Exersise 7

For any convergent sequence (p_n), let p be the limit of (p_n). We can choose $\epsilon = 1$ and from the definition of convergent there exists N such that $\delta(p_n, p) < 1$ for all $n > N$, the number of numbers $\delta(p_i, p)$s with $i < N$ is finite therefore we can choose the p_k with the largest $\delta(p_i, p)$. Therefore we have that (p_n) is bounded by $B = 1 + \delta(p_k, p)$ around p since for any p_i if $i > N$ then $\delta(p_i, p) < 1 < B$ and if $i \leq N$ then from how p_k was chosen we know $\delta(p_i, p) \leq \delta(p_k, p) < B$.

Exersise 8

a. For any $\epsilon > 0$ we have by definition there exists a limit x and a $N > 0$ such that $|x - x_n| < \epsilon$ for all $n > N$. We have that $||x| - |x_n|| = |x| - |x_n|$ or $|x_n| - |x|$ depending on if $x > x_n$ or $x \leq x_n$. From the triangle ineq we have $|x| - |x_n| \leq |x - x_n|$ and $|x_n| - |x| \leq |x - x_n|$ thus $||x| - |x_n|| \leq |x - x_n|$. Therefore for all $n > N$ we have $||x| - |x_n|| < \epsilon$ and thus $(|x_n|)$ converges to $|x|$

b. If $(|x_n|)$ converges in $\mathbb{R}$ then (x_n) converges in $\mathbb{R}$

c. This is not true. Consider the sequence $(a_n) = (-1)^n$. We have that $(|a_n|) = (1) \to 1$ while for any $N > 0$ we can choose $\epsilon = 1$ and there exists a_n, a_{n+1} with $n > N$ and $|a_n - a_{n+1}| > \epsilon$ thus (a_n) is not Cauchy and so not convergent

Exersise 14

a. If we have the isometry $f : M \to N$, for any open set $U \subseteq N$ and any point in the preimage $x \in f^{-1}(U)$, since U is open we know there exists $r \in \mathbb{R}^+$ such that $B_r(f(x)) \subset U$. I claim that $B_r(x) \subseteq f^{-1}(U)$ and thus $f^{-1}(U)$ is open so f is continuous. The argument is the following:
We have that for any point $p \in B_r(x)$ that $d_M(x, p) < r$ and we have that $d_M(x, p) = d_N(f(x), f(p)) < r$ thus

$$f(p) \in B_r(f(x)) \Rightarrow f(p) \in U \Rightarrow p \in f^{-1}(U) \Rightarrow B_r(x) \subseteq f^{-1}(U)$$

b. Notice that since f is bijective, f^{-1} is a well defined function. f^{-1} is also an isometry since we have that for any $x, y \in N$ there exists $p, q \in M$ where $f(p) = x, f(q) = y$ so

$$d_N(x, y) = d_N(f(p), f(q)) = d_M(p, q) \Rightarrow d_M(f^{-1}(x), f^{-1}(y)) = d_M(p, q) = d_N(x, y)$$

Therefore f^{-1} is continuous as we have proven in (a) so f is a homeomorphism as it fits the topological definition of a homeomorphism.

c. If there exists an isometry $f : [0,1] \to [0,2]$ then since f is surjective there exists $a, b \in [0,1]$ where $f(a) = 0, f(b) = 2$ however we have that $\delta(a,b) \leq 1$, however $\delta(f(a), f(b)) = \delta(0,2) > 1$. Therefore $\delta(a,b) \neq \delta(f(a), f(b))$ which contradicts f being an isometry.

Exersise 25

The only possible sequence of points in the singleton set $\{p\}$ is the constant sequence $(p_n)n : p_n = p \forall n$, which converge to p. Thus a singlton set contains all its limit points and so is closed.
Every finite set of points is a finite union of singletons which are closed, since the finite union of closed sets is closed, the finite set of points is closed.

Exersise 26

If none of Us points are limits of its complement, then its complement contains all of its limit points, and thus is closed. The complement of a closed set is open so U must be open.
Conversly if U is open then the complement of U is closed so the complement contains all of its limit points, since U and its complement are disjoint, this means that U does not contain any of its complement's limit points.

Exersise 27

(a) For any point $p \in \bar{S}$, we have that that there exists a sequence $(p_n)_n$ contained in S such that $(p_n)_n \to p$ since $S \subset T$, $(p_n)_n$ is a sequence of points in T as well and thus p is a limit point in T, therefore $p \in \bar{T}$ so $\bar{S} \subset \bar{T}$
(b) For any point $p \in S^{\text{o}}$, we have that there exists an r such that $B_r(p) \subset S$ since $S \subset T$ we have that $B_r(p) \subset T$ and therefore $p \in T^{\text{o}}$ so $S^{\text{o}} \subset T^{\text{o}}$

Exersise 19

$\mathbb{Q}$ is not homeomorphic to $\mathbb{N}$.
We have that every subset of $\mathbb{N}$ is open. The reason is because for any point in any subset $p \in T \subset \mathbb{N}$ we let $r = 1$ then $B_r(p) = \{p\} \subset T$.
Therefore for any bijection $f : \mathbb{Q} \to \mathbb{N}$ the inverse image of a singleton is a singlton: $f^{-1}(\{p\}) = \{q\}$ and in $\mathbb{N}$ $\{p\}$ is open, but singletons in $\mathbb{Q}$ are not open, so the inverse image of an open set is not open, therefore f is not continuous so not a homeomorphism.

Exersise 30

If there exists a metric δ that defined $\mathfrak{T}$ then by the axioms of metrics, we know $\delta(a,b) \neq 0$ since $a \neq b$ thus for $r = \delta(a,b)$, from the way topologies are defined by a metric we have that $B_r(a) \in \mathfrak{T}$ however $B_r(a) = \{a\}$ since $\delta(a,b) \not< r$ so $b \notin B_r(a)$ thus $\{a\} \in \mathfrak{T}$ which is a contradiction.