12.17 Since every element has order 2 and commutes with every other element, any injective mapping from K to K is an automorphim as long as the identity maps to itself. To count the number of injections, we have a can map to 3 elements, then b can map to the remaining 2 elements, and c must map to whats left. Therefore there are $3 \cdot 2 = 6$ automorphisms

12.20

a. For any elements $x, y \in G$ we have

$$\varphi(xy) = (xy)^n$$

Since G is abelian

$$= x^n y^n = \varphi(x)\varphi(y)$$

We can show φ is a injective which would imply it is bijective since it is a mapping from G to G. If $x^n = y^n$, we have

$$y^n x^{-n} = e$$

since G is abelian we have

$$(x^{-1}y)^n = e$$

since n is relatively prime to $|G|$, we know that n cannot be a multiple of the order of $x^{-1}y$ since the order must divide $|G|$ by lagranges thm, unless $x^{-1}y = e$. This implies $x = y$ since inverses are unique

b. Since φ is surjective, for any $x \in G$ there is a $y \in G$ such that

$$\varphi(y) = y^n = x$$

12.23

a. We know that the mapping $\varphi(h) = ghg^{-1}$ is an automorphim of G for any $g \in G$. Applying these mappings to H we have

$$\varphi(H) = gHg^{-1}$$

And so if for all these mappings we have $\varphi(H) \subseteq H$ then we have

$$gHg^{-1} \subseteq H$$

Which is means H is normal

b. Consider $G =$ the Klien's 4group. Let $H = \{e, a\}$ and let φ be the automorphim with $\varphi(e) = e, \varphi(a) = b, \varphi(b) = c, \varphi(c) = a$. We have

$$\varphi(H) = \{e, b\} \nsubseteq H$$

12.31 No, consider $G =$ the Klien's 4group. $\psi(a) = b, \psi(b) = c$, and $\psi(c) = a$. If we let $H = \{e, a\}$ we have $\varphi(a) = \psi(a) = b$, but $\varphi(b) = b$ so φ is not injective so not an automorphim.

12.34 Letting H bet the set of inner automorphisms of G, we have for any $A(x) = axa^{-1}, B(x) = bxb^{-1}, C(x) = cxc^{-1} \in H$.

$$A \circ B = abxb^{-1}a^{-1} \in H$$

since $ab \in G$. Since H is closed under the group operation and since Aut(G) is finite, that is sufficient to show H is a subgroup. To check if normal we have for any $\varphi \in$Aut(G):

$$\varphi \circ H \circ \varphi^{-1} = \{A(x) = \varphi(a\varphi^{-1}(x)a^{-1}) : a \in G\}$$

$$= \{A(x) = \varphi(a)x\varphi(a^{-1}) : a \in G\}$$

and since φ is an automorphim on G,

$$\{A(x) = bxb^{-1} : b \in G\} = H$$

So H is normal

13.2 H consists of only an identity element and an element of order 2. Lets call this element a and the identity e. If there existed a homomorphism $\varphi : Q_8 \to H$ then we know $\varphi(I) = e$ since $(\varphi(I))^2 = \varphi(I)$. If there is some element $q \in Q_8$ such that $\varphi(q) = a$. A property of Q_8 is that for any element $q \in Q_8$, there is an element j such that $j^2 = q$. Therefore we have

$$(\varphi(j))^2 = \varphi(q) = a$$

But there is no element $k \in H$ such that $k^2 = a$ so there is nothing that $\varphi(j)$ can map to.

13.6 We have

$$\{0(3\mathbb{Z}/12\mathbb{Z}), 1(3\mathbb{Z}/12\mathbb{Z}), 2(3\mathbb{Z}/12\mathbb{Z})\}$$

13.8 We can define a homomorphism $\varphi : G \to (\mathbb{Z}, +)$ such that for a given $\frac{a}{b} \in G$ with a and b in their most reduced state (relatively prime to each other) we have $\varphi(\frac{a}{b}) = m(a) - m(b)$

where $m(x)$ is the number of times 2 divides x (note that since a and b are relatively prime, $m(a)$ and/or $m(b)$ is zero). φ is a homomorphism since for any $\frac{a}{b}, \frac{c}{d}$,

$$\varphi(\frac{a}{b}) + \varphi(\frac{c}{d}) = m(a) - m(d) + m(c) - m(b) = m(ac) - m(bd) = \varphi(\frac{ac}{bd})$$

The kernal of φ would be $\frac{a}{b} \in G$ such that $m(a) = m(b) = 0$ which is precisely H. Lastly φ is surjective since for any $z \in \mathbb{Z}$ we have $\varphi(\frac{2^z}{1}) = z$ if $z \geq 0$ and $\varphi(\frac{1}{2^z}) = z$ if $z < 0$. Therefore by the Fundamental Theorem we have our desired result

13.9 Let $\varphi : G \to \{\mathbb{R} - \{0\}, \cdot\} \times \{\mathbb{R} - \{0\}, \cdot\}$ with

$$\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c)$$

φ is a homomorphism since

$$\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \varphi \begin{pmatrix} i & j \\ 0 & k \end{pmatrix} = (ai, ck) = \varphi \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} i & j \\ 0 & k \end{pmatrix} \right) = \varphi \begin{pmatrix} ai & aj + bk \\ 0 & ck \end{pmatrix}$$

We know $(1, 1)$ is the identity of $\{\mathbb{R} - \{0\}, \cdot\} \times \{\mathbb{R} - \{0\}, \cdot\}$, and (a, c) is precisely when the input matrix is in H so $\ker(\varphi) = H$. It is clear φ is surjective since we can choose a, c to be anything in the matrix. Therefore by the Fundamental Theorem we have our desired result.

$$G/H = \{\mathbb{R} - \{0\}, \cdot\} \times \{\mathbb{R} - \{0\}, \cdot\}$$