16.1 This is a special case of thm 16.1 ii:
We have

$$(-1)a + a = (-1+1)a = 0 \cdot a = 0$$

and so subtracting a on both sides yields

$$(-1)a = -a$$

16.7 Since F is a field we know there is $a^{-1} \in F$ such that $aa^{-1} = 1$. Therefore if we let $x = a^{-1}(-b)$ we satisfy the equation:

$$a(a^{-1}(-b)) + b = (aa^{-1})(-b) + b = -b + b = 0$$

We get that first equality since $\cdot$ is associative

16.11

a. The only unit is $(1,1)$ since for any $a, b \in \mathbb{Z}$, $ab = 1 \Leftrightarrow a = 1, b = 1$. The only zero-divisor is $(0,0)$ since for any $a, b \in \mathbb{Z}$, $ab = 0 \Leftrightarrow a = 0$ and/or $b = 0$. Since the set of nilpotents elements is a subset of zero-divisors, it follows that the only nilpotent is also $(0,0)$.

b. From previous knowledge of groups we know every element in $\mathbb{Z}_3$ has an inverse under the group operation of multiplication modulo 3, therefore we know for any $(a,b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ there is a $(a^{-1}, b^{-1}) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ such that $(a,b)(a^{-1}, b^{-1}) = (1,1)$ and so every element in $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is a unit. Since 3 is prime there is no two numbers that can multiply together to be a multiple of 3 unless one of the two numbers is already a multiple of 3, only $(0,0)$ is a zero-divisor and from that it follows (since the set of nilpotents is a subset of zero-divisors) that $(0,0)$ is the only nilpotent

c. The units are $(1,1), (1,5), (3,1), (3,5)$ with respective inverses $(1,1), (1,5), (3,1), (3,5)$. The zero-divisors are all the rest of the elements: $(0,2), (0,3), (0,4), (2,2), (2,3), (2,4)$. The nilpotents are $(0,0), (2,0)$.

16.13

a. If there were two multiplicative identities: $1 \neq 1'$ we would have by definition of the multiplicative identity

$$1 = 1 \cdot 1' = 1'$$

and so $1 = 1'$

b. If there were two multiplicative inverses, let β and α be multiplicative inverses of a. We have

$$\beta = \beta(a\alpha) = (\beta a)\alpha = \alpha$$

And so $\beta = \alpha$

A From the definition we know that the center is abelian and from the definition of a division ring we know every element is a unit. Now all we need to show is that the center is closed under multiplication and addition. Given any $a, b \in$ the center of R we have for any $x \in R$

$$(a + b)x = ax + bx = xa + xb = x(a + b)$$

and so $a + b$ is in the center. We also have

$$(ab)x = axb = x(ab)$$

and so ab is in the center. Therefore the center is a field.

B $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain. Consider any $a, b \in \mathbb{Z}/\{0\}$

$$(a, 0) \cdot (0, b) = (0, 0)$$

and so $(a, 0)$ and $(0, b)$ are non-zero zero-divisors.

C

D

E