Exersise 1.1

We have the isomorphism

$$\phi : \mathbb{R}[x]/(x^2 + x + 1) \to \mathbb{R}[\zeta_3]$$

$$x \to \zeta_3$$

Where $\zeta_3 = 1/2 + \frac{\sqrt{3}}{2}i$ is the third root of unity. This is an isomorphism since $x^2 + x + 1$ is the minimal polynomial of ζ_3 over $\mathbb{R}$.
We have that $\mathbb{R}[\zeta_3] \cong \mathbb{C}$ since by definition $\mathbb{C} = \mathbb{R}[i]$, $\zeta_3 \in \mathbb{C}$ so $\mathbb{R}[\zeta_3] \subseteq \mathbb{C}$ and $i = (\zeta_3 - 1/2)\frac{2}{\sqrt{3}}$ so $\mathbb{C} \subseteq \mathbb{R}[\zeta_3]$

Exersise 1.2

Let $\alpha = \sqrt{2} + \sqrt{3}$. It is clear $\alpha \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ so $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. We have

$$\frac{\alpha^3 - 9\alpha}{2} = \frac{11\sqrt{2} + 9\sqrt{3} - 9(\sqrt{2} + \sqrt{3})}{2} = \sqrt{2}$$

$$\sqrt{3} = \alpha - \frac{\alpha^3 - 9\alpha}{2}$$

So $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\alpha) \Rightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$, thus $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\alpha)$ so α is a primitive element

Exersise 1.3

We have the factorization

$$x^5 + x^2 - x - 1 = (x + 1)(x - 1)(x^2 + x + 1)$$

Where $x^2 + x + 1$ is irriducible since the roots are $\pm\zeta_3 \notin \mathbb{Q}$. Thus either $\alpha = \pm 1$ which yields a degree 1 extension $\mathbb{Q}[\alpha] = \mathbb{Q}$, or $\alpha = \pm\zeta_3$ which yields a degree 2 extension since 2 is the degree of the minimal polynomial of α: $x^2 + x + 1$

Exersise 1.4

For any choice of a, we have that α is a root of the following

$$m_0(x) = (x - \alpha)(x + \alpha)(x - \frac{1}{\alpha})(x + \frac{1}{\alpha}) = (x^2 - \alpha^2)(x^2 - \frac{1}{\alpha^2}) = x^4 - (4a + 2)x^2 + 1 \in \mathbb{Q}$$

Thus the minimal polynomial must divide m_0 (while having α as a root), which yields the possiblities besides m_0

$$m_1(x) = (x - \alpha)(x \pm \frac{1}{\alpha}) = x^2 - (\alpha \pm \frac{1}{\alpha})x \pm 1$$

$$m_2(x) = (x - \alpha)(x + \alpha) = x^2 - \alpha^2$$

$$m_3(x) = x - \alpha$$

The minimal polynomial of α is the smallest degree polynomial of the ones listed above with coefficents in $\mathbb{Q}$. Each polynomial is possible.
If $\alpha \in \mathbb{Q}$ then $m_\alpha = m_3$. Such is the case when $a = 0$.
If $\alpha^2 \in \mathbb{Q}$ and conditions were not met above, then $m_\alpha = m_2$ this is the case iff $\sqrt{a^2 + a} \in \mathbb{Q}$ since $\alpha^2 = 2a + 2\sqrt{a^2 + a} + 1$. This is possible for example when $a = 1/3$
For $\alpha - \frac{1}{\alpha}$, notice that $1/\alpha = \sqrt{a+1} - \sqrt{a}$ so $\alpha \pm 1/\alpha = \sqrt{a}$ or $\sqrt{a+1}$. Thus if either a or $a + 1$ are squares in $\mathbb{Q}$ and conditions were not met above then $m_\alpha = m_1$.
And finally if none of the above were true then $m_\alpha = m_0$

Exersise 1.5

We know that $\alpha^2 \in k(\alpha)$ so $k(\alpha^2) \subseteq k(\alpha)$
Since $[k(\alpha) : k]$ is odd, the minimal polynomial over k, m_α, has odd degree $(2n - 1)$:

$$m_\alpha(\alpha) = \alpha^{2n-1} + c_{2n-2}\alpha^{2n-2} + \cdots + c_2\alpha^2 + c_1\alpha + c_0 = 0$$

Multiplying by α on both sides in K yields

$$\alpha^{2n} + c_{2n-2}\alpha^{2n-1} + \cdots + c_2\alpha^3 + c_1\alpha^2 + c_0\alpha = 0$$

Subtracting all odd degree terms:

$$\alpha^{2n} + \cdots + c_1\alpha^2 = -c_{2n-2}\alpha^{2n-1} - \cdots - c_2\alpha^3 - c_0\alpha$$

Factoring out α and relabeling constants $k_i = -c_i$:

$$\alpha^{2n} + \cdots + c_1\alpha^2 = \alpha(k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0)$$

We have α in terms of a ratio of polynomials in α^2:

$$\alpha = \frac{\alpha^{2n} + \cdots + c_1\alpha^2}{k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0} \in k(\alpha^2)$$

We know that this is well defined, ie $k_{2n-2}\alpha^{2n-2} + \cdots + k_0 \neq 0$ is invertable, since it is a non-zero polynomial $f(\alpha) = k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0$ of degree less than m_α and therefore cannot be zero otherwise we would contradict minimality of m_α. f is nonzero since $k_0 = -c_0$ is nonzero since if $c_0 = 0$ then

$$x | m_\alpha(x) = x^{2n-1} + c_{2n-2}x^{2n-2} + \cdots + c_2x^2 + c_1x$$

which contradicts m_α being irriducible.
Thus $k(\alpha) \subseteq k(\alpha^2)$ so $k(\alpha) = k(\alpha^2)$

Exersise 1.6

Since A is a subring of K we know A is an integral domain. All we must show is that for any $\alpha \in A$, $\alpha^{-1} \in A$.

We have that $k[\alpha] \subseteq A$ where $k[\alpha]$ is the smallest subring of K to contain k and α. It turns out that $k[\alpha] = k(\alpha)$ and therefore $\alpha^{-1} \in k(\alpha) \subseteq A$, so A is a field.
The reason $k[\alpha] = k(\alpha)$ is since $\alpha \in K$ and K algebraic over k, there is a minimal polynomial for α, $m_\alpha(x) \in k[x]$. $k[\alpha]$ must contain all linear powers of α over k, with $m_\alpha(\alpha) = 0$. From this we have the isomorphism $k[\alpha] \cong k[x]/(m_\alpha(x))$ (this isomorphism is established more rigorously in Dummit and Foote's section of Field Theory) which is a field since $(m_\alpha(x))$ is maximal. Thus $k[\alpha]$ is a field so $k(\alpha) \subseteq k[\alpha]$. Since $k(\alpha)$ is a ring we know $k[\alpha] \subseteq k(\alpha)$, so $k[\alpha] = k(\alpha)$