Math 506, SPRING 2018 ***HOMEWORK #3*** ***Peter Gylys-Colwell***

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We have the following commutative diagram

$$\begin{tikzcd}
 & P' \arrow[dl, shift left, "\alpha'"] \arrow[d, two heads, "\phi'"] & \ker\phi' \arrow[l, hook', "i'"] \\
P \arrow[ur, shift left, "\alpha"] \arrow[r, two heads, "\phi"] & M & \\
\ker\phi \arrow[u, hook, "i"] & &
\end{tikzcd}$$

We get the mappings α and α' from projectivity since $\phi : P \to M$ surjects $\exists\alpha' : P' \to P$ such that $\phi = \phi' \circ \alpha'$ and $\exists\alpha : P \to P'$ such that $\phi' = \phi \circ \alpha$.
Since $\phi \circ \alpha' \circ i' = \phi' \circ i' = 0$, $\exists! s' : \ker\phi' \to \ker\phi$ with $\alpha' \circ i' = i \circ s'$
Thus we have the mappings

$$\begin{tikzcd}
P \arrow[d, "\alpha"] & & \ker\phi' \arrow[d, "s'"] \\
P' \arrow[dr] & & \ker\phi \arrow[dl] \\
 & P' \oplus \ker\phi &
\end{tikzcd}$$

which induces a unique mapping $f : P \oplus \ker\phi' \to P' \oplus \ker\phi$. Swapping the labeling of the above argument would also yield a unique mapping $g : P' \oplus \ker\phi \to P \oplus \ker\phi'$. I am stuck on where to go from here

An example where the condition fails is with the $\mathbb{Z}$ modules $P = \mathbb{Z}, P' = \mathbb{Z}/(4)$ and $M = \mathbb{Z}/(2)$. We then have the canonical mappings with kernels

$$\phi : P \to M, \phi' : P' \to M$$

$$\ker\phi = 2\mathbb{Z}, \ker\phi' = 2\mathbb{Z}/(4)$$

Then

$$\ker\phi' \oplus P = 2\mathbb{Z}/(4) \oplus \mathbb{Z} \ncong 2\mathbb{Z} \oplus \mathbb{Z}/(4) = \ker\phi \oplus P'$$

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Since P is free of infinite rank and P' is finitly generated we can surject $\phi : P \to P'$.
Thus letting $M = P'$ we have the maps $\phi : P \to M$, $\mathrm{id} : P' \to M$ which yields

$$P \simeq P' \oplus \ker\phi$$

We have that any module that is the direct summand of a free module is projective. This is because if $F = M \oplus N$ is free the functor

$$\mathrm{Hom}(F, -) = \mathrm{Hom}(M, -) \oplus \mathrm{Hom}(N, -)$$

is exact which means the summands must be exact. We did not need the hypothesis to apply for every M, just the case when $P' = M$

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4

From Maschke's Theorem we know that any $\mathbb{C}[G]$ module is semisimple, thus for any injection

$$\phi : I \to M$$

we have that $M \simeq I \oplus \operatorname{coker} \phi$ and thus ϕ has the left inverse defined by

$$\phi^{-1} \oplus 0 : I \oplus \operatorname{coker} \phi \to I$$

Where ϕ^{-1} denotes the inverse of ϕ over $\operatorname{im} \phi$. A well known equivalent definition of an injective module is that every injection has a right inverse, thus we have shown I to be injective.

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For any embedding $N \subseteq M$ of R-modules and R-module homomorphism $\alpha : N \to I$ we can consider the collection of pairs

$$\mathcal{C} = (D, \beta)$$

Where $D \subseteq M$ are submodules containing N and $\beta : D \to I$ are mappings which restricted to N yield the equality $\beta|_N = \alpha$

We have that $\mathcal{C}$ forms a partially ordered set where $(D_1, \beta_1) < (D_2, \beta_2)$ iff $D_1 \subset D_2$ and $\beta_1 = \beta_2|_{D_2}$. We have that any ascending chain

$$(D_1, \beta_1) \leq (D_2, \beta_2) \leq (D_3, \beta_3) \leq \dots$$

has the upper bound

$$\left(\bigcup_{i=1}^{\infty} D_i, \bigcup_{i=1}^{\infty} \beta_i \right)$$

Thus by Zorns lemma there is a maximal element (M', α') We have that $M' = M$ and thus α' is a mapping $M \to I$ where $\alpha'|_N = \alpha$ thus establishing I to be injective. We have $M' = M$ as follows

Suppose for contradiction there is $m \in M \backslash M'$, then we have the ideal

$$J = \{r \in R | rm \in M'\}$$

We can restrict α' to $\alpha'|_{Jm} : Jm \to I$. Thus this mapping extends to a mapping $\alpha''|_{Rm} : Rm \to I$. Thus we have a new pair $(M' \cup Rm, \alpha'')$ which is strictly larger than (M', α'), contradicting maximality.

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$(\Rightarrow)$: If an abelian group G is divisible, then as a $\mathbb{Z}$ module, we have that G satisfies Baer's criterion and thus is injective.
For any ideal

$$J = \langle n \rangle \subset \mathbb{Z}$$

with a morphism $\phi : J \to G$ by divisiblity there exists y such that $ny = \phi(n)$ and thus we have the extension

$$\psi : \mathbb{Z} \to G$$

defined by $\psi(1) = y$

$(\Leftarrow)$: Suppose G was injective. Given any element $y \in G$ and $n \in \mathbb{N}$ consider the subgroup $\langle y \rangle = Y \subseteq G$ which has canonical inclusion map $i : Y \to G$. We will define the group

$$H = \mathbb{Z}/(n \cdot |Y|)$$

(if $|Y| = \infty$ then $H = \mathbb{Z}$). We have the injective map

$$f : Y \to H$$

where $f(a) = n$, $f(ka) = kn$ thus we have the diagram

$$\begin{tikzcd} & I \\ Y \arrow[ur, "i"] \arrow[r, "f"] & H \end{tikzcd}$$

thus from injectivity there is an induced map $h : H \to I$ with $h \circ f = i$. Thus $n \cdot h(1) = h(n) = h(f(a)) = i(a) = a$ so $h(1)$ is a solution to $ng = y$ so G is divisible