4.9 If the positive rationals over multiplication were cyclic, there would be some generator $x \in \mathbb{Q}$ such that for any $y \in \mathbb{Q}$, $x^n = y$ where $n \in \mathbb{Z}$. This is not possible, consider the number

$$y = \frac{x+1}{2} \in \mathbb{Q}^+$$

we know $y \in (1, x)$ but for any $n \in \mathbb{Z}$, $x^n \notin (1, x)$. And so $(\mathbb{Q}^+, \cdot)$ cannot be cyclic.

4.10

a. We can write any integer in the form $7n + r$, where r is the number modulo 7. And so for any two numbers we have:

$$(7k_1 + k_2)(7g_1 + g_2) = 7(k_1 + g_1) + g_2 + k_2 = (7g_1 + g_2)(7k_1 + k_2)$$

and so the operation is commutative. Similarly since multiplication is associative, we know multiplication modulo a number will also be associative.
We also know 1 is still an identity element
Looking at each element, we have

$$2 \odot 4 = 1, \quad 3 \odot 5 = 1, \quad 6 \odot 6 = 1$$

and so every element has an inverse. Therefore the collection under $\odot$ is a group

b. The group is cyclic, consider 3:

$$3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5, 3^6 = 1$$

All elements are accounted for.

4.14 Consider an infite set X, with $P(X)$ being its powerset.
An example of an infinite group where all elements have finite order is

$$(P(X), \triangle)$$

As we have proven earlier, powersets under symmetric difference are groups.
The order of any $A \in P(X)$ is 2 since $A \triangle A \triangle A = A$, but $P(X)$ is infinite.

4.24 $o(y) = 2$ implies $y = y^{-1}$.
We have

$$yxy = x^2$$

$$yxyyxy = x^2x^2$$
$$y(x^2)y = x^4$$
$$yyxyy = x^4$$

And so

$$x = x^4$$

so the order of x must be 3

4.32 Let $z = xy$.
Observe that since G is abelian, $z^k = x^k y^k$. Therefore

$$z^k = e \Leftrightarrow x^k y^k = e$$

$x^k y^k = e$ implies one of two cases. The first case is that $x = y^{-1}$ which means $m = n$ so their greatest common multiple is $m = n$, so we would be done since x, y have order m. The other case is $x^k = e, y^k = e$ which is if and only if k is a multiple of both n and m, and so the least common multiple of m, n would be the order of z.

4.33

a. We have for any $x = (a_x, b_x, c_x)$ in the group described:

$$x^3 = (a_x + a_x + a_x, b_x + b_x + b_x, c_x + (c_x + c_x + a_x b_x) + (a_x + a_x)b_x) \mod 3$$
$$= (3a_x, 3b_x, 3c_x + 3a_x b_x) \mod 3$$

Since all these terms are multiples of 3, we have:

$$x^3 = (0, 0, 0) = e$$

Therefore for any x, y in the group, we have

$$(xy)^3 = e = e\dot{e} = x^3y^3$$

And similarly

$$(xy)^4 = (xy)^3xy = xy = xx^3yy^3 = x^4y^4$$

However the group is not abelian, consider

$$(1, 0, 0)(1, 1, 0) = (2, 1, 0) \neq (2, 1, 1) = (1, 1, 0)(1, 0, 0)$$

b. We have the following logic for any x, y in the group described:

$$(xy)^{n+2} = x^{n+2}y^{n+2}$$
$$(xy)^{n+1}xy = x(x^{n+1}y^{n+1})y$$

Since $(xy)^{n+1} = x^{n+1}y^{n+1}$

$$x^{n+1}y^{n+1}xy = x(xy)^{n+1}y$$

Applying x^{-1} on the left and y^{-1} on the right for both sides:

$$x^n y^{n+1} x = (xy)^{n+1}$$

$$x^n y^n yx = (xy)^n xy$$

And since $x^n y^n = (xy)^n$, they have the same inverse. Applying this inverse to the left for both sides yeilds the property that G must be commutative:

$$yx = xy$$

5.6

a. If $m|n$, let x be the genorator of G. We know

$$x^{n/m}$$

is an element of G since $n/m \in \mathbb{Z}$, and the element has order m.
For the other direction, if there is some element y with order m,
We know for some k we have $x^k = y$ (where x is the genorator of G) and so if we consider the cyclic subgroup $H =< y >$, we know

$$|H| = o(y) = \frac{n}{(k,n)} = m$$

(where (k,n) denotes the gcd of k,n) and so $m(k,n) = n$, which means m divides n.

b. As established in part a, there is an element of order $m \in G$ iff m divides $|G|$. So the possible orders are

$$\{1, 2, 4, 5, 8, 10, 20, 40\}$$

5.7 If $(m,n) = 1$ then x^m has order n since $(x^m)^k = e$ if and only if n divides k, and so x^m is a genorator of a subgroup H that is the same size as G, so $H = G$.
Conversly if x^m has order n, since both n and m divides $(m,n)n$, $(m,n)n$ must be the order of x^m, and so $(m,n)n = n$ so $(m,n) = 1$

5.17 Counterexample:
Consider $G = \mathbb{Z}$ mod 9 under addition. So

$$G = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

and let

$$H = \{0, 1, 2, 7, 8\}$$

We have that H is closed under inverses since $1 + 8 = 0$ and $2 + 7 = 0$. However H is not closed under addition so cannot be a group: $1 + 2 \notin H$

5.18

a. If H, J are proper subgroups of G such that $H \cup J = G$, then we will reach the following contradiction:
We know that $J/H \neq \emptyset$ and $H/J \neq \emptyset$ since that would mean $H \cup J = H$ or $H \cup J = J$ which means H or J are not proper subgroups
Therefore we know there is some elements $h \in H : h \notin J$, and $j \in J : j \notin J$
Now we have $hj \in G$ so $hj \in H \cup J$, however if $hj \in H$ then $h^{-1}hj \in H$ so $j \in H$ which is not possible, and similarly if $hj \in J$ then $hjj^{-1} \in J$ so $h \in J$ which is not possible.
Therefore we have a contradiction since hj must be in $H \cup J$ but it cannot be in either H or J.

b. The Klien's four group is an example:

$$\{e, a, b, c\}$$

such that $a^2 = b^2 = c^2 = e$ Therefore the union of the proper subgroups:

$$\{e, a\}, \{e, b\}, \{e, c\}$$

is equal to the original group.

8.7 Consider the elements $(1, 2)$ and $(2, 3)$ which are well defined in S_n for all $n \geq 3$. We have

$$(1, 2) \circ (2, 3) = (1, 2, 3) \neq (2, 3) \circ (1, 2) = (1, 3, 2)$$

And so S_n is not commutative for $n \geq 3$

8.10

a. Since these cycles are disjoint, we know they commute. Therefore we know that we can reorder f^k as so:

$$f^k = f_1^k f_2^k \dots f_m^k$$

And we know each of these terms can be the identity if and only if k is a multiple of the orders of each of the cycles. Therefore, the least common multiple of the orders of each of the cycles must be the order of f

b. This permutation written in cycle notation is

$$(1, 6, 4, 9)(2, 7, 11)(3, 5, 8)(10, 12)$$

Since we know the order of this permutation is the lcm of the order of each of these cycles, we only need to calculate each of the cycles orders.
We know that the order of a cycle is the length of a cycle, so the orders of each of the cycles above are $\{4, 3, 3, 2\}$ and the lcm of these 4 numbers is 12 so the order of the original permutation is 12

8.11

a. Let

$$x = (1, 2, 3, 4, 5)$$
$$y = (1, 6, 7, 8, 9)$$

We have that

$$xy = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

Which means that $o(x) = o(y) = 5$ and $o(xy) = 9$

b. Each element must have order equal to the lcm of the disjoint cycles that make up the element. We know that the order of the cycle is equal to the size of the cycle. Therefore we are looking for the maximum lcm of a collection of sizes that add up to ≤ 9. We will call this collection S
We notice that each of the numbers in S must be relatively prime to each other, since otherwise we could factor out a factor from two of these numbers to get a smaller sum, but the same lcm.
Potential options from this reasoning yields that S can be

$$\{2, 3\}, \{4, 3\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{2, 7\}$$

And from calculating the lcm of each we find that $S = \{4, 3\}$ with the lcm $= 20$. And so the maximum order of an element in S_9 is 20.