Math 380a, Fall 2016 ***HOMEWORK #7*** ***Peter Gylys-Colwell***

1. If we foil out the product we have

$$\prod_{i=1}^{n}(1+x^{2^i}) = 1+x^2+x^4+x^6\dots x^{2^n-2}$$

We know this since every number has a unique binary representation, and by multiplying by either a 2^{2^i} or by 1 we are either choosing a 1 or a 0 for the ith digit of the binary representation of the power, since i starts at 1 and not zero, we know only even powers show up

The series we get is a geometric series whose limit is:

$$= \frac{1-(x^2)^{n+1}}{1-x^2} \to \frac{1}{1-x^2}$$

2. We know that the limit must satisfy

$$L = \frac{e^L - 1}{2}$$

and so L is either 0, or a number w, where $2 > w > 1$ such that

$$2w + 1 = e^w$$

Analysing the function $f(x) = \frac{e^x-1}{2} - x$ we have $f(x) > 0$ for $x < 0$, $f(x) < 0$ for $w > x > 0$ and $f(x) > 0$ for $x > w$

Therefore we know the sequence must diverge for $\alpha = 2$ since $x_0 = 2 > w$ and $x_{n+1} - x_n = f(x_n)$, x_n would be an increasing unbounded sequence.

For $\alpha = .5$, the sequence converges to 0. We know that if $x_n \in (0, w)$ then $f(x_n) < 0$ so $x_n > x_{n+1}$. We also know that if $x_n > 0$ then $x_{n+1} > 0$ since $e^{x_n} > 1$. Therefore x_n is a decreasing sequence bounded from below by 0

3. The limit goes to infinity.

We know that the limit must satisfy

$$L = L + 10^{-10^L}$$

so

$$10^{-10^L} = 0$$

so $L \to \infty$

4. We can rewrite the sum as

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1+\frac{j^2}{n^2}}$$

This is a riemann sum of the function

$$\frac{1}{1+x^2}$$

over the interval $(0,1)$.
Therefore by the fundamental thm of calculus we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1+\frac{j^2}{n^2}} = \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{2}$$