10.11

11.1 We know that the determinant is multiplicative, We also know for any $A \in GL(2,\mathbb{R})$, where $|A|$ signifies the determinant, we have $|A| = \frac{1}{|A^{-1}|}$.
Therefore for any $B \in SL(2,\mathbb{R})$:

$$|ABA^{-1}| = |A|1\frac{1}{|A|} = 1$$

so

$$ABA^{-1} \in S(2,\mathbb{R}$$

Therefore

$$SL(2,\mathbb{R}) \triangleleft GL(2,\mathbb{R})$$

11.2 H is not a normal group of $GL(2,\mathbb{R})$, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1} \in GL(2,\mathbb{R}) \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in H$$

We have

$$ABA^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \notin H$$

11.3 We know that $H = \{e, a\}$ where e is the identity and $a = a^{-1}$.
It is known that e commutes with every element in G so $e \in Z(G)$. Since H is normal we have for any $g \in G$

$$gag^{-1} \in \{e, a\}$$

If $gag^{-1} = e$ then applying g on the right yields $ga = g$ so $a = e$ which is not true. Therefore $gag^{-1} = a$, and so applying g on the right yields $ga = ag$ and so $a \in Z(G)$, so $H \subseteq Z(G)$

11.11

11.16 NOT DONNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNN
Given any element $a \in \mathbb{Q}$ and $z \in \mathbb{Z}$ we have since addition is commutative for $aza^{-1} \in a\mathbb{Z}a^{-1}$

$$aza^{-1} = aa^{-1}z = z \in \mathbb{Z}$$

so we know $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}$. Therefore $\mathbb{Q}/\mathbb{Z}$ is a group.
We know $\mathbb{Q}/\mathbb{Z}$ is infinite, if it were not we could list the group:

$$\{a_1\mathbb{Z}, a_2\mathbb{Z}, a_3\mathbb{Z}, \ldots a_n\mathbb{Z}\}$$

and come up with an element $a_{n+1}\mathbb{Z} \in \mathbb{Z}$

11.17 Since G is abelian, for any $a, b \in G$ we have

$$aHbH = \{ah_1bh_2 : h_1, h_2 \in H\} = \{bh_2ah_1 : h_1, h_2 \in H\} = bHaH$$

so G/H is abelian

11.18 We know every element of G has the form x^n where x is the generator of G, therefore every element of G/H also has the form Hx^n
We have

$$Hx^kHx^j = Hx^{k+j}$$

Therefore Hx is the generator of G/H since every term in G/H is of the form $(Hx)^n = Hx^n$ which means G/H is cyclic.

11.26 We will show there is a bijection between the elements of gHg^{-1} and H which would imply $|gHg^{-1}| = |H|$.
We will define this bijection as $f : H \to gHg^{-1}$, $f(h) = ghg^{-1}$. We have that if

$$f(h_1) = f(h_2)$$

then

$$gh_1g^{-1} = gh_2g^{-1}$$
$$g^{-1}gh_1g^{-1}g = g^{-1}gh_2g^{-1}g$$
$$h_1 = h_2$$

so f is one-to-one. We also know f is onto since $\forall ghg^{-1} \in gHg^{-1}$, $h \in H$ so $f(h) = ghg^{-1}$. Therefore f is a bijection so $|gHg^{-1}| = |H|$.