Exercise 9

f has to be constant. Given any $x, y \in \mathbb{R}$ for any $\epsilon > 0$ by equicontinuity there exists $\delta > 0$ so that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| = |f(nx) - f(ny)| < \epsilon$ for all n. We can do a change of variables: $a = nx, b = ny$ to have that

$$\frac{|a-b|}{n} < \delta \Rightarrow |f(a) - f(b)| < \epsilon$$

Thus we get for all $a, b \in \mathbb{R}$ it must be the case (by choosing n sufficiently large) that $|f(a) - f(b)| < \epsilon$ for all $\epsilon > 0$. Thus $f(a) - f(b) = 0$ so f must be constant

Exercise 12

We will show each limit point satisfy the same equicontinuity conditions as every other $f \in \mathcal{E}$. Consider a sequence of functions $(f_n) \in \mathcal{E}$ which uniformly converge to f. Given $\epsilon > 0$. By equicontinuity of $\mathcal{E}$, given $x \in M$ there exists $\delta > 0$ so that for all $y \in M$ with $d(x, y) < \delta$ we have $d(g(x), g(y)) < \epsilon/3$ for all $g \in \mathcal{E}$ (In particular this is true for each f_n) thus since we can choose N so that for $n > N$ we have $d_u(f_n, f) < \epsilon/3$ (where d_u is the uniform metric). From the triangle inequality we have that this δ works for any limit point

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f(y), f_n(y)) < \epsilon$$

By choosing n large enough we have $d(f(x), f_n(x)), d(f(y), f_n(y)) < \epsilon/3$

Exercise 13

(a) Consider the covering of $\mathbb{R}$ by the compact sets $U_k = [-k, k]$. We have that $f_n|_{U_k}$ is uniformly bounded and equicontinuous (in lecture we established pointwise equicontinuous is the same as equicontinuous over a compact set). From this it follows for U_1 there is a subsequence $f_{1,1}, f_{2,1}, f_{3,1} \ldots$ which is uniformly convergent restricted to U_1

For each U_k there is a subsequence of the previous sequence $f_{1,k}, f_{2,k} \ldots$ which, when taking the restriction to U_k, uniformly converges to a continuous function over U_k. We can make a new subsequence $f_{k,k}$ by taking the diagonal of the matrix of subsequences

$$\begin{gathered} f_{1,1}, f_{2,1}, f_{3,1}, f_{4,1} \ldots \\ f_{1,2}, f_{2,2}, f_{3,2}, f_{4,2} \ldots \\ f_{1,3}, f_{2,3}, f_{3,3}, f_{4,3} \ldots \\ f_{1,4}, f_{2,4}, f_{3,4}, f_{4,4} \ldots \\ \vdots \end{gathered}$$

We have that this subsequence converges pointwise to a continuous function as follows.

If we fix $x \in \mathbb{R}$ there is a k so that $x \in U_k$. For $n > k$ we have that $f_{n,n}|_{U_k}$ is a Cauchy sequence of continuous functions under the uniform norm and thus the limit of $f_{n,n}|_{U_k}$ is continuous at x. This limit is the restriction of the limit f of $f_{n,n}$ to U_k and thus f is continuous at x.

(b) We don't necessarily have uniform convergence. Consider the sequence of functions

$$f_n(x) = \begin{cases} 1 - |x-n| & x \in [n-1, n+1] \\ 0 & x \notin [n-1, n+1] \end{cases}$$

The sequence is pointwise bounded and equicontinuous since f_n is just a horizontal shift of f_1 which is a bounded continuous function. f_n converges pointwise to 0 since $\lim_{n\to\infty} f_n(x) = 0$ for any x however it does not converge uniformly since $|f_n - 0|_u = 1$ for all n

Exercise 15

(a) ($\Rightarrow$) If f is uniformly continuous, we define our modulus of continuity to be

$$\mu(s) = \sup\{|f(x) - f(y)| : x, y \in [a,b], |x-y| < s\}$$

We have that $\mu(s) \to 0$ as $s \to 0$ since by uniform continuity we can choose $\epsilon > 0$ and then choose a $\delta > 0$ so that

$$|f(x) - f(y)| < \epsilon \quad \forall x, y \in [a,b] : |x-y| < \delta$$

Thus $\mu(\delta) < \epsilon$ we can always choose $\delta \to 0$ arbitrarily small as well to get $\lim_{s\to 0} \mu(s) < \epsilon$. Since ϵ can be arbitrarily small,

$$\lim_{s\to 0} \mu(s) = 0$$

($\Leftarrow$) If f has a modulus of continuity, given $\epsilon > 0$ we can choose $\delta > 0$ by continuity of μ at 0 so that

$$|s-t| < \delta \Rightarrow \mu(|s-t|) < \epsilon$$

Thus since $|f(s) - f(t)| < \mu(|s-t|)$ we have the conditions for uniform continuity

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$$

(b) ($\Rightarrow$) If $\mathcal{E}$ is equicontinuous then

$$\mu(s) = \sup\{|f_n(x) - f_n(y)| : \forall n \in \mathbb{N}, x, y \in [a,b], |x-y| < s\}$$

We have that $\mu(s) \to 0$ as $s \to 0$ since by equicontinuity we can choose $\epsilon > 0$ and then choose a $\delta > 0$ so that

$$|f_n(x) - f_n(y)| < \epsilon \quad \forall n \in \mathbb{N}, \forall x, y \in [a,b] : |x-y| < \delta$$

Thus $\mu(\delta) < \epsilon$ we can always choose $\delta \to 0$ arbitrarily small as well to get $\lim_{s\to 0}\mu(s) < \epsilon$. Since ϵ can be arbitrarily small,

$$\lim_{s\to 0}\mu(s) = 0$$

($\Leftarrow$) If $\mathcal{E}$ has a common modules of continuity $\mu(s)$ then given $\epsilon > 0$ we can choose $\delta > 0$ by continuity of μ at 0 so that

$$|s-t| < \delta \Rightarrow \mu(|s-t|) < \epsilon$$

Thus since for any n $|f_n(s) - f_n(t)| < \mu(|s-t|)$ we have the conditions for equicontinuity

$$|s-t| < \delta \Rightarrow \forall n |f_n(s) - f_n(t)| < \epsilon$$

Exercise 19

M is totally bounded thus we have a finite covering of M of $\delta/2$ balls. Let $x_1, \dots x_n$ be the centers of these balls. If these points are all in A then we are done. Otherwise by definition of dense there is $a_1, \dots a_n \in A$ such that

$$x_1 \in B_{\delta/2}(a_1), x_2 \in B_{\delta/2}(a_1), \dots x_n \in B_{\delta/2}(a_n)$$

(this is because M is the limit points of A)
We have that $B_{\delta/2}(x_k) \subset B_\delta(a_k)$ and thus $B_\delta(a_k)$ is a covering of M

Exercise Additional Problem 1

Notice that

$$\left|f^{(m)}(x)\right| = \left|\sum_{k=m}^{\infty} a_k \frac{k!}{(k-m)!} x^{k-m}\right| \leq \sum_{k=m}^{\infty} \frac{Ck!}{R^k (k-m)!}|x|^{k-m}$$

Notice that if we let $g(x) = (1 - x/R)^{-1}$ then we have the following series expansion

$$g^{(m)}(|x|) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \frac{|x|^{k-m}}{R^k}$$

Thus

$$|f^(m)(x)| \leq Cg^{(m)}(x)$$

The following inductive argument shows $g^{(m)} = \frac{m!}{R^m}(g(x))^{m+1}$.
Base case:

$$g'(x) = R^{-1}\frac{1}{(1-\frac{x}{R})^2} = 1!R^{-1}g(x)^2$$

From the inductive hypothesis if

$$g^{(m)}(x) = \frac{m!}{R^m}\left(1 - \frac{x}{R}\right)^{-m-1}$$

differentiating yields the desired result

$$g^{(m+1)}(x) = \frac{(m+1)!}{R^{m+1}}\left(1 - \frac{x}{R}\right)^{-m-2}$$

Thus from our above equalities we have

$$\left|f^{(m)}(x)\right| \le \frac{Cm!}{R^m}(g(|x|))^{m+1} = \frac{Cm!}{R^m}\left(1 - \frac{|x|}{R}\right)^{-m-1}$$