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Checking all conditions:

For any $p = (x_0, y_0, z_0)$, we have the homeomorphism $f : U \subset \mathbb{R}^2 \to C$ where C denotes the cylinder. We define f as

$$f(\theta, y) = (\cos(\theta), \sin(\theta), z_0 + y)$$

And for $x_0 \neq -1, y_0 \neq 0$, $U = (-\pi, \pi) \times (-1, 1)$, otherwise $U = (0, 2\pi) \times (-1, 1)$. This map is smooth since each component is differentiable. f has a continuous inverse since f is bijective and the inverse of the compenents are continuous since locally, the inverse is equal to $(\sin^{-1}(y), z - z_0)$ or $(\cos^{-1}(x), z - z_0)$. The jacobian is

$$df_{(\theta, y)} = \begin{bmatrix} -\sin(\theta) & 0 \\ \cos(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

Which is linearly independent for any $(\theta, y) \in U$

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$C = \{(x, y, z) \in \mathbb{R}^3; z = 0, x^2 + y^2 \leq 1\}$ is not regular. The reason for this is because the point $(0, 1, 0) \in C$ cannot have a neighborhood V such that there is a homeomorphism $f : U \subset \mathbb{R}^2 \to V \cap C$. The reason for this is because $V \cap C$ is homeomorphic to $C' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ by the map $\pi_{12}(x, y, z) = (x, y)$ (which has the continuous inverse map $\pi_{12}^{-1}(x, y) = (x, y, 0)$). We have that C' is a closed set in $\mathbb{R}^2$. Thus composing homeomorphisms, we would get $\pi_{12} \circ f : U \to C'$ is a homeomorphism from closed unit disc in $\mathbb{R}^2$ to the open unit disc in $\mathbb{R}^2$. We know that such a mapping is not possible since removing a point on the boundary of the closed unit disc still yields a null homotopic set, while removing any point on the unit open disc yields a set which is not null homotopic (we proved this in Math 441)

For $D = \{(x, y, z) \in \mathbb{R}^3; z = 0, x^2 + y^2 < 1\}$ we have the homeomorphism

$$f : \{x, y\} \in \mathbb{R}^2; x^2 + y^2 < 1\} \to D$$

Where $f(x, y) = (x, y, 0)$ which is clearly smooth with a continuous inverse $f^{-1}(x, y, 0) = (x, y)$ and linearly independent jacobian:

$$df_{(x, y)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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(a) Calculating the Jacobian:

$$df_{(x,y,z)} = 2(x + y + z - 1)(1, 1, 1)$$

Thus the critical points is the simplex $x + y + z = 1$. Evaluating f at any point in the simplex yields the critical value 0.

(b) We have that $f(\mathbb{R}^3) = \mathbb{R}^+$. All $\mathbb{R}^3 \setminus \{0\}$ are regular points and thus from Prop 2, $f^{-1}(c)$ for $c > 0$ is regular. We have that $f^{-1}(0)$ is regular too since

$$f^{-1}(0) = \{(x, y, z) : x + y + z = 1\}$$

is a plane (which we know is regular)

(c) Calculating the Jacobian:

$$df_{(x,y,z)} = (yz^2, xz^2, 2xyz)$$

The critical points is the plane $z = 0$ union the line $y = 0, x = 0$. Evaluating f at any critical point yields the critical value 0.

For $c \in f(\mathbb{R}^3) \setminus \{0\}$, c is regular so $f^{-1}(c)$ yields a regular surface. For $c = 0$, $f^{-1}(0)$ is not regular since it is the union of three normal planes $x = 0, y = 0, z = 0$ which intersect at $(0, 0, 0)$. There is no well defined tangent vector at the point $(0, 0, 0)$

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We know that dx_q is one to one if and only if $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ are linearly independent. From the definition of '$\wedge$' we know that $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = 0$ if and only if the vectors are linearly dependent. Thus dx_q is one-to-one iff $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} \neq 0$

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Letting c be the speed of the points, for a given t we have the positions

$$p(t) = (0, 0, ct), q(t) = (a, ct, 0)$$

Thus the line containing $p(t), q(t)$ parameterized by s is described as

$$p(t) + s(q(t) - p(t)) = (0, 0, ct) + s(a, ct, -ct)$$

So for x, y, z on the line we have

$$\frac{x}{a} = \frac{y}{ct} = \frac{z - ct}{-ct} = s$$

$$ctx = ay = act - az$$

Letting t vary in $\mathbb{R}^+$, we will show this is the same set as $y(x - a) + zx = 0$.
To show $\subseteq$, notice $y = \frac{ctx}{a}, z = ct - \frac{ctx}{a}$ so we have

$$y(x - a) + zx = \frac{ctx}{a}(x - a) + \frac{act - ctx}{a}x = 0$$

Thus any point (x, y, z) satisfying one of the line equations satisfies the closed form equation.
To show $\supseteq$, notice that for any fixed x, by choosing s so that $as = x$, we get the line $y(as - a) + asz = 0$. Thus we can choose t to get any y, z combination satisfying the equation.
This establishes a continuous mapping from the parameterization to the surface:

$$f : \mathbb{R}^2 \to S$$

$$f(t, s) = p(t) + s(q(t) - p(t)) = (sa, ct, ct - cts)$$

Calculating the jacobian:

$$d_f = \begin{bmatrix} 0 & a \\ sc & ct \\ c - sc & -ct \end{bmatrix}$$

Since $c, a \neq 0$ we have that the jacobian is surjective for all s, c, and thus f is a smooth function with nonsingular jacobian. f is one to one since s uniquely determines the x value of a point and then t will uniquely determine the y, z values. Thus f establishes S to be regular