10.2

a. We know that 42 must divide $32|H|$ since $|H|$ is the order of 32 and adding 32 H is the same as multiplying 32 by H, therefore

$$|H| = \frac{LCM(32,48)}{32} = \frac{32 * 3 * 7}{32} = 21$$

And by Lagrange's thm we have

$$|H||G : H| = |G| = 48$$

so $|G : H| = 2$

b. Using the same logic above we have

$$|H| = \frac{LCM(24,54)}{24} = \frac{24 * 3 * 3}{24} = 9$$

and so

$$|G : H| = \frac{|G|}{|H|} = \frac{54}{9} = 6$$

c. Using the same logic:

$$|H| = \frac{LCM(100,112)}{100} = \frac{2 * 2 * 7 * 100}{100} = 28$$

So

$$|G : H| = \frac{|G|}{|H|} = \frac{112}{28} = 4$$

10.5 Given any element $a \in G$, by Lagrange's thm we have $|\langle a\rangle|$ divides $|G| = 8$. Since G is not cyclic we know $\langle a\rangle \neq G$ so $|\langle a\rangle| \neq 8$. Therefore the only options for $|\langle a\rangle|$ are $1, 2, 4$. Since all these numbers divide 4, we know that $a^4 = e$

10.6 We know that the intersection of two subgroups is a group. Therefore if we let $A = H \cap K$, we have that A is a subgroup of both H and K so by Lagrange's thm we have that $|A|$ divides both $|H| = 12$ and $|K| = 5$. The only number that divides both 12 annd 5 is 1, so $|A| = 1$ so $A = \{e\}$

10.14

a. This is because we know for the element x of order 6, $|\langle x\rangle| = 6 = |G|$ and a subgroup of G that is the same size of G is equivalent to G. Therefore $G = \langle x\rangle$ is cyclic

b. By Lagrange's thm for any element $a \in G$, $|\langle a\rangle|$ divides $|G| = 6$ $|\langle a\rangle|$ cannot equal 6 since G is not cyclic, so a must have either order 1, 2, or 3. We know that only e has order 1. We cannot have all the elements have order 2, otherwise we would have two elements a, b, then we would have ab, each having order 2 so they are their own inverses. So we have $\{e, a, b, ab\}$ which is closed with size 4, so we must introduce another element c, which would bring us to $\{e, a, b, c, ab, ac, bc, abc\}$ which is too large. Therefore there is some element a of order 3

c. If we take some element $b \in G : b \notin \langle a\rangle$, we already know $e, a, a^2 \in G$. We know since $b \notin \langle a\rangle$ that $ab, a^2b \notin \langle a\rangle$ since b is not equal to any power of a. Looking at ab, we can deduce $ab \neq a^2b$, since applying b^{-1} on the right yields $a \neq a^2$ which is true. Therefore we have

$$\{e, a, a^2, b, ab, a^2b\} \subseteq G$$

Are all unique elements

d. We cannot have b^2 be a seperate element in the above set since $|G| = 6$, if $b^2 = a$ then $b = a^2$ which is not true, if b^2a^2 then $b = a$ which is not true, if $b^2 = ab$ then applying b^{-1} on the right yields $b = a$ which is not true, and finally if $b^2 = a^2b$ then $b = a^2$ which is not true. Therefore $b^2 = e$.

e. We have $(ba)^{-1} = a^{-1}b^{-1} = a^2b$ but since we concluded a^2b has order 2 $(a^2b)^{-1} = a^2b = ba$. Similarly $(ba^2)^{-1} = ab = (ab)^{-1}$

10.15 By Lagrange's thm we have $|G| = |G : H||H|$ and $|H| = |H : K||K|$ so $|K| = \frac{|H|}{|H:K|}$. We also have

$$|G : K| = \frac{|G|}{|K|}$$

Substituting the equalities for $|G|$ and $|K|$ yields

$$|G : K| = \frac{|H : K||G : H||H|}{|H|} = |G : H||H : K|$$

10.16 Because $|G|$ is odd we know the order of none of the elements except for the identity can be 2.
If the order of some element $a \in G$ was 2 then $|\langle a\rangle| = 2$ but by Lagrange's thm $|\langle a\rangle|$ should divide $|G|$ which cannot happen since $|G|$ is odd.
Therefore for all elements $a \in G$, we know $a^{-1} \neq a$
Therefore if we take the product of all the elements in G, we know that for every element in that product, it's inverse is present in that product as well. Since G is abelian we can

rearrange the product so that each element and it's inverse present in the product cancel out, to be left with e

12.1

a.

b.

c.

12.9

12.12

12.13