Exersise 3.7

Given $x_0 \in \mathbb{R}^n$ and any $\epsilon > 0$, let $\delta = \epsilon$. For any $x \in B(x_0, \delta)$ we have $\|x - x_0\| < \delta = \epsilon$. We also have $|f(x) - f(x_0)| = |\|x\| - \|x_0\||$.
We know that we have $|\|x\| - \|x_0\|| = \|x\| - \|x_0\|$ or $-(\|x\| - \|x_0\|) = \|x_0\| - \|x\|$. By the triangle inequality we know both $\|x\| - \|x_0\| \leq \|x - x_0\|$ and $\|x_0\| - \|x\| \leq \|x_0 - x\| = \|x - x_0\|$.
And so $|f(x) - f(x_0)| = |\|x\| - \|x_0\|| \leq \|x - x_0\| < \epsilon$
Thus f is continuous

Exersise 3.9

a. If there exists some N such that $x_j = x_k$ for all $j, k > N$ then $\delta(x_j, x_k) = 0 < \epsilon$ for all $\epsilon > 0$, and so by definition x_n converges. Conversly if x_n converges, let $\epsilon = 1/2$. We have that for some N, $\delta(x_j, x_k) < 1/2$ for all $j, k > N$. Since $\delta(x_j, x_k) > \epsilon$ if and only if $x_j \neq x_k$, we have that $x_j = x_k$ for all $j, k > N$

b. For any $x_0 \in X$ and any $\epsilon > 0$, let $\delta = 1/2$. We have that $\delta(x, x_0) < \delta$ if and only if $x = x_0$, by definition of the discrete metric. Therefore $B_(x_0, \delta) = \{x_0\}$ and as one of the properties of the metric, we have $d(f(x_0), f(x_0)) = 0 < \epsilon$. Therefore by definition f is continuous

Exersise 3.11

For a given $\epsilon > 0$, f continuous means for that given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, x_i) < \delta$ implies $\rho(f(x), f(x_i)) < \epsilon$. $x_n \to x$ means there is a $N > 0$ such that for $k > N$, $d(x_k, x) < \delta$ and therefore for $k > N$, $\rho(f(x), f(x_k)) < \epsilon$. Thus $f(x_n) \to f(x)$

Exersise 3.14

For any $\theta_0 \in [0, 2\pi)$, given $\epsilon > 0$ let $\delta =$. For any $\theta \in [0, 2\pi)$ with $|\theta - \theta_0| < \delta$ we have

$$\|f(\theta) - f(\theta_0)\| = \sqrt{(\cos(\theta) - \cos(\theta_0))^2 + (\sin(\theta) - \sin(\theta_0)^2}$$

$$= \sqrt{\cos^2(\theta) - 2\cos(\theta_0)\cos(\theta) + \cos^2(\theta_0) + \sin^2(\theta) - 2\sin(\theta_0)\sin(\theta) + \sin^2(\theta_0)}$$

$$= \sqrt{2(1 - (\cos(\theta_0)\cos(\theta) + \sin(\theta_0)\sin(\theta)))}$$

Using the sum formula $(\cos(a - b) = \cos a \cos b + \sin a \sin b)$ we have:

$$= \sqrt{2(1 - \cos(\theta - \theta_0))}$$

A common property of sin is that $|\sin x| < |x|$ since $|x|$ is the arc length while sin is the vertical length of point on the unit circle. Therefore $\sin^2(\theta - \theta_0) = 1 - \cos^2(\theta - \theta_0) < (\theta - \theta_0)^2 < \delta^2$. And so we have

$$\|f(\theta) - f(\theta_0)\| <$$

Exersise 3.17

a. By definition of open for metric spaces, we have that for any $a \in \emptyset$, for any $\epsilon > 0$, $B(a,\epsilon)$ is itself the empty set since a does not exist so $B(a,\epsilon) \subseteq \emptyset$. Thus the empty set is open

b. For any $a \in X$ and $\epsilon > 0$ we have that $B(a,\epsilon) = \{x \in X : \delta(x,a) < \epsilon\}$ thus $B(a,\epsilon) \subseteq X$ and so X is open

c. For any $a \in B(x,\epsilon)$, let $\epsilon' = \epsilon - \delta(x,a)$. Thus we have for any $y \in B(a,\epsilon')$ we have $\delta(y,a) < \epsilon' = \epsilon - \delta(x,a)$. Thus from the triangle inequality we have:

$$\delta(y,x) \leq \delta(x,a) + \delta(a,y) < \epsilon$$

Thus $y \in B(x,\epsilon)$, so $B(a,\epsilon') \subseteq B(x,\epsilon)$. Thus $B(x,\epsilon)$ is open

d. For any $x \in U_1 \cap \cdots \cap U_k$, since each U_i is open there exists for each U_i $\epsilon_i > 0$ where $B(x,\epsilon_i) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots \epsilon_k\}$. We have that $B(x,\epsilon) \subseteq B(x,\epsilon_i)$ for all i. This is because we have for any $a \in B(x,\epsilon)$ we have that $\delta(a,x) < \epsilon \leq \epsilon_i$ and thus $a \in B(x,\epsilon_i)$. Therefore $B(x,\epsilon) \subseteq U_i$ for all i, so $B(x,\epsilon) \subseteq U_1 \cap U_2 \cap \ldots U_k$. Thus $U_1 \cap \ldots U_k$ is open

Exersise §13, 3

In example 4 we have $X - X = \emptyset$ which is countable so $X \in \mathfrak{T}_c$, and $X - \emptyset = X$ so $\emptyset \in \mathfrak{T}_c$. For any collection of sets $A \subseteq \mathfrak{T}_c$ we have from Demorgans laws:

$$X - \left(\bigcup_{U \in A} U\right) = \bigcap_{U \in A} (X - U)$$

Intersections of countable sets are countable, therefore $\left(\bigcup_{U \in A} U\right) \in \mathfrak{T}_c$.
For a finite collection $A \subset \mathfrak{T}_c$ we have from Demorgans laws:

$$X - \left(\bigcap_{U \in A} U\right) = \bigcup_{U \in A} (X - U)$$

Finite unions of countable sets are countable. Therefore $\bigcup_{U \in A}(X - U) \in \mathfrak{T}_c$. Thus all the axioms of a topology are satisfied, so $\mathfrak{T}_c$ is a topology.
However we have $\mathfrak{T}_\infty$ is not necessarily a topology:
Let $X = \mathbb{Z}$. Let $U = \{x \in \mathbb{Z} : x < 0\}$ and $V = \{x \in \mathbb{Z} : x > 0\}$. We have that $X - U = \{x \in \mathbb{Z} : x \geq 0\}$ is an infinite set and $X - V = \{x \in Z : x \leq 0\}$ is an infinite set, so $U, V \in \mathfrak{T}_\infty$. However we have

$$X - (V \cup U) = \{0\}$$

Is not infinite. Thus $U \cup V \notin \mathfrak{T}_\infty$. So $\mathfrak{T}_\infty$ does not satisfy the axioms of a topology.

Exersise §13, 4

a. We have that $X, \emptyset \in \mathfrak{T}_\alpha$ for all α, so $X, \emptyset \in \bigcap \mathfrak{T}_\alpha$
We have that for any collection of sets $A \subseteq \bigcap \mathfrak{T}_\alpha$, we have that for each $\mathfrak{T}_\alpha$, $A \subseteq \mathfrak{T}_\alpha$, and thus since $\mathfrak{T}_\alpha$ is a topolgy

$$\bigcup_{U \in A} U \in \mathfrak{T}_\alpha$$

so $\bigcup_{U \in A} U \in \bigcap \mathfrak{T}_\alpha$.
For a finite collection $A \subseteq \bigcap \mathfrak{T}_\alpha$, we again have that for each $\mathfrak{T}_\alpha$, $A \subseteq \mathfrak{T}_\alpha$, and thus since $\mathfrak{T}_\alpha$ is a topolgy, we have that

$$\bigcap_{U \in A} U \in \mathfrak{T}_\alpha$$

So $\bigcap_{U \in A} U \in \bigcap \mathfrak{T}_\alpha$

b.

c.