1

We have that
$$k(\alpha) \cong k[x]/(f)$$
We have
$$K \otimes_k k(\alpha) \simeq K \otimes_k k[x]/(f) \simeq K[x]/(f)$$
as K algebras. The second equivalence comes from the general fact that for rings R, S with $R \subset S$ and R-module M then as S algebras
$$S \otimes_R M \simeq M_S$$
Where M_S is the module M extended as an S module (when such an extention is possible)

2

For any $a \in A, x \in F$ ($a \neq 0$ is not a zero divisor and $x \neq 0$)
We have the module homomorphism $\phi_a : A \to A$ given by
$$\phi_a(r) = ar$$
is injective thus we have the exact sequence
$$0 \longrightarrow A \xrightarrow{\phi_a} A$$
which corresponds to the exact sequence
$$0 \longrightarrow A \otimes_A F \xrightarrow{\phi_a \otimes \mathrm{id}} A \otimes_A F$$
We have that
$$\phi_a \otimes \mathrm{id}(1 \otimes x) = a \otimes x = 1 \otimes ax$$
Since $\phi_a \otimes \mathrm{id}$ is injective it must have a trivial kernel and thus $1 \otimes ax \neq 0 \Rightarrow ax \neq 0$

3

($\Rightarrow$):
For any ideal $I \subset A$ we have the natural embedding
$$0 \to I \to A$$
which by definition of flatness induces the embedding
$$0 \to I \otimes_A F \to A \otimes_A F$$
We have that $A \otimes_A F \cong F$ as A-modules by the isomorphism
$$a \otimes x \to ax$$

thus we have the desired exact sequence

$$0 \to I \otimes_A F \to F$$

($\Leftarrow$):
Since the Tensor product is left adjoint it is right exact, so we must only show left exactness. For some exact sequence of A-modules

$$0 \to X \to Y$$

We have that Y is isomorphic to a quotient of a free module

$$Y \cong \left(\bigoplus_{i \in S} A_i \right) / Q$$

From our hypothesis we know F is A-flat (tensoring preserves injective maps into A). We will show F is Y-flat (and thus conclude F is flat) by showing that if F is M-flat then F is flat for every direct sum and quotient module of M. Since Y is a direct sum and then quotient of A this implies F is Y-flat.

Assuming F is M-flat we have

$$F \otimes \left(\bigoplus_{i \in S} M_i \right) = \bigoplus_{i \in S} F \otimes M_i$$

and thus for an exact sequence

$$0 \to N \to \bigoplus_{i \in S} M_i$$

the injection can be factored into a direct sum of the mapping into each component M_i. From flatness of F each component is an injective mapping when tensoring with F and thus the mapping is still injective

$$0 \to N \otimes F \to \bigoplus_{i \in S} F \otimes M_i$$

Suppose now we have the quotient Q where F is M-flat

$$0 \to I \to M \to Q \to 0$$

Let Q' be a submodule of Q and M' its inverse image in M. This yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & M' \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & M \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

Tensoring with F yields

$$\begin{tikzcd}
 & & 0 \arrow[d] & K \arrow[d] & \\
 & F \otimes I \arrow[r] \arrow[d] & F \otimes M' \arrow[r] \arrow[d] & F \otimes Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes I \arrow[r] \arrow[d] & F \otimes M \arrow[r] & F \otimes Q & \\
 & 0 & & &
\end{tikzcd}$$

where K is the kernel of the mapping from $F \otimes Q' \to F \otimes Q$. The snake lemma yields the short exact sequence

$$0 \to K \to 0$$

and thus $K = 0$ so we have the exact sequence

$$0 \to F \otimes Q' \to F \otimes Q$$

establishing F to be Q-flat. Thus we are done.

4

From problem 2 we get the implication ($\Rightarrow$).
($\Leftarrow$) From problem 3 we know it is sufficient to show that for ever ideal $\langle a \rangle \subset A$ there is an embedding

$$\langle a \rangle \otimes_A F \to F$$

If we consider the kernel of the natural mapping

$$\langle a \rangle \otimes_A F \to F$$

$$ar \otimes x \to arx$$

we have $ar \otimes x \to arx = 0$ can be zero if and only if $ar = 0$ or $x = 0$. Thus the kernel is trivial and we have an embedding

An example of a torsion free module that is not flat is the ideal

$$I = \langle x, y \rangle \subset R = k[x, y]$$

We have the exact sequence is not preserved

$$0 \to I \to R$$

$$0 \to I \otimes I \to I \otimes R$$

Since

$$0 \neq x \otimes y - y \otimes x \to x \otimes y - y \otimes x$$

and in $I \otimes R$ since $1 \in R$:

$$x \otimes y - y \otimes x = xy \otimes 1 - xy \otimes 1 = 0$$

5

Given F flat and the short exact sequence

$$0 \to N \to M \to F \to 0$$

Given any A-module L since the tensor is right exact we must only show exactness around $N \otimes L$.
We have that L can be written as a quotient of a flat L' with the exact sequence

$$0 \to L'' \to L' \to L \to 0$$

(This is since every module can be written as the quotient of a free module and all free modules are flat)
Thus we have the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & L'' \otimes N \arrow[r] \arrow[d] & L'' \otimes M \arrow[r] \arrow[d] & L'' \otimes F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L' \otimes N \arrow[r] \arrow[d] & L' \otimes M \arrow[r] \arrow[d] & L' \otimes F & \\
 & L \otimes N \arrow[r] \arrow[d] & L \otimes M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The Snake Lemma yields the exact sequence

$$0 \to L \otimes N \to L \otimes M$$

Since the cokernels of the left two morphisms are $L \otimes N$ and $L \otimes M$. Thus we are done

6

Since The Tensor product is left adjoint, we know that it is right exact. Thus exactness will be implied by left exactness. In other words that any exact sequence of A -modules of the form

$$0 \to X \to Y$$

is preserved
We have the following commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & X \otimes N \arrow[r] \arrow[d] & X \otimes M \arrow[r] \arrow[d] & X \otimes F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \otimes N \arrow[r] & Y \otimes M \arrow[r] & Y \otimes F &
\end{tikzcd}
$$

From flatness of F we have the third vertical map is an injection. If N is flat then the first vertical map is an injection and from the Four Lemma it must be the case that the map $X \otimes M \to Y \otimes M$ is an injection. Conversly if M is flat then since the mapping $X \otimes N \to X \otimes M \to Y \otimes M$ is injective and is equal to the mapping $X \otimes N \to Y \otimes N$ composed with an injective mapping, it must be the case that the mapping $X \otimes N \to Y \otimes N$ is injective