3.2 22

As the hint suggests, for any $\epsilon > 0$ define

$$E_n = \{x \in [a,b] | f(x) - f_n(x) < \epsilon\}$$

Notice that E_n is the preimage of f_n of a ball around $f(x)$: $E_n = f_n^{-1}(B_\epsilon(f(x)))$ and thus is open. Next notice that if $m > n$ then $E_n \subseteq E_m$. This is true since $f(x) \geq f_m(x) > f_n(x)$ so if $f(x) - f_n(x) > \epsilon$ then $f(x) - f_m(x) > \epsilon$

Finally notice that the collection of E_ns cover $[a,b]$. This is true since for every $x \in [a,b]$ there is N such that $f(x) - f_N(x) < \epsilon$ so $x \in E_N$. Thus since $[a,b]$ is compact we have a finite subcovering

$$E_{n_1} \dots E_{n_k}$$

Since the largest indexed term E_N in the subcovering contains all the other E_{n_i} we have that $[a,b] \subseteq \bigcup E_{n_i} \subseteq E_N$. Thus for all $n > N$ we have the desired result $E_n \supseteq E_N \supseteq [a,b]$ and $f(x) - f_n(x) < \epsilon$ for all $n > N$

3.2 23

Given measurable function $f(x)$, define $E_1 = f^{-1}([0,\infty]), E_2 = f^{-1}([-\infty, 0))$. We can now define the nonegative measurable functions $g_1(x) = \chi_{E_1}(x)f(x)$, $g_2(x) = -\chi_{E_2}(x)f(x)$. Since E_1 and E_2 are disjoint and cover all the domain of f we get

$$f(x) = g_1 - g_2$$

From what was proven about nonegative measurable functions we have a sequence of simple functions with pointwise convergence $\phi_n \to g_1, \psi_n \to g_2$ and thus we have the sequence of sums of simple functions (which is again simple)

$$\phi_n + \psi_n \to f$$

3.3 25

We can express $\mathbb{R} \backslash F$ as the countable disjoint union of open intervals I_n.

We will extend f to all of $\mathbb{R}$ by using the following definition on each $I_n = (a_n, b_n)$, for $x \in I_n$ define

$$f(x) = \frac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) + f(a_n)$$

Notice this is the equation for a line with points $(a_n, f(a_n)), (b_n, f(b_n))$.

If $I_n = (-\infty, p)$ or $= (p, \infty)$ define $f(x) = f(p)$ on I_n.

We have that this extension is continuous on all of $\mathbb{R}$. It is clearly continuous on the interior of each I_n and the interior of F (since locally f is a continous function on F^i and I_n). On any boundary point $b \in F$ we have that b is the boundary of some I_n. Thus either $I_n = (b, p)$ or (p, b) either way, from our definition of the extension notice that $\lim_{x \to b^+} f(x) = f(b)$ and $\lim_{x \to b^-} f(x) = f(b)$ and thus f is continous at b

3.3 29

We can write E as a countable disjoint union of measurable sets:

$$E = \bigcup_{n \in \mathbb{Z}} E \cap (n, n+1]$$

Calling E_n each of these measurable sets, since E_n has finite measure we can use Lusin's result in the finite measure case to say there is a g_n defined on $F_n \subset E_n$ where $f = g_n$ on F_n and $m(E_n \backslash F_n) \leq \epsilon / 2^n$ we can define the desired F as

$$F = \bigcup F_n$$

We have that

$$m(E \backslash F) = m(E \backslash \cup F_n) = \sum m(E_n \backslash F_n) \leq \sum \epsilon / 2^n = \epsilon$$

with the desired g defined as $g(x) = g_n(x)$ for $x \in E_n$. All that is left is to show F is closed. For any limit point $p_n \to p$ of F, $p \in (n, n+1]$ for some n, if $p \in (n, n+1)$ then it must be the case that for sufficiently large n, $p_n \in F_n$ so $p \in F_n$. If $p = n + 1$ then the subsequence of p_n contained in F_n converges to p so is contained in p_n.

3.3 31

We can use Egoroff's Theorem to obtain for $k > 2$, measurable $E_k \subset E$ where $m(E \backslash E_k) < 1/k$ and f_n converges uniformly on E_k. We will define

$$E_1 = E \backslash \bigcup_{k=2}^{\infty} E_k$$

We have that

$$m(E_1) = m\left(\bigcap_{k=2}^{\infty} E \backslash E_k\right) = \lim_{k \to \infty} 1/k = 0$$

and finally it is clear from how E_1 was constructed $E = \cup_{k=1}^{\infty} E_k$.

4.2 12

Let $F \subset E$ be the set where $f(x) = g(x)$ and $m(E \backslash F) = 0$. Letting $F' = E \backslash F$, we have $f(x) = \chi_F g(x) + \chi_{F'} f(x)$. Integration yields

$$\int_E f = \int_F g + \int_{F'} f$$

Since f is bounded there is some value B where $|f(x)| < B$ for all $x \in F'$ and thus

$$\left| \int_{F'} f \right| \leq \int_{F'} B = Bm(F') = 0$$

Similarly since g is bounded there is some B' where $|g(x)| < B'$ and

$$\int_E g = \int_F g + \int_{F'} g$$

$\left| \int_{F'} g \right| \leq B'm(F') = 0$ and thus

$$\int_E f = \int_F g = \int_E g$$

4.2 16

Let $F = \{x \in E | f(x) \neq 0\}$ and define $F_n = \{x \in E | f(x) > 1/n\}$. We have that $F = \cup_{n=1}^{\infty} F_n$ and thus

$$m\left(F\right) = m\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} m(F_n)$$

Thus $m(F) = 0$ iff $m(F_n) = 0$ for all n. If some $m(F_n) > 0$ then we have

$$0 = \int_E f = \int_{E \setminus F_n} f + \int_{F_n} f$$

$\int_{E \setminus F_n} f \geq 0$ since $f \geq 0$ and so

$$\geq \int_{F_n} f \geq \int_{F_n} \frac{1}{n} = m(F_n)\frac{1}{n} > 0$$

which is a contradiction, thus $m(F) = 0$

4.3 24

(i) From the simple aproximation Theorem we can get an increasing sequence of simple functions to converge pointwise to f on E:

$$\varphi_n \to f$$

For each φ_n we have an increasing sequence of simple functions $\psi_{kn} \to \varphi_n$ with finite support defined as

$$\psi_{kn} = \chi_{B_k} \varphi_n$$

where B_k is the ball of radius k (thus the support can be at most $2k$)
We have that the sequence of simple functions with finite support defined as ψ_{nn} converges to f pointwise. For any x we have that there is a N such that $f(x) - \varphi_n(x) < \epsilon$ for all $n > N$ and for K sufficiently large, $x \in B_K$ so for $M = \max\{N, K\}$ we have that $f(x) - \psi_{m,m}(x) < \epsilon$

for all $m > M$

(ii) We have that

$$\int_E f = \sup\left\{\int_E h | h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f\right\}$$

thus since we are taking a sup over a larger set

$$\int_E f \geq \sup\left\{\int_E \varphi | \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f\right\}$$

From (i) we have a sequence φ_n of simple and finite support functions that converge pointwise to f. From Fatous Lemma we have

$$\int_E f \leq \liminf \int_E \varphi_n \leq \sup\left\{\int_E \varphi | \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f\right\}$$

And thus we have equality

4.3 26

consider the sequence of functions over $E = \mathbb{R}$

$$f_n(x) = 1/n|x|$$

This is a decreasing sequence of functions which converge pointwise to 0, however

$$\int_{\mathbb{R}} f_n = \infty$$

for all n so

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \infty \neq \int_{\mathbb{R}} 0$$