Math 506, SPRING 2018 ***HOMEWORK #8*** ***Peter Gylys-Colwell***

1

If we consider a morphism ϕ with kernel k

$$K \xrightarrow{k} A \xrightarrow{\phi} B$$

For any morphisms $\varphi_1, \varphi_2 : C \to K$ where $k \circ \varphi_1 = k \circ \varphi_2$ which we will call f. We have that

$$\phi \circ k \circ \varphi_1 = \phi \circ k \circ \varphi_2 = \phi \circ f = 0_{CB}$$

And thus from the universal property of the kernel there is a unique φ so that the diagram commutes

$$\begin{tikzcd}
& A \arrow[r, "\phi"] & B \\
C \arrow[ur, "f"] \arrow[r, dashed, "\exists!\varphi"] & K \arrow[u, "k"] \arrow[ur, "0"] &
\end{tikzcd}$$

Thus since φ_1, φ_2 both are morphisms in the place of φ that make the diagram commute, they must be equal. Therefore k is a monomorphism

2

It is the case every monomorphism $\phi : A \to B$ is the kernel of $\operatorname{coker}\phi$. To prove this, we will show that A is isomorphic to the image and thus by definition the kernel of the cokernel. As proven in problem 4 we know that the kernel of a monomorphism is the zero morphism. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\operatorname{coim}\phi"] & B \arrow[r, "\operatorname{coker}\phi"] & \operatorname{coker}\phi \\
& \operatorname{coim} \arrow[r, "v"] & \operatorname{im}\phi \arrow[u, "\operatorname{im}\phi"'] &
\end{tikzcd}$$

From the axioms of Abelian Categories v is an isomorphism. $\operatorname{coim}\phi = \operatorname{coker}\ker\phi$ and since $\mathrm{id} \circ 0 = 0$ from the universal property

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\mathrm{id}"] \arrow[d] & A \\
& \operatorname{coim}\phi \arrow[ur, "\exists!\alpha"'] &
\end{tikzcd}$$

There is a unique α such that $\operatorname{coim}\phi \circ \alpha = \mathrm{id}$. Thus $\operatorname{coim}\phi$ is an isomorphism and we have the isomorphism $\operatorname{coim}\phi \circ v : A \to \operatorname{im}\phi$

3

($\Rightarrow$) If $\phi : A \to B$ is a monomorphism yet had nontrivial kernel K (not injective) then we would have the inclusion map $\pi : K \to A$ and zero map $0 : K \to A$ compose to the same zero map:

$$0 = \phi \circ \pi = \phi \circ 0$$

which contradicts ϕ be a monomorphism since $0 \neq \pi$
($\Leftarrow$) If ϕ is injective then if we have

$$\phi \circ \varphi_1 = \phi \circ \varphi_2$$

Then for any element g in the domain of φ_1, φ_2 we have

$$\phi(\varphi_1(g)) = \phi(\varphi_2(g))$$

Since ϕ is injective this means that $\varphi_1(g) = \varphi_2(g)$ and thus $\varphi_1 = \varphi_2$ so ϕ is a monomorphism

4

The image of a zero morphism $0 \to A$ is precisely the same morphism thus we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & A \arrow[r, "\phi"] & B \\
 & 0 \arrow[u] \arrow[r] & \ker\phi \arrow[ul, "k"]
\end{tikzcd}$$

($\Rightarrow$) If the mapping $0 \to \ker\phi$ is an isomorphism then $\ker\phi = 0$.
If it is the case

$$\phi \circ \varphi_1 = \phi \circ \varphi_2$$

Then since we are in an additive category

$$0 = \phi \circ \varphi_1 - \phi \circ \varphi_2 = \phi \circ (\varphi_1 - \varphi_2)$$

Thus $\varphi_1 - \varphi_2$ must factor through $\ker\phi$ but since the only morphism to $0 = \ker\phi$ is the zero morphism it must be the case $\varphi_1 - \varphi_2 = 0$ which means $\varphi_1 = \varphi_2$ so ϕ is a monomorphism
($\Leftarrow$) If ϕ is a monomorphism then we have

$$\phi \circ 0_{\ker\phi, A} = \phi \circ k = 0_{\ker\phi, B}$$

Thus it must be the case $k = 0$ which means $\ker\phi = 0$ so $0 \to \ker\phi$ is an isomorphism

5

1. Let $\mathscr{C}$ be a category with a zero object. The cokernel of a morphism, if it exists, is an epimorphism
2. In an abelian category every epimorphism is the cokernel of a morphism. Thus we can conclude in an abelian category a morphism $\phi : A \to B$ is an epimorphism if and only if $\phi = \operatorname{coim}\phi$

3. A morphism $\phi : A \to B$ in $\mathcal{A}b$ is a epimorphism if and only if it is surjective
4. Let $\mathscr{A}$ be an abelian category and $\phi : A \to B$ a morphism in $\mathscr{A}$. We have that ϕ is an epimorphism if and only if the sequence of morphisms $A \xrightarrow{\phi} B \longrightarrow 0$ is exact

6

$(i) \Leftrightarrow (ii)$:
If we consider the sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$$

From problem 4 we know the sequence is exact around A if and only if ϕ is a monomorphism. From the dual statment of problem 4 we get that the sequence is exact around B if and only if ϕ is an epimorphism

$(i) \Rightarrow (iii)$
From problem 2 we know that since ϕ is a monomorphism ϕ is the kernel of $\operatorname{coker}\phi$ thus is the equalizer of $\operatorname{coker}\phi$ and 0. It is the case that any epimorphism that is an equalizer is an isomorphism and thus ϕ is an isomorphism. The reason for this is as follows:
If $\phi : K \to A$ is an equalizer of $\varphi_1, \varphi_2 : A \to B$, then by the definition of equalizer and epimorphism

$$\phi \circ \varphi_1 = \phi \circ \varphi_2 \Rightarrow \varphi_1 = \varphi_2$$

Thus ϕ trivially equalizes φ_1, φ_2. Since the trivial equalizer is the identity $\mathrm{id}_A : A \to A$ and equalizers are unique up to isomorphism, it must be the case ϕ is an isomorphism

$(iii) \Rightarrow (i)$
If ϕ is an isomorphism then there exists a ϕ^{-1}. Thus for some morphisms φ_1, φ_2

$$\varphi_1 \circ \phi = \varphi_2 \circ \phi$$

it must be the case

$$\varphi_1 \circ \phi \circ \phi^{-1} = \varphi_2 \circ \phi \circ \phi^{-1}$$
$$\Downarrow$$
$$\varphi_1 = \varphi_2$$

and similarly if

$$\phi \circ \varphi_1 = \phi \circ \varphi_2$$
$$\phi^{-1} \circ \phi \circ \varphi_1 = \phi^{-1} \circ \phi \circ \varphi_2$$
$$\Downarrow$$
$$\varphi_1 = \varphi_2$$

And thus ϕ is both a monomorphism and epimorphism