Exercise 51

If $f(x) < g(x)$ for all $x \in [a, b]$ then we have that $h(x) = g(x) - f(x)$ is a sum of riemann integrable functions, and thus riemann integrable. Leting p be a point of continuity for h, we can choose a partition $P = \{x_0 = a, x_1, \ldots x_n = b\}$ such that $p \in (x_1, x_2)$ and $h(x_1, x_2) \subset B_{h(p)/2}(h(p))$. (in other words, $h(x) > h(p)/2$ for all $x \in (x_1, x_2)$). We have the inequality

$$\int_a^b h(x)\ dx \geq \sum_{i=1}^{n} m_i(x_i - x_{i-1}) > 0$$

We have that $m_i \geq 0$ for all i, and $m_1 \geq h(p)/2 > 0$. Thus we have strict inequality.
We know that integration preserves addition and subtraction:

$$\int_a^b g(x)\ dx - \int_a^b f(x)\ dx = \int_a^b h(x)\ dx > 0$$

Thus

$$\int_a^b g(x)\ dx > \int_a^b f(x)\ dx$$

Exercise 53

We have that the discontinuity points $D(\max(f, g))$, $D(\min(f, g))$ are subsets of $D(f) \cup D(g)$. Since the finite union of zero sets is a zero set, this implies riemann integrablility.
Showing the set inclusion above is equivalent to showing that if f, g are continuous at x, then $\max(f, g), \min(f, g)$ are continuous at x. This was proven in homework for Math 424 using the epsilon delta definition of continuity

Exercise 62

$$2^k a_{2^k} = a_{2^k} + a_{2^k} + \cdots + a_{2^k} \leq a_{2^{k-1}+1} + a_{2^{k-1}+2} + a_{2^{k-1}+3} + \ldots a_{2^k} = \sum_{i=2^k+1}^{2^k} a_i$$

Thus

$$\sum_{i=1}^{2^n} a_i = \sum_{k=1}^{n} \sum_{j=2^{k-1}+1}^{2^k} a_j \geq \sum_{k=1}^{n} 2^k a_k$$

So by comparison $\sum_{k=1}^n 2^k a_k$ converges if $\sum_{i=1}^{2^n} a_i$ converges which converges iff $\sum a_i$ converge. Conversly

$$2^k a_{2^k} = a_{2^k} + a_{2^k} + \cdots + a_{2^k} \geq a_{2^k} + a_{2^k+2} + a_{2^k+3} + \ldots a_{2^{k+1}-1} = \sum_{i=2^k}^{2^{k+1}-1} a_i$$

so now we have

$$\sum_{i=1}^{2^n-1} a_i = \sum_{k=1}^{n} \sum_{j=2^k}^{2^{k+1}-1} a_j \leq \sum_{k=1}^{n} 2^k a_k$$

Thus $\sum_{i=1}^{2^n-1} a_i$ converge if $\sum_{k=1}^{n} 2^k a_k$ converge.

Exercise Additional Problem 1

For any $x_j > x_{j-1} \in (a,b)$, from the mean value theorem there exists $x \in (x_j, x_{j-1})$ where

$$f'(x)(x_j - x_{j-1}) = f(x_j) - f(x_{j-1})$$

Using the standard definition of m_j, M_j established in lecture, we have $m_j \leq f'(x) \leq M_j$. Thus

$$m_j(x_j - x_{j-1}) \leq f(x_j) - f(x_{j-1}) \leq M_j(x_j - x_{j-1})$$

Thus for any partition of (a,b) $P = \{x_0 = a, x_1, \dots x_{n-1}, x_n = b\}$ we have

$$\underline{I}(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = f(b) - f(a)$$

$$\overline{I}(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = f(b) - f(a)$$

Thus since $\int_a^b f'(x)\, dx = \sup_P\{\underline{I}(P)\} = \inf_P\{\overline{I}(P)\}$. The first inequality yields $\int_a^b f'(x)\, dx \leq f(b) - f(a)$ and the second yields $\int_a^b f'(x)\, dx \geq f(b) - f(a)$. So $\int_a^b f'(x)\, dx = f(b) - f(a)$

Exercise Additional Problem 2

(a) Since $k > 0$, we can choose $x_0 > 0$ large enough so $x_0^k > y$. By the intermediate value theorem there exists $x \in (0, x_0)$ such that $x^k = y$ since $y \in (0^k, x_0^k)$. There is only one such x since if there existed $x_1, x_2 > 0$ where $x_1^k = x_2^k = y$ then by the mean value theorem there exists $x \in (x_1, x_2)$ where

$$\frac{d}{dx}x^k = \frac{x_2^k - x_1^k}{x_2 - x_1} = 0$$

$$kx^k = 0$$

but this is the case iff $x = 0$ which is not true

(b) Suppose for contradiction $\lim_{k\to\infty} y^{1/k} \neq 1$. This would mean there exists $r \neq 1$ such that for some K, $1 > r > y^{1/k}$ or $y^{1/k} > r > 1$ for all $k > K$

Taking k powers (since taking a $k \in \mathbb{N}$ power of positive numbers preserves inequalities)

$$1 > r^k > y \text{ or } y > r^k > 1$$

For all $k > K$. This is a contradiction since we know $\lim_{k\to\infty} r^k = 0$ or ∞ which cannot be the case for y