Math 441, Fall 2017 **HOMEWORK #6** *Peter Gylys-Colwell*

Exercise §23, 3

We have that
$$A \cup \left(\bigcup A_\alpha \right) = \bigcup (A \cup A_\alpha)$$
We know that $A \cap A_\alpha$ is nonempty for each α so from Thm 23.3 each $A \cup A_\alpha$ is connected. Since A is nonempty we have that there exists $p \in A$ so
$$p \in \bigcap (A \cup A_\alpha)$$
Thus from thm 23.3 we know that
$$\bigcup (A \cup A_\alpha)$$
is connected.

Exercise §23, 5

If X has the discrete topology and $A \subseteq X$ is a subspace with more than two points $a, b \in A$, $a \neq b$, we have the seperation $A = \{a\} \cup (A - \{a\}$ where $A - \{a\}$ is nonempty since $b \in A - \{a\}$. Both these sets are open since every subset of the discrete topology is open. It is clear that one-point sets are connected since in order for $\{x\} = U \cup V$ where $U \cap V = \emptyset$, either U or V must be empty.
$\mathbb{Q}$ is totally disconnect since for any subspace $A \subseteq \mathbb{Q}$ with two points $a, b \in A$, since the irrationals are dense there exists an irrational $r \in [a, b]$, and thus we have the seperation
$$A = (A \cap (-\infty, r)) \cup (A \cap (r, \infty))$$

Exercise §23, 9

Let us choose a point $(x, y) \in (X \times Y) - (A \times B)$ where $x \notin A, y \notin B$. We define the set $T = (\{x\} \times Y) \cup (X \times \{y\})$. This set is connected since $\{x\} \times Y$, $X \times \{y\}$ are homeomorphic to Y and X respectively, and thus since the intersection of these sets is (x, y) (nonempty) from thm 23.3, T is connected.
Now for any $(a, b) \in (X \times Y) - (A \times B)$ we have that either $a \notin A$ or $b \notin B$. Therefore we have that $T_a = \{a\} \times Y$ or $T_b = X \times \{b\}$ is contained in $(X \times Y) - (A \times B)$. Define $T_{(a,b)}$ as one of these sets which is contained in $(X \times Y) - (A \times B)$. Letting $M = (X \times Y) - (A \times B)$, we have that
$$M = \bigcup_{(a,b) \in M} T_{(a,b)}$$
This is clear since every $(a, b) \in M$ is contained in a $T_{(a,b)}$ and every $T_{(a,b)}$ is contained in M. It is clear that each $T_{(a,b)}$ is connected since it is homeomorphic to X or Y. Finally we have that for each (a, b), $T_{(a,b)} \cap T = (x, b)$ or (a, y) and thus is not empty. Therefore we can use problem 23.3 to conclude that M is connected.

Exercise §24, 1

(a) If there exists a homeomorphism f from $[0,1]$ or $(0,1]$ to $(0,1)$, then $f(1) = a$ for some $a \in (0,1)$. However then we have that f restricted to $[0,1]/\{1\}$ and $(0,1]/\{1\}$ is a homeomorphism to $(0,1)/\{a\}$. However $[0,1]/\{1\}$ and $(0,1]/\{1\}$ are connected while $(0,a) \cup (a,1)$ is disconnected which is a contradiction since homeomorphisms conserve connectivity. Similarly if there exists a homeomorphism f from $[0,1]$ to $(0,1]$ then we have that $f(0) \neq f(1)$ so we have either $f(0)$ or $f(1) \neq 1$. WLOG we will say $f(0) = a \neq 1$. Then we have that the restriction of f from $[0,1]/\{0\}$ to $(0,a) \cup (a,1]$ is a homeomorphism. However it does not preserve connectivity, which is a contradiction.
(b) Let $X = (0,1)$, $Y = [0,1]$. We already know X is not homeomorphic to Y. however X is homeomorphic to any open interval contained in Y (we proved in lecture all open intervals of $\mathbb{R}$ are homeomorphic) and thus there exists an imbedding from X to Y. Similarly we know Y is homeomorphic to any closed interval in X and thus there exists an imbedding from Y to X
(c) We have that if there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$ then letting $a \in \mathbb{R}^n$ be the point where $f(a) = 0$ then the restriction $f : \mathbb{R}^n/\{a\} \to \mathbb{R}/\{0\}$. However we have that $\mathbb{R}/\{0\}$ is disconnected while $\mathbb{R}^n/\{a\}$ is not which is a contradiction. $\mathbb{R}^n/\{a\}$ is not disconnected since we can apply Exercise 23.9 with $X = \mathbb{R}$, $Y = \mathbb{R}$, $A = \{a_x\}$ and $B = \{a_y\}$

Exercise §24, 2

We can define the continous map $g(x) = f(x) - f(-x)$. This is continous since it is the composition and sum of continous functions.
Notice that $g(x) = 0 \Rightarrow f(x) = f(-x)$, and thus if we show g must equal zero at some point, we are done. Also notice that $g(x) = -g(-x)$. Since g maps connected sets to contected sets, we have that the connected set $A = S^1 \cap \{x + iy \in \mathbb{C}, x \geq 0$ gets mapped to a connected set. Therefore we have that $g(i)$ is either positive or negative (or zero in which case we are done). Then $g(-i)$ is also in the image and the opposite sign of $g(i)$. Thus we have a positive value and a negative value in $g(A)$. If $0 \notin g(A)$ then we get the seperation $g(A) = (g(A) \cap (-\infty, 0)) \cup ((0,\infty) \cap g(A))$ which contradicts $g(A)$ being connected. Thus $0 \in g(A)$ so $f(x) = f(-x)$ for some $x \in A$.

Exercise §24, 10

We have proven the general result for $\mathbb{R}^n$ in lecture. Therefore it is true for $n = 2$