10.2

a. We know that 42 must divide $32|H|$ since $|H|$ is the order of 32 and adding 32 H is the same as multiplying 32 by H, therefore

$$|H| = \frac{LCM(32,48)}{32} = \frac{32 * 3 * 7}{32} = 21$$

And by Lagrange's thm we have

$$|H||G:H| = |G| = 48$$

so $|G:H| = 2$

b. Using the same logic above we have

$$|H| = \frac{LCM(24,54)}{24} = \frac{24 * 3 * 3}{24} = 9$$

and so

$$|G:H| = \frac{|G|}{|H|} = \frac{54}{9} = 6$$

c. Using the same logic:

$$|H| = \frac{LCM(100,112)}{100} = \frac{2 * 2 * 7 * 100}{100} = 28$$

So

$$|G:H| = \frac{|G|}{|H|} = \frac{112}{28} = 4$$

10.5 Given any element $a \in G$, by Lagrange's thm we have $|\langle a \rangle|$ divides $|G| = 8$. Since G is not cyclic we know $\langle a \rangle \neq G$ so $|\langle a \rangle| \neq 8$. Therefore the only options for $|\langle a \rangle|$ are $1, 2, 4$. Since all these numbers divide 4, we know that $a^4 = e$

10.6 We know that the intersection of two subgroups is a group. Therefore if we let $A = H \cap K$, we have that A is a subgroup of both H and K so by Lagrange's thm we have that $|A|$ divides both $|H| = 12$ and $|K| = 5$. The only number that divides both 12 annd 5 is 1, so $|A| = 1$ so $A = \{e\}$

10.14

a. This is because we know for the element x of order 6, $|\langle x\rangle| = 6 = |G|$ and a subgroup of G that is the same size of G is equivalent to G. Therefore $G = \langle x\rangle$ is cyclic

b. By Lagrange's thm for any element $a \in G$, $|\langle a\rangle|$ divides $|G| = 6$ $|\langle a\rangle|$ cannot equal 6 since G is not cyclic, so a must have either order 1, 2, or 3. We know that only e has order 1. We cannot have all the elements have order 2, otherwise we would have two elements a, b, then we would have ab, each having order 2 so they are their own inverses. So we have $\{e, a, b, ab\}$ which is closed with size 4, so we must introduce another element c, which would bring us to $\{e, a, b, c, ab, ac, bc, abc\}$ which is too large. Therefore there is some element a of order 3

c. If we take some element $b \in G : b \notin \langle a\rangle$, we already know $e, a, a^2 \in G$. We know since $b \notin \langle a\rangle$ that $ab, a^2b \notin \langle a\rangle$ since b is not equal to any power of a. Looking at ab, we can deduce $ab \neq a^2b$, since applying b^{-1} on the right yields $a \neq a^2$ which is true. Therefore we have

$$\{e, a, a^2, b, ab, a^2b\} \subseteq G$$

Are all unique elements

d. We cannot have b^2 be a seperate element in the above set since $|G| = 6$, if $b^2 = a$ then $b = a^2$ which is not true, if b^2a^2 then $b = a$ which is not true, if $b^2 = ab$ then applying b^{-1} on the right yields $b = a$ which is not true, and finally if $b^2 = a^2b$ then $b = a^2$ which is not true. Therefore $b^2 = e$.

e. We have $(ba)^{-1} = a^{-1}b^{-1} = a^2b$ but since we concluded a^2b has order 2 $(a^2b)^{-1} = a^2b = ba$. Similarly $(ba^2)^{-1} = ab = (ab)^{-1}$

10.15 By Lagrange's thm we have $|G| = |G : H||H|$ and $|H| = |H : K||K|$ so $|K| = \frac{|H|}{|H:K|}$. We also have

$$|G : K| = \frac{|G|}{|K|}$$

Substituting the equalities for $|G|$ and $|K|$ yields

$$|G : K| = \frac{|H : K||G : H||H|}{|H|} = |G : H||H : K|$$

10.16 Because $|G|$ is odd we know the order of none of the elements except for the identity can be 2.
If the order of some element $a \in G$ was 2 then $|\langle a\rangle| = 2$ but by Lagrange's thm $|\langle a\rangle|$ should divide $|G|$ which cannot happen since $|G|$ is odd.
Therefore for all elements $a \in G$, we know $a^{-1} \neq a$
Therefore if we take the product of all the elements in G, we know that for every element in that product, it's inverse is present in that product as well. Since G is abelian we can

rearrange the product so that each element and it's inverse present in the product cancel out, to be left with e

12.1

a. This is an epimorphism. We have its a homomorphism since

$$|xy| = |x||y|$$

for every $x \in \mathbb{R}^+$ we have $x = |y|$ where $y = x$ with $y \in \mathbb{R} - \{0\}$. However both y and $-y$ map to x so it is not one-to-one

b. This is an isomorphism. We know the function $f(x) = \sqrt{x}$ is one-to-one and onto on the positive real line.
It is a homomorphism since

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

c. This is a epimorphism. We know it is not injective since $\varphi((x-1)) = \varphi((x-1)(x-2)) = 0$. We do know it is surjective since given any $r \in \mathbb{R}$ we have $\varphi(rx) = r$. It is homomorphic since $\varphi(P_1(x) + P_2(x)) = P_1(1) + P_2(1) = \varphi(P_1(1)) + \varphi(P_2(1))$

d. This is also an epimorphism. We know it is not injective since $\varphi(x+1) = \varphi(x+2) = 1$. We do know it is surjective since every polinomial's antiderivative is a polinomial. Finally we know it is a homomorphism since

$$\varphi(P_1(x) + P_2(x)) = P_1'(x) + P_2'(x) = \varphi(P_1) + \varphi(P_2)$$

e. This is the same as applying the element $A \in G$ on the left hand side, we know G is commutative (proved in previous hw that symetric difference is commutative), which means for any $BC \in G$ we have $\varphi(BC) = ABC \neq ABAC = BC = \varphi(B)\varphi(C)$. Therefore φ is not a homomorphism

12.9 There is no isomorphic subgroup of Q_8 to V. The reason is because V has 4 elements of order 2 but Q_8 only has one: $-I$. We know that isomorphisms preserve orders so no such isomorphism can exist

12.12 For any group G of order 8, by Lagrange's thm we know elements can only have order $1, 2, 4, 8$ (because the size of their cyclic subgroup equals the order of its element).
The first case is if an element a has order 8. In this case $G = \langle a \rangle$ is a cyclic group.
If an element a has order 4 then for an element $b \notin \langle a \rangle$ we have two cases:
The first case is that $\langle b \rangle$ and $\langle a \rangle$ do not intersect, in which case we have the cosets $e\langle b \rangle, a\langle b \rangle, a^2\langle b \rangle, a^3\langle b \rangle$.
Therefore $|\langle b \rangle| = 2$ since we need the sum of the sizes of the cosets to be $= G$. So b has order 2. So we have $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. This is isomorphic to D_8

For the case where $\langle b \rangle$ and $\langle a \rangle$ do intersect, we can narrow it down to show only $b^2 = a^2$ is possible. We know that $a \neq b$, and since $(b^3)^{-1} = b$ and $(a^3)^{-1} = a$ if $a^3 = b^3$ that would imply $a = b$, so $a^3 \neq b^3$.
Looking at other terms we have $(ab)(ba) = a(b^2)a = a^4 = e$. We know that ab can have either order 4 or 2, if order 2, we have $ab = ba$ so a, b commute. Which would mean we have

$$G = \{e, a, b, a^2, a^3, b^3, ab, a^3b = ab^3\}$$

As for the other case when ab has order 4, we have

$$G = \{e, a, b, a^2, a^3, b^3, ab, (ab)^2 = (ba)^2, ba\}$$

This is isomorphic to the quaternians (where $a = J$, $b = K$, $ab = L$, and $a^2 = -I$)
That is all the cases where an element has order 4. The rest of the cases are where all elements have order 2.
We can add on two elements a, b so we have $\{e, a, b, ab\}$ and since ab has order 2, $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. The set is closed so we add another element c, which gives us

$$\{e, a, b, c, ab, ac, bc, abc\}$$

Using the same logic as before we know $ac = ca$ and $bc = cb$. So all elements commute. There is no other possible changes to made to the elements of order 2, so we are done. In total we have counted 5 possible subgroups, each with different properties, which means they are not isomorphic.

12.13

a. Given any $x, y \in G$ we have since H is abelian

$$\varphi(yx) = \varphi(y)\varphi(x) = \varphi(x)\varphi(y) = \varphi(xy)$$

Since φ is one to one, we know there exists an inverse mapping φ^{-1} from the image of φ back to G. Applying the inverse shows that G is abelian:

$$\varphi^{-1}(\varphi(xy)) = \varphi^{-1}(\varphi(yx))$$

$$xy = yx$$

So G is abelian

b. Given any $x, y \in H$, since φ is onto we know there is some $x', y' \in G$ such that $\varphi(x') = x, \varphi(y') = y$. Therefore since G is abelian we have

$$xy = \varphi(x)\varphi(y) = \varphi(x'y') = \varphi(y')\varphi(x') = \varphi(y')\varphi(x') = yx$$

So H is abelian since $xy = yx$

c. As shown in problem a we know φ being an isomorphism and H abelian $\Rightarrow$ G abelian. And in problem b we showed the other way, H abelian $\Leftarrow$ G abelian