**1**

($\Rightarrow$) If an $R$-module $M$ has no nontrivial essential extensions then we know we can embed $M$ into an injective module $I$. If we consider the set $S$ of submodules $N$ of $I$ where $N \cap M = 0$ we can apply Zorns Lemma to get a maximal module $D \in S$. To check Zorns Hypothesis: $0 \in S$ (so $S \neq \emptyset$). For any chain

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

there is a largest element $N = \bigcup N_i$ which still has the property $N \cap M = \bigcup N_i \cap M = 0$
We have that $I/D$ is an essential extension of $M$ as follows.
For any $N \subset I/D$, if it were the case that $N \cap M = 0$ then the pre image of $N$ from the mapping $I \to I/D$ is a module in $S$ containing $D$ which contradicts maximality of $D$
Therefore we can conclude $M = I/D$. This along with the fact $M \cap D = 0$ implies $I = M \oplus D$. We know that the summand of an injective module is injective and thus $M$ is injective.

($\Leftarrow$) If an $R$-module $I$ is injective suppose for contradiction $I$ has a nontrivial essential extension $M$. We have from injectivity that the mapping

$$0 \to I \to M$$

splits. Thus there is some submodule $N \subset M$ where $M = I \oplus N$ so $I \cap N = 0$ which is a contradiction of $M$ essential

**2**

We can use Zorns Lemma. If we consider the set $\mathcal{C}$ of essential extensions of $M$, for any chain $M_1 \subset M_2 \subset \dots$ of essential extensions there is a maximal element $\mathcal{M} = \bigcup M_i$. We have that $\mathcal{M}$ is an essential extension since if $L \subset \mathcal{M}$ then $L = \bigcup L \cap M_i$ and so one $L \cap M_i \neq 0$ so $0 \neq (L \cap M_i) \cap M \subseteq L \cap M$. Thus from Zorns Lemma there is a maximal element $E(M)$

We have that if $E(M) \subset I$ has a nontrivial essential extension then $I$ is essential extension of $M$ which is a contradiction of maximality. This is the case since for any $L \subset I$, $L \cap E(M)$ is a submodule of $E(M)$ and thus $0 \neq (L \cap E(M)) \cap M \subseteq L \cap M$. Therefore $E(M)$ has no nontrivial extensions and so from problem 1 is injective.
$E(M)$ is minimal among the injective modules containing $M$ since for any injection $f : M \to I$, the natural inclusion $i : M \to E(M)$ and injectivity of $E(M)$ yields mapping $\phi : E(M) \to I$ such that $f = \phi \circ i = \phi|_M$. We have that $\ker \phi|_M = \ker \phi \cap M = 0$ since $f$ is an injection and thus from being an essential extension $\ker \phi = 0$ so $\phi$ is an injection

**3**

We have the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \to m} \mathbb{Z} \xrightarrow{1 \to 1} \mathbb{Z}/(m) \longrightarrow 0$$

Applying the Hom functor yields

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \xrightarrow{1\to 1} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \xrightarrow{1\to m} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \longrightarrow 0$$

It is the case $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \simeq \mathbb{Z}/(n)$ and $\mathrm{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \simeq \mathbb{Z}/(d)$ where $d = \gcd(m, n)$. Thus the Hom sequence is isomorphic to

$$0 \longrightarrow \mathbb{Z}/(d) \xrightarrow{1\to n/d} \mathbb{Z}/(n) \xrightarrow{1\to m} \mathbb{Z}/(n) \longrightarrow 0$$

Thus we have

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \mathbb{Z}/(d)$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \mathbb{Z}/(d)$$
$$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong 0, \forall n \geq 2$$

**4**

We have the same projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1\to n} \mathbb{Z} \xrightarrow{1\to 1} \mathbb{Z}/(n) \longrightarrow 0$$

Applying the $\mathbb{Z}/(m) \otimes -$ functor yields

$$0 \longrightarrow \mathbb{Z}/(m) \otimes \mathbb{Z} \xrightarrow{1\otimes n} \mathbb{Z}/(m) \otimes \mathbb{Z} \xrightarrow{1\otimes n} \mathbb{Z}/(m) \otimes \mathbb{Z}/(n) \longrightarrow 0$$

We have $\mathbb{Z}/(m) \otimes \mathbb{Z} \simeq \mathbb{Z}/(m)$ and $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) \simeq \mathbb{Z}/(d)$ where $d = \gcd(m, n)$
Thus we have

$$\mathrm{Tor}^0_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \mathbb{Z}/(d)$$
$$\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \ker(1 \otimes n) \cong \mathbb{Z}/(d)$$
$$\mathrm{Tor}^n_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong 0, \forall n \geq 2$$

**5**

Let $R = \mathbb{Z}/(4)$. We have the $R$ module $\mathbb{Z}/(2)$. We have the projective resolution

$$\cdots \longrightarrow \mathbb{Z}/(4) \xrightarrow{1\to 2} \mathbb{Z}/(4) \xrightarrow{1\to 2} \mathbb{Z}/(4) \xrightarrow{1\to 2} \mathbb{Z}/(4) \xrightarrow{1\to 1} \mathbb{Z}/(2) \longrightarrow 0$$

Taking the Hom Functor

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \xrightarrow{1\to 1} \mathrm{Hom}(\mathbb{Z}(4), \mathbb{Z}/(2)) \xrightarrow{1\to 2} \mathrm{Hom}(\mathbb{Z}(4), \mathbb{Z}/(2)) \xrightarrow{1\to 2} \cdots$$

We have that $\mathrm{Hom}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \simeq \mathrm{Hom}(\mathbb{Z}/(4), \mathbb{Z}/(2)) \simeq \mathbb{Z}/(2)$. Thus we have the isomorphic sequence

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{1\to 1} \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \xrightarrow{0} \cdots$$

Thus we have that

$$\mathrm{Ext}^n_{\mathbb{Z}/(4)\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$$

for all $n \geq 0$

**6**

Notice that the example in 5 shows that the statment is not true