1 If there is a given bipartite graph $G = X \cup Y$ with the given degree sequence. With X, Y the vertix sets that are not connected within eachother. We know that the sum of the degrees of verticies in X must equal the sum of the degrees of verticies in Y. However this cannot be the case, since the sum of degrees of verticies that contains the degree 5 will be 2 modulo 3 since all the other degrees are 0 mod 3, and the sum from the other set will be 0 mod 3. Because the two sums are unequal mod 3, they must be unequal in general. And so no bipartite graph with the given degree sequence can exist.

2 We can induct on d:
The base case is where $d = 1$, in which case there is 1 perfect matching (as proven in class. For the inductive step, given a bipartite graph $G = (X, Y)$ with all verticies having degree $d + 1$. We have proved in class that such a graph has a perfect matching M. If we consider the graph $H = G - M$ where we remove all the edges in M from G, by the definition of a perfect matching, every vertex in G is connected to exactly 1 edge in M, therefore removing M would lower the degree of every vertex by 1. Therefore H is a bitartite graph with all verticies having degree d, which by the inductive hypothesis has d distinct perfect matchings. Since M is does not intersect with the set of edges in H, M cannot intersect with any of the perfect matchings of H. Therefore these matchings of H and M make up $d + 1$ distinct perfect matchings of G

3 For any nonempty subset $S \subseteq X$, since every vertex has degree ≥ 4, we know that $|N(S)| \geq 4$ since all verticies in X has 4 neighbors. Therefore the Hall thm conditions is satisified for $|S| < 5$. For $|S| = 5$ we have $S = X$. Since all verticies in Y have degree ≥ 1 we know that $|N(X)| = |Y| > |X|$. Therefore the Hall thm is satisified for all subsets of X which means there exists a perfect matching of X to Y

4 Letting $G = (M, W)$ be the bipartite graph where M are the nodes of men, W the nodes of women, and an edge represents whether the man and woman know each other. If we make a new graph G' by adding two verticies to W and edges between these new two verticies and every node in M, then the conditions in the problem imply that for every $2 \leq k \leq N$, every k nodes in X have at least k neighbors in G' (we add two since every node in X is connected to the two new added nodes in W). Every subset of size 1 of X in G' has a neighbor set of size ≥ 1 as well since every node in X is connected to the added nodes. Therefore the conditions of Halls thm are satisified for G' and so there exists a perfect matching M from X to $W \cup \{v_1, v_2\}$ where v_1, v_2 are the added verticies. When we remove v_1, v_2 and the edges connected to them from M, we also remove two verticies in X from M, and so we are left with a matching M' with $N - 2$ verticies in X and edges and verticies contained within G, which corresponds to the desired result: a matching of $N - 2$ of the men to the women they know.

5

a.

b.

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