Exersise 9.1

(a) For any ideal $I \subset R \times S$, we have the projection maps $\pi_R : R \times S \to R, \pi_S : \mathbb{R} \times S \to S$ where $\pi_R(r,s) = r$ and $\pi_S(r,s) = s$. We have that I is the intersection of the ideals $P = \pi_R(I) \times S$ and $Q = R \times \pi_S(I)$. It is clear that $R \times S = P + Q$ since $R \times \{0\} \subset Q$ and $\{0\} \times S \subset P$. Thus we can use the chinese remainder theorem.

$$(R \times S)/I \cong (R \times S)/P \oplus (R \times S)/Q$$

We have that $(R \times S)/P = (R \times S)/(\pi_R(I) \times S) \cong R/\pi_R(I)$ and similarly $(R \times S)/Q \cong S/(\pi_S(I)$. Thus

$$(R \times S)/I \cong R/\pi_R(I) \oplus S/\pi_S(I)$$

Since R, S are semi-simple, we know that the short exact sequences

$$0 \to \pi_R(I) \to R \to R/(\pi_R(I)) \to 0$$

$$0 \to \pi_S(I) \to S \to S/(\pi_S(I)) \to 0$$

split, and we get

$$R \cong R/(\pi_R(I)) \oplus \pi_R(I), S \cong S/(\pi_S(I)) \oplus \pi_S(I)$$

So

$$R \times S \cong R/(\pi_R(I)) \oplus \pi_R(I) \times S/(\pi_S(I)) \oplus \pi_S(I)$$

from our chinese remainder theorem identity:

$$R \times S \cong (R \times S)/I \oplus \pi_R(I) \oplus \pi_S(I)$$

And $I = \pi_R(I) \oplus \pi_S(I)$ thus $R \times S$ is semi-simple.

(b) If a is square free, since a PID is a UFD we can factor a as a produt of prime elements $a = p_1 p_2 \dots p_n$. Thus from the chinese remainder theorem.

$$R/(a) \cong R/(p_1) \oplus R/(p_2) \oplus \cdots \oplus R/(p_n)$$

each $R/(p_i)$ is a field (since the p_is are irriducible elements) and thus simple. Thus $R/(a)$ is a direct sum of simple modules over R and thus semi-simple.
Conversly if $p^2|a$ for some prime p then the chinese remainder theorem results in

$$R/(a) \cong R/(p^n) \oplus R/(b)$$

where $p \not| \, b$ and $n \geq 2$. We have the ideal generated by $(p, 0)$ in $R/(p^n) \oplus R/(b)$ quotients to

Exersise 9.2

(a) We have the following isomorphism

$$\mathbb{C}[\mathbb{Z}/n] \cong \mathbb{C}[x]/(x^n - 1)$$

We have from the chinese remainder theorem

$$\mathbb{C}[x]/(x^n - 1) = \mathbb{C}[x]/\left(\prod_{m=0}^{n-1}(x - e^{\frac{2\pi i m}{n}})\right) \cong \prod_{m=0}^{n-1} \mathbb{C}[x]/(x - e^{\frac{2\pi i m}{n}})$$

Each $\mathbb{C}[x]/(x - e^{\frac{2\pi i m}{n}})$ is a free $\mathbb{C}$ module of rank 1 and thus $\mathbb{C}[x]/(x - e^{\frac{2\pi i m}{n}}) \cong \mathbb{C}$. So we have

$$\mathbb{C}[\mathbb{Z}/n] \cong \prod_{m=0}^{n-1} \mathbb{C}$$

Exersise 9.3

$(iii) \Rightarrow (ii) \Rightarrow (i)$:

From the identities we established leading up to the Artin Wedernburn theorem, we know that $R \cong \text{End}_{R^{op}}(R^{op}) \cong M_n(D)$, where $M_n(D) \cong \text{End}(M_1 \oplus M_2 \oplus \ldots M_n)$. This comes from the identity $R^{op} \cong \oplus_{i=1}^{n} M_n$ where M_n are simple modules. Since each M_i is isomorphic to each other we only have R is the direct sum of only one matrix ring over a division ring. Thus $(iii) \Rightarrow (ii)$. In problem 7.6(1) we have shown $M_n(D)$ is simple. $M_n(D)$ as an R module is isomorphic to $D^n \oplus D^n \oplus \cdots \oplus D^n$ which are simple $M_n(D)$ modules. Thus as we have established in lecture, $M_n(D)$ is Artinian since every every submodule is of the form $D^n \oplus D^n \oplus D^n \cdots \oplus D^n$ and so a decsending chain must terminate on some number of D^n as a direct sum. Thus $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (iii)$:

R contains a simple submodule M as illustrated by the following process. If R is simple, we are done, otherwise R must contain some proper module M_1, either M_1 is simple or M_1 contains proper M_2. Continuing this logic yields a decsending chain of proper modules $R \supset M_1 \supset M_2 \ldots$ since R is Artinian there exists M_n which terminates the chain, and thus must be simple.

We know that $\text{Ann}_R(M)$ is a two sided ideal of R. Thus since R is simple, we know $\text{Ann}_R(M) = 0$. We can now construct an isomorphism $\phi : R \to M^n$ for some n and thus proving R is semi-simple with every simple submodule of R isomorphic to M. Since in lecture we established every simple R module is a submodule of R, we have shown (iii). We construct this isomorphism as follows:

Let $\phi_0 : R \to M$ where $1 \to m_0$ for some $m_0 \in M$. Either $\ker \phi_0 = 0$ or $\exists r_0 \in \ker \phi_0$. Since $\text{Ann}_R(M) = 0$ there exists $m_1 \in M$ such that $r_1 m_1 \neq 0$. We define the new homorphism $\phi_1 : R \to M^2$ where $1 \to (m_0, m_1)$ thus $r_1 \notin \ker \phi_1$ since $\phi_1(r_1) = (r_1 m_0, r_1 m_1) \neq (0, 0)$, either $\ker \phi_2 = 0$ or there exists $r_2 \in \ker \phi_1$ and $m_2 \in M$ with $r_2 m_2 \neq 0$. Thus we can define $\phi_2 : R \to M^3$ where $1 \to (m_0, m_1, m_2)$ and thus $r_2 \notin \ker \phi_2$. Continuing this process we get a decsending chain of submodules of R

$$\ker \phi_0 \supset \ker \phi_1 \supset \ker \phi_2 \supset \ldots$$

Thus since R is Artinian this chain terminates, which means there exists $\phi_k = \phi : R \to M^n$ with $\ker \phi = 0$. So ϕ is an imbedding into M^n. Thus R is a submodule of M^n and thus $R \cong M^k$ for some $k \leq n$.

Exersise 9.4

(a)

Exersise 9.5

(a) Letting $A(t) = \sum_{i\geq 0} h_i t$ and $B(t) = \sum_{i \geq 0} e_i t$, we can using combinatorial reasoning to write these series in a new form.
When we multiply out

$$\prod_{i=1}^{n}(1 + x_i t) = (1 + x_1 t)(1 + x_2 t) \dots (1 + x_n t)$$

We get $B(t)$. The reasoning for this is because for each k, a t^k only shows up in the product by choosing k $x_i t$ terms and multiplying by 1 for the other terms. Thus every t^k term is of the form $x_{j_1} x_{j_2} \dots x_{j_k}$ where $1 \leq j_1 < j_2 < \dots j_n \leq n$, and conversely all $x_{j_1} x_{j_2} \dots x_{j_k}$ show up uniquely as a coefficient of one of the t^k terms by choosing $x_{j_1} x_{j_2} \dots x_{j_k}$ and multiplying out by 1 for the other terms. Summing up all these terms we get the symmetric polynomials:

$$\sum_{1 \leq j_1 < j_2 < \dots j_k \leq n} x_{j_1} \dots x_{j_k} t^k = e_k t^k$$

Thus $B(t) = \prod_{i=1}^{n}(1 + x_i t)$ since each coefficient of t^k is the same in both polynomials.
For $A(t)$ we have the following product of the closed form of the geometric series

$$\prod_{i=1}^{n} \frac{1}{1 - x_i t} = \prod_{i=1}^{n} \left(1 + x_i t + (x_i t)^2 + (x_i t)^3 \dots + (x_i t)^k \dots\right)$$

When we factor out this product we get $A(t)$. The reasoning for this is because for each k, a t^k only shows up in the product if we choose $(x_{j_1} t)^{n_1}, (x_{j_2} t)^{n_2}, \dots (x_{j_l} t)^{n_l}$ so that $n_1 + n_2 \dots n_l = k$ and multiply by the 1 term for every other term in the product. Thus every t^k term is one of the terms in h_k. We have that every term of h_k shows up uniquely as a coefficient of one of the t^k since any monomial $x_{j_1}^{n_1} x_{j_2}^{n_2} \dots x_{j_l}^{n_l}$ of total degree k shows up only by choosing the terms $(x_{j_1} t)^{n_1} (x_{j_2} t)^{n_2} \dots (x_{j_l} t)^{n_l}$ and 1s in the other terms. Thus when we sum up all the t^k terms we get $h_k t^k$.

(b) From our product identities we have the equality

$$A(t)B(-t) = \prod_{i=1}^{n}(1 - x_i t) \prod_{i=1}^{n} \frac{1}{1 - x_i t} = 1$$

By factoring out $A(t)B(-t)$ we get the constant term $e_0 h_0 = 1$, thus subtracting the constant term on both sides we get the sum of nonconstant terms is 0. Thus for each $k \geq 1$ the coefficient of t^k is zero. We have that every t^k coefficient term is of the form $h_n(-1)^m e_m$ where $m+n=k$. Thus the sum of the coefficients of the t^k terms is $h_k - h_{k-1}e_1 + h_{k-2}e_2 - \cdots + (-1)^k e_k$. This coefficient must be zero, thus we have Newtons identity

$$h_k - h_{k-1}e_1 + h_{k-2}e_2 - \cdots + (-1)^k e_k = 0$$

(c) From Newtons identity we can that $\Lambda_n = \mathbb{Z}[h_1, \dots h_n]$. We from lecture that $\Lambda_n = \mathbb{Z}[e_1, \dots e_n]$ thus if we show $\mathbb{Z}[h_1, \dots h_n] = \mathbb{Z}[e_1, \dots e_n]$ we are done. By showing that $h_1, \dots h_n$ linearly spans $e_1 \dots e_n$ we are done (we already know $e_1, e_2 \dots e_n$ spans $h_1 \dots h_n$ since $e_1, \dots e_n$ generate all symmetric polynomials). Using induction we have the base case $h_0 = e_0$. From Newtons identity:

$$(-1)^{k-1}(h_k - h_{k-1}e_1 + h_{k-2}e_2 - \dots e_{k-1}h_1) = e_k$$

We have that each e_k is a linear sum of e_i and h_j where $i < k$ thus from our inductive hypothesis each e_i is a linear sum of h_js and thus e_k is a linear sum of h_js.