6.3 No, if there were some generator $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ we have that $(a,b)^n = (na, nb)$ but there is no possible power $n \in \mathbb{Z}$ such that $(a, b-1) = (na, nb) = (a,b)^n$ since $a = na$ implies that $n = 1$ but then $nb \neq b - 1$.

6.5 For any element $(a,b) \in A \times B$ we have $(a,b) \circ (a^{-1}, b^{-1}) = (e_a, e_b)$, and $(a^{-1}, b^{-1}) \in A \times B$ since, A, B are groups. Therefore every element has an inverse. We already know the operations are assosiative since crossing two associative operations is an associative operation, and finally we know $A \times B$ is closed under these operations since we just apply the operations component wise and A, B are closed under their respective operation. Therefore $A \times B$ is a subgroup of $G \times H$

6.10 We have

$$\{(0,0)\}, \langle(1,0)\rangle, \langle(1,1)\rangle, \langle(0,1)\rangle, \langle(0,2)\rangle, \langle(1,2)\rangle$$

For a total of 6 subgroups

6.12 (i). If (a,b) is a generator of $G \times H$, then for any $g \in G$ and $h \in H$ we have for some $n \in \mathbb{Z}$, since $G \times H$ is a cyclic for $(g,h) \in G \times H$ we have $(a,b)^n = (a^n, b^n) = (g,h) \Leftrightarrow a^n = g, b^n = h$ and so a, b are generators of G, H respectively
(ii). For any subgroup $A \times B$ of $G \times H$ we know that for any $(a,b) \in A \times B$, $(a,b)^{-1} = (a^{-1}, b^{-1}) \in A \times B$, we know the group operations must be closed and assosiative as well. Therefore A and B satisify all the conditions to be subgroups of G, H respectively since the inverse of every element in A, B is contained in A, B respectively and the sets are closed under their respective group operation.

13.10

a. If G is abelian then $G \times G$ is abelian. We know that any subgroup of an abelian group is normal, and so this would imply D is normal. Conversly if G was not abelian, we can take elements $a, b \in G$ that dont commute, we have for $(b,b) \in D$

$$(a,b)(b,b)(a,b)^{-1} = (aba^{-1}, bbb^{-1}) = (aba^{-1}, b)$$

Since $ab \neq ba$ we know $(ab)a^{-1} \neq baa^{-1} = b$ which means

$$(a,b)(b,b)(a,b)^{-1} = (aba^{-1}, b) \notin D$$

b. Let $\varphi : G \times G$ be defined as $\varphi(a,b) = ab^{-1}$, we have

$$\varphi(a,b)\varphi(c,d) = ab^{-1}cd^{-1} = ac(bd)^{-1} = \varphi(ac, bd)$$

So φ is a homomorphism. D is precisely the kernel of φ since $\varphi(a, b) = e \Leftrightarrow ab^{-1} = e \Leftrightarrow a = b$. Therefore by the fundamental theorem we have

$$(G \times G)/D \cong G$$

13.11

a. We can define a homomorphism $\varphi : G \to G/H \times G/K$ with $\varphi(g) = (gH, gK)$. To show it is a homomorphism we have for $a, b \in G$:

$$\varphi(a)\varphi(b) = (aH, aK)(bH, bK) = (abH, abK) = \varphi(ab)$$

We know that $\ker(\varphi) = H \cap K$ since $\varphi(g) = (H, K) \Leftrightarrow g \in K$ and $g \in H$. Therefore by the Fundamental Theorem we have

$$G/(H \cap K) \cong \varphi(G)$$

We know $\varphi(G)$ must be a subgroup of $G/H \times G/K$ since the image of a homomorphism is a group. And so we are done

b. If $G = HK$ we can show the φ from part a is surjective which would imply $\varphi(G) = G/H \times G/K$. For any $(aH, bK) \in G/H \times G/K$. Since $G = HK$, $a = h_a k_a, b = h_b k_b$ where $h_a, h_b \in H, k_a, k_b \in K$. Now since H, K are normal:

$$(h_a k_a H, h_b k_b K) = (h_a H k_a, h_b K) = (k_a H, h_b K)$$

and so we have

$$\varphi(k_a h_b) = (k_a H, h_b K)$$

And so φ is surjective.

13.16 We can define $\varphi : G \times H \to G/A \times H/B$ with $\varphi(g, h) = (gA, hB)$. We have that φ is a homomorphism since $\varphi(g_1, h_1)\varphi(g_2, h_2) = (g_1 A, h_1 B)(g_2 A, h_2 B)$ since A, B are normal, $= (g_1 g_2 A, h_1 h_2 B) = \varphi(g_1 g_2, h_1 h_2)$. We also know $\ker(\varphi) = A \times B$ since $\varphi(a, b) = (A, B)$ iff $a \in A, b \in B$. We know φ is surjective since for any (gA, hB) we know there is a $g \in G, h \in H$ such that $\varphi(g, h) = (gA, hB)$. Therefore by the fundamental thm we have our result:

$$\frac{G \times H}{A \times B} \cong G/A \times H/B$$

13.20 We can commpose φ with the canonical homomorphism $\rho : K \to K/J$. The composition of homomorphisms is a homomorphism so $\varphi \circ \rho$ is a homomorphism. Now we can use the Fundamental Theorem, letting $f = \varphi \circ \rho$ we have $f : G \to K/J$ is a homomorphism and is surjective since ρ is surjective and φ is surjective.

$$G/\ker(f) \cong K/J$$

And $\ker(f)$ is some normal subgroup H of G.

13.25

a. We have for D_3, the symmetry group of the triangle which is not abelian, we established in class $H = \{e, FR\}$ is a normal subgroup and H is abelian since there is only two elements and one of them is e. We also know D_3/H is abelian since $D_3/H = \{eH, RH, R^2H\}$ and all the Rs commute. So D_3 is metablelian.

b. Let H be the subgroup of G that is abelian along with G/H being abelian. We know $\varphi(H)$ is an abelian subgroup of K that is normal since the image of an abelian group is an abelian group for any homomorphism and from thm 13.3 we get normality. We will call $\varphi(H)$ J for convienence. We have that K/J is abelian since for any $aJ, bJ \in K/J$ since φ is surjective there is some $a_g, b_g \in G : \varphi(a_g) = a, \varphi(b_g) = b$ and so we have

$$aJbJ = abJ = \varphi(a_g)\varphi(b_g)J = \varphi(a_g b_g H)$$

and since G/H is abelian

$$= \varphi(b_g a_g H) = baJ = bJaJ$$

And so K is metablelian