Exersise 28

a. This can be proven with induction on the number of points:
Base case, for 2 points w_1, w_2 we have that any point on the the convex combination is of the form $w_1t + w_2(t-1)$ where $t \in [0,1]$. Therefore by definition of convex, we know that E contains all of these points.
Inductive step:
For $w_1, \ldots w_{k+1}$ we have that any point w in the convex combination is of the form

$$w = s_1w_1 + \ldots s_{k+1}w_{k+1}$$

Let $t = \frac{-1}{s_{k+1}-1}$. We have that

$$m = (1-t)w_{k+1} + tw = \frac{1}{1-s_{k+1}}(s_1w_1 + \ldots s_kw_k + (s_{k+1} - s_{k+1})w_{k+1})$$

Since w is in the convex combination we have that $s_1 + \ldots s_k = 1 - s_{k+1}$. Thus we have that the coeffecients of m add up to 1 and the w_{k+1} term is zero, so m is in the convex combination of the k points $w_1 \ldots w_k$, thus by inductive hypothesis is in E. We have that

$$w = \frac{1}{t}m - \frac{1-t}{t}w_{k+1}$$

where $\frac{1}{t} = 1 - s_{k+1} \leq 1$ and $\frac{1-t}{t} = 1 - \frac{1}{t}$. Thus w lies on the line from m to w_{k+1} and thus must be in E since E is convex

b. The converse is fairly obvious because if a set contains the convex combination of any two points then we have that any point on the the convex combination is of the form $w_1t + w_2(t-1)$ where $t \in [0,1]$, which by definition means that E is convex

Exersise 29

a. E is the unit ball for the dot product defined by $\langle (x,y,z),(x',y',z') \rangle = \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}$. It is clear that the axioms of a inner product are satisfied:

$$(x,y,z)\cdot(x',y',z') = \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = (x',y',z')\cdot(x,y,z)$$

$$(x,y,z)\cdot(sx',sy',sz') = \frac{sxx'}{a^2} + \frac{syy'}{b^2} + \frac{szz'}{c^2} = s(x,y,z)\cdot(x',y',z')$$

$$(x,y,z)\cdot(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} > 0 \text{ and } = 0 \Leftrightarrow x = y = z = 0$$

Thus using the norm defined by $\|x\| = \sqrt{\langle x, x\rangle}$, we have $E = \{x \in \mathbb{R}^3 : \|x\| \le 1$. Thus we have that for any $a, b \in E$ with the point $c = at + b(1-t)$ for some $t \in [0,1]$ we can use the triangle inequality

$$\|c\| = \|at + b(1-t)\| \le \|at\| + \|b(1-t)\| = t\,\|a\| + (1-t)\,\|b\|$$

and thus since $\|a\| \le 1, \|b\| \le 1$

$$\|c\| \le t + 1 - t = 1$$

so $c \in E$, thus E is convex

b. Letting $E = [a_1, b_1] \times [a_2, b_2] \times \dots [a_m, b_m]$ we have that for any $x = (x_1, x_2 \dots x_m), y = (y_1, y_2 \dots y_m) \in E$ with the point $z = xt + z(1-t)$ for some $t \in [0,1]$ we have that with $z = (z_1, z_2 \dots z_m)$, for each i, $a_i \le x_i, y_i \le b_i$, so

$$ta_i + (1-t)a_i \le tx_i + (1-t)y_i \le tb_i(1-t)b_i$$

so since $z - i = tx_i + (1-t)y_i$, we have

$$a_i \le z_i \le b_i$$

For all i. Thus $z \in E$, so E is convex

Exersise 30

a. for any $v = (v_1, v_2), w = (w_1, w_2) \in S$ with the point $u = tv + (1-t)w = (u_1, u_2)$, we have that

$$f(u_1) = f\left(tv_1 + (1-t)w_1\right) \le tf(v_1) + (1-t)f(w_1) \le tv_2 + (1-t)w_2 = u_2$$

Thus S is convex if f is convex. Conversly, if there exists $x, y \in (a,b)$ with $s, t \in [0,1]$ and $s + t = 1$ where

$$f(sx + ty) > sf(x) + tf(y)$$

then we have that for $(x, f(x)), (y, f(y)) \in E$, we have the point on the line between them $(sx + ty, sf(x) + tf(y))$ with

$$f(sx + ty) > sf(x) + tf(y)$$

Therefore we have $(sx + ty, sf(x) + tf(y)) \notin E$ so E would not be convex

c. Suppose we have $a < x < x' < u < u' < b$. We can write

$$\sigma(x, u) = \frac{f(u) - f(x)}{u - x}$$

We can write

$$u = \frac{u-x}{u'-x}u' + \left(1 - \frac{u-x}{u'-x}\right)x$$

Letting $t = \frac{u-x}{u'-x}$ we have $0 \leq t \leq 1$. And so

$$\frac{f(u)-f(x)}{u-x} = \frac{f(tu'+(1-t)x)-f(x)}{tu'+(1-t)x-x}$$

using convexity of f we have

$$\leq \frac{tf(u')+(1-t)f(x)-f(x)}{tu'+(1-t)x-x} = \frac{tf(u')-tf(x)}{tu'-tx}$$

$$= \frac{f(u')-f(x)}{u'-x}$$

Similarly letting $t = \frac{x'-x}{u-x}$ we have $0 \leq t \leq 1$. We have $x' = ut + (1-t)x$ So we have

$$\frac{f(u)-f(x')}{u-x'} = \frac{f(u)-f(tu+(1-t)x)}{u-(tu+(1-t)x)}$$

using convexity of f we have

$$\geq \frac{f(u)-(tf(u)+(1-t)f(x))}{u-(tu+(1-t)x)} = \frac{tf(u)-tf(x)}{tu-tx}$$

$$= \frac{f(u)-f(x)}{u-x}$$

d. If f is convex we can use the previous result:

$$f''(x) = \lim_{h\to 0}\lim_{k\to 0}\frac{\sigma(x+h-k,x+h+k)-\sigma(x-h-k,x-h+k)}{h}$$

from our result in part c we have that $\sigma(x+h-k,x+h+k) \geq \sigma(x-h-k,x-h+k)$.
Thus $f''(x) \geq 0$.
Conversly if $f''(x) \geq 0 \forall x \in (a,b)$,

Exersise 39

(a) We have that every algebraic number is a root of some polinomial over $\mathbb{Z}$ therefore there is a surjective mapping from the set of polinomials to the set of algebraic numbers. Thus showing there are a countable number of polinomials will show the algebraic numbers are countable.
We can define a surjective mapping ϕ from the denumerable union of finite cartesian products of denumerable sets:

$$S = \bigcup_{i\in\mathbb{Z}} \mathbb{Z}^i$$

Which by corollary 18 and thm 17 is denumerable, to the set of polinomials.
We define for any $(a_0, a_1, a_2, \dots a_n) \in \mathbb{Z}^n$, $\phi(a_0, a_1, a_2, \dots a_n) = a_n x^n + a_{n-1}x^{n-1} \dots + a_0$. ϕ is surjective since for any polinomial $p = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$ we have $\phi(a_0, a_1, a_2, \dots a_n) = p$. Thus the set of polinomials is countable, and thus the set of algebraic numbers is countable

Exersise 47

(a) We can use the bilinearity of the dot product:

$$|x+y|^2+|x-y|^2 = (x+y)\cdot(x+y)+(x-y)\cdot(x-y) = x\cdot(x+y)+y\cdot(x+y)+x\cdot(x-y)-y\cdot(x-y)$$

$$= x \cdot x + 2x \cdot y + y \cdot y + x \cdot x - 2x \cdot y + y \cdot x = |x|^2 + |y|^2$$

Exersise 1

a. This follows directly from the triangle inequality, which follows from the Cauchy-Shwartz inequality which applies to all inner products:

$$|(1-t)x + ty| \leq |(1-t)x| + |ty| = (1-t)|x| + t|y| = 1 - t + t = 1$$

b. For the first norm, let $x = (1, 0, 0, 0 \dots 0), y = (0, 1, 0, 0 \dots 0)$, $|x| = |y| = 1$. We have $|1/2x + 1/2y| = |(1/2, 1/2, 0, 0 \dots 0)| = 1 \not< 1$. For the second norm we have $x = (1, 0, 0, 0 \dots 0), y = (1, 1, 0, 0 \dots 0)$, we have $|x| = |y| = 1$ and $|1/2x + 1/2y| = |(1, 1/2, 0, 0, \dots 0)| = 1 \not< 1$ Thus the norms cannot be of the form $\sqrt{\langle \cdot, \cdot \rangle}$ since they are not strictly convex