Math 402, Fall 2016 *HOMEWORK #3* *Peter Gylys-Colwell*

2.1

a. $\mathbb{R}^+$ is not a group since none of the elements have inverses besides $\{0\}$. The inverse of $a \in \mathbb{R}^+$ is $-a \notin \mathbb{R}^+$

b. This is a group, 0 is the identity, the inverse of $a \in 3\mathbb{Z}$ is $-a$ and 3 divides $-a$ since 3 divides a so $-a \in 3\mathbb{Z}$, and addition is assosiative. Therefore $3\mathbb{Z}$ under addition satisfies all the requirements of a group

c. This is not a group since there is no identity. $a * b \geq 0$ for all $a, b \in \mathbb{R}$, therefore if $a < 0$, $a * b \neq a$ for all $b \in \mathbb{R}$

d. This is a group since 1 is the identity, -1 and 1 are their own inverses and multiplication is assosiative.

e. This is a group since for any $a, b \in \mathbb{Q}$ such that $\sqrt{a}, \sqrt{b} \in \mathbb{Q}$, $\sqrt{ab} \in \mathbb{Q}$. The inverse of a would be $\frac{1}{a}$ which by the way the square root works, $\sqrt{\frac{1}{a}} \in \mathbb{Q}$. The identity would be 1 and multiplication is associative. Therefore all the conditions are met for this set under multiplication to be a group

f. This is not a group since the operation is not associative. Subtraction is not associative.

g. This is a group, we have $(0, 1)$ be the identity, for a given (a, b) in the set, $(-a, \frac{1}{b})$ is its inverse, and addition and multiplication are associative and so $*$ is associative too

h. This is a group, 0 is the identity. The operation is accociative:

$$a * (b * c) = a * (b + c - bc) = a + b + c - a(b + c - bc)$$

$$= a + b + c - ab - ac + abc = (a + b - ab) + c - c(a + b - ab) = (a * b) * c$$

And the inverse of any $a \in \mathbb{R} - \{1\}$ is $\frac{a}{a-1}$ since

$$a * \frac{a}{a-1} = \frac{1}{a-1}(a(a-1) + a - a^2) = 0$$

which is well defined since $a - 1 \neq 0$. Therefore all the conditions of being a group are satisfied

i. This is a group, 1 is the identity, addition is associative, and the inverse of a given $a \in \mathbb{Z}$ is $-a$: $a - 1 - (a - 1) = 0$

2.3 Under intersection $P(X)$ is a group, but not under union. As established in an earlier homework assignments, for a given $A \in P(X)$, the operation

$$F(B) = A \cup B$$

is not injective and therefore does not have an inverse, and therefore the element A cannot have an inverse under the union operation.
For intersection, we have proven in a previous homework that the intersection operation is assosiative. We have the empty set be the identity since

$$\emptyset \cap A = A \cap \emptyset = \emptyset$$

and for the inverse of a given element A we have

$$A^{-1} = X - A$$

since

$$A \cap (X - A) = (X - A) \cap A = \emptyset$$

2.5 No, c has no inverse, there is no element c^{-1} in the table such that $c^{-1}ca = a$

2.7

	a	b
a	a	b
b	b	a

a is the identity, and b is it's own inverse

2.8 Yes it forms a group. The identity function is $f(x) = 1$ since

$$1g(x) = g(x)1 = g(x)$$

for all functions $g(x)$. For any function $f(x)$ where $f(x) \neq 0 \ \forall x \in \mathbb{R}$, we define its inverse to be $\frac{1}{f(x)}$. Since $f(x) \neq 0$, the inverse is well defined. And multiplication is associative, so the operation of multiplying these functions is associative. Therefore all the criteria for being a group are met.

2.10 Matrix multiplication is associative, the Identity matrix would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which as established in linear algebra, changes nothing under matrix multiplication. Since the determinanant of these matricies is non-zero, they all have inverses:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

which are also of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Therefore all the requirements to be a group are met

3.3 Example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Some straight forward calculations yield $AB = BC$ even though $A \neq C$

3.4 Because G is a group, $\exists x^{-1} \in G$. Therefore if we apply x^{-1} to both sides of the equation we have:

$$x \cdot g = x$$

$$x^{-1} \cdot xg = x^{-1} \cdot x$$

$$e \cdot g = e$$

and by the definition of the identity, this can only be the case if $g = e$

3.7 If an element occured more than once in a row or column of a table, that is equivalent to saying there exists $a \in G$ such that

$$\exists\ b, c \in G : b \neq c, \text{ and either } ab = ac, \text{ or } ba = ca$$

However we know $a^{-1} \in G$, therefore

$$a^{-1}ab = a^{-1}ac \text{ or } baa^{-1} = caa^{-1}$$

and so

$$b = c$$

which is a contradicion. Therefore the maping of elements in G to the elements in the rows and columns must be injective. And because the number of elements in each row is the same as the number of elements in G, we know that the mapping is surjective. Therefore every element occurs precisely once in each row and column.

3.11 For a given $x, y \in G$, with $x \neq y$ we have for the element xy:

$$(xy)^{-1} = xy$$

however

$$(xy)(yx) = x(yy)x = e$$

as well, therefore they are both the inverse of xy, which is unique. So

$$(xy)^{-1} = xy = yx$$

3.14 First we will show there is an Identity element. We know some element $a \in G$ there is an element $e \in G$ such that

$$ae = a$$

from this, we have $\forall y \in G$

$$(ae)y = ay = a(ey)$$

And therefore $ey = y$ for all y, Therefore for $\forall x \in G$:

$$x(ey) = xy = (xe)y$$

and so $x = xe$. Therefore e satisfies all the properties of the identity element for G.
From there we have for any $a \in G$:

$$\exists a^{-1} \in G : aa^{-1} = e$$

And so we have

$$a(a^{-1}a) = a = (a^{-1}a)a$$

and so $a^{-1}a = e$. Therefore a^{-1} satisfies the properties to be the inverse of a.
Theres an identity, all elements have an inverse, and G is closed under an assosicative operation, therefore G is a group.

3.15 We will let the cardinality of G be n. First we will show that there must be an Identity element.
If we take some $a \in G$ we look at the set

$$\{a, a^2, a^3, a^4, \dots a^n, a^{n+1}\}$$

Since there are n possible values in G but there are $n+1$ elements in this set, by the pigeonhole principle, two of these values must be the same: $a^i = a^j$ for some $i < j$.
Therefore we have $a^i = a^i a^{j-i}$. We will relabel a^{j-i} as e. We now have for all $b \in G$

$$(a^i e)b = a^i(eb) = a^i b$$

and so $eb = b$. Similarly for any $c \in G$

$$c(eb) = (ce)b = cb$$

so $ec = c$. Therefore e satisfies all the properties to be the identity of G.
From the following property for any $x, a, b \in G$

$$a \neq b \Rightarrow ax \neq bx$$

We know the mapping of $f(a) = a \cdot x$ is injective, and therefore since it maps from G to G, it is surjective. Therefore we know there is some $x^{-1} \in G$ such that $x^{-1}x = e$. Also that

$$x(x^{-1}x) = (xx^{-1})x = x$$

and so $xx^{-1} = e$. So x^{-1} satisfies all the properties of the inverse of x. Therefore all elements in G have an inverse.
All the properties required to be a group are satisfied and so G is a group

3.16 Consider $G = \mathbb{N}$ under addition.
The conditions are met since for $a, b, c \in \mathbb{N}$:

$$a + b = c + b \Leftrightarrow a = c$$

and

$$a + b \in \mathbb{N} \Leftrightarrow a, b \in \mathbb{N}$$

However there is no identity or inverses over G.