PETER IS GREAT **10.11**

a. Given any two left cosets aH, bH, we can construct a bijection $f : aH \to bH$ as $f(ah) = bh$. Which would imply the sets have the same size
f is one-to-one since if $f(ah_1) = f(ah_2)$ then $bh_1 = bh_2$, applying b^{-1} on the left yields $h_1 = h_2$ so $ah_1 = ah_2$.
f is onto since for any bh, we can choose the same h for ah and we have $f(ah) = bh$
Therefore f is a bijection

b. Given any right coset and left coset Ha, bH, we can construct a bijection $f : Ha \to bH$ where $f(ha) = bh$. Which would imply the sets have the same size.
f is one-to-one since if $f(ha_1) = f(ah_2)$, then $bh_1 = bh_2$, applying b^{-1} on the left yields $h_1 = h_2$.
f is onto since for any bh, we can choose the same h for ha, and we would have $f(ha) = bh$.
Therefore f is a bijection

11.1 We know that the determinant is multiplicative, We also know for any $A \in GL(2,\mathbb{R})$, where $|A|$ signifies the determinant, we have $|A| = \frac{1}{|A^{-1}|}$.
Therefore for any $B \in SL(2,\mathbb{R})$:

$$|ABA^{-1}| = |A|1\frac{1}{|A|} = 1$$

so

$$ABA^{-1} \in S(2,\mathbb{R}$$

Therefore

$$SL(2,\mathbb{R}) \triangleleft GL(2,\mathbb{R})$$

11.2 H is not a normal group of $GL(2,\mathbb{R})$, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1} \in GL(2,\mathbb{R}) \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in H$$

We have

$$ABA^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \notin H$$

11.3 We know that $H = \{e, a\}$ where e is the identity and $a = a^{-1}$.
It is known that e commutes with every element in G so $e \in Z(G)$. Since H is normal we have for any $g \in G$

$$gag^{-1} \in \{e, a\}$$

If $gag^{-1} = e$ then applying g on the right yields $ga = g$ so $a = e$ which is not true. Therefore $gag^{-1} = a$, and so applying g on the right yields $ga = ag$ and so $a \in Z(G)$, so $H \subseteq Z(G)$

11.11 We can go through each subgroup of D_4 and check which is normal
List of all subgroups:

$$\{I\}, \{I, R^2\}, \{I, F\}, \{I, FR\}, \{I, FR^2\}, \{I, FR^3\}$$
$$\{I, R, R^2, R^3\}, \{I, R^2, F, FR^2\}, \{I, R^2, FR, FR^3\}, D_4$$

We know that every subgroup H of size 4 must be normal since there are only two possible cosets (since D_4 is size 8), H, xH where $x \notin H$, Therefore $xH = \{a \in G : a \notin H\} = Hx$.
As for the groups of size 2, none of them are normal: $F\{I, R^2\} \neq \{I, R^2\}F$, $R\{I, F\} \neq \{I, F\}R$, $R\{I, FR^2\} \neq \{I, FR^2\}R$, $R\{I, FR^3\} \neq \{I, FR^3\}R$.
Since D_4 and $\{I\}$ are included, total there are 6 normal subgroups

11.16 Given any element $a \in \mathbb{Q}$ and $z \in \mathbb{Z}$ we have since addition is commutative for $aza^{-1} \in a\mathbb{Z}a^{-1}$

$$aza^{-1} = aa^{-1}z = z \in \mathbb{Z}$$

so we know $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}$. Therefore $\mathbb{Q}/\mathbb{Z}$ is a group.
Next we can show every element

$$\frac{a}{b}\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

has finite order. We can show this by seeing that

$$(\frac{a}{b}\mathbb{Z})^b = (\frac{a}{b} + \frac{a}{b} + \dots \frac{a}{b})\mathbb{Z} = a\mathbb{Z} = e\mathbb{Z}$$

Since $a \in \mathbb{Z}$.
Finally we can show $\mathbb{Q}/\mathbb{Z}$ is infinite, if it were not we could list the group:

$$\{a_1\mathbb{Z}, a_2\mathbb{Z}, a_3\mathbb{Z}, \dots a_n\mathbb{Z}\}$$

Where $a_1, \dots, a_n$ are all $\in (0, 1)$. We can always choose our a_i in such a way since there is always an element $\in (0, 1)$ in each coset because we can add or subtract the integer part of any element in the coset to be only left with the fractional portion of that number
Since there are infinite rational numbers in $(0, 1)$, we can choose $a_{n+1} \in (0, 1)$ such that $a_{n+1} \notin \{a_1, \dots a_n\}$. We know there is no integer z such that $a_{n+1} + z = a_i$ for any $i : 0 < i \leq n$ since $|a_{n+1} - a_i| < 1$ so $a_{n+1} \notin a_i\mathbb{Z}$ therefore

$$a_{n+1}\mathbb{Z} \notin \{a_1\mathbb{Z}, a_2\mathbb{Z}, a_3\mathbb{Z}, \dots a_n\mathbb{Z}\}$$

which is a contradiction

11.17 Since G is abelian, for any $a, b \in G$ we have

$$aHbH = \{ah_1bh_2 : h_1, h_2 \in H\} = \{bh_2ah_1 : h_1, h_2 \in H\} = bHaH$$

so G/H is abelian

11.18 We know every element of G has the form x^n where x is the generator of G, therefore every element of G/H also has the form Hx^n
We have since H is normal

$$Hx^kHx^j = Hx^{k+j}$$

Therefore Hx is the generator of G/H since every term in G/H is of the form $(Hx)^n = Hx^n$ which means G/H is cyclic.

11.26 We will show there is a bijection between the elements of gHg^{-1} and H which would imply $|gHg^{-1}| = |H|$.
We will define this bijection as $f : H \to gHg^{-1}$, $f(h) = ghg^{-1}$. We have that if

$$f(h_1) = f(h_2)$$

then

$$gh_1g^{-1} = gh_2g^{-1}$$
$$g^{-1}gh_1g^{-1}g = g^{-1}gh_2g^{-1}g$$
$$h_1 = h_2$$

so f is one-to-one. We also know f is onto since $\forall ghg^{-1} \in gHg^{-1}$, $h \in H$ so $f(h) = ghg^{-1}$. Therefore f is a bijection so $|gHg^{-1}| = |H|$.