16.1 This is a special case of thm 16.1 ii:
We have

$$(-1)a + a = (-1+1)a = 0 \cdot a = 0$$

and so subtracting a on both sides yields

$$(-1)a = -a$$

16.7 Since F is a field we know there is $a^{-1} \in F$ such that $aa^{-1} = 1$. Therefore if we let $x = a^{-1}(-b)$ we satisfy the equation:

$$a(a^{-1}(-b)) + b = (aa^{-1})(-b) + b = -b + b = 0$$

We get that first equality since $\cdot$ is associative

16.11

a. The only unit is $(1,1)$ since for any $a, b \in \mathbb{Z}$, $ab = 1 \Leftrightarrow a = 1, b = 1$. The only zero-divisor is $(0,0)$ since for any $a, b \in \mathbb{Z}$, $ab = 0 \Leftrightarrow a = 0$ and/or $b = 0$. Since the set of nilpotents elements is a subset of zero-divisors, it follows that the only nilpotent is also $(0,0)$.

b. From previous knowledge of groups we know every element in $\mathbb{Z}_3$ has an inverse under the group operation of multiplication modulo 3, therefore we know for any $(a,b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ there is a $(a^{-1}, b^{-1}) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ such that $(a,b)(a^{-1}, b^{-1}) = (1,1)$ and so every element in $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is a unit. Since 3 is prime there is no two numbers that can multiply together to be a multiple of 3 unless one of the two numbers is already a multiple of 3, only $(0,0)$ is a zero-divisor and from that it follows (since the set of nilpotents is a subset of zero-divisors) that $(0,0)$ is the only nilpotent

c. The units are $(1,1), (1,5), (3,1), (3,5)$ with respective inverses $(1,1), (1,5), (3,1), (3,5)$. The zero-divisors are all the rest of the elements: $(0,2), (0,3), (0,4), (2,2), (2,3), (2,4)$. The nilpotents are $(0,0), (2,0)$.

16.13

a. If there were two multiplicative identities: $1 \neq 1'$ we would have by definition of the multiplicative identity

$$1 = 1 \cdot 1' = 1'$$

and so $1 = 1'$

b. If there were two multiplicative inverses, let β and α be multiplicative inverses of a. We have

$$\beta = \beta(a\alpha) = (\beta a)\alpha = \alpha$$

And so $\beta = \alpha$

A From the definition we know that the center is abelian and from the definition of a division ring we know every element is a unit. Now all we need to show is that the center is closed under multiplication and addition. Given any $a, b \in$ the center of R we have for any $x \in R$

$$(a+b)x = ax + bx = xa + xb = x(a+b)$$

and so $a+b$ is in the center. We also have

$$(ab)x = axb = x(ab)$$

and so ab is in the center. Therefore the center is a field.

B $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain. Consider any $a, b \in \mathbb{Z}/\{0\}$

$$(a, 0) \cdot (0, b) = (0, 0)$$

and so $(a, 0)$ and $(0, b)$ are non-zero zero-divisors.

C $\mathbb{Z}_{10}$ is not an integral domain. Consider

$$2 \cdot 5 = 0$$

and so 2 and 5 are non-zero zero-divisors. Observing that S is the set of all even integers in R we know that S is closed under addition and multiplication since multiplying or adding to even numbers yields an even number. Addition is still commutative in S. Therefore S is a subring of R.
S is an integral domain since for any $s \in S$ in order for $s \cdot a = 0$, 10 must divide sa and so $2 \cdot 5$ must divide sa. However since s is even if it also has a factor of 5 then it is a multiple of 10 since it has a factor of both 5 and 2. If $s \neq 0$ mod 10 then, a must have a factor of 5 and so if $a \in S$ then a is 0 since a would have a factor of 5 and a factor of 2. Therefore there is no non-zero term $a \in S$ such that $sa = 0$.
S is a field since S is commutative (since R is commutative) and each term is a unit with 6 the multiplicative identity: $2 \cdot 8 = 6$, $4 \cdot 4 = 6$, $6 \cdot 6 = 6$, $6 \cdot 2 = 2$, $6 \cdot 4 = 4$, and $6 \cdot 8 = 8$.

D We in order for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be in the center we have for any $w, x, y, z \in \mathbb{R}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} wa+xc & wb+xd \\ ya+zc & yb+zd \end{bmatrix}$$

Equating the top left and bottom right corners gives us $by = cx$. The only way for those quantities be equal for any x, y is if $b = c = 0$. From there, equating the top right and bottom left corners gives us $ax = xd$ and $dy = ya$. Dividing by x for the first equation or y for the second equation yields $a = d$. Therefore the center consisits of all matricies of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

With $a \in \mathbb{R}$

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