2-3, 1

A is continuous since its components are continuous functions. Given any point $p \in S^2$ we can choose parametarizations $x_1 : U_1 \subset \mathbb{R}^2 \to S^2$, $x_2 : U_2 \subset \mathbb{R}^2 \to S^2$ such that $p \in x_1(U_1)$, $A(p) \in x_2(U_2)$.
From class we established there are the following parametarizations we can choose from:

$$x_1, x_2 \in \{(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi), (\cos\theta\sin\varphi + \pi, \sin\theta\sin\varphi + \pi, \cos\varphi + \pi),$$

$$(\cos\theta + \pi\sin\varphi, \sin\theta + \pi\sin\varphi, \cos\varphi)\ldots\}$$

We must show that

$$x_2^{-1} \circ A \circ x_1$$

is differentiable at p. Notice that $A = A^{-1}$ and thus by showing A is differentiable we have shown its inverse to be differentiable. Hence concluding A is a diffeomorphism.
It is clear this composition is differentiable since each component

$$x_2^{-1} \circ A \circ x_1(p) = (x_{2,1}^{-1}(-x_{1,1}(p_1)), x_{2,2}^{-1}(-x_{1,2}(p_2)), x_{2,3}^{-1}(-x_{1,3}(p_3)))$$

Is a composition of differentiable functions and thus differentiable regardless of choice of x_1, x_2.

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We have the diffeomorphism $f : \mathbb{R}^2 \to P$ with $f(x, y) = (x, y, z^2)$. To establish f is a diffeomorphism, notice that f is also a parametarization. Thus the conditions of being a diffeomorphism rely on asking whether over any open set if

$$f^{-1} \circ f \circ \mathrm{id}$$

is differntiable. This mapping is the identity mapping so clearly differntiable.

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The diffeomorphism $f : S^2 \to E$ is defined by

$$f(x, y, z) = (ax, by, cz)$$

We have that this is a diffeomorphism since notice that for any parametarization $x_1 : U \subset \mathbb{R}^2 \to S^2$ around a point $p \in S^2$ we have a parametarization $x_2 = f \circ x_1$ of E around $f(p)$. Thus we have

$$x_1^{-1} \circ f \circ x_2 = x_1^{-1} \circ f^{-1} \circ f \circ x_1 = \mathrm{id}$$

is differentiable

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For diffeomorphism $f : S_1 \to S_2$, suppose we have the parametarizations around a point $p \in S_1$ and $f(p) \in S_2$

$$x_1 : U \subset \mathbb{R}^2 \to S_1, x_2 : V \subset \mathbb{R}^2 \to S_2$$

$$y_1 : U \subset \mathbb{R}^2 \to S_1, y_2 : V \subset \mathbb{R}^2 \to S_2$$

We need to show that if the parametarization with x_1, x_2 establishes f to be differentiable then so does y_1, y_2

We can use the change of Parametarization Theorem. Letting

$$W_1 = x_1(U) \cap y_1(U), W_2 = x_2(V) \cap y_2(V)$$

We have the diffeomorphisms h_1, h_2

$$h_1 = x_1^{-1} \circ y_1 : y_1^{-1}(W_1) \to x_1^{-1}(W_1)$$

$$h_2 = x_2^{-1} \circ y_2 : y_2^{-1}(W_2) \to x_2^{-1}(W_2)$$

Notice that $p \in W_1$ and $f(p) \in W_2$ and thus when we restrict to the open set W_2 such that

$$W_3 = x_1^{-1}(f^{-1}(W_2)) \cap y^{-1}(f^{-1}(W_2)) \subset W_1$$

$$y_2^{-1} \circ f \circ y_1|_{W_3} = h_2^{-1} \circ x_2^{-1} \circ f \circ x_1 \circ h_1$$

Thus we have that y_1, y_2 establish f to be a diffeomorphism under this restriction since $y_2^{-1} \circ f \circ y_1|_{W_3}$ is a composition of the diffeomorphic functions $h_2^{-1}, x_2^{-1} \circ f \circ x_1, h_1$

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Checking the three conditions of an equivalence relation:

Reflexivity:

We have that the identity

$$\mathrm{id} : S \to S$$

is a diffeomorphism and thus $S \sim S$

Transitivity:

If we have the diffeomorphisms

$$f_1 : S_1 \to S_2, f_2 : S_2 \to S_3$$

Then it is the case that

$$f_3 = f_2 \circ f_1 : S_1 \to S_3$$

is a diffeomorphism. The reason for this is because for parametarizations

$$x_1 : U_1 \to S_1, x_2 : U_2 \to S_2, x_3 : U_3 \to S_3$$

establishing that f_1, f_2 is diffeomorphic, we have that for a sufficientnly small domain around our point $p \in U_1$ in question

$$x_3^{-1} \circ f_3 \circ x_1 = x_3^{-1} \circ f_2 \circ x_2 \circ x_2^{-1} \circ f_1 \circ x_1$$

This is the composition of differentiable functions with differentiable inverses $x_2^{-1} \circ f_1 \circ x_1, x_3^{-1} \circ f_2 \circ x_2$ and therefore is itself a differentiable function with differentiable inverse. Thus f_3 is a diffeomorphism so $S_1 \sim S_3$

Symmetry:

If there exists the diffeomorphism

$$f : S_1 \to S_2$$

then it is the case that f^{-1} is a diffeomorphism as well and thus

$$f^{-1} : S_2 \to S_1$$

establishes $S_2 \sim S_1$