1. We can prove this by induction:
Base Case: for $n = 1$, we have $\frac{a_1}{a_1} \geq 1$ is true.
For the inductive step we look at

$$((a_1 + \cdots + a_n) + a_{n+1})((\frac{1}{a_1} + \cdots + \frac{1}{a_n}) + \frac{1}{a_{n+1}})$$

$$= (a_1 + \ldots a_n)(\frac{1}{a_1} + \cdots + \frac{1}{a_n}) + a_{n+1}(\frac{1}{a_1} + \cdots + \frac{1}{a_n}) + (a_1 + \ldots a_n)\frac{1}{a_{n+1}} + 1$$

By the inductive hypothesis

$$\geq n^2 + a_{n+1}(\frac{1}{a_1} + \cdots + \frac{1}{a_n}) + (a_1 + \ldots a_n)\frac{1}{a_{n+1}} + 1$$

If we look term for term at

$$a_{n+1}(\frac{1}{a_1} + \cdots + \frac{1}{a_n}) + (a_1 + \ldots a_n)\frac{1}{a_{n+1}}$$

We have

$$\sum_{i=1}^{n} \frac{a_{n+1}}{a_i} + \frac{a_i}{a_{n+1}} = \sum_{i=1}^{n} \frac{a_i^2 + a_{n+1}^2}{a_i a_{n+1}}$$

now consider

$$2a_i a_{n+1} < 2a_i a_{n+1} + a_i^2 + a_{n+1}^2 = (a_i + a_{n+1})^2$$

And by the cauchy shwartz ineq:

$$\leq a_i^2 + a_{n+1}^2$$

Therefore

$$1 = \frac{a_i^2 + a_{n+1}^2}{a_i^2 + a_{n+1}^2} < \frac{a_i^2 + a_{n+1}^2}{2a_i a_{n+1}}$$

And so

$$\sum_{i=1}^{n} \frac{a_i^2 + a_{n+1}^2}{a_i a_{n+1}} > \sum_{i=1}^{n} 2 = 2n$$

So we have

$$(a_1 + \cdots + a_n + a_{n+1})(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}}) \geq n^2 + 2n + 1 = (n+1)^2$$

Which is the desired result

2.

a. Using the two number form of the AM-GM inequality we have

$$\frac{a_1+a_2+a_3+a_4}{4} = \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2} \geq \sqrt{\frac{a_1+a_2}{2}\frac{a_3+a_4}{2}}$$

We can use the inequality again since $ab \leq cd \Rightarrow \sqrt{ab} \leq \sqrt{cd}$ for any $a, b, c, d \geq 0$.

$$\geq \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} = \sqrt[4]{a_1a_2a_3a_4}$$

b. For three numbers a_1, a_2, a_3 we have the following from the AM-GM result for four numbers:

$$\frac{a_1+a_2+a_3+\frac{a_1+a_2+a_3}{3}}{4} \geq \sqrt[4]{a_1a_2a_3}\sqrt[4]{\frac{a_1+a_2+a_3}{3}}$$

We observe that

$$\frac{a_1+a_2+a_3+\frac{a_1+a_2+a_3}{3}}{4} = \frac{3a_1+3a_2+3a_3+a_1+a_2+a_3}{12} = \frac{a_1+a_2+a_3}{3}$$

Therefore if we divide $\sqrt[4]{\frac{a_1+a_2+a_3}{3}}$ on both sides of our AM-GM inequality we get

$$\left(\frac{a_1+a_2+a_3}{3}\right)^{3/4} \geq \sqrt[4]{a_1a_2a_3} = (\sqrt[3]{a_1a_2a_3})^{3/4}$$

If we take both sides to the $4/3$ power we get the desired result

c. We can prove the AM-GM in general with a modified induction. We will prove that the n case implies the $2n$ case and then prove that the n case implies the $n-1$ case. This would Imply the AM-GM for all n since we prove it for every power of two, and then prove it for all numbers less than a power of two. The base case is the $n = 2$ which was proven in class.
To prove the $2n$ case, given

$$\{a_1, a_2 \ldots a_n, a_{n+1} \ldots a_{2n}\}$$

We will define A as the Arithmetic mean and G the Geometric mean of $\{a_1 \ldots a_n\}$ and A' as the Arithmetic mean and G' the Geometric mean of $\{a_{n+1} \ldots a_{2n}\}$
Using the AM-GM for the $n = 2$ case we have

$$\frac{A+A'}{2} \geq \sqrt{A}\sqrt{A'}$$

By the inductive hypothesis we have $G \leq A$, $G' \leq A'$ so

$$\sqrt{A}\sqrt{A'} \geq \sqrt{G}\sqrt{G'}$$

So

$$\frac{A+A'}{2} \geq \sqrt{G}\sqrt{G'}$$

We observe that

$$\sqrt{G}\sqrt{G'} = \sqrt{(a_1 \dots a_n)^{1/n}}\sqrt{(a_{n+1} \dots a_{2n})^{1/n}} = (a_1 \cdot a_2 \dots a_{2n})^{1/2n} = G_{2n}$$

Where G_{2n} is the Geometric mean of $\{a_1 \dots a_{2n}\}$.
We also have that

$$\frac{A + A'}{2} = \frac{a_1 + \dots a_n + a_{n+1} + \dots a_{2n}}{2n} = A_{2n}$$

where A_{2n} is the Arithmetic mean of $\{a_1 \dots a_{2n}\}$.
So we have

$$A_{2n} \geq G_{2n}$$

Now to prove the $n-1$ case.
for a given $\{a_1, \dots a_{n-1}\}$ we define $a_n = A_{n-1}$ where A_{n-1} is the Arithmetic mean of the set. By our inductive hypothesis we have

$$A_n \geq G_n$$

Where A_n and G_n are the Arithmetic and Geometric means of $\{a_1 \dots a_n\}$
We have that

$$A_n = \frac{a_1 + \dots a_{n-1}}{n} + \frac{a_1 + \dots a_{n-1}}{n(n-1)} = \frac{(n-1)(a_1 + \dots a_{n-1}) + a_1 + \dots a_{n-1}}{n(n-1)} = A_{n-1}$$

We also have

$$G_n = \sqrt[n]{a_1 \dots a_{n-1}}\,\sqrt[n]{A_{n-1}} = G_{n-1}^{(n-1)/n} A_{n-1}^{1/n}$$

If we substitute these equations into our original inequality we have

$$A_{n-1} \geq G_{n-1}^{(n-1)/n} A_{n-1}^{1/n}$$

divident the $A_{n-1}^{1/n}$ on both sides yields

$$A_{n-1}^{(n-1)/n} \geq G_{n-1}^{(n-1)/n}$$

And so

$$A_{n-1} \geq G_{n-1}$$

3. We can prove this by induction:
Base Case $(n = 1)$ is true since there is only one element in the set $\{\frac{a_1}{b_1}\}$
Now looking at proving

$$\frac{a_1 + \dots a_{n+1}}{b_1 + \dots b_{n+1}} \leq \frac{A_{n+1}}{B_{n+1}}$$

where A_{n+1}, B_{n+1} are the numerator and denominator respectively of the max of $\{\frac{a_1}{b_1}, \dots \frac{a_{n+1}}{b_{n+1}}\}$. We can assume as the inductive hypothesis

$$\frac{a_1 + \dots a_n}{b_1 + \dots b_n} \leq \frac{A_n}{B_n}$$

where A_n, B_n are the numerator and denominator respectively of the max of $\{\frac{a_1}{b_1}, \dots \frac{a_n}{b_n}\}$. This is the case if and only if

$$B_n \sum_{i=1}^{n} a_i \leq A_n \sum_{i=1}^{n} b_i$$

We also know

$$\frac{A_n}{B_n} \leq \frac{A_{n+1}}{B_{n+1}}$$

since $\{\frac{a_1}{b_1}, \dots \frac{a_n}{b_n}\}$ is contained in $\{\frac{a_1}{b_1}, \dots \frac{a_{n+1}}{b_{n+1}}\}$, so

$$\frac{B_{n+1}}{B_n} \leq \frac{A_{n+1}}{A_n}$$

so

$$B_{n+1} \sum_{i=1}^{n} a_i \leq A_{n+1} \sum_{i=1}^{n} b_i$$

similarly we have

$$\frac{A_{n+1}}{B_{n+1}} \geq \frac{a_{n+1}}{b_{n+1}} \Rightarrow A_{n+1} b_{n+1} \geq B_{n+1} a_{n+1}$$

so

$$B_{n+1} \sum_{i=1}^{n} a_i + B_{n+1} a_{n+1} \leq A_{n+1} \sum_{i=1}^{n} b_i + A_{n+1} b_{n+1}$$

$$\Updownarrow$$

$$\frac{a_1 + \dots a_{n+1}}{b_1 + \dots b_{n+1}} \leq \frac{A_{n+1}}{B_{n+1}}$$

This proves that the fraction of the sums is always less than or equal to the max of the ratios. To prove that it is always greater than or equal to the min of the ratios, we just swap the a_is and b_is. Then the max of the ratios of these terms would be the min of the ratio of our original ratios, so we have

$$\frac{b_1 + \dots b_n}{a_1 + \dots a_n} \leq \frac{\beta_n}{\alpha_n}$$

So

$$\frac{a_1 + \dots a_n}{b_1 + \dots b_n} \geq \frac{\alpha_n}{\beta_n}$$

Where $\frac{\alpha_n}{\beta_n}$ is the min of the ratios

4. We can prove this by induction:
Base Case: for $n = 2$ we have:

$$(1 + a_1)(1 + a_2) = 1 + a_1 + a_2 + a_1 a_2$$

And since a_1, a_2 are the same sign and nonzero, we know $a_1 a_2 > 0$ so the assersion holds.
Now we have

$$(1 + a_1)(1 + a_2) \dots (1 + a_{n+1}) = (1 + a_1) \dots (1 + a_n) + (1 + a_1) \dots (1 + a_n) a_{n+1}$$

By the inductive hypothesis we know that

$$> 1 + 2 + \dots a_n + (1 + a_1) \dots (1 + a_n) a_{n+1}$$

Now we go through cases, if all the a_i terms are positive, then $1 + a_i > 1$ so

$$(1 + a_1) \dots (1 + a_n) a_{n+1} > a_{n+1}$$

so we have the desired result

$$(1 + a_1)(1 + a_2) \dots (1 + a_{n+1}) > 1 + 2 + \dots a_n + a_{n+1}$$

And the other case is a_i terms are negative but > -1 then $1 > a_i + 1 > 0$. Therefore

$$0 < (1 + a_1) \dots (1 + a_n) < 1$$

so

$$a_{n+1}(1 + a_1) \dots (1 + a_n) > a_{n+1}$$

so we have the desired result

$$(1 + a_1)(1 + a_2) \dots (1 + a_{n+1}) > 1 + 2 + \dots a_n + a_{n+1}$$

5. We can use the AM-GM inequality for $\{1, \dots n\}$.
We have

$$AM = \frac{1 + 2 + \dots n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

and

$$GM = \sqrt[n]{1 \cdot 2 \dots n} = \sqrt[n]{n!}$$

And since the terms of $\{1, \dots n\}$ are not equal for $n > 1$, we know that the AM-GM is a strict inequality:

$$\frac{n+1}{2} > \sqrt[n]{n!}$$

$$\left(\frac{n+1}{2}\right)^n > \sqrt{n!}$$