

Problem 1.1

We have that $|G| = 385 = 11 \cdot 7 \cdot 5$. Therefore from Sylow's theorems we know there is an element of order 11, an element of order 7 and another of order 5. We have that the number of Sylow 11-subgroups, n_{11} , is equal to $1 + 11k$ for $k \geq 0$ and n_{11} divides $7 \cdot 5$ since its the index of the normalizer. Since $7, 5, 35 \not\equiv 1 \pmod{11}$ we know that $n_{11} = 1$. Therefore the subgroup of size 11 is normal. Thus

$$G \cong \mathbb{Z}/11\mathbb{Z} \rtimes H$$

Where H is a subgroup of order 35. We have already classified all groups of order pq in lecture (as well as being an example in Dummit and Foote, pg 181). Thus since 5 does not divide $7 - 1$, we know that $H \cong \mathbb{Z}/35\mathbb{Z}$. Now we can consider all the homomorphisms $\varphi : \mathbb{Z}/11\mathbb{Z} \rightarrow \text{Aut}(H)$. We have that ϕ must map the generator of $\mathbb{Z}/11\mathbb{Z}$ to an element of order 11. However the order of $\text{Aut}(H)$ is 34 and so no element has order 11 except the identity. Therefore the only possible homomorphism is the trivial $\varphi(g) = 0 \ \forall g \in \mathbb{Z}_{11}$. Thus the only possibility is

$$G \cong \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/35\mathbb{Z} \cong \mathbb{Z}/385\mathbb{Z}$$

Problem 1.2

(1) We have that $C_G(H) = H$, it is a commonly known fact that the diagonal matrices over a field commute since multiplying by a diagonal matrix scales the columns or rows by the diagonal.

To show that no other matrices are in the center, for any $A \in G$, if $A_{i,j} \neq 0$ for some $i \neq j$, then we choose $D \in H$ such that $D_{i,i} \neq D_{j,j}$. We have that

$$(DA)_{i,j} = D_{i,i}A_{i,j} \neq D_{j,j}A_{i,j} = (AD)_{i,j}$$

Thus $DA \neq AD$.

(2) The normalizer is all the permutation matrices multiplied by an element of H :

$$N_G(H) = \{PD \in G : D \in H, \exists \sigma \in S_3, P_i = e_{\sigma(i)}\} = S$$

Permutation matrices normalize H since left multiplication by a permutation matrix permutes the rows while while inverse right multiplication applies the permutation to the columns. Thus applying PDP^{-1} to $D \in H$ for $P \in S$ will permute the diagonals:

$$(PDP^{-1})_{j,j} = D_{\sigma(j),\sigma(j)}$$

While every other zero entry will stay zero. Thus we get another Diagonal matrix. No other matrix can be in the normalizer since if we have a matrix A where $A_{i,j} \neq 0$ and $A_{i,k} \neq 0$ with $j \neq k$, then for any $D \in H$, we have that

$$(DA)_{i,j} = D_{i,i}A_{i,j}, (DA)_{i,k} = D_{i,i}A_{i,k}$$

However if we choose $D' \in H$ with $D'_{j,j} \neq D'_{k,k}$ we have

$$(AD')_{i,j} = D'_{j,j}A_{i,j}, (AD')_{i,k} = D'_{k,k}A_{i,k}$$

Thus it $AD' \neq DA$ for all $D \in H$ since the entries of $(AD')_{i,j}$ and $(AD')_{i,k}$ are not $A_{i,j}$ and $A_{i,k}$ scaled by the same amount as the the same entries of every DA must satisfy.

(3) This quotient is isomorphic to S_3 .

Consider the homomorphism $\phi : N_G(H) \rightarrow S_3$ where for any $PD \in N_G(H)$ with $PD = \{D \in H, \sigma \in S_3, P_i = e_{\sigma(i)}\}$, $\phi(PD) = \sigma$. This is a homomorphism since multiplying by two permutation matrices is the same as composing the permutations of the matrices. ϕ is clearly surjective since we can choose a matrix $P \in N_G(H)$ for any $\sigma \in S_3$ where $P_i = e_{\sigma(i)}$ and so $\phi(P) = \sigma$. The kernel of ϕ is clearly the diagonal matrices H since they are the elements of $N_G(H)$ where they do not permute the columns at all. Thus we have that ϕ induces an isomorphism from $N_G(H)/H \rightarrow S_3$

Problem 1.3

Letting r signify the rotation element and s signify the reflexion, we have that the only normal subgroups of D_{12} are $\langle r \rangle$, $\langle r^2 \rangle$, $\langle r^3 \rangle$, $\langle s, r^2 \rangle$, and $\langle rs, r^2 \rangle$. We have that $\langle r^2 \rangle$ and $\langle r^3 \rangle$ are not maximal since they are contained in $\langle r \rangle$. Thus the composition series are

$$1 \triangleleft \langle r^3 \rangle \triangleleft \langle r \rangle \triangleleft D_{12}$$

$$1 \triangleleft \langle r^2 \rangle \triangleleft \langle r \rangle \triangleleft D_{12}$$

$$1 \triangleleft \langle r^2 \rangle \triangleleft \langle s, r^2 \rangle \triangleleft D_{12}$$

$$1 \triangleleft \langle r^2 \rangle \triangleleft \langle rs, r^2 \rangle \triangleleft D_{12}$$

(I checked the subgroups of $\langle s, r^2 \rangle$ and $\langle rs, r^2 \rangle$ to find the only normal subgroup is $\langle r^2 \rangle$)

Problem 1.4

For any group G with $|G| = p^2q$ we have from Sylows theorems there exists a subgroup N_{p^2} of order p^2 . We know that the index of the normalizer of the p -group, n_p , satisfies $n_p = 1 + pk, k \geq 0$ and $n_p|q$. Since q is prime n_p is either 1 or q . If n_p is 1, the p -group is normal.

Thus we have that $G \cong \mathbb{Z}/q \rtimes N_{p^2}$ where N_{p^2} is the group of size p^2 , thus we have that $|G/N_{p^2}| = q$ and so must be cyclic, hence abelian. In lecture we established that all groups of order p^2 are abelian. Thus we have the series

$$1 \triangleleft N_{p^2} \triangleleft G$$

If $n_p = q$ then we have that $q \equiv 1 \pmod{p}$. From Sylows Theorems we know that there is a q -subgroup N_q of order q with the index of the normalizer, n_q , satisfying $n_q \equiv 1 \pmod{q}$ and $n_q|p^2$. Thus n_q is either 1, p , or p^2

n_q cannot be p since $q \equiv 1 \pmod{p}$ and $p \equiv 1 \pmod{q}$ is not possible for any primes p, q . (This is because one has to be smaller than the other and thus will themselves mod the

other)

$n_q = p^2$ leads to a simple case from similar reasoning. We have that $p^2 \equiv 1 \pmod{q}$ so $p \equiv \pm 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$. We know that $p \equiv 1 \pmod{q}$ is not possible, if $p \equiv -1 \pmod{q}$. If $p < q$ then $p = q - 1$ so p must be even, so $p = 2$, and $q = 3$. If $q < p$ then $q = 1$ which is a contradiction. We will handle the $p = 2, q = 3$ case in the end.

If n_q is 1 then $N_q \triangleleft G$ so we have the composition series

$$1 \triangleleft N_q \triangleleft G$$

N_q is of prime order so abelian, G/N_q is of order p^2 . All groups of order p^2 are abelian as is established in lecture

Finally if $q = 3, p = 2$ then G is a group of order 12. Dummit and Foote classify all groups of order 12, two are abelian (and thus solvable). The rest are the following: D_{12} which has the composition series with abelian factors

$$1 \triangleleft \langle r \rangle \triangleleft D_{12}$$

A_4 which has the composition series

$$1 \triangleleft \langle (12)(34), (13)(24) \rangle \triangleleft A_4$$

And finally the Dicyclic group $D = \langle a, b, c | a^3 = b^2 = c^2 = abc \rangle$ which has the composition series

$$1 \triangleleft \langle b, c \rangle \triangleleft D$$

Problem 1.5

(1) We have that $|GL_3(\mathbb{F}_2)| = (2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 3 \cdot 2^3$ therefore a Sylow 7-group has order 7. The size of $GL_3(\mathbb{F})$ was determined by counting all the possible nonzero vectors for the first column. There are 2^3 possible vectors and 1 zero vector. Then counting all the nonzero vectors in the 2nd column that is not a linear combination of the first vector. There are 2 possible vectors that is zero or the previous vector. Finally counting all the independent vectors for the third column. There are 4 vectors that are a linear combination of the two previous vectors.

We have that the elt

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has order 7. I have verified that $A^7 = 1$ and thus $\langle A \rangle$ is a 7-subgroup.

(2) We can consider the action of $G = GL_3(\mathbb{F}_2)$ on $S = \mathbb{F}_2^3 / \{0\}$ where the action is applying the linear transformation to the vector in S . Thus we know that G is a subgroup of S_7 since it permutes 7 objects. We have that the elt A from (1) is of order 7 and thus is isomorphic

to an element of order 7 in S_7 . The only elts of order 7 in S_7 are the 7 cycles so A would map under isomorphism to $(1\ 2\ 3\ 4\ 5\ 6\ 7)$.

From Cauchy's theorem we know that there is an element B of order 2 in G as well. B cannot be in the group generated by A since no elt in $\langle A \rangle$ has order 2. Thus we can add the image of B as a generator.

If B maps to a two cycle, then we get the group generated by a seven cycle and a two cycle which has order 5040 and so is much larger than G .

If B maps to three disjoint 2 cycles then again we get that the group generated by a seven cycle and three disjoint two cycles has order again 5040.

If B maps to two disjoint two cycles we have four nonisomorphic options. Either B maps to $(1\ 4)(2\ 3)$, $(1\ 7)(2\ 3)$, $(1\ 2)(4\ 5)$ or $(1\ 5)(2\ 3)$. For the $(1\ 4)(2\ 3)$, $(1\ 7)(2\ 3)$, $(1\ 2)(4\ 5)$ cases, each of the resulting subgroups have order 2520 which is much larger than G , finally we have that the group generated with $(1\ 2)(4\ 5)$ is the same size as G . Therefore the only option for G to be isomorphic to is the group generated by $(1\ 2)(4\ 5)$ and $(1\ 2\ 3\ 4\ 5\ 6\ 7)$.

The way I calculated the sizes of these groups was with programming in Sage.

Problem 1.6

(1) We can show that $k[x, y, z, w]/(xy - zw)$ is an integral domain, and thus $(xy - zw)$ is prime

Consider the homomorphism $\phi : k[x, y, z, w] \rightarrow k[\frac{zw}{y}, y, z, w]$. Where $k[\frac{zw}{y}, y, z, w]$ is the ring of polynomials in $k[x, y, z, w]$ with $\frac{zw}{y}$ plugged into x in every polynomial. We define $\frac{zw}{y}$ by localizing $k[x, y, z, w]$ at y . Localizations of integral domains are integral domains, so we know that $k[\frac{zw}{y}, y, z, w]$ is an integral domain.

We define ϕ with the cononical mapping $\phi(x) = \frac{zw}{y}$, $\phi(y) = y$, $\phi(z) = z$, $\phi(w) = w$ and with ϕ the identity on k . ϕ is clearly a homomorphism since

$$\phi(f(x, y, z, w)g(x, y, z, w)) = f(\frac{zw}{y}, y, z, w)g(\frac{zw}{y}, y, z, w) = \phi(f)\phi(g)$$

$$\phi(f(x, y, z, w) + g(x, y, z, w)) = f(\frac{zw}{y}, y, z, w) + g(\frac{zw}{y}, y, z, w) = \phi(f) + \phi(g)$$

ϕ is surjective since every monomial $\left(\frac{zw}{y}\right)^{n_1} y^{n_2} z^{n_3} w^{n_4}$ is mapped to by $x^{n_1} y^{n_2} z^{n_3} w^{n_4}$. If we show the kernel of ϕ is the ideal $(xy - zw)$ then we have shown that $k[x, y, z, w]/(xy - zw)$ is an integral domain.

We have that for any $p(x, y, z, w) \in k[x, y, z, w]$ with $\phi(p) = 0$ then

$$p(\frac{zw}{y}, y, z, w) = 0$$

We can write p as

$$p = xyp_1(x, y, z, w) + zw p_2(x, y, z, w) + c_1x + c_2y + c_3z + c_4w + c_5$$

Plugging in $x = \frac{zw}{y}$, in order for $p(x, y, z, w) = 0$, we have that $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ since no other terms in $xyp_1(x, y, z, w)$ and $zw p_2(x, y, z, w)$ will cancel with these terms since

each of term of $xyp_1(x, y, z, w)$ and $zwp_2(x, y, z, w)$ has a higher degree in (zw) .

Thus we have $p = xyp_1(x, y, z, w) + zwp_2(x, y, z, w)$. Now when plugging in $x = \frac{zw}{y}$ we get $zwp_1 + zwp_2 = 0$ so $zw(p_1 + p_2) = 0$ and therefore $p_1 = -p_2$ and thus $p = p_1(xy - zw)$ so $(xy - zw)$ divides. Thus the ideal $(xy - zw)$ is the kernel. p

(2) x is irreducible since when considering the degree of a polynomial with respect to any of the variables, degrees of a product of polynomials is equal to the sum of the degrees. In order for $x = f(x, y, z, w)g(x, y, z, w)$ It would have to be the case that one of the polynomials was degree zero with respect to all of the variables, and thus is an elt of k so a unit.

x is not prime since x divides zw in the quotient since $xy - (xy - zw) = zw$, however x does not divide z or w . x does not divide z or w because the argument for showing x is irreducible works for showing that both z and w are irreducible.