Exercise 91

Given any $\epsilon > 0$, consider the covering of N by $\epsilon/2$- neighborhoods $B = \{B_{\epsilon/2}(q) : q \in N\}$ and the preimage $P = \{f^{-1}(S) : S \in B\}$. Since $\cup_{S \in B} S = N$, we have that $\cup_{U \in P} U = M$. Thus P covers M (and P is a collection open sets since f is continuous and we are taking preimages of open sets) so from the lebesgue number lemma there exists $\lambda > 0$ such that for any $m \in M$ there is a $U \in P$ such that $B_\lambda(m) \subset U$. Thus for any $x, y \in N$ where $d(x, y) < \lambda$ we have that $x, y \in B_\lambda(x)$, thus from what we have shown there is a $m \in M$ such that $B_\lambda(x) \subset f^{-1}(B_{\epsilon/2}(m))$ so $f(x), f(y) \in B_{\epsilon/2}(m)$. Thus from the triangle ineq, $d_M(f(x), f(y)) < d_M(f(x), m) + d_M(f(y), m) \leq \epsilon$. Thus f is uniformly continuous.

Exercise 93

We can consider the complements. Let $\mathcal{U} = \{U = M - C : C \in \mathcal{C}\}$. The finite intersection property translates to for any finite collection $U_1, U_2, \ldots U_n \in \mathcal{U}$, we have that from Demorgans law:

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} M - C_i = M - \bigcap_{i=1}^{n} C_i \neq M$$

Thus $\mathcal{U}$ does not contain a finite subcovering of M. Thus it must be the case that M is not covered by $\mathcal{U}$ or we contradict covering compact. Thus from Demorgans law

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{C \in \mathcal{C}} M - C = M - \bigcap_{C \in \mathcal{C}} C \neq M$$

which is only the case if $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$

Exercise 94

For any collection of open sets $\mathcal{U}$ which covers M, if the finite intersection property holds, consider the complements $\mathcal{C} = \{C = M - U : U \in \mathcal{U}\}$. Since $\mathcal{U}$ covers M we have

$$M = \bigcup_{U \in \mathcal{U}} U = \bigcup_{C \in \mathcal{C}} M - C = M - \bigcap_{C \in \mathcal{C}} C$$

Thus $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Thus $\mathcal{C}$ must not satisfy the finite intersection property so there exists $C_1, C_2, \ldots C_n$ such that

$$\bigcap_{i=1}^{n} C_i = \emptyset \Rightarrow M = M - \bigcap_{i=1}^{n} C = \bigcup_{i=1}^{n} M - C_i = \bigcup_{i=1}^{n} U_i$$

Thus we have a finite subcover.

Exercise 96

From the definition of dense we have that $B \subset \overline{A}$, thus $\overline{B} \subset \overline{A}$ since $\overline{B}$ is contained in every closed set which contains B. Since B is dense in C we have $C \subset \overline{B} \subset \overline{A}$. Thus A is dense in C

Exercise Additional Problem 1

Given any sequence $x_n \in K$ we can define the chain $A_1 \supset A_2 \supset \dots$ of relatively closed sets in K as $A_n = \overline{B_n} \cap K$ with $B_n = \{x_j : j \geq n\}$. It is clear $A_n \supset A_{n+1}$ since $B_n \supset B_{n+1}$. Thus we have from assumption

$$p \in \bigcap A_n \neq \emptyset$$

We have that p is the limit of some subsequence of x_n (and thus K is compact). We can construct this subsequence inductively as follows (letting $n_k = 1$):
We have that $p \in \overline{B}_n$ for all n, thus for $\epsilon = \frac{1}{k}$ there exists $x_{n_k} \in B_{1+n_{k-1}}$ so that $d(p, x_{n_k}) < \epsilon$. We thus have that $n_k > n_{k-1}$ since $x_{n_k} \in B_{1+n_{k-1}}$ and all the indicies in $B_{1+n_{k-1}}$ are greater than n_{k-1} and thus we have a subsequence. Thus we have the subsequence $(x_{n_k})_k \to p$ since $d(x_{n_k}, p) \to 0$