Additional Problem 1

We know that the limit of the Rieman sums is equal to the integral. Thus we have

$$\left| \int_a^b F(s)ds \right| = \left| \lim_{N \to \infty} \sum_{i=0}^{N} F(x_i) \Delta x_N \right|$$

where $\Delta x_N = \frac{b-a}{N}$ and $x_i = a + i\Delta x_N$. Since $| \cdot |$ is continuous we can interchange the limit and the norm and apply the triangle inequality to conclude

$$\leq \lim_{N \to \infty} \sum_{i=0}^{N} |F(x_i)\Delta x_N| = \int_a^b |F(s)|\, ds$$

Additional Problem 2

(a) Notice the following

$$|\gamma_{j+1}(t) - \gamma_j(t)| = \left| \int_0^t F(\gamma_j(s)) - F(\gamma_{j-1}(s))ds \right|$$

from problem 1

$$\leq \int_0^t |F(\gamma_j(s)) - F(\gamma_{j-1}(s))|\, ds$$

By Lipshitz condition

$$\leq \int_0^t L\, |\gamma_j(s) - \gamma_{j-1}(s)|\, ds$$

Now we can consisely state our inductive argument:
For the base case, notice that

$$\gamma_0(t) = p$$

$$\gamma_1(t) = p + \int_0^t F(p)ds = p + Mt$$

Thus from the inequality established above

$$|\gamma_2(t) - \gamma_1(t)| \leq L \int_0^t |\gamma_1(s) - \gamma_0(s)|ds$$

$$= L \int_0^t Ms\ ds = \frac{MLt^2}{2}$$

For the inductive step we have

$$|\gamma_{j+1}(t) - \gamma_j(t)| \leq \int_0^t L\, |\gamma_j(s) - \gamma_{j-1}(s)|\, ds$$

which by the inductive hypothesis

$$\leq \int_0^t L\frac{ML^{j-1}t^j}{j!} = \frac{ML^jt^{j+1}}{(j+1)!}$$

(b) Notice that we have a taylor series which by Taylors Theorem converges

$$\lim_{N\to\infty}\sum_{j=1}^{N}\frac{ML^jT^{j+1}}{(j+1)!} = MTe^{LT}$$

we have for $t \in [0,T]$

$$\sum_{j=1}^{N}|\gamma_{j+1}(t)-\gamma_j(t)| \leq \sum_{j=1}^{N}\frac{ML^jt^{j+1}}{(j+1)!} \leq \sum_{j=1}^{N}\frac{ML^jT^{j+1}}{(j+1)!}$$

by the Weierstrass M-test we thus have uniform convergence on $[0,T]$

(c) We have

$$\gamma(s) = \lim_{n\to\infty}\gamma_n(s) = \lim_{n\to\infty}\int_0^s F(\gamma_{n-1}(t))\ dt$$

Since we have uniform convergence we can interchange order of limits and integration

$$= \int_0^s \lim_{n\to\infty} F(\gamma_{n-1}(t))\ dt = \int_0^s F(\gamma(t))dt$$

Additional Problem 3

We have

$$|T_Ax|_{\max} = \max_i |A_{i1}x_1 + A_{i2}x_2 + \ldots A_{im}x_m|$$

For each i we have

$$|A_{i1}x_1 + A_{i2}x_2 + \ldots A_{im}x_m| \leq |A_{i1}x_1| + |A_{i2}x_2| + \ldots |A_{im}x_m| \leq L|x|_1$$

Thus

$$|T_Ax|_{\max} \leq L|x|_1$$

L is the smallest value to establish this inequality since letting $A_{kl} = L$, letting $x \in \mathbb{R}^m$ be such that $x_l = 1$ and $x_i = 0$ for $i \neq l$, we have that $|x|_1 = 1$ and

$$|A_{l1}x_1 + A_{l2}x_2 + \ldots A_{lk}x_l + \ldots A_{lm}x_m| = L = L|x|_1$$

and thus for this choice of x

$$|T_Ax|_{\max} = L|x|_1$$

For the second part

$$|T_Ax|_E = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{m} A_{ij}x_j\right)^2}$$

$$\leq \sqrt{n}|T_Ax|_{\max} \leq \max A_{ij}\sqrt{nm}|x|_1 \leq \max A_{ij}\sqrt{nm}|x|_E = L|x|_E$$

For an example where we can get a stronger bound let T_A be the identity for $m = n > 1$. We have

$$|T_Ax|_E = |x|_E < \sqrt{mn}|x|_E = L|x|_E$$

Thus we can choose $L = 1$

For an example where L is the best bound notice that for $m = n = 1$ T_Ax takes the form ax for $a \in \mathbb{R}$ and thus

$$|ax|_E = |a||x| = L|x|$$

Additional Problem 4

($\Rightarrow$) If A is invertable then assume for contradiction we can choose $x_n \neq 0$ where we can scale $|x_n|_E = 1$ so that

$$|T_Ax_n|_E \leq \frac{1}{n}|x_n|_E$$

Notice that the unit ball $\{x \in \mathbb{R}^m : |x|_E = 1\}$ is compact and thus we get a convergent subsequence with a limit $x \neq 0$. We have that

$$|T_Ax|_E < \frac{1}{n}\forall n \Rightarrow T_Ax = 0$$

which contradicts T_A have trivial nullspace

($\Leftarrow$) If A is not invertable then the kernel is nontrivial so there is some $x \neq 0 \in \mathbb{R}^m$ such that

$$|T_Ax|_E = 0 \ngeq c|x|_E$$

for all c (proved by contrapositive)

If (*) is true then we have

$$c|T_{A^{-1}}x|_E \leq |T_AT_{A^{-1}}x|_E = |x|_E$$

Thus

$$|T_{A^{-1}}x|_E \leq c^{-1}|x|_E$$