2.4 18

(i) By definition of outer measure (where it is the infimum of measure of all open interval coverings) for any $\epsilon > 0$ there exists a covering of E by open intervals I_k

$$E \subseteq \bigcup I_k = U$$

with

$$m^*(U) - m^*(E) < \epsilon$$

Let U_ϵ denote such an open set. We have the G_δ set

$$G = \bigcap_{n=1}^{\infty} U_{1/n}$$

Notice that since $E \subseteq U_{1/n}$ for all n, $E \subseteq G$. Also notice $G \subseteq U_{1/n}$ for all n. Thus for every $n > 0$

$$m^*(E) \leq m^*(G) \leq m^*(E) + \frac{1}{n}$$

Thus $m^*(G) = m^*(E)$

(ii) ($\Rightarrow$): From Theorem 11 there exists F_σ set F where

$$F \subseteq E$$

$$m^*(E \backslash F) = 0$$

Since F is a measurable set from disjoint union additivity (which holds if only one of the disjoint sets is measurable) we have the desired result

$$m^*(E) = m^*(F \cup (E \backslash F)) = m^*(F) + m^*(E \backslash F) = m^*(F)$$

($\Leftarrow$): If there exists an F_σ set $F \subseteq E$ where $m^*(F) = m^*(E)$ then since F is measurable

$$m^*(E) = m^*(F \cup (E \backslash F)) = m^*(F) + m^*(E \backslash F)$$

and thus $m^*(E \backslash F) = 0$. We have that all sets of outer measure 0 are measurable and thus E is the union of measurable sets (and thus measurable)

$$E = F \cup (E \backslash F)$$

2.4 19

From problem 18 we know that there is a G_δ set $G = \cap_{k=1}^{\infty} U_k$ where

$$\lim_{N\to\infty} m^*\left(\bigcap_{k=1}^{N} U_k\right) = m^*(G) = m^*(E)$$

Thus for any $\epsilon > 0$ and sufficiently large N we have

$$m^*\left(\bigcap_{k=1}^{N} U_k\right) < m^*(E) + \epsilon$$

Letting $\mathcal{O}_N = \bigcap_{k=1}^{N} U_k$ be this open set we have

$$m^*\left(\mathcal{O}_N\right) - m^*(E) < \epsilon$$

From subadditivity

$$m^*(\mathcal{O}_N \backslash E) \geq m^*(\mathcal{O}_N) - m^*(E)$$

If we had equality for every $\mathcal{O}_N$ however, for every $\epsilon > 0$ we can choose a $\mathcal{O}_N$ as described above so that

$$m^*(\mathcal{O}_N \backslash E) < \epsilon$$

which from Theorem 11 establishes E to be measurable. Thus equality cannot be the case for each $\mathcal{O}_N$

2.4 20

($\Leftarrow$): This follows directly from the definition of measurable with the set in question (a, b)
($\Rightarrow$): From the definition of outer measure for any $\epsilon > 0$ we can cover E by a countable collection of open intervals I_k such that

$$\sum_{k=1}^{\infty} m^*(I_k) < m^*(E) + \epsilon$$

From hypothesis

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} m^*(E \cap I_k) + m^*(I_k \backslash E) \geq m^*\left(\bigcup_{k=1}^{\infty} E \cap I_k\right) + m^*\left(\bigcup_{k=1}^{\infty} I_k \backslash E\right)$$

Denoting $U = \bigcup_{k=1}^{\infty} I_k$ the open set, the above statement is equal to

$$m^*(U \cap E) + m^*(U \backslash E)$$

since $E \subseteq U$, $U \cap E = E$ so tracing back from our first inequality yields

$$m^*(E) + \epsilon > m^*(E) + m^*(U \backslash E)$$

Thus $m^*(U \backslash E) < \epsilon$ which from Theorem 11 establishes E to be measurable

2.5 25

Consider the example

$$B_1 = \mathbb{R}$$

$$B_n = (n, \infty)$$

We have that

$$\bigcap_{n=1}^{\infty} B_n = \emptyset \Rightarrow m^*\left(\bigcap_{n=1}^{\infty} B_n\right) = 0$$

while for each B_n, $m^*(B_n) = \infty$ so

$$\lim_{n\to\infty} m^*(B_n) = \infty$$

2.5 26

We have that

$$A \cap \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A \cap E_k$$

From subadditivity we have

$$m^*\left(\bigcup_{k=1}^{\infty} A \cap E_k\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

We have that for all $N > 0$,

$$m^*\left(\bigcup_{k=1}^{\infty} A \cap E_k\right) \geq m^*\left(\bigcup_{k=1}^{N} A \cap E_k\right) = \sum_{k=1}^{N} m^*(A \cap E_k)$$

The second equality follows from Proposition 6. Since this is the case for all N, we can conclude

$$m^*\left(\bigcup_{k=1}^{\infty} A \cap E_k\right) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

Thus since we have light inequalities both ways, we can conclude equality.

2.5 28

For any countable collection of disjoint measurable sets $\{E_1, E_2, \dots\}$ we can form an ascending chain of measurable sets by setting

$$A_k = \bigcup_{i=1}^{k} E_i$$

We have that

$$m^*\left(\bigcup_{s=1}^{k} E_s\right) = m^*\left(\bigcup_{s=1}^{k} A_s\right) = m^*(A_k) = \sum_{i=1}^{k} m^*(E_i)$$

And thus from the continuity property

$$m^*\left(\bigcup_{s=1}^{\infty} E_s\right) = m^*\left(\bigcup_{s=1}^{\infty} A_s\right) = \lim_{k\to\infty} m^*(A_k) = \sum_{i=1}^{\infty} m^*(E_i)$$