/m \otimes K = 0$. Therefore $mK = K$. From Nakayama's Lemma Corollary 2 (section 2 of Reid) this implies $K = 0$ and thus $M = A^n$

9

We can proceed by induction, if $\mathrm{pd}_{A/xA}(M/xM) = 0$ then M/xM is a projective A/xA module and thus $M/xM \oplus K = F$ where F is a free A/xA module. Tensoring with A yields as A modules

$$M/xM \oplus (K \otimes_{A/xA} A) = F/xF$$

where F now denotes a free A module
Thus we have the projective resolution

$$0 \to xF \oplus (K \otimes_{A/xA} A) \to F/xF \to M/xM$$

Thus $\mathrm{pd}_A(M/xM) \leq 1$ and it cannot be 0 since M/xM is not torsion free and so not projective.
We of course have $\mathrm{pd}_A(M) \geq 0$ as well.
Let P be any projective A module projecting onto M we have the exact sequence

$$0 \to K \to P \to M \to 0$$

by inductive hypothesis $\mathrm{pd}_{A/xA}(K/xK) = \mathrm{pd}_A(K/xK) - 1 \leq \mathrm{pd}_A(K)$. Notice that $\mathrm{pd}_A(K) \leq \mathrm{pd}_A(M)$ since any projective resolution of K will show up above as a projective resolution to $P \to M$. When tensoring with $\otimes_A A/xA$ we get the exact sequence

$$\cdots \to \mathrm{Tor}^1_A(M, A/xA) \to K/xK \to P/xP \to M/xM \to 0$$

We have that $\mathrm{Tor}^1_A(M, A/xA) = 0$. This is calculated with the projective resolution

$$0 \to xA \to A \to A/xA \to 0$$

and thus $\mathrm{Tor}_1^A(M, A/xA) = \ker(xA \to A)$ which is 0 since x is not a zero divisor. Thus we have

$$0 \to K/xK \to P/xP \to M/xM \to 0$$

This was the case for any K so letting K/xK be the kernel which realizes the smallest projective resolution of M/xM we have the desired result (by either viewing the mapping $K/xK \to M/xM$ as A linear or A/xA linear)

$$\mathrm{pd}_A(M/xM) - 1 = \mathrm{pd}_{A/xA}(M/xM) = \mathrm{pd}_{A/xA}(K/xK) + 1$$

And since $\mathrm{pd}_{A/xA}(K/xK) + 1 \le \mathrm{pd}_A(K) \le \mathrm{pd}_A(M)$ we get the final result

$$\mathrm{pd}_A(M/xM) - 1 = \mathrm{pd}_{A/xA}(M/xM) \le \mathrm{pd}_A(M)$$

10