Exercise 22

It does follow. For any Cauchy sequence $(a_n)_n$ with the limit a (if it exists), we have that the set $S = (a_n)_n \cup \{a\}$ is closed and bounded (We know that every Cauchy sequence is bounded). From our midterm problem we know that any subsequence of $(a_n)_n$ converges to a. Therefore in order for S to be compact, a must exist in S and so exists in M

Exercise 23

We know that $(0, 1)$ is open in $\mathbb{R}$ as proven in class. For any r, we have that $(1/2, r/2) \in B_r(1/2, 0)$ but $(1/2, r/2) \notin (0, 1) \times \{0\}$ therefore $B_r(1/2, 0) \not\subseteq (0, 1) \times \{0\}$. Therefore $(0, 1) \times \{0\}$ is not open.

Exercise 28

(a) Not necessarily. Consider the map from the unit circle $f : S^1 \to [0, 2\pi)$ where $f(\cos(\theta), \sin(\theta)) = \theta$. We have that the inverse image of the open set $[0, \epsilon)$ is the closed set $\{(\cos(\theta), \sin(\theta)) : \theta \in [0, \epsilon)\}$
(b) Yes. Since f has a continous inverse mapping f^{-1} for any open set $U \subseteq M$ we have that the pullback of f^{-1} of an open set is open. Thus $f(U) = (f^{-1})^{-1}(U)$ is open.
(c) Yes. Since f is bijective it has an inverse f^{-1}. For any open set U, the pullback of f^{-1} of U is just $f(U)$ which is open. Thus f^{-1} is continous. So f is a homeomorphism.
(d) Not necessarily. Consider the map $f(x) = \frac{1}{3}x^3 - x$. We know all polinomials are continous, and f is clearly surjective. The 'humps' where the derivative is zero is $1, -1$, thus the open sets $(-1 - \epsilon, 1 + \epsilon)$ for small ϵ will be mapped to the closed set $[-\frac{2}{3}, \frac{2}{3}]$

Exercise 32

For any point $p \in \mathbb{N}$ we have that for $r = 1$, by definition the set $B_r(p) = \{p\}$ is open. Therefore singletone points are open in $\mathbb{N}$, so any set $S \subseteq \mathbb{N}$ is open since

$$S = \bigcup_{s \in S} \{s\}$$

And we know arbitrary unions of open sets are open. Therefore we have that S^c is open as well. The complement of an open set is closed so we know that S is closed as well. Therefore every set $S \subseteq \mathbb{N}$ is clopen.
This means that every function $f : \mathbb{N} \to M$ is continous since the inverse image of any open set $U \subseteq M$ will be open.

Exercise 34

For any closed set $L \subset N$ with N closed from the inheritance principle we know $L = C \cap N$ for some closed set $C \subset M$. Intersections of closed sets are closed. Thus L is closed in M. Conversly if L is closed in M then $L = N \cap L$ so L is closed in N
Similarly if $U \subset N$ is open and N is open, then from the inheritance principle $U = V \cap N$

where V is open in M. Finite intersections of open sets are open, thus U is open in M. Conversly if L is open in M then $L = N \cap L$ so L is open in N

Exercise 38

For d_E:
Checking all the axioms of metrics:
$d_E(x, y) \geq 0$ since $\sqrt{a^2 + b^2}$ is clearly nonegative for all $a, b \in \mathbb{R}$
For $d_E(x, y) = 0$, $\sqrt{d_X(a_x, b_x)^2 + d_Y(a_y, b_y)^2} = 0$ iff $d_X(a_x, b_x) = d_Y(a_y, b_y) = 0$ iff $a = b$
It is clear $d_E(x, y) = d_E(y, x)$
For $d_E(a, c) \leq d_E(a, b) + d_E(b, c)$ we have

$$\sqrt{d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2} \leq \sqrt{(d_X(a_x, b_x) + d_X(b_x, c_x))^2 + (d_X(a_x, b_x) + d_X(b_x, c_x))^2}$$

Iff

$$\begin{gathered} d_X(a_x, c_x)^2 + d_Y(a_y, c_y)^2 \\ \leq d_X(a_x, b_x)^2 + d_Y(a_y, b_y)^2 + d_X(b_x, c_x)^2 + d_Y(b_y, c_y)^2 \\ +2\sqrt{(d_X(a_x, b_x)^2 + d_Y(a_y, b_y)^2)(d_X(b_x, c_x)^2 + d_Y(b_y, c_y)^2)} \end{gathered}$$

From the Cauchy shwartz it follows that this inequality is true. Since this is the same as taking the standard inner product on $\mathbb{R}$
For $d_{\max}$:
Checking all the axioms of metrics:
$d_{\max}(x, y) \geq 0$ since $\max(|a|, |b|)$ is clearly nonegative for all $a, b \in \mathbb{R}$
For $d_{\max}(x, y) = 0$, $\max(|d_X(a_x, b_x)|, |d_Y(a_y, b_y)|) = 0$ iff $d_X(a_x, b_x) = d_Y(a_y, b_y) = 0$ which is the case iff $a = b$.
It is clear $d_{\max}(x, y) = d_{\max}(y, x)$
For $d_{\max}(a, c) \leq d_{\max}(a, b) + d_{\max}(b, c)$ we have

$$\max(|d_X(a_x, c_x)|, |d_Y(a_y, b_y)|) \leq \max(|d_X(a_x, b_x) + d_X(b_x, c_x)|, |d_Y(a_y, b_y) + d_Y(b_y, c_y)|)$$

$$\leq \max(d_X(a_x, b_x), d_Y(a_y, b_y)) + \max(d_X(b_x, c_x), d_Y(b_y, c_y))$$

For d_{sum}:
Checking all the axioms of metrics:
$d_{\text{sum}}(x, y) \geq 0$ since $a + b$ is clearly nonegative for all $a, b \in \mathbb{R}^+$
For $d_{\text{sum}}(x, y) = 0$, $d_X(a_x, b_x) + d_Y(a_y, b_y) = 0$ iff $d_X(a_x, b_x) = d_Y(a_y, b_y) = 0$ which is the case iff $a = b$.
It is clear $d_{\text{sum}}(x, y) = d_{\text{sum}}(y, x)$
For $d_{\text{sum}}(a, c) \leq d_{\text{sum}}(a, b) + d_{\text{sum}}(b, c)$ we have

$$d_X(a_x, c_x) + d_Y(a_y, b_y) \leq d_X(a_x, b_x) + d_X(b_x, c_x) + d_Y(a_y, b_y) + d_Y(b_y, c_y)$$

Exercise 52

(a) We know that the intersection cannot contain 2 or more points since if there exists x, y in the intersection with $x \neq y$ then $d(x, y) > 0$, we have that every interval must contain these two points. Therefore from the definition of the diameter, we have that the diameter of every interval is $\geq d(x, y) > 0$ which contradicts the diameter converging to 0 (we set $\epsilon = d(x, y)$ and then for all N we have that $\text{dia}(I_n) \geq \epsilon$ for all $n > N$)
Now we show its nonempty. For each I_n we choose a point a_n in I_n. We have that the sequence of points $(a_n)_n$ is Cauchy since for any $\epsilon > 0$ we have that there exists N such that $\text{diam}(I_n) < \epsilon$ for $n > N$ so for all $a_k, a_n,\ n, k > N$ we have that $a_k, a_n \in I_{\min(k,n)}$ so $d(a_n, a_k) \leq \text{dia}(I_{\min(k,n)}) \leq \epsilon$.
We have that every interval contains the limit point a of the Cauchy sequence since for each I_k we have the subsequence of $(a_n)_n$ starting at k which is also Cauchy and since I_k is closed, it contains the limit point. Therefore a is in the intersection, so the intersection is non-empty.

Exercise Additional Problem 1

Since S is bounded we have that there exists $r > 0, x \in S$ with $S \subseteq B_r(x)$, where B_r is the closed ball of radius r. We have that the lim S is contained B_r since we know that it is contained in every closed set that contains S. Thus it is bounded