Math 380a, Fall 2016 **HOMEWORK #9** *Peter Gylys-Colwell*

1. I will say the cliff is to the right of the drunk for convention

a. If we let $X(p)$ be the probablity that the drunk will eventually end up one square to his left (and fall from his original position) we have $X(p) = X(p) + (1-p)X(p)^2$ since there is a $X(p)$ probablity he will just step right and fall, otherwise he will step left with probablity $(p-1)$ and from there can end up back where he started with probablity $X(p)$ and end up off the cliff with probablity $X(p)$ again. Therefore solving the quadratic we have $X(p)$ is either 1 or $\frac{p}{1-p}$. We know with $p=1$, $X(1)=1$ and with $p=0$, $X(0)=0$. Assuming $X(p)$ is a continuous function of p, it would follow that

$$X(p) = \begin{cases} 1 & \frac{1}{2} \leq p \leq 1 \\ \frac{p}{1-p} & 0 \leq p \leq \frac{1}{2} \end{cases}$$

Since $1 = \frac{p}{1-p}$ at $p = \frac{1}{2}$ and from the bounds established at $p = 0, p = 1$ we know what $X(p)$ will be on either side of $\frac{1}{2}$

b. We can reduce the number of different paths he can take to fall on precisely the kth step (P_k) to a recursive formula based on previous steps.

$$P_k = P_1 P_{k-2} + P_3 P_{k-4} + \ldots P_{k-2} P_1$$

The explination for this recursision formula is because after he takes his first step to the left, we can look at the number of ways he can first return to his original position. There are P_i ways he first returns on precisely the ith step, once he is there, he has $k-i-1$ steps remaining and so there are precisely P_{k-i-1} ways for him to fall on the kth step, now we just sum up all these ways. Notice that P_k is 0 if k is even since his displacement from his original position to the cliff is an odd number of steps and an even total number of steps would result in an even displacement.
This recersive form is equivalent to the Catalyn numbers defenition with some reindexing: $C_k = P_{2k-1}$. Catalyn numbers have the closed form

$$P_k = C_{(k+1)/2} = \frac{1}{(k+1)/2+1}\binom{2(k+1)/2}{(k+1)/2} = \frac{1}{k/2+3/2}\binom{k+1}{(k+1)/2}$$

We know that in total he must take $\frac{k+1}{2}$ steps right and $\frac{k-1}{2}$ steps left, so the total probablity of falling after k steps is

$$\frac{1}{k/2+3/2}\binom{k+1}{(k+1)/2} p^{\frac{k+1}{2}}(1-p)^{\frac{k-1}{2}}$$

2. We can simply integrate the density of 1 over the area of the regions where if the center of the coin landed, the coin would be inside a square, and divide by the total area of the chess board. C has 8×8 squares, so 64 squares, each d^2 area so $64d^2$ area total. There is a small square with side length $d-a$ sharing the center point of each square of C such that if the center of the coin lands in any of these squares, the coin would be contained entirely in a square of C. If the center of the coin is not in one of these squares, it will intersect with the boundary of one of the squares of C. There are 64 of these squares, so the total area the coin's center can land on is $64(d-a)^2$. So the probability is

$$\frac{64(d-a)^2}{64d^2} = \frac{(d-a)^2}{d^2}$$

3. We have already established in class that the expected value of the minimum of a set of n random numbers in $[0,1]$ is $\frac{1}{n+1}$. If we shift all the values over by -1, we would shift the expected value by -1 as well and it would be the expected value of the min of the n terms in $[-1,0]$. And finally if we multiply each term by -1, we would multiply the expected value by -1, and we would be measureing the maximum value of the n terms in $[0,1]$, which is what we want. Therefore our expected value is

$$-(\frac{1}{n+1} - 1) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

4. For a given P, the probablity that our rectangle is inside C is the area of the rectangle inscribed in C with one of the corners being P and its sides parallel to the axis divided by the area of the whole circle. Any Q outside of this rectangle would create a rectangle that has a side that extends beyond the bounds of C and any Q inside this rectangle would create a rectangle contained in the inscribed rectangle which is contained in C. Therefore to find the whole probability, we can integrate uniformly over the domain of P the areas of these inscribed rectangles divided by the area of C and divided by the arclength we are integrating over. Using polar coordinates we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|2\cos(\theta)2\sin(\theta)|}{\pi}\, d\theta = \frac{1}{2\pi}4\int_0^{\pi/2} \frac{4\cos(\theta)\sin(\theta)}{\pi}\, d\theta$$

Using a u-substitution for $u = \sin(\theta), du = \cos(\theta)$:

$$= \frac{4}{2\pi}\frac{4\sin^2(\pi/2)}{2\pi} = \frac{4}{\pi^2}$$

EC. We simulate in the following way. We flip the coin twice if the coin pattern is H,T then we say our balanced coin result would be T, if the pattern is T,H then we say H. Any other pattern and we ignore it and repeat the process. We have that the probablity of H,T is $p(1-p)$ while the probablity of T,H is $(1-p)p$, and so the probabilities are equal, therefore we are simulating a balanced coin. We know that this process must terminate since the probablility of not terminating after n times is $(2(1-p)p)^n = (2(p-p^2))^n < 1^n$ and so as $n \to \infty$, $(2(1-p)p)^n \to 0$. (The reason $2(p-p^2) < 1$ is because the function $f(x) = x - x^2$ on $[0,1]$ is $< \frac{1}{2}$, we can use the derivative to find the critical point at $\frac{1}{2}$ to conclude this)