Exersise 7.1

Notice that S_3 is the group generated by $x = (123), y = (12)$ where $x^3 = y^2 = (xy)^2$ and thus the answer in problem 6 yields all the representations of S_3

Exersise 7.2

We have that Q_8 is generated by i, j with $i^4 = j^4 = k^4 = 1$ and $ij = k$. For the 1 dimensional case i, j must map to $\{1, \zeta_4, -1, -\zeta_4\}$. We have the representations

$$i \to -1, j \to -1, k \to 1$$

$$i \to -1, j \to 1, k \to -1$$

$$i \to 1, j \to -1, k \to -1$$

We cannot have $i \to \zeta_4$ since then $i^2 = j^2 \to -1$ so $j \to \{\zeta_4, -\zeta_4\}$ but then we have $ij = -ji$ map to different elements: $ij \to -1, -ji \to 1$. This covers all the possiblities of one degree representations
For degree 2 we have the irreducible representation

$$i \to \begin{bmatrix} \zeta_4 & 0 \\ 0 & -\zeta_4 \end{bmatrix}, j \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k \to \begin{bmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{bmatrix}$$

This is a complete list of representations since the sum of the degrees of each of these representations is equal to 8 which matches with the degree of $\mathbb{C}[Q_8]$

Exersise 7.3

Letting R, F be the generators of D_n where $R^n = F^2 = RFRF = 1$ we have the irreducible faithful representation

$$R \to \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}$$

$$F \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This representation is irreducible since when viewed as a $\mathbb{C}[G]$ module we have that the module is isomorphic to the two by two matricies $M_2(\mathbb{C})$ (since the images of R and F generate $M_2(\mathbb{C})$ as $\mathbb{C}$ algebras) and not a direct sum of matrix rings.

Exersise 7.4

Fixing g, we have the representation

$$T = \rho(g) \in GL_n(\mathbb{C})$$

is a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$. We can choose a basis so that T is in Jordan canonacal form

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

We have that the minimal polynomial is seperable (and thus the matrix is fully diagonalizable) since $T^m = 1$ where m is such that $g^m = 1$, and thus the minimal divides the seperable polynomial $x^m - 1$. Each λ_i is a root of unity and thus $\mathrm{Tr}(\rho) = \mathrm{Tr}(T) = \sum_{i=1}^{n} \lambda_i$ is a sum of roots of unity

Exersise 7.5

If we have a faithful irreducible representation

$$\rho : \mathbb{C}[G] \to M_n(\mathbb{C})$$

We know that ρ must map $Z(G)$ to the center of $M_n(\mathbb{C})$ since $Z(G)$ is in the center of $k[G]$. The center of $M_n(\mathbb{C})$ is the set of diagonal matricies $D_n(\mathbb{C})$ which is isomorphic to $\mathbb{C}$ as rings. This induces the group homomorphism

$$\rho \circ \pi : Z(G) \to \mathbb{C}^\times$$

where $\pi : D_n(\mathbb{C}) \to \mathbb{C}$ is the ring isomorphism
Since G is finite each $g \in Z(G)$ has a power m such that $g^m = 1$. Thus $g \to \zeta_m$ maps to some mth root of unity. Thus we have

$$Z(G) \to \langle \zeta_{m_1}, \zeta_{m_2}, \dots \zeta_{m_n} \rangle \subset \mathbb{C}^\times$$

It is the case that

$$\langle \zeta_{m_1}, \zeta_{m_2}, \dots \zeta_{m_n} \rangle \subseteq \langle \zeta_m \rangle \cong \mathbb{Z}/(\varphi(m))$$

where $m = \gcd(m_1, m_2, \dots m_n)$. Thus

$$\mathbb{Z}(G) \subseteq \mathbb{Z}/(\varphi(m))$$

is cyclic

Exersise 7.6

Aside from the trivial representation, there are two 1 dimensional representations of G. Since $x^3 = y^2 = 1$ it must be the case that

$$x \to \{1, \zeta_3, \zeta_3^2\}, y \to \pm 1$$

if $x \not\to 1$ then we would have $xy \to \{\zeta_6, \zeta_6^5, \zeta_3, \zeta_3^2\}$ none of which square to 1, thus $x \to 1$ and we have the nontrivial representation $y \to -1$

For the two dimensional case we can choose a basis so that the representation of x is in the Jordan Canonical form. If the form was

$$x \to \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \text{ then } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda^3 & 3\lambda^3 \\ 0 & \lambda^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

would mean $3\lambda^3 = 0$ which is not possible. Thus

$$x \to \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Where λ_1, λ_2 have order 3 in $\mathbb{C}$ and thus $\lambda_1, \lambda_2 \in \{1, \zeta_3, \zeta_3^2\}$. Thus we have the options

$$x \to \begin{bmatrix} \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix}, x \to \begin{bmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{bmatrix}$$

Since we have $xyxy = 1$, $y = y^{-1}$ it must be the case $xy \neq x^{-1}y = yx$, we have that the first option for x is not possible since the image of x is in the center of $M_2(\mathbb{C})$ and so would commute with the image of y which is not the case in G
Thus the only possiblity is the second option, which leads to the image for y:

$$y \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There can be no larger degree irreducible representations than 2 since $\mathbb{C}[G]$ is a degree 6 vector space over $\mathbb{C}$ and any $n \times n$ matrix ring is degree ≥ 9 for $n \geq 3$