Exersise 8.1

Notice that the set of left proper ideals of R form a partialy ordered set P with inclusion as the ordering relation ($K \leq J \Leftrightarrow K \subseteq J$). We know that P is not empty since $I \in P$.
If we show that every chain in P has an upper bound in P since then by Zorn's Lemma P has a maximal element (which is a maximal ideal). Considering any chain of proper ideals.

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

we have that

$$U = \bigcup_{i=1}^{\infty} I_i \in P$$

U is an ideal since for any $x, y \in U, r \in R$, there exists I_n such that $x, y \in I_n$ then $x + y \in I_n \subseteq U, rx \in I_n \subseteq R$. U is proper since $1 \notin I_i \forall i$ so $1 \notin U$. We have that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \subseteq U$$

So every chain is bounded. Thus we are done.

Exersise 8.2

(a) For any $a \in D$, if we consider the set $1, a, a^2, a^3, \dots a^n$ where n is the dimension of D over k. We have $n+1$ elements and thus they are linearly dependent. So there exists a nonzero polinomial

$$f(a) = k_n a^n + k_{n-1} a^{n-1} + \dots k_1 a + k_0 = 0$$

(b) $D = k$ since for any $a \in D$ and $f \in k[x]$, $f(a) = 0$ we can factor f completely since k is completely

$$f(a) = (a - a_n)(a - a_{n-1}) \dots (a - a_0)$$

Where $a_n, a_{n-1}, \dots a_0 \in k$. Since D is a domain we know $a = a_i$ for one of the a_i and thus $a \in k$

Exersise 8.3

If M is some $k[G]$ module with submodule $N \subset M$, we have the surjective homomorphism $\pi : M \to M/N$. Since k is a subring of $k[G]$ M and M/N are k vectorspaces. We have a k linear section $s : M/N \to M$ such that $\pi \circ s = \mathrm{id}$. The reason for this is because M/N has a basis B as a vectorspace so for each $b \in B$ there is some $m \in M$ with $\pi(m) = b$ then we define $s(b) = m$. s is a fully defined k linear map from where it sends its basis. We have that

$$s'(x) = \frac{1}{|G|} \sum_{g \in G} e_g s(e_{g^{-1}} x)$$

is a $k[G]$ module homomorphism. Checking the properties:
$s'(x+y) = s'(x) + s'(y)$, we can use the fact that $s(x+y) = s(x) + s(y)$

$$s'(x+y) = \frac{1}{|G|}\sum_{g\in G} e_g s(e_{g^{-1}}(x+y)) = \frac{1}{|G|}\sum_{g\in G} e_g s(e_{g^{-1}}x) + e_g s(e_{g^{-1}}y)) = s'(x) + s'(y)$$

For $s'(rx) = rs'(x)$ for $r \in k[G]$ we have that $r = e_{g_1}k_1 + e_{g_2}k_2 + \ldots e_{g_n}k_n$ so

$$s'(rx) = s'(e_{g_1}k_1x + e_{g_2}k_2x + \ldots e_{g_n}k_nx) = k_1s'(e_{g_1}x) + k_2s'(e_{g_2}x) + \ldots k_ns'(e_{g_n}x)$$

We know s' is k linear since

$$s'(kx) = \frac{1}{|G|}\sum_{g\in G} e_g s(e_{g^{-1}}kx) = \frac{1}{|G|}\sum_{g\in G} e_g ks(e_{g^{-1}}x) = ks'(x)$$

Thus we must only check that $s'(e_hx) = e_hs'(x)$.

$$s'(e_hx) = \frac{1}{|G|}\sum_{g\in G} e_g s(e_{g^{-1}h}x)$$

We can relabel $z = h^{-1}g$ and $z^{-1} = g^{-1}h$. Since $h^{-1}G = G$ we have the same sum

$$\frac{1}{|G|}\sum_{z\in G} e_{hz}s(e_{z^{-1}}x) = \frac{e_h}{|G|}\sum_{z\in G} e_z s(e_{z^{-1}}x) = e_hs'(x)$$

Thus s' is a $k[G]$ module homomorphism.
Letting $Q = s'(M/N)$ we have that $M = N \oplus Q$ and thus M is semisimple. We can show $M = N \oplus Q$ by showing $Q \cap N = 0$ and $Q + N = M$ thus from the chinese remainder theorem $M \cong M/N \oplus M/Q$

Exersise 8.4

Consider $R = \mathbb{Z}$ for some prime p we have the chain of R modules

$$0 \to \mathbb{Z}/(p) \to \mathbb{Z}/(p^2) \to (\mathbb{Z}/(p^2))/(\mathbb{Z}/(p)) \cong \mathbb{Z}/(p) \to 0$$

We know $\mathbb{Z}/(p)$ is simple, yet $\mathbb{Z}/(p^2) \ncong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$ since the generator must map to an element of order p^2, so $\mathbb{Z}/(p^2)$ is not simple

Exersise 8.5

(a)
$(i \Rightarrow ii)$
This follows directly from the definition. Letting $N = P$, $\pi = p$, $f =$id. By the definition of a projective there exists $s : P \to M$ with $p \circ s =$id.
$(ii \Rightarrow iii)$

$(iii \Rightarrow i)$
For any R module M, N, surjective homomorphism $\pi : M \to N$ and homomorphism f :

$P \to N$ we can extend f as $f : (P \oplus Q) \to N$ by setting $f(0 \oplus Q) = 0$. We have that $P \oplus Q$ is free so has some generators $g_1, g_2, \dots$. Since π is surjective, there exists $m_1, m_2, \dots \in M$ where $\pi(m_1) = f(g_1), \pi(m_2) = f(g_2) \dots$. Thus we can define a homomorphism using the universal propertiy of free modules

$$g : P \oplus Q \to M \text{ where } g_1 \to m_1, g_2 \to m_2 \dots$$

We have that $\pi(g(g_i)) = g_i$ and thus since homomorphisms from free modules are entirely determined by the image of the generators, $\pi \circ g = \text{id}$. Thus if we restrict g to the submodule P we get the mapping showing P is projective.

(b)
$(i \Rightarrow ii)$
This follows directly from the definition. To use the same notation in the assignments description of injective, letting $M = I$, $N = M$, $\pi = s$, $f =$id it follows from the definition of injective there exists $p : M \to I$ with $p \circ s =$id.
$(ii \Rightarrow i)$

Exersise 8.6

(a) (b) (c)

Exersise 8.7

$(i \Rightarrow ii)$
$(ii \Rightarrow iii)$
$(iii \Rightarrow i)$