1.4 36

We have that any σ-algebra which contains intervals of the form $[a, b)$ must contain all open intervals and conversley any σ-algebra which contains the open intervals must contain all intervals of the form $[a, b)$. Thus since the Borel sets are the smallest σ-algebra to contain the open intervals, it is also the smallest σ-algebra to contain intervals of the form $[a, b)$
To show this containment:
$(\subseteq)$: if $\mathcal{A}$ is σ-algebra containing the open intervals (and thus by closure under complements the closed intervals) then for any $[a, b)$ we have $(-\infty, a), [b, \infty) \in \mathcal{A}$ and thus

$$[a, b) = ((-\infty, a) \cup [b, \infty))^C \in \mathcal{A}$$

$(\supseteq)$: if $\mathcal{A}$ is σ-algebra containing the intervals $[x, y)$ then for any (a, b) we have the collection $[a + 1/n, b) \in \mathcal{A}$ and we have

$$(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b) \in \mathcal{A}$$

2.1 3

We can split up the union into a union of disjoint sets:

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} \left(E_k \backslash \bigcup_{i=1}^{k-1} E_i \right)$$

from countable additivity over disjoint sets we have

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} \left(E_k \backslash \bigcup_{i=1}^{k-1} E_i \right)\right)$$

$$= \sum_{k=1}^{\infty} m\left(E_k \backslash \bigcup_{i=1}^{k-1} E_i \right)$$

Since

$$m(E_k) = m\left(E_k \cap \left(\bigcup_{i=1}^{k-1} E_i\right)\right) + m\left(E_k \backslash \bigcup_{i=1}^{k-1} E_i\right)$$

we get that

$$m(E_k) \geq m\left(E_k \backslash \bigcup_{i=1}^{k-1} E_i\right)$$

and thus

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k)$$

2.1 4

Countably Additive:
For disjoint sets $E_1, E_2 \dots E_k \dots$, since the sets are disjoint the number of elements in the union is equal to the sum of the number of elements in each set. Thus

$$c\left(\bigcup E_k\right) = \sum c(E_k)$$

Translation Invariant:
If a set E is translated by y we have that $E + y$ has the same number of elements and thus

$$c(E) = c(E + y)$$

2.2 6

We have the covering $(0, 1)$ of A where $\ell((0, 1)) = 1$ and thus $m^*(A) \leq 1$.
We also have from countable subadditivity

$$1 \leq m^*([0, 1] \backslash A) + m^*(A)$$

It is the case $[0, 1] \backslash A = \mathbb{Q} \cap [0, 1]$ is a countable set and thus has outer measure zero thus

$$1 \leq m^*(A)$$

So $m^*(A) = 1$

2.2 9

Since $B \subseteq A \cup B$ we have

$$m^*(B) \leq m^*(A \cup B)$$

by subadditivity

$$m^*(A \cup B) \leq m^*(B) + m^*(A \backslash (A \cap B))$$

Since $A \backslash (A \cap B) \subseteq A$

$$0 \leq m^*(A \backslash (A \cap B)) \leq m^*(A) = 0$$

$$m^*(A \cup B) \leq m^*(B)$$

And thus $m^*(B) = m^*(A \cup B)$

2.3 14

If we consider the sequence of balls $B_1, B_2, \ldots$ where

$$B_n = \{x \in \mathbb{R} : |x| \leq n\}$$

we have that

$$E = \bigcup_{n=1}^{\infty} E \cap B_n$$

from subadditivity

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E \cap B_n)$$

since $m^*(E) > 0$ it must be the case $\sum_{n=1}^{\infty} m^*(E \cap B_n) > 0$ which means one of the terms $m^*(E \cap B_N)$ in the sum must be > 0. $E \cap B_N$ is a bounded subset of E with positive measure

2.3 15

Consider the set of intervals

$$\mathcal{I} = \{\ldots [-2\epsilon, -\epsilon), [-\epsilon, 0), [0, \epsilon), [\epsilon, 2\epsilon) \ldots\}$$

Notice that $\mathcal{I}$ covers $\mathbb{R}$ and so we have

$$E = \bigcup_{I_k \in \mathcal{I}} E \cap I_k$$

Since $m(I_k) = \epsilon$ and $E \cap I_k \subseteq I_k$ we have the desired property

$$m(E \cap I_k) \leq \epsilon$$

Since each I_k and E is measurable from additivity

$$m(E) = \sum_{I_k \in \mathcal{I}} m(E \cap I_k)$$

thus since $m(E)$ is finite the sum converges which means for sufficiently large index, the sum of the rest of the terms is less than ϵ. Let $E \cap I_N, E \cap I_{N+1}, E \cap I_{N+2}, \ldots$ denote this tail of the sum. Letting $T = \bigcup_{n=N}^{\infty} E \cap I_n$ with

$$m(T) = \sum_{n=N}^{\infty} m(E \cap I_n) < \epsilon$$

And thus we have E is a finite union of measurable sets with measure less than ϵ

$$E = (E \cap I_1) \cup (E \cap I_2) \cup (E \cap I_3) \cup \ldots (E \cap I_{N-1}) \cup T$$