Exersise 3.7

Given $x_0 \in \mathbb{R}^n$ and any $\epsilon > 0$, let $\delta = \epsilon$. For any $x \in B(x_0, \delta)$ we have $\|x - x_0\| < \delta = \epsilon$. We also have $|f(x) - f(x_0)| = |\|x\| - \|x_0\||$.
We know that we have $|\|x\| - \|x_0\|| = \|x\| - \|x_0\|$ or $-(\|x\| - \|x_0\|) = \|x_0\| - \|x\|$. By the triangle inequality we know both $\|x\| - \|x_0\| \leq \|x - x_0\|$ and $\|x_0\| - \|x\| \leq \|x_0 - x\| = \|x - x_0\|$.
And so $|f(x) - f(x_0)| = |\|x\| - \|x_0\|| \leq \|x - x_0\| < \epsilon$
Thus f is continuous

Exersise 3.9

a. If there exists some N such that $x_j = x_k$ for all $j, k > N$ then $\delta(x_j, x_k) = 0 < \epsilon$ for all $\epsilon > 0$, and so by definition x_n converges. Conversly if x_n converges, let $\epsilon = 1/2$. We have that for some N, $\delta(x_j, x_k) < 1/2$ for all $j, k > N$. Since $\delta(x_j, x_k) > \epsilon$ if and only if $x_j \neq x_k$, we have that $x_j = x_k$ for all $j, k > N$

b. For any $x_0 \in X$ and any $\epsilon > 0$, let $\delta = 1/2$. We have that $\delta(x, x_0) < \delta$ if and only if $x = x_0$, by definition of the discrete metric. Therefore $B_(x_0, \delta) = \{x_0\}$ and as one of the properties of the metric, we have $d(f(x_0), f(x_0)) = 0 < \epsilon$. Therefore by definition f is continuous

Exersise 3.11

For a given $\epsilon > 0$, f continuous means that there exists a $\delta > 0$ such that $d(x, x_i) < \delta$ implies $\rho(f(x), f(x_i)) < \epsilon$. $x_n \to x$ means there is a $N > 0$ such that for $k > N$, $d(x_k, x) < \delta$. Therefore for $k > N$ we have $\rho(f(x), f(x_k)) < \epsilon$. Thus $f(x_n) \to f(x)$

Exersise 3.14

For any $\theta_0 \in [0, 2\pi)$, given $\epsilon > 0$ let $\delta = \min(\epsilon, 1)$. For any $\theta \in [0, 2\pi)$ with $|\theta - \theta_0| < \delta$ we have

$$\|f(\theta) - f(\theta_0)\| = \sqrt{(\cos(\theta) - \cos(\theta_0))^2 + (\sin(\theta) - \sin(\theta_0))^2}$$

$$= \sqrt{\cos^2(\theta) - 2\cos(\theta_0)\cos(\theta) + \cos^2(\theta_0) + \sin^2(\theta) - 2\sin(\theta_0)\sin(\theta) + \sin^2(\theta_0)}$$

$$= \sqrt{2(1 - (\cos(\theta_0)\cos(\theta) + \sin(\theta_0)\sin(\theta)))}$$

Using the sum formula ($\cos(a - b) = \cos a \cos b + \sin a \sin b$) we have:

$$= \sqrt{2(1 - \cos(\theta - \theta_0))}$$

We can use the Taylor series of $\cos\theta$ to get a bound

$$= \sqrt{2\left(1 - \left(1 - \frac{(\theta - \theta_0)^2}{2} + \frac{(\theta - \theta_0)^4}{4!} - \dots\right)\right)}$$

Since we know $|\theta-\theta_0| < 1$ (since $\delta \leq 1$) we know that $(\theta-\theta_0)^2 \geq \left(\frac{(\theta-\theta_0)^4}{12} - \ldots\right)$ since a convergent alternating series terms are always greater than the sum of the terms following it. Therefore we have

$$= \sqrt{\left((\theta-\theta_0)^2 - \frac{(\theta-\theta_0)^4}{12} + \ldots\right)} \leq \sqrt{(\theta-\theta_0)^2} = |\theta-\theta_0| < \delta \leq \epsilon$$

And so F is continous.
F is surjective since for any $p = (x, y) \in S^1$, we let θ be the angle that p makes with the origin and the point $(1, 0)$ and from the geometric definition of sin and cos we have that $\sin\theta = y$, $\cos\theta = x$ and so $F(\theta) = p$. F is injective since if $F(\theta) = F(\alpha)$, then we have $\sin\theta = \sin\alpha$, since $\theta, \alpha \in [0, 2\pi)$ we know that $\theta = \pi - \alpha$ or $\theta = \alpha$, then we have that $\cos\theta = \cos\alpha$ which means that $\theta = 2\pi - \alpha$ or $\theta = \alpha$. We cannot have $\pi - \alpha = \theta = 2\pi - \alpha$ since then $0 = \pi$ therefore $\theta = \alpha$.
Therefore F is bijective and so has an inverse mapping F^{-1}.
However we have that $F^{-1}(1, 0) = 0$, but for $\epsilon = \pi$ and any $\delta > 0$, letting $a < \min\{1/2, \delta\}$ we have that for

$$p = \left(-\sqrt{1 - \frac{a^2}{4}}, \frac{a}{2}\right) \in S^1$$

with $\|p - 0\| < \delta$ and $F^{-1}(p) = \theta$ where

$$\theta = \sin^{-1}\left(-\sqrt{1 - \frac{a^2}{4}}\right), \theta = \cos^{-1}\left(\frac{a}{2}\right)$$

We have that $\theta > \pi$ since $\left(-\sqrt{1 - \frac{a^2}{4}}\right) < 0$ and $\sin^{-1}\alpha > \pi$ for any $\alpha \in (0, -1)$.
Therefore $|F^{-1}(p) - F^{-1}(0)| > \epsilon$. So F^{-1} is not continous

Exersise 3.17

a. By definition of open for metric spaces, we have that for any $a \in \emptyset$, for any $\epsilon > 0$, $B(a, \epsilon)$ is itself the empty set since a does not exist so $B(a, \epsilon) \subseteq \emptyset$. Thus the empty set is open

b. For any $a \in X$ and $\epsilon > 0$ we have that $B(a, \epsilon) = \{x \in X : \delta(x, a) < \epsilon\}$ thus $B(a, \epsilon) \subseteq X$ and so X is open

c. For any $a \in B(x, \epsilon)$, let $\epsilon' = \epsilon - \delta(x, a)$. Thus we have for any $y \in B(a, \epsilon')$ we have $\delta(y, a) < \epsilon' = \epsilon - \delta(x, a)$. Thus from the triangle inequality we have:

$$\delta(y, x) \leq \delta(x, a) + \delta(a, y) < \epsilon$$

Thus $y \in B(x, \epsilon)$, so $B(a, \epsilon') \subseteq B(x, \epsilon)$. Thus $B(x, \epsilon)$ is open

d. For any $x \in U_1 \cap \cdots \cap U_k$, since each U_i is open there exists for each U_i $\epsilon_i > 0$ where $B(x, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots \epsilon_k\}$. We have that $B(x, \epsilon) \subseteq B(x, \epsilon_i)$ for all i. This is because we have for any $a \in B(x, \epsilon)$ we have that $\delta(a, x) < \epsilon \leq \epsilon_i$ and thus $a \in B(x, \epsilon_i)$. Therefore $B(x, \epsilon) \subseteq U_i$ for all i, so $B(x, \epsilon) \subseteq U_1 \cap U_2 \cap \ldots U_k$. Thus $U_1 \cap \ldots U_k$ is open

Exersise §13, 3

In example 4 we have $X - X = \emptyset$ which is countable so $X \in \mathfrak{T}_c$, and $X - \emptyset = X$ so $\emptyset \in \mathfrak{T}_c$. For any collection of sets $A \subseteq \mathfrak{T}_c$ we have from Demorgans laws:

$$X - \left(\bigcup_{U \in A} U \right) = \bigcap_{U \in A} (X - U)$$

Intersections of countable sets are countable, therefore $\left(\bigcup_{U \in A} U \right) \in \mathfrak{T}_c$.
For a finite collection $A \subset \mathfrak{T}_c$ we have from Demorgans laws:

$$X - \left(\bigcap_{U \in A} U \right) = \bigcup_{U \in A} (X - U)$$

Finite unions of countable sets are countable. Therefore $\bigcup_{U \in A} (X - U) \in \mathfrak{T}_c$. Thus all the axioms of a topology are satisfied, so $\mathfrak{T}_c$ is a topology.
However we have $\mathfrak{T}_\infty$ is not necessarily a topology:
Let $X = \mathbb{Z}$. Let $U = \{x \in \mathbb{Z} : x < 0\}$ and $V = \{x \in \mathbb{Z} : x > 0\}$. We have that $X - U = \{x \in \mathbb{Z} : x \geq 0\}$ is an infinite set and $X - V = \{x \in Z : x \leq 0\}$ is an infinite set, so $U, V \in \mathfrak{T}_\infty$. However we have

$$X - (V \cup U) = \{0\}$$

Is not infinite. Thus $U \cup V \notin \mathfrak{T}_\infty$. So $\mathfrak{T}_\infty$ does not satisfy the axioms of a topology.

Exersise §13, 4

a. We have that $X, \emptyset \in \mathfrak{T}_\alpha$ for all α, so $X, \emptyset \in \bigcap \mathfrak{T}_\alpha$
We have that for any collection of sets $A \subseteq \bigcap \mathfrak{T}_\alpha$, we have that for each $\mathfrak{T}_\alpha$, $A \subseteq \mathfrak{T}_\alpha$, and thus since $\mathfrak{T}_\alpha$ is a topolgy

$$\bigcup_{U \in A} U \in \mathfrak{T}_\alpha$$

so $\bigcup_{U \in A} U \in \bigcap \mathfrak{T}_\alpha$.
For a finite collection $A \subseteq \bigcap \mathfrak{T}_\alpha$, we again have that for each $\mathfrak{T}_\alpha$, $A \subseteq \mathfrak{T}_\alpha$, and thus since $\mathfrak{T}_\alpha$ is a topolgy, we have that

$$\bigcap_{U \in A} U \in \mathfrak{T}_\alpha$$

So $\bigcap_{U \in A} U \in \bigcap \mathfrak{T}_\alpha$.
$\bigcup \mathfrak{T}_\alpha$ is not necessarily a topology:
Consider $X = \{a, b, c\}$. We have the topolgies

$$\mathfrak{T}_1 = \{\emptyset, X, \{a\}, \{a, c\}\}, \mathfrak{T}_2 = \{\emptyset, X, \{b\}, \{b, c\}\}$$

$$\mathfrak{T}_1 \cup \mathfrak{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$$

However $\{a, c\} \cap \{b, c\} = \{c\} \notin \mathfrak{T}_1 \cup \mathfrak{T}_2$. So $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is not a topology

b. The largest topology contained in all $\mathfrak{T}_\alpha$ is

$$\bigcap \mathfrak{T}_\alpha$$

This follows directly from the conditions. $\bigcap \mathfrak{T}_\alpha$ is contained in all $\mathfrak{T}_\alpha$ and any topology that is contained in all $\mathfrak{T}_\alpha$ must be contained in their intersection.
The smallest topology that contains each $\mathfrak{T}_\alpha$ is the Topology $\mathfrak{T}$ that consists of all the sets of $\bigcup \mathfrak{T}_\alpha$ as well as the sets obtained by taking any sequence of unions and finite intersections of sets in $\bigcup \mathfrak{T}_\alpha$. From this construction it is clear that $\mathfrak{T}$ contains all $\mathfrak{T}_\alpha$. It is also clear $\mathfrak{T}$ is a topology as well since taking any union or finite intersection of sets of $\mathfrak{T}$ will again produce a set obtained by taking a sequence of unions and finite intersections of sets in $\bigcup \mathfrak{T}_\alpha$
The reasoning that $\mathfrak{T}$ is the smallest of the desired Topologies, and is unique is as follows:
If there exists a smaller topology $G \subset \mathfrak{T}$ where $\mathfrak{T}_\alpha \subset G$ for each $\mathfrak{T}_\alpha$ then there must be some $A \in \mathfrak{T}$ with $A \notin G$. We know A has the form

$$A_1 \cup A_2 \cdots \cap A_i \cap \cdots \cup A_k \ldots$$

Where each $A_i \in \mathfrak{T}_\alpha$ for some $\mathfrak{T}_a$. However, this means that G is not a topology since we would have that A is union of sets in G that is not in G. This also implies that if there is a topology F that contains each $\mathfrak{T}_\alpha$ then $F \cap \mathfrak{T} = \mathfrak{T}$ since for any $A \in \mathfrak{T}$ with $A \notin F$ we have the same problem. Thus $\mathfrak{T}$ must be unique, since for any $F \neq \mathfrak{T}$ we have $\mathfrak{T} \subset F$ so $\mathfrak{T}$ is smaller than F.

c. The smallest topology containing $\mathfrak{T}_1, \mathfrak{T}_2$ would be

$$\mathfrak{T}_1 \cup \mathfrak{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\{b, c\}\}$$

The largest topology containing $\mathfrak{T}_1, \mathfrak{T}_2$ would be

$$\mathfrak{T}_1 \cap \mathfrak{T}_2 = \{\emptyset, X, \{a\}\}$$