Math 504, Fall 2017 ***MIDTERM*** ***Peter Gylys-Colwell***

Problem 1.1

We have that $|G| = 385 = 11 \cdot 7 \cdot 5$. Therefore from Sylow's theorems we know there is an element of order 11, an element of order 7 and another of order 5. We have that the number of Sylow 11-subgroups, n_{11}, is equal to $1 + 11k$ for $k \geq 0$ and n_{11} divides $7 \cdot 5$ since its the index of the normalizer. Since $7, 5, 35 \not\equiv 1 \pmod{11}$ we know that $n_{11} = 1$. Therefore the subgroup of size 11 is normal. Thus

$$G \cong Z/11\mathbb{Z} \rtimes H$$

Where H is a subgroup of order 35. We have already classified all groups of order pq in lecture. Thus since 5 does not divide $7-1$, we know that $H \cong Z/35\mathbb{Z}$. Now we can consider all the homomorphisms $\varphi : Z/11\mathbb{Z} \to \mathrm{Aut}(H)$. We have that ϕ must map the generator of $\mathbb{Z}/11\mathbb{Z}$ to an element of order 11. However the order of $\mathrm{Aut}(H)$ is 34 and so no element has order 11 except the identity. Therefore the only possible homomorphism is the trivial $\varphi(g) = 0 \ \forall g \in \mathbb{Z}_{11}$. Thus the only possibility is

$$G \cong \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/35\mathbb{Z} \cong \mathbb{Z}/385\mathbb{Z}$$

Problem 1.2

(1) We have that $C_G(H) = H$, it is a commonly known fact that the diagonal matricies over a field commute since multiplying by a diagonal matrix scales the columns or rows by the diagonal.

To show that no other matricies are in the center, for any $A \in G$, if $A_{i,j} \neq 0$ for some $i \neq j$, then we choose $D \in H$ such that $D_{i,i} \neq D_{j,j}$. We have that

$$(DA)_{i,j} = D_{i,i}A_{i,j} \neq D_{j,j}A_{i,j} = (AD)_{i,j}$$

Thus $DA \neq AD$.

(2) The normalizer is all the permutation matricies multiplied by an element of H:

$$N_G(H) = \{PD \in G : D \in H, \exists \sigma \in S_3, P_i = e_{\sigma(i)}\} = S$$

Permutation matricies normalize H since left multiplication by a permutation matrix permutes the rows while while inverse right multiplication applys the permutation to the columns. Thus applying PDP^{-1} to $D \in H$ for $P \in S$ will permute the diagonals:

$$(PDP^{-1})_{j,j} = D_{\sigma(j),\sigma(j)}$$

While every other zero entry will stay zero. Thus we get another Diagonal matrix. No other matrix can be in the normalizer since if we have a matrix A where $A_{i,j} \neq 0$ and $A_{i,k} \neq 0$ with $j \neq k$, then for any $D \in H$, we have that

$$(DA)_{i,j} = D_{i,i}A_{i,j}, (DA)_{i,k} = D_{i,i}A_{i,k}$$

However if we choose $D' \in H$ with $D'_{j,j} \neq D'_{k,k}$ we have

$$(AD')_{i,j} = D'_{j,j}A_{i,j}, (AD')_{i,k} = D'_{k,k}A_{i,k}$$

Thus it $AD' \neq DA$ for all $D \in H$ since the entries of $(AD')_{i,j}$ and $(AD')_{i,k}$ are not $A_{i,j}$ and $A_{i,k}$ scaled by the same amount as the the same entries of every DA must satisfy.

(3) This quotient is isomorphic to S_3.
Consider the homomorphism $\phi : N_G(H) \to S_3$ where for any $PD \in N_G(H)$ with $PD = \{D \in H, \sigma \in S_3, P_i = e_{\sigma(i)}\}$, $\phi(PD) = \sigma$. This is a homomorphism since multiplying by two permutation matricies is the same as composing the permutations of the matricies. ϕ is clearly surjective since we can choose a matrix $P \in N_G(H)$ for any $\sigma \in S_3$ where $P_i = e_{\sigma(i)}$ and so $\phi(P) = \sigma$. The kernel of ϕ is clearly the diagonal matricies H since they are the elements of $N_G(H)$ where they do not permute the columns at all. Thus we have that ϕ induces an isomorphism from $N_G(H)/H \to S_3$

Problem 1.3

Letting r signify the rotation element and s signify the reflextion, we have that the only normal subgroups of D_{12} are $\langle r\rangle$, $\langle r^2\rangle$, $\langle r^3\rangle$, $\langle s, r^2\rangle$, and $\langle rs, r^2\rangle$. We have that $\langle r^2\rangle$ and $\langle r^3\rangle$ are not maximal since they are contained in $\langle r\rangle$. Thus the composition series are

$$1 \lhd \langle r^3\rangle \lhd \langle r\rangle \lhd D_{12}$$

$$1 \lhd \langle r^2\rangle \lhd \langle r\rangle \lhd D_{12}$$

$$1 \lhd \langle r^2\rangle \lhd \langle s, r^2\rangle \lhd D_{12}$$

$$1 \lhd \langle r^2\rangle \lhd \langle rs, r^2\rangle \lhd D_{12}$$

(I checked the subgroups of $\langle s, r^2\rangle$ and $\langle rs, r^2\rangle$ to find the only normal subgroup is $\langle r^2\rangle$)

Problem 1.4

For any group G with $|G| = p^2q$ we have from Sylows theorems there exists a subgroup N_{p^2} of order p^2. We know that the index of the normalizer of the p-group, n_p, satisfies $n_p = 1 + pk, k \geq 0$ and $n_p|q$. Since q is prime n_p is either 1 or q If n_p is 1, the p-group is normal.
Thus we have that $G \cong \mathbb{Z}/q \rtimes N_{p^2}$ where N_{p^2} is the group of size p^2, thus we have that $|G/N_{p^2}| = q$ and so must be cyclic, hence abelian. In lecture we established that all groups of order p^2 are abelian. Thus we have the series

$$1 \lhd N_{p^2} \lhd G$$

If $n_p = q$ then we have that $q \equiv 1 \pmod p$. From Sylows Theorems we know that there is a q-subgroup N_q of order q with the index of the normalizer, n_q, satisfying $n_q \equiv 1 \pmod q$ and $n_q|p^2$. Thus n_q is either $1, p$, or p^2
n_q cannot be p since $q \equiv 1 \pmod p$ and $p \equiv 1 \pmod q$ is not possible for any primes p, q. (This is because one has to be smaller than the other and thus will themselves mod the

other)
$n_q = p^2$ leads to a simple case from similar reasoning. We have that $p^2 \equiv 1 \pmod q$ so $p \equiv \pm 1 \pmod q$ and $q \equiv 1 \pmod p$. We know that $p \equiv 1 \pmod q$ is not possible, if $p \equiv -1 \pmod q$. If $p < q$ then $p = q - 1$ so p must be even, so $p = 2$, and $q = 3$. If $q < p$ then $q = 1$ which is a contradiction. We will handle the $p = 2, q = 3$ case in the end.

If n_q is 1 then $N_q \lhd G$ so we have the composition series

$$1 \lhd N_q \lhd G$$

N_q is of prime order so abelian, G/N_q is of order p^2. All groups of order p^2 are abelian as is established in lecture
Finally if $q = 3, p = 2$ then G is a group of order 12. Dummit and Foote classify all groups of order 12, two are abelian (and thus solvable). The rest are the following: D_{12} which has the composition series with abelian factors

$$1 \lhd \langle r \rangle \lhd D_{12}$$

A_4 which has the composition series

$$1 \lhd \langle (12)(34), (13)(24) \rangle \lhd A_4$$

And finally the Dicyclic group $D = \langle a, b, c | a^3 = b^2 = c^2 = abc \rangle$ which has the composition series

$$1 \lhd \langle b, c \rangle \lhd D$$

Problem 1.5

(1) We have that $|GL_3(\mathbb{F}_2)| = (2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 3 \cdot 2^3$ therefore a Sylow 7-group has order 7. The size of $GL_3(\mathbb{F})$ was determined by counting all the possible nonzero vectors for the first column. There are 2^3 possible vectors and 1 zero vector. Then counting all the nonzero vectors in the 2nd column that is not a linear combination of the first vector. There are 2 possible vectors that is zero or the previous vector. Finally counting all the independent vectors for the third column. There are 4 vectors that are a linear combination of the two previous vectors.
We have that the elt

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has order 7. I have verified that $A^7 = 1$ and thus $\langle A \rangle$ is a 7-subgroup.

(2) We can consider the action of $G = GL_3(\mathbb{F}_2)$ on $S = \mathbb{F}_2^3/\{0\}$ where the action is applying the linear transformation to the vector in S. Thus we know that G is a subgroup of S_7 since it permutes 7 objects. We have that the elt A from (1) is of order 7 and thus is isomorphic

to an element of order 7 in S_7. The only elts of order 7 in S_7 are the 7 cycles so A would map under isomorphism to (1 2 3 4 5 6 7).
From Euler's theorem we know that there is an element B of order 2 in G as well. B cannot be in the group generated by A since no elt in $\langle A \rangle$ has order 2. Thus we can add the image of B as a generator.
If B maps to a two cycle, then we get the group generated by a seven cycle and a two cycle which has order 5040 and so is much larger than G.
If B maps to three disjoint 2 cylces then again we get that the group generated by a seven cycle and three disjoint two cycles has order again 5040.
If B maps to two disjoint two cycles we have four nonisomorpphic options. Either B maps to $(1\ 4)(2\ 3), (1\ 7)(2\ 3), (1\ 2)(4\ 5)$ or$(1\ 5)(2\ 3)$. For the $(1\ 4)(2\ 3), (1\ 7)(2\ 3), (1\ 2)(4\ 5)$ cases, each of the resulting subgroups have order 2520 which is much larger than G, finally we have that the group generated with (1 2)(4 5) is the same size as G. Therefore the only option for G to be isomorphic to is the group generated by (1 2)(4 5) and (1 2 3 4 5 6 7).
The way I calculated the sizes of these groups was with programming in Sage.

Problem 1.6

(1) We can show that $k[x, y, z, w]/(xy - zw)$ is an integral domain, and thus $(xy - zw)$ is prime
Consider the homomorphism $\phi : k[x, y, z, w] \to k[\frac{zw}{y}, y, z, w]$. Where $k[\frac{zw}{y}, y, z, w]$ is the ring of polinomials in $k[x, y, z, w]$ with $\frac{zw}{y}$ plugged into x in every polinomial. We define $\frac{zw}{y}$ by localizing $k[x, y, z, w]$ at y. Localizations of integral domains are integral domains, so we know that $k[\frac{zw}{y}, y, z, w]$ is an integral domain.
We define ϕ with the cononical mapping $\phi(x) = \frac{zw}{y}, \phi(y) = y, \phi(z) = z, \phi(w) = w$ and with ϕ the identity on k. ϕ is clearly a homomorphism since

$$\phi(f(x, y, z, w)g(x, y, z, w)) = f(\frac{zw}{y}, y, z, w)g(\frac{zw}{y}, y, z, w) = \phi(f)\phi(g)$$

$$\phi(f(x, y, z, w) + g(x, y, z, w)) = f(\frac{zw}{y}, y, z, w) + g(\frac{zw}{y}, y, z, w) = \phi(f) + \phi(g)$$

ϕ is surjective since every monomial $\left(\frac{zw}{y}\right)^{n_1} y^{n_2}z^{n_3}w^{n_4}$ is mapped to by $x^{n_1}y^{n_2}z^{n_3}w^{n_4}$. If we show the kernel of ϕ is the ideal $(xy - zw)$ then we have shown that $k[x, y, z, w]/(xy - zw)$ is an integral domain.
We have that for any $p(x, y, z, w) \in k[x, y, z, w]$ with $\phi(p) = 0$ then

$$p(\frac{zw}{y}, y, z, w) = 0$$

We can write p as

$$p = xyp_1(x, y, z, w) + zwp_2(x, y, z, w) + c_1x + c_2y + c_3z + c_4w + c_5$$

Plugging in $x = \frac{zw}{y}$, in order for $p(x, y, z, w) = 0$, we have that $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ since no other terms in $xyp_1(x, y, z, w)$ and $zwp_2(x, y, z, w)$ will cancel with these terms since

they have higher degrees in terms of y, z, w respectivly.
Thus we have $p = xyp_1(x, y, z, w) + zwp_2(x, y, z, w)$. Now when plugging in $x = \frac{zw}{y}$ we get $zwp_1 + zwp_2 = 0$ so $zw(p_1 + p_2) = 0$ and therefore $p_1 = -p_2$ and thus $p = p_1(xy - zw)$ so $(xy - zw)$ divides. Thus the ideal $(xy - zw)$ is the kernel. p

(2) x is irreducible since when considering the degree of a polinomial with respect to any of the variables, degrees of a product of polinomials is equal to the sum of the degrees. In order for $x = f(x, y, z, w)g(x, y, z, w)$ It would have to be the case that one of the polinomials was degree zero with respect to all of the variables, and thus is an elt of k so a unit.
x is not prime since x divides zw in the quotient since $xy - (xy - zw) = zw$, however x does not divide z or w. x does not divide z or w because the argument for showing x is irreducible works for showing that both z and w are irreducible.