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For any mapping $k[t] \to k[x,y]/(1-xy)$ the mapping is fully defined by the image of t:

$$t \to s(x,y) \in k[x,y]/(1-xy)$$

Notice that we can split up s by x and y terms $s(x,y) = f(x) + g(y)$ since for any $x^m y^n$ term, the term reduces to a monomial of either x or y depending on which power is larger (since $xy = 1$)
This mapping cannot be surjective and thus is not an isomorphism since WLOG if $\deg f(x) > 0$ then there is no $p(t) \in k[t]$ where $p(t) \to y$.
The reason for this is the degree with respect to x of the image of $p(t)$ will be $\deg p(t) \cdot \deg f(x)$.
Thus either $p(t) \in k$ or $p(t)$ maps to something with x (which cannot be equal to y)

We have the degree equality above since $\deg(f(x) + g(y))^n = n \deg f(x)$ ($g(y)$ is just a constant with respect to x) and so the leading term of $p(t)$ will map to the leading term of the image with degree $\deg p(t) \cdot \deg f(x)$

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3

$(\supseteq)$: It is clear $\ker\phi \supseteq (z_{00}z_{11} - z_{01}z_{10})$ since

$$z_{00}z_{11} - z_{01}z_{10} \to x_0y_0x_1y_1 - x_0y_1x_1y_0 = 0$$

$(\subseteq)$: for any $f(z_{00}, z_{10}, z_{01}, z_{11}) \in \ker\phi$ we can write

$$f = q(z_{00}, z_{10}, z_{01}, z_{11})(z_{00}z_{11} - z_{01}z_{10}) + r(z_{00}, z_{10}, z_{01}, z_{11})$$

so that no terms show up in r where all $z_{00}z_{01}z_{10}z_{11}$ variables are present . It must be the case that $(z_{00}z_{11} - z_{01}z_{10})$ divides r as follows.
We know that $r \in \ker\phi$ since $f - q(z_{00}z_{11} - z_{01}z_{10}) \in \ker\phi$. If r had some nonzero term

$$cz_{00}^{n_{00}} z_{11}^{n_{11}} z_{01}^{n_{01}} z_{10}^{n_{10}}$$

The image would be

$$c(x_0y_0)^{n_{00}}(x_1y_1)^{n_{11}}(x_0y_1)^{n_{01}}(x_1y_0)^{n_{10}}$$

Thus we have corresponding powers

x_0	x_1	y_0	y_1
$n_{00} + n_{01}$	$n_{10} + n_{11}$	$n_{00} + n_{10}$	$n_{11} + n_{01}$

There must be another term in r which maps to the same term in order to cancel out this term (since r is in the kernel)
This is equivalent to finding multiple solutions to the equation

$$Am = x$$

where $m = (m_{00}, m_{10}, m_{01}, m_{11})^T$, x is the powers described above, and

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Computing the nullspace of A over $\mathbb{R}$ yields the vector space spanned by $v = (1, -1, -1, 1)$, and thus there must be another term in r with powers $= n + kv$ for some $k \in \mathbb{Z}$. Since r has no terms where all the powers are nonzero, the only way this is possible is if the powers are of the form $(0, n, n, 0)$ and $(n, 0, 0, n)$ which corresponds to terms of the form

$$cz_{00}^n z_{11}^n - cz_{01}^n z_{10}^n$$

And thus r must be divisible by $(z_{00}z_{11} - z_{01}z_{10})$

4

We have that in $R = k[x, y, z, t]/I$, $zy^2 = x^2y = ztx$ and so

$$(y^2 - xt)z = 0$$

and thus R is not an integral domain so I is not prime. We have that $z \neq 0$ in R and $y^2 - xt \neq 0$ in R since when viewed as polynomials in y the leading terms can never match since x and z will never cancel out

$$y^2 - xt \neq f(y)(-yz + x^2) + g(y)(xy - zt)$$

$$\forall f, g \in k[x, z, t][y]$$

5

We know that $\mathbb{Z}[x_1 \dots x_n]/\mathfrak{m}$ is a field with some finite charactersitic p. It can be viewed as a $\mathbb{Z}/(p)$ algebra generated by $x_1, \dots x_n$. We have that each x_i is integral over $\mathbb{Z}[x_1 \dots x_{i-1}]/\mathfrak{m}$ and thus we can conclude $\mathbb{Z}[x_1 \dots x_n]/\mathfrak{m}$ is a finitely generated $\mathbb{Z}/(p)$ module and thus finite.

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