Exercise §26, 3

If $\mathcal{A} = \bigcup A_n$ is a finite union of compact sets, for any covering $A = \bigcup U_\alpha$, we have that this is also a covering of each A_n since $A_n \subset A \subseteq \bigcup U_\alpha$. Thus for each A_n there is a finite subcovering $\bigcup U_{k,n}$. Since there is a finite number of $U_{k,n}$ for each A_n and there is a finite number of A_n, we have that the collection of every $U_{k,n}$ for every A_n is finite. Thus we have the finite subcovering for $\mathcal{A}$:

$$\mathcal{A} \subseteq \bigcup U_{n,k}$$

This is true since for every $a \in A$ we have that $a \in A_n$ for some A_n and thus $n \in U_{n,k}$ for some $U_{n,k}$

Exercise §26, 4

Every metric space is Hausdorf, thus from theorem 26.3 we know that every compact subspace is closed. Every compact subspace is bounded since if we choose some $x \in C$ where C is our compact subspace, we have the covering

$$C \subseteq \bigcup_{n \in \mathbb{N}} B_n(x)$$

Thus since C is compact we have a finite union $\bigcup B_{n_k}(x)$ containing C. Since $B_n(x) \subseteq B_m(x)$ for all $n, m \in \mathbb{N}$ we have that $C \subseteq \bigcup B_{n_k}(x) = B_{\max n_k}(x)$. Thus C is bounded.
If we consider $\mathbb{R}$ with the discrete metric topology, then $\mathbb{R}$ is closed and bounded since $d(x,y) \leq 1 \ \forall x, y \in \mathbb{R}$. $\mathbb{R}$ is not compact since

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$$

And clearly no finite subunion equals $\mathbb{R}$

Exercise §26, 5

For any $a \in A$ from lemmma 26.4 we can create a neighborhood U_a and open set V_a disjoint where $a \in U_a, B \subseteq V_a$. Thus we have the covering

$$A \subseteq \bigcup_{a \in A} U_a$$

Since A is compact there exists a finite subcovering (we will call U):

$$A \subseteq \bigcup_{a_n \in A} U_{a_n} = U$$

Thus we have that $V = \bigcap_{a_n \in A} V_{a_n}$ is open since it is a finite intersection of open sets. $B \subseteq V$ since $B \subseteq V_{a_n}$ for all a_n, and $V \cap U = \emptyset$ since if $v \in V$ then $v \in V_{a_n}$ for all a_n since, $V_{a_n} \cap U_{a_n} = \emptyset$ we have that $v \notin U_{a_n}$ for all a_n and thus $v \notin U$