1-5, 15

We get $|\tau|$ from the identity $b' = \tau n$

$$|b'(s)| = |\tau(s)|$$

We get κ as follows:

$$t = n \wedge b = \frac{b'}{\tau} \wedge b$$

We know that κ is invariant of orientation, so $\kappa = |t'| = |(-t)'|$. We have

$$\kappa = |(\pm t)'| = \left| \frac{d}{dt} \left(\frac{b'}{|\tau|} \wedge b \right) \right|$$

1-5 16

Using the Frenet Equations and the fact $t \cdot b = 0$ we get $\kappa^2 + \tau^2$

$$n' \cdot n' = (\kappa t + \tau b) \cdot (\kappa t + \tau b) = \kappa^2 + \tau^2$$

We have

$$n \wedge n' = n \wedge (-\tau b - \kappa t) = -\tau t + \kappa b$$

We also have

$$-n'' = \kappa' t + \kappa t' + \tau' b + \tau b = (\kappa^2 + \tau^2) n + \kappa' t + \tau' b$$

Thus

$$(n \wedge n') \cdot n'' = \tau \kappa' - \kappa \tau'$$

Thus we have the function $f(s)$

$$f(s) = \frac{(n \wedge n') \cdot n''}{\kappa^2 + \tau^2} = \frac{\tau \kappa' - \kappa \tau'}{\kappa^2 + \tau^2} = \frac{(\tau \kappa' - \kappa \tau')/\tau^2}{\frac{\kappa^2}{\tau^2} + 1} = \frac{\left(\frac{\kappa}{\tau}\right)'}{\frac{\kappa^2}{\tau^2} + 1}$$

Thus integrating this function (determined entirely from n)

$$\int f(s)\ ds = \arctan \frac{\kappa}{\tau}$$

Thus taking tan on both sides gives us κ/τ. This together with knowing $\kappa > 0$ and $\kappa^2 + \tau^2$ lets us determine κ, τ since it is a system of two equations and two unknown variables.

1-5 17

(a) $(\Rightarrow)$ If α is a helix, let v be the direction vector such that $t \cdot v = c$ is constant. Differentiating yields

$$\kappa n \cdot v = 0$$

So either $\kappa = 0$ (which yields κ/τ is constant) or n is perpendicular to v. Thus for constants c_1, c_2

$$v = c_1 t + c_2 b$$

Differentiating:

$$0 = v' = c_1 \kappa n + c_2 \tau n$$

thus

$$-\frac{c_2}{c_1} = \frac{\kappa}{\tau}$$

($\Leftarrow$) Choosing c_1, c_2 so they satisfy the same equality

$$-\frac{c_2}{c_1} = \frac{\kappa}{\tau}$$

We let $v = c_1 t + c_2 b$ and we have

$$\kappa n \cdot v = \kappa n \cdot (c_1 t + c_2 b) = 0$$

So $t \cdot v$ is constant (since its derivative is 0). v is a constant vector since

$$v' = c_1 \kappa n + c_2 \tau n = 0$$

thus α is a helix

(b) ($\Rightarrow$) Using the same v as in (a), we already established that $n \cdot v = 0$ and thus all normal lines are parallel to the plane generated by v.
($\Leftarrow$) If $n \cdot v = 0$, then $\frac{d}{ds} t \cdot v = 0$ so $t \cdot v$ is constant.

(c) ($\Rightarrow$) from (a) and (b) we know t, n make a constant angle with v. Since b is always perpendicular to t, n this means that b must also make a constant angle with v.
($\Leftarrow$) if $b \cdot v$ is constant, then by differentiating we get $n \cdot v = 0$ and thus from (b) we know α is a helix

(d) calculating t:

$$t(s) = \left(\frac{a}{c} \sin \theta(s), \frac{a}{c} \cos \theta(s), \frac{b}{c} \right)$$

Let $v = e_3$ we have

$$t \cdot e_3 = \frac{b}{c}$$

Thus α is a helix. As established above we have that $v = c_1 t + c_2 b$ where $\frac{\kappa}{\tau} = \frac{-c_1}{c_2}$, we have $c_1 = v \cdot t = \frac{b}{c}$ and

$$c_2 = |v - c_1 t| = |\frac{a}{c}(\sin \theta(s), \cos \theta(s), 0)| = \frac{a}{c}$$

So $\frac{\kappa}{\tau} = \frac{a}{b}$

1-7 1

No such curve exists since it violates the isoperimetric inequality:

$$l^2 - 4\pi A = 36 - 4(3)\pi < 0$$

1-7 2

Consider the circle with AB as a chord, and semicircle s_1 from A to B with arclength l. There is the other semicircle on the other side of s_1 from A to B we will label s_2. For any curve C from A to B, we have the closed curve $C \cup s_2$. The area of this curve is precisely the area bounded by C and $\overline{AB}$ plus the area bounded by s_2 and $\overline{AB}$: (here $A_{N,M}$ will denote the area of the region bounded by the curves N, M)

$$A_{C,s_2} = A_{s_2,AB} + A_{C,AB}$$

Similarly

$$A_{s_1,s_2} = A_{s_1,AB} + A_{s_2,AB}$$

Notice that the arclengths of $s_1 \cup s_2$, $C \cup s_2$ are the same, So we can use the isoperimetric inequality for the circle $s_1 \cup s_2$ to conclude

$$A_{C,s_2} \leq A_{s_1,s_2}$$

canceling $A_{s_2,AB}$ on both sides:

$$A_{C,AB} \leq A_{s_1,AB}$$

So area is maximized by the semicircle s_1.