Math 461, Fall 2016 **HOMEWORK #3** *Peter Gylys-Colwell*

1. This number is equivalent to counting the number of surjective functions from the set of 12 places in a string to the set of $\{A, B, C, D\}$. The size of the domain of these functions is 12 and the size of the set the functions map to is 4, as established in class we know from this that the number of these functions is:

$$4^{12} - 3^{12}\binom{1}{4} + 2^{12}\binom{2}{4} - \binom{3}{4}$$

2. We can calculate this by calculating the number of elements in the complement of the set of ways to add up positive $x_1 \ldots x_{10}$ to 100 such that at least one of the terms is > 30. By the inclusion excultion principle we know the way to calculate such a number is:

$$A - \sum_{1 \leq i \leq 10} |A_i| + \sum_{1 \leq i < j \leq 10} |A_i \cap A_j| - \sum_{1 \leq i < j < k \leq 10} |A_i \cap A_j \cap A_k| \ldots$$

Where A is the set of all positive $x_1 \ldots x_{10}$ such that they add up to 100, and A_i is the set of all positive $X_1 \ldots x_{10}$ such that they add up to 100 and that $x_i > 30$.
As established in class, we know the way to count $|A|$ is

$$\binom{99}{9}$$

As for the other terms, we know that the count of $|A_i|$ is the same as the count of all the different ways for positive $x_1 \ldots x_i - 30, \ldots x_{10}$ to add up to 70. We can treat $x_i - 30$ as no different from any of the other terms and so our count will be

$$|A_i| = \binom{69}{9}$$

Using the same trick for $|A_i \cap A_j|$ we are counting the ways for positive $x_1 \ldots x_i - 30 \ldots x_j - 30 \ldots x_{10}$ to add up to 40. And so

$$|A_i \cap A_j| = \binom{39}{9}$$

Using the same logic with a new term we have

$$|A_i \cap A_j \cap A_k| = \binom{9}{9} = 1$$

And lastly we have that

$$|A_i \cap A_j \cap A_k \cap A_l| = 0$$

since 4 of the terms are greater than 30 so the sum of the terms must be > 100
We know there are $\binom{10}{1}$ of the $|A_i|$ terms, $\binom{10}{2}$ of the $|A_i \cap A_j|$ terms, and $\binom{10}{3}$ of the $|A_i \cap A_j \cap A_k|$ terms, and so for our final calculation, we know the answer is

$$\binom{99}{9} - \binom{10}{1}\binom{69}{9} + \binom{10}{2}\binom{39}{9} - \binom{10}{3}$$

3. We will prove that the two quantities on either side of the equation are counting the same thing.
If we consider the set $Z = X \cup Y$ where $|X| = n$ and $|Y = m|$ and $|X \cap Y| = 0$, and we want to count the number of k element subsets of Z consisting of only elements from Y.
One way to count this would be to just count k element subsets of Y since if a subset of Z consists entirely of elements of Y, it must also be a subset of Y. Counting the quantity this way would yield

$$\binom{m}{k}$$

when $m \geq k$ and 0 otherwise, which is the value on the rhs of the equation.
Another way to count these k element subsets would be to use the inclusion exclusion principle. We find the cardinality of the complement of the union of k subsets of Z that contain at least one element in X. This would be calculated as

$$Z_k - \sum_{1 \leq i \leq n} |X_i| + \sum_{1 \leq i < j \leq n} |X_i \cap X_j| \dots$$

Where X_i refers the set of k subsets of Z with the ith term of X in each of the subsets and Z_k refers to the set of all k subsets of Z
We know that $Z_k = \binom{|Z|}{k} = \binom{m+n}{k}$, and we have $|X_i| = \binom{n+m-1}{k-1}$ and there are $\binom{n}{1}$ of the $|X_i|$ terms. More generally we can say for the ith term of the series described above:

$$\sum_{1 \leq l_1 < l_2 \dots l_i \leq n} |X_{l_1} \cap X_{l_2} \cdots \cap X_{l_n}|$$

We have $|X_{l_1} \cap X_{l_2} \cdots \cap X_{l_i}| = \binom{n+m-i}{k-i}$ since there are $n+m-i$ elements left in Z to choose from and there are $k-i$ elements still needing to be chosen. There are $\binom{n}{i}$ of these terms we are summing.

$$\sum_{1 \leq l_1 < l_2 \dots l_i \leq n} |X_{l_1} \cap X_{l_2} \cdots \cap X_{l_n}| = \binom{n+m-i}{k-i}\binom{n}{i}$$

Depending on weather i is even or odd we are adding or subtracting the terms. When the sum we are calculating is equivalent to

$$\sum_{i=0} n(-1)^i \binom{n}{i}\binom{m+n-i}{k-i}$$

And so we are done

4. We can count this the following way. When we choose two of $\{e_1, e_2, e_3, e_4\}$, the other two must be unchosen. Therefore we have

$$2^{\binom{10}{2}-4}$$

Choices for choosing the remaining edges. The number of ways to choose two edges from $\{e_1, e_2, e_3, e_4\}$ is $\binom{4}{2}$. Therefore the total number of unequal graphs containing exactly 2 of the four edges is

$$2^{\binom{10}{2}-4}\binom{4}{2}$$