1. For the exponential generating function $f(x)$ of a_n, we have

$$f(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} x^i, f'(x) = \sum_{i=0}^{\infty} \frac{a_{i+1}}{i!} x^i$$

Since $a_{n+1} = a_n + na_{n-1}$ we have

$$f'(x) = \sum_{i=1}^{\infty} \frac{a_{i+1}}{i!} x^i + a_1 = \sum_{i=1}^{\infty} \frac{a_i}{i!} x^i + \sum_{i=1}^{\infty} \frac{ia_{i-1}}{i!} x^i + a_1$$

$$= \sum_{i=0}^{\infty} \frac{a_i}{i!} x^i - a_0 + \sum_{i=1}^{\infty} \frac{a_{i-1}}{(i-1)!} x^i + a_1$$

$$= f(x) - a_0 + xf(x) + a_1 = f(x) + xf(x)$$

solving the differential equation we get

$$\frac{f'(x)}{f(x)} = 1 + x$$

$$\frac{d}{dx} \log(f(x)) = 1 + x$$

$$f(x) = ce^{x+x^2}$$

Since $f(0) = 1$, we have $c = 1$

We can use the product of generating functions to get

$$f(x) = e^x e^{2x} = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{x^{2j}}{j!} \right) = \sum_{n=0}^{\infty} \sum_{\{k+2j=n\}} \frac{x^n}{j!k!}$$

And so we have

$$a_n = n! \sum_{\{k+2j=n\}} \frac{1}{j!k!}$$

2. We have the generating functions

$$(x+1)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

And

$$(x+1)^m = \sum_{k=0}^{m} \binom{m}{k} x^k, (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Since $(x+1)^m(x+1)^n = (x+1)^{m+n}$, we have

$$\left(\sum_{k=0}^{m}\binom{m}{k}x^k\right)\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) = \sum_{k=0}^{m+n}\binom{m+n}{k}x^k$$

When taking the product of two ordernary generating functions with sequences a_n, b_n we know the product has generating sequence $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Applying this to the product of generating functions on the left we have:

$$\sum_{k=0}^{m+n}\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}x^k = \sum_{k=0}^{m+n}\binom{m+n}{k}x^k$$

Since these are equal polinomials, the coefficents must be equal, so

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$$

3.

a. We have

$$f(x) = \sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$$

Differentiating k times we have all the terms where $n < k$ dissapear and the rest have the form:

$$f^{(k)}(x) = \sum_{n=k}^{\infty}\frac{a_n n(n-1)(n-2)\dots(n-k+1)}{n!}x^{(n-k)}$$

Therefore when we plug in $x = 0$, all the terms are zero except the term where x has a zero power which is the term when $n = k$:

$$f^{(k)}(0) = \frac{a_k k(k-1)(k-2)\dots 2\cdot 1}{k!} = \frac{a_k k!}{k!} = a_k$$

b. Consider the generating function:

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If we take a product of k of these functions we get

$$f(x)f(x)\dots f(x) = (f(x))^k = \sum_{n=0}^{\infty} a_n x^n$$

Where the a_n counts all the ways to choose $x_1, x_2, \ldots x_k$ exponents from the terms in each $f(x)$ in order for them to add up to n. Therefore a_n is the sequence described in the problem.
We can now calculate a_n by taking the nth derivative and plugging in 0, we have

$$\frac{d}{dx}(f(x))^k = \frac{d}{dx}(1-x)^{-k} = k(1-x)^{-k-1}$$

And so we have

$$\frac{d^n}{dx^n}(x-1)^{-k} = k(k+1)(k+2)\ldots(k+n-1)(1-x)^{-k-n}$$

Plugging in zero we get

$$a_n n! = k(k+1)\ldots(k+n-1)$$

And so

$$a_n = \frac{k(k+1)\ldots(k+n-1)}{n!} = \binom{n+k-1}{k-1}$$

4. There is such a subset defined as follows: $S = \{n : \text{the base 4 representation of } n \text{ only contains 0's and 1's}\}$.
We know that every positive integer has a unique base 4 representation. For a given $n > 0$ we can look at any digit place d. We will signify $d(n)$ to be the dth digit of n in base 4
We know that for any $x, y \in S$ none of the digits will carry in $x+2y$ since $d(x) \leq 1, d(2y) \leq 2$ so $d(x+2y) = d(x) + 2d(y)$
If $d(n) = 0$, then there is only one possible way to have $d(n) = d(x)+d(2y)$ which is $d(x) = 0$ and $d(y) = 0$, if $d(n) = 1$, then the only possibility is $d(x) = 1, d(y) = 0$, if $d(n) = 2$ then $d(x) = 0, d(y) = 1$, and finally if $d(n) = 3$ then $d(x) = 1$ and $d(y) = 1$.
Therefore for every digit of n, the digits of x and y are uniquely determined, and satisfy $d(n) = d(x+2y)$, and so for any n, x, y are uniquely determined and satisfy $n = x + 2y$ So S has the desired property