16.24 Only do a,b,c

a.

b.

c.

17.1

a. This is a subring since it is closed under multiplication:

$$\begin{matrix} 0 & b \\ c & d \end{matrix}$$

b.

c.

17.20 If $aR = R$ then since $1 \in R$ there must be $1 \in aR$ which means there must be some a^{-1} such that $aa^{-1} = 1$ which means a is a unit. For implication in the other direction, we have for any $x \in R$, assuming a is a unit with multiplicative inverse a^{-1}, we have $a^{-1}x \in R$ and $a(a^{-1}x) = x \in aR$. Therefore every element of R is an element of aR and so $R \subseteq aR$, and since R is closed under multiplication, for any $x \in R$, $ax \in R$, so $aR \subseteq R$ and so it follows

$$R = aR$$

A

a. We have

$$a^2 = a \Rightarrow a^2 - a = 0 \Rightarrow a(a-1) = 0$$

Since R is an integral domain, $a(a-1) = 0$ if and only if either a or $a-1$ is zero, and since the additive inverse is unique, that means a is either 1 or 0.

b. The idempotents are 1, 5, and 6.

c. For any $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ we have

$$(a,b)(a,b) = (a,b) \Rightarrow (a^2 - a, b^2 - b) = (0,0) \Rightarrow a^2 - a = 0, b^2 - b = 0$$

And since $\mathbb{Z}$ is an integral domain, that means $a, b \in \{0,1\}$ and so the idempotents are $(0,0), (1,1), (1,0), (0,1)$

B We can deduce the set of idempotents in S is a subset of the idempotents in R since $s \in S \Rightarrow s \in R$ and the conditions in either set is the same: $s^2 = s$.
As shown in problem Aa, the only idempotents in R are 1_R and 0_R
Subrings of an integral domain is an integral domain as well so S also has the property that the idempotents in S are 1_S and 0_R. Therefore we have.

$$\{0_S, 1_S\} \subseteq \{0_R, 1_R\}$$

From basic group theory we know the identity of a subgroup is equal to the identity of the containing group. Therefore $0_S = 0_R$ since 0 is the identity of the groups R, S over additition. So we have $1_S \neq 0_S \Rightarrow 1_S \neq 0_R$. The only other element in $\{0_R, 0_S\}$ that 1_S can be is 1_R

C

D True:
Consider the subring

$$S = 5\mathbb{Z}_{25} = \{0, 5, 10, 15, 20\}$$

This is a subring since we have $x|5 \Leftrightarrow x \in S$ and for any $a, b \in S$, $ab|5$ so $ab \in S$. We also have $a+b|5$ so $a+b \in S$. Thererfore S is closed under addition and multiplication and is finite, so it is a subring. S is isomorphic to $\mathbb{Z}_5$, let $\varphi : S \to \mathbb{Z}_5$ with $\varphi(5x) = x$. We have

$$\varphi(5x)\varphi(5y) = xy = \varphi(5xy)$$

and

$$\varphi(5x) + \varphi(5y) = x + y = \varphi(5(x+y))$$

So φ is a homomorphism. We have $\varphi(0) = 0, \varphi(5) = 1, \varphi(10) = 2, \varphi(15) = 3, \varphi(20) = 4$, and so φ is a bijections so an isomorphism.

E