Exersise 3.1

(1) We can define the following isomorphism: $\phi : H \rtimes_{\varphi} K \to H \rtimes_{\varphi\circ\lambda} K$
With $\phi(h,k) = (h, \lambda^{-1}(k))$. It is clear ϕ is bijective since we can define the inverse mapping $\phi^{-1}(h,k) = (h, \lambda(k))$. ϕ is a homomorphism as follows:

$$\phi\left((h,k)(h',k')\right) = \phi(h\varphi(k)(h'), kk') = (h\varphi(k)(h'), \lambda^{-1}(kk'))$$

$$= (h\varphi(\lambda(\lambda^{-1}(k)))(h'), \lambda^{-1}(kk')) = (h, \lambda^{-1}(k))(h', \lambda^{-1}(k')) = \phi(h,k)\phi(h',k')$$

Thus ϕ is an isomorphism and thus $H \rtimes_{\varphi} K \cong H \rtimes_{\varphi\circ\lambda} K$

(2) We can define the isomorphism: $\phi : H \rtimes_{\varphi} K \to H \rtimes_{\psi\circ\varphi\circ\psi^{-1}} K$
With $\phi(h,k) = (\psi(h), k)$. It is clear ϕ is bijective since we can define the inverse mapping $\phi^{-1}(h,k) = (\psi^{-1}(h), k)$. ϕ is a homomorphism as follows:

$$\phi\left((h,k)(h',k')\right) = \phi(h\varphi(k)(h'), kk') = (\psi(h\varphi(k)(h')), kk')$$

$$= (\psi(h)\psi(\varphi(\psi^{-1}(\psi(k))))\psi(h'), kk') = (\psi(h), k)(\psi(h'), k') = \phi(h,k)\phi(h',k')$$

Thus ϕ is an isomorphism and thus $H \rtimes_{\varphi} K \cong H \rtimes_{\psi\circ\varphi\circ\psi^{-1}} K$

Exersise 3.2

(1) Consider the canonical homomorphism: $\pi : R \to R/I_1 \times \cdots \times R/I_k$ where $\pi(r) = r + I_1 \times \cdots \times r + I_k$.
We have that $r \in \ker(\pi)$ iff $r \in I_1 \cap \cdots \cap I_k$. Thus if we can show that π is surjective and $I_1 \cap \cdots \cap I_k = I_1 \ldots I_k$ Then we have proven the claim. We will use induction on the number of ideals:
Base case, $(k = 2)$:
It is clear that $I_1 I_2 \subseteq I_1 \cap I_2$ for the other direction we have that since $I_1 + I_2 = R$ there exists $x \in I_1$ and $y \in I_2$ with $x + y = 1$. Thus for any $a \in I_1 \cap I_2$ we have that $a = ax + ay \in I_1 I_2$ thus $I_1 I_2 = I_1 \cap I_2$. Additionaly we have with the same x, y, for any $(a,b) \in R/I_1 \times R/I_2$, $\pi(x+y) = \pi(1) = (1,1)$, $\pi(x) + \pi(y) = (1,1)$, since $x \in I_1$ we know $\pi(x)$ is zero in the I_1 component, same goes with y for the I_2 component and thus $\pi(x) = (0,1)$, $\pi(y) = (1,0)$. So

$$\pi(bx + ay) = (a,b)$$

So π is surjective, proving the base case.
We can reduce the $k+1$ step to the k step in the following manner:
Define $\mathcal{I} = I_1 I_2 \ldots I_k$. We have to show that $\mathcal{I} + I_{k+1} = R$ and then we can apply our base case reasoning. $I_i + I_{k+1} = R$ for each $I_i \neq I_{k+1}$ and so there exists $x_i \in I_i, y_i \in I_{k+1}$ with $x_i + y_i = 1$, we have that

$$1 = (x_1 + y_1)(x_2 + y_2)\ldots(x_k + y_k)$$

Factoring the product on the right we have each term is multiplied by an x_i and thus an element of I_{k+1} except for the $y_1y_2\dots y_k$ term wich is an elt of $\mathcal{I}$. Thus $1 \in \mathcal{I} + I_{k+1}$, and since if an Ideal contains 1 it is R we have $\mathcal{I} + I_{k+1} = R$.
Thus we have from our inductive hypothesis

$$R/I_1 \cap \dots \cap I_{k+1} = R/\mathcal{I} \cap I_{k+1} \cong R/\mathcal{I} \times R/I_{k+1} \cong R/I_1 \times R/I_2 \dots \times R/I_{k+1}$$

(2) We can use the chinese remainder theorem:
Let $I_i = p_i^{a_i}$ for each i. For any I_i, I_j with $i \neq j$ we have that the $\gcd(p_i^{a_i}, p_j^{a_j}) = 1$ and thus from the euclidean algorithm we know there exists $n, m \in \mathbb{Z}$ with $np_i^{a_i} + mp_j^{a_j} = 1$ and thus $1 \in I_i + I_j \Rightarrow I_i + I_j = \mathbb{Z}$. The last thing to check is that $n\mathbb{Z} = I_1 \cap I_2 \cap \dots \cap I_k$. Since for each i, $p_i^{a_i} | n$, it is clear $n\mathbb{Z} \subseteq I_1 \cap I_2 \cap \dots \cap I_k$. For any $x \in I_1 \cap I_2 \cap \dots \cap I_k$ we have that $p_i^{a_i} | x$ for all i and so $n|x \Rightarrow I_1 \cap I_2 \cap \dots \cap I_k \subseteq n\mathbb{Z}$. Thus $I_1 \cap I_2 \cap \dots \cap I_k = n\mathbb{Z}$, and so all the conditions of the chinese remainder thm are satisfied:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/I_1 \times \dots \times \mathbb{Z}/I_k$$

(3) If we have $\pi : R \to R_1 \times R_2$ is an isomorphism, let $e_1 = \pi^{-1}(1,0), e_2 = \pi^{-1}(0,1)$. We have in $R_1 \times R_2$ that $(1,0)^2 = (1,0)$, $(0,1)^2 = (0,1)$, $(0,1)(1,0) = (0,0)$, $(1,0)+(0,1) = (1,1)$ and so composing π^{-1} with these equalities yields $e_1^2 = e_1, e_2^2 = e_2, e_1e_2 = 0$, $e_1 + e_2 = 1$.
Conversly if $\exists e_1, e_2 \in R$ with the described properties, then consider the ideals $I_1 = e_1R, I_2 = e_2R$. We have that $e_1 + e_2 = 1 \in I_1 + I_2$ and so $I_1 + I_2 = R$. We also have that

$$I_1I_2 = \{\sum_{k=1}^{n} e_1x_ke_2y_k : x_k, y_k \in R, n \in \mathbb{Z}^+\}$$

Since $e_1x_ke_2y_k = e_1e_2x_ky_k = 0$ we have $I_1I_2 = \{0\}$ and thus from the chinese remainder thm we have

$$R \cong R/I_1I_2 \cong R_1 \times R_2$$

where $R_1 = R/I_1$, $R_2 = R/I_2$

Exersise 3.3

We will define $\lambda : S^{-1}R \to A$ as $\lambda(a,b) = \psi(a)\psi(b)^{-1}$. We must check that λ is well defined, is a homomorphism, commutes as $\varphi \circ \lambda = \psi$, and is unique.
Checking λ is well defined:
We have for $(a,b) = (c,d) \in S^{-1}R$, then for some $t \in S$ $t(ad-bc) = 0$ so $\psi(t)(\psi(ad)-\psi(bc)) = 0$, since $\psi(t)$ is a unit, we have $\psi(ad) = \psi(bc)$ and so $\lambda(a,b) = \psi(a)\psi(b)^{-1} = \psi(c)\psi(d)^{-1} = \lambda(c,d)$
Checking λ is homomorphism:
For any $(a,b),(c,d) \in S^{-1}R$ we have

$$\lambda((a,b)(c,d)) = \lambda(ac,bd) = \psi(ac)\psi(bd)^{-1} = \lambda(a,b)\lambda(c,d)$$

$$\lambda((a,b)+(c,d)) = \lambda(ad+bc,bd) = \psi(ad+bc)\psi(bd)^{-1} = \psi(ad)\psi(bd)^{-1}+\psi(bc)\psi(bd)^{-1} = \lambda(ad,bd)+\lambda(bc,bd)$$

We have $adb - adb = 0$ so $(ad, bd) \sim (a, b)$ and $(bc, bd) \sim (c, d)$ so

$$= \lambda(a, b) + \lambda(c, d)$$

Thus λ is a homomorphism
Checking $\lambda \circ \varphi = \psi$:
We have that for any $r \in R$, $\lambda(\varphi(r)) = \lambda(r, 1) = \psi(r)\psi(1)^{-1} = \psi(r)$. Thus $\lambda \circ \varphi = \psi$
Checking uniqueness:
If there exists $\lambda' : S^{-1}R \to A$ such that $\lambda' \circ \varphi = \psi$, we have that for any $r \in R$, $\lambda' \circ \varphi(r) = \lambda'(r, 1) = \psi(r)$. Thus $\lambda'(r, 1) = \lambda(r, 1) \forall r \in R$. However we have that $1r - r1 = 0$ so $\lambda'(1, r)\lambda'(r, 1) = \lambda'((1, r)(r, 1)) = \lambda'(r, r) = \lambda'(1, 1) = 1$ Thus we have $\lambda'(1, r) = \lambda'(r, 1)^{-1} = \psi(r)^{-1} = \lambda(1, r)$. Thus for any $(a, b) \in S^{-1}R$, $\lambda'(a, b) = \lambda'(a, 1)\lambda'(1, b) = \lambda(a, 1)\lambda(1, b) = \lambda(a, b)$. So $\lambda' = \lambda$ and so λ is unique

Exersise 3.4

(1) If we consider the canonical homomorphism $\pi : R \to R/I$, with $\pi(r) = r + I$, we have the bijection from prime ideals in R containing I to prime ideals of R/I by taking $P \subset R$ to $\pi(P)$. First we have to check this mapping actually takes prime ideals to prime ideals:
If P is a prime ideal of R we have that for any $p+I \in \pi(P)$ (letting $p \in P$), $a+I, b+I \in R/I$, $(p+I)(a+I) = pa+I \in \pi(P)$ since $pa \in P$ so $\pi(P)$ is an ideal. If $(a+I)(b+I) = ab+I \in \pi(P)$ then we have that $ab + i = p$ for some $i \in I$, $p \in P$. Since $I \subseteq P$ we have $ab = p - i \in P$ and so either a or b is in P. Thus $\pi(P)$ is prime.
Now we have to check injectivity and surjectivity.
If we have prime ideals $P, Q \subseteq R$ containing I with $\pi(P) = \pi(Q)$ then for any $p \in P$ we have $p + I \in \pi(Q)$ so $p + i = q$ for some $i \in I, q \in Q$. We have $q - i \in Q$ so $p \in Q$, thus $P \subseteq Q$, this argument could be repeated switching the labels of P and Q to get $Q \subseteq P$ and thus $P = Q$ so our mapping is injective.
If Q is a prime ideal of R/I then if we consider the pullback $P = \pi^{-1}(Q)$, if we prove that P is a prime ideal containing I, then by definition of the pullback we have $\pi(P) = Q$ and so our mapping is surjective. For any $p, a, b \in R$ with $p \in P$ and $a, b \in R$ we have that $\pi(pa) = \pi(p)\pi(a) \in Q$ since $\pi(p) \in Q$, thus $pa \in P$ and so P is an ideal. If $ab \in P$ then$\pi(ab) = \pi(a)\pi(b) \in Q$ and since Q is prime, either $\pi(a)$ or $\pi(b)$ is in Q so a or b is in P, so P is prime. Finally we have that $\pi(I) = 0 + I \in Q$ so $I \subseteq P$ and thus P is a prime ideal containing I with $\pi(P) = Q$. So we have surjectivity.

(2) If we consider the homomorphism $\pi : R \to S^{-1}R$ with $\pi(r) = (r, 1)$ then we have the bijective mapping from prime ideals of R to prime ideals of $S^{-1}R$ by taking P to the extension $(S^{-1}R)\pi(P)$(the ideal generated by $\pi(P)$). I will use the notation P^e to signify this extension, and for an Ideal $I \subset S^{-1}R$, I will say I^c to signify the contraction $\pi^{-1}(I)$. This is the same notation used in Dummit and Foot.
We have to check that this mapping actually takes prime ideals to prime ideals, We will also show that there is an inverse mapping (and thus our mapping is bijective).
First we will show for prime ideal $P \subseteq R$ with $P \cap S = \emptyset$, we have $P^e = \{(x, y) : x \in P, y \in S\}$. For any elt $(a, b) \in P^e$ we have from definition that $(a, b) = (p_1, 1)(c_1, d_1) + (p_2, 1)(c_2, d_2) +$

$\dots (p_n, 1)(c_n, d_n)$ with $p_i \in P$ for each p_i, thus using the definition of addition in $S^{-1}R$ we get

$$a = p_1c_1(d_2d_3 \dots d_n) + p_2c_2(d_1d_3 \dots d_n) + \dots + p_nc_n(d_1 \dots d_{n-1})$$

Therefore $a \in P$ since it is a sum of terms in P, and thus $P^e = \{(x, y) : x \in P, y \in S\}$
Now we can show P^e is prime. We have that for any $(a, b), (c, d) \in S^{-1}R$, if $(a, b)(c, d) \in P^e$ then $(ac, bd) \in P^e$ we have that $ac \in P$ since $P^e = \{(x, y) : x \in P, y \in S\}$. Therefore either $a \in P$ or $c \in P$, so either $(a, b) \in P^e$ or $(c, d) \in P^e$ so P^e is prime.
The inverse mapping of taking the extensions is taking the contraction. By showing $(P^e)^c = P$ we have completed our proof. For any $p \in P$ we have that $\pi(p) \in P^e$ and so $p \in \pi^{-1}(P^e)$, thus $p \in (P^e)^c$ so $P \subseteq (P^e)^c$. For any $r \in (P^e)^c$ we have that $\pi(r) \in P^e$ and thus $(r, 1) \in P^e$. Thus we have that

$$(r, 1) \sim (p, s)$$

For some $p \in P$, $s \in S$. So for some $t \in S, t(rs - p) = 0$. Thus we have $r(ts) = pt$. $pt \in P$ so $r(ts) \in P$, since $ts \in S$ and $S \cap P = \emptyset$ we have that $ts \notin P$ so $r \in P$. Thus $(P^e)^c \subseteq P$ so $P = (P^e)^c$

(3) In part (1) we do have a bijection: for any maximal ideal M containing I we have that $\pi(M)$ is maximal in R/I. The reasoning is that if we follow the proof written for part (1), it follows that for any ideal I' containing I, $\pi(I')$ is an ideal. Therefore $\pi(M)$ is an ideal, if there exists an ideal S with $\pi(M) \subset S \subset R/I$ then $\pi^{-1}(S)$ is an ideal $\subset R$ and $M \subset \pi^{-1}(S)$, which is contradicts M maximal, thus $\pi(M)$ is maximal. This is because for any $a \in R, b \in \pi^{-1}(S)$ we have that $\pi(ab) = \pi(a)\pi(b) \in S$ so $ab \in \pi^{-1}(S)$ so $\pi^{-1}(S)$ is an ideal. For $s \in S, s \notin \pi(M)$ we have that $\pi^{-1}(s) \not\subset M$ and for $m \in M$, $\pi(m) \in S$ so $m \in \pi^{-1}(S)$ so $M \subset \pi^{-1}(S)$.
Our proof of injectivity for maximal ideals is the same as proven in part (1) since primality was never used. For surjectivity we have shown that the pullback of a maximal ideal is an ideal already. We also know it is maximal since if S is maximal in R/I then if $\pi^{-1}(S) \subset M \subset R$ then $S \subset \pi(M) \subset R/I$.
For part (2) we also have a bijection.
All maximal ideals are prime ideals, so we have already illustrated our inverse mapping is well defined for maximal ideals (and thus we have bijectivity) we just have to check our mapping maps maximal ideals to maximal ideals:
For any Maximal ideal M of R with $M \cap S = \emptyset$ we have that M^e is maximal in $S^{-1}R$. If there exists a maximal ideal $A \subset S^{-1}R$ with $M^e \subset A$ then we have from our established correspondence that A^c is a prime ideal in R. We already know then that $\pi^{-1}(M^e) = M \subseteq \pi^{-1}(A)$ since $M^e \subset A$. We have that $M \subset A^c$ since $\exists (a, b) \in A$ where $(a, b) \notin M^e$ we have that $(a, 1) = (a, b)(b, 1) \in A$ and $(a, 1) \notin M^e$ since M^e is prime yet (a, b) and $(b, 1) \notin M^e$ we know that $a = \pi^{-1}(a, 1) \subset A^c$ but $\pi^{-1}(a, 1) \not\subset M$. This contradicts M maximal. So M^e must be maximal.

Exersise 3.5

(1) We have by definition that $Z(R)$ is commutative and since R is a division ring every

non-zero elt is a unit. Thus all we have to check is that $Z(R)$ is closed under multiplcation, addition, and inverses.
We have for any $x, y \in Z(R)$ and arbitrary $r \in R$

$$(x+y)r = xr + yr = rx + ry = r(x+y) \Rightarrow x+y \in Z(R)$$

$$(xy)r = r(xy) \Rightarrow xy \in Z(R)$$

$$x^{-1}r = x^{-1}rxx^{-1} = rx^{-1} \Rightarrow x^{-1} \in Z(R)$$

$$(x+(-x))r = 0, r(x+(-x)) = 0 \Rightarrow xr + (-x)r = 0, rx + r(-x) = 0$$

$$\Rightarrow (-x)r = -(rx) = r(-x) \Rightarrow -x \in Z(R)$$

Thus $Z(R)$ is a field

(2) The center consists of all $Z(M_n(R)) = \{aI : a \in Z(R)\}$ where I is the identity matrix. It is clear that these matricies commute with $M_n(R)$ since left multiplication by aI is the same as scaling each row by a on the left, while right multiplcation scales each column by a on the right. The result in both cases is the same: multiplying each entry of the multiplied matrix by a (multiplying by a on the left vs the right is the same since $a \in Z(R)$).
To show that no other matricies are in the center, we first show elts of $Z(M_n(R))$ must be diagonal. For any $D \in M_n(R)$, if $D_{i,j} \neq 0$ for some $i \neq j$, then we choose $A \in M_n(R)$ where $A_{j,i} = 1$ and every other entry of A is 0. We have

$$(DA)_{i,i} = \sum_{k=1}^{n} D_{i,k}A_{k,i} = D_{i,j}$$

While

$$(AD)_{i,i} = \sum_{k=1}^{n} A_{i,k}D_{k,i} = 0 \neq (DA)_{i,i}$$

Thus $DA \neq AD$, so $D \notin Z(M_n(R))$.
Now we must show that only diagonal matricies of the form aI can be in the center.
If $D \in M_n(R)$ is a diagonal matrix with $D_{i,i}neqD_{j,j}$ then choose $A \in M_n(R)$ with $A_{i,j} = 1$ and every other entry 0. We then have

$$(DA)_{i,j} = \sum_{k=1}^{n} D_{i,k}A_{k,j} = D_{i,i}$$

$$(AD)_{i,j} = \sum_{k=1}^{n} A_{i,k}D_{k,j} = D_{j,j} \neq (DA)_{i,j}$$

Thus $DA \neq AD$ so $D \notin Z(M_n(R))$. Thus all the diagonal entries of matricies of the center must be the same elt of R.
Finally we know that if $aI \in Z(M_n(R))$ then $a \in Z(R)$ since if $ar \neq ra$ for some $r \in R$ then we can choose $rI \in M_n(R)$ and we have

$$(aI)(rI) = arI \neq raI = (rI)(aI)$$

Exersise 3.6

(1) We have that

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Since

$$(1+x)\sum_{n=0}^{\infty}(-1)^n x^n = 1 + (x-x) + (-x^2+x^2) + (x^3-x^3)\cdots = 1$$

(2) For any $A = \sum_{n=0}^{\infty} a_n x^n \neq 0$, $B = \sum_{n=0}^{\infty} b_n x^n \neq 0$, let k be the first index where $a_k \neq 0$ and j the first index where $b_j \neq 0$.
If we have $AB = \sum_{n=0}^{\infty} c_n x^n$ then we have

$$c_{j+k} = \sum_{i=0}^{j+k} a_i b_{j+k-i}$$

We have that for $i < k$, $a_i = 0$ and for $i > k$, $b_{j+k-i} = 0$. So we have that

$$C_{j+k} = a_k b_j$$

and since R is an integral domain, $c_{j+k} \neq 0$ so $AB \neq 0$ so $R[[x]]$ is an integral domain.