Exersise §17, 11

If we have the Hausdorff spaces X, Y, for each pair $(x, y), (x', y') \in X \times Y$, since X, Y are Hausdorff there are neighborhoods U_1, U_2 of x and x' respectivly and V_1, V_2 of y, y' respectivly that are disjoint. Thus we have $(x, y) \in U_1 \times V_1$ and $(x', y') \in U_2 \times V_2$ (by definition of the product topology these sets are neighborhoods) where $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint. Thus $X \times Y$ is Hausdorff.

Exersise §17, 12

If X is Hausdorff and we have a subspace $A \subseteq X$, then for any pair of points $x, y \in A$, since X is Hausdorff we have neighborhoods U, V of x, y respectivly in X. Thus we have that $U \cap A$, $V \cap A$ are neighborhoods of x, y in A. Since U is disjoint from V, we know that $U \cap A$ is disjoint from $V \cap A$, and thus A is Hausdorff

Exersise §17, 13

If X is Hausdorff then for any $(x, y) \in X \times X$ where $x \neq y$ we have that there exists open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $U \cap V = \emptyset$ we know that $(U \times V) \cap \Delta = \emptyset$ since U and V do not have any points that are the same. By definition we know $U \times V$ is open in $X \times X$ as well. Thus we can take the following arbitrary union where U_x and V_y denote the open sets described above for each (x, y)

$$W = \bigcup_{\{(x,y) \in X \times X : x \neq y\}} U_x \times U_y$$

W is an open set since it is the union of open sets. We also have that $W = (X \times X) - \Delta$. It is clear $W \subseteq (X \times X) - \Delta$ since W is a union of open sets that do not intersect with Δ. We also have that $(X \times X) - \Delta \subseteq W$ since for each $(x, y) \in X \times X$ there is a $U_x \times V_y$ with $(x, y) \in U_x \times V_y \subseteq W$. Thus Δ is the complement of the open set W, so Δ is closed

Conversly we know that the basis that generates the topology of $X \times X$ is all sets of the form $U \times V$ where U and V are open sets of X. Therefore letting $W = (X \times X) - \Delta$, for any $(x, y) \in W$, since W is open we know there exists a basis element $U \times V$ contained in W which contains (x, y). Thus since $W \cap \Delta = \emptyset$ we have that $U \times V \cap \Delta = \emptyset$ and $U \cap V = \emptyset$. Thus X is Hausdorff since for any x, y we have found open sets U, Vsuch that $x \in U, y \in V$ and $U \cap V = \emptyset$

Exersise §18, 4

The case for g is equivalent to the case for f if we just relabel the sets X and Y, thus this only needs to be proven for f.

For any open set $M \subseteq X \times Y$, since the topology of $X \times Y$ is generated by the product of open sets of X and Y we know that $M = \bigcup U_i \times V_i$ where U_i, V_i are open sets of X, Y

respectivly. Thus we have that

$$f^{-1}(M) = f^{-1}\left(\bigcup U_i \times V_i\right) = \bigcup f^{-1}(U_i \times V_i)$$

We have that $f^{-1}(U_i \times V_i) = U_i$ if $y_0 \in V_i$ and $f^{-1}(U_i \times V_i) = \emptyset$ if $y_0 \notin V_i$. Thus we have

$$f^{-1}(M) = \bigcup U_i \text{ or } = \emptyset$$

Thus $f^{-1}(M)$ is open and so f is continous.
It is clear that f is injective since $f(a) = f(b) \Rightarrow (a, y_0) = (b, y_0) \Rightarrow a = b$
Finally we must check $f^{-1} : f(X) \to X$ is continous. We have $(f^{-1})^{-1} = f$. For open set $U \subseteq X$ we have that $f(U) = U \times y_0$. We have that $f(X) = X \times y_0$ and so

$$U \times y_0 = (U \times Y) \cap (X \times y_0) = (U \times Y) \cap f(X)$$

$U \times Y$ is an open set, and thus $(U \times Y) \cap f(X)$ is an open set in $f(X)$, so f^{-1} is continous on $f(X)$. Thus f is a homeomorphism

Exersise §18, 7

(a) It suffices to show that the inverse image of any basis element is open then f is continous. For $\mathbb{R}$ the basis elements can be the open intervals (a, b). For any open interval (a, b) we have that

$$f^{-1}(a, b) = \{x \in \mathbb{R} : a < f(x) < b\}$$

If $f^{-1}(a, b) = \emptyset$ then $f^{-1}(a, b)$ is open, otherwise, for each $p \in f^{-1}(a, b)$, since $\lim_{x \to p^+} f(x) = f(p)$, letting $\epsilon = \min\{|f(p) - a|/2, |f(p) - b|/2\} > 0$ we know there exists a $\delta > 0$ such that for all $x \in [p, p + \delta)$, we have that $|f(x) - f(p)| < \epsilon$ and thus

$$-\epsilon + f(p) < f(x) < f(p) + \epsilon$$

From how we have chosen ϵ we have that $a \leq -\epsilon + f(p) < \epsilon + f(p) \leq b$. Thus $f(x) \in (a, b)$. Thus we have that $[p, p + \delta) \subseteq f^{-1}(a, b)$.
For each point $p \in f^{-1}(a, b)$ we can construct the interval $I_p = [p, p + \delta)$ described above. We have that

$$f^{-1}(a, b) = \bigcup_{p \in f^{-1}(a,b)} I_p$$

As we have shown, $I_p \subseteq f^{-1}(a, b)$ for each p so $\bigcup_{p \in f^{-1}(a,b)} I_p \subseteq f^{-1}(a, b)$. We also have for every $p \in f^{-1}(a, b)$, $p \in I_p$ so the other direction of containment is also true.
In R_l the I_ps are open sets, and thus $f^{-1}(a, b)$ is an open set. Therefore f is continous.