Exersise 3.1

(1) We can define the following isomorphism: $\phi : H \rtimes_\varphi K \to H \rtimes_{\varphi\circ\lambda} K$
With $\phi(h,k) = (h, \lambda^{-1}(k))$. It is clear ϕ is bijective since we can define the inverse mapping $\phi^{-1}(h,k) = (h, \lambda(k))$. ϕ is a homomorphism as follows:

$$\phi\left((h,k)(h',k')\right) = \phi(h\varphi(k)(h'), kk') = (h\varphi(k)(h'), \lambda^{-1}(kk'))$$

$$= (h\varphi(\lambda(\lambda^{-1}(k)))(h'), \lambda^{-1}(kk')) = (h, \lambda^{-1}(k))(h', \lambda^{-1}(k')) = \phi(h,k)\phi(h',k')$$

Thus ϕ is an isomorphism and thus $H \rtimes_\varphi K \cong H \rtimes_{\varphi\circ\lambda} K$

(2) We can define the isomorphism: $\phi : H \rtimes_\varphi K \to H \rtimes_{\psi\circ\varphi\circ\psi^{-1}} K$
With $\phi(h,k) = (\psi(h), k)$. It is clear ϕ is bijective since we can define the inverse mapping $\phi^{-1}(h,k) = (\psi^{-1}(h), k)$. ϕ is a homomorphism as follows:

$$\phi\left((h,k)(h',k')\right) = \phi(h\varphi(k)(h'), kk') = (\psi(h\varphi(k)(h')), kk')$$

$$= (\psi(h)\psi(\varphi(\psi^{-1}(\psi(k))))\psi(h'), kk') = (\psi(h), k)(\psi(h'), k') = \phi(h,k)\phi(h',k')$$

Thus ϕ is an isomorphism and thus $H \rtimes_\varphi K \cong H \rtimes_{\psi\circ\varphi\circ\psi^{-1}} K$

Exersise 3.2

(1) Consider the canonical homomorphism: $\pi : R \to R/I_1 \times \cdots \times R/I_k$ where $\pi(r) = r + I_1 \times \cdots \times r + I_k$.
We have that $r \in \ker(\pi)$ iff $r \in I_1 \cap \cdots \cap I_k$. Thus if we can show that π is surjective and $I_1 \cap \cdots \cap I_k = I_1 \dots I_k$ Then we have proven the claim. We will use induction on the number of ideals:
Base case, $(k = 2)$:
It is clear that $I_1 I_2 \subseteq I_1 \cap I_2$ for the other direction we have that since $I_1 + I_2 = R$ there exists $x \in I_1$ and $y \in I_2$ with $x + y = 1$. Thus for any $a \in I_1 \cap I_2$ we have that $a = ax + ay \in I_1 I_2$ thus $I_1 I_2 = I_1 \cap I_2$. Additionaly we have with the same x, y, for any $(a,b) \in R/I_1 \times R/I_2$, $\pi(x+y) = \pi(1) = (1,1)$, $\pi(x) + \pi(y) = (1,1)$, since $x \in I_1$ we know $\pi(x)$ is zero in the I_1 component, same goes with y for the I_2 component and thus $\pi(x) = (0,1)$, $\pi(y) = (1,0)$. So

$$\pi(bx + ay) = (a,b)$$

So π is surjective, proving the base case.
We can reduce the $k+1$ step to the k step in the following manner:
Define $\mathcal{I} = I_1 I_2 \dots I_k$. We have to show that $\mathcal{I} + I_{k+1} = R$ and then we can apply our base case reasoning. $I_i + I_{k+1} = R$ for each $I_i \neq I_{k+1}$ and so there exists $x_i \in I_i, y_i \in I_{k+1}$ with $x_i + y_i = 1$, we have that

$$1 = (x_1 + y_1)(x_2 + y_2) \dots (x_k + y_k)$$

Factoring the product on the right we have each term is multiplied by an x_i and thus an element of I_{k+1} except for the $y_1 y_2 \dots y_k$ term wich is an elt of $\mathcal{I}$. Thus $1 \in \mathcal{I} + I_{k+1}$, and since if an Ideal contains 1 it is R we have $\mathcal{I} + I_{k+1} = R$.
Thus we have from our inductive hypothesis

$$R/I_1 \cap \cdots \cap I_{k+1} = R/\mathcal{I} \cap I_{k+1} \cong R/\mathcal{I} \times R/I_{k+1} \cong R/I_1 \times R/I_2 \cdots \times R/I_{k+1}$$

(2) We can use the chinese remainder theorem:
Let $I_i = p_i^{a_i}$ for each i. For any I_i, I_j with $i \neq j$ we have that the $\gcd(p_i^{a_i}, p_j^{a_j}) = 1$ and thus from the euclidean algorithm we know there exists $n, m \in \mathbb{Z}$ with $np_i^{a_i} + mp_j^{a_j} = 1$ and thus $1 \in I_i + I_j \Rightarrow I_i + I_j = \mathbb{Z}$. The last thing to check is that $n\mathbb{Z} = I_1 \cap I_2 \cap \cdots \cap I_k$. Since for each i, $p_i^{a_i} | n$, it is clear $n\mathbb{Z} \subseteq I_1 \cap I_2 \cap \cdots \cap I_k$. For any $x \in I_1 \cap I_2 \cap \cdots \cap I_k$ we have that $p_i^{a_i} | x$ for all i and so $n | x \Rightarrow I_1 \cap I_2 \cap \cdots \cap I_k \subseteq n\mathbb{Z}$. Thus $I_1 \cap I_2 \cap \cdots \cap I_k = n\mathbb{Z}$, and so all the conditions of the chinese remainder thm are satisfied:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/I_1 \times \cdots \times \mathbb{Z}/I_k$$

Exersise 3.3

Exersise 3.4

Let R be a ring

Exersise 3.5

(1) We have by definition that $Z(R)$ is commutative and since R is a division ring every non-zero elt is a unit. Thus all we have to check is that $Z(R)$ is closed under multiplcation, addition, and inverses.
We have for any $x, y \in Z(R)$ and arbitrary $r \in R$

$$(x+y)r = xr + yr = rx + ry = r(x+y) \Rightarrow x + y \in Z(R)$$

$$(xy)r = r(xy) \Rightarrow xy \in Z(R)$$
$$x^{-1}r = x^{-1}rxx^{-1} = rx^{-1} \Rightarrow x^{-1} \in Z(R)$$
$$(-x)r = (-1)xr = r(-x) \Rightarrow -x \in Z(R)$$

Thus $Z(R)$ is a field

(2) The center of $M_n(R)$ consists of all diagonal matricies with entries in $Z(R)$.

Exersise 3.6

(1) We have that

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Since

$$(1+x)\sum_{n=0}^{\infty}(-1)^n x^n = 1 + (x - x) + (-x^2 + x^2) + (x^3 - x^3)\cdots = 1$$

(2)