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For any $W = ax_u + bx_v \in T_p(S)$, we have that $I_p(W) = Ea^2 + 2Fab + Gb^2$ where E, F, G is given as follows:

(a)

$$E = \langle x_u, x_u \rangle = (a\cos u \cos v)^2 + (b\cos u \sin v)^2 + (c\sin u)^2$$
$$G = \langle x_v, x_v \rangle = (a\sin u \sin v)^2 + (b\cos u \sin v)^2$$
$$F = \langle x_u, x_v \rangle = b^2\cos^2 u \sin^2 v - a^2 \sin u \cos u \sin v \cos v$$

(b)

$$E = \langle x_u, x_u \rangle = (a\cos v)^2 + (b\sin v)^2 + 4u^2$$
$$G = \langle x_v, x_v \rangle = a^2u^2\sin^2 v + b^2u^2\cos^2 v$$
$$F = \langle x_u, x_v \rangle = b^2u\cos v\sin v - a^2u\cos v\sin v$$

(c)

$$E = \langle x_u, x_u \rangle = (a\cosh v)^2 + (b\sinh v)^2 + 4u^2$$
$$G = \langle x_v, x_v \rangle = a^2u^2\sinh^2 v + b^2u^2\cosh^2 v$$
$$F = \langle x_u, x_v \rangle = b^2u\cosh v\sinh v - a^2u\cosh v\sinh v$$

(d)

$$E = \langle x_u, x_u \rangle = (a\cosh u\cos v)^2 + (b\cosh u\sin v)^2 + (c\sinh u)^2$$
$$G = \langle x_v, x_v \rangle = (a\sinh u\sin v)^2 v + (b\sinh u\cos v)^2$$
$$F = \langle x_u, x_v \rangle = b^2\cosh u\sinh u\cos v\sin v - a^2\cosh u\sinh u\cos v\sin v$$
$$= (b^2 - a^2)\cosh u\sinh u\cos v\sin v$$

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Parameterize the surface by

$$g(x, y) = (x, y, f(x, y))$$

we have

$$g_x = (1, 0, f_x)$$
$$g_y = (0, 1, f_y)$$
$$|g_x \wedge g_y| = |(-f_x, -f_y, 1)| = \sqrt{1 + f_x^2 + f_y^2}$$

Thus by definition of area

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}$$

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($\Rightarrow$) For any coordinate curve with respect to u $x(u, v_0) : [u_1, u_2] \to S$ where v_0 is fixed, we have that the length of the curve is

$$\int_{u_1}^{u_2} |x_u(u, v_0)| \, du = \int_{u_1}^{u_2} \sqrt{E} du$$

We have that for any choice of v this length must be the same since we can form the quadralateral with verticies $(u_1, v_0), (u_2, v_0), (u_1, v_1), (u_2, v_1)$ and conclude that the arc length of the curve $x(u, v_0) : [u_1, u_2] \to S$ is the same as the curve $x(u, v) : [u_1, u_2] \to S$. Thus

$$\frac{d}{dv} \int_{u_1}^{u_2} \sqrt{E} du = 0 \Rightarrow \frac{d}{dv} E = 0$$

By swapping the labeling of u and v this argument also concludes that $\frac{d}{du} G = 0$
($\Leftarrow$) for any quadralateral with the verticies $(u_1, v_1), (u_2, v_1), (u_1, v_2), (u_2, v_2)$, if we have $\frac{d}{dv} E = 0$ then

$$\frac{d}{dv} \int_{u_1}^{u_2} \sqrt{E} du = 0$$

and thus the arc length from u_1 to u_2 is constant with respect to v.
Similarly $\frac{d}{du} G = 0$ implies the arc length from v_1 to v_2 is constant with respect to u. Thus the lengths of the curves on opposite sides of the quadralateral are equal

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We can reparametrize $\mathbb{R}^2$:

$$f(u, v) = \int \frac{1}{\sqrt{E}} \, du, \;\; g(u, v) = \int \frac{1}{\sqrt{G}} \, dv$$

(we can choose any integration constant for the indefinate integral)
now we have the parametrization of S

$$y(u, v) = x(f(u, v), g(u, v))$$

Since $\frac{d}{dv} E = 0$ and $\frac{d}{du} G = 0$ we have $g_u = 0, f_v = 0$ and thus by the chain rule

$$y_u = x_u f_u + x_v g_u = x_u \frac{1}{\sqrt{E}}$$

$$y_v = x_u f_v + x_v g_v = x_v \frac{1}{\sqrt{G}}$$

Thus with our new parametrization

$$E_y = \langle y_u, y_u \rangle = \frac{1}{E} \langle x_u, x_u \rangle = 1$$

$$G_y = \langle y_v, y_v \rangle = \frac{1}{G} \langle x_v, x_v \rangle = 1$$

we have the identity $F_y = \frac{\cos\theta}{|E||G|}$ and thus $F_y = \cos\theta$

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$$x_\rho = (\cos\theta, \sin\theta, 0)$$
$$x_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$$

and thus we get

$$E = \cos^2\theta + \sin^2\theta = 1$$
$$G = \rho^2(\cos^2\theta + \sin^2\theta) = \rho^2$$
$$F = \rho\cos\theta\sin\theta - \rho\cos\theta\sin\theta = 0$$