Exercise 19

(a) Consider any limit point x of D_k. We will show $\text{osc}(x) \geq 1/k \Rightarrow x \in D_k$ so D_k is closed. (We will use the diameter definition of osc, where $\text{osc}(x) = \lim_{r\to 0} \text{diam } f(B_r(x)))$
For any $r > 0$, since x is a limit point of D_k, there exists $y \in B_r(x) \cap D_k$. Since $B_r(x)$ is open there exists $B_s(y) \subset B_r(x)$. Since y in D_k we know that $\text{diam } f(B_s(y)) \geq 1/k$. Since $B_s(y) \subset B_r(x)$ we have

$$\text{diam } f(B_r(x)) \geq \text{diam } f(B_s(y)) \geq 1/k$$

Thus for every $r > 0$ $\text{diam } f(B_r(x)) \geq 1/k$ so we know

$$\text{osc}(x) = \lim_{r\to 0} \text{diam } f(B_r(x)) \geq 1/k$$

(b) Since every discontinuity point has osc > 0, the discontinuity set can be written as a countable union of D_k where each D_k is closed as proven in part a.

$$D = \bigcup_{k=1}^{\infty} D_k$$

(c) Since the continuity set is the complement of the discontinuity set, we have

$$C = [a, b] \backslash \left(\bigcup_{k=1}^{\infty} D_k \right)$$

From Demorgans law

$$C = \bigcap_{k=1}^{\infty} ([a, b] \backslash D_k)$$

Which is a countable intersection of open sets (since the complement of closed sets are open so $[a, b] \backslash D_k$ is open)

Exercise 27

(b) Consider the indicator function on the rationals $\chi_{\mathbb{Q}} : [0, 1] \to \mathbb{R}$. We have that for any $n \in \mathbb{N}$,

$$x_k^* = \frac{2a + (2k - 1)(b - a)}{2n} \in \mathbb{Q}$$

Thus $\chi_{\mathbb{Q}}(x_k^*) = 1$ for all k, n so our calc limit yields 1, while $\chi_{\mathbb{Q}}$ is not Riemann integrable since it is continuous nowhere

Exercise 28

$(i \Rightarrow ii)$: This follows directly from the definition of a zero set. If Z is a zero set, then for each $\epsilon > 0$ there is a countable couvering of Z by open intervals (a_i, b_i) with total length $\sum b_i - a_i < \epsilon$. We can replace (a_i, b_i) with $[a_i, b_i]$ and since $(a_i, b_i) \subset [a_i, b_i]$ this is still a covering of Z, and the lengths are the same.
$(ii \Rightarrow i)$: Given $\epsilon > 0$ from ii there exists a countable covering $C_i = [a_i, b_i]$ with total length $\sum b_i - a_i < \epsilon/2$. Since the covering is countable we can replace each $C_i = [a_i, b_i]$ with $U_i = (a_i - \frac{\epsilon}{2}\frac{1}{4^i}, b_i + \frac{\epsilon}{2}\frac{1}{4^i})$. The U_i make up a covering since each $C_i \subset U_i$ so $Z \subset \bigcup C_i \subset \bigcup U_i$. The total length is

$$\sum_{i=1}^{\infty} b_i + \frac{\epsilon}{2}\frac{1}{4^i} - (a_i - \frac{\epsilon}{2}\frac{1}{4^i}) = \sum b_i - a_i + \frac{\epsilon}{2}\sum_{i=1}^{\infty}\frac{1}{2^i} = \frac{\epsilon}{2} + \sum b_i - a_i < \epsilon$$

Exercise Additional Problem 1

For any $\epsilon > 0$ and $x \in [0,1] \backslash \mathbb{Q}$ we have that there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. We have that there are only finitely many $\frac{p}{q} \in \mathbb{Q}$ where $q < k$ (we can bound the number of these rational points from above by the sum of denominators less than k which is finite). Thus there exists a minimum distance $d > 0$ from these numbers and x (this is because there are a finite number of distances, each of which are > 0 since $x \notin \mathbb{Q}$ so not equal to any of these numbers). We have that $\forall y \in B_d(x)$, $f(y) < \epsilon$ and thus f is continuous at x. The reason for this is as follows, if $y \notin \mathbb{Q}$ then $f(y) = 0$ and we are done. If $y \in \mathbb{Q}$ then we have $|y - x| < d$ thus y cannot have denominator $< k$ since otherwise we would contradict the minimality of d. Thus $f(y) \leq \frac{1}{k} < \epsilon$ so $f(y) < \epsilon$.

Exercise Additional Problem 2

If $x \in \partial S$, then for all $r > 0$, we have $B_r(x) \cap S, B_r(x) \cap S^c \neq \emptyset$. Thus there exists $y, z \in B_r(x)$ where $\chi_S(y) = 0$ and $\chi_S(z) = 1$. Thus diam $\chi_S(B_r(x)) \geq 1$ since χ_S cannot take any values besides 1 and 0, we have equality: diam $\chi_S(B_r(x)) = 1$ for all $r > 0$

$$\text{osc}(x) = \lim_{r \to 0} \text{diam } \chi_S(B_r(x)) = 1$$

If $x \notin \partial S$ then there exists $r > 0$ such that $B_r(x) \subset S$ or S^c. We thus have $\chi_S(B_r(x)) = \{1\}$ or $\{0\}$ which are sets with diameter 0. Thus $\text{osc}(x) = 0$