9.1

a. This is not an equivalence relation:
Consider $1, 2$, we have $2 - 1 \geq 0$ but $1 - 2 \not\geq 0$

b. This is an equivalence relation: for any $a \in \mathbb{Z}$, if $|a| = |b|$ then $b = a$ or $b = -a$ so $|b| = |a|$, and if for some $c \in \mathbb{Z}$ and $c\ R\ b$ then $c = b$ or $c = -b$ so $c = a$ or $c = -a$ so $c\ R\ a$

c. This is not an equivalence relation:
Consider $1, 0, -1$. We have $1\ R\ 0$ and $-1\ R\ 0$ but we dont have $1\ R\ -1$

d. This is not an equivalence relation:
Consider $1, 0, -1$. We have $1\ R\ 0$ and $-1\ R\ 0$ but we dont have $1\ R\ -1$

9.3 All points that belong to the same equivalence class have the property

$$y - x = a$$

for some $a \in \mathbb{R}$ therefore the equivalence classes are lines of the form

$$y = a + x$$

9.5

a. We have $\langle J \rangle = \{I, J, -I, -J\}$, and so we have

$$\langle J \rangle I = \langle J \rangle$$

$$\langle J \rangle K = \{K, L, -K, -L\}$$

b. We have $\langle -I \rangle = \{I, -I\}$, and we have

$$\langle -I \rangle J = \{J, -J\}$$

$$\langle -I \rangle K = \{K, -K\}$$

$$\langle -I \rangle L = \{L, -L\}$$

$$\langle -I \rangle I = \langle -I \rangle$$

9.6 We have

$$HI = H$$
$$Hf = \{f, fg\}$$
$$Hf^2 = \{f^2, g\}$$
$$Hf^3 = \{f^3, f^3g\}$$

As for the left cosets we have

$$IH = H$$
$$fH = \{f, f^3g\}$$
$$f^2H = \{f^2, g\}$$
$$f^3H = \{f^3, fg\}$$

9.10 We have $H = \{I, \{1\}, \{1,2\}, \{2\}\}$, and so

$$HI = H$$

$$H\{1,2,3,4\} = \{\{1,2,3,4\}, \{2,3,4\}, \{3,4\}, \{1,3,4\}\}$$
$$H\{1,2,3\} = \{\{1,2,3\}, \{2,3\}, \{3\}, \{1,3\}\}$$
$$H\{1,2,4\} = \{\{1,2,4\}, \{2,4\}, \{4\}, \{1,4\}\}$$

9.11 For any sets A, B, C we have

$$A\ R\ A$$

since we can create a bijective map from each element of A back to itself. We also have

$$A\ R\ B \Rightarrow B\ R\ A$$

Since every bijective function from A to B has a bijective inverse function from B to A. Finally

$$A\ R\ B \text{ and } B\ R\ C \Rightarrow A\ R\ C$$

Since we the composition of bijective functions is bijective and so if we compose the bijective function from A to B and the bijective function from B to C we get a bijective function from A to C

9.14 R is not an equivalence relation for any non-abelian group G. Consider $a, b, e \in G$ such that e is the identity and a and b do not commute. We have

$$a\ R\ e, b\ R\ e$$

but it is not the case that

$$a \; R \; b$$

9.16

a. For any a, b, c in G we have

$$a^{-1}a = e \in H$$

so $a \; R \; a$, we have

$$a^{-1}b \in H \Rightarrow (a^{-1}b)^{-1} = b^{-1}a \in H$$

so $a \; R \; b \Rightarrow b \; R \; a$. Finally we have

$$a^{-1}b, b^{-1}c \in H \Rightarrow a^{-1}bb^{-1}c = a^{-1}c \in H$$

so $a \; R \; b$ and $b \; R \; c \Rightarrow a \; R \; c$

b. let $\bar{a}$ denote the equivalence class of a, we have

$$\bar{a} = aH$$

Since

$$x \in \bar{a} \text{ iff } a^{-1}x \in H \text{ iff } x \in aH$$

c.

$$aH = bH \Leftrightarrow a^{-1}aH = H = a^{-1}bH \Leftrightarrow a^{-1}b \in H$$

9.18

$$Hx = Ky \Leftrightarrow H = Kyx^{-1}$$

In order for $e \in H$, from the above equality we know $(yx^{-1})^{-1} \in K$ so $yx^{-1} \in K$ so

$$Kyx^{-1} = K = H$$

9.19 Consider $f_3 = x + 1$, $f_3 \in H$ so $f_2 \circ f_3 \in f_2H$.

$$f_2 \circ f_3 = 2(x+1) = 2x + 2$$

however there is no $f \in H$ such that

$$f_1 \circ f = 2x + 2$$

The proof for this is that we know all $f \in H$ have the form $x + n$, so $f_1 \circ f$ has the form $2(x+n) + 1 = 2x + 2n + 1$, $2n + 1$ must be an odd number, and so there is no $n \in \mathbb{Z}$ such

that $2n + 1 = 2$
This proves that $f_2 H \neq f_1 H$ since there is an element in $f_2 H$ not in $f_1 H$
To show $Hf_2 = f_1 H \cup f_2 H$, we wll show every element of Hf_2 is in $f_1 H \cup f_2 H$ and then every element of $f_1 H \cup f_2 H$ is in Hf_2
For any $f_3 \in H$ we have $f_3 = x + n$ for some $n \in \mathbb{Z}$ so we have

$$f_3 \circ f_2 = 2x + n$$

if n is even

$$= 2x + 2k = 2(x + k) \in f_2 H$$

and if n is odd

$$= 2x + 2k + 1 = 2(x + k) + 1 \in f_1 H$$

For some $k \in \mathbb{Z}$
therefore no matter what

$$f_3 \circ f_2 \in f_1 H \cup f_2 H$$

And

$$f_1 \circ f_3 = 2x + 2n + 1 = (2x) + (2n + 1) \in Hf_2$$

and

$$f_2 \circ f_3 = 2x + 2n = (2x) + (2n) \in Hf_2$$

so

$$Hf_2 = f_1 H \cup f_2 H$$