**7**

(c) Consider the sequence of functions

$$f_n(x) = x^n$$

Notice that $|f_n|_{C^0} = 1$ and $|f_n|_{L^1} = \int_0^1 x^n dx = \frac{1}{n+1}$. This sequence establishes that $|\cdot|_{C^0}$ and $|\cdot|_{L^1}$ are not comparable since

$$|f_n|_{L^1} = \frac{|f_n|_{C^0}}{n+1}$$

becomes an arbitrarily small ratio

**8**

(a) $T$ is linear since we know integration is linear. $T$ is continous since for any convergent sequence $f_n$ under the uniform norm

$$\lim_{n\to\infty} \int_0^x f_n(t)dt = \int_0^x \lim_{n\to\infty} f_n(t)dt$$

and thus $T$ is sequentially continous.
We have that the norm of $T$ is given by

$$|T| = \sup_{f\in C^0} \frac{\left|\int_0^x f(t)dt\right|_{C^0}}{|f|_{C^0}}$$

For arbitrary $f \in C^0$ and letting $M = |f|_{C^0}$ we have

$$\frac{\left|\int_0^x f(t)dt\right|_{C^0}}{|f|_{C^0}} \leq \max_{x\in[0,1]} \frac{\int_0^x M dt}{M} = \max_{x\in[0,1]} x\frac{M}{M} = 1$$

Thus $|T| \leq 1$ notice that when $f$ is constant we get equality and thus $|T| = 1$
(b)

$$T(\cos(nt)) = \int_0^x \cos(nt)\ dt = \frac{\sin(nx)}{n}$$

(c) $K$ is bounded since $|f_n|_{C^0} \leq 1$ for all $n$.
$K$ is closed as follows:
We will show the only possible Cauchy sequence in $K$ is eventually constant and thus $K$ is closed. For any sequence $f_{n_k}$ we have two possibilities: (1) $n_k < N$ stays bounded or (2) $n_k$ becomes arbitrarily large.
(1) If $n_k < N$ then we have

$$\int_0^1 \cos nt - \cos mt\ dt = \frac{1}{n} - \frac{1}{m}$$

thus from the intermediate value theorem we can conclude

$$|\cos nt - \cos mt|_{C^0} \geq \left|\frac{1}{n} - \frac{1}{m}\right|$$

Thus we have for $n \neq m < N$

$$|f_n - f_m|_{C^0} = |\cos nt - \cos mt|_{C^0} \geq \min_{1 \geq m \neq n < N} \left|\frac{1}{n} - \frac{1}{m}\right| > 0$$

Thus in order to be Cauchy, the tail must be constant $(n = m)$
(2)

**17**
(a) We have

$$f(q) - f(p) = (\cos 2\pi, \sin 2\pi) - (\cos \pi, \sin \pi) = (1, 0) - (-1, 0) = (2, 0)$$

We have that

$$Df_\theta = (-\sin\theta, \cos\theta)$$

In order to satisfy $Df_\theta(q - p) = f(q) - f(p)$ we have the second coordinate equality

$$\cos\theta = 0$$

which can only happen if $\theta = 3\pi/2$. Plugging in $\theta = 3\pi/2$ does not yield the correct equality however

$$Df_\theta(q - p) = (\pi, 0) \neq (2, 0)$$

**18**
(a) It follows directly from the definition of differentiability. If the limit (letting $h \in \mathbb{R}^m$)

$$\lim_{h \to 0} \frac{f(p + h) - f(p)}{|h|}$$

exists (definition of differentiability), then letting $h = tu$ we have the limit

$$\lim_{t \to 0} \frac{f(p + tu) - f(p)}{|tu|} = \lim_{t \to 0} \frac{f(p + tu) - f(p)}{t}$$

exists
(b) Letting $u = (a, b)$ we have

$$\Delta_{(0,0)} f(u) = \lim_{t \to 0} \frac{(at)^3 bt}{(at)^4 + (bt)^2} = \lim_{t \to 0} \frac{a^3 b t^4}{a^4 t^4 + b^2 t^2} = \lim_{t \to 0} \frac{a^3 b}{a^4 + b^2 \frac{1}{t^2}} = 0$$

$f$ is not differentiable at $(0,0)$ however since letting $x = y^2$ we have

$$\lim_{x\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^6 y}{y^8 + y^2} = \lim_{y\to 0} \frac{1}{y + \frac{1}{y^5}} \to \infty$$

does not exist

**Additional Problem 1**

Notice that $\det(A)$ is a continous map. Also notice that a matrix $A$ is invertable if and only if $\det(A) \neq 0$. Thus

$$\mathcal{M} = \det^{-1}(\mathbb{R} \setminus \{0\})$$

$\mathbb{R} \setminus \{0\}$ is an open set, thus since the continous preimage of an open set is open, $\mathcal{M}$ is open $\mathcal{M}$ is dense since if we consider any noninvertable $A \in \mathbb{R}^{n^2} - \mathcal{M}$, we can choose a basis so that $A$ is upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

Since $A$ is singular we know there are some 0s on the diagonal
For any ball of radius $r$ around $A$ we can choose $\epsilon_1, \dots \epsilon_s$ such that we replace each 0 on a diagonal with an $\epsilon_i$ to get a new matrix $A'$. This new matrix is invertable since it has no zeros on its upper triangular form and the distance from $A$ to $A'$ is less than $r$ by choosing $\epsilon_1, \dots \epsilon_s$ small enough. Thus any ball centered around a matrix in the complement of $\mathcal{M}$ must intersect $\mathcal{M}$ so $\mathcal{M}$ is dense