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Checking all conditions:
For any $p = (x_0, y_0, z_0)$, we have the homeomorphism $f : U \subset \mathbb{R}^2 \to C$ where C denotes the cylinder. We define f as

$$f(\theta, y) = (\cos(\theta), \sin(\theta), z_0 + y)$$

And for $x_0 \neq -1, y_0 \neq 0$, $U = (-\pi, \pi) \times (-1, 1)$, otherwise $U = (0, 2\pi) \times (-1, 1)$. This map is smooth since each component is differentiable. f has a continuous inverse since f is bijective and the inverse of the compenents are continuous since locally, the inverse is equal to $(\sin^{-1}(y), z - z_0)$ or $(\cos^{-1}(x), z - z_0)$. The jacobian is

$$df_{(\theta,y)} = \begin{bmatrix} -\sin(\theta) & 0 \\ \cos(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

Which is linearly independent for any $(\theta, y) \in U$

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$C = \{(x, y, z) \in \mathbb{R}^3; z = 0, x^2 + y^2 \leq 1\}$ is not regular. The reason for this is because the point $(0, 1, 0) \in C$ cannot have a neighborhood V such that there is a homeomorphism $f : U \subset \mathbb{R}^2 \to V \cap C$. The reason for this is because $V \cap C$ is homeomorphic to $C' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ by the map $\pi_{12}(x, y, z) = (x, y)$ (which has the continuous inverse map $\pi_{12}^{-1}(x, y) = (x, y, 0)$). We have that C' is a closed set in $\mathbb{R}^2$. Thus composing homeomorphisms, we would get $\pi_{12} \circ f : U \to C'$ is a homeomorphism from closed unit disc in $\mathbb{R}^2$ to the open unit disc in $\mathbb{R}^2$. We know that such a mapping is not possible since removing a point on the boundary of the closed unit disc still yields a null homotopic set, while removing any point on the unit open disc yields a set which is not null homotopic (we proved this in Math 441)

For $D = \{(x, y, z) \in \mathbb{R}^3; z = 0, x^2 + y^2 < 1\}$ we have the homeomorphism

$$f : \{x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\} \to D$$

Where $f(x, y) = (x, y, 0)$ which is clearly smooth with a continuous inverse $f^{-1}(x, y, 0) = (x, y)$ and linearly independent jacobian:

$$df_{(x,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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(a) Calculating the Jacobian:

$$df_{(x,y,z)} = 2(x+y+z-1)(1,1,1)$$

Thus the critical points is the simplex $x+y+z=1$. Evaluating f at any point in the simplex yields the critical value 0.

(b) We have that $f(\mathbb{R}^3) = \mathbb{R}^+$. All $\mathbb{R}^3 \setminus \{0\}$ are regular points and thus from Prop 2, $f^{-1}(c)$ for $c > 0$ is regular. We have that $f^{-1}(0)$ is regular too since

$$f^{-1}(0) = \{(x,y,z) : x+y+z=1\}$$

is a plane (which we know is regular)

(c) Calculating the Jacobian:

$$df_{(x,y,z)} = (yz^2, xz^2, 2xyz)$$

The critical points is the plane $z=0$ union the line $y=0, x=0$. Evaluating f at any critical point yields the critical value 0.

For $c \in f(\mathbb{R}^3) \setminus \{0\}$, c is regular so $f^{-1}(c)$ yields a regular surface. For $c=0$, $f^{-1}(0)$ is not regular since it is the union of three normal planes $x=0, y=0, z=0$ which intersect at $(0,0,0)$. There is no well defined tangent vector at the point $(0,0,0)$

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We know that dx_q is one to one if and only if $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ are linearly independent. From the definition of '$\wedge$' we know that $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = 0$ if and only if the vectors are linearly dependent. Thus dx_q is one-to-one iff $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} \neq 0$

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Letting c be the speed of the points, for a given t we have the positions

$$p(t) = (0,0,ct), q(t) = (a, ct, 0)$$

Thus the line containing $p(t), q(t)$ parameterized by s is described as

$$p(t) + s(q(t) - p(t)) = (0,0,ct) + s(a, ct, -ct)$$

So for x, y, z on the line we have

$$\frac{x}{a} = \frac{y}{ct} = \frac{z - ct}{-ct} = s$$

$$ctx = ay = act - az$$

Letting t vary in $\mathbb{R}^+$, we will show this is the same set as $y(x-a) + zx = 0$.