Exersise 2.1
We have
$$x^5 + x^2 - x - 1 = (x-1)(x+1)(x^3 + x + 1)$$
The roots are
$$\pm 1, \frac{\alpha}{3^{2/3}} - \frac{1}{3^{1/3}\alpha}, \zeta_3^2\frac{1}{3^{1/3}\alpha} - \zeta_3\frac{\alpha}{3^{2/3}}, \zeta_3\frac{1}{3^{1/3}\alpha} - \zeta_3^2\frac{\alpha}{3^{2/3}}$$
where $\zeta_3 = \frac{1}{2} + \frac{\sqrt{3}}{2}$ is the third root of unity and
$$\alpha = \sqrt[3]{\frac{\sqrt{93} - 9}{2}}$$
(I found these values using wolfram alpha for the roots of $x^3 + x + 1$, then plugged in the values to verify). From this we get the splitting field is
$$\mathbb{Q}(\alpha, \sqrt[3]{3}, \zeta_3)$$
Since $\sqrt[3]{3} = 2(\zeta_3 - 1/2) \in \mathbb{Q}(\zeta_3)$,
$$= \mathbb{Q}(\alpha, \zeta_3)$$
we have
$$[\mathbb{Q}(\alpha, \zeta_3) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \zeta_3) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}]$$
$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ since the degree of the irreducible polynomial is 3. $\zeta_3 \notin \mathbb{Q}(\alpha)$ since $\mathbb{Q}(\alpha) \subset \mathbb{R}$ while $\zeta_3 \notin \mathbb{R}$, so $[\mathbb{Q}(\zeta_3, \alpha) : \mathbb{Q}(\alpha)] \geq 2$. Over $\mathbb{Q}$ the irreducible polynomial of ζ_3 is $x^2 + x + 1$, thus we have $2 = [\mathbb{Q}(\zeta_3) : \mathbb{Q}] \geq [\mathbb{Q}(\zeta_3, \alpha) : \mathbb{Q}(\alpha)]$, So $[\mathbb{Q}(\zeta_3, \alpha) : \mathbb{Q}(\alpha)] = 2$.
Thus
$$[\mathbb{Q}(\alpha, \zeta_3) : \mathbb{Q}] = 6$$

Exersise 2.2
For any element $a \in D$, since D is finite dimensional over k (lets say of degree n) the $n+1$ vectors $1, a, a^2 \ldots a^n$ are linearly dependent and thus there exists $k_0, k_1, \ldots k_n \in k$ where
$$k_0 + k_1 a + \ldots k_n a^n = 0$$
Thus a is algebraic over k. Since k is algebraically closed, this means $a \in k$. Thus $D \subseteq k$ so $D = k$

Exersise 2.3

Exersise 2.4

($\Rightarrow$) If K is a splitting field for some polynomial $f \in k[x]$, then for any irreducible polynomial $p(x) \in k[x]$ with a root $\alpha \in K$ we can consider the ideal $I = \{h \in k[x] : h(\alpha) = 0\} \subset k[x]$. Since $k[x]$ is a PID and p is irreducible and in I, we have that $I = (p)$. It must be the case that
This follows from the Primitive Element Theorem. The Primitive Element Theorem establishes that there exists $\alpha \in K$ such that $K = k(\alpha)$. Thus for any

Exersise 2.5

We can consider the group structure of multiplication over the units of $\mathbb{F}_p$. By Lagrange's Theorem, for any unit $\alpha \in \mathbb{F}_p$, $\alpha^p = \alpha$, thus α is a root of $x^p - x$. Thus all p elements of $\mathbb{F}_p$ (including 0 since $0^p - 0 = 0$) are roots of $x^p - x$. Since $x^p - x$ can have at most p roots (since polynomials have at most their degree number of roots), there can be no multiple roots since it has p distinct roots.

Exersise 2.6

($\Rightarrow$) suppose α is a root of multiplicity ≥ 2. We have that $\alpha^n = 1$, by Lagranges theorem applied to the group of units with multiplication in the prime field of k we know that either $\alpha = 1$ or $n|p$.
If $\alpha \neq 1$ and $n|p$ then we have $x - \alpha$ divides $x^n - 1$ which yields

$$x^n - 1 = (x - \alpha)(x^{n-1} + \alpha x^{n-2} + \alpha^2 x^{n-3} + \dots \alpha^{n-1})$$

α must be a root of the second polynomial, which means

$$\alpha^{n-1} + \alpha^{n-1} + \dots \alpha^{n-1} = n\alpha^{n-1} = 0$$

Since $\alpha^n = 1, \alpha^{n-1} = \alpha^{-1} \neq 0$. Thus it must be the case that $n = 0 \Rightarrow p|n$

If $\alpha = 1$ then $x - 1$ divides $x^n - 1$ yielding

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \dots 1)$$

In order for 1 of multiplicity ≥ 2 it must be a root of

$$x^{n-1} + x^{n-2} + x^{n-3} + \dots 1$$

plugging in 1 yields a sum of n 1s. In order for that sum to be zero, it must be the case that $p|n$
($\Leftarrow$) If $p|n$, again we have

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$

1 is a root of multiplicity ≥ 2 since 1 is a root of $x - 1$ and a root of $x^{n-1} + \dots + x + 1$. Again this is because plugging in 1 we get a sum of n 1s and since the characterstic divides n, that sum is 0