Exersise 20 If $a_n \to a$ and $a_n \to b$ with $b \neq a$, let $\epsilon = \frac{|b-a|}{2} > 0$. By definition there exists N such that for all $k > N$, $|a_k - a| < \epsilon$. However by triangle inequality, for any $k > N$ we have

$$|b-a| = |b - x_k + x_k - a| \leq |b - x_k| + |x_k - a|$$

and so since $|a - x_k| < \frac{|b-a|}{2} > 0$

$$|b-a| - \frac{|b-a|}{2} < |b-a| - |x_k - a| \leq |b - x_k|$$

So we have $|b - x_k| > \frac{|b-a|}{2} = \epsilon$ for all $k > N$. Thus $x_n \not\to b$, which is a contradiction

Exersise 1 Let $x \cdot 1^* = A|A'$, $1^* = B|B'$ and $x = C|C'$. By definition

$$A = \{bc : b, c \geq 0, b \in B, c \in C \text{ or } a : a < 0, a \in B \cup C\}$$

Let $a \in A$ if $a < 0$ then $a \in C$ since $x > 1^* > 0^*$ we have $S = \{a \in \mathbb{Q} : a < 0\} \subset B \subset C$
If $a \geq 0$ then $a = bc$ for some $b \in B, c \in C$. By definition of 1^* we know $b < 1$ and so $bc < c$ and therefore $bc = a \in C$ since $a \in \mathbb{Q}$ and if $a \in \mathbb{Q} - C = C'$ then $a < c$ which contradicts the axiom that elements of C' are greater than elements of C. Therefore we have $A \subseteq C$
For any $c \in C$, if $c < 0$ we know $c \in B$ and $c \in C$ so $c \in A$.
If $c \geq 0$ we have that since C does not contain an upper bound, there must exist a $c' \in C$ such that $c < c'$ therefore $\frac{c}{c'} \in \mathbb{Q}$ and < 1. Therefore $\frac{c}{c'} \in B$ and we have $c' \frac{c}{c'} = c \in A$. Therefore $C \subseteq A$. Thus we have equality, $C = A \Rightarrow x \cdot 1^* = x$

Exersise 2

a. If a_n is Cauchy then we know that a_n is bounded, so $\forall n, a_n < B$ for some $B \in \mathbb{R}$. For a given $\epsilon > 0$ there is a $N > 0$ such that $j, k > N$ implies $|a_j - a_k| < \frac{\epsilon}{2B}$. We have that (since $|a_j + a_k| < 2M$)

$$|a_j^2 - a_k^2| = |a_j - a_k||a_j + a_k| < \frac{\epsilon}{2B} 2B$$

And thus $|a_j^2 - a_k^2| < \epsilon$, so a_n^2 is Cauchy

b. If a_n^2 is Cauchy, it is bounded. We have for some $C \in \mathbb{R}$, $C^2 > a_n^2 \geq c^2$. Given $\epsilon > 0$, there is a N such that for any $j, k > N$ we have $|a_j^2 - a_k^2| < 2C\epsilon$. Again we have

$$|a_j^2 - a_k^2| = |a_j - a_k||a_j + a_k| < \frac{\epsilon}{2C} |a_j + a_k|$$

and we have $|a_j + a_k| < 2C$, so

$$|a_j - a_k| < \epsilon$$

Thus a_n is Cauchy

Exersise 3 Base Case:
$n=1$, $1+c=1+c$
Inductive Step:
assuming nth case we have

$$(1+c)^{n+1}=(1+c)^n+c(1+c)^n\geq 1+nc+c(1+c)^n\geq 1+nc+c=1+(n+1)c$$

And thus the $n+1$ case is true

Exersise 4

a. Since $r>0$ we can write r as $r=1+c$ with $c>0$. Suppose such an upper bound x existed, then we have $x\leq 1+\lceil (x-1)/c\rceil c$. However as proven in problem 3, if we let $n=\lceil (x-1)/c\rceil+1$ then $r^n\geq 1+nc>1+(n-1)c=x$

b. Since $r>0$, we know $\frac{1}{r^n}>0$. For $\epsilon>0$, we know from part a that there exists N such that $r^N>\frac{1}{\epsilon}$, and for all $n>N$, $r^n>\frac{1}{\epsilon}$ since $r^{n-N}>1$ and $r^n=r^Nr^{n-N}$. Therefore we have $|r^n|>|\frac{1}{\epsilon}|$, and so $|\frac{1}{r^n}|<\epsilon$. Thus $\frac{1}{r^n}\to 0$

Exersise 5 From the Triangle Ineq:

$$|y|+|x-y|\geq|y+x-y|=|x|$$

So

$$|x-y|\geq|x|-|y|$$

This argument works relabeling x and y, so $|y-x|\geq|y|-|x|$. Depending on which is larger, we know $||x|-|y||=|x|-|y|$ or $|y|-|x|$, either way, we have

$$|x-y|=|y-x|\geq||x|-|y||$$

Exersise 6 If $x=\lambda y$, we know that $|\langle\lambda y,y\rangle|=|\lambda\langle y,y\rangle|=|\lambda|y|^2|=|x||y|$.

If $|\langle x,y\rangle|=|x||y|$, we can define $Q(t)=\langle x+ty,x+ty\rangle$. By bilinear properties of the inner product we have

$$\begin{aligned}Q(t)=\langle x+ty,x+ty\rangle&=\langle x,x+ty\rangle+\langle ty,x+ty\rangle\\&=\langle x,x\rangle+t\langle x,y\rangle+t\langle y,x\rangle+t^2\langle y,y\rangle\end{aligned}$$

By assumtion that $|\langle x,y\rangle|=|x||y|$

$$\begin{aligned}&=|x|^2+2t|x||y|+t^2|y|^2\\&=(|x|+t|y|)^2\end{aligned}$$

Letting $t=-\frac{|x|}{|y|}$ we have

$$Q(-\frac{|x|}{|y|})=\left(|x|+-\frac{|x|}{|y|}|y|\right)^2=0$$

Therefore we have

$$\langle x+-\frac{|x|}{|y|}y,x+-\frac{|x|}{|y|}y\rangle=0$$

Which is the case if and only if $x-\frac{|x|}{|y|}y=0\Rightarrow x=\frac{|x|}{|y|}y$