Exersise 5.1

(1) We have that for any of the generators $e_q \in \mathbb{R}[Q_8]$ that $(e_1+e_{-1})e = e_q+e_{-q} = e(e_1+e_{-1})$ in other words, a commutes with every generator in $\mathbb{R}[Q_8]$ and thus commutes with every element in $\mathbb{R}[Q_8]$ thus $a\mathbb{R}[Q_8] = \mathbb{R}[Q_8]a$.
We have that $\mathbb{H} \cong \mathbb{R}[Q_8]/a\mathbb{R}[Q_8]$ since we can relabel the cosets of e_1 as 1, e_i as i, e_j as j and e_k as k to get $\mathbb{H}$
(2) Let $a = e_{()} + e_{(12)}$ we have that if we apply a to $e_{(13)}$ on the left we get

$$e_{(13)} + e_{(132)}$$

However there is no possible elt $b \in \mathbb{C}[S_3]$ where $ba = e_{(13)}a$ thus $a\mathbb{C}[S_3] \neq \mathbb{C}[S_3]a$. The reason no b exists is because in order for $be_{()} + be_{(12)} = ba = e_{(13)} + e_{(132)}$ b must have either $e_{(13)}$ or $e_{(132)}$ as a term so that $be_{()} = e_{(13)}$ or $e_{(132)}$ however

$$e_{(13)}a = e_{(13)} + e_{(123)}$$

and

$$e_{(132)}a = e_{(132)} + e_{(23)}$$

Both of which produce terms that do not show up in $e_{(13)}a$

Exersise 5.2

We have that for any $a, b \in \mathbb{Z}$

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We know that homomorphisms must map nilpotent elts to nilpotent elts, the only nilpotent elt in $\mathbb{Z}$ is 0 thus

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \to 0, \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \to 0$$

We have that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus we have that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to 0$$

Therefore the only homomorphism is the zero homomorphism.

Exersise 5.3

(1) For any $a, b \in R$ we have $a + I, b + I \in R/I$, if

$$(a + I)(b + I) = ab + I = 0 + I$$

We have $ab+I=0 \Leftrightarrow ab \in I$. If R/I is an integral domain then either $a+I=0$ or $b+I=0 \Rightarrow a \in I$ or $b \in I$ so I is prime. Conversly if I is prime then $ab+I=0 \Rightarrow ab \in I \Rightarrow a \in I$ or $b \in I$ so $a+I=0$ or $b+I=0$ thus R/I is an integral domain

(2) If I is maximal then if there exists $a+I \in R/I$ where $a \notin I$ that is not invertable then we can define a new ideal $M=(a)+I$. Since a is not invertable in R/I we have that $1 \notin M$ since if $1 \in M$ then $1=r_1a+r_2s$ where $s \in I$ and thus r_1+I would be the inverse of $a+I$ in R/I which is a contradiction. But then we have that M is a proper ideal which contains I and is larger that I since $a \in M, a \notin I$ and thus I's maximality is contradicted. Thus every nonzero elt of R/I has a multiplicative inverse.
Conversly if R/I is a field yet I is not maximal then there exists an ideal $M \neq R, I$ with $I \subset M$ then choose $a \in M$ where $a \notin I$. We have that $a+I$ is invertable so there exists $b \in R$ such that $ab+rs=1$ where $r \in R, s \in I$. Notice that $a, s \in M$ and thus $ab+rs \in M$ which means $1 \in M \Rightarrow M=R$ which contradicts M proper. Thus I is maximal.

(3) By definition $R_p = S^{-1}R$ where $S=R-P$. From last weeks homework we have proven there is a bijective correspondence between prime ideals in R not meeting S and R_P. From how S is definied prime ideals in R not meeting S are prime ideals contained in P. Thus we have our bijective correspondence.

Exersise 5.4

(1) We use one of the standard norms $N(a+ib)=a^2+b^2$. We have to check that N lets the euclidean algorithm work: $\forall a, b \neq 0 \in \mathbb{C}[i]$, $\exists d, r \in \mathbb{C}[i]$ such that $a=db+r$ where $N(r)<N(b)$ or $r=0$.
For any $\alpha, \beta \in \mathbb{C}[i]$, since $\mathbb{C}$ is a field we know $(a+bi)=\alpha/\beta \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Thus we have

$$\alpha = (a+bi)\beta$$

We can choose integers x, y such that $|a-x| \leq \frac{1}{2}$ and $|b-y| \leq \frac{1}{2}$. Thus

$$\alpha = (x+iy)\beta + ((a-x)+(b-y)i)\,\beta$$

Notice that $\alpha, (x+iy)\beta \in \mathbb{C}[i]$ and thus $((a-x)+(b-y)i)\beta \in \mathbb{C}[i]$ since $\mathbb{C}[i]$ is closed under addition.
We have that either $(a-x)+(b-y)i=0$ or

$$N(((a-x)+(b-y)i)\beta) = ((a-x)^2+(b-y)^2)N(\beta) \leq \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) N(\beta)$$

$$= \frac{1}{2}N(\beta) < N(\beta)$$

Thus letting $\gamma = a+iy$ and $\rho(a-x)+(b-y)i$ we have

$$\alpha = \gamma\beta + \rho$$

with $N(\rho) < N(\beta)$ or $\rho = 0$. Thus $\mathbb{C}[i]$ is a Euclidean domain.
(2) We have

$$6 = -i \cdot (1+i)^2 \cdot 3$$

i is a unit, $1+i$ is irreducible since $N(1+i) = 2$ is prime and so if $ab = 1+i$ then $N(ab) = N(a)N(b) = 2 \Rightarrow N(a)$ or $N(b) = 1 \Rightarrow a$ or b is a unit. 3 is irreducible since $N(3) = 9$ so if $ab = 3$ then $N(ab) = N(a)N(b) = 9$, if $N(a)$ or $N(b) = 1$ then a or b is a unit, otherwise if $N(a) = N(b) = 3$ then $\exists x, y \in \mathbb{Z}$ where $a = x + iy$ and $x^2 + y^2 = 3$, a simple check of possible numbers less than 3 shows there is no solutions and thus this is not possible.

Exersise 5.5

(1) We have

$$9 = 3^2 = (2+\sqrt{-5})(2-\sqrt{-5})$$

We now have to show $3, 2+\sqrt{-5}$ and $2-\sqrt{-5}$ are irreducible and thus this is two different factoraizations.
We will use the traditional norm defined over the complex numbers $|x+iy| = x^2+y^2$ which satisfy all the axioms of norms. First we have in $\mathbb{Z}[\sqrt{-5}]$ that $|x+y\sqrt{-5}| = 1 \Rightarrow x+y\sqrt{-5} = \pm 1$. The reason is because $1 = |x+y\sqrt{-5}| = x^2+5y^2$, and since $x, y \in \mathbb{Z}$ we know $y = 0, x = \pm 1$. Second we know that there exists no $x+y\sqrt{-5}$ with $|x+y\sqrt{-5}| = 3$ since $x^2+5y^2 > 3$ if $y \neq 0$ and other wise we have $x^2+5y^2 = x^2+0 \neq 3$ for all $x \in \mathbb{Z}$ since 3 is not a square in $\mathbb{Z}$
Therefore we have that if $3 = \alpha\beta$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ then

$$9 = |3| = |\alpha||\beta|$$

So since no norms can be 3, one of the norms must be 1 and thus a unit. The argument is the same for $2+\sqrt{-5}, 2-\sqrt{-5}$:

$$|2+\sqrt{-5}| = |2-\sqrt{-5}| = 9$$

and thus any product equal to either of these terms must be the product of a unit and an elt of norm 9.
(2) Let $I = \langle 3, 2+\sqrt{-5}\rangle$. If $I = \langle\lambda\rangle$ for some $\lambda \in \mathbb{Z}[\sqrt{-5}]$ then $3 = x\lambda$ and $2+\sqrt{-5} = y\lambda$ for some $x, y \in \mathbb{Z}[\sqrt{-5}]$.
However we have already shown that 3 and $2+\sqrt{-5}$ are irreducible. Thus λ is either a unit or x, y is a unit. The only units are $1, -1$ so if lambda is not a unit then $3 = \pm(2+\sqrt{-5})$ which is obviously not the case. Hence λ must be a unit so $\langle\lambda\rangle = \mathbb{Z}[\sqrt{-5}] \Rightarrow 1 \in I$. However this is not possible. If we have

$$3\alpha + (2+\sqrt{-5})\beta = 1$$

Then multiplying by $2-\sqrt{-5}$ yields

$$3\alpha(2-\sqrt{-5}) + 9 = 2-\sqrt{-5}$$

Which means 3 divides $2-\sqrt{-5}$, but $2-\sqrt{-5}$ is irreducible and $3,-3$ are the only possible values of 3 multiplied with a unit so this is impossible

Exersise 5.6

We have that the number of elements in the qotient is 8:

$$R = \mathbb{Z}[x]/(2, x^3+1) \cong \mathbb{Z}_2[x]/(x^3+1) = \{x^2, x^2+x, x^2+x+1, x^2+1, x+1, x, 1, 0\}$$

Every ideal of R is generated be a combination of these elements. However since R is finite, all non-nilpotent elements of R are units and thus generate all of R. We have that the only nilpotent elements are $(x+1), (x^2+x+1), (x+1)^2 = x^2+1, (x+1)^3 = x^2+x$ since $x^3+1 = (x+1)(x^2-x+1)$. We know that these are the only nilpotents since $\mathbb{Z}_2[x]$ is a UFD so we know that any elt that is nilpotent must share an irreducible divisor with (x^3+1) in $\mathbb{Z}_2[x]$ in order to divide a multiple of (x^3+1) by a factor not divisible by (x^3+1).
Thus we have the proper Ideals

$$(x{+}1), (x^2{+}x{+}1), ((x{+}1)^2), ((x{+}1)^3), (x{+}1, x^2{+}x{+}1), ((x{+}1)^2, x^2{+}x{+}1), ((x{+}1)^3, x^2{+}x{+}1)$$