Exercise 22

It does follow. For any Cauchy sequence $(a_n)_n$ with the limit a (if it exists), we have that the set $S = (a_n)_n \cup \{a\}$ is closed and bounded (We know that every Cauchy sequence is bounded). From our midterm problem we know that any subsequence of $(a_n)_n$ converges to a. Therefore in order for S to be compact, a must exist in S and so exists in M

Exercise 23

We know that $(0,1)$ is open in $\mathbb{R}$ as proven in class. For any r, we have that $(1/2, r/2) \in B_r(1/2, 0)$ but $(1/2, r/2) \notin (0,1) \times \{0\}$ therefore $B_r(1/2, 0) \not\subseteq (0,1) \times \{0\}$. Therefore $(0,1) \times \{0\}$ is not open.

Exercise 28

(a) Not necessarily. Consider the map from the unit circle $f : S^1 \to [0, 2\pi)$ where $f(\cos(\theta), \sin(\theta)) = \theta$. We have that the inverse image of the open set $[0, \epsilon)$ is the closed set $\{(\cos(\theta), \sin(\theta)) : \theta \in [0, \epsilon)\}$
(b) Yes. Since f has a continous inverse mapping f^{-1} for any open set $U \subseteq M$ we have that the pullback of f^{-1} of an open set is open. Thus $f(U) = (f^{-1})^{-1}(U)$ is open.
(c) Yes. Since f is bijective it has an inverse f^{-1}. For any open set U, the pullback of f^{-1} of U is just $f(U)$ which is open. Thus f^{-1} is continous. So f is a homeomorphism.
(d) Not necessarily. Consider the map $f(x) = \frac{1}{3}x^3 - x$. We know all polinomials are continous, and f is clearly surjective. The 'humps' where the derivative is zero is $1, -1$, thus the open sets $(-1-\epsilon, 1+\epsilon)$ for small ϵ will be mapped to the closed set $[-\frac{2}{3}, \frac{2}{3}]$

Exercise 32

For any point $p \in \mathbb{N}$ we have that for $r = 1$, by definition the set $B_r(p) = \{p\}$ is open. Therefore singletone points are open in $\mathbb{N}$, so any set $S \subseteq \mathbb{N}$ is open since

$$S = \bigcup_{s \in S} \{s\}$$

And we know arbitrary unions of open sets are open. Therefore we have that S^c is open as well. The complement of an open set is closed so we know that S is closed as well. Therefore every set $S \subseteq \mathbb{N}$ is clopen.
This means that every function $f : \mathbb{N} \to M$ is continous since the inverse image of any open set $U \subseteq M$ will be open.

Exercise 34

For any closed set $L \subset N$ with N closed from the inheritance principle we know $L = C \cap N$ for some closed set $C \subset M$. Intersections of closed sets are closed. Thus L is closed in M. Conversly if L is closed in M then $L = N \cap L$ so L is closed in N
Similarly if $U \subset N$ is open and N is open, then from the inheritance principle $U = V \cap N$

where V is open in M. Finite intersections of open sets are open, thus U is open in M. Conversly if L is open in M then $L = N \cap L$ so L is open in N

Exercise 38

For d_E:
Checking all the axioms of metrics:
$\sqrt{a^2+b^2}$ is clearly nonegative for all $a, b \in \mathbb{R}$
$\sqrt{d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2} = 0$ iff $d_X(a_x, c_x) = d_Y(a_y, c_y) = 0$ iff $x = c$ It is clear $d_E(x, y) = d_E(y, x)$
For $d_E(a, c) \leq d_E(a, b) + d_E(b, c)$ we have $\sqrt{d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2}$
$\leq \sqrt{(d_X(a_x, b_x) + d_X(b_x, c_x))^2 + (d_X(a_x, b_x) + d_X(b_x, c_x))^2} \leq \sqrt{d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2} + \sqrt{d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2}$
For $d_{\max}$:

For d_{sum}:

Exercise 52

(a) Letting

Exercise Additional Problem 1