Exercise 41

Let us define the metric $d(x,y) = |x-y|$. Then $B = \{x \in \mathbb{R}^m : d(x,0) \leq 1\}$. For any convergent sequence $(a_n)_n \in B$ with limit a, we have that $a \in B$ since if $d(a,0) = 1+\epsilon$ then for any $N > 0$ and $n > N$ we have that $d(a_n,a) \leq d(a_n,0)+d(a,0) \Rightarrow d(a_n,a)-d(a_n,0) \geq d(a,0)$ so $d(a_n,a) \geq \epsilon$ which cotradicts convergence. Thus B is closed. B is clearly bounded since $d(x,0) \leq 1 \forall x \in B$. Thus since B is closed, bounded, and a subset of $\mathbb{R}^m$, it is compact.

Exercise 42

The problem is that it is not necessarily true that the convergent subsequences in $(a_n)_n$ and $(b_n)_n$ will have the same indicies. So we dont necessarily have that (a_{n_k}, b_{n_k}) exists as a convergent subsequence of (a_n, b_n)

Exercise 43

For any sequences $(a_n)_n \in A$, $(b_n)_n \in B$, since $A \times B$ is compact we have that the sequence (a_n, b_n) has a convergent subsequence (a_{n_k}, b_{n_k}). We know that a sequence in $M \times N$ converges iff the components in M and N converge. Therefore a_{n_k} and b_{n_k} are convergent subsequences for $(a_n)_n, (b_n)_n$ respectivly. Thus A, B are compact.

Exercise 44

(a) For any convergent sequence $(m_n, y_n)_n \in G, m_n \in M, y_n \in \mathbb{R}$ where G is the graph of f with the limit $(m_n, y_n) \to (m, y)$, we have that $y_n = f(m_n)$. Thus since f is continuous we have that $f(m) = f(y)$ and thus $(y, m) \in G$, so G is closed

(b) For any sequence $(m_n, y_n)_n \in G, m_n \in M, y_n \in \mathbb{R}$, since M is compact, there exists a convergent subsequence (m_{n_k}) of $(m_n)_n$, and thus $(m_{n_k}, y_{n_k}) \to (m, y)$, we have that $y_n = f(m_n)$. Thus since f is continuous we have that $f(m) = f(y)$ and thus $(y, m) \in G$, so G is compact

(c) Suppose for contradiction there is a convergent sequence $m_n \in M$ with limit m where $f(m_n)$ does not converge. Thus there exists a $\epsilon > 0$ where we can choose a subsequence $y_{n_k} = f(m_{n_k})$ such that $d(y_{n_k}, f(m)) > \epsilon$. However no subsequence of (m_{n_k}, y_{n_k}) converges since if we have any convergent subsequence $(m_{n_{k_l}}, y_{n_{k_l}})_l$ then we have that $m_{n_{k_l}} \to m$ yet $y_{n_{k_l}} \to y \neq f(m)$. Therefore G would not contain the limit point of $(m_{n_{k_l}}, y_{n_{k_l}})_l$ which would contradict G being closed (which is implied by compactness). Thus we contradict compactness.

(d) We can define the discontinous function

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$$

We have the graph is the union of three closed sets in $\mathbb{R}^2$ the singleton $\{0\}$, and two curves $\{(x, y) : y = \frac{1}{x}, x > 0\}$ and $\{(x, y) : y = \frac{1}{x}, x < 0\}$ which are closed, thus the graph is closed.

Exercise 46

We have that $A \times B$ is the product of compact sets and thus compact. We know that the distance function d is continuous, and thus $d : A \times B \to \mathbb{R}$ maps to a compact set. Thus $d(A \times B)$ is compact so it contains its smallest value. Thus we have that $\exists (a, b) \in A \times B$ with $d(a, b) \leq d(a_0, b_0)$ for all $(a_0, b_0) \in A \times B$

Exercise 53

This is true. For each K_n choose two points points $a_n, b_n \in K_n$ where $d(a_n, b_n) =$diam K_n. We have the sequence $(a_n, b_n)_n \in K_1^2$. Since K_1 is compact we know that there exists a subsequence $(a_{n_k}, b_{n_k})_k$ which converges to (a, b). We have the limits for the components (since a sequence conveges iff its components converge) $a, b \in K$ since each K_i contains the tail of the subsequences a_{n_k}, b_{n_k} for $n_k > i$ (which has the same limit) and since each K_i is closed, it must contain the limit thus each K_i contians a, b. Now we have that $d(a_{n_k}, b_{n_k})$ is a convergent sequence converging to $d(a, b)$ since d is continuous. Since $d(a_{n_k}, b_{n_k}) \geq \mu$ we know that its limit $d(a, b) \geq \mu$. Thus diam $K \geq \mu$

Exercise 55

(a) If p is a limit, then we have a sequence $(p_n)_n \in S$ where for each $\epsilon > 0$ we can choose a p_n where $d(p_n, p) < \epsilon$ and the $\inf\{d(p_n, p)\} = 0$ we have $\inf\{d(p_n, p)\} \geq \operatorname{dist}(S, p) \geq 0$, thus $\operatorname{dist}(S, p) = 0$. Conversely if $\operatorname{dist}(S, p) = 0$ then for $\epsilon = \frac{1}{n}$ we can choose $p_n \in S$ such that $d(p_n, p) < \epsilon$. Thus we have that the sequence $(p_n)_n$ converges to p.

(b) For any $\epsilon > 0$ let $\delta = \epsilon$. For any p, q with $d(p, q) < \delta$ we have that for any $s \in S$

$$d(q, s) \leq d(p, q) + d(p, s) \Rightarrow d(q, s) - d(p, s) \leq d(p, q) < \epsilon$$

relabeling q and p yields the other inequality: $d(p, s) - d(q, s) \leq \epsilon$. Thus we have that $|d(p, s) - d(q, s)| < \epsilon$ for all $s \in S$. We have that

$$|\operatorname{dist}(p, S) - \operatorname{dist}(q, S)| = |\inf\{d(p, s), s \in S\} - \inf\{d(q, s), s \in S\}| \leq \inf\{|d(p, s) - d(q, s)|, s \in S\} \leq \epsilon$$

We have the first inequality since for any $d(p, s_1) - d(q, s_2)$ we have that $|d(p, s_1) - d(q, s_2)| \leq |d(p, s_1) - d(q, s_1)|$. Thus we have that $\operatorname{dist}(p, S)$ is uniformly continuous.

Exercise Additional Problem 1

Suppose for contradiction that $A/(U_1 \cup \ldots U_n)$ is not empty for all n. Then we can choose a sequence $(a_n)_n$ where $a_n \in A/(U_1 \cup \ldots U_n)$. Since A is compact, there exists a subsequence $(a_{n_k})_k \in A$ which converges to a. We have that $a \in U_N$ for some N since A is a union of the U_is. Since U_N is open there is a r such that $B_r(a) \subset U_N$. However this is a contradiction of convergence of a_{n_k} since for all $n_k > N$ we have that $a_{n_k} \notin U_N$ and thus not in $B_r(a)$. Thus we have that $A/(U_1 \cup \dots \cup U_N) = \emptyset$ so $A \subset (U_1 \cup \dots \cup U_N)$.

Exercise Additional Problem 2

We know the norm is continuous, thus $|f(x)|$ is a continuous function mapping to $\mathbb{R}$. We know the preimage of open sets are open. We can use these facts as follows:
Suppose for contradiction f was unbounded. We have that

$$\mathbb{R} = (-1,1) \cup (-2,2) \cup (-3,3) \cup \dots (-n,n) \cup \dots$$

. Thus if we take the preimage of f we get a union of open sets

$$A = U_1 \cup U_2 \cup \dots \cup U_n \cup \dots$$

We have that $A/(U_1 \cup U_2 \cup \dots U_N) \neq \emptyset$ for all N and thus A is not a finite union of open sets which is a contradiction. We know that $A/(U_1 \cup U_2 \cup \dots U_N) \neq \emptyset$ since f is unbounded there exists $a \in A$ such that $|f(a)| > N$ and thus $f(a) \notin (-N, N)$ which means a is not in the preimages $U_1 \dots U_N$