1.

1. If n is one digit it is clear that the sequence remains constant since $S(n) = n$. If n has multiple digits we can show that $S(n)$ will always yield a number smaller than n and is therefore decreasing, which would imply after repeated applications of S, the output would eventually be one digit and therefore constant.
   To show this we simply observe if d_i is the digit of n in the ith place from the right, we have
   $$n = d_k 10^k + d_{k-1} 10^{k-1} \dots d_1 10 + d_0$$
   while
   $$S(n) = d_k + d_{k-1} \cdots + d_1 + d_0$$
   We have for $i \geq 1$, $d_i < 10^i d_i$, and therefore if n has more than one digit, $S(n) < n$

2. First we can use the useful property of modular arithmetic mod 9:
   $$n \equiv S(n) \bmod 9$$
   Therefore
   $$n \equiv S(S(\dots S(n)\dots)) \bmod 9$$
   And so the digital sum of n is equivalent to n mod 9
   We also can say twin primes have the form $6n-1, 6n+1$ for some $n \in \mathbb{N}$ since any other form has one of the primes divisible by either 2 or 3.
   Therefore the product of these primes is
   $$36n^2 - 1$$
   Which is 8 mod 9. Therefore the digital sum of any prime twins must be 8

2.

1. $$9^{121} + 13^{1331} = (9^{11})^{11} + ((13^{11})^{11})^{11}$$
   by Fermats Little Thm
   $$(9^{11})^{11} + ((13^{11})^{11})^{11} \equiv (9 + 13) \bmod 11 \equiv 0 \bmod 11$$
   Since both $(9^{11})^{11}$ and $((13^{11})^{11})^{11}$ are odd, $(9^{11})^{11} + ((13^{11})^{11})^{11}$ must be even.
   Therefore since both 2 and 11 divide the sum, 22 must divide it as well

2. We have $31|(30^{239}+239^{30})$ and here is why:
$30 \equiv -1 \bmod 31$, so since 239 is odd:

$$30^{239} \equiv -1 \bmod 31$$

and by Fermat's Little Thm

$$239^{30} \equiv 1 \bmod 31$$

Therefore

$$30^{239}+239^{30} \equiv -1+1 \equiv 0 \bmod 31$$

And so $31|(30^{239}+239^{30})$

3. There is no such value. To start with, if a_0 is not prime the condition fails right away. If a_0 is prime, we can find a composite term of the sequence as follows:

We can rewrite the term of a_n in terms of a_0:

$$a_n = 2(2(\dots 2(2a_0+1)+1)\dots)+1)+1$$

$$a_n = 2^n a_0 + 2^{n-1} + 2^{n-2} + \dots 2^2 + 2 + 1$$

The series on the right can be rewritten with the identity:

$$2^{n-1} + 2^{n-2} + \dots 2^2 + 2 + 1 = 2^n - 1$$

And so

$$a_n = 2^n a_0 + 2^n - 1$$

Since a_0 is prime we can use Fermats Little Thm, we set $n = a_0 - 1$:

$$2^n \equiv 1 \bmod a_0$$

And so

$$a_n \equiv 0 \bmod a_0$$

Which means $a_0|a_n$

4. We know $d_i \geq i$ for all $i : 1 \leq i \leq k$ since $d_1 = 1$ and d_i increases by at least 1 every time i is incrimented.
We also know the following:

$$d_1 = \frac{n}{d_k},\ d_2 = \frac{n}{d_{k-2}},\ d_3 = \frac{n}{d_{k-3}} \dots d_i = \frac{n}{d_{k+1-i}}$$

Since factoring out the ith smallest factor out of n yields the ith largest factor of n and vice versa.
Therefore since $d_i \leq i$, we have:

$$d_1 \leq \frac{n}{k},\ d_2 \leq \frac{n}{k-1},\ d_3 \leq \frac{n}{k-2} \dots d_i \leq \frac{n}{k+1-i}$$

And so

$$\sum_{i=1}^{k-1} d_i d_{i+1} \leq \sum_{i=1}^{k-1} \frac{n^2}{(k+1-i)(k-i)}$$

reordering the indecies with $j = k - i$ yields:

$$= \sum_{j=1}^{k-1} \frac{n^2}{j(j+1)} = n^2 \sum_{j=1}^{k-1} \frac{1}{j(j+1)}$$

We can telescope this sum:

$$\sum_{j=1}^{k-1} \frac{1}{j(j+1)} = \sum_{j=1}^{k-1} \frac{1}{j} - \frac{1}{j+1} = 1 - \frac{1}{k} < 1$$

so

$$\sum_{i=1}^{k-1} d_i d_{i+1} < n^2$$

EC Question. If d_1 is 2, we have

$$d_1 - d_0 = 1$$

and so incrimenting by 1 each time should yield a number relatively prime to n. Therefore every number in

$$\{1, 2, 3, \ldots n - 1\}$$

is relatively prime to n and so n must be prime.
If $d_1 = 3$, we have

$$d_1 - d_0 = 2$$

And so incrimenting by 2 each time should yield a number relatively prime to n. This would be all the odd numbers less than n and therefore n can only have the prime factor of 2 since all other primes are odd. And so n must be a power of 2.

If $d_1 > 3$, we find it impossible for n to have the property described in the problem.
All numbers below d_1 are not relatively prime with n and so d_1 must be prime, otherwise if d_1 had a factor, it would have to share that factor with n. It follows from this that all primes less than d_1 must divide n.
Since $d_1 - d_0 = d_1 - 1$ we know all numbers relatively prime to n that are less than n must be of the form

$$(d_1 - 1)k + 1$$

In order for the property to hold.
However if

$$d_1 \equiv 1 \bmod 3$$

then $2|d_1$ so d_1 is not relatively prime to n, similarly if

$$d_1 \equiv 0 \bmod 3$$

then $3|d_1$ and so d_1 is not relatively prime to n, if

$$d_1 \equiv 2 \bmod 3$$

then for $k = 2$:

$$(d_1 - 1)k + 1 \equiv 0 \bmod 3$$

so $(d_1 - 1)k + 1$ is not relatively prime to n.
We know $(d_1 - 1)2 + 1$ must be less than n since $(d_1 - 1)!$ divides n and $d_1 > 4$ (since d_1 cannot be even) so $(d_1 - 1)! > 2d_1$.