

2.4 18

(i) By definition of outer measure (where it is the infimum of measure of all open interval coverings) for any $\epsilon > 0$ there exists a covering of E by open intervals I_k

$$E \subseteq \bigcup I_k = U$$

with

$$m^*(U) - m^*(E) < \epsilon$$

Let U_ϵ denote such an open set. We have the G_δ set

$$G = \bigcap_{n=1}^{\infty} U_{1/n}$$

Notice that since $E \subseteq U_{1/n}$ for all n , $E \subseteq G$. Also notice $G \subseteq U_{1/n}$ for all n . Thus for every $n > 0$

$$m^*(E) \leq m^*(G) \leq m^*(E) + \frac{1}{n}$$

Thus $m^*(G) = m^*(E)$

(ii) (\Rightarrow): From Theorem 11 there exists F_σ set F where

$$F \subseteq E$$

$$m^*(E \setminus F) = 0$$

Since F is a measurable set from disjoint union additivity (which holds if only one of the disjoint sets is measurable) we have the desired result

$$m^*(E) = m^*(F \cup (E \setminus F)) = m^*(F) + m^*(E \setminus F) = m^*(F)$$

(\Leftarrow): If there exists an F_σ set $F \subseteq E$ where $m^*(F) = m^*(E)$ then since F is measurable

$$m^*(E) = m^*(F \cup (E \setminus F)) = m^*(F) + m^*(E \setminus F)$$

and thus $m^*(E \setminus F) = 0$. We have that all sets of outer measure 0 are measurable and thus E is the union of measurable sets (and thus measurable)

$$E = F \cup (E \setminus F)$$

2.4 19

From problem 18 we know that there is a G_δ set $G = \bigcap_{k=1}^{\infty} U_k$ where

$$\lim_{N \rightarrow \infty} m^* \left(\bigcap_{k=1}^N U_k \right) = m^*(G) = m^*(E)$$

Thus for any $\epsilon > 0$ and sufficiently large N we have

$$m^* \left(\bigcap_{k=1}^N U_k \right) < m^*(E) + \epsilon$$

Letting $\mathcal{O}_N = \bigcap_{k=1}^N U_k$ be this open set we have

$$m^*(\mathcal{O}_N) - m^*(E) < \epsilon$$

From subadditivity

$$m^*(\mathcal{O}_N \setminus E) \geq m^*(\mathcal{O}_N) - m^*(E)$$

If we had equality for every \mathcal{O}_N however, for every $\epsilon > 0$ we can choose a \mathcal{O}_N as described above so that

$$m^*(\mathcal{O}_N \setminus E) < \epsilon$$

which from Theorem 11 establishes E to be measurable. Thus equality cannot be the case for each \mathcal{O}_N

2.4 20

(\Leftarrow): This follows directly from the definition of measurable with the set in question (a, b)

(\Rightarrow): From the definition of outer measuer for any $\epsilon > 0$ we can cover E by a countable collection of open intervals I_k such that

$$\sum_{k=1}^{\infty} m^*(I_k) < m^*(E) + \epsilon$$

From hypothesis

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} m^*(E \cap I_k) + m^*(I_k \setminus E) \geq m^* \left(\bigcup_{k=1}^{\infty} E \cap I_k \right) + m^* \left(\bigcup_{k=1}^{\infty} I_k \setminus E \right)$$

Denoting $U = \bigcup_{k=1}^{\infty} I_k$ the open set, the above statement is equal to

$$m^*(U \cap E) + m^*(U \setminus E)$$

since $E \subseteq U$, $U \cap E = E$ so tracing back from our first inequality yields

$$m^*(E) + \epsilon > m^*(E) + m^*(U \setminus E)$$

Thus $m^*(U \setminus E) < \epsilon$ which from Theorem 11 establishes E to be measurable

2.5 25

Consider the example

$$B_1 = \mathbb{R}$$

$$B_n = (n, \infty)$$

We have that

$$\bigcap_{n=1}^{\infty} B_n = \emptyset \Rightarrow m^* \left(\bigcap_{n=1}^{\infty} B_n \right) = 0$$

while for each B_n , $m^*(B_n) = \infty$ so

$$\lim_{n \rightarrow \infty} m^*(B_n) = \infty$$

2.5 26

We have that

$$A \cap \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A \cap E_k$$

From subadditivity we have

$$m^* \left(\bigcup_{k=1}^{\infty} A \cap E_k \right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

We have that for all $N > 0$,

$$m^* \left(\bigcup_{k=1}^{\infty} A \cap E_k \right) \geq m^* \left(\bigcup_{k=1}^N A \cap E_k \right) = \sum_{k=1}^N m^*(A \cap E_k)$$

The second equality follows from Proposition 6. Since this is the case for all N , we can conclude

$$m^* \left(\bigcup_{k=1}^{\infty} A \cap E_k \right) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

Thus since we have light inequalities both ways, we can conclude equality.

2.5 28

For any countable collection of disjoint measurable sets $\{E_1, E_2, \dots\}$ we can form an ascending chain of measurable sets by setting

$$A_k = \bigcup_{i=1}^k E_i$$

We have that

$$m^* \left(\bigcup_{s=1}^k E_s \right) = m^* \left(\bigcup_{s=1}^k A_s \right) = m^*(A_k) = \sum_{i=1}^k m^*(E_i)$$

And thus from the continuity property

$$m^* \left(\bigcup_{s=1}^{\infty} E_s \right) = m^* \left(\bigcup_{s=1}^{\infty} A_s \right) = \lim_{k \rightarrow \infty} m^*(A_k) = \sum_{i=1}^{\infty} m^*(E_i)$$