Exersise 4.1
We have the root $\alpha = \sqrt[4]{-2} = \zeta_8 \sqrt[4]{2}$ and every other root is of the form $\zeta_4^n \alpha$ where $\zeta_4 = i, \zeta_8 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ are roots of unity.

Exersise 4.2

Exersise 4.3

Exersise 4.4
We have that $K = \mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m)$ is an extentsion of $\mathbb{Q}$. From the correspondences of Galois theory, we know that $\mathrm{Aut}(K/\mathbb{Q})$ must be the a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ as well as $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. However we have

$$\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong C_m, \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong C_n$$

Every subgroup of C_m is of the form C_d where $d|m$ thus the only possible subgroup of C_m, C_n is C_1

Exersise 4.5
We can use strong induction, first establishing a base case:
For $n = 1$ we have

$$\Phi_1(-x) = -x - 1 = -\Phi_2(x)$$

for $n = 3$:

$$\Phi_3(-x) = x^2 - x + 1 = \Phi_6(x)$$

For the inductive step we use the well established identity:

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

We can reorder the product for $2n$ since each divisor of n must be odd:

$$x^{2n} - 1 = \prod_{d|2n} \Phi_d = \prod_{d|n} \Phi_d(x)\Phi_{2d}(x)$$

We also have the factorization $x^{2n} - 1 = (x^n - 1)(x^n + 1)$. Since n is odd, $x^n + 1 = -((-x)^n - 1)$:

$$= -(x^n - 1)((-x)^n - 1) = -\prod_{d|n} \Phi_d(x) \prod_{d|n} \Phi_d(-x)$$

So we have

$$\prod_{d|n} \Phi_d(x)\Phi_{2d}(x) = -\prod_{d|n} \Phi_d(x) \prod_{d|n} \Phi_d(-x)$$

From our inductive hypothesis, for each $d < n, d \neq 1$ we have $\Phi_d(-x) = \Phi_{2d}(x)$, thus we can divide on both sides

$$\Phi_{2n}(x)\Phi_1(x) \prod_{d|n} \Phi_d(x) \prod_{d|n, 1<d<n} \Phi_d(-x) = -\prod_{d|n} \Phi_d(x) \prod_{d|n} \Phi_d(-x)$$

$$\Phi_{2n}(x)\Phi_1(x) = -\Phi_2(-x)\Phi_n(-x)$$

Since $\Phi_1(x) = -\Phi_2(-x)$ we get our equality

$$\Phi_{2n}(x) = \Phi_n(-x)$$

Exersise 4.6