Math 441, Fall 2017 **HOMEWORK #9** **Peter Gylys-Colwell**

Exercise 1

Checking equivalence relation axioms for any paths $f, g, h : [a, b] \to Y$ from x to y in Y:
Reflexivity:
For f, we define $H : [a, b] \times I \to Y$ as $H(s, t) = f(s)$. H is continous since it is equal to the composition of the continous maps $\mathrm{id}_{[a,b]} : [a, b] \times I \to [a, b]$ and $f : [a, b] \to Y$. We have that H is a path homotopy from f to f since $H(s, 0) = f(s) = H(s, 1)$, $H(a, t) = f(a) = x, H(b, t) = f(b) = y$ and thus $f \sim f$
Symmetry:
If $f \sim g$ then there exists $H : [a, b] \times I \to Y$ where $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$, $H(a, t) = x, H(b, t) = y$. We can define $H' : [a, b] \times I \to Y$ where $H' = H \circ (\mathrm{id}_{[a,b]}, 1 - \mathrm{id}_I)$. H' is continous since $(\mathrm{id}_{[a,b]}, 1 - \mathrm{id}_I)$ is component-wise continous and so continous and thus H' is the composition of continous functions. We have that H' is a path homotopy from g to f since $H'(s, 0) = H(s, 1 - 0) = g(s)$, $H'(s, 1) = H(s, 1 - 1) = f(s)$ and $H'(a, t) = H(a, 1 - t) = x, G'(b, t) = H(b, 1 - t) = y$. Thus $g \sim f$
Transitivity:
If there exists a path homotopy H from f to g and path homotopy G from g to h ($f \sim g, g \sim h$) we can define the homotopy $F : [a, b] \times I \to Y$ using the pasting lemma as follows. Consider $H' : [a, b] \times [0, 1/2] \to Y$ as $H' = H \circ (\mathrm{id}_{[a,b]}, \frac{1}{2}\mathrm{id}_I)$ and $G' : [a, b] \times [1/2, 1] \to Y$ as $G' = G \circ (\mathrm{id}_{[a,b]}, \frac{1}{2} + \frac{1}{2}\mathrm{id}_I)$. Both these mappings are continous since they are the composition of continous mappings, and their domains intersect on $S = [a, b] \times \{1/2\}$. We have that $H'(S) = G'(S)$ since $H'(s, 1/2) = H(s, 1) = g(s) = G(s, 0) = G'(s, 1/2)$. Thus we define $F : [a, b] \times I \to Y$ using the pasting lemma. F is a path homotopy from f to h since $F(s, 0) = H'(s, 0) = H(s, 0) = f(s)$ and $F(s, 1) = G'(s, 1) = G(s, 1) = h(s)$. Also $F(a, t) = H(a, 1/2t) = x$ or $= G(a, 1/2 + 1/2t) = x$ and $F(b, t) = F(b, t) = H(b, 1/2t) = y$ or $= G(b, 1/2 + 1/2t) = y$. Thus $f \sim h$

Exercise 2

If there exists $\theta : S^1 \to \mathbb{R}$ such that $p \circ \theta = \mathrm{id}_{S^1}$, from Exercise §24, 2 we know there exists $t \in S^1$ such that $\theta(t) = \theta(-t)$. Then we have $p(\theta(t)) = p(\theta(-t))$ which is a contradiction since that implies $t = -t$.

Exercise 3

We have the map $f : [0, 2\pi] \to S^1$ with $f(\theta) = (\cos(\theta), \sin(\theta))$. We know that f is continous, and $[0, 2\pi]$ is simply connected, however S^1 is not simply connected. Conversly we have the constant map $g : S^1 \to \{0\}$ where $g(s) = 0$. g is continous and $\{0\}$ is simply connected, while S^1 is not.

Exercise 4

We can assume S_1 is the circle with center $\{(0, 0)\}$ not in the set since shifting and scaling $\mathbb{R}^2$ are homeomorphisms, thus A is homeomorphic to such a set. We have the inclusion

mapping $i : S^1 \to A$ which is the the continous identity mapping of $S^1 \subset A$. There exists the retraction $\rho : A \to S^1$ with $\rho(x) = \frac{x}{|x|}$. ρ satisfies $\rho \circ i = \text{id}$ since every $x \in S^1$ has norm 1 so $\frac{x}{|x|} = x$. We have that any loop $f : [a, b] \to S^1$ based at $x \in S^1$ that is not null homotopic invokes a loop based at $i(x)$ that is not null homotopic in A and thus A is multiply connected since S^1 multiply connected implies there exists non null homotopic loops in A. We get this loop in A as $f' = i \circ f$. We have that f' cannot be null homotopic since if there existed a homotopy $H' : [a, b] \times I \to A$ from f' to the constant loop, then we would have the homotopy $H : [a, b] \times I \to S^1$ from f to the constant loop defined as $H(s, t) = \rho(H'(s, t))$ which would be a contradiction. Checking H is the described homotopy:
$H(a, t) = \rho(H'(a, t)) = x = H(b, t) = \rho(H'(b, t)) = H(b, t)$
$H(s, 0) = \rho(H'(s, 0)) = \rho(i(f(s))) = f(s)$
$H(s, 1) = \rho(H'(s, 1)) = \rho(i(x)) = x$
And thus we are done

Exercise 7

($\Rightarrow$) We already know the direct product of connected spaces is connected, thus $X_1 \times \cdots \times X_n$ is connected.
For any loop $f : I \to X_1 \times \cdots \times X_n$ based at $x = (x_1, \ldots x_n)$, we will show f is null homotopic and thus $X_1 \times \cdots \times X_n$ is simply connected. If we consider the compotnents of $f : f_1, f_2, \ldots f_n$, these are loops in $X_1, X_2, \ldots X_n$ based at $x_1, x_2, \ldots x_n$ respectively. Thus since $X_1, X_2, \ldots X_n$ are simply connected there exists path homotopies $H_1, H_2, \ldots H_n : [a, b] \times I \to X_1, X_2, \ldots X_n$ from $f_1, f_2 \ldots f_n$ to the constant loops. We have that $H = (H_1, H_2, \ldots H_n)$: is a path homotopy from f to the constant loop and thus f is null homotopic. Checking H is a path homotopy:
$H(a, t) = (H_1(a, t), H_2(a, t), \ldots H_n(a, t)) = x = (H_1(b, t), H_2(b, t), \ldots H_n(b, t)) = H(b, t),$
$H(s, 0) = (H_1(s, 0), H_2(s, 0), \ldots H_n(s, 0)) = (f_1(s), f_2(s), \ldots f_n(s)) = f(s)$
$H(s, 1) = (H_1(s, 1), H_2(s, 1), \ldots H_n(s, 1)) = (x_1, x_2, \ldots x_n) = x$
($\Leftarrow$) If $X_1 \times X_2 \cdots \times X_n$ is simply connected, for any loop $f : I \to X_i$, we can extend the loop to $X_1 \times \ldots X_n$ by letting each component $\neq i$ be constant. Thus there is a homotopy from the extention of f to the constant loop. The ith component of this homotopy will be a homotopy from f to to the constant loop. Thus X_i is simply connected.

Exercise 8

We can define the continous map $f : S^{n-1} \times \mathbb{R}^+ \to R^n \setminus \{0\}$ with $f(x, r) = rx$. Notice this mapping is the same as the polar coordinate representation of R^n. f has the continous inverse $f^{-1}(x) = (\frac{x}{|x|}, |x|)$ and thus is a homeomorphism. Thus since S^{n-1} and $\mathbb{R}^+$ are simply connnected for $n \geq 3$ we know that $\mathbb{R}^n$ is simply connected.

Exercise 9

If we have some homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^n$ for $n \geq 3$, we can consider the restriction $f' : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$. This restriction must be a homeomorphism, however this is not possible since $\mathbb{R}^2 \setminus \{0\}$ is not simply connected yet $\mathbb{R}^n \setminus \{f(0)\} \cong \mathbb{R}^n \setminus \{0\}$ is simply connected as proven in Exercise 8.