Math 504, Fall 2017 *HOMEWORK #6* *Peter Gylys-Colwell*

Exersise 6.1

(1) We know that degree 3 polynomials are irreducible iff they have a root. We can use the rational root test to verify which polynomials have roots.
A root of $x^3 + nx + 2$ must be $1, -1, 2$ or -2. We have that $1 + n + 2 = 0 \Rightarrow n = -3$, $-1 - n + 2 = 0 \Rightarrow n = -1$, $2^3 + 2n + 2 = 0 \Rightarrow n = -5$ and $-2^3 - 2n + 2 = 0 \Rightarrow n = -3$. Thus $x^3 + nx + 2$ is irreducible iff $n \neq 1, -3, -5$

(2) Over $\mathbb{Z}$, $x^8 - 1 = (x-1)(x+1)(x^2+1)(x^4+1)$. We know that $x^2 + 1$ is irreducible since it has no roots, $x^4 + 1$ is irreducible since $(x-1)^4 + 1 = x^4 - 4x^3 + 6x^2 - 4x + 6$ is irreducible by eisenstein criteria.
Over $\mathbb{Z}/2$, we have $x^8 - 1 = (x+1)^8$
Over $\mathbb{Z}/3$, $x^8 - 1 = (x-1)(x+1)(x^2+1)(x^2+x+2)(x^2+2x+2)$ (a simple check of roots shows these deg 2 polynomials are irreducible)

Exersise 6.2

(1) If $n = m$ it is clear that $R^n \cong R^m$ since we can bijectively map the basis to each other. Since each element is a unique sum of the basis, our map will be bijective.
If $n < m$, we can show that a surjective map $\varphi : R^n \to R^m$ is not possible. φ is fully defined from where φ sends the basis $g_1, g_2, \dots g_n$ to R^m. If R^m has the basis $b_1, b_2, \dots b_m$, we have $\varphi(g_k) = \sum_i^m r_{i,k} b_i$ for $r_{i,k} \in R$, we can write this as an n by m matrix:

$$A_{i,k} = [r_{i,k}]$$

We can extend the domain of φ from R^n to R^m by setting $\varphi(R^{m-n}) = 0$. Thus we would have A extends to an $m \times m$ matrix. However, since φ is surjective, there exists a righthand inverse: $\rho : R^m \to R^n$ such that $\varphi \circ \rho = 1$. Thus our extension with $\rho : R^m \to R^m$, $\varphi : R^m \to R^m$ also satisfies $\varphi \circ \rho = 1$. If we consider the square matricies B, A for ρ and φ we have

$$\det(A)\det(B) = \det(\mathrm{id}) = 1$$

However $\det(A) = 0$ since it has a row of zeros and thus not a unit. Thus we have a contradiction.

(2) Let g_1, g_2 be the generators of R^2 and h the generator of R. We have that any $T \in \mathrm{End}_k(V)$ is of the form

$$T(k_1 b_1 + k_2 b_2 + \dots k_n b_n + \dots) = k_1 T(b_1) + k_2 T(b_2) + \dots k_n T(b_n) + \dots$$

Where the b_is are the basis of V and k_is $\in k$.
We can define the following isomorphism:

$$\varphi(g_1) = T_1 h, \varphi(g_2) = T_2 h$$

Where $T_1(b_n) = b_n \forall n \in 2\mathbb{Z}$ and $T_1(b_n) = 0\ \forall n \in 1 + 2\mathbb{Z}$, similarly $T_2(b_n) = b_n \forall n \in 1 + 2\mathbb{Z}$ and $T_2(b_n) = 0\ \forall n \in 2\mathbb{Z}$.
We have surjectivity since for any $Th \in R$ we have

$$T = TT_1 + TT_2$$

Thus $\varphi(Tg_1 + Tg_2) = Th$. We have injectivity since $TT_1 + TT_2 = FT_1 + FT_2 \Rightarrow T(b_1) = F(b_1), T(b_2) = F(b_2) \dots T(b_i) = F(b_i) \dots \Rightarrow T = F$

Exersise 6.3

(1) We can use the fact $M'' \cong M/M'$. If M is noetherian it is clear that M' is noetherian since it is isomorphic to a submodule of M. Submodules of noetherian modules are noetherian since any ascending chain of M' is an ascending chain of M. Then we have that M'' is noetherian since any ascending chain in M/M' has a corresponding ascending chain obtained from the cononical mapping $\pi : M/M' \to M$. We know that for any two submodules $N, N' \subset M/M'$ we have that $N \subset N' \Leftrightarrow \pi(N) \subset \pi(N')$, and thus a chain in $M'' = M/M'$ terminates iff its image from π terminates.
Conversly, if M'' and M' are noetherian then we have that any submodule of M/M' is finitely generated and any submodule of M' is finitely generated. We have that any submodule S of M is generated by the generators of $S \cap M'$ and the elements obtained by mapping generators of $S + M' \subset M/M'$ to one of their coset representatives.
This is a generating set of S since for any $s \in S$ we can write $s = m + m'$ for $m \notin M', m' \in M'$ then we have that m can be writen as a sum of generators obtained by coset representatives of generators of $S + M$ with a difference of some elements in $S \cap M'$. Then we have the remaining elements are only in $S \cap M'$ and thus can be written as a sum of generators in $S \cap M'$. Since this set of generators is finite (since $S \cap M'$ and $S + M$ are submodules of noetherian modules and thus finitely generated), M is noetherian.

(2) We can induct on the rank of the R-Modules. For rank = 1, we know that the only possible modules are ideals of R, which are noetherian.
For a rank $n + 1$ R-Module M, we have that for a rank 1 sub module R, M/R is a rank n R-Module. Thus from our inductive hypothesis, since R and M/R are noetherian, we know that M is noetherian.

Exersise 6.4

(1) If $rk(M) = n$, then we have a linear independent set $A = \{g_1, \dots g_n\} \subset M$. The submodule generated by A is a free module R^n. If we consider the quotient M/R^n, every element m not in R^n (and thus not zero in M/R^n) cannot be written as a linear sum of elements in A. However we know that $\{m\} \cup A$ cannot be linearly independent since it would contradict $rk(M) = n$, therefore we have $r_m m = r_1 g_1 + \dots r_n g_n$. Thus $r_m m = 0$ in M/R^n. So M/R^n is torsion
Conversly if M/R^n is torsion then we know that R^n is a submodule of M. Therefore there exists a set of n independent elements in M which are the generators of R^n. We have that there cannot exist any set of $n + 1$ linearly independent elts of M since they are all torsion in M/R^n, and thus when we multiply by appropriate $r \in R$ for each element we get

$n+1$ elts in R^n. Since we know that any $n+1$ elements in R^n are linearly dependent when R is a PID we know we can find a non-zero linear combination of these elements to equal zero.

(2) We have that sets of linearly independent elts $A = \{a_1, \dots a_n\} \subset M, B = \{b_1, \dots b_m\} \subset M'$ are linearly independent in $M \oplus M'$ as $A \times \{0\} \cup \{0\} \times B$. Thus $rk(M \oplus M') \geq rk(M) + rk(M')$. We have that $rk(M \oplus M') \leq rk(M) + rk(M')$ since if we have a set $G = \{(a_1, b_1), (a_2, b_2) \dots (a_{m+n+1}, b_{m+n+1})\} \subset M \oplus M'$, since rank of $M < n+1$ there exists r_is $\in R$ such that

$$r_1 a_1 + \dots r_n a_n + r_{n+1} a_{m+n+1} = 0$$

Let us define $g = r_1(a_1, b_1) + \dots r_n(a_n, b_n) + r_{n+1}(a_{m+n+1}, b_{m+n+1})$. We have that $g = (0, b)$ for some $b \in M'$. Now in M' we have that

$$r_{n+1} b_{n+1} + \dots r_{n+m} b_{n+m} + r_{m+n+1} b = 0$$

Thus we can define $h = (r_{n+1}(a_{n+1}, b_{n+1}) + \dots r_{n+m}(a_{n+m}, b_{n+m}) + r_{m+n+1} g$ with $h = (a, 0)$ for some $a \in M$. Thus we have that $g \cdot h = 0$ which is a linear combination of $(a_1, b_1) \dots (a_{n+m+1}, b_{n+m+1})$

(3) We have that any $(m, m') \in M \oplus M'$ we have that for any $r \in R$ we have that $r(m, m') = 0$ iff $rm = 0, rm' = 0$. Thus $(m, m') \in \mathrm{Tor}_R(M \oplus M')$ iff $m \in \mathrm{Tor}_R(M)$, $m' \in \mathrm{Tor}_R(M')$. So we have the cononical mapping $\pi : \mathrm{Tor}_R(M \oplus M') \equiv \mathrm{Tor}_R(M) \oplus \mathrm{Tor}_R(M')$ where $\pi(m, m') = (m, m')$ is an isomorphism.

Exersise 6.5

(1) Let the generators of M be $g_1, \dots g_n$. Since M is torsion there exists $r_1, \dots r_n \in R, r_i \neq 0$ where $r_1 g_1 = r_2 g_2 = \dots r_n g_n = 0$. Thus for any $m \in M$ we have $(r_1 r_2 r_3 \dots r_n) m = 0$ since m is linear sum of elts of $g_1, \dots g_n$ and since R is commutative we can rearrange so that r_i multiplies with g_i in each term. Thus $(r_1 r_2 \dots r_n) \in \mathrm{Ann}(M)$, and $(r_1 r_2 \dots r_n) \neq 0$ since R is an ID

(2) Let $R = \mathbb{Z}$ and

$$S = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \dots \mathbb{Z}/2^n \oplus \dots$$

M is defined as the set of finite tuples in S. Thus we have that every element in M is torsion since for any $m \in M$, there is an N such that for $n > N$ the nth component of m is zero. Thus $2^N m = 0$. However $\mathrm{ann}(R) = 0$ since for any $2^n \in R$ we have that the element m with $n+1$ component 1 multiplies with 2^n to yield $2^n \neq 0$ in the $n+1$ th component.

Exersise 6.6

If we consider the free module R, we know that every submodule is an ideal and thus from our assumption every ideal $I \subset R$ is free. Therefore for each I there is a generating set A. A can only have one element since the rank of R is 1 so every submodule has rank 1. Thus $I = (a)R$ so R is a PID