Exersise 1.1

We have the isomorphism

$$\phi : \mathbb{R}[x]/(x^2 + x + 1) \to \mathbb{R}[\zeta_3]$$

$$x \to \zeta_3$$

Where $\zeta_3 = 1/2 + \frac{\sqrt{3}}{2} i$ is the third root of unity. This is an isomorphism since $x^2 + x + 1$ is the minimal polynomial of ζ_3 over $\mathbb{R}$.
We have that $\mathbb{R}[\zeta_3] \cong \mathbb{C}$ since by definition $\mathbb{C} = \mathbb{R}[i]$, $\zeta_3 \in \mathbb{C}$ so $\mathbb{R}[\zeta_3] \subseteq \mathbb{C}$ and $i = (\zeta_3 - 1/2)\frac{2}{\sqrt{3}}$ so $\mathbb{C} \subseteq \mathbb{R}[\zeta_3]$

Exersise 1.2

Let $\alpha = \sqrt{2} + \sqrt{3}$. It is clear $\alpha \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ so $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. We have

$$\frac{\alpha^3 - 9\alpha}{2} = \frac{11\sqrt{2} + 9\sqrt{3} - 9(\sqrt{2} + \sqrt{3})}{2} = \sqrt{2}$$

$$\sqrt{3} = \alpha - \frac{\alpha^3 - 9\alpha}{2}$$

So $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\alpha) \Rightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$, thus $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\alpha)$ so α is a primitive element

Exersise 1.3

We have the factorization

$$x^5 + x^2 - x - 1 = (x+1)(x-1)(x^2 + x + 1)$$

Where $x^2 + x + 1$ is irridducible since the roots are $\pm\zeta_3 \notin \mathbb{Q}$. Thus either $\alpha = \pm 1$ which yields a degree 1 extension $\mathbb{Q}[\alpha] = \mathbb{Q}$, or $\alpha = \pm\zeta_3$ which yields a degree 2 extension since 2 is the degree of the minimal polynomial of α: $x^2 + x + 1$

Exersise 1.4

If $\sqrt{a}, \sqrt{a+1} \in \mathbb{Q}$ then $\alpha \in \mathbb{Q}$ and $m_\alpha(x) = x - \alpha$
If $\sqrt{a} \in \mathbb{Q}, \sqrt{a+1} \notin \mathbb{Q}$, then $m_\alpha(x) = (x - \sqrt{a})^2 - a - 1$. The only roots of m_α are $\sqrt{a} \pm \sqrt{a+1} \notin \mathbb{Q}$ and thus since m_α is degree 2, m_α is irridducible and so minimal over $\mathbb{Q}$.
If $\sqrt{a} \notin \mathbb{Q}, \sqrt{a+1} \in \mathbb{Q}$, then $m_\alpha(x) = (x - \sqrt{a+1})^2 - a$. The only roots of m_α are $\pm\sqrt{a} + \sqrt{a+1} \notin \mathbb{Q}$ and thus since m_α is degree 2, m_α is irridducible and so minimal over $\mathbb{Q}$.
If $\sqrt{a}, \sqrt{a+1} \notin \mathbb{Q}$, $m_\alpha(x) = x^4 - (4a+2)x^2 + 1$.
m_α is irridducible since the roots are $\pm\alpha, \pm\frac{1}{\alpha} \notin \mathbb{Q}$ and no degree 2 polynomial can divide m_α as follows.
If we can write $m_\alpha = p(x)q(x)$ where p, q are degree 2 polynomials $\in \mathbb{Q}[x]$ then the roots of p must be a subset of $\pm\alpha, \pm\frac{1}{\alpha}$ but none of the options of $(x \pm \alpha)(x \pm \frac{1}{\alpha}), (x \pm \frac{1}{\alpha})(x \pm \frac{1}{\alpha}), (x \pm \alpha)(x \pm \alpha)$ yield a polynomial in $\mathbb{Q}[x]$

Exersise 1.5

We know that $\alpha^2 \in k(\alpha)$ so $k(\alpha^2) \subseteq k(\alpha)$
Since $[k(\alpha) : k]$ is odd, the minimal polynomial over k, m_α, has odd degree $(2n-1)$:

$$m_\alpha(\alpha) = \alpha^{2n-1} + c_{2n-2}\alpha^{2n-2} + \cdots + c_2\alpha^2 + c_1\alpha + c_0 = 0$$

Multiplying by α on both sides in K yields

$$\alpha^{2n} + c_{2n-2}\alpha^{2n-1} + \cdots + c_2\alpha^3 + c_1\alpha^2 + c_0\alpha = 0$$

Subtracting all odd degree terms:

$$\alpha^{2n} + \cdots + c_1\alpha^2 = -c_{2n-2}\alpha^{2n-1} - \cdots - c_2\alpha^3 - c_0\alpha$$

Factoring out α and relabeling constants $k_i = -c_i$:

$$\alpha^{2n} + \cdots + c_1\alpha^2 = \alpha(k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0)$$

We have α in terms of a ratio of polynomials in α^2:

$$\alpha = \frac{\alpha^{2n} + \cdots + c_1\alpha^2}{k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0} \in k(\alpha^2)$$

We know that this is well defined, ie $k_{2n-2}\alpha^{2n-2} + \cdots + k_0 \neq 0$ is invertable, since it is a non-zero polynomial $f(\alpha) = k_{2n-2}\alpha^{2n-2} + \cdots + k_2\alpha^2 + k_0$ of degree less than m_α and therefore cannot be zero otherwise we would contradict minimality of m_α. f is nonzero since $k_0 = -c_0$ is nonzero since if $c_0 = 0$ then

$$x | m_\alpha(x) = x^{2n-1} + c_{2n-2}x^{2n-2} + \cdots + c_2x^2 + c_1x$$

which contradicts m_α being irriducible.
Thus $k(\alpha) \subseteq k(\alpha^2)$ so $k(\alpha) = k(\alpha^2)$

Exersise 1.6

Since A is a subring of K we know A is an integral domain. All we must show is that for any $\alpha \in A$, $\alpha^{-1} \in A$.
We have that $k[\alpha] \subseteq A$ where $k[\alpha]$ is the smallest subring of K to contain k and α. It turns out that $k[\alpha] = k(\alpha)$ and therefore $\alpha^{-1} \in k(\alpha) \subseteq A$, so A is a field.
The reason $k[\alpha] = k(\alpha)$ is since $\alpha \in K$ and K algebraic over k, there is a minimal polynomial for α, $m_\alpha(x) \in k[x]$. $k[\alpha]$ must contain all linear powers of α over k, with $m_\alpha(\alpha) = 0$. From this we have the isomorphism $k[\alpha] \cong k[x]/(m_\alpha(x))$ (this isomorphism is established more rigorously in Dummit and Foote's section of Field Theory) which is a field since $(m_\alpha(x))$ is maximal. Thus $k[\alpha]$ is a field so $k(\alpha) \subseteq k[\alpha]$. Since $k(\alpha)$ is a ring we know $k[\alpha] \subseteq k(\alpha)$, so $k[\alpha] = k(\alpha)$