Character Table of A_4

There are 4 Conjugacy classes of A_4:

$$C_1 = \{1\}, \quad C_2 = \{g(123)g^{-1}\}, \quad C_3 = \{g(124)g^{-1}\}, \quad C_4 = \{g(12)(34)g^{-1}\}$$

With sizes

$$|C_1| = 1 \quad |C_2| = 4 \quad |C_3| = 4 \quad |C_4| = 3$$

$g \in A_4$. Thus there are 4 irreducible representations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ (there is always the trivial 1 dim representation which we will call φ_1)

We know that

$$12 = |A_4| = \dim_{\mathbb{C}} \mathbb{C}[A_4] = \dim^2 \varphi_1 + \dim^2 \varphi_2 + \dim^2 \varphi_3 + \dim^2 \varphi_3$$

with $\dim^2 \varphi_1 = 1$. The only way to have the above equality is with $\dim^2 \varphi_2 = \dim^2 \varphi_3 = 1, \dim^2 \varphi_4 = 9$. Thus we have two nontrivial 1 dimensional irreducible representations and one 3 dimensional irreducible representation

For the 1 dimensional case we know that the Character is equal to the representation which is a class function and thus fully defined by the image of Conjugacy class representatives

We know that $(123), (124)$ has order 3 so must map to a 3rd root of unity and $(12)(34)$ has order 2 so must map to a 2nd root of unity.

From these considerations we arrive at representations

$$\varphi_2((123)) = \zeta_3 \quad \varphi_2(124) = \zeta_3^2 \quad \varphi_2((12)(34)) = 1$$

$$\varphi_3((123)) = \zeta_3^2 \quad \varphi_3(124) = \zeta_3 \quad \varphi_3((12)(34)) = 1$$

Using these representation along with the fact for any character $\chi_i(1) = \dim \varphi_i$ we can arrive at the partially completed character table

	1	(123)	(124)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ζ_3	ζ_3^2	1
χ_3	1	ζ_3^2	ζ_3	1
χ_4	3	x_1	x_2	x_3

We can solve for x_1, x_2, x_3 by using the orthogonality relations:

$$1 + \zeta_3 + \zeta_3^2 + 3x_1 = 0 \Rightarrow x_1 = 0$$

$$1 + \zeta_3 + \zeta_3^2 + 3x_2 = 0 \Rightarrow x_2 = 0$$

$$1 + 1 + 1 + 3x_3 = 0 \Rightarrow x_3 = -1$$

and thus complete our Character table

	1	(123)	(124)	$(12)(34)$
χ_1	1	1	1	1
χ_2	1	ζ_3	ζ_3^2	1
χ_3	1	ζ_3^2	ζ_3	1
χ_4	3	0	0	-1