UNIVERSITY OF CHEMISTRY AND TECHNOLOGY, PRAGUE
Faculty of Chemical Engineering
Department of Chemical Engineering

Heat transfer

Lectures and seminars

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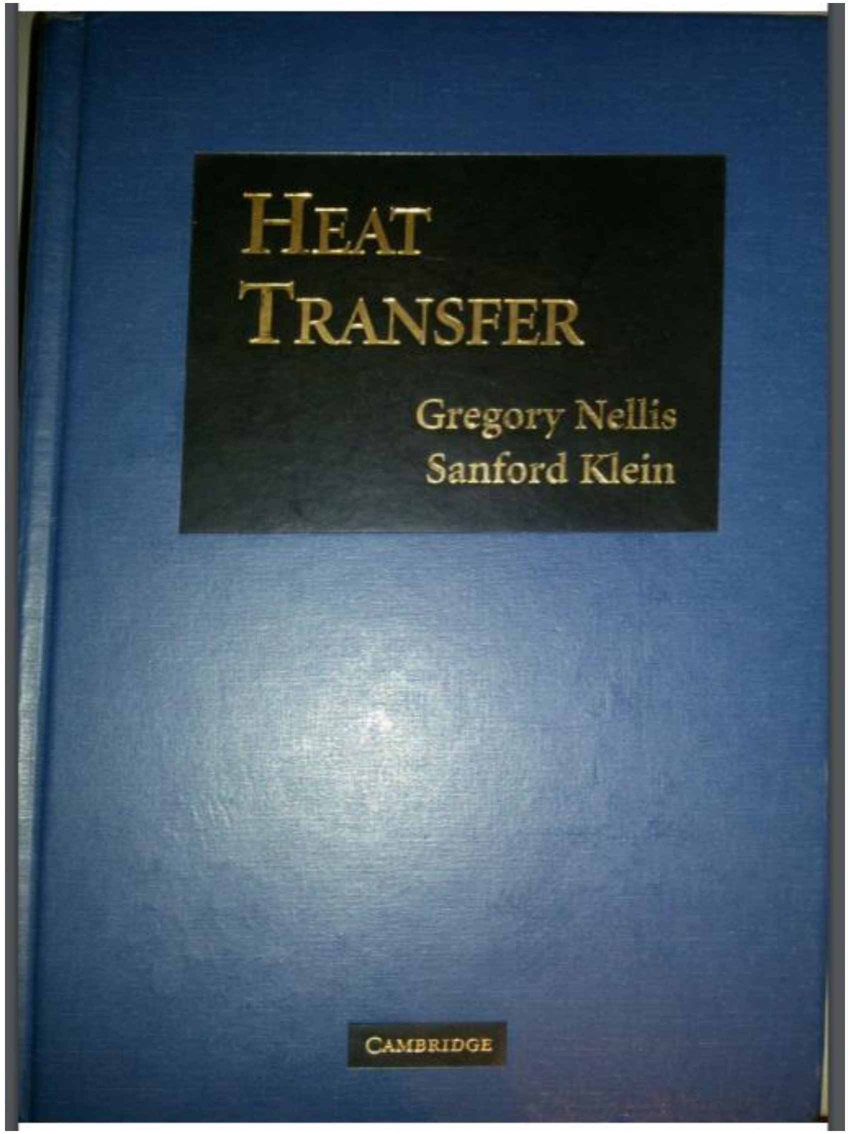
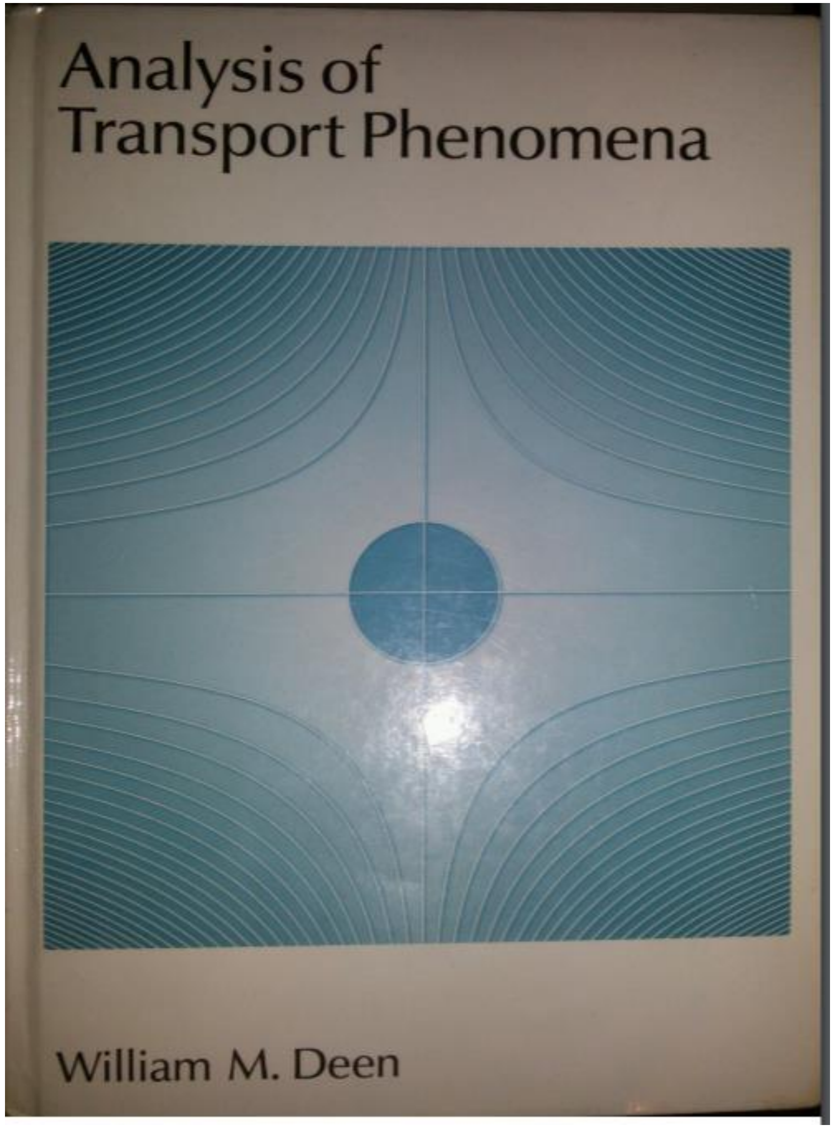
Recommended books





Problems will be solved in Maple and COMSOL

Lecture 1

- Scalar, vector and tensor quantities
- Scalar product, vector product, vector differential operators, material derivatives, volumetric and surface integrals, mass balance in general volume…
- Transformation of equations into dimensionless shape, scaling of quantities Characteristic heat conduction time

Scalar, vector and tensor

$$\boldsymbol{a}$$

Scalar:
An element of a field, usually a real **number**
- It is not spatially oriented
- In the selected space (time) it can be expressed by one value
- Typical scalars: temperature, pressure, concentration

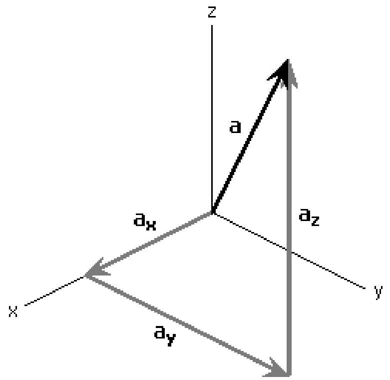
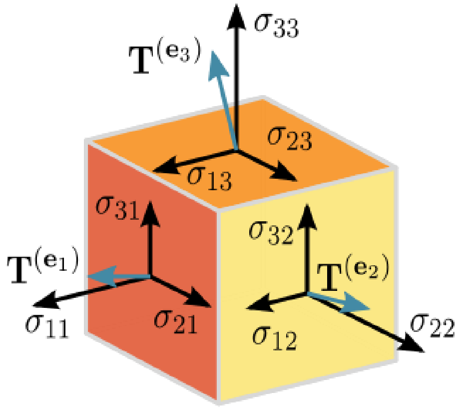


Figure 1

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Vector: A vector has **magnitude** (size) and **direction**
- has a direction
- in 3D space it can be characterized by three values
- typical vectors: velocity, gradient of pressure, gradient concentration



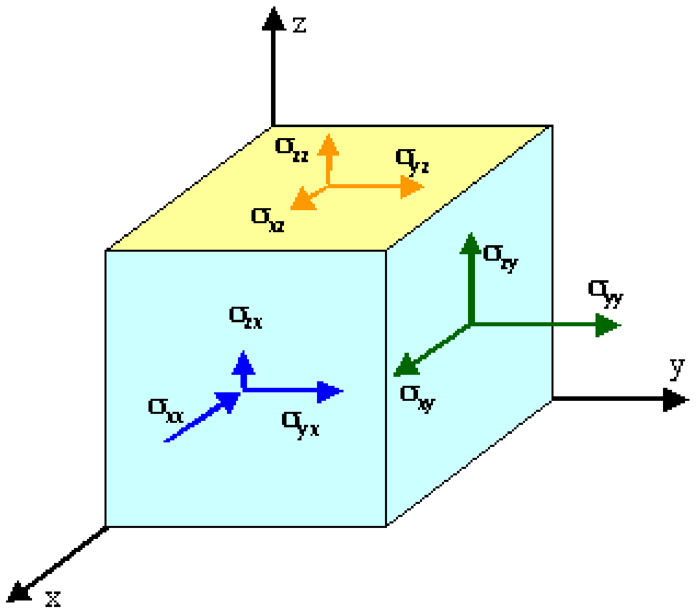
$$\boldsymbol{a} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}$$

Tensor of 2nd order: geometric objects that describe linear relations between geometric vectors, scalars and other tensors.
- has a direction
- In 3D space it can be characterized by nine values
- Typical tensor: velocity deformation tensor in the liquid

Example: cube of material subjected to an arbitrary load → measure the stress on it in various directions ↔ measurements form a second rank tensor; the stress tensor

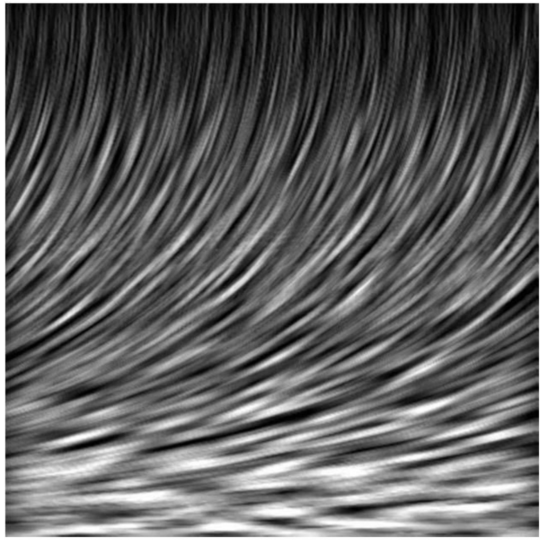
$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

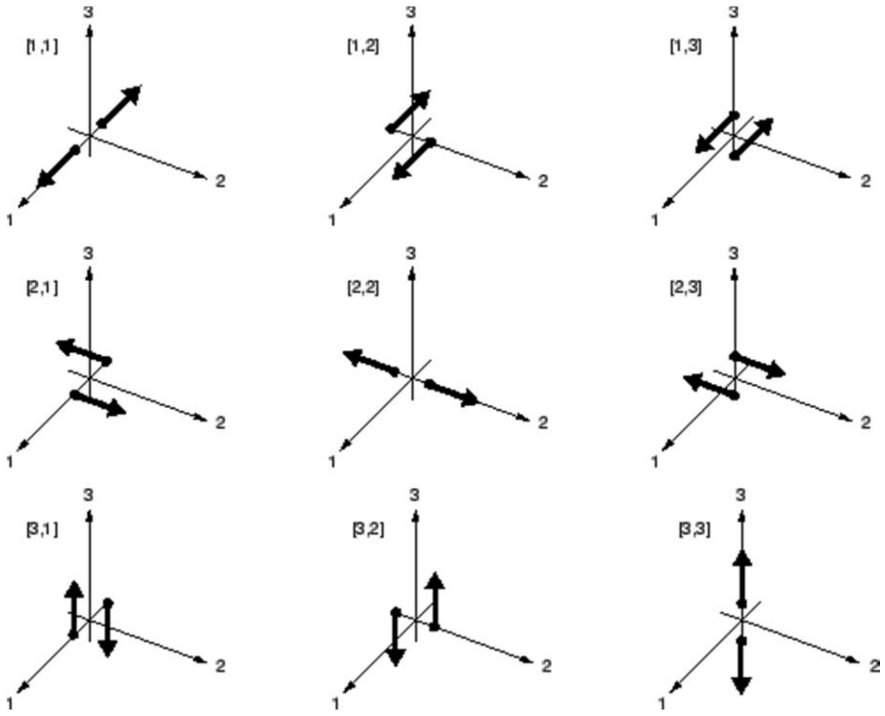
Tensors



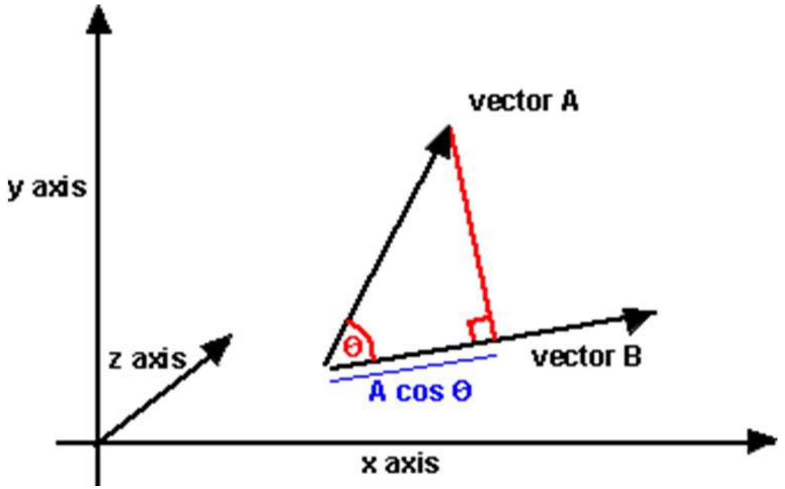
Tensors express tensions in the liquid or solid media.

The tensor can describe **what changes of characteristic property** (the change in velocity in liquids or the change of shape in solid materials) in the direction perpendicular to some surface which are caused (initiated) by applying tangential or normal forces to that area.





Scalar product

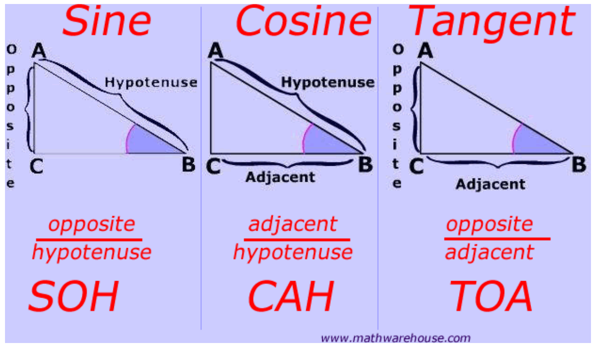


- **The product of two vectors = scalar**

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z = \sum_i a_i b_i$$

The scalar product is the product of the vector size ***b*** projected into vector ***a*** and vector size ***a*** and / or vice versa.

- **Properties of the scalar product of vectors**



-Commutative

$$\boldsymbol{a}\cdot\boldsymbol{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \sum_i a_i b_i = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \cdot \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \boldsymbol{b}\cdot\boldsymbol{a}$$

-Distributive

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \left(\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \right) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \begin{bmatrix} b_x + c_x \\ b_y + c_y \\ b_z + c_z \end{bmatrix} = \sum_i a_i (b_i + c_i) = \sum_i a_i b_i + \sum_i a_i c_i = (\boldsymbol{a}\cdot\boldsymbol{b}) + (\boldsymbol{a}\cdot\boldsymbol{c})$$

Scalar product

- Product of vector and tensor = vector

$$\boldsymbol{a}\cdot\boldsymbol{\tau} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} \sum_i a_i \tau_{ix} & \sum_i a_i \tau_{iy} & \sum_i a_i \tau_{iz} \end{bmatrix}$$

$$\boldsymbol{\tau}\cdot\boldsymbol{a} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \cdot \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sum_i \tau_{xi} a_i & \sum_i \tau_{yi} a_i & \sum_i \tau_{zi} a_i \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix}$$

and $\mathbf{x} = (2, 1, 0)$, then

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\cdot 1 - 1\cdot 1 + 0\cdot 2 \\ 2\cdot 0 - 1\cdot 3 + 0\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \end{aligned}$$

- The property of the scalar product of the vector and the tensor

Associative

$$(\boldsymbol{a}\cdot\boldsymbol{\tau})\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\boldsymbol{\tau}\cdot\boldsymbol{b})$$

Proof will be done by students separately!

Vector differential operators

- Gradient operator

The **gradient** is a multi-variable generalization of the derivative.
While a **derivative** can be defined as a **function of a single variable**, for functions of several variables, the gradient takes its place. The gradient is a vector-valued function, as opposed to a **derivative**, which is **scalar-valued**.
The given vector must be differential to apply the gradient phenomenon.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

- Gradient of scalar field = vector field:

whose magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field. If the vector is resolved, its components **represent the rate of change of the scalar field with respect to each directional component.**

$$\nabla a = \operatorname{grad} a = \begin{bmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \end{bmatrix}$$

- Gradient of vector field = tensor

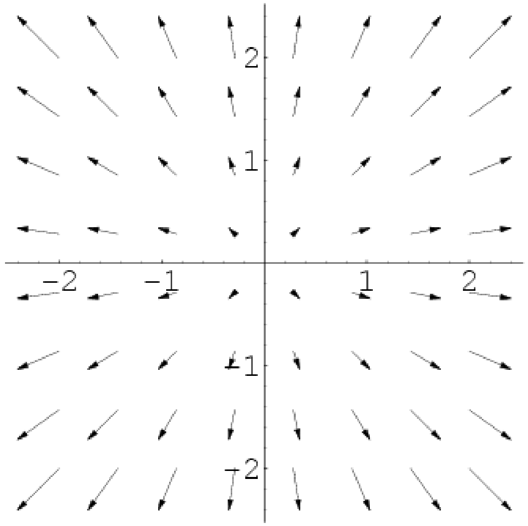
$$\nabla \boldsymbol{a} = \begin{bmatrix} \frac{\partial a_x}{\partial x} & \frac{\partial a_y}{\partial x} & \frac{\partial a_z}{\partial x} \\ \frac{\partial a_x}{\partial y} & \frac{\partial a_y}{\partial y} & \frac{\partial a_z}{\partial y} \\ \frac{\partial a_x}{\partial z} & \frac{\partial a_y}{\partial z} & \frac{\partial a_z}{\partial z} \end{bmatrix}$$

Divergence of a Vector Field

The divergence represents the **volume density of the outward flux of a vector field from an infinitesimal volume around a given point.**

Example:

- Consider air as it is heated.
- The velocity of the air at each point defines a vector field F.
- When the air is heated in a region, it expands in all directions, and thus the velocity field F points outward from that region.
- This expansion of fluid flowing with velocity field F is captured by the divergence of F
- The divergence of the velocity field in that region should have a positive value.



Divergence of a Vector Field

The divergence of a vector field F = <P,Q,R> is defined as the partial derivative of P with respect to x plus the partial derivative of Q with respect to y plus the partial derivative of R with respect to z.

$$\boxed{divF = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

$\nabla \cdot A = divA$

In Cartesian

$\nabla \cdot A \equiv \partial Ax/\partial x + \partial Ay/\partial y + \partial Az/\partial z$

In Cylindrical

$\nabla \cdot A \equiv \partial(r \cdot Ay)/(r \cdot \partial r) + \partial A\phi/(r \cdot \partial\phi) + \partial Az/\partial z$

In Spherical

$\nabla \cdot A \equiv \partial(R2 \cdot AR)/(R2 \cdot \partial R) + \partial(A\phi \cdot \sin\theta)/(R \cdot \sin\theta \cdot \partial\theta) + \partial A\phi/(R \cdot \sin\theta \cdot \partial\phi)$

Vector differential operators

- Vector field divergence = scalar

$$\nabla \cdot \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \sum_i \frac{\partial a_i}{\partial i}$$

- Laplace operator = operator divergence

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \sum_i \frac{\partial^2}{\partial i^2}$$

$$\nabla^2 \mathbf{H} = \nabla^2 \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \\ \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} \\ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \end{bmatrix}$$

- Divergence of tensor field = vector

$$\nabla \cdot \tau = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} \sum_i \frac{\partial}{\partial i} \tau_{ix} & \sum_i \frac{\partial}{\partial i} \tau_{iy} & \sum_i \frac{\partial}{\partial i} \tau_{iz} \end{bmatrix}$$

Vector differential operators

Exercise - prove it is true

$$\nabla \cdot \left[\nabla \boldsymbol{a} + (\nabla \boldsymbol{a})^t \right] = \nabla^2 \boldsymbol{a} + \nabla(\nabla \cdot \boldsymbol{a})$$

Material derivative

The material derivative computes the **time rate of change of any quantity** such as heat (temperature) or momentum (velocity, (which gives acceleration for a portion of a material moving with a velocity v). If the material is a fluid, then the movement is simply the flow field.

The material derivative indicates the **rate of change of a spatial variable** as it is perceived by an observer moving along with the fluid.

Using a differential function (scalar or vector) we can monitor the approximate increment of this function around the selected point. For calculating the differential function it is necessary to know the derivative (tangents) of this function at the selected point for all independent variables (spatial coordinates and time).

$$da = \frac{\partial a}{\partial t}dt + \frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz$$

da: differential change or total differential of, a of function of several variables (t, x, y, z)

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial x}v_x + \frac{\partial a}{\partial y}v_y + \frac{\partial a}{\partial z}v_z = \frac{\partial a}{\partial t} + v \cdot \nabla a$$

By dividing the differential by time increment, we obtain a material derivative of the function where v is the flow velocity vector

Local part

Convection part-
Non-zero even in steady state

Integral transformation

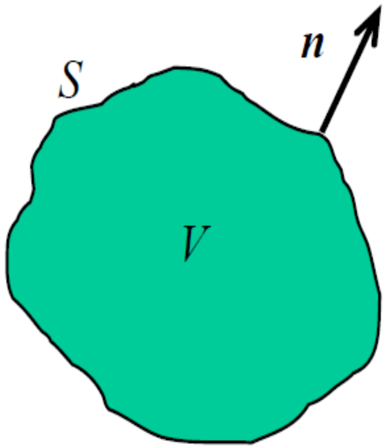
Express **conservation laws for transportable quantities** in the control volume V, which is surrounded by the control surface *S*

- The control volume still contains the same amount of fluid and at the fluid flow is deformed
- *n* is a normal vector perpendicular to the *dS* area, $\|n\| = \vec{1}$, $n_x^2 + n_y^2 = 1$ ($\vec{n}_x = (1,0,0)$ and $\vec{n}_y = (0,1,0)$ in 2D)
- The amount of physical quantity (property) that increases / decreases (accumulates) in the control volume *V* is equal to the amount (measured) of the quantity (property) that enter / exit through the S-area of the control volume

For scalar type variables, vector (tensor):

$$\int_V \nabla a dV = \int_S \boldsymbol{n} a dS$$

$$\int_V \nabla \cdot \boldsymbol{a} dV = \int_S \boldsymbol{n} \cdot \boldsymbol{a} dS$$



E – a simple solid region with boundary surface...

S – given the positive (outward) orientation, and

$\mathbf{F}$ – a vector field whose components have continuous derivatives in an open region of $\mathbf{R}^3$ containing E

The Divergence (or Gauss's) Theorem:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Continuity equation – mass conservation law

Continuity equation in physics is an equation that describes the transport of some quantity

- The flow of mass over the control volume is equal to mass accumulation in the control volume.

$$\dot{m} = -\frac{dm}{dt}$$

- Mass flow across the system boundary can be written as product of density, velocity and area boundary system.

$$\int_S \rho \boldsymbol{n} \cdot \boldsymbol{v} dS = -\frac{\partial}{\partial t}\int_V \rho dV$$

- The surface integral can be converted to a volume integral using integral transformation.

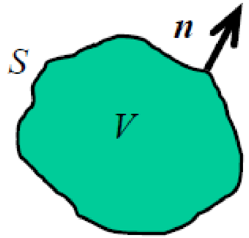
$$\int_V \rho \nabla \cdot \boldsymbol{v} dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

- Equations of continuity in differential form

$$\rho \nabla \cdot \boldsymbol{v} = -\frac{\partial \rho}{\partial t}$$

- For constant density fluids, the equation can be further simplified

$$\nabla \cdot \boldsymbol{v} = 0$$



Transformation of equations into dimensionless form

- The method serves to reduce the number of parameters
- The following equations can be used for any system of physical units (SI or other)

Procedure:

1-identify all dependent, independent variables: - Dependent (temperature, velocity, pressure)
- Independent (time, space coordinates)

2- For each variable we choose a characteristic magnitude that is the same dimension as this variable

3- We introduce dimensionless variables by dividing the dimensional by characteristic variables.

$$\tilde{X} = \frac{\overset{\nearrow \text{ dimensional variable}}{X}}{\underset{\searrow \text{ characteristic variable}}{X_0}}$$

4- We substitute the dimensionless variables in the equation and we divide the equation by the constant before a selected term of equation

5- We obtain equations in dimensionless shapes. Coefficients before the members are also dimensionless – Dimensionless criteria

Example: Transform the following equation into dimensionless number:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

Independent variables: $\boldsymbol{t, x, y}$ (time, coordinates x, y)
Dependent variables: $\boldsymbol{v_x}$, $\boldsymbol{v_y}$, $\boldsymbol{p}$ ($\boldsymbol{x, y,}$ component of vector velocity, pressure)
Parameters (constants): η, ρ (dynamic viscosity, density)

Introducing dimensionless variables

$$\tilde{t}=\frac{t}{t_0} \qquad \tilde{x}=\frac{x}{x_0} \qquad \tilde{y}=\frac{y}{y_0} \qquad \tilde{v}_x=\frac{v_x}{v_0} \qquad \tilde{v}_y=\frac{v_y}{v_0} \qquad \tilde{p}=\frac{p}{p_0}$$

We derive, how derivatives of dimensional variables depend on dimensionless derivatives

$$\frac{\partial v_x}{\partial t}=\frac{\partial v_x}{\partial \tilde{v}_x}\frac{\partial \tilde{v}_x}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t}=\frac{v_0}{t_0}\frac{\partial \tilde{v}_x}{\partial \tilde{t}}$$

$$\frac{\partial v_x}{\partial x}=\frac{\partial v_x}{\partial \tilde{v}_x}\frac{\partial \tilde{v}_x}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x}=\frac{v_0}{x_0}\frac{\partial \tilde{v}_x}{\partial \tilde{x}}, \text{ podobně } \frac{\partial v_x}{\partial y}=\frac{v_0}{y_0}\frac{\partial \tilde{v}_x}{\partial \tilde{y}}$$

$$\frac{\partial^2 v_x}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}\right)=\frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x}\left(\frac{v_0}{x_0}\frac{\partial \tilde{v}_x}{\partial \tilde{x}}\right)=\frac{\partial}{\partial \tilde{x}}\frac{1}{x_0}\left(\frac{v_0}{x_0}\frac{\partial \tilde{v}_x}{\partial \tilde{x}}\right)=\frac{v_0}{x_0^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}$$

$$\frac{\partial^2 v_x}{\partial y^2}=\frac{v_0}{y_0^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\frac{\partial p}{\partial x}=\frac{p_0}{x_0}\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

And we substitute them in the original equation:

$$\rho\left(\frac{v_0}{t_0}\frac{\partial\tilde{v}_x}{\partial\tilde{t}}+\tilde{v}_x v_0\frac{v_0}{x_0}\frac{\partial\tilde{v}_x}{\partial\tilde{x}}+\tilde{v}_y v_0\frac{v_0}{y_0}\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right)=-\frac{p_0}{x_0}\frac{\partial\tilde{p}}{\partial\tilde{x}}+\eta\left(\frac{v_0}{x_0^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}+\frac{v_0}{y_0^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}\right)$$

So far we used general notation for scaling factors. At this point, let's define some of them:

v_0 = U … average flow velocity
x0 = y0 = d … tube (pipe) diameter
t0 = d/U … convective time (length/velocity)

$$\frac{\rho U^2}{d}\left(\frac{\partial\tilde{v}_x}{\partial\tilde{t}}+\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right)=-\frac{p_0}{d}\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{\eta U}{d^2}\left(\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}\right)$$

We divide the equation by a factor $\frac{\eta U}{d^2}$

$$\boxed{\frac{\rho\, d\, U}{\eta}}\left(\frac{\partial\tilde{v}_x}{\partial\tilde{t}}+\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right)=-\frac{p_0\, d}{\eta\, U}\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}$$

Dimensionless Reynolds criterion Re

dimensionless form of equation

Furthermore, we define characteristic pressure as $p_0 = \frac{\eta U}{d}$. Pa.s.m.s^{-1}.m^{-1}= Pa

Then:

Inertial term

Viscous term

$$\boxed{\mathrm{Re}\left(\frac{\partial \tilde{v}_x}{\partial \tilde{t}} + \tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}}$$

Pressure drop

Fluide flow

Navier-Stokes equation

1 parameter instead of two

Re is the number that tells us whether a flow is turbulent (inertial forces dominate) or not

$$\boxed{\mathrm{Re} = \rho\, d\, U / \eta}$$

We can use scaling argument to get the ratio

ratio of inertial forces to viscous forces

Re<<1 → velocity very low and viscosity high → inertial term close to 0

Re>>1 → viscosity very low → inertial forces high → viscous term can be neglected

Scaling

A special case - dimensionless-characteristic properties, in other words scaling factors, are defined in a way that dimensionless variables (both dependent and independent) and their amount of changes are equal to values in the order of ~ 1.

If $\tilde{f} \sim 1 \qquad \Delta\tilde{f} \sim 1 \qquad \tilde{x} \sim 1 \qquad \Delta\tilde{x} \sim 1$

we obtain

$$\left.\begin{aligned} \frac{\mathrm{d}\tilde{f}}{\mathrm{d}\tilde{x}} &\doteq \frac{\Delta\tilde{f}}{\Delta\tilde{x}} \sim \frac{1}{1} \sim 1 \\ \frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}\tilde{x}^2} &\doteq \frac{\Delta\tilde{f}}{\Delta\tilde{x}^2} \sim \frac{1}{1^2} \sim 1 \end{aligned}\right\}$$

Derivatives also acquire values in order of 1

- In many cases, it is rather difficult to find scaling factors.
- If the equation is well scaled, the values of dimensionless criteria determine the weight of particular terms of the equation. Some terms can thus be neglected.
- Scaling is an important aid (tool) in the derivation of the theoretical criterion equation for calculating Nusselt or Sherwood numbers

Example: Perform scaling of the Fourier equation for heat conduction

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

T… temperature in K
t … time in s
x … spatial coordinate in m
a … thermal diffusivity

First, we transform the equation to dimensionless one:

$$\Theta = \frac{T}{T_0} \quad , \quad \tilde{t} = \frac{t}{t_0} \quad , \quad \tilde{x} = \frac{x}{x_0}$$

$$\frac{\partial T}{\partial t} = \frac{T_0}{t_0}\frac{\partial \Theta}{\partial \tilde{t}} \quad , \quad \frac{\partial^2 T}{\partial x^2} = \frac{T_0}{x_0^2}\frac{\partial^2 \Theta}{\partial \tilde{x}^2}$$

$$\frac{\cancel{T_0}}{t_0}\frac{\partial \Theta}{\partial \tilde{t}} = \frac{a\cancel{T_0}}{x_0^2}\frac{\partial^2 \Theta}{\partial \tilde{x}^2}$$

Now we have to define the scaling factors for t_0 and x_0. Usually the size of the system is known, for example the wall thickness, tube diameter, … For example, for pipeline the characteristic dimension is the diameter *d*. If we set x_0 = d, then it is assured that $\tilde{x} \in <0,1>$ (and that's what we want)

$$\frac{1}{t_0}\frac{\partial \Theta}{\partial \tilde{t}} = \frac{a}{d^2}\frac{\partial^2 \Theta}{\partial \tilde{x}^2} \qquad \Big| * t_0$$

$$\frac{\partial \Theta}{\partial \tilde{t}} = \frac{t_0\, a}{d^2}\frac{\partial^2 \Theta}{\partial \tilde{x}^2}$$

As we assume that $\Theta \sim 1, \Delta\Theta \sim 1$

- We did not have to define scaling of T_0 but it is usually the difference between the maximum and minimum tempereture of the system.

Furthemore, we know that

$$\tilde{x} \sim 1, \Delta\tilde{x} \sim 1 \quad \text{and} \quad \frac{\partial^2\Theta}{\partial\tilde{x}^2} \sim \frac{\Delta\Theta}{(\Delta x)^2} \sim \frac{1}{1^2} \sim 1$$

What remains at this point is to define the scaling for time to hold true:

$$\tilde{t} \sim 1, \Delta\tilde{t} \sim 1 \quad \text{and} \quad \frac{\partial\Theta}{\partial t} \sim \frac{\Delta\Theta}{\Delta t} \sim \frac{1}{1} \sim 1$$

$$\underbrace{\frac{\partial\Theta}{\partial\tilde{t}}}_{\sim 1} = \underset{\substack{\Downarrow \\ \frac{t_0 a}{x_0^2} \sim 1}}{\frac{t_0 a}{x_0^2}} \underbrace{\frac{\partial^2\Theta}{\partial\tilde{x}^2}}_{\sim 1}$$

We identify time scaling $\boxed{t_0 = \frac{d^2}{a}}$

Diffusion/conductive time

$$t_0 = \frac{d^2}{a}$$

- Time during which the heat is transferred at a distance *d* by heat conductivity.
- Time is proportinal to the square of this distance.
- It is rather a rough estimate of the order than its exact value.

Lecture 2

- Heat control, Fourier's law
- Fourier equation, derivation for general control volume
- boundary conditions
- Biot's number
- Steady heat conduction in the thickness variable plate (seminar)
- Steady rods in a circular cross section (seminar)
- Thermal resistance
- Mechanism of heat conduction

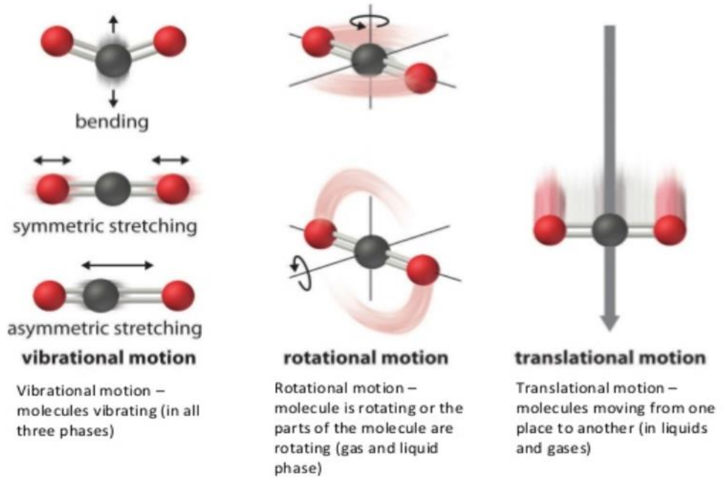
Heat transfer: Basic concepts

Three modes of molecular motion:

Particles/Molecules can:

- **vibrate:** wiggle from a fixed position
- **translate:** move from one location to another
- **rotate:** revolve on an imaginary axis

→ These motions give to the particles/molecules kinetic energy.



- Solids can not move through space. They only vibrate.
- Liquids and gases are free to move around in space. They can have all three modes of motion.

https://chem.libretexts.org

Heat transfer: Basic concepts

- Temperature is a measure of the average amount of kinetic energy possessed by the particles in a sample of matter.
- The more the particles vibrate, translate and rotate, the greater the temperature of the object.

- Heat transfer is a transfer of kinetic energy of molecules

- It is the temperature difference (temperature gradient) between the two neighboring objects that causes this heat transfer.

- Heat flows in direction of decreasing temperatures since higher temperatures are associated with higher molecular energy.

- The heat transfer continues until the two objects have reached thermal equilibrium and are at the same temperature

- Heat transfer can be grouped into three broad categories: conduction, convection, and radiation.

Heat transfer modes

Conduction Convection Radiation

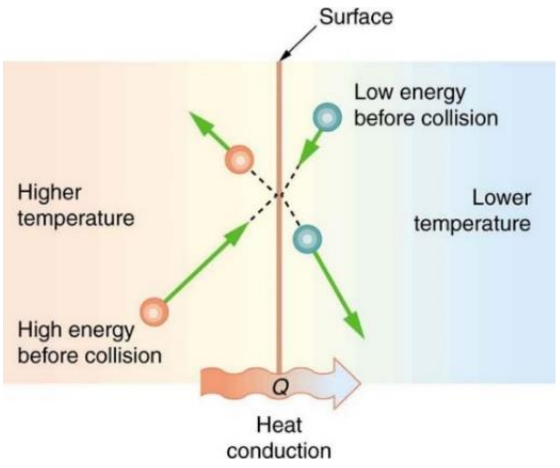
Conduction

Heat transfer: Basic concepts

- Conduction transfers heat via direct molecular collision without any motion of the material as a whole.
- An area of greater kinetic energy will transfer thermal energy to an area with lower kinetic energy.
- Heat transfer by conduction applies in solid, liquid and gaseous materials, in systems at rest as well in motion
- Conduction is the most common form of heat transfer and occurs via physical contact.
- Examples would be to place your hand against a window or place metal into an open flame.

Example:

In commercial heat exchange equipment, heat is conducted through a solid wall (often a tube wall) that separates two fluids having different temperatures.

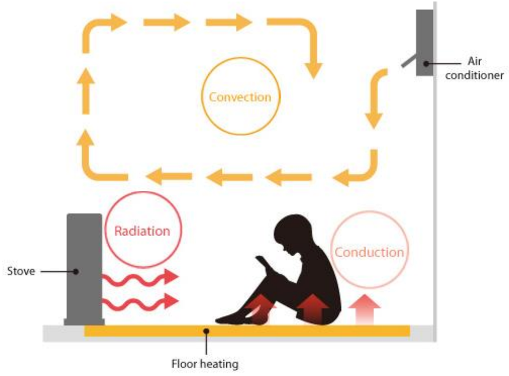


Convection

Convective heat transfer occurs when a gas or liquid flows past a solid surface whose temperature is different from that of the fluid.

Example:

- When a fluid (e.g. air or a liquid) is heated and then travels away from the source of heat, it carries the thermal energy along.
- The fluid above a hot surface expands, becomes less dense, and rises.
- As the immediate hot fluid rises, it pushes denser, colder fluid down causing convection currents which transport energy



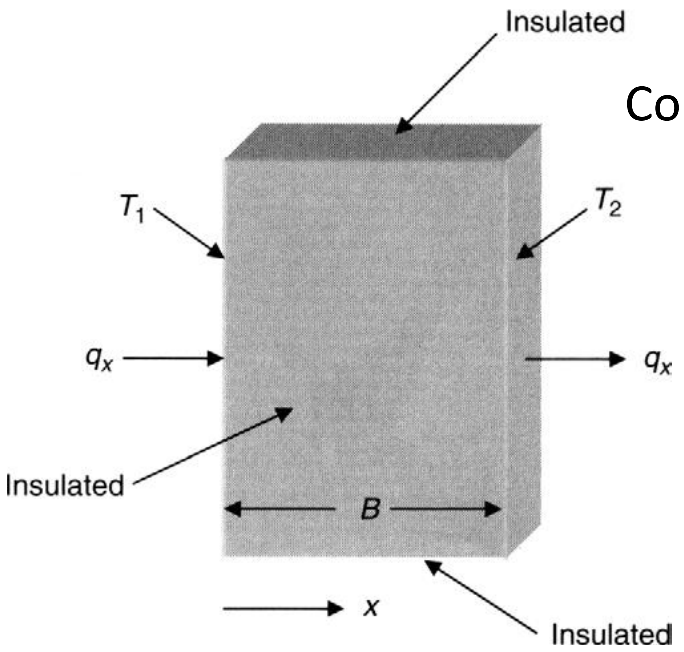
Forced convection ⟷ Natural convection

Forced convection ↓ Fluid motion is caused by an external agent such as a pump or blower.

Natural convection ↓ Fluid motion is the result of buoyancy forces created by temperature differences within the fluid.

https://phys.org

Conductive Heat Transfer



Conductive heat transfer can be expressed with "**Fourier's Law**"

$$q_x = -\lambda \frac{\mathrm{d}T}{\mathrm{d}x}$$

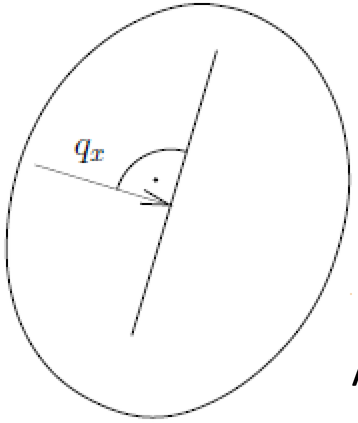
Temperature gradient: Driving force for heat conduction (negative)

Fourier's Law of Heat Conduction

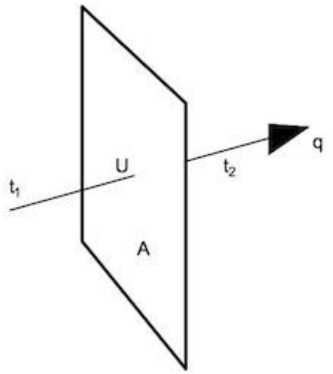
The time rate of heat flow (or heat transfer) is proportional to the temperature gradient. The constant of proportionality is a coefficient of thermal conductivity λ.

q_x ... time rate of heat flow [W m^2] – heat transferred per unit of time through the cross-sectional area
λ ... coefficient of thermal conductivity [W m^{-1} K^{-1}]: depends on the thermodynamic state of the material
T ... temperature [K]
x ... spatial

Fourier's law is valid in this form only if thermal conductivity can be assumed constant.



A - area



Heat flow Q_x [W] in the perpendicular direction to the area A $\dot{Q}_x = q_x\, A$

q_x ... time rate of heat flow [W/m^2]
A ... heat transfer area m^2

In a general case (form) – heat conduction can occur in all directions:

Partial derivatives used: temperature varies in all three directions.

$$\vec{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = -\lambda \begin{bmatrix} \frac{\mathrm{d}T}{\mathrm{d}x} \\ \frac{\mathrm{d}T}{\mathrm{d}y} \\ \frac{\mathrm{d}T}{\mathrm{d}z} \end{bmatrix} = -\lambda \nabla T$$

∇ T ... temperature gradient - difference - over the material (^{o}C, ^{o}F)

λ - coefficient of thermal conductivity – an important property of materials – W m^{-1} K^{-1}
Metals – good thermal conductors - $\lambda \sim 10^1$-10^2 W m^{-1} K^{-1}
Thermal insulators (cork, foam plastic, cotton) – 10^{-2} Wm^{-1} K^{-1}

bricks $\lambda \sim 1$ W m^{-1} K^{-1}
water $\lambda \sim 0.6$ W m^{-1} K^{-1}
Air $\lambda \sim 0.025$ W m^{-1} K^{-1}

Heat flow $\dot{Q} = \frac{\mathrm{d}Q}{\mathrm{d}t}$ [W]

From the first law of thermodynamics, it follows that for isobaric system (constant pressure) performing only volume work the change of enthalpy is equal to the heat exchanged between the system and its surrounding

$$\mathrm{d}Q = \mathrm{d}H$$
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}t}$$

We will express the change of enthalpy of the system using specific enthalpy:

$$\mathrm{d}H = \mathrm{d}(mh) = \mathrm{d}(\rho V h)$$

h … specific enthalpy [J kg^{-1}]
V … system volume [m^3]
ρ … density [kg m^{-3}]

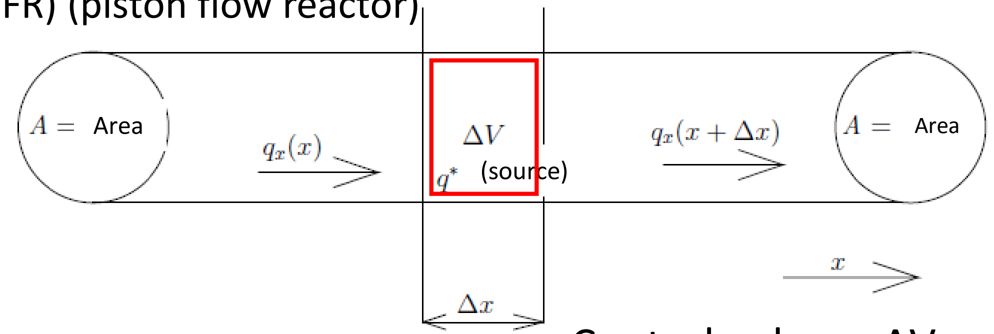
In the case when the system volume as well as density (or mass) are constant, then

$$\begin{aligned} \mathrm{d}H &= V\rho\,\mathrm{d}h = V\rho\,c_\mathrm{p}\,\mathrm{d}T \\ c_\mathrm{p} &\equiv \frac{\mathrm{d}h}{\mathrm{d}T} \Rightarrow \mathrm{d}h = c_\mathrm{p}\,\mathrm{d}T \end{aligned}$$

Cp … specific heat capacity – an important property of materials [J kg^{-1} K^{-1}]
(the amount of heat required to increase the temperature of 1 kg of a material by 1 K (1°C)).

Non-stationary heat conduction in one dimensional (1D) system

Example - Plug flow reactor model (PFR) (piston flow reactor)



Control volume ΔV

q* - volume source of heat [W m^{-3}] (Joule heat (electric current through conductor), reaction heat (e.g. nuclear reaction)…)

Heat energy balance in the element ΔV

INPUT + SOURCE = OUTPUT + ACCUMULATION

$$q_x(x)A + q^*\Delta V = q_x(x+\Delta x)A + \frac{\mathrm{d}H}{\mathrm{d}t}$$

$$\Delta x \to 0$$

$$\cancel{q_x(x)A} + q^*\cancel{\Delta V} = \cancel{q_x(x)A} + \frac{\partial q_x}{\partial x}\cancel{\Delta x A} + \cancel{\Delta V}\rho\, c_\mathrm{p}\frac{\partial T}{\partial t} \qquad \Big| -q_x(x)A;\ /\Delta V$$

$$q^* = \frac{\partial q_x}{\partial x} + \rho\, c_\mathrm{p}\frac{\partial T}{\partial t} \qquad q_x = -\lambda\frac{\partial T}{\partial x}$$

$$q^* = \frac{\partial}{\partial x}(-\lambda\frac{\partial T}{\partial x}) + \rho\, c_\mathrm{p}\frac{\partial T}{\partial t} \qquad [\lambda]$$

$$q^* = -\lambda\frac{\partial^2 T}{\partial x^2} + \rho\, c_\mathrm{p}\frac{\partial T}{\partial t} \qquad \Big| +\lambda\frac{\partial^2 T}{\partial x^2}$$

$$q^* + \lambda\frac{\partial^2 T}{\partial x^2} = \rho\, c_\mathrm{p}\frac{\partial T}{\partial t} \qquad \Big| /\rho\, c_\mathrm{p}$$

$$\frac{q^*}{\rho\, c_\mathrm{p}} + \boxed{\frac{\lambda}{\rho\, c_\mathrm{p}}}\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

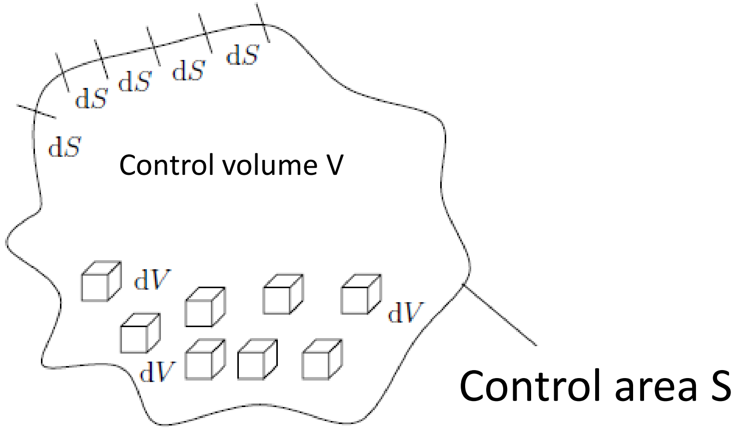
thermal diffusivity [m^{2}s^{-1}] $\quad a = \frac{\lambda}{\rho\, c_\mathrm{p}}\ [\mathrm{m^2\ s^{-1}}]$

$$\frac{q^*}{\rho\, c_\mathrm{p}} + a\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Fourier equation

$$\boxed{\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + \frac{q^*}{\rho\, c_\mathrm{p}}}$$

Derivation of the Fourier equation for a general volume element



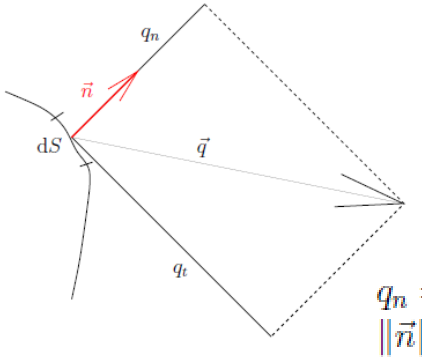
Balance of thermal energy in infinitesimal element

INPUT + SOURCE = OUTPUT + ACCUMULATION

$\underbrace{\text{INPUT} - \text{OUTPUT}}_{\text{Total heat transfer across the boundaries of the system}} + \text{SOURCE} = \text{ACCUMULATION}$

$$\mathrm{d}\dot{Q} = q_n \, \mathrm{d}S = -\vec{n} \cdot \vec{q} \, \mathrm{d}S$$

Total heat transfer across the boundaries of the system



q_n ...the normal component of the heat flow intensity across the boundaries of the system

$$q_n = \vec{n} \cdot \vec{q}$$
$$\|\vec{n}\| = 1 \quad \text{Unit vector}$$

Accumulation in the volume element

$$\frac{\mathrm{d}(\mathrm{d}H)}{\mathrm{d}t} = \frac{\mathrm{d}(\rho\,\mathrm{d}V\,h)}{\mathrm{d}t} = \frac{d(\rho\,\mathrm{d}V\,c_\mathrm{p}(T - T_{ref}))}{\mathrm{d}t} = \rho\,c_\mathrm{p}\,\mathrm{d}V\frac{\partial T}{\partial t}$$

$$c_\mathrm{p} = \left(\frac{\mathrm{d}h}{\mathrm{d}T}\right)_p \Rightarrow h = \int\limits_{T_{ref}}^{T} c_\mathrm{p}\,\mathrm{d}T = c_\mathrm{p}(T - T_{ref})$$

We assume ρ and Cp are constant

Balance:

$$\int\limits_S -\vec{n}\cdot\vec{q}\,\mathrm{d}S + \int\limits_V q^*\,\mathrm{d}V = \int\limits_V \rho\,c_\mathrm{p}\frac{\partial T}{\partial t}\,\mathrm{d}V$$

q^* - volume source of energy [W m^{-3}]

Gauss transform:

$$\int\limits_V -\nabla\cdot\vec{q}\,\mathrm{d}V + \int\limits_V q^*\,\mathrm{d}V = \int\limits_V \rho\,c_\mathrm{p}\frac{\partial T}{\partial t}\,\mathrm{d}V$$

The balance must hold also for elementary volume dV, then:

$$-\nabla\cdot\vec{q} + q^* = \rho\,c_\mathrm{p}\frac{\partial T}{\partial t}$$

$$\rho\, c_\mathrm{p} \frac{\partial T}{\partial t} = - \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} -\lambda \frac{\partial T}{\partial x} \\ -\lambda \frac{\partial T}{\partial y} \\ -\lambda \frac{\partial T}{\partial z} \end{bmatrix} + q^*$$

$$\rho\, c_\mathrm{p} \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q^*$$

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho\, c_\mathrm{p}} \nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}$$

Thermal diffusivity: $a \equiv \frac{\lambda}{\rho\, c_\mathrm{p}}\ [\mathrm{m}^2\ \mathrm{s}^{-1}]$

$$\boxed{\frac{\partial T}{\partial t} = a \nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}}$$

General form for all coordinate systems

Fourier equation for heat conduction

Example to solve

$$\nabla \cdot \left[\nabla \boldsymbol{a} + (\nabla \boldsymbol{a})^t \right] = \nabla^2 \boldsymbol{a} + \nabla(\nabla \cdot \boldsymbol{a})$$

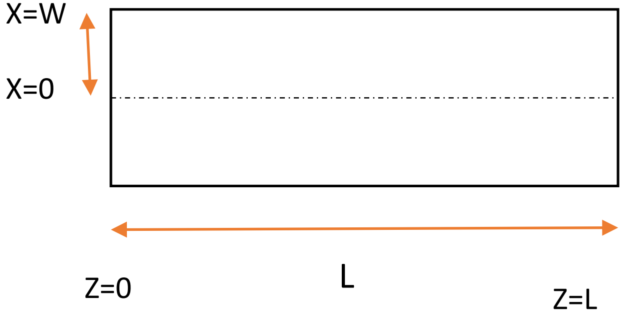
Seminar II to solve

Typical boundary conditions

First-type (Dirichlet) boundary condition

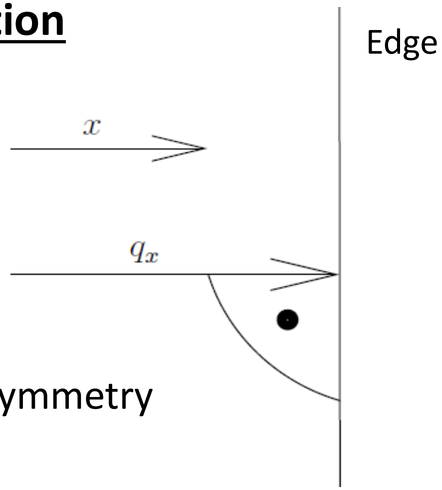
The value of temperature on the edge is defined as:

At Z=0 → $T_{edge} = T_0$



Second-type (Neumann) boundary condition

1) Zero heat flow over the edge



$$q_x = 0$$

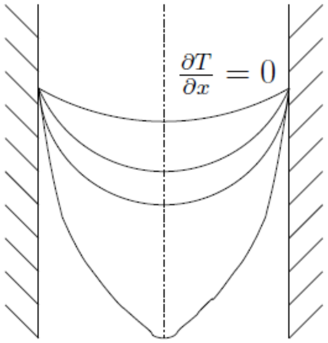
$$-\lambda \frac{dT}{dt} = 0$$

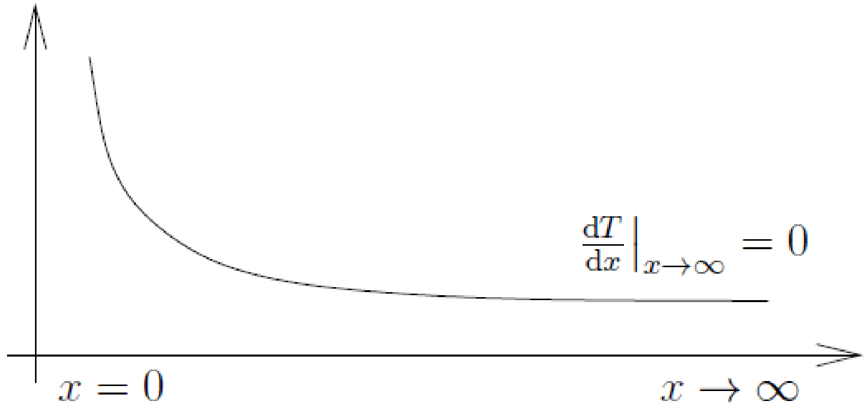
$$\boxed{\frac{dT}{dt} = 0}$$

Zero flux

2) Use of Neumann's boundary conditions at the symmetry axis and in the semi-infinite domains

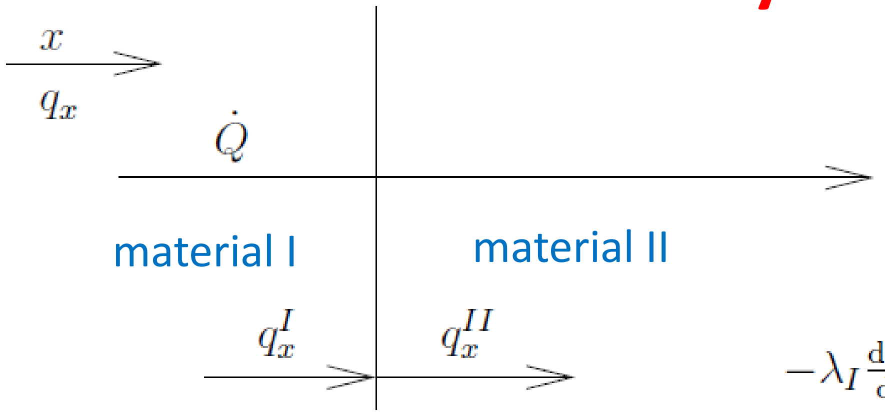
Usage – axis of symmetry





Semi-infinite domains

Condition of continuity of heat flow on the phase interface

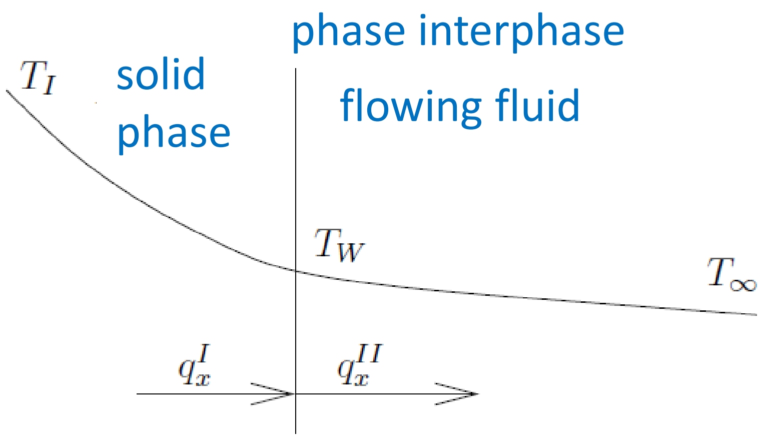


No accumulation of the heat at the interface.
Flux continuity. Used for all system coordinates

$$q_x^I = q_x^{II}$$

$$-\lambda_I \frac{\mathrm{d}T_I}{\mathrm{d}x} = -\lambda_{II} \frac{\mathrm{d}T_{II}}{\mathrm{d}x}$$

<u>Convective boundary condition,</u> condition of continuity of heat flow on the phase-interface (one of the phases conveys heat by conduction and convection = heat transfer),
<u>Robin boundary condition (third type boundary condition)</u>



From courses of CHI

$$q_x^I = q_x^{II}$$

$$-\lambda_I \frac{\mathrm{d}T_I}{\mathrm{d}x}\Big|_W = \alpha(T_W - T_\infty)$$

α - coefficient of heat transfer [$\mathrm{Wm^{-2}K^{-1}}$]

α depends on - geometry
- flow type
- material properties of the fluid

Steady heat conduction

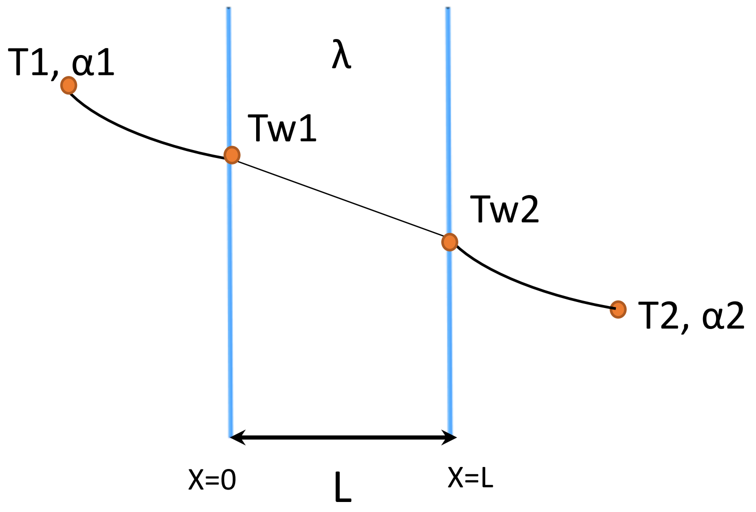
For one-dimensional heat conduction (temperature depending on one variable only), we can devide a basic description of the process.

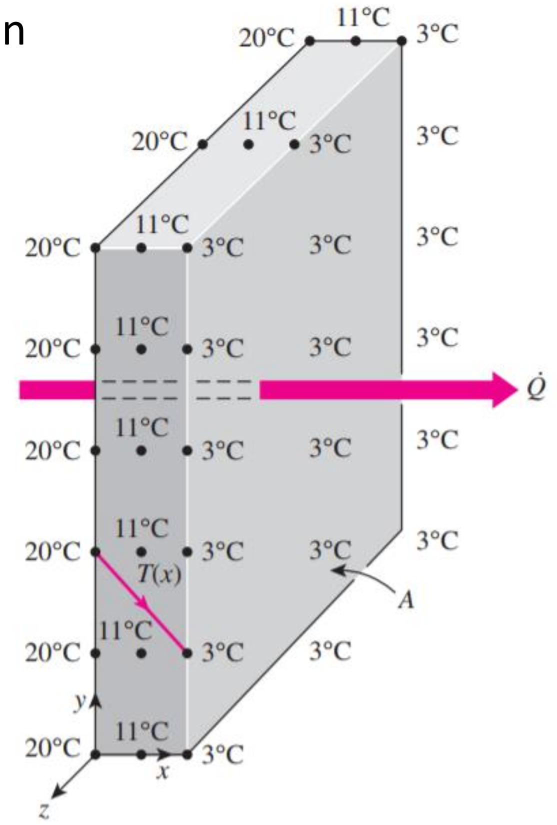
Modelling of the heat transfer through the walls as:
- steady state, - one dimensional

Measure the temperatures of an exposed surface of a plane wall

For 1 D heat transfer through a plane wall of thickness L, specified temperature Boundary conditions are expressed as: T(x=0) = Tw1

T(x=L) = Tw2





Heat transfer through a wall is one-dimensional when the temperature of the wall varies is one direction only

Biot's number

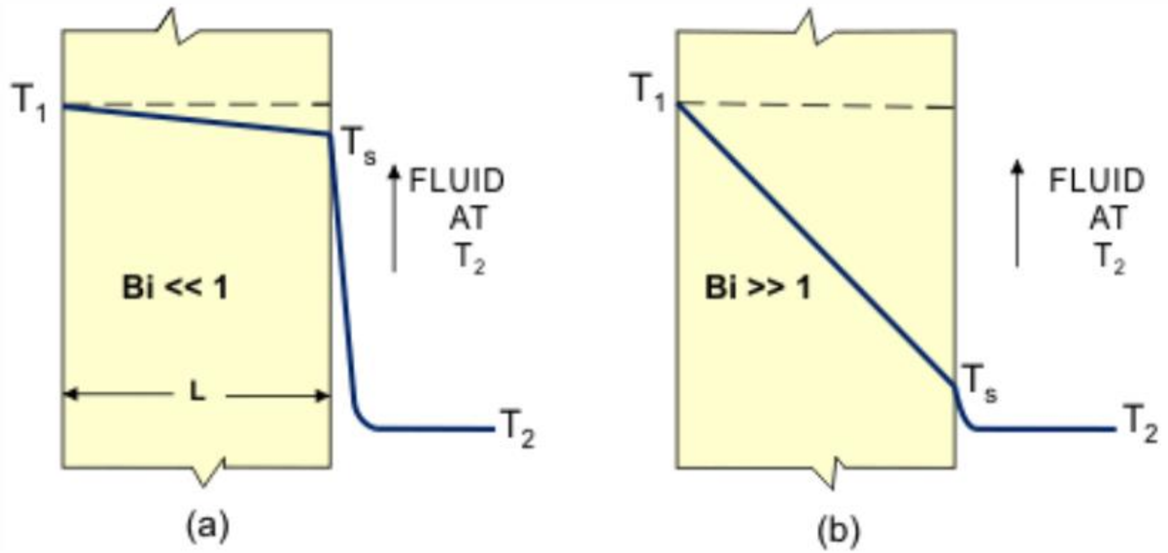
Biot number shows **how convection and conduction heat transfer phenomena are related**.
Small values of this number shows that the conduction is the main heat transfer method, while high values of this number indicates that the convection is the main heat transfer mechanism.

$$\text{Biot number} = \frac{\text{Internal conductive resistance within the body}}{\text{External convective resistance at the surface of the body}}$$

$$\boxed{\begin{aligned} Bi &= (L/\lambda)/(1/\alpha) \\ &= \alpha.L/\lambda \end{aligned}}$$

α: Heat transfer coefficient: Intensity of sharing the heat by transport by convection from the surface to the surrounding
λ: Thermal conductivity: Intensity of heat transport inside the solid by conduction to the surface

Bi > > 1External resistance is very small
Bi < <1: Internal resistance is very small: high conductive
0.1<Bi<100: Internal and external resistances are high



Thermal resistance

The concept of thermal resistance is based on the observation that many diverse physical phenomena can be described by a general rate equation that may be stated as follows:

$$\text{Flow rate} = \frac{\text{Driving force}}{\text{resistance}}$$

The quantity that flows is heat (thermal energy) and the driving force is the temperature difference. The resistance to heat transfer is termed the thermal resistance, and is denoted by $\boldsymbol{R_{th}}$. Thus, the general rate equation may be written as:

$$q = \frac{\Delta T}{R_{th}}$$

In rectangular system:

$$q_x = \frac{\lambda A(T_1 - T_2)}{B} = \frac{\Delta T}{R_{th}} = \frac{T_1 - T_2}{R_{th}}$$

$$R_{th} = \frac{B}{kA}$$

A: cross-sectional area, across which the heat flows
T_1 - T_2: temperature difference
B: thickness of the material.

The thermal resistance concept permits some relatively complex heat-transfer problems to be solved in a very simple manner. The reason is that thermal resistances can be combined in the same way as electrical resistances. Thus, for resistances in series, the total resistance is the sum of the individual resistances:

Resistance in series: $R_{Tot} = \sum_i R_i$ Resistance in parallel: $R_{Tot} = \left(\sum_i 1/R_i\right)^{-1}$

Mechanisms of Heat Conduction

Processes responsible for conduction take place at the molecular or atomic level.

Heat conduction: random molecular motion
Thermal energy is the energy associated with translational, vibrational, and rotational motions of the molecules comprising a substance.

high-energy molecule moves from a high-temperature region of a fluid toward a region of lower temperature (and, hence, lower thermal energy), it carries its thermal energy along with it.

When a high-energy molecule collides with one of lower energy, there is a partial transfer of energy to the lower-energy molecule

→ Molecular motions and interactions is a net transfer of thermal energy from regions of higher temperature to regions of lower temperature.

- In solids: result of vibrations of the solid lattice and of the motion of free electrons in the material.
- In metals: free electrons are plentiful, thermal energy transport by electrons predominates.
- In non-metallic solids: thermal energy transport occurs primarily by lattice vibrations.
- More regular the lattice structure of a material is, the higher its thermal conductivity (e.g. Quartz).
- Materials that are poor electrical conductors may nevertheless be good heat conductors (diamond).
- Insulating materials, both natural and man-made, owe their effectiveness to air or other gases trapped in small compartments → relatively low thermal conductivity of air (and other gases), thereby imparts a low effective thermal conductivity to the material as a whole.

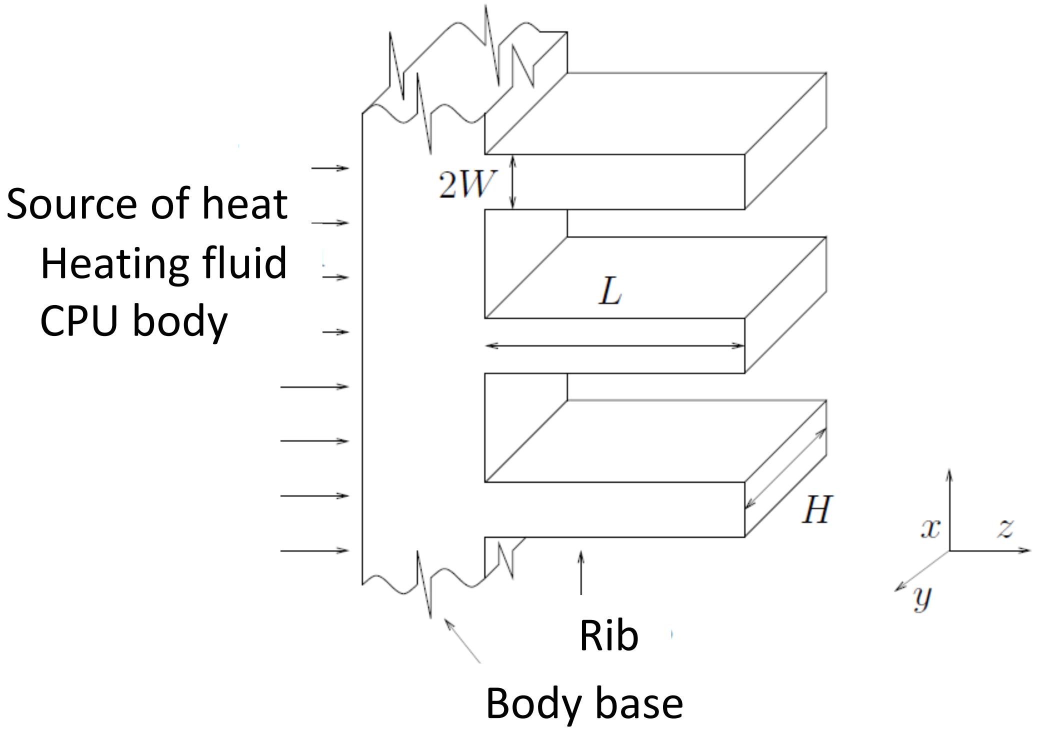
Lecture 3

- Heat transfer in ribbed surface
- Thin film approximation
- Heat exchange efficiency over
- Ribbed surface

Heat conduction over a ribbed surface

Significance: → heaters, heating elements
→ engine coolers
→ CPU coolers (Central Processing Unit coolers)

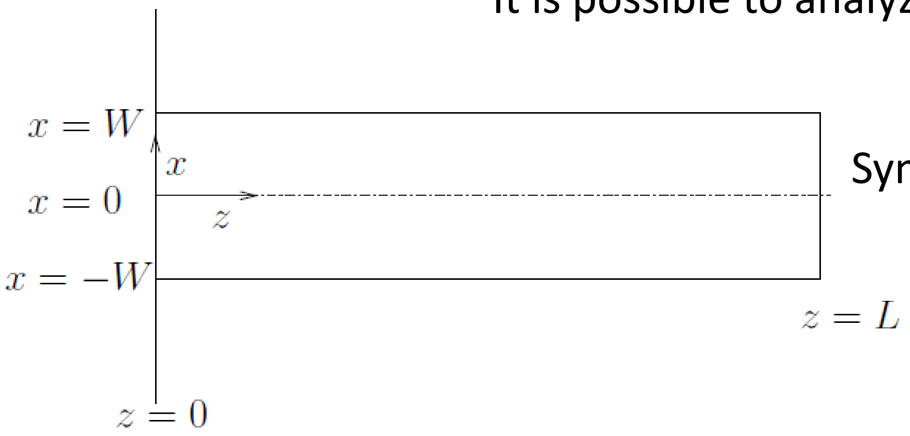
Schematic figure



Assumptions:

- H>>L → changes in the direction of *y* are negligible
- Rib material is an excellent heat conductor → temperature changes in the base of the body in the *z* direction are negligible
- Ribs are sufficiently distant → no mutual influence of heat transfer between individual ribs
- The system is in steady state and inside ribs there is no source of heat

It is possible to analyze heat transfer for each rib separately



Symmetry area

<u>Fourier equation</u>

$$\cancel{\frac{\partial T}{\partial t}} = a\nabla^2 T + \cancel{\frac{q^*}{\rho\, c_p}}$$

$$a\nabla^2 T = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0$$ Laplace equation

4 boundary conditions:

$x = 0,\ z$ symmetry $\dfrac{\partial T}{\partial x} = 0$

$x = W,\ z$ Transfer conduction/convection $-\lambda \dfrac{\partial T}{\partial x} = \alpha (T - T_\infty)$

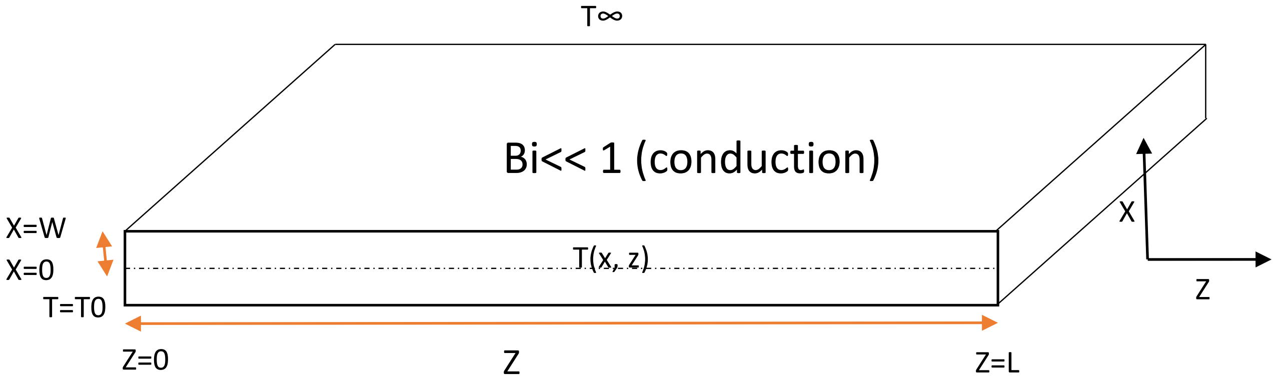
Temperature of surrounding far away from ribs

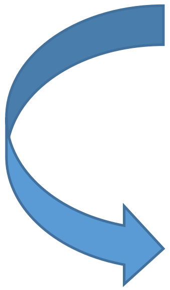
$x,\ z = 0$ fixed temperature $T = T_0$

$x,\ z = L$ transfer/conduction $-\lambda \dfrac{\partial T}{\partial z} = \alpha (T - T_\infty)$

Thin layer approximation of the ribbed surface

- Consider steady heat transfer from an extended surface or "fin" to the surrounding (e.g. air).
- L and W are such L/W>>1.
- It is assumed that Bi<<1.
- It is assumed that y direction is large enough to make the problem 2D.
- Thus we assume that T=T(x,z).
- Due to the symmetry of the rib→ Consider half of the object.
- Difference between a fin and fully submerged object is that the temperature at one end of the fin is fixed.
- Though the small Bi does not make the fin isotherm, it allows us to eliminate one of the independent variables: Importance of resulting "**fin approximation**" **is that it is prototype for reducing 2D model into a 1D one.**
- T is not function of x (No change of T in x axis at constant Z).
- Given that the temperature field is approximately 1D, the **local value can be replaced by the <u>cross-sectional average</u>** (at constant Z).





Problem 4

Rib efficiency Ω

$$\Omega = \frac{\text{heat flow from the surface of the rib}}{\text{heat flow from the surface of the rib to the surroundings at the maximum driving force, i.e. the surface temperature of the rib is everywhere } T_0 \text{ and the driving force is } (T_0 - T\infty)}$$

$$\Omega = \frac{\alpha H \int_0^L (\overline{T} - T_\infty)\, \mathrm{d}z}{\alpha H L (T_0 - T_\infty)}$$

We will express it using dimensionless quantities

$\Theta = (T-T_\infty)/(T_0-T)$

$\eta = L/W$

~~Z~~=Z/W

Problem 4

$$\Omega = \frac{\cancel{\alpha H} \int_0^\eta \cancel{(T_0 - T_\infty)} \Theta W \, \mathrm{d}\tilde{z}}{\cancel{\alpha H} L \cancel{(T_0 - T_\infty)}}$$

$$\Omega = \frac{1}{\eta} \int_0^\eta \Theta \, \mathrm{d}\tilde{z}$$

$$\Omega = \frac{1}{\eta} \int_0^\eta \left(C_1 e^{\sqrt{\mathrm{Bi}}\tilde{z}} + C_2 e^{-\sqrt{\mathrm{Bi}}\tilde{z}} \right) \mathrm{d}\tilde{z}$$

$$\Omega = \frac{1}{\eta} \left\{ \left[\frac{C_1}{\sqrt{\mathrm{Bi}}} e^{\sqrt{\mathrm{Bi}}\tilde{z}} \right]_0^\eta - \left[\frac{C_2}{\sqrt{\mathrm{Bi}}} e^{-\sqrt{\mathrm{Bi}}\tilde{z}} \right]_0^\eta \right\}$$

$$\Omega = \frac{1}{\eta\sqrt{\mathrm{Bi}}} \left\{ C_1 e^{\sqrt{\mathrm{Bi}}\tilde{z}} - C_1 - C_2 e^{-\sqrt{\mathrm{Bi}}\tilde{z}} + C_2 \right\}$$

$$\Omega = \frac{1}{\eta\sqrt{\mathrm{Bi}}} \left\{ C_1 \left[e^{\sqrt{\mathrm{Bi}}\tilde{z}} - 1 \right] - C_2 \left[e^{-\sqrt{\mathrm{Bi}}\tilde{z}} - 1 \right] \right\}$$

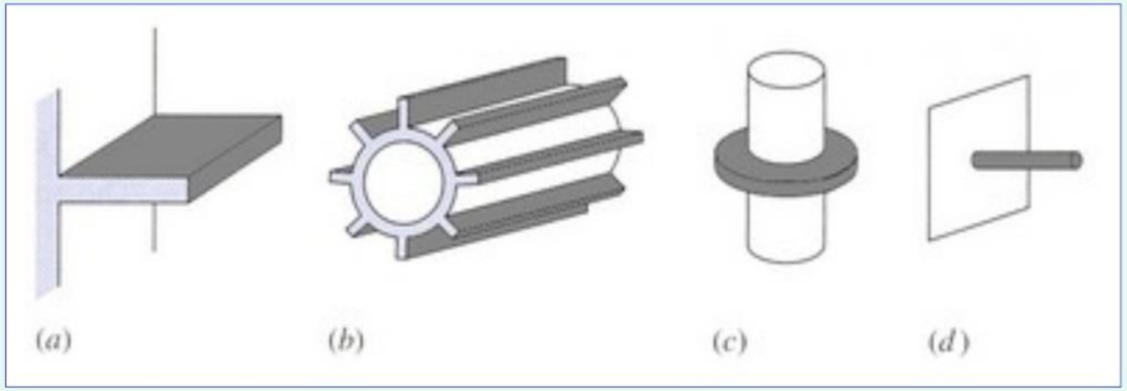
Fourier equation

$$\boxed{\frac{\partial T}{\partial t} = a\nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}}$$

Only for systems where there is no conduction (stationary)

(y direction)

Fin configurations



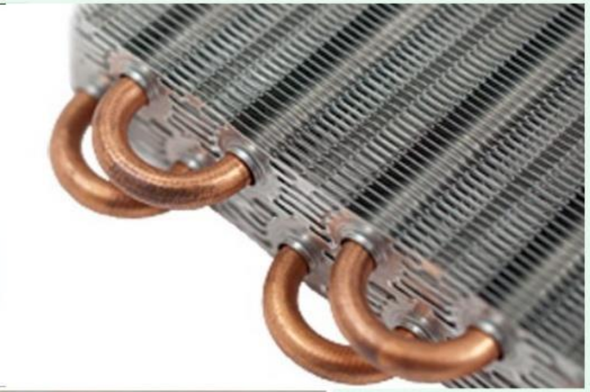
(a) straight fin of uniform cross-section on plane wall

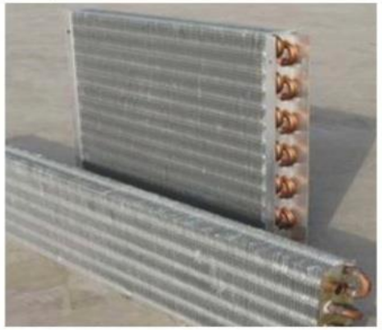
(b) straight fin of uniform cross-section on circular tube

(c) annular fin

d) straight pin fin

Fins Applications





LECTURE 4

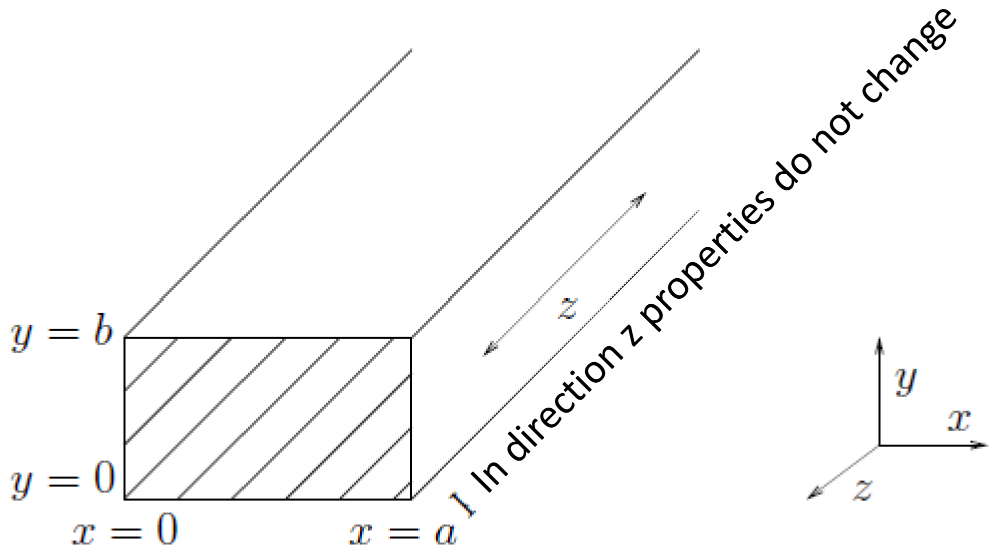
Steady heat conduction in multiple dimensions (rectangular cross-section body) and uninterrupted heat conduction (heat transfer over a membrane)

Steady heat conduction in multiple spatial dimensions

Fourier equation

$$0 = a\nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}$$

In the simplest case, a long rectangular cross-section (beam, wire, ...) can be considered, which is placed in an environment with constant properties



In the body, heat can be released due to the passage of electric current (resistance wire) or due to chemical reaction (type of plug/piston reactor).
The nature of the solution [T (x, y)] depends on the choice of boundary conditions. At all edges, for example, the constant temperature T_0 can be considered.

Fourier equation + boundary conditions:

$$0 = a\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + \frac{q^*}{\rho\, c_{\mathrm{p}}} \quad \Big| \frac{\rho\, c_{\mathrm{p}}}{\lambda}$$

$$0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q^*}{\lambda}$$

Edges $\quad y = 0, x \quad T = T_0 \qquad x = 0, y \quad T = T_0$

$\qquad\quad y = b, x \quad T = T_0 \qquad x = a, y \quad T = T_0$

Before the solution, it is appropriate to modify the equation to make the boundary condition homogeneous.
So we define: $\hat{T} = T - T_0$

Boundary condition $\boxed{\widehat{T} = 0}$ In the edges

Equation $$0 = \frac{\partial^2(\widehat{T} + T_0)}{\partial x^2} + \frac{\partial^2(\widehat{T} + T_0)}{\partial y^2} + \frac{q^*}{\lambda}$$

$$\boxed{0 = \frac{\partial^2 \widehat{T}}{\partial x^2} + \frac{\partial^2 \widehat{T}}{\partial y^2} + \frac{q^*}{\lambda}}$$

Analytical solution can be obtained by finite Fourier transform (FFT) method or other method

$$\widehat{T}(x, y) = \sum_{n=1}^{\infty} T_n(y)\psi_n(x)$$

$$T_n(y) = \frac{\sqrt{2}(-1+(-1)^n)\sqrt{\frac{1}{a}}\alpha \exp(-\lambda_n y)(-1+\exp(\lambda_n y))[\exp(\lambda_n y) - \exp(b\lambda_n)]}{(1+\exp(b\lambda_n))\lambda_n^3}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\lambda_n x)$$

$$\lambda_n = \frac{n\pi}{a}$$

The FFT method will not be discussed in the basic course and will not part of the examination.

Transient heat conduction in a spatially distributed system

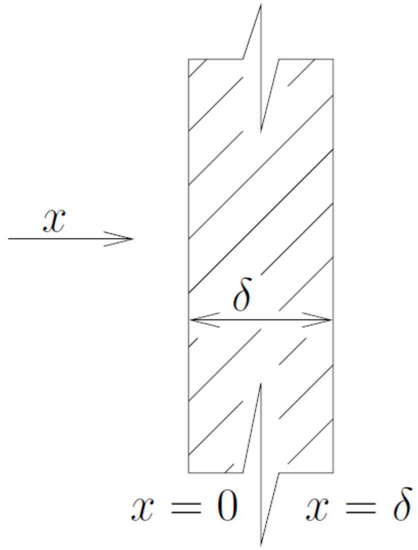
Transient heat conduction

The temperature of a body, in general, varies with time as well as position

We consider the variation of temperature with time and position in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere, e.g. membrane, planar heat transfer surface, brick wall…

Let us consider spatially 1D system without heat source

Large flat plate of the thickness δ



$$\frac{\partial T}{\partial t} = a\nabla^2 T + \frac{q^*}{\rho\, c_{\mathrm{p}}}$$

$$1\mathrm{D},\ q^* = 0 \qquad \frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

For the solution of the problem we need two boundary conditions and one initial condition

Let's consider that everywhere inside the plate, initial temperature is T_0

$$t = 0, x \qquad T = T_0$$

At time $t > 0$, we will increase the temperature on left edge to a value of T1. On the right edge we will keep the temperature T_0.

$$\begin{aligned} t > 0, x = 0 \qquad & T = T_1 \\ t > 0, x = \delta \qquad & T = T_0 \end{aligned}$$

By solution of the Fourier equation we will obtain the temperature as a function of time and coordinate x.

The student should be able at this point to answer:

1) What temperature profile will be established in the plate?

2) Order estimate of the time required to establish the temperature profile.

We will transform model equations to a dimensionless form

a: Thermal diffusivity ($m^2\ s^{-1}$)

$$\Theta = \frac{T - T_0}{T_1 - T_0}$$

$$\tilde{x} = \frac{x}{\delta}$$

$$\tilde{t} = \frac{t}{t_0} = \frac{t\,a}{\delta^2}$$

$$\cancel{\frac{(T_1 - T_0)a}{\delta^2}} \frac{\partial \Theta}{\partial \tilde{t}} = \cancel{a \frac{(T_1 - T_0)}{\delta^2}} \frac{\partial^2 \Theta}{\partial \tilde{x}^2}$$

$$\boxed{\frac{\partial \Theta}{\partial \tilde{t}} = \frac{\partial^2 \Theta}{\partial \tilde{x}^2}}$$

Initial condition

$\tilde{t} = 0, \tilde{x} \qquad \Theta = 0$

boundary conditions

$\tilde{t} > 0, \tilde{x} = 0 \qquad \Theta = 1$

$\tilde{t} > 0, \tilde{x} = 1 \qquad \Theta = 0$

The exact solution can be found for example by FFT method:

$$\boxed{\Theta(\tilde{x}, \tilde{t}) = 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} (1 - \exp[-n^2 \pi^2 \tilde{t}\,]) \sin(n\pi\tilde{x})}$$

Lecture 5

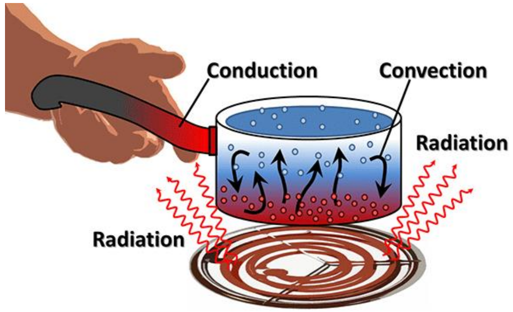
- Combined heat by conduction and convection
- Deriving the Fourier-Kirchhoff equation for general control volume
- Péclet's number

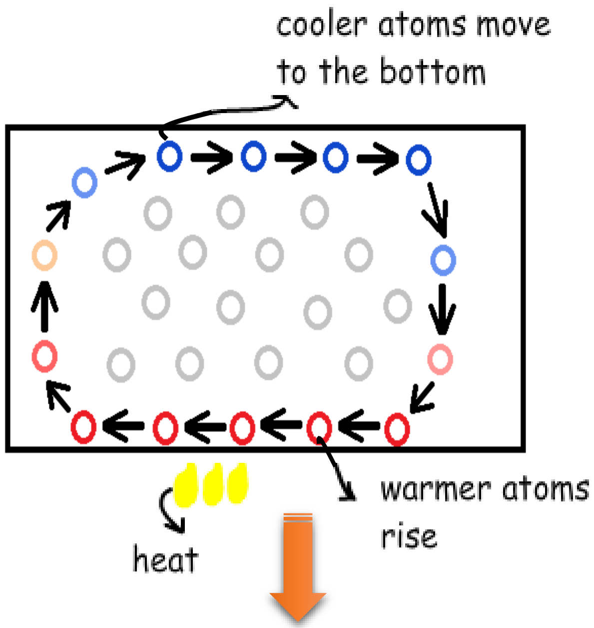
Heat transfer by Convection

Convective heat transfer occurs when a gas or liquid flows past a solid surface whose temperature is different from that of the fluid.

Example:

- When a fluid (e.g. air or a liquid) is heated and then travels away from the source of heat, it carries the thermal energy along.
- The fluid above a hot surface expands, becomes less dense, and rises.
- As the immediate hot fluid rises, it pushes denser, colder fluid down causing convection currents which transport energy





Heat transfered by macroscopic movement of matter (fluids)

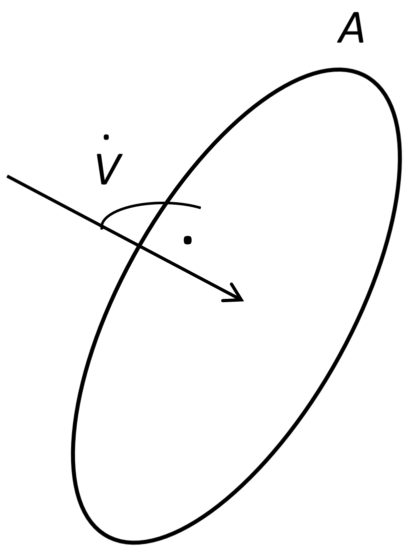
Forced convection ↔ Natural convection

Forced convection: Fluid motion is caused by an external agent such as a pump or blower.

Natural convection: Fluid motion is the result of buoyancy forces created by temperature differences within the fluid.

Convective heat transfer

Let us assume that the volume flow of the substance is perpendicular to the plane



From the first law of thermodynamics, it follows that for isobaric system (constant pressure) performing only volume work the change of enthalpy is equal to the heat exchanged between the system and its surrounding

$$dQ = dH$$

$$\frac{dQ}{dt} = \frac{dH}{dt}$$

Each mass carries a certain thermal content Q.
In a usual case: heat flow is equal to enthalpy flow ($\dot{Q} = \dot{H}$)

$$\dot{Q} = \dot{m}h = \dot{V}\rho\, c_p(T - Tref) \text{ [W]}$$

h - specific enthalpy [kg/s] [J/kg]

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \Rightarrow h = \int_{T_{ref}}^{T} c_p\, dT \doteq \langle c_p \rangle (T - T_{ref})$$

Average specific heat capacity [J kg^{-1} K^{-1}]

We will further assume that C_p is constant (i.e. C_p is not a function of temperature in a given temperature range T_{ref} to T)

The intensity of the heat flow through the convection of the surface A is thus:

$$q^k = \frac{\dot{Q}}{A} = \frac{\dot{V} \rho \, c_\mathrm{p} (T - T_{ref})}{A} = v \rho \, c_\mathrm{p} (T - T_{ref})$$

v - Velocity of convective flow [m s^{-1}]

The intensity of the heat flow is oriented in space:

$$\vec{q}^k = \begin{bmatrix} q_x^k \\ q_y^k \\ q_z^k \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \rho \, c_\mathrm{p} (T - T_{ref})$$

$$\boxed{\vec{q}^k = \vec{v} \rho \, c_\mathrm{p} (T - T_{ref})}$$

The transfer by conduction and convection typically takes place simultaneously.

The overall intensity of heat flow q is a sum of that by conduction q^v and convection q^k

$$\vec{q} = \vec{q}^{\,v} + \vec{q}^{\,k}$$

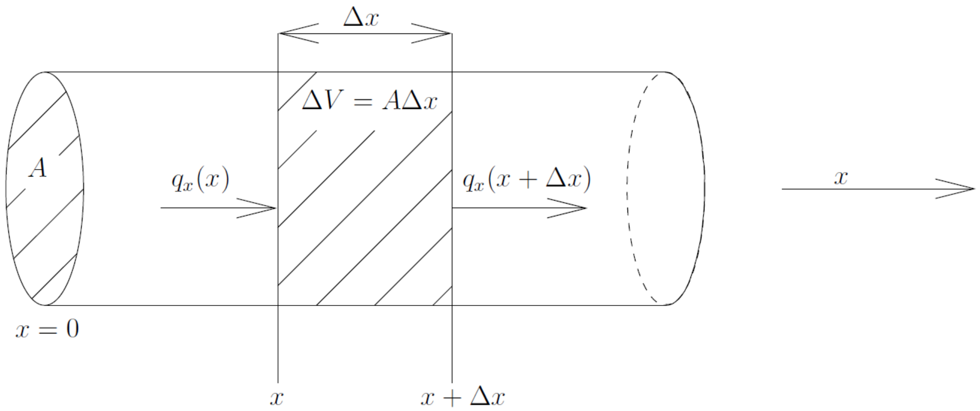
Generally:

$$\boxed{\vec{q} = -\lambda \nabla T + \vec{v} \rho \, c_{\mathrm{p}} (T - T_{ref})}$$

For coordinate *x:*

$$q_x = -\lambda \frac{\mathrm{d}T}{\mathrm{d}x} + v_x \rho \, c_{\mathrm{p}} (T - T_{ref})$$

Transient heat transfer by conduction and convection in 1D system



We balance the heat energy in the control volume ΔV

INPUT + SOURCE = OUTPUT + ACCUMULATION

Volumetric source of heat

$$
\begin{aligned}
q_x(x)A + q^* A\Delta x &= q_x(x+\Delta x)A + \frac{\mathrm{d}H}{\mathrm{d}t} \\
\cancel{q_x(x)A} + q^* \cancel{A\Delta x} &= \cancel{q_x(x)A} + \frac{\partial q_x}{\partial x}\cancel{A\Delta x} + \frac{\partial(\rho \cancel{A\Delta x}\, c_p (T - T_{ref}))}{\partial t} \\
q^* &= \frac{\partial q_x}{\partial x} + \rho\, c_p \frac{\partial T}{\partial t} \qquad \rho, c_p \text{ constants} \\
\frac{\partial T}{\partial t} &= -\frac{1}{\rho\, c_p}\frac{\partial q_x}{\partial x} + \frac{q^*}{\rho\, c_p} \\
\frac{\partial T}{\partial t} &= -\frac{1}{\rho\, c_p}\frac{\partial}{\partial x}\left[-\lambda \frac{\partial T}{\partial x} + v_x \rho\, c_p (T - T_{ref})\right] + \frac{q^*}{\rho\, c_p}
\end{aligned}
$$

The continuity equation applies. If we consider an incompressible flow (liquid), then

$$
\begin{aligned}
\nabla \cdot \vec{v} &= 0 \quad \textbf{Slide 15} \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} &= 0 \quad \text{3D-system} \\
\frac{\partial v_x}{\partial x} &= 0 \quad \text{1D-system}
\end{aligned}
$$

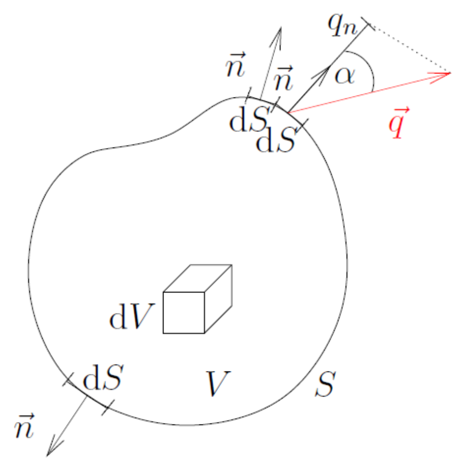
Incompressible flow implies that the density remains constant within a parcel of fluid that moves with the flow velocity → divergence of flow velocity is zero

It follows that v_x is constant in the spatial 1D system. If the cross-section of the system along the axis changed, v_x would also change. However, it would be a 2D system!

$$
\boxed{\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} - v_x\frac{\partial T}{\partial x} + \frac{q^*}{\rho\, c_\mathrm{p}}}
$$

Fourier-Kirchhoff equation in 1D-system

Derivation of the Fourier-Kirchhoff equation in general form



V - control volume
S - area enclosing the control volume

- The heat flux *Q* passes through the boundary of the system. The flow leaving (entering) the system flow is the flow perpendicular to the surface.
- Thus, the heat flux through the area *dS* is: $q_n.dS$
- q_n - the normal component of the vector $\boldsymbol{q}$

$$q_n = -\vec{n} \cdot \vec{q} = -\|n\| \cdot \|q\| \cdot \cos\alpha \qquad \|n\| = 1$$

INPUT - OUTPUT + SOURCE = ACCUMULATION

$$\int_S -\vec{n} \cdot \vec{q} \, \mathrm{d}S + \int_V q^* \, \mathrm{d}V = \frac{\mathrm{d}H}{\mathrm{d}t}$$

Sum of flows across all boundaries

Sum of sources over the entire volume

Accumulation of thermal energy in the whole system

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial}{\partial t}(H) = \frac{\partial}{\partial t}\int_V h\,\underbrace{\rho\,\mathrm{d}V}_{\text{mass of element dV}} = \underbrace{\frac{\partial}{\partial t}\int_V \rho\, c_\mathrm{p}(T - T_{ref})\,\mathrm{d}V}_{\text{sum of accumulations of heat in the entire volume}}$$

We will write the balance:

$$\int_S -\vec{n}\cdot\vec{q}\,\mathrm{d}S + \int_V q^*\,\mathrm{d}V = \frac{\partial}{\partial t}\int_V \rho\, c_\mathrm{p}(T - T_{ref})\,\mathrm{d}V$$

If the volume element does not depend on time, we can change the order of integration and derivation in the accumulation term:

$$\int_S -\vec{n}\cdot\vec{q}\,\mathrm{d}S + \int_V q^*\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}\left(\rho\, c_\mathrm{p}(T - T_{ref})\right)\mathrm{d}V$$

$$\int_V -\nabla\cdot\vec{q}\,\mathrm{d}V + \int_V q^*\,\mathrm{d}V = \int_V \rho\, c_\mathrm{p}\frac{\partial T}{\partial t}\,\mathrm{d}V$$

The balance holds also for the volume *dV*: $$-\nabla\cdot\vec{q} + q^* = \rho\, c_\mathrm{p}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho\, c_{\mathrm{p}}} \nabla \cdot \vec{q} + \frac{1}{\rho\, c_{\mathrm{p}}} q^*$$

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho\, c_{\mathrm{p}}} \underbrace{\nabla}_{\vec{a}} \cdot \left[-\underbrace{\lambda \nabla T}_{\vec{b}} + \underbrace{\vec{v} \rho\, c_{\mathrm{p}} (T - T_{ref})}_{\vec{c}} \right] + \frac{q^*}{\rho\, c_{\mathrm{p}}}$$

the scalar product is distributive

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Both quantities depend on spatial coordinates

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho\, c_{\mathrm{p}}} \nabla \cdot \nabla T - \nabla \cdot \left[\underbrace{\vec{v}}_{\vec{a}} \underbrace{(T - T_{ref})}_{f} \right] + \frac{q^*}{\rho\, c_{\mathrm{p}}}$$

It holds true that:

$$\nabla \cdot (f \vec{a}) = \nabla f \cdot \vec{a} + f \nabla \cdot \vec{a}$$

f – scalar function
a – vector function

$$\frac{\partial T}{\partial t} = a \nabla^2 T - \nabla (T - T_{ref}) \cdot \vec{v} + (T - T_{ref}) \nabla \cdot \vec{v} + \frac{q^*}{\rho\, c_{\mathrm{p}}}$$

$$\nabla (T - T_{ref}) = \nabla T$$ - Proof as a homework

$$\nabla \cdot \vec{v} = 0$$ - Continuity equation for an incompressible fluid

$$\frac{\partial T}{\partial t} = a\nabla^2 T - \vec{v} \cdot \nabla T + \frac{q^*}{\rho\, c_\mathrm{p}}$$

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = a\nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}$$

$$\frac{\partial}{\partial t} + \vec{v} \cdot \nabla = \frac{\mathrm{D}}{\mathrm{D}t}$$

→ operator of the material derivative

Material derivative describes the time rate of change of some physical quantity (like heat or momentum) of a material element that is subjected to a space-and-time-dependent macroscopic velocity field variations of that physical quantity

$$\boxed{\frac{\mathrm{D}T}{\mathrm{D}t} = a\nabla^2 T + \frac{q^*}{\rho\, c_\mathrm{p}}}$$

Fourier-Kirchhoff equation

In Cartesian coordinates, the equation can be written as:

$$\boxed{\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q^*}{\rho\, c_\mathrm{p}}}$$

Transformation of Fourier-Kirchhoff (FK) equation into dimensionless form

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = a \nabla^2 T + \frac{q^*}{\rho \, c_\mathrm{p}}$$

$$\Theta = \frac{T}{T_0} \qquad \tilde{t} = \frac{t}{t_0} \qquad \tilde{v} = \frac{v}{v_0} \qquad \tilde{x} = \frac{x}{x_0} \qquad \tilde{\nabla} = x_0 \nabla$$

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_0} \frac{\partial}{\partial \tilde{x}} \\ \frac{1}{x_0} \frac{\partial}{\partial \tilde{y}} \\ \frac{1}{x_0} \frac{\partial}{\partial \tilde{z}} \end{bmatrix} = \frac{1}{x_0} \tilde{\nabla} \qquad \tilde{\nabla} = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}} \\ \frac{\partial}{\partial \tilde{y}} \\ \frac{\partial}{\partial \tilde{z}} \end{bmatrix}$$

$$\frac{T_0}{t_0} \frac{\partial \Theta}{\partial \tilde{t}} + \frac{v_0 T_0}{x_0} \tilde{\vec{v}} \cdot \tilde{\nabla} \Theta = \frac{a T_0}{x_0^2} \tilde{\nabla}^2 \Theta + \frac{q^*}{\rho \, c_\mathrm{p}}$$

Let $\boxed{t_0 = \frac{x_0}{v_0}}$ **- convective time**

$$\frac{v_0 T_0}{x_0} \left[\frac{\partial \Theta}{\partial \tilde{t}} + \tilde{\vec{v}} \cdot \tilde{\nabla} \Theta \right] = \frac{a T_0}{x_0^2} \tilde{\nabla}^2 \Theta + \frac{q^*}{\rho \, c_\mathrm{p}} \qquad \Bigg| \, \frac{x_0^2}{a \, T_0}$$

$$\boxed{\frac{v_0 x_0}{a}} \left(\frac{\partial \Theta}{\partial \tilde{t}} + \tilde{\vec{v}} \cdot \tilde{\nabla} \Theta \right) = \tilde{\nabla}^2 \Theta + \frac{q^* x_0^2}{\rho \, c_\mathrm{p} \, a \, T_0}$$

Pe – Péclet number

Let $\tilde{q}^* = \frac{q^* \, x_0^2}{\rho \, c_\mathrm{p} \, a \, T_0}$ $\qquad \frac{\mathrm{W \; m^2 \; m^3 \; kg \; K \; s}}{\mathrm{m^3 \; kg \; J \; m^2 \; K}} = 1$

q* -dimensionless volume source of heat

Physical significance of Péclet number (Pe)

$$\boxed{\mathrm{Pe}\left(\frac{\partial \Theta}{\partial \tilde{t}} + \tilde{\vec{v}} \cdot \tilde{\nabla}\Theta\right) = \tilde{\nabla}^2\Theta + \tilde{q}^*}$$

Fourier-Kirchhoff equation

$$\mathrm{Pe} = \frac{v_0\, x_0}{a} = \frac{v_0}{\frac{a}{x_0}}$$

velocity of heat sharing by convection (velocity of convection)

Velocity of heat sharing by conduction (velocity of conduction)

$V_0 = u$
$X_0 = L$

When **a/x_0** tends to **0**: Pe tends to **infinite**: **Convection mechanism dominates**
If **V_0** tends to **0**, **Pe** tends to **0**. Almost no flow of the fluid: **Conduction mechanism dominates**

Conductive time: $$t_0 = \frac{x_0^2}{a}$$

Conductive velocity: $$v_c = \frac{x_0}{t_0} = \frac{x_0 a}{x_0^2} = \frac{a}{x_0}$$

Lecture 6

Transient Heat transfer

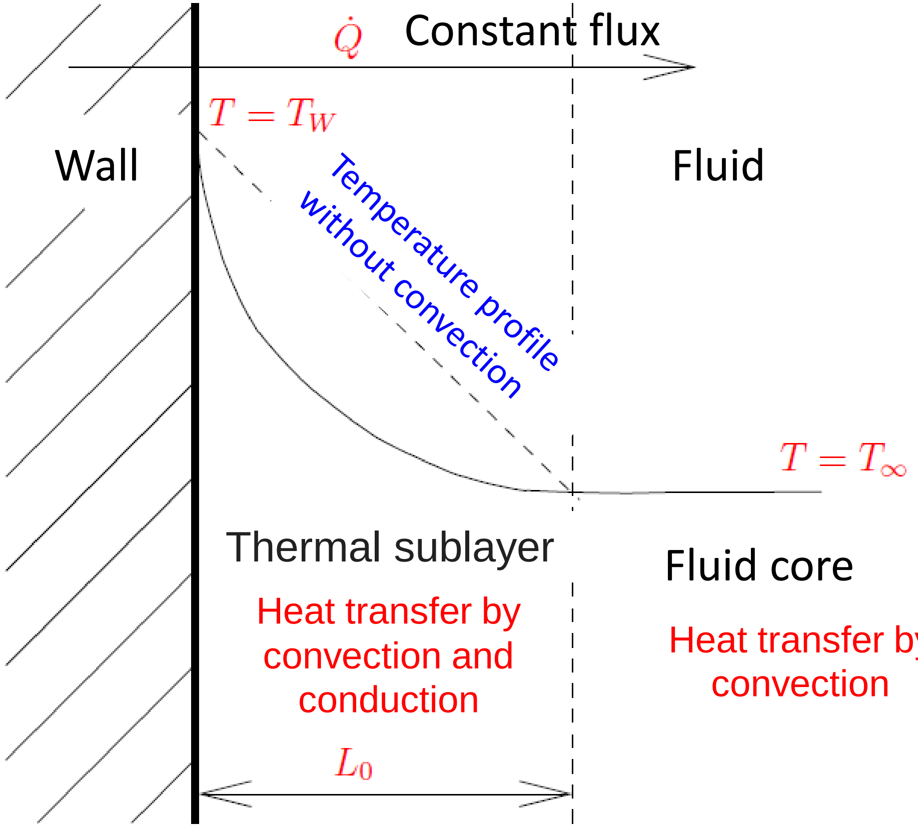
- Combined heat transfer by conduction and convection
- Newton's law of cooling
- Nusselt criterion
- Nusselt criterion in body wrapping
- Qualitative Behavior of Nusselt's Criterion in a Limited Area in Laminar Flow
- Graetz problem

Transient Heat transfer

Combined heat transfer by conduction and convection - very common

Typical examples:

- heat transfer between the heat exchange surface and fluid in recurrent exchangers
- flow around particles (particle drying)
- free convection (natural convection: by density differences in the fluid occurring due to temperature gradients.) (gas heating over heating elements)



Typical temperature distribution for fluid flow along at the surface

- The importance of conduction increases in a direction towards the wall as the flow velocity approaches 0 at the wall.

- At the layer closest to the wall the fluid velocity is zero, hence the heat transfer at the phase interface takes place only by conduction.

- At steady state, the heat flow in the most immediate layer must be equal to the heat flow over the entire thermal sublayer, therefore:

$$-\lambda \left.\frac{dT}{dx}\right|_{\text{wall}} = \underbrace{\alpha(T_W - T_\infty) = q_x}_{\text{Newton's law of cooling}}$$

thermal conductivity of fluid

heat transfer coefficient

Equations can be transformed to dimensionless forms

$$\Theta = \frac{T}{T_0}, \qquad \tilde{x} = \frac{x}{L_0}$$

L_0 → Characteristic dimension of the system such as tubing diameter

$$-\frac{\lambda T_0}{L_0} \left.\frac{\mathrm{d}\Theta}{\mathrm{d}\tilde{x}}\right|_{\text{wall}} = \alpha T_0 (\Theta_W - \Theta_\infty)$$

After re-arrangement:

$$\boxed{\frac{-\left.\frac{\mathrm{d}\Theta}{\mathrm{d}\tilde{x}}\right|_{\text{wall}}}{\Theta_W - \Theta_\infty} = \frac{\alpha L_0}{\lambda} \equiv \mathrm{Nu}}$$

Nusselt criterion

Dimensionless form of heat transfer coefficient

In heat transfer at a boundary (surface) within a fluid, Nu: Ratio of convective to conductive heat transfer across (normal to) the boundary.

Nu - how many times the heat transfer (conduction + convection) is more intense than in the case of stationary fluid (where heat transfer is only due to conduction).

For stationary fluid, it holds that:

$$-\lambda \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{\text{wall}} = +\lambda \frac{T_W - T_\infty}{L_0}.$$

For stationary fluid, it holds that: $$-\lambda \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{\text{wall}} = +\lambda \frac{T_W - T_\infty}{L_0}.$$

$$+\lambda \frac{\cancel{T_W - T_\infty}}{L_0} = \alpha \cancel{(T_W - T_\infty)}$$

$$\boxed{\mathrm{Nu} = 1 = \frac{\alpha L_0}{\lambda}}$$

The Nusselt number depends on the nature of the fluid flow, the fluid properties and the geometric arrangement.

$$\mathrm{Nu} = \mathrm{Nu}\left(\mathrm{Re}, \mathrm{Pr}, \Gamma, \text{ position }\right)$$

geometric simplexes

The nature of fluid flow: $\mathrm{Re} = \dfrac{v L_0}{\nu}$ ν kinematic viscosity [$m^2 s^{-1}$] **Reynolds number** (ratio of inertial forces to viscous forces)

Properties of the fluid: $\mathrm{Pr} = \dfrac{\nu}{a}$ **Prandtl number** (ratio of momentum diffusivity to thermal diffusivity)

In some cases, the dependency of the Nusselt criterion can be written:

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Pe},\ \Gamma,\ \text{Position})$$

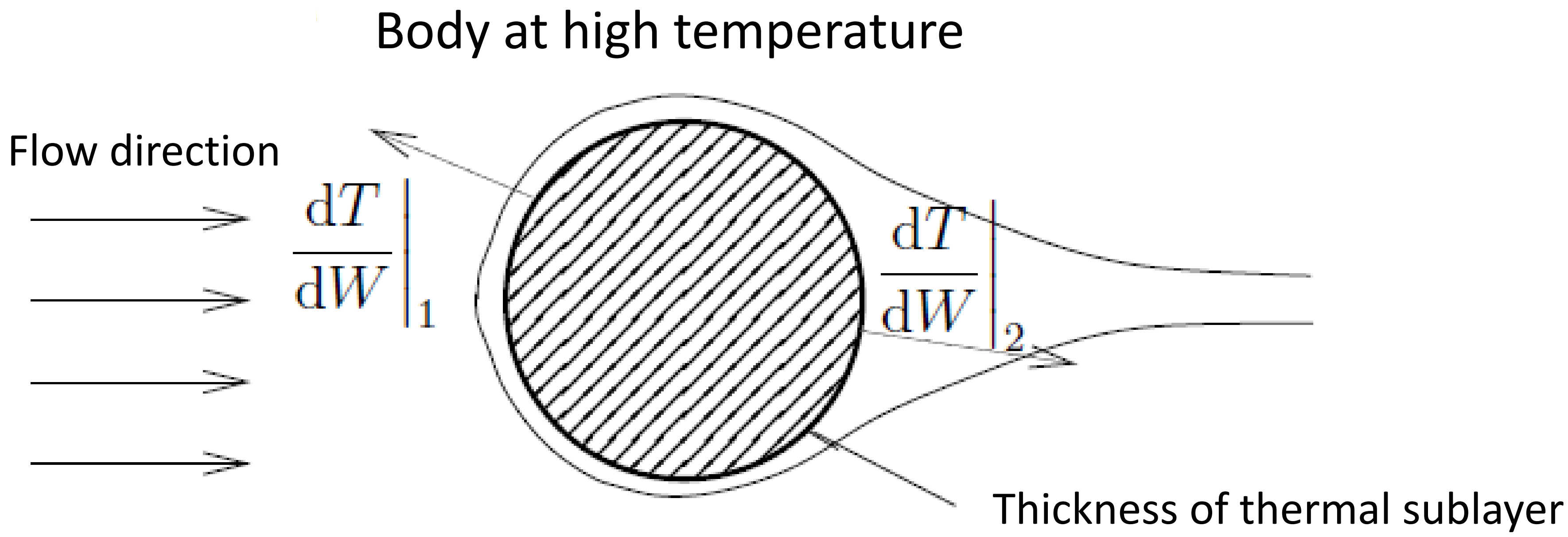
$$\mathrm{Pe} = \mathrm{Re}\ \mathrm{Pr} = \frac{vL_0}{a}$$

To calculate the heat transfer coefficient, it is necessary to find a suitable dependence for calculating the Nusselt criterion

Dependencies can be obtained:

- By the solution of the Fourier-Kirchhoff equation and possibly other transport equations
- empirically

The value of the Nusselt criterion is position dependent. For example, in case of body wrapping, the Nusslet criterion is different at each surface location - the value of the normal temperature derivative changes to the surface.



$$-\left.\frac{dT}{dW}\right|_1 \gg -\left.\frac{dT}{dW}\right|_2$$

It is therefore advantageous to define the average value of the Nusselt number on the entire surface of the object.

$$\overline{\mathrm{Nu}} = \frac{1}{S}\int_S \mathrm{Nu}\,\mathrm{d}S = \frac{1}{S}\int_S \frac{-\vec{n}\cdot\nabla\Theta}{\Theta_W - \Theta_\infty}\,\mathrm{d}S$$

The scalar product expresses $\vec{n}\cdot\nabla\Theta$ the temperature derivative value in the direction of the normal vector $\vec{n}$, a vector perpendicular to the body surface.

The relationships for calculating Nu can be found in the form:

$$\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(\mathrm{Pe}, \mathit{\Gamma})$$

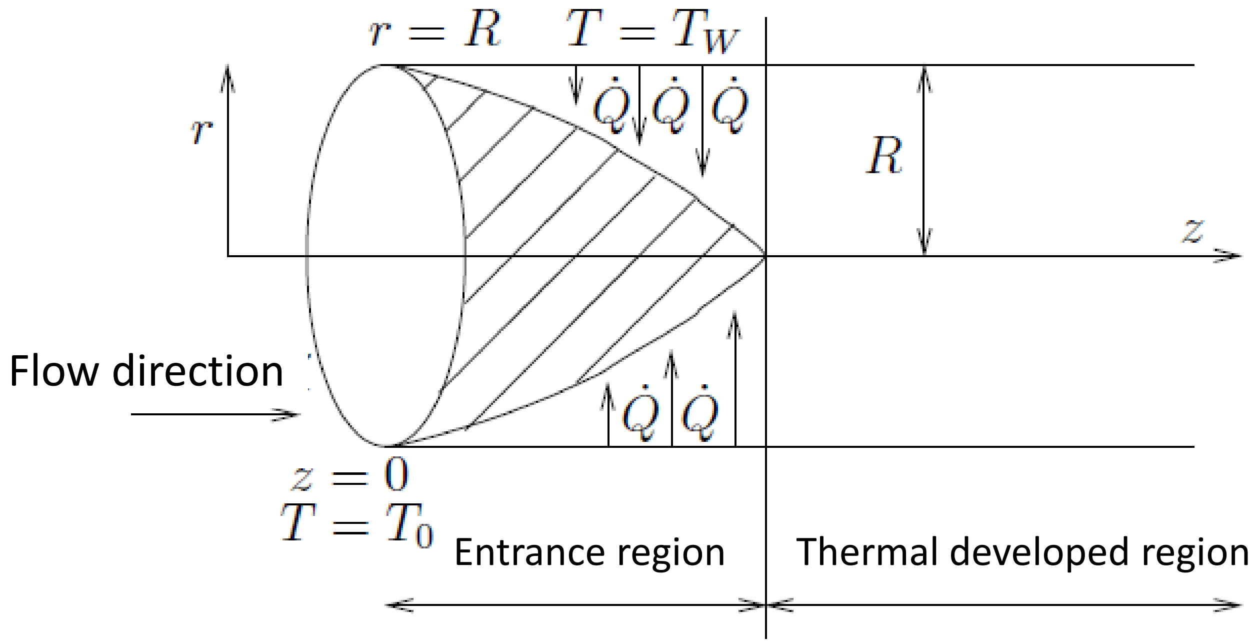
$$\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(\mathrm{Re},\ \mathrm{Pr},\ \mathit{\Gamma})$$

$$\ldots$$

Qualitative Behavior of Nusselt's Criterion in a Limited Area in Laminar Flow

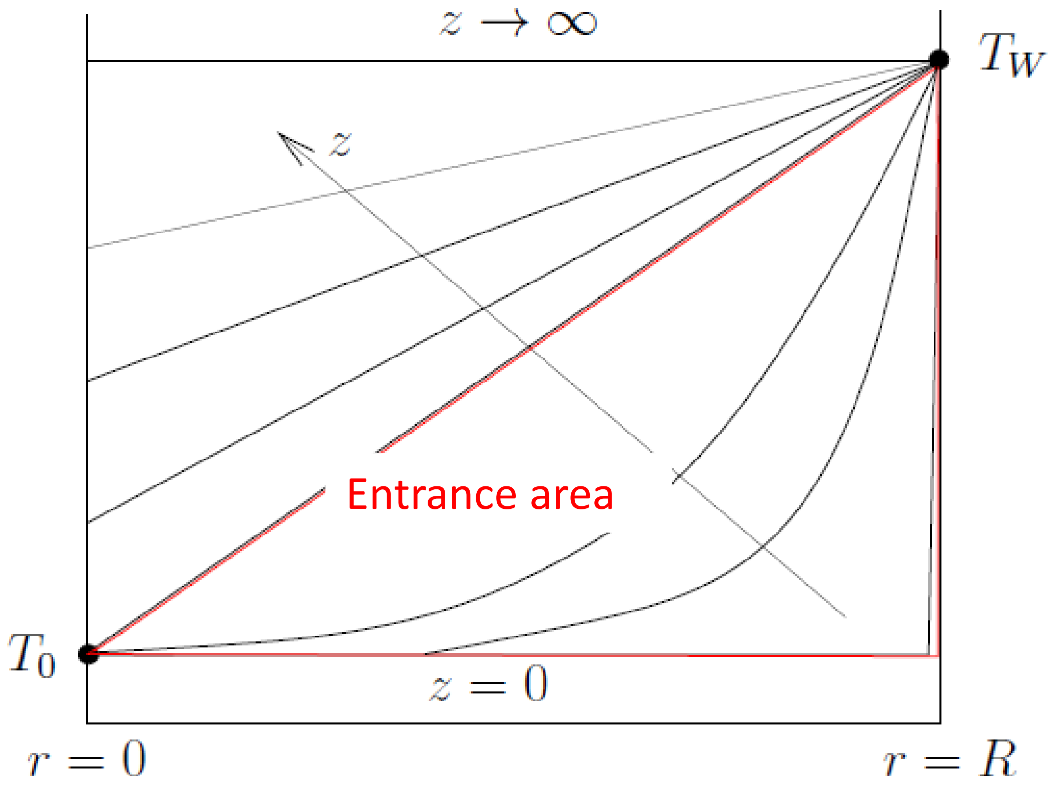
- Piping systems
- Flow between flat plates
- Flowing liquid films

Let's consider pipe with radial coordinates *r* and axial *z.* We also consider that the fluid in the pipeline flows in laminar regime according to the axis *z,* the fluid has a temperature of T_0 at the inlet and the walls have constant temperature T_W.



The core of the fluid (the hatched area) remains at a distance from unheated input - not yet affected by the heat flow from the walls conduit. This is the so-called entry area. Once the heat from the walls arrives to the middle part, the entire volume of fluid is affected by heat (warmed/ cooled). Then we talk about so-called thermal developed area.

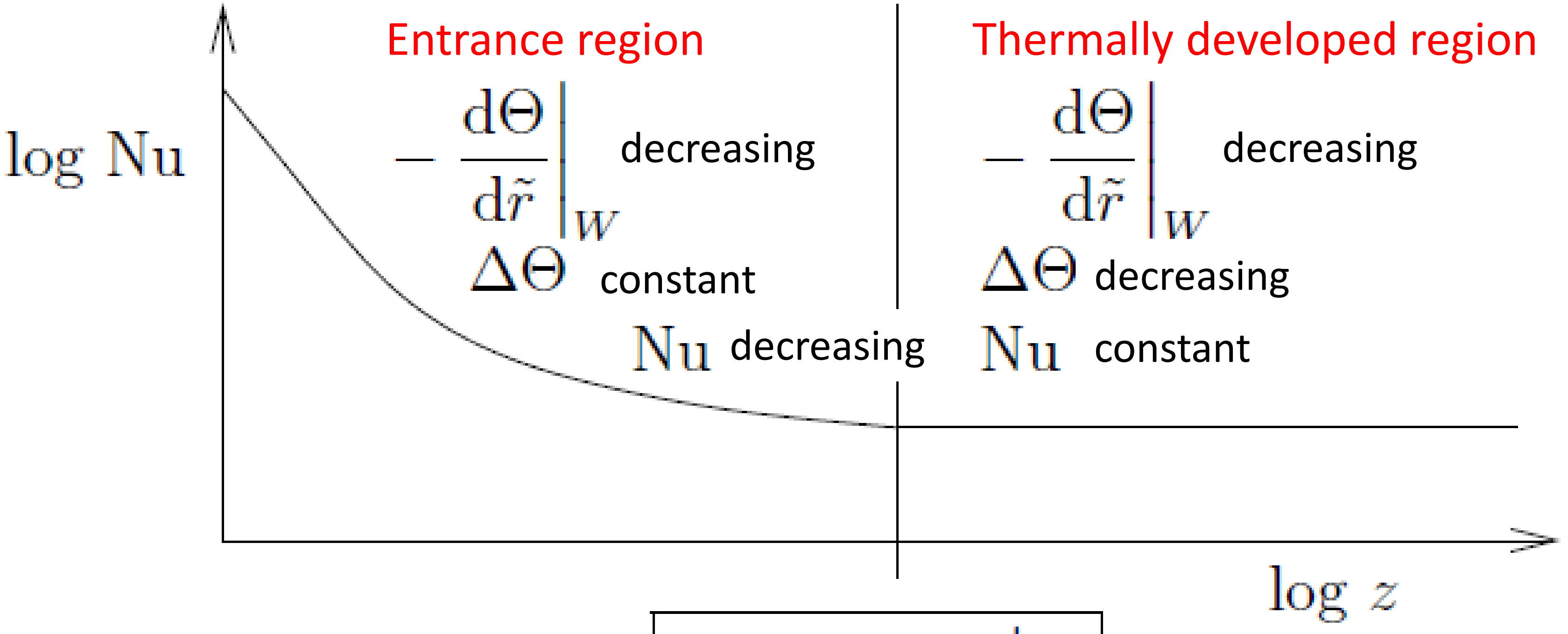
Qualitative character of the temperature field



$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=R}$$ along the *z* coordinate decreases

The value T_W - T_0 remains constant throughout the entrance region

The value Nu in the entrance region decreases along the axis z



$$\boxed{\mathrm{Nu} = \frac{-\left.\dfrac{d\Theta}{d\tilde{r}}\right|_W}{\Delta\Theta}}$$

- Nu is very large at z=0 and declines with increasing z
- Typically, Nu initially varies as some inverse power of z, so that a log-log plot of Nu(z) is linear at small z.
- For long enough tubes or films, Nu approaches a constant, even though the temperature may continue to depend on z.
- The position at which Nu becomes essentially constant separates the thermal entrance region from the thermally fully developed region.

Graetz's problem

Steady heat transfer, steady laminar flow without heat inside (no heat source)

Fourier-Kirchhoff's (FK) equation

$$\cancel{\frac{\partial T}{\partial t}} + \vec{v} \cdot \nabla T = a\nabla^2 T + \cancel{\frac{q^*}{\rho\, c_p}}$$

$$v_z = 2U\left[1 - \left(\frac{r}{R}\right)^2\right] \quad \rightarrow$$

solution of the Navier-Stokes equation for the laminar flow of incompressible fluid in a circular cross section called "Poiseville flow"

U is the average flow rate.

Differential operations in cylindrical coordinates

$$\cancel{v_r \frac{\partial T}{\partial r}} + v_z \frac{\partial T}{\partial z} = a \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \cancel{a \frac{\partial^2 T}{\partial z^2}}$$

$v_r = 0$ → in laminar flow the fluid moves only in the direction of z

since Pe >> 1 in *z* direction (assumption)

Convection mechanism dominates
Neglecting axial conduction

$$v_z \frac{\partial T}{\partial z} = a \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$$

3 boundary conditions:

$z = 0, r$	$T = T_0$	Temperature at pipe inlet
$z, r = 0$	$\frac{\partial T}{\partial r} = 0$	Symmetry, the heat does flow through the center in the direction *r*
$z, r = R$	$T = T_W$	Pipe wall temperature

Transformation to dimensionless form:

$$\tilde{r} = \frac{r}{R} \qquad \Theta = \frac{T - T_0}{T_W - T_0} \qquad \tilde{z} = \frac{z}{R\,\mathrm{Pe}} \qquad \mathrm{Pe} = \frac{2UR}{a}$$

U is the mean velocity (velocity in the middle of the pipeline)
R: tube radius

→ Called Graetz problem

$$\tilde{r} = \frac{r}{R} \qquad \Theta = \frac{T - T_0}{T_W - T_0} \qquad \tilde{z} = \frac{z}{R\,\mathrm{Pe}} \qquad \mathrm{Pe} = \frac{2UR}{a}$$

- The noteworthy aspect of the independent variables is that Pe has been embedded in the axial coordinate.
- This choice of $\tilde{z} = \frac{z}{R\,\mathrm{Pe}}$ is motivated by the fact that when all terms in equation are made dimensionless, z and Pe appear only as a ratio.
- Pe is based only on mean velocity and the diameter, which is usual convention for circular tubes.

$$2U\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = a\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

$$2U\left[1-\tilde{r}^2\right]\frac{\cancel{T_W - T_0}}{R\,\mathrm{Pe}}\frac{\partial \Theta}{\partial \tilde{z}} = \frac{a\,\cancel{(T_W - T_0)}}{R^2}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \Theta}{\partial \tilde{r}}\right) \qquad \Big| \frac{R^2}{a}$$

$$\frac{2U\,R}{a\,\mathrm{Pe}}\left[1-\tilde{r}^2\right]\frac{\partial \Theta}{\partial \tilde{z}} = \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \Theta}{\partial \tilde{r}}\right), \qquad \frac{2U\,R}{a\,\mathrm{Pe}} = 1,$$ See definition of Pe

$$\boxed{\left(1-\tilde{r}^2\right)\frac{\partial \Theta}{\partial \tilde{z}} = \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \Theta}{\partial \tilde{r}}\right)}$$

Transformation of boundary conditions to dimensioless form:

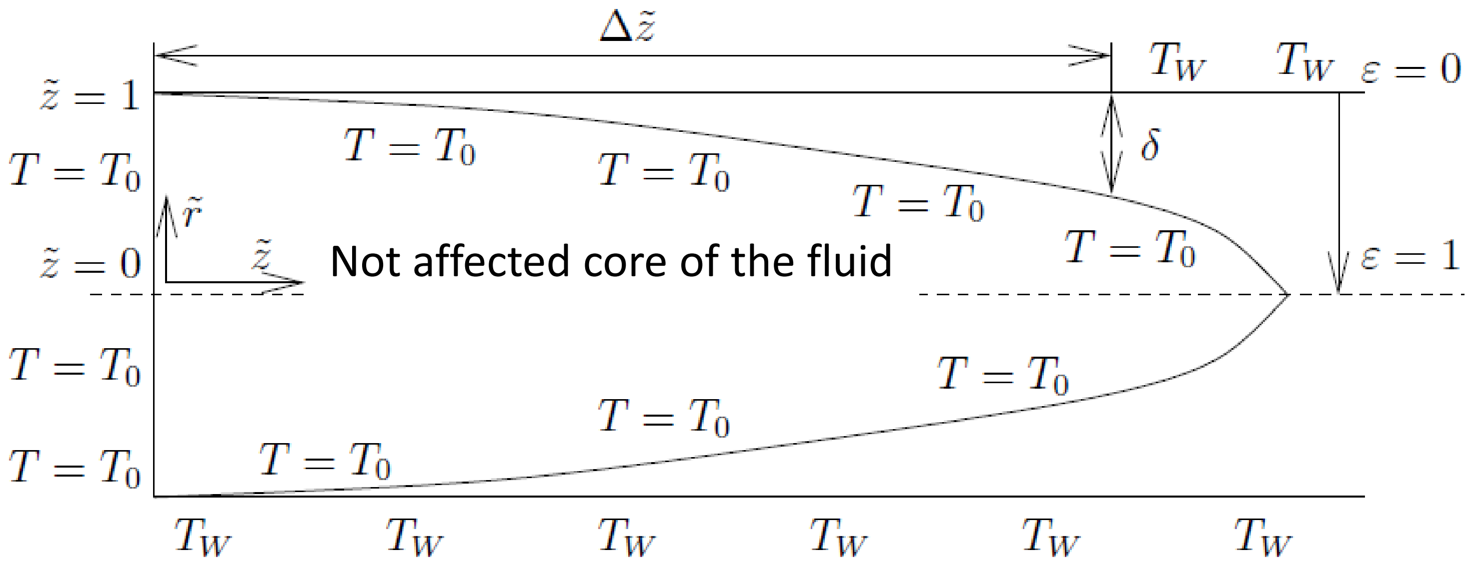
$$\begin{array}{llcll} z=0, r & T=T_0 & \Rightarrow & \tilde{z}=0 & \Theta=0 \\ r=0, z & \dfrac{\partial T}{\partial r}=0 & \Rightarrow & \tilde{r}=0 & \dfrac{\partial \Theta}{\partial \tilde{z}}=0 \\ r=R, z & T=T_W & \Rightarrow & \tilde{r}=1 & \Theta=1 \end{array}$$

Because the solution to Graetz's problem is quite complicated, it will be solved numerically during the seminar. The main conclusion of Graetz analytical solution is that the Nu value in the thermally developed region is constant. Specifically, in a circular cross - section tube for $\tilde{z} \to +\infty \quad \mathrm{Nu} = 3,657$

Here, an asymptotic (approximate) solution allowing finding the value of the Nusselt criterion in the entrance region will be presented:

$$\left(1-\tilde{r}^2\right)\frac{\partial\Theta}{\partial\tilde{z}} = \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\Theta}{\partial\tilde{r}}\right) = \frac{1}{\tilde{r}}\frac{\partial\Theta}{\partial\tilde{r}} + \frac{\partial^2\Theta}{\partial\tilde{r}^2}$$

$$\left(1-\tilde{r}^2\right)\frac{\partial\Theta}{\partial\tilde{z}} = \frac{1}{\tilde{r}}\frac{\partial\Theta}{\partial\tilde{r}} + \frac{\partial^2\Theta}{\partial\tilde{r}^2}$$



A substitution $\underline{\varepsilon = 1 - \tilde{r}}$ will be introduced so that the coordinate axis has a beginning on the pipe wall and is equal to one at the center of the pipe

$$\tilde{r} = 1 - \varepsilon \qquad \frac{\partial \Theta}{\partial \tilde{r}} = \frac{\partial \Theta}{\partial \varepsilon} \overset{=-1}{\frac{\partial \varepsilon}{\partial \tilde{r}}} = -\frac{\partial \Theta}{\partial \varepsilon}$$

$$\tilde{r}^2 = 1 - 2\varepsilon + \varepsilon^2 \quad \frac{\partial^2 \Theta}{\partial \tilde{r}^2} = \frac{\partial}{\partial \tilde{r}} \left(\frac{\partial \Theta}{\partial \tilde{r}} \right) = \frac{\partial}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \tilde{r}} \left(-\frac{\partial \Theta}{\partial \varepsilon} \right) = \frac{\partial^2 \Theta}{\partial \varepsilon^2}$$

$$(1 - 1 + 2\varepsilon - \varepsilon^2) \frac{\partial \Theta}{\partial \tilde{z}} = -\frac{1}{1 - \varepsilon} \frac{\partial \Theta}{\partial \varepsilon} + \frac{\partial^2 \Theta}{\partial \varepsilon^2}$$

$$\boxed{(2\varepsilon - \varepsilon^2) \frac{\partial \Theta}{\partial \tilde{z}} = \frac{1}{\varepsilon - 1} \frac{\partial \Theta}{\partial \varepsilon} + \frac{\partial^2 \Theta}{\partial \varepsilon^2}}$$

Boundary conditions:

$$\boxed{\begin{array}{ll} \tilde{z} = 0 & \Theta = 0 \\ \varepsilon = 1 & \dfrac{\partial \Theta}{\partial \tilde{r}} = 0 \\ \varepsilon = 0 & \Theta = 1 \end{array}}$$

In place of $\Delta\tilde{z}$ from the tube entry, the heat was transferred to a distance of δ from the wall. Derivatives will be replaced by differences and an estimate of the size of the individual members of the equation will be made.

$$(2\varepsilon - \varepsilon^2)\frac{\partial\Theta}{\partial\tilde{z}} \doteq (2\delta - \delta^2)\frac{\Delta\Theta}{\Delta\tilde{z}} \doteq 2\delta\frac{1}{\Delta\tilde{z}} \doteq \frac{\delta}{\Delta\tilde{z}}$$

$$\delta^2 \ll 2\delta, \Delta\Theta = 1$$

This is the order estimate, therefore the coefficient 2 will be neglected

$$\frac{1}{\varepsilon - 1}\frac{\partial\Theta}{\partial\varepsilon} \doteq \frac{1}{\delta - 1}\frac{\Delta\Theta}{\delta} \doteq -\frac{\Delta\Theta}{\delta} \doteq -\frac{1}{\delta}$$

$$\delta \ll 1 \qquad \Delta\Theta = 1$$

$$\frac{\partial^2\Theta}{\partial\varepsilon^2} \doteq \frac{\Delta\Theta}{\delta^2} \doteq \frac{1}{\delta^2}$$

$$\Delta\Theta = 1$$

Now all three of the resulting terms will be put into the original equation.

$$\frac{\delta}{\Delta\tilde{z}} = -\frac{1}{\delta} + \frac{1}{\delta^2} \qquad \overset{\times}{|}(\delta^2\Delta\tilde{z})$$

$$\frac{\delta}{\Delta\tilde{z}} = -\frac{1}{\delta} + \frac{1}{\delta^2} \qquad \delta - \text{ small}$$

$$\frac{\delta}{\Delta\tilde{y}} \approx \frac{1}{\delta^2} \qquad \frac{1}{\delta} \ll \frac{1}{\delta^2}$$

$$\delta^3 = -\delta\Delta\tilde{z} + \Delta\tilde{z}$$

Because δ is very small $\delta\Delta\tilde{z} \ll \Delta\tilde{z}$ $\boxed{\delta^3 \sim \Delta\tilde{z}}$

We will calculate the Nusselt criterion:

$$\mathrm{Nu} = \frac{-\dfrac{\mathrm{d}\Theta}{\mathrm{d}\varepsilon}}{\Delta\Theta} \doteq \frac{\dfrac{\Delta\Theta}{\delta}}{\Delta\Theta} = \frac{1}{\delta}$$

$$\mathrm{Nu} \approx \Delta\tilde{z}^{-\frac{1}{3}} \qquad \tilde{z} = \frac{z}{R\,\mathrm{Pe}} \qquad \boxed{\mathrm{Nu} \approx \left(\frac{z}{R}\right)^{-\frac{1}{3}} \mathrm{Pe}^{\frac{1}{3}} = \left(\frac{R\,\mathrm{Pe}}{z}\right)^{\frac{1}{3}}}$$

Example: Calculate the Nusselt criterion

$$\mathrm{Nu} = \frac{-\frac{\mathrm{d}\Theta}{\mathrm{d}\varepsilon}}{\Delta\Theta} \doteq \frac{\frac{\Delta\Theta}{\delta}}{\Delta\Theta} = \frac{1}{\delta}$$

$$\mathrm{Nu} \approx \Delta\tilde{z}^{-\frac{1}{3}} \qquad \tilde{z} = \frac{z}{R\,\mathrm{Pe}} \qquad \boxed{\mathrm{Nu} \approx \left(\frac{z}{R}\right)^{-\frac{1}{3}} \mathrm{Pe}^{\frac{1}{3}} = \left(\frac{R\,\mathrm{Pe}}{z}\right)^{\frac{1}{3}}}$$

Exact relationship for Nu for given boundary conditions and geometry.

$$\boxed{\mathrm{Nu} = 1,357 \left(\frac{z}{R}\right)^{-\frac{1}{3}} \mathrm{Pe}^{\frac{1}{3}}}$$

Estimated length of the inlet/entrance region:

$$\frac{L_T}{R} \sim 1 + 0,1\,\mathrm{Pe}$$, where $$\mathrm{Pe} = \frac{2UR}{a}$$

L_T - The distance from the pipe inlet, when Nu reaches a constant value.

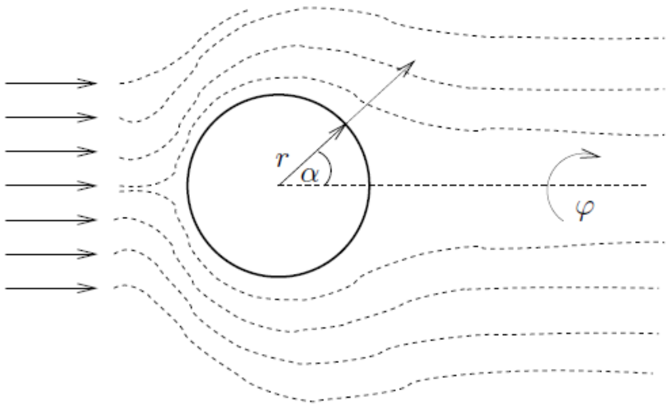
Lecture 7

Laminar flow along a solid object

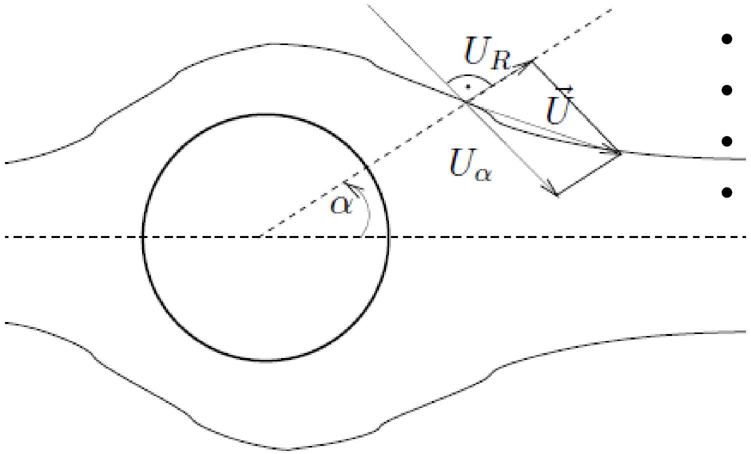
Laminar flow along a solid object - Heat transfer

- Flow along a solid object → heat transfer coefficient α
- Heat transfer fluid-solid wall depends on flow-object orientation
- Velocity profile 2D or 3D
- Pe » 1 → temperature edge sublayer: temperature change from surface (then constant temperature in bulk)

Solid object = sphere → velocity profile, heat transfer



Shape of velocity profile, spherical coordinates



Decomposition of average velocity U

- Velocity profile dependent on r, α coordinates
- Velocity profile symmetry: based on rotational axis φ
- Velocity U composed of two parts
- Based on Navier-Stokes:

Radius

$$U_R = U\cos\alpha\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

$$U_\alpha = -U\sin\alpha\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]$$

Average velocity far from sphere

- FK equation in a steady state without heat source: $\vec{U}\cdot\nabla T = a\nabla^2 T$
- FK equation in spherical coordinates (transformation formula (1) and (4) in page 85):

$$U_R\frac{\partial T}{\partial r} + U_\alpha\frac{1}{r}\frac{\partial T}{\partial \alpha} = a\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha\frac{\partial T}{\partial \alpha}\right)\right]$$

- Temperature boundary conditions:

$$\begin{aligned} &r = R, \alpha && T = T_0 \\ &r \to +\infty, \alpha && T = T_\infty \end{aligned}$$

- Scaling factors:

$$\Theta = \frac{T - T_\infty}{T_0 - T_\infty} \qquad \xi = \frac{r}{R} \qquad \tilde{v} = \frac{\vec{U}}{U}$$

→ So far, angle α is not scaled

$$U\tilde{v}_R\frac{\cancel{T_0 - T_\infty}}{\cancel{R}}\frac{\partial\Theta}{\partial\xi} + U\tilde{v}_\alpha\frac{1}{\cancel{R}}\frac{\cancel{T_0 - T_\infty}}{\xi}\frac{\partial\Theta}{\partial\alpha} =$$

$$= a\left[\frac{1}{\cancel{R^2}}\frac{1}{\underset{R}{\xi^2}}\frac{\partial}{\partial\xi}\left(\cancel{R^2}\xi^2\frac{(\cancel{T_0 - T_\infty})\partial\Theta}{R\ \partial\xi}\right) + \frac{1}{\cancel{R^2}\xi^2\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{(\cancel{T_0 - T_\infty})\partial\Theta}{\partial\alpha}\right)\right]\ \Bigg|\ \frac{R}{T_0 - T_\infty}$$

→ $\cdot(R/a)$

Pe = $(U.R)/a$

$$\boxed{\mathrm{Pe}\left[\tilde{v}_R\frac{\partial\Theta}{\partial\xi} + \tilde{v}_\alpha\frac{1}{\xi}\frac{\partial\Theta}{\partial\alpha}\right] = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{\xi^2\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial\Theta}{\partial\alpha}\right)}$$

$$\begin{aligned} &\xi = 1, \alpha && \Theta = 1 \\ &\xi \to +\infty, \alpha && \Theta = 0 \end{aligned}$$

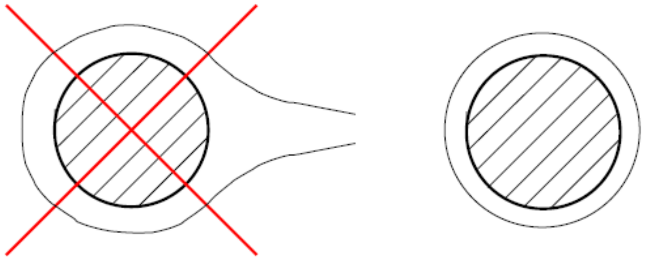
Heat transfer for Stokes flow (creeping flow) (inertial forces are small compared with viscous forces) (Re → 0): Pe ≪ 1

- For Pe $\xrightarrow{<< 1}$ 0 heat transfer by convection is negligible
- Heat transfer occurs primarily by conduction
- Simplified form of FK equation:

$$0 = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right) + \frac{1}{\xi^2 \sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha \frac{\partial \Theta}{\partial \alpha}\right)$$

- Since convection is negligible, fluid flow does not deform the temperature profile around sphere → symetrical according to α cooridanate

Pe ≪ 1



- Another simplification:

$$0 = \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)$$

$$\boxed{\begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right) = 0 & \\ \xi = 1 & \Theta = 1 \\ \xi \to +\infty & \Theta = 0 \end{array}}$$

Solution:

$$\begin{aligned} \xi^2 \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} &= C_1 \\ \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} &= \frac{C_1}{\xi^2} \\ \mathrm{d}\Theta &= \frac{C_1}{\xi^2}\mathrm{d}\xi \\ \Theta &= -\frac{C_1}{\xi} + C_2 \end{aligned}$$

$$\begin{array}{llll} \xi \to +\infty & \Theta = 0 & \Rightarrow & C_2 = 0 \\ & \Theta & = & -\frac{C_1}{\xi} \\ \xi = 1 & \Theta = 1 & \Rightarrow & C_1 = -1 \end{array}$$

$$\boxed{\Theta = \frac{1}{\xi}}$$

Nu criterion for sphere with symetrical temperature profile:

$$\mathrm{Nu} = \frac{-\left.\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right|_{\xi=1}}{\Delta\Theta} = \frac{+\left.\frac{1}{\xi^2}\right|_{\xi=1}}{1} = 1$$

- If: $\mathrm{Nu} \equiv \frac{\alpha R}{\lambda}$ $\longrightarrow$ Nu = 1

- Or: $\mathrm{Nu} \equiv \frac{\alpha D}{\lambda}$ $\longrightarrow$ Nu = 2

Heat transfer for Stokes flow (Re → 0): Pe ≫ 1

$$\begin{matrix} \mathrm{Pe} \gg 1 \\ \mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr} \end{matrix} \quad \Rightarrow \quad \begin{matrix} \mathrm{Re} \ll 1 \\ \mathrm{Pr} \gg 1 \end{matrix} \qquad \mathrm{Pr} \equiv \frac{\nu}{a} \gg 1$$

ν: Kinematic viscosity [$m^2\,s^{-1}$]

a: thermal diffusivity [$m^2\,s^{-1}$]

Thus:

- High viscosity fluids → velocity profile is developed quite far from sphere
- Low thermal diffusivity → temperature changes only realy close to the surface = Temperature sublayer
- Temperature and heat conductivity are linearly dependent

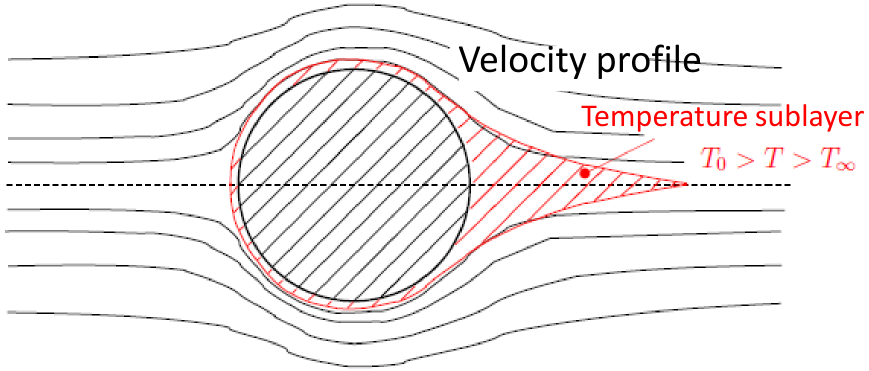


Figure: Shape of velocity and temperature profiles.

- Even though Pe ≫ 1, heat conduction IS NOT NEGLIGIBLE!
- Heat conduction is the only mechanism of heat transport from sphere surface to fluid surroundings
- Velocity U = 0 on the surface
- (Mechanism of radiation will be discussed later)

- Two parts of solution: INNER and OUTER

1) OUTER solution

- Description of temperature outside the sublayer
- No meaning for calculation of Nu criterion
- Without conduction parts + BC

$$\xi \to +\infty \quad \Theta = 0$$

$$\tilde{v}_R \frac{\partial \Theta}{\partial \xi} + \tilde{v}_\alpha \frac{1}{\xi} \frac{\partial \Theta}{\partial \alpha} = 0$$

$$\xi \to +\infty$$

$$\begin{aligned} \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} &= 0 \\ \Theta &= K \\ \Theta &= 0 \end{aligned}$$

Temperature is not a function of α angle.

Outer solution: $\boxed{\Theta = 0}$

1) INNER solution

- Description of temperature inside the sublayer
- Necessary for calculation of Nu criterion
- Both effects – conduction and convection
- Scaling → all variables value of appros. 1
- FK equation:

$$\mathrm{Pe}\left[\tilde{v}_R \frac{\partial \Theta}{\partial \xi} + \tilde{v}_\alpha \frac{1}{\xi}\frac{\partial \Theta}{\partial \alpha}\right] = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right) + \frac{1}{\xi^2 \sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha \frac{\partial \Theta}{\partial \alpha}\right)$$

- Temperature θ, velocities v_R, $v_\alpha \approx 1$

$$\tilde{v}_R = \frac{U_R}{U} = \cos\alpha\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

$$\boxed{\tilde{v}_R = \cos\alpha\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]}$$

$$\tilde{v}_\alpha = \frac{U_\alpha}{U} = -\sin\alpha\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]$$

$$\boxed{\tilde{v}_\alpha = -\sin\alpha\left[1 - \frac{3}{4}\xi^{-1} - \frac{1}{4}\xi^{-3}\right]}$$

A) Scaling of α (0-π)

(0, 180) $\eta = \cos\alpha$

$$\tilde{v}_R = \eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right] \qquad \frac{\partial\eta}{\partial\alpha} = -\sin\alpha$$

$$\mathrm{Pe}\left\{\eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]\frac{\partial\Theta}{\partial\xi} - \sin\alpha\left[1 - \frac{3}{4}\xi^{-1} - \frac{1}{4}\xi^{-3}\right]\frac{1}{\xi}\frac{\partial\Theta}{\partial\eta}\frac{\partial\eta}{\partial\alpha}\right\} =$$

+ $\frac{\partial\eta}{\partial\alpha} = -\sin\alpha,\ \sin^2\alpha = 1 - \cos^2\alpha = 1 - \eta^2$

$$= \mathrm{Pe}\left\{\eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]\frac{\partial\Theta}{\partial\xi} + (1-\eta^2)\frac{1}{\xi}\left[1 - \frac{3}{4}\xi^{-1} - \frac{1}{4}\xi^{-3}\right]\frac{\partial\Theta}{\partial\eta}\right\}$$

Left side of F.K. equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{\sin\alpha}\frac{1}{\xi^2}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial\Theta}{\partial\alpha}\right) =$$

$$= \frac{2}{\xi}\frac{\partial\Theta}{\partial\xi} + \frac{\partial^2\Theta}{\partial\xi^2} + \frac{1}{\sin\alpha}\frac{1}{\xi^2}\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial\alpha}\left(\sin\alpha\frac{\partial\Theta}{\partial\eta}\frac{\partial\eta}{\partial\alpha}\right) =$$

$$= \frac{2}{\xi}\frac{\partial\Theta}{\partial\xi} + \frac{\partial^2\Theta}{\partial\xi^2} + \frac{1}{\xi^2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right]$$

Right side of F.K. equation

$$\boxed{\begin{aligned}&\eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]\frac{\partial\Theta}{\partial\xi} + (1-\eta^2)\frac{1}{\xi}\left[1 - \frac{3}{4}\xi^{-1} - \frac{1}{4}\xi^{-3}\right]\frac{\partial\Theta}{\partial\eta} =\\ &= \mathrm{Pe}^{-1}\left\{\frac{2}{\xi}\frac{\partial\Theta}{\partial\xi} + \frac{\partial^2\Theta}{\partial\xi^2} + \frac{1}{\xi^2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right]\right\}\end{aligned}}$$

/divided by Pe criterion

F.K. equation

Now scaling of radial coordinates should be performed ξ from <1, 1+x> to the interval <0,1>
x: sublayer thickness

$$Y \equiv (\xi - 1)\mathrm{Pe}^{b}$$

New radial coordinate <0,1>

When $\xi = 1 \rightarrow Y = 0$

- With increasing Pe, sublayer width is decreasing: conduction velocity very low compared to convection velocity. Therefore *Y* depends on Pe
- *b* is unknown constant. It must be positive in order to make new radial between <0,1>

B) Scaling of ξ (1;1+sublayer width)

$$Y \equiv (\xi - 1)\mathrm{Pe}^b$$ ✚

$$\frac{\partial \Theta}{\partial \xi} = \frac{\partial \Theta}{\partial Y}\frac{\partial Y}{\partial \xi} = \frac{\partial \Theta}{\partial Y}\mathrm{Pe}^b$$

$$\frac{\partial^2 \Theta}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial \Theta}{\partial \xi}\right) = \frac{\partial}{\partial Y}\frac{\partial Y}{\partial \xi}\left(\mathrm{Pe}^b\frac{\partial \Theta}{\partial Y}\right) = \mathrm{Pe}^{2b}\frac{\partial^2 \Theta}{\partial Y^2}$$

- New Y coordinate
- ξ = 1, Y = 0
- With increasing Pe, sublayer width is decreasing: conduction velocity very low compared to convection velocity
- b constant is positive

Value comparison of individual parts of FK

1. part

$$\eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]\frac{\partial \Theta}{\partial \xi}$$

$$\eta\left[1 - \frac{3}{2}\xi^{-1} + \frac{1}{2}\xi^{-3}\right]\frac{\partial \Theta}{\partial Y}\mathrm{Pe}^b = \eta \mathrm{Pe}^b\frac{\partial \Theta}{\partial Y}\left[1 - \frac{3}{2}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1+3x+3x^2+x^3}\right] =$$

$$= \eta \mathrm{Pe}^b\frac{\partial \Theta}{\partial Y}\left[1 - \frac{3}{2} + \frac{3}{2}x - \frac{3}{2}x^2 + \frac{1}{2} - \frac{3}{2}x + 3x^2\right] \doteq$$

$$\doteq \eta \mathrm{Pe}^b\frac{\partial \Theta}{\partial Y}\frac{3}{2}x^2 = \eta \mathrm{Pe}^b\frac{\partial \Theta}{\partial Y}\frac{3}{2}Y^2\mathrm{Pe}^{-2b} = \mathrm{Pe}^{-b}\,\eta\frac{3}{2}Y^2\frac{\partial \Theta}{\partial Y}$$

Size ~ 1

$$Y = (\underbrace{\xi}_{1+x} - 1)\mathrm{Pe}^b = x\mathrm{Pe}^b.$$

$x = Y/\mathrm{Pe}^b$

≈ Pe^{-b}

1st term is in the order of

Value comparison of individual parts of FK

2. part

$$\frac{1-\eta^2}{\xi}\left[1-\frac{3}{4}\xi^{-1}-\frac{1}{4}\xi^{-3}\right]\frac{\partial\Theta}{\partial\eta}=$$

$$=(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\frac{1}{Y\,\mathrm{Pe}^{-b}+1}\left[1-\frac{3}{4}+\frac{3}{4}x-\frac{3}{4}x^2-\frac{1}{4}+\frac{3}{4}x-\frac{6}{4}x^2\right]\doteq$$

$$\mathrm{Pe}\gg 1, Y\sim 1 \quad\Rightarrow\quad Y\,\mathrm{Pe}^{-b}\ll 1 \quad\Rightarrow\quad \frac{1}{Y\,\mathrm{Pe}^{-b}+1}\sim 1$$

$$\doteq(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\left[\frac{3}{2}x-\frac{9}{4}x^2\right]\doteq(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\frac{3}{2}x=\underbrace{(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\frac{3}{2}Y}_{\sim 1}\,\mathrm{Pe}^{-b} \qquad \approx \mathrm{Pe}^{-b}$$

In thermal sublayer ξ is close to1, therefore x is close to 0 → $\hat{x}^2 \ll x$

3. part

$$\mathrm{Pe}^{-1}\frac{2}{\xi}\frac{\partial\Theta}{\partial\xi}=\mathrm{Pe}^{-1}\underbrace{\frac{2}{Y\mathrm{Pe}^{-b}+1}}_{\longrightarrow 1}\frac{\partial\Theta}{\partial Y}\mathrm{Pe}^{b}=2\underbrace{\frac{\partial\Theta}{\partial Y}}_{\sim 1}\,\mathrm{Pe}^{b-1} \qquad \approx \mathrm{Pe}^{b-1}$$

Value comparison of individual parts of FK

4. part

$$\mathrm{Pe}^{-1}\frac{\partial^2\Theta}{\partial\xi^2} = \mathrm{Pe}^{-1}\mathrm{Pe}^{2b}\underbrace{\frac{\partial^2\Theta}{\partial Y^2}}_{\sim 1}$$

≈ Pe^{2b-1}

5. part

$$\mathrm{Pe}^{-1}\frac{1}{\xi^2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right] = \mathrm{Pe}^{-1}\underbrace{\frac{1}{(Y\mathrm{Pe}^{-b}+1)^2}}_{\sim 1}\underbrace{\left[(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right]}_{\sim 1}$$

≈ Pe^{-1}

<u>All parts:</u>

$$\mathrm{Pe}^{-b} + \mathrm{Pe}^{-b} = \mathrm{Pe}^{b-1} + \mathrm{Pe}^{2b-1} + \mathrm{Pe}^{-1}$$

Pe ≫ 1, b > 0 → only largest parts on both sides

$\mathrm{Pe}^{-b} = \mathrm{Pe}^{2b-1}$ → $-b = 2b-1$ → $b = 1/3$! $\boxed{Y = (\xi - 1)\mathrm{Pe}^{\frac{1}{3}}}$!

- We can express Nu criterion as a function of α:

$$\mathrm{Nu}(\alpha) = \frac{-\dfrac{\partial\Theta}{\partial\xi}}{\Delta\Theta} = \underbrace{\frac{-\dfrac{\partial\Theta}{\partial Y}}{\Delta\Theta}}_{\text{Constant size} \sim 1} \frac{\partial Y}{\partial\xi} = K\,\mathrm{Pe}^{\frac{1}{3}}$$

- *K* is function of object shape and position:

$$\mathrm{Nu}(\alpha) = K(\alpha)\,\mathrm{Pe}^{\frac{1}{3}}$$

- Average Nu criterion of all surface:

$$\overline{\mathrm{Nu}} = \frac{1}{A}\int_A K\,\mathrm{Pe}^{\frac{1}{3}}\,\mathrm{d}A = \overline{K}\,\mathrm{Pe}^{\frac{1}{3}}$$

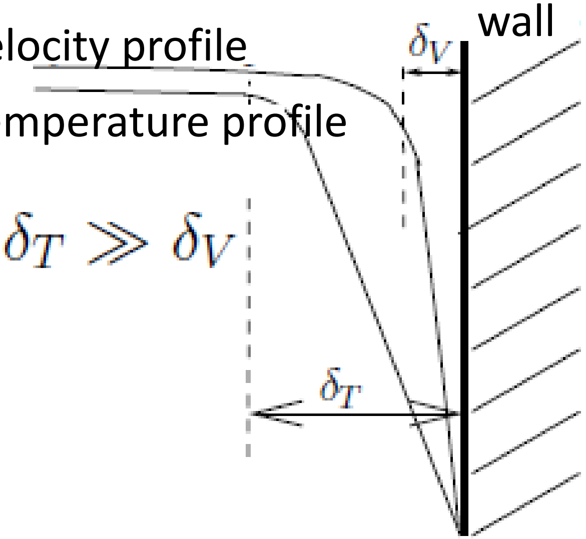
- For sphere (Pe ≫ 1):

$$\overline{\mathrm{Nu}} = 1{,}249\,\mathrm{Pe}^{\frac{1}{3}}$$

Heat transfer in a laminar sublayer: Re ≫ 1, Pe ≫ 1

- For Re ≫ 1, Pe ≫ 1 both <u>temperature and velocity sublayer</u>
- Width may be various

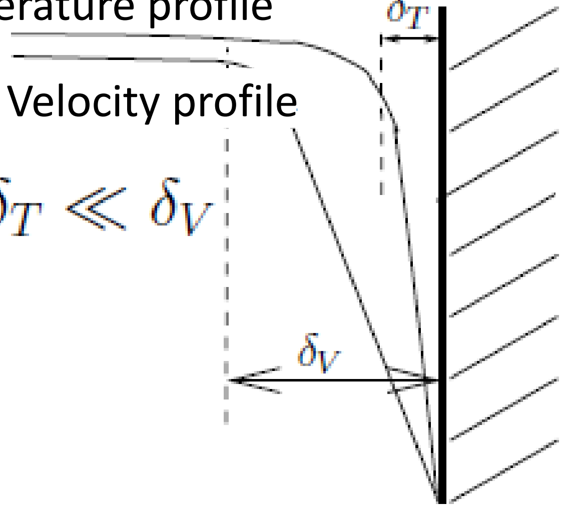
$$Pe = Re\ Pr \qquad Pr = \frac{\nu}{a}$$



$$\delta_T \gg \delta_V$$

- Pr ≪ 1
- $\nu \to 0$, $a \to \infty$

$$\boxed{Nu = k\, Pr^{\frac{1}{2}} Re^{\frac{1}{2}}}$$



$$\delta_T \ll \delta_V$$

- Pr ≫ 1
- $\nu \to \infty$, $a \to 0$

$$\boxed{Nu = k\, Pr^{\frac{1}{3}} Re^{\frac{1}{2}}}$$

✚ Pr ≈ 1, $\nu \approx a$ → $\delta_T = \delta_V$

Nu criterion for laminar flow

The dependence is always in the form
and must hold Pe >> 1

$$\mathrm{Nu} = k\,\mathrm{Re}^a\,\mathrm{Pr}^b$$

$\mathrm{Re} \ll 1 \quad \mathrm{Pr} \gg 1$ Stokes flow around solid object (small particles, aerosol…)

$\mathrm{Re} \sim 1$

$$\mathrm{Nu} = k\,\mathrm{Pe}^{\frac{1}{3}} = k\,\mathrm{Re}^{\frac{1}{3}}\mathrm{Pr}^{\frac{1}{3}}$$

$\mathrm{Re} \gg 1 \quad \mathrm{Pr} \gg 1$ Laminar flow around solid object

$$\mathrm{Nu} = k\,\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$$

$\mathrm{Re} \gg 1 \quad \mathrm{Pr} \ll 1$ Laminar flow along solid object

$$\mathrm{Nu} = k\,\mathrm{Pe}^{\frac{1}{2}} = k\,\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{2}}$$

$\underbrace{\mathrm{Re},\ \mathrm{Pr}}_{\text{arbitrary}} \quad \mathrm{Pe} \gg 1$ One liquid along another (emulsions, bubbles…)

$$\mathrm{Nu} = k\,\mathrm{Pe}^{\frac{1}{2}} = k\,\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{2}}$$

Lecture 8

Heat transfer by Natural (Free) convection

Heat transfer by Natural (Free) convection

- Liquid temperature change → density change (temperature dilatation) → influences velocity and pressure profile
- FK eq. + NS ew. (2D or 3D) + continuity eq. solved together
- Balance of momentum is given by NS:

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=\rho\vec{g}-\nabla p+\eta\,\nabla^2\,\vec{v}$$

- ρ = ρ (T), η = η(T)
- 3 parts of NS equation influenced by temperature change!

- Definition of dynamic pressure (including gravity) for reference temperature T_0 and density ρ_0:

$$\nabla P=\nabla p-\rho_0 g$$

- Dynamic pressure combined with NS eq.:

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla P+g\,\underbrace{(\rho-\rho_0)}+\eta\,\nabla^2\,\vec{v}$$

Driving force of natural convection

Inertial term

Viscous term

$$\boxed{\mathrm{Re}\left(\frac{\partial\tilde{v}_x}{\partial\tilde{t}}+\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right)=-\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}}$$

Fluide flow
Navier-Stokes equation

- **Assumption:** Liquid density is influenced only by temperature change.
  It is not dependent on concentration of compounds.

- Then, we are allowed to use **Boussinesq approximation** (buoyancy) (2 parts):

1) Density(T) changes lineary around T_0 temperature. We can use Taylor series:

$$\rho(T) = \rho_0(T_0) + \left(\frac{\partial \rho}{\partial T}\right)_{T_0} (T - T_0)$$

Definition of temperature dilatation coefficient (coefficient of expansion of the fluid) β:

$$\boxed{\rho - \rho_0 = \beta\, \rho\, (T\text{-}T_0)} \qquad \beta \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{T_0}$$

Combination of Taylor series with formula of temperature dilatation coefficient:

$$\boxed{\frac{\rho - \rho_0}{\rho} = -\beta \cancel{\frac{\rho}{\rho}}(T - T_0)}$$

Then, combined with NS and divided by density:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla P + \vec{g}\frac{1}{\rho}(\rho - \rho_0) + \underbrace{\frac{1}{\rho}\eta}_{\text{kinematic viscosity}}\ \nabla^2 \vec{v}$$

$$\frac{\eta}{\rho} = \nu \qquad [\mathrm{m^2 s^{-1}}]$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla P - \vec{g}\beta(T - T_0) + \nu \nabla^2 \vec{v}$$

Boussinesq approximation (buoyancy)

Boussinesq approximation:

The density is assumed to be constant in all the conservation equations except in the body force term in Y-momentum equation, where the temperature dependent density that drives the flow in natural convection is captured by taking Boussinesq approximation into account:

$$\boxed{\rho - \rho_0 = \beta\,\rho\,(T - T_0)}$$

This approximation is accurate as long as changes in actual density are small

- Then, we are allowed to use **Boussinesq approximation** (2 parts):

2) Density change is negligible around reference temperature: $\rho(T) \approx \rho(T_0)$.

Density change is negligible if: $\beta \cdot \Delta T \ll 1$ or $\frac{\Delta\rho}{\rho} \ll 1$ or $\frac{a^2}{L^2 c_p \Delta T} \ll 1$

For gaz B can be close to 0

In liquids delta rho can be very small

Then:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho_0}\nabla P - \vec{g}\beta(T - T_0) + \nu_0\nabla^2\vec{v}}$$

Characteristic properties are constants: A, rho 0, nu 0

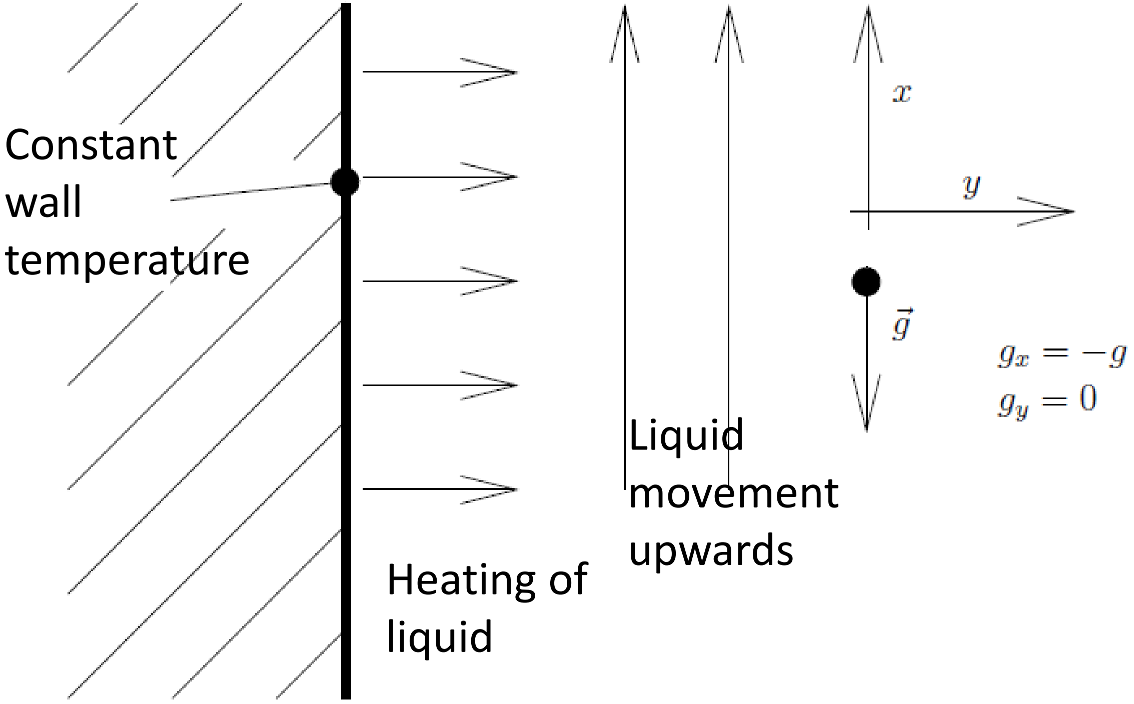
Temperature dilatation coefficient for ideal gas:

$$\boxed{\beta = \frac{1}{T}\ [\mathrm{K}^{-1}]}$$

$$pV = nRT = \frac{m}{M}RT \Rightarrow \rho = \frac{pM}{RT} \Rightarrow \frac{\partial\rho}{\partial T} = -\frac{pM}{RT^2}$$

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{T_0} = -\frac{RT}{pM}\frac{-pM}{RT^2} = \frac{1}{T}$$

Heat transfer by Natural convection near a vertical infinite wall



- Heat transport influences density

→ change of velocity and pressure profile

- FK eq. + NS eq. + continuity eq.:

→ for velocity, pressure, temperature

Steady state without heat source:

$$
\begin{array}{lrcl}
\text{FK} & v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} & = & a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \\
\text{NS} & v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} & = & -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g\beta(T - T_0) + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \\
 & v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} & = & -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \\
\text{Continuity eq.} & \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} & = & 0
\end{array}
$$

Heat transfer by Natural convection in a vertical infinite wall

- Characteristic variables in value interval <0,1>:

$$\theta = \frac{T - T_0}{\Delta T} \sim 1$$

T_0 – reference temperature, ΔT – maximum temperature difference

$$\tilde{v}_x = \frac{v_x}{U_b},\ \tilde{v}_y = \frac{v_y}{U_b}$$

U_b – reference velocity of natural convection (now unknown)

$$\tilde{x} = \frac{x}{L},\ \tilde{y} = \frac{y}{L}$$

L – characteristic length (often unknown)

- Natural convection:

Intertial+pressure+free convection parts of NS eq. approx. the same value (magnitude)

The viscosity part of NS is negligible. Far from the wall

$$v_x \frac{\partial v_x}{\partial x} \sim \frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\tilde{v}_x U_b \frac{U_b}{\cancel{L}}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} \sim \frac{1}{\rho_0}\frac{P_0}{\cancel{L}}\frac{\partial \tilde{p}}{\partial \tilde{x}} \quad \Bigg| \frac{L}{U_b^2}$$

$$\underbrace{\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}}}_{\sim 1} \sim \underbrace{\frac{P_0}{\rho_0 U_b^2}}\ \underbrace{\frac{\partial \tilde{p}}{\partial \tilde{x}}}$$

This has to be set. $\sim 1 \Rightarrow \boxed{P_0 = \rho_0 U_b^2}$

$$\tilde{P} = \frac{P}{\rho_0 U_b^2}$$

Nondimensionalization of NS eq.:

$$\frac{U_b^2}{L}\left(\tilde{v}_x\frac{\partial \tilde{v}_x}{\partial \tilde{x}}+\tilde{v}_y\frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right)=-\frac{\rho_0}{\rho_0}\frac{U_b^2}{L}\frac{\partial \tilde{P}}{\partial \tilde{x}}+g\beta\Delta T\Theta+\frac{\eta\, U_b}{L^2}\left(\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}+\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}\right)\quad\Bigg|\,\frac{L}{U_b^2}$$

$$\underset{\sim 1}{\tilde{v}_x\frac{\partial \tilde{v}_x}{\partial \tilde{x}}}+\underset{\sim 1}{\tilde{v}_y\frac{\partial \tilde{v}_x}{\partial \tilde{y}}}=\underset{\sim 1}{-\frac{\partial \tilde{P}}{\partial \tilde{x}}}+\underset{\sim 1}{\frac{g\beta L\Delta T}{U_b^2}}\Theta+\frac{\nu}{U_b\, L}\left(\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}+\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}\right)$$

- For natural convection, all above mentioned parts have approx. the same value
- Then:

$$\frac{g\beta L\Delta T}{U_b^2}=1 \qquad \boxed{U_b=\sqrt{gL\beta\Delta T}}$$

- Viscosity part of NS eq. is multiplied by $\dfrac{\nu}{U_b L}$
- Reynolds criterion: $\dfrac{U_b L}{\nu}$

$$\mathrm{Gr}=\left(\frac{U_b L}{\nu}\right)^2=\frac{g\beta\Delta T L^3}{\nu^2}=\mathrm{Re}_b^2$$

Grashof criterion

NS eq. with Grashof criterion:

$$\boxed{\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \Theta + \mathrm{Gr}^{-\frac{1}{2}} \left(\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2} \right)}$$

By analogy, dimensionless form of NS eq. in case of vy is:

$$\boxed{\tilde{v}_x \frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = -\frac{\partial \tilde{P}}{\partial \tilde{y}} + \mathrm{Gr}^{-\frac{1}{2}} \left(\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2} \right)}$$

Continuity eq.:

$$\frac{U_b}{L} \left(\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial \tilde{v}_y}{\partial \tilde{y}} \right) = 0$$

$$\boxed{\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0}$$

FK eq.:

$$\frac{U_b\cancel{\Delta T}}{\cancel{L}}\left(\tilde{v}_x\frac{\partial\Theta}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\Theta}{\partial\tilde{y}}\right) = \frac{a\cancel{\Delta T}}{L^{\cancel{2}}}\left(\frac{\partial^2\Theta}{\partial\tilde{x}^2}+\frac{\partial^2\Theta}{\partial\tilde{y}^2}\right) \quad \Bigg| \frac{L}{U_b\Delta T}$$

$$\tilde{v}_x\frac{\partial\Theta}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\Theta}{\partial\tilde{y}} = \frac{a}{U_b L}\left(\frac{\partial^2\Theta}{\partial\tilde{x}^2}+\frac{\partial^2\Theta}{\partial\tilde{y}^2}\right)$$

$$\boxed{\frac{a}{U_b L}=\frac{1}{\mathrm{Pe}_b}}$$ Péclet criterion for natural convection

$$\frac{LU_b}{a}=\frac{LU_b}{\nu}\frac{\nu}{a}=\mathrm{Gr}^{\frac{1}{2}}\mathrm{Pr}$$

$$\boxed{\tilde{v}_x\frac{\partial\Theta}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\Theta}{\partial\tilde{y}}=\mathrm{Gr}^{-\frac{1}{2}}\mathrm{Pr}^{-1}\left(\frac{\partial^2\Theta}{\partial\tilde{x}^2}+\frac{\partial^2\Theta}{\partial\tilde{y}^2}\right)}$$

- For description of natural convection, Rayleigh number is used:

$$\mathrm{Ra}=\mathrm{Gr}\,\mathrm{Pr}=\frac{gL^3\beta\Delta T}{a\,\nu}=$$ Destabilizing forces (natural convection) / Stabilizing forces (viscosity forces)

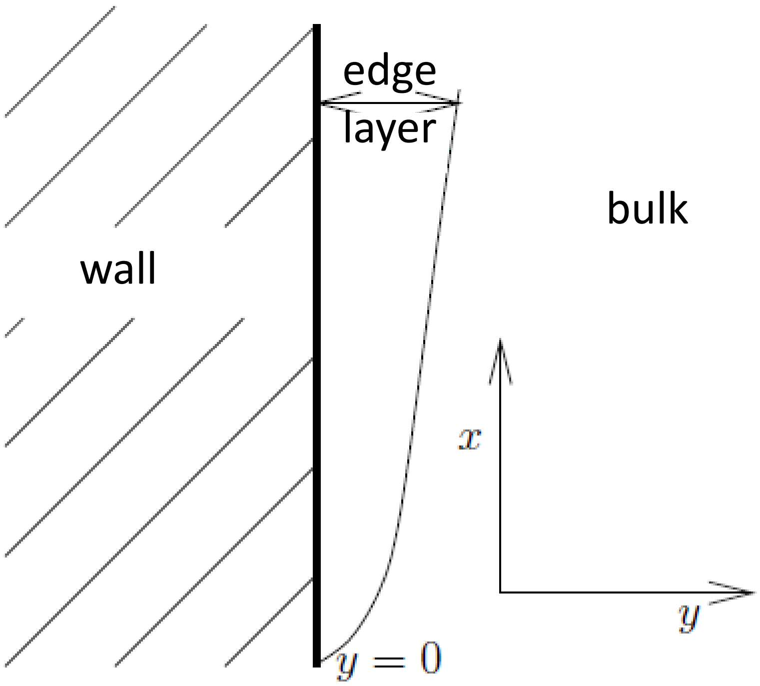
- The goal of calculations is **Nusselt criterion**:

$\mathrm{Nu}=\mathrm{Nu}(\mathrm{Gr},\,\mathrm{Pr},$ Geometry simplex, Position $)$

$\overline{\mathrm{Nu}}=\overline{\mathrm{Nu}}(\mathrm{Gr},\,\mathrm{Pr},$ Geometry simplex $)$

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

- For $\mathrm{Gr}^{\frac{1}{2}} = \mathrm{Re}_b \gg 1$ → velocity sublayer occurs (similar to laminar flow around solid objects)



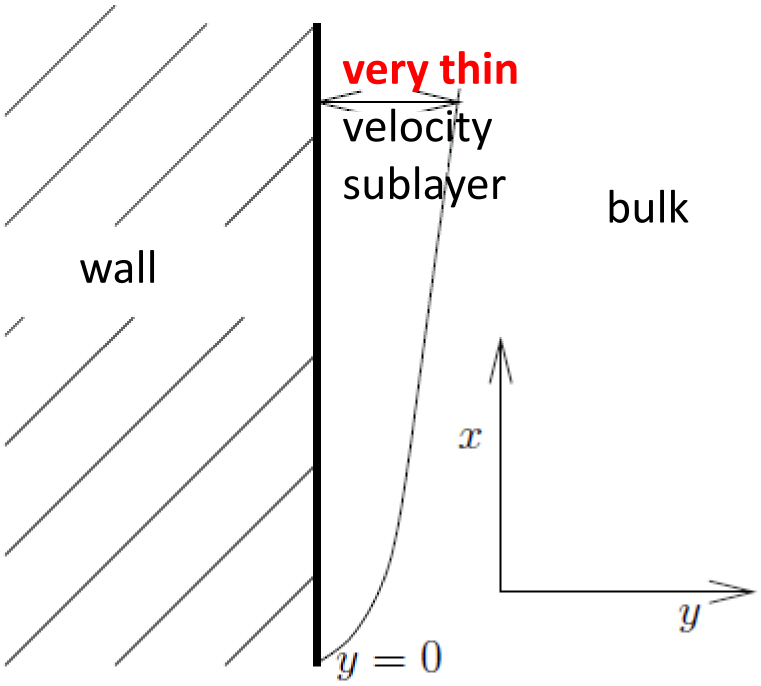
1) INNER solution: in edge layer.
   - intertial+viscosity+free convection parts of NS eq. approx. the same value
2) OUTER solution: bulk.
   - intertial+free convection parts of NS eq. approx. the same value
   - viscosity part of NS is negligible

INNER solution is necessary or Nu criterion calculation: $$\mathrm{Nu} = \frac{-\left.\dfrac{\partial \Theta}{\partial \tilde{y}}\right|_{y=0}}{\Delta \Theta}$$

↓

Simplification → lower number of used equations

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$



1) Second derivatives in x direction → 0 near infinite wall.

$$\frac{\partial^2 \Theta}{\partial \tilde{x}^2}, \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}, \frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} \to 0$$

2) In sublayer: pressure is not changing in y direction.

$$\frac{\partial P}{\partial y} \to 0 \qquad P = P(x)$$

1)+2) Similar approximation for flow between two infinite walls.

- We know pressure P=P(x) = inserted pressure Δp → we do not have to calculate it

→ **we need 3 eq. for $\tilde{v}_x, \tilde{v}_y, \Theta$: NS, FK, continuity eq.**

* It is difficult to prove simplifications 1),2) mentioned previously.
In general, it is allowed to reduce one equation: P=P(x).

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

- Final equations:

$$
\begin{array}{rcl}
\dfrac{\partial \tilde{v}_x}{\partial \tilde{x}} + \dfrac{\partial \tilde{v}_y}{\partial \tilde{y}} & = & 0 \\
\tilde{v}_x \dfrac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \dfrac{\partial \tilde{v}_x}{\partial \tilde{y}} & = & -\dfrac{\partial \tilde{P}}{\partial \tilde{x}} + \Theta + \mathrm{Gr}^{-\frac{1}{2}} \dfrac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2} \\
\sim 1 \qquad\quad \sim 1 & & \sim 1 \quad \sim 1 \quad \to 0 \quad \boxed{\to +\infty} \text{ !!!} \\
\tilde{v}_x \dfrac{\partial \Theta}{\partial \tilde{x}} + \tilde{v}_y \dfrac{\partial \Theta}{\partial \tilde{y}} & = & \mathrm{Gr}^{-\frac{1}{2}} \mathrm{Pr}^{-1} \dfrac{\partial^2 \Theta}{\partial \tilde{y}^2} \\
\sim 1 \qquad\quad \sim 1 & & \to 0 \quad \boxed{\to +\infty} \text{ !!!}
\end{array}
$$

$$\mathrm{Gr}^{\frac{1}{2}} \gg 1, \text{ tj. } \mathrm{Gr}^{-\frac{1}{2}} \to 0$$

For $\dfrac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$ a $\dfrac{\partial^2 \Theta}{\partial \tilde{y}^2} \to 1$ it is necessary to rescale **y coordinate** and **velocity v_y** (Gr cannot reach infinity)

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

↓

It is necessary to rescale **y coordinate** and **velocity v_y** (Gr cannot reach infinity):

$$Y \equiv \tilde{y}\,\mathrm{Gr}^{\frac{1}{4}} \qquad V_Y \equiv \tilde{v}_y\,\mathrm{Gr}^{\frac{1}{4}}$$

To eliminate Gr

$$\boxed{\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\cancel{\mathrm{Gr}^{-\frac{1}{4}}}}{\cancel{\mathrm{Gr}^{-\frac{1}{4}}}}\frac{\partial V_Y}{\partial Y} = 0}$$

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \cancel{\mathrm{Gr}^{-\frac{1}{4}}} V_Y \frac{\partial \tilde{v}_x}{\partial Y}\cancel{\mathrm{Gr}^{\frac{1}{4}}} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \Theta + \cancel{\mathrm{Gr}^{-\frac{1}{2}}}\frac{\partial^2 \tilde{v}_x}{\partial Y^2}\cancel{\mathrm{Gr}^{\frac{1}{2}}}$$

$$\boxed{\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + V_Y \frac{\partial \tilde{v}_x}{\partial Y} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \Theta + \frac{\partial^2 \tilde{v}_x}{\partial Y^2}}$$

$$\tilde{v}_x \frac{\partial \Theta}{\partial \tilde{x}} + V_Y \cancel{\mathrm{Gr}^{-\frac{1}{4}}}\cancel{\mathrm{Gr}^{\frac{1}{4}}}\frac{\partial \Theta}{\partial Y} = \cancel{\mathrm{Gr}^{-\frac{1}{2}}}\mathrm{Pr}^{-1}\frac{\partial^2 \Theta}{\partial Y^2}\cancel{\mathrm{Gr}^{\frac{1}{2}}}$$

$$\boxed{\tilde{v}_x \frac{\partial \Theta}{\partial \tilde{x}} + V_Y \frac{\partial \Theta}{\partial Y} = \mathrm{Pr}^{-1}\frac{\partial^2 \Theta}{\partial Y^2}}$$

→ all parts have approx. the same value (approx. same magnitude)

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

Sublayer properties are influnced by Pr criterion too: $\mathrm{Pr} = \dfrac{\nu}{a}$

$\mathrm{Pr} \ll 1$ $\rightarrow$ Thin velocity sublayer (low viscosity)

$\rightarrow$ Wide temperature sublayer (high thermal conductivity)

$\mathrm{Pr} \gg 1$ $\rightarrow$ Wide velocity sublayer (high viscosity)

$\rightarrow$ Thin temperature sublayer (low thermal conductivity)

↓

What are the values of individual parts of all used equations? → For limit values of Pr criterion: Pr → 0

Pr → ∞

- Thickness of the thermal sublayer is influenced by Pr, but it should stay in interval from 0 to 1

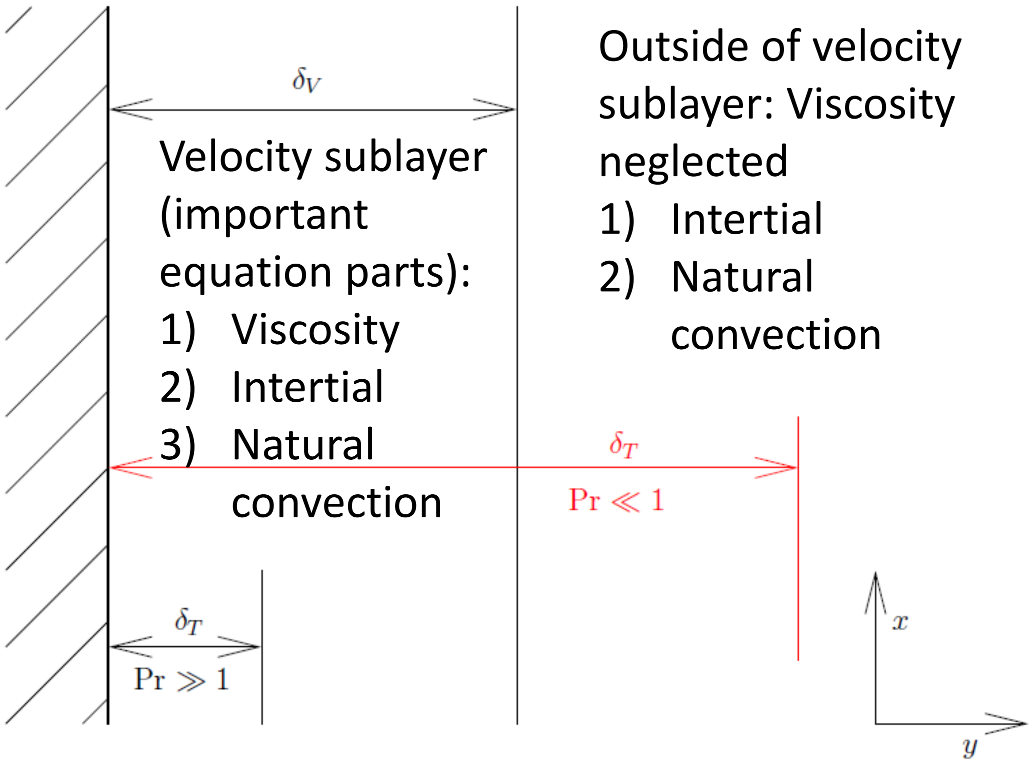
a) Dimensionless Y coordinate:

$$\widehat{Y} = Y\,\mathrm{Pr}^m$$ → Unknown index

b) Rescaled velocity (to keep velocity between values 0 and 1 in tempertature sublayer):

$$\widehat{v}_x = \tilde{v}_x \mathrm{Pr}^n \qquad \widehat{v}_y = V_Y \mathrm{Pr}^p$$

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$



Thermal and velocity sublayers near to infinity wall.

Rescaled equations:

$$\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial V_Y}{\partial Y} = 0$$

$$\mathrm{Pr}^{-n} \underbrace{\boxed{\frac{\partial \widehat{v}_x}{\partial \tilde{x}}}}_{\sim 1} + \mathrm{Pr}^{m-p} \underbrace{\boxed{\frac{\partial \widehat{v}_y}{\partial \widehat{Y}}}}_{\sim 1} = 0$$

Therefore: $\Rightarrow$ $\boxed{-n = m - p}$

N-S equation $\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + V_Y \frac{\partial \tilde{v}_x}{\partial Y} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \Theta + \frac{\partial^2 \tilde{v}_x}{\partial Y^2}$

$$\boxed{\mathrm{Pr}^{-2n}\, \widehat{v}_x \frac{\partial \widehat{v}_x}{\partial \tilde{x}} + \mathrm{Pr}^{-p-n+m}\, \widehat{v}_y \frac{\partial \widehat{v}_x}{\partial \widehat{Y}} = -\frac{\partial \widehat{P}}{\partial \tilde{x}} + \Theta + \mathrm{Pr}^{2m-n} \frac{\partial^2 \widehat{v}_y}{\partial \widehat{Y}^2}}$$

$\mathrm{Pr} \to 0$ $\sim \mathrm{Pr}^{-2n}$ $\sim \mathrm{Pr}^{-p-n+m}$ $\sim \mathrm{Pr}^{0}$

$\mathrm{Pr} \to \infty$ $\sim \mathrm{Pr}^{0}$ $\sim \mathrm{Pr}^{2m-n}$

In the thermal sub-layer

$$\tilde{v}_x \frac{\partial \Theta}{\partial \tilde{x}} + V_Y \frac{\partial \Theta}{\partial Y} = \frac{\partial^2 \Theta}{\partial Y^2} \mathrm{Pr}^{-1}$$

$$\boxed{\mathrm{Pr}^{-n}\, \widehat{v}_x \frac{\partial \Theta}{\partial \tilde{x}} + \mathrm{Pr}^{m-p}\, \widehat{v}_y \frac{\partial \Theta}{\partial \widehat{Y}} = \mathrm{Pr}^{2m-1} \frac{\partial^2 \Theta}{\partial^2 \widehat{Y}^2}}$$

$\mathrm{Pr} \to 0$ $\sim \mathrm{Pr}^{-n}$ $\sim \mathrm{Pr}^{m-p}$ $\sim \mathrm{Pr}^{2m-1}$

$\mathrm{Pr} \to \infty$ $\sim \mathrm{Pr}^{-n}$ $\sim \mathrm{Pr}^{m-p}$ $\sim \mathrm{Pr}^{2m-1}$

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

$\boxed{\mathrm{Pr} \gg 1 \quad \text{or} \quad \mathrm{Pr} \to \infty}$

In momentum balance, viscosity and natural convection prevail.

From NS $\quad 2m - n = 0 \Rightarrow n = 2m$

From FK $\quad 2m - 1 = -n, \; 2m - 1 = -2m$

$$\boxed{m = \frac{1}{4}, \quad n = \frac{1}{2}}$$

From continuity equation $\quad -n = m - p$

$$p = \frac{1}{4} + \frac{1}{2} \Rightarrow \boxed{p = \frac{3}{4}}$$

$$\widehat{Y} = Y\mathrm{Pr}^{\frac{1}{4}}$$

$$\widehat{v}_x = \tilde{v}_x Pr^{\frac{1}{2}}$$

$$\widehat{v}_y = V_Y \mathrm{Pr}^{\frac{3}{4}}$$

$\boxed{\mathrm{Pr} \ll 1 \quad \text{or} \quad \mathrm{Pr} \to 0}$

From

NS $\quad -2n = 0 \Rightarrow n = 0$

$-p - n + m = 0 \Rightarrow m = p$

From continuity equation not usable $\quad m = p$

From

FK $\quad 2m - 1 = -n = 0$

$$\boxed{m = \frac{1}{2}, \quad p = \frac{1}{2}, \quad n = 0}$$

$$\widehat{Y} = Y\mathrm{Pr}^{\frac{1}{2}}$$

$$\widehat{v}_x = \tilde{v}_x$$

$$\widehat{v}_y = V_Y \mathrm{Pr}^{\frac{1}{2}}$$

Heat transfer by Natural convection near a vertical infinite wall: $Gr^{1/2} \gg 1$

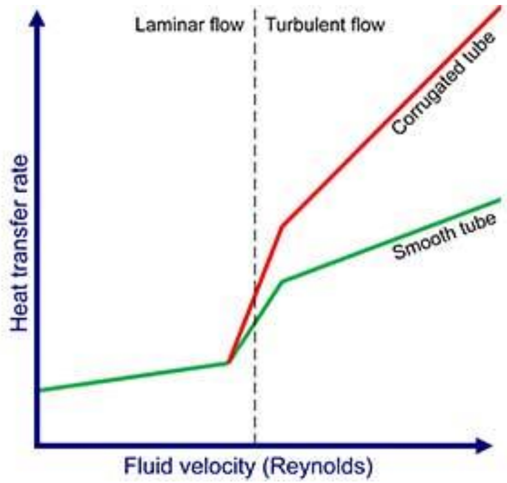
$\boxed{\mathrm{Pr} \gg 1 \quad \text{or} \quad \mathrm{Pr} \to \infty}$	$\boxed{\mathrm{Pr} \ll 1 \quad \text{or} \quad \mathrm{Pr} \to 0}$
Now, we can calculate Nu criterion near to wall surface: $\mathrm{Nu} = \frac{-\left.\frac{\partial \Theta}{\partial \tilde{y}}\right\|_{y=0}}{\Delta\Theta} = -\left.\frac{\partial \Theta}{\partial \tilde{y}}\right\|_{y=0} \quad , \Delta\Theta = 1$ $\mathrm{Nu} = -\frac{\partial \Theta}{\partial \hat{Y}} \frac{\partial \hat{Y}}{\partial Y} \frac{\partial Y}{\partial \tilde{y}}$	$\overline{\mathrm{Nu}} = \frac{1}{A} \int \mathrm{Nu}\, \mathrm{d}A$ $Y = \mathrm{Gr}^{\frac{1}{4}} \tilde{y}$
$\mathrm{Nu} = -\boxed{\frac{\partial \Theta}{\partial \hat{Y}}} \mathrm{Pr}^{\frac{1}{4}}\, \mathrm{Gr}^{\frac{1}{4}}$ Local value near to the surface is approx. ≈ 1 $\boxed{\overline{\mathrm{Nu}} = K \mathrm{Pr}^{\frac{1}{4}}\, \mathrm{Gr}^{\frac{1}{4}}}$ Nu criterion for vertical wall $\overline{\mathrm{Nu}} = 0,6703\, \mathrm{Pr}^{\frac{1}{4}}\, \mathrm{Gr}^{\frac{1}{4}}$	$\mathrm{Nu} = -\frac{\partial \Theta}{\partial \hat{Y}} \mathrm{Pr}^{\frac{1}{2}}\, \mathrm{Gr}^{\frac{1}{4}}$ ~ 1 $\boxed{\overline{\mathrm{Nu}} = K \mathrm{Pr}^{\frac{1}{2}}\, \mathrm{Gr}^{\frac{1}{4}}}$ Nu criterion for vertical wall $\overline{\mathrm{Nu}} = 0,8005\, \mathrm{Pr}^{\frac{1}{2}}\, \mathrm{Gr}^{\frac{1}{4}}$

Lecture 9

Heat transfer in turbulent and restricted environment

Heat transfer in turbulent and restricted environment

- Character of flow is described by Rynolds number: $Re = \dfrac{F_{intertial}}{F_{viscosity}}$



1) $Re < Re_c$: laminar flow (pipe with circular cross section $Re_c \approx 2\,300$)
2) $Re > Re_c$: transient flow = laminar flow is changing with turbulent flow
3) $Re \gg Re_c$: only turbulent flow

Characterization of turbulent flow:

- Fluctuation of velocity, temperature and other parameters
- Time dependent **3D** (always) flow
- Swirling of fluid increases mass and heat transfer
- Pressure drop is increasing

Velocity and pressure at a point fluctuate with time in a random manner

Fanning friction coefficient: ratio between the local shear stress and the local flow kinetic energy density

$$f \equiv \frac{\frac{\Delta p}{L}}{\frac{\rho\, U_0^2}{R}} = \frac{\Delta p\, R}{L\, \rho\, U_0^2}$$

- Describes dimensionless pressure drop
- NS equation:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \nabla P + \eta(\nabla^2\,\vec{v})$$

Inertial part

Pressure part

Viscosity part

Dynamic pressure

$$\nabla P \equiv \nabla p - \rho\, g$$

http://www.fponthenet.net/article/135747/Heat-exchanger-considerations.aspx

- Inertial forces ≫ viscosity forces (Re ≫ 1)

→ left and right sides of NS equation equal → inertial part ≈ pressure part

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \nabla P + \eta(\nabla^2\vec{v})$$

Inertial part

Pressure part

Negligible in the centre,
↑ near to the wall

$$\rho\,\vec{v}\cdot\nabla\vec{v} \sim \nabla P$$

Rescaling/Dimensionless formula:

$$\frac{\rho U_0^2}{R}\vec{v}^*\cdot\tilde{\nabla}\vec{v}^* \sim \frac{\Delta P_0}{L}\tilde{\nabla}P^*,$$ where:

U_0	Average flow velocity
R	Tube radius
L	Tube length
ΔP	Pressure drop over tube length L

$$\rho\cdot v_y\cdot\frac{\partial v_x}{\partial y} \approx -\frac{\partial p}{\partial x}$$

$$\widetilde{v_x} = \frac{v_x}{U} \quad \tilde{x} = \frac{x}{L} \quad \widetilde{v_y} = \frac{v_y}{U} \quad \tilde{y} = \frac{y}{D} \quad \tilde{p} = \frac{p}{\Delta p}$$

$$\rho\cdot U\cdot\widetilde{v_y}\cdot\frac{U}{D}\cdot\frac{\partial\widetilde{v_x}}{\partial\tilde{y}} \approx \frac{\Delta p}{L}\cdot\frac{\partial\tilde{p}}{\partial\tilde{x}}$$

$$\left(\frac{\frac{\partial\tilde{p}}{\partial\tilde{x}}}{\frac{\partial\widetilde{v_x}}{\partial\tilde{y}}\cdot\widetilde{v_y}}\right)^{-1} = \frac{\Delta p\cdot D}{L\cdot\rho\cdot U^2} = 2\cdot f$$

Fanning friction coefficient:

- Dimensionless pressure drop

$$f \equiv \frac{\frac{\Delta p}{L}}{\frac{\rho U_0^2}{R}} = \frac{\Delta p\,R}{L\,\rho\,U_0^2}$$

Darcy friction coefficient (Chemical Engineering I, II):

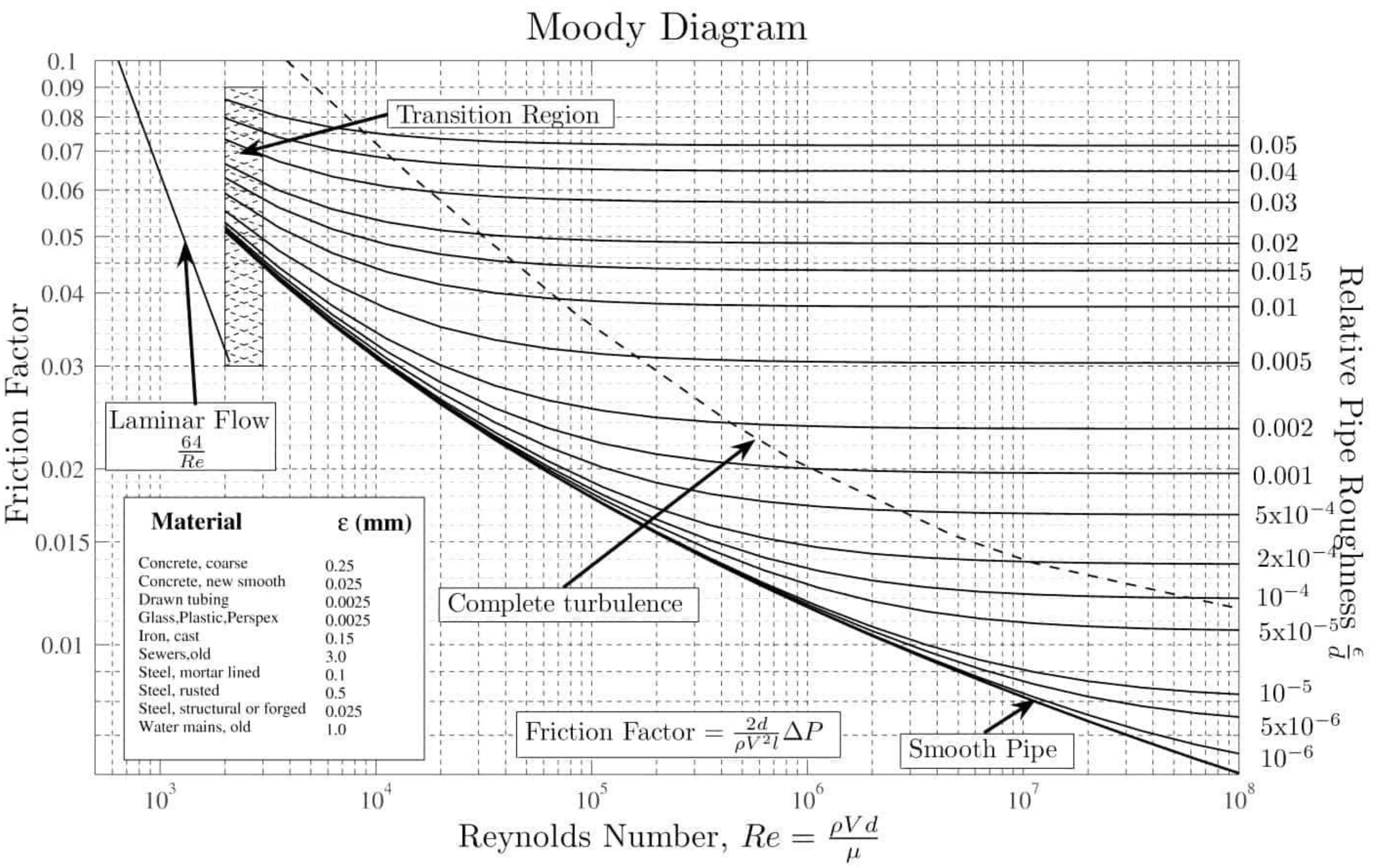
(friction coefficient)

$$\lambda_f = 4f$$

Fanning friction factor

- **The Fanning friction factor**, named after John Thomas Fanning, is a dimensionless number, that is **one-fourth of the Darcy friction factor**.
- Attention must be paid to note which one of these is used as the friction factor. This the only difference between these two factors.
- In all other aspects they are identical, and by applying the conversion factor of 4 the friction factors may be used interchangeably.

$$\lambda_f = 4f$$



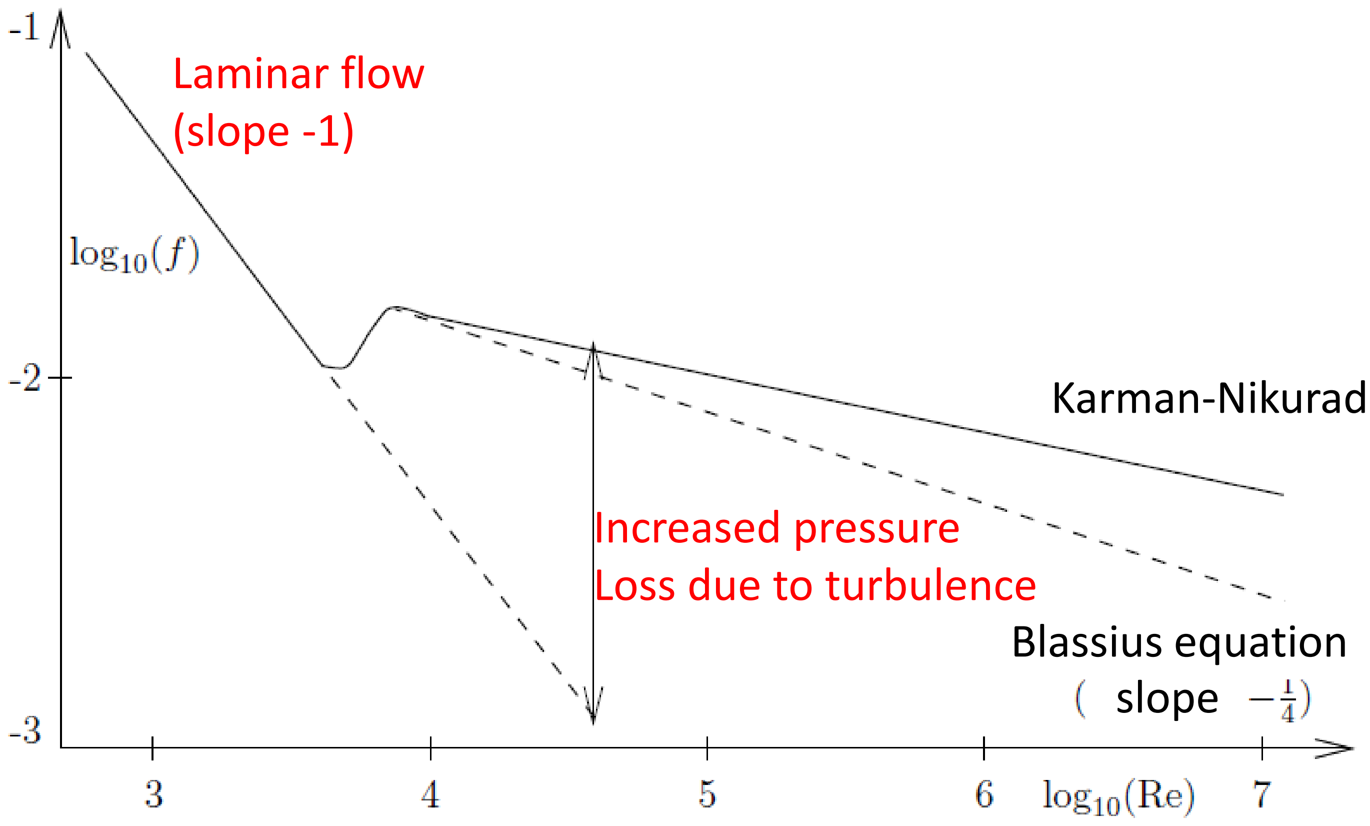
Pressure drop in tube

Circular cross section (laminar flow): $f = \frac{16}{\mathrm{Re}}$ or $\lambda_f = \frac{64}{\mathrm{Re}}$ $\quad \mathrm{Re} \leqq 2300$

Turbulent flow: Empirical formula

Karman-Nikuradse: $\frac{1}{\sqrt{f}} = 4\log_{10}(\mathrm{Re}\sqrt{f}) - 0,4$ $\quad \mathrm{Re} > 3000$

Blassius equation: $f = 0{,}791\,\mathrm{Re}^{-\frac{1}{4}}$ $\quad 3000 < \mathrm{Re} < 100000$



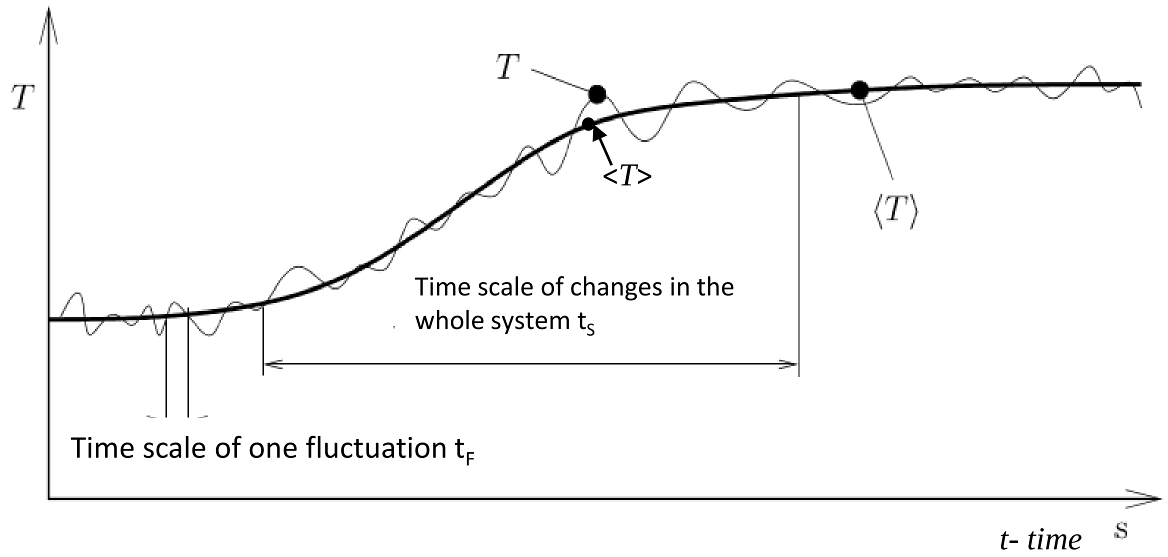
Dependence of friction factor f on Re criterion.

Time fluctuation of temperature

Time fluctuation of temperature in a particular position x,y,z:

- Average temperature <T> is an integral temperature
- Integration in an interval from 0 to t_a: $t_f < t_a < t_s$

$$\langle T\rangle = \frac{1}{t_a}\int_0^{t_a} T(t)\,\mathrm{d}t \longrightarrow$$ Reynolds averaging



Temperature fluctuations at two different scales

- Temperature in time t is the addition of average temperature and temperature fluctuation: T = <T> + θ
- Average velocity has the similar expression: $\vec{v} = \langle\vec{v}\rangle + \vec{U}$

Average velocity in the interval t_a

Vector of velocity fluctuation in time t

Properties of averaged quantities (T = <T> + θ)

1) The mean value of fluctuations in t_a interval is equal to zero: <θ> = 0

2) RMS (root mean square) fluctuation = quality indicator of fluctuation size: $\Theta_{\mathrm{RMS}} = \sqrt{\langle \Theta^2 \rangle} > 0$ Intensity of fluctuation

3) Next averaging of averaged quantity does not change the value:

$$\langle T \rangle = \langle \langle T \rangle + \Theta \rangle = \frac{1}{t_a} \int\limits_{t}^{t+t_a} (\langle T \rangle + \Theta)\mathrm{d}t = \frac{1}{t_a} \int\limits_{t}^{t+t_a} \langle T \rangle \mathrm{d}t + \frac{1}{t_a} \int\limits_{t}^{t+t_a} \Theta \mathrm{d}t =$$

$$= \langle \langle T \rangle \rangle + \langle \Theta \rangle = \langle \langle T \rangle \rangle,$$ Based on definition: $\langle \Theta \rangle = 0$

4) Order of averaging and derivation may be interchanged:

$$\begin{aligned} \langle \nabla \cdot \vec{v} \rangle &= \nabla \cdot \langle \vec{v} \rangle \\ \langle \nabla^2 T \rangle &= \nabla^2 \langle T \rangle \\ \left\langle \frac{\partial T}{\partial t} \right\rangle &= \frac{\partial \langle T \rangle}{\partial t} \\ \langle \nabla T \rangle &= \nabla \langle T \rangle \end{aligned}$$

Averaging

$$\langle \nabla T \rangle = \frac{1}{t_a} \int\limits_{t}^{t+t_a} \nabla T \mathrm{d}t = \frac{1}{t_a} \int\limits_{t}^{t+t_a} \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} \mathrm{d}t = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \underbrace{\boxed{\frac{1}{t_a} \int\limits_{t}^{t+t_a} T \mathrm{d}t}}_{= \langle T \rangle} = \nabla \langle T \rangle$$

For simplification only x and y spatial coordinates

A) Averaging of continutiy equation:

$$\nabla\cdot\vec{v}=0$$

$$\langle\nabla\cdot\vec{v}\rangle=\boxed{\nabla\cdot\langle\vec{v}\rangle=0}$$ Averaged continuity equation

Other properties: $$\nabla\cdot\vec{v}=\nabla\cdot(\langle\vec{v}\rangle+\vec{U})=\underbrace{\nabla\cdot\langle\vec{v}\rangle}_{0}+\nabla\cdot\vec{U}=0\Rightarrow\boxed{\nabla\cdot\vec{U}=0}$$

Divergence of vector of fluctuation = 0

B) Averaging of FK equation without source:

$$a=\frac{\lambda}{\rho\,c_p}$$

$$\frac{\partial T}{\partial t}+\vec{v}\cdot\nabla T = a\nabla^2 T \qquad \Big|\rho\,c_p$$

$$\rho\,c_p\frac{\partial T}{\partial t}+\rho\,c_p\,\vec{v}\cdot\nabla T = \lambda\nabla^2 T$$

Every part individually averaged:

$$\left\langle\rho\,c_p\frac{\partial T}{\partial t}\right\rangle = \rho\,c_p\left\langle\frac{\partial T}{\partial t}\right\rangle = \rho\,c_p\frac{\partial\langle T\rangle}{\partial t}$$

$$\langle\lambda\nabla^2 T\rangle = \lambda\langle\nabla^2 T\rangle = \lambda\nabla^2\langle T\rangle$$

Only in 2D to simplify the proplem:

$$\vec{v}\cdot\nabla T=\left(\langle\vec{v}\rangle+\vec{U}\right)\cdot\nabla\left(\langle T\rangle+\Theta\right)=\left(\langle v_x\rangle+U_x\right)\left(\frac{\partial\langle T\rangle}{\partial x}+\frac{\partial\Theta}{\partial x}\right)+$$

$$+\left(\langle v_y\rangle+U_y\right)\left(\frac{\partial\langle T\rangle}{\partial y}+\frac{\partial\Theta}{\partial y}\right)=$$

$$\underbrace{\langle v_x\rangle\frac{\partial\langle T\rangle}{\partial x}+\langle v_y\rangle\frac{\partial\langle T\rangle}{\partial y}}_{\langle\vec{v}\rangle\cdot\nabla\langle T\rangle}+\underbrace{\langle v_x\rangle\frac{\partial\Theta}{\partial x}+\langle v_y\rangle\frac{\partial\Theta}{\partial y}}_{\langle\vec{v}\rangle\cdot\nabla\Theta}+\underbrace{U_x\frac{\partial\langle T\rangle}{\partial x}+U_y\frac{\partial\langle T\rangle}{\partial y}}_{\vec{U}\cdot\nabla\langle T\rangle}+\underbrace{U_x\frac{\partial\Theta}{\partial x}+U_y\frac{\partial\Theta}{\partial y}}_{\vec{U}\cdot\nabla\Theta}=$$

$$\langle\vec{v}\cdot\nabla T\rangle=\left\langle\langle\vec{v}\rangle\cdot\nabla\langle T\rangle+\langle\vec{v}\rangle\cdot\nabla\Theta+\vec{U}\cdot\nabla\langle T\rangle+\vec{U}\cdot\nabla\Theta\right\rangle=$$

Next slide

B) Averaging of FK equation without source:

$$= \langle \vec{v} \rangle \cdot \nabla \langle T \rangle + \langle \vec{v} \rangle \cdot \nabla \underbrace{\langle \Theta \rangle}_{=0} + \underbrace{\langle \vec{U} \rangle}_{=0} \cdot \nabla \langle T \rangle + \left\langle \vec{U} \cdot \nabla \Theta \right\rangle =$$

Mean value of
Fluctuations =0

$$= \langle \vec{v} \rangle \cdot \nabla \langle T \rangle + \left\langle \vec{U} \cdot \nabla \Theta \right\rangle$$

Then, FK:

$$\rho\, c_p \frac{\partial \langle T \rangle}{\partial t} + \rho\, c_p \langle \vec{v} \rangle \cdot \nabla \langle T \rangle = \lambda \nabla^2 \langle T \rangle - \rho\, c_p \left\langle \vec{U} \cdot \nabla \Theta \right\rangle$$

$$\lambda \nabla^2 \langle T \rangle = \lambda \nabla \cdot \nabla \langle T \rangle = -\nabla \cdot \underbrace{(-\lambda \nabla \langle T \rangle)}_{\text{Fourier law}} = -\nabla \cdot \underbrace{\langle \vec{q} \rangle}_{\text{Averaged intensity of heat flow by conduction}}$$

B) Averaging of FK equation without source:

Now, we will analyse multiplication:

$$\nabla \cdot \left(\vec{U}\Theta\right) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} U_x\Theta \\ U_y\Theta \end{bmatrix} = \frac{\partial U_x}{\partial x}\Theta + U_x\frac{\partial \Theta}{\partial x} + \frac{\partial U_y}{\partial y}\Theta + U_y\frac{\partial \Theta}{\partial y} =$$

$$= \Theta \underbrace{\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}\right)}_{\nabla \cdot \vec{U} = 0} + \underbrace{U_x\frac{\partial \Theta}{\partial x} + U_y\frac{\partial \Theta}{\partial y}}_{\vec{U} \cdot \nabla\Theta} = \vec{U} \cdot \nabla\Theta$$

Definition of heat flow caused by turbulent flow: $\vec{q}^{\,*} \equiv \rho\, c_{\mathrm{p}} \left(\vec{U}\Theta\right)$ Accelerated heat conduction by turbulent flow

Final form of averaged FK eq.: $$\boxed{\rho\, c_{\mathrm{p}} \frac{\partial \langle T \rangle}{\partial t} + \rho\, c_{\mathrm{p}} \langle \vec{v} \rangle \cdot \nabla \langle T \rangle = -\nabla \cdot \left(\langle \vec{q} \rangle + \vec{q}^{\,*}\right)}$$

Describes fluctuation
Estimated from averaged values

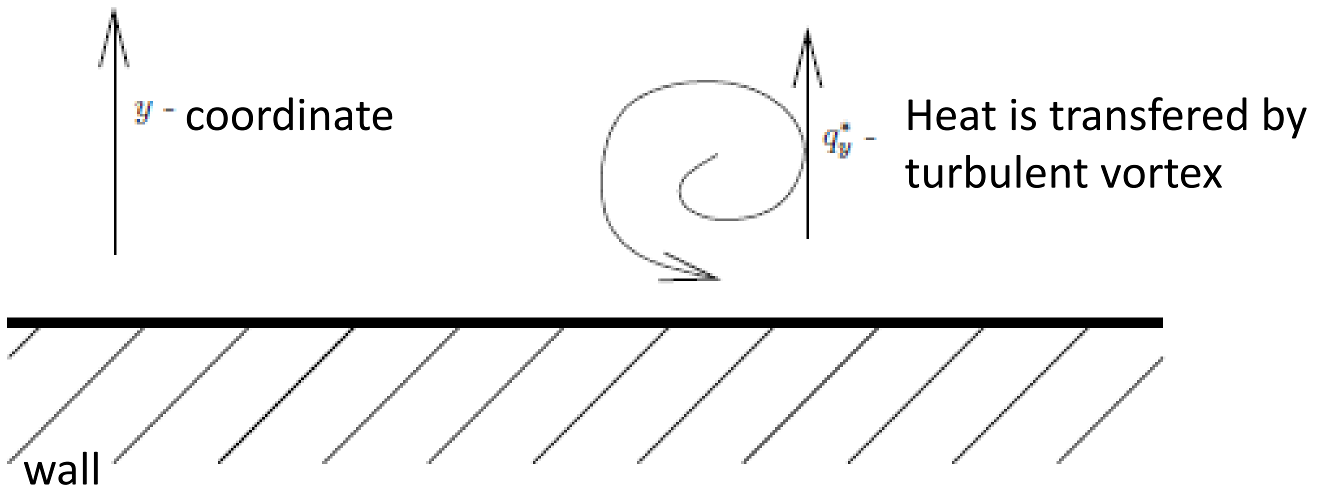
$$\frac{<\partial T>}{\partial t} + <\vec{v}> \nabla <T> = a \cdot \nabla^2 <T> - \overbrace{<\vec{u} \cdot \nabla\Theta>}^{\nabla \cdot <\vec{u}\Theta> \text{ ... Momentum flow}}$$

$$\nabla \cdot (\vec{u}\Theta) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} u_x\Theta \\ u_y\Theta \end{bmatrix} = \frac{\partial}{\partial x}(u_x\Theta) + \frac{\partial}{\partial y}(u_y\Theta) =$$

$$= \underbrace{u_x \frac{\partial\Theta}{\partial x} + u_y \frac{\partial\Theta}{\partial y}}_{\vec{u} \cdot \nabla\Theta} + \underbrace{\Theta \frac{\partial u_x}{\partial x} + \Theta \frac{\partial u_y}{\partial y}}_{\Theta\nabla \cdot \vec{u} = 0} = \vec{u} \cdot \nabla\Theta$$

Diffusivity model

Constitutive equation for heat flow caused by turbulence:



Turbulent vortex near to a wall + heat transfer.

Consitutive equation:

$$\boxed{q_y^* = -\rho\, c_p\, \varepsilon_H \frac{\partial \langle T \rangle}{\partial y}}$$

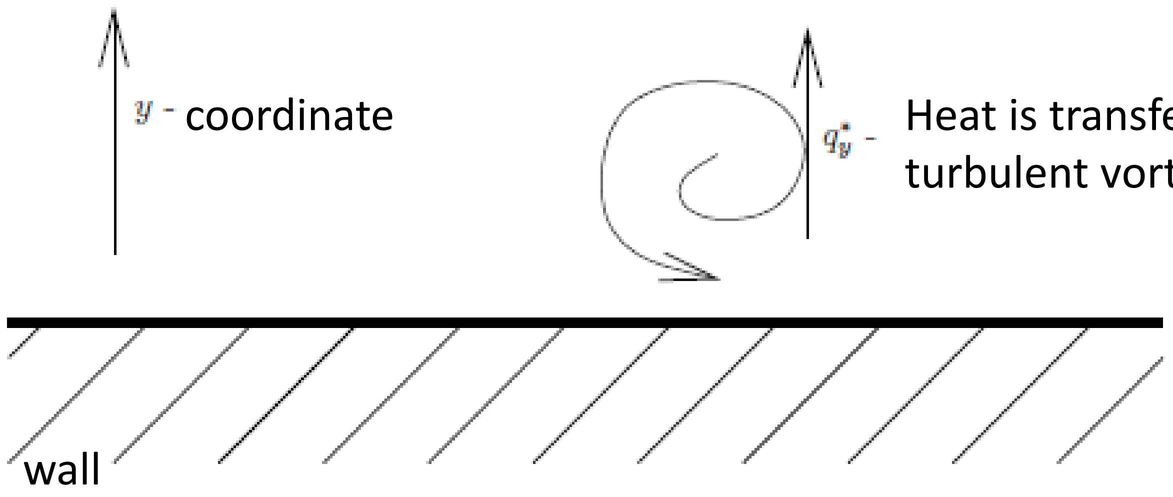
Diffusivity model

ε_H – diffusivity of heat vortex [$m^2 s^{-1}$]

$\varepsilon_H \approx \varepsilon_M$ (diffusivity of heat vortex is comparable to momentum diffusivity)

- Turbulent vortex transfers heat from place with higher temperature to place with lower temperature
- Turbulent heat transfer = accelerated heat conduction → heat flow is proportional to negative gradient of averaged temperature
- Both heat and momentum are carried by the same vortex
- ε_H is not constant – it is dependent on the distance to the wall
- Near to wall: $\varepsilon_H \to 0$ (no vortex)
- Usually: $\varepsilon_M \approx 0.85 * \varepsilon_H$

Constitutive equation for heat flow caused by turbulence:



Turbulent vortex near to a wall + heat transfer

Consitutive equation:

$$\boxed{q_y^* = -\rho\, c_p\, \varepsilon_H \frac{\partial \langle T \rangle}{\partial y}}$$

$$\langle \overrightarrow{q_C} \rangle = -\lambda \cdot \Delta \langle T \rangle \quad \text{Conduction}$$

$$\langle \overrightarrow{q_T} \rangle = f(\langle T \rangle, \langle \vec{v} \rangle) \longrightarrow \text{Turbulence}$$

$$\langle \overrightarrow{q_T} \rangle \approx q^*$$

ε_H – difusivity of heat vortex [m^2s^{-1}]

$\varepsilon_H \approx \varepsilon_M$ (diffusivity of heat vortex is comparable to diffusivity of momentum)

How to find the turbulent difusivity of vortex?

→ Prandtl method:

Diffusivity of momentum:

$$\varepsilon_M = \kappa^2 y^2 \left| \frac{\partial \langle v_z \rangle}{\partial y} \right|$$

Diffusivity of heat vortex:

$$\varepsilon_H = l^2 \boxed{\left| \frac{\partial \langle v_z \rangle}{\partial y} \right|}$$

Normal (perpendicular) velocity derivative related to tube surface

- $\varepsilon_M / \varepsilon_H = 0.85$
- Firstly, we calculate ε_M → then ε_H
- l – distance of mixed volume
- l = κ*y (y is perpendicular distance from tube)
- Usually κ ≈ 0.4

Nu criterion for turbulent flow in tube

Correlation from table data (properties of flow, type of liquid, geometry)

$\longrightarrow \boxed{\mathrm{Nu} = \frac{f}{2}\,\mathrm{Re}\,\mathrm{Pr}} \longrightarrow f = f(\mathrm{Re})$ ref. Blassius eq. $\left(f = 0,791\,\mathrm{Re}^{-\frac{1}{4}}\right)$

Eq. is valid for: $\mathrm{Pr} \approx 1$ & $10^4 < \mathrm{Re} < 10^6$

Colburn eq.:

$\mathrm{Nu} = 0,023\,\mathrm{Re}^{0,8}\,\mathrm{Pr}^{\frac{1}{3}}$ Valid for $\mathrm{Pr} \approx 1$ & $10^4 < \mathrm{Re} < 10^6$

Bhatti and Shah eq.:

$$\mathrm{Nu} = \frac{(f/2)(\mathrm{Re} - 1000)\mathrm{Pr}}{1 + 12,7(f/2)^{\frac{1}{2}}(\mathrm{Pr}^{\frac{2}{3}-1})},$$

Eq. is valid for: $0,5 \leq \mathrm{Pr} < 2000$ & $2300 \leq \mathrm{Re} \leq 5 \cdot 10^6$, Accuracy: $\pm 10\%$

Lecture 10

Heat transfer in boiling liquid

Heat transfer in boiling liquid

- During boiling → **phase transition** liquid-vapour
- Significant **change of density** – approx. 3 orders of magnitude (1 000 kg/m^3 → 1 kg/m^3)
- Density change causes flow of both liquid phase and vapour phase (similar to natural convection)
- **Coefficient of heat transfer is much higher** due to much higher difference in densities (compared to natural convection)

Boiling curve

- Describes intensity of heat flow between liquid phase and heating plate on their temperature difference

$$\Delta T_e = T_s - T_b$$

Temperature difference (overheating)

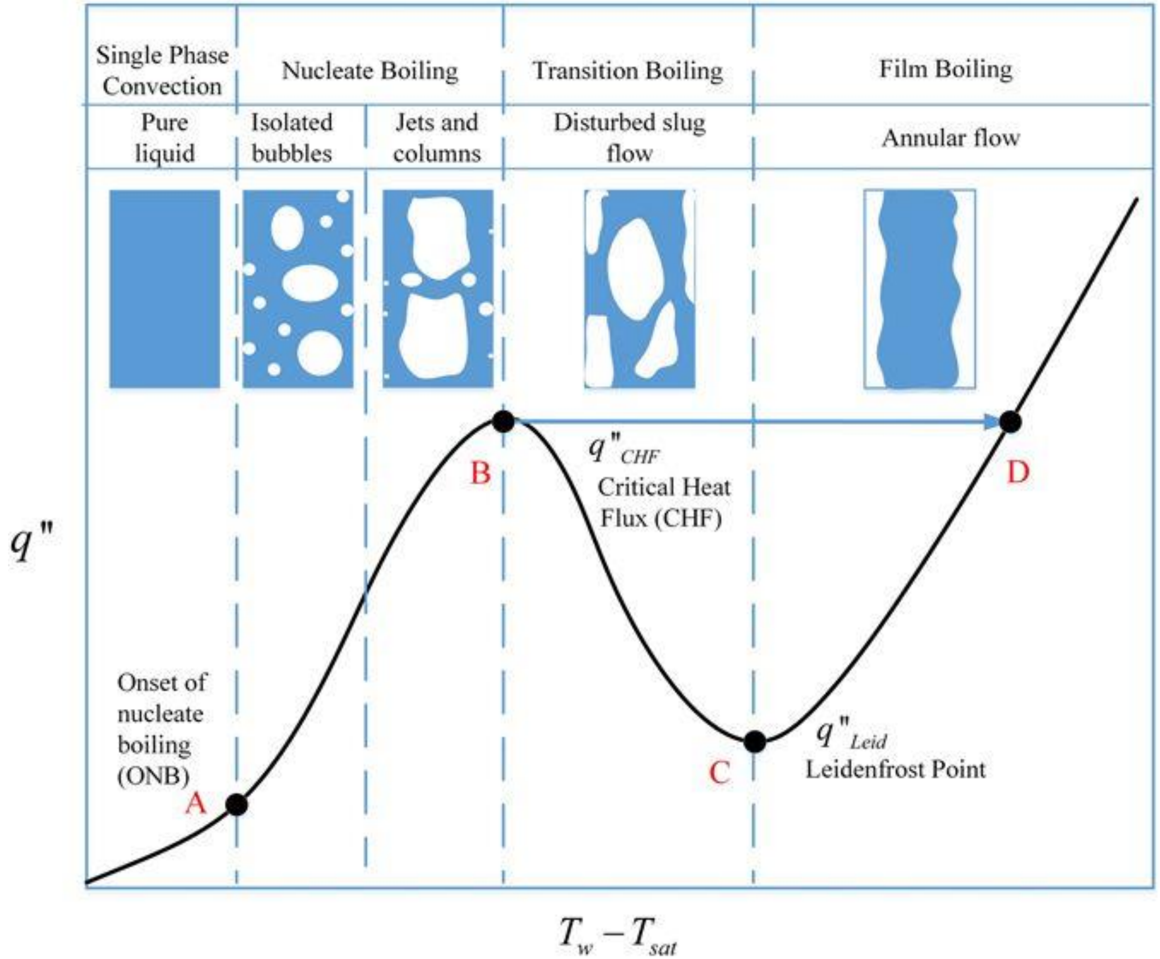
Temperature of heating plate

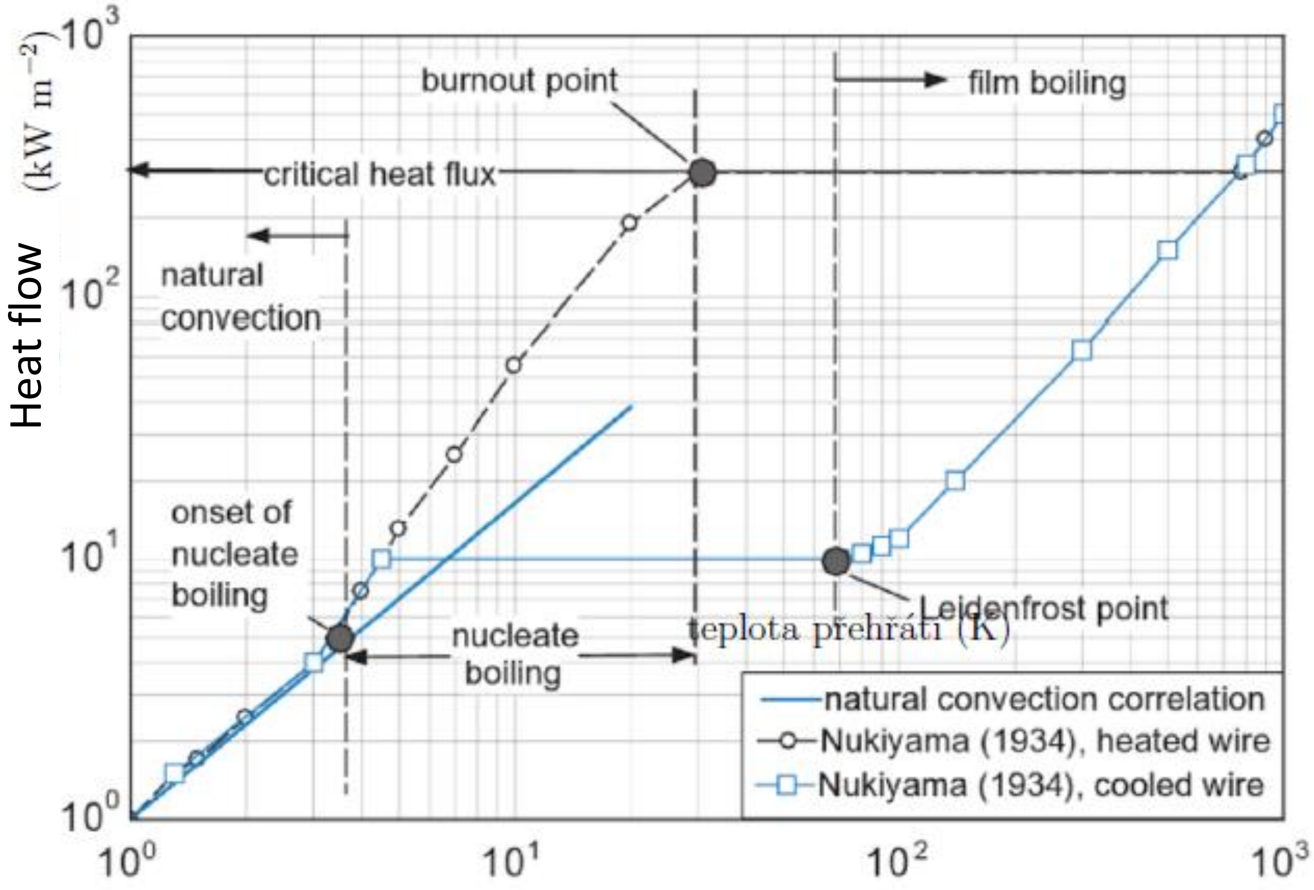
Temperature of boiling point (for specific pressure)

Boiling phases

We can describe 4 phases of boiling:

Boiling phase is dependent on the temperature difference between heating plate and bulk liquid.





S. Nukiyama, Maximum and minimum values of heat q transmitted from metal to boiling water under atmospheric pressure. J. Soc. Mech. Eng. Jpn. 37 (1934) 53-54, 367-374

Hu H, Xu C, Zhao Y, Ziegler KJ, Chung JN 2017. Boiling and quenching heat transfer advancement by nanoscale surface modification. Scientific Reports 7(1):6117.

Individual boiling phases ($\Delta T_e > 0$)

1) **Natural convection ($\Delta T_e \geq 0$)**

- liquid is warming up near to heating plate
- liquid is circulating due to changes in density → new cold liquid is getting near to heating plate
- there si no occurring vapour phase
- temperature of liquid is increasing
- Nu criterion → correlation for natural convection: $\alpha \approx 10^3\ Wm^{-2}K^{-1}$

2) **Nucleate boiling ($\Delta T_e > 0$)**

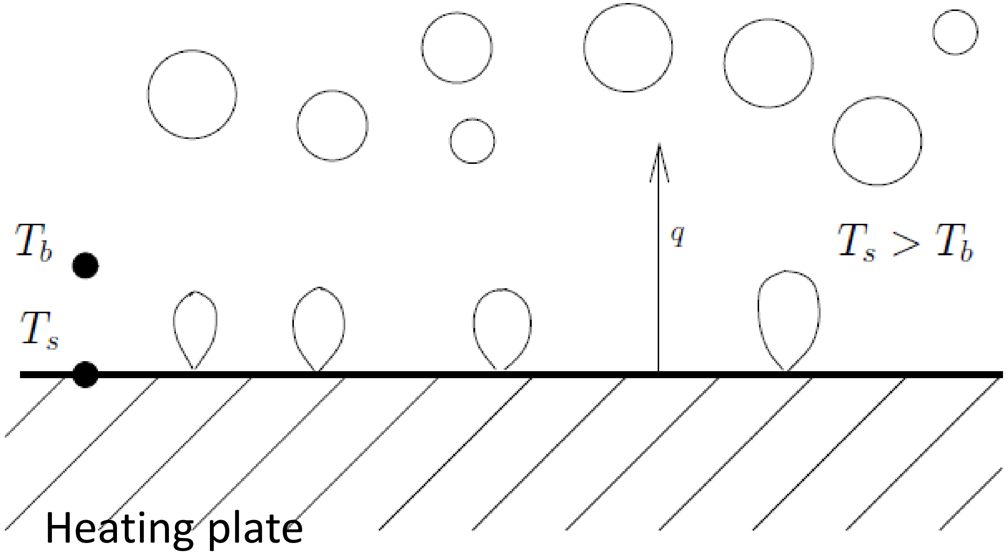
- if the temperature difference is sufficient, small bubbles starts to occur on the heating plate
- bubbles occur on the nucleation spots → grow → rise upwards
- properties of bubble flow are dependent on the value of temperature difference
- more intensive flow/vortexing causes better heat transfer → coefficient α dramatically increases (with increasing ΔT_e)

$$q = \alpha\, \Delta T_e = \alpha(T_s - T_b)$$

$\alpha \sim 10^3 - 10^4\ \mathrm{W\ m^{-2}\ K^{-1}} \longrightarrow$ useful in heat exchangers

Individual boiling phases ($\Delta T_e > 0$)

2) Nucleation boiling ($\Delta T_e > 0$)



3) Transient boiling phase ($\Delta T_e \gg 0$)

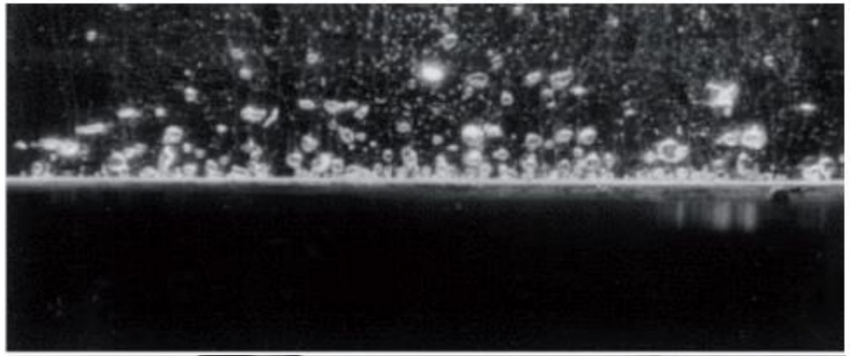
- high temperature difference → heating plate is fully covered with occuring bubbles → colder liquid cannot be in contact with heating plate
- heat conductivity of vapour is significantly lower than conductivity of water
- „boiling crisis“ → it may damage heating plate/exchanger

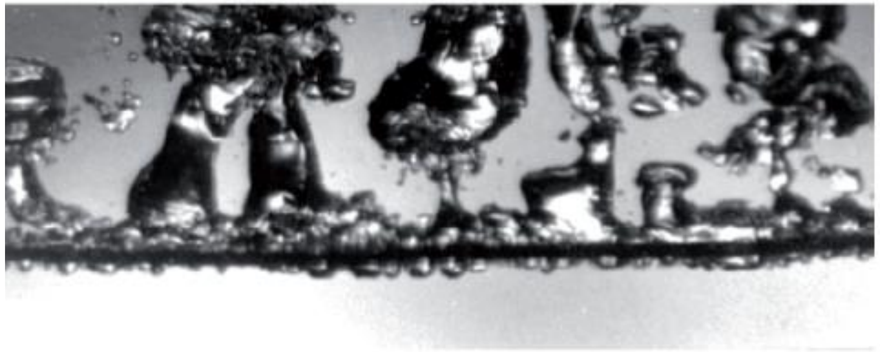
$\Delta T_e \approx 10^2$ K → α goes from 10^4 to 10^2 $Wm^{-2}K^{-1}$

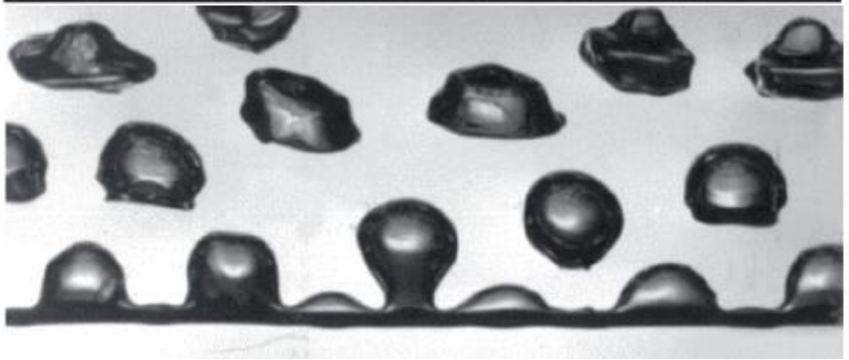
Individual boiling phases ($\Delta T_e > 0$)

4) Film boiling (ΔT_e > burning temperature)

- the whole surface of heating plate is covered with vapour
- vapour from the whole surface starts getting away in the same time → cold water gets near to heating plate again
- heat transfer coefficient is minimal in the Leidenfrost point ($\alpha \approx 10^2\ Wm^{-2}K^{-1}$)
- with further temperature increase → α is increasing too
- film boiling is not desirabled in exchangers







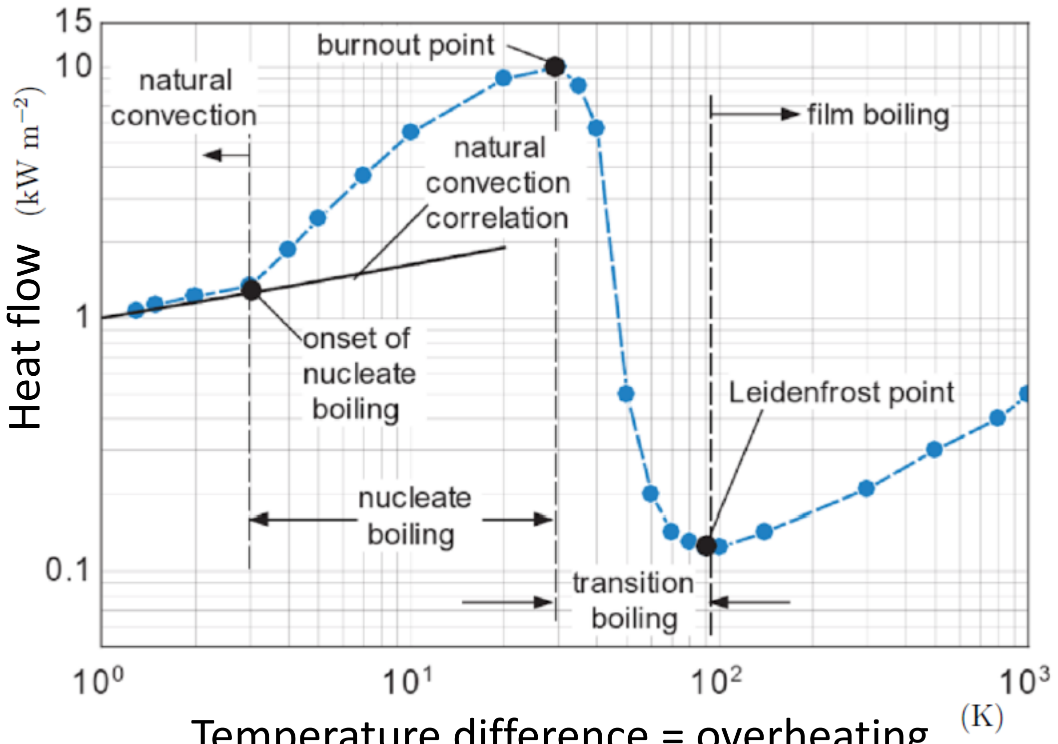
Bubble creation near to a heating plate (Nucleate boiling)

Dependence of heat transfer coefficient on the overheating (ΔT_e) :

- For **nucleate boiling → Rohensow correlation:**

$$q = \mu_l \, \Delta h_S \sqrt{\frac{g(\rho_l - \rho_v)}{\sigma}} \left(\frac{c_{pl} \, \Delta T_e}{C_{nb} \Delta h_S \, \mathrm{Pr}_l^n} \right)^3 \; [\mathrm{W \ m^{-2}}]$$

Calculated for boiling/condensation temperature.



G. Nellis, S. Klein,
Heat transfer, Cambridge University Press (2009), 782

μ_l	$[\mathrm{Pa\ s}]$	Dynamic viscosity of liquid
ρ_l	$[\mathrm{kg\ m^{-3}}]$	Density of liquid
ρ_v	$[\mathrm{kg\ m^{-3}}]$	Density of vapour
σ	$[\mathrm{N\ m^{-1}}]$	Surface tension liquid-vapour
c_{pl}	$[\mathrm{J\ kg^{-1}\ K^{-1}}]$	Specific heat capacity of liquid
Pr_l		Prandtl number of liquid
n		Exponent – usually 1-1.7
g	$[\mathrm{m\ s^{-2}}]$	Gravitational acceleration
Δh_S	$[\mathrm{J\ kg^{-1}}]$	Enthalpy difference equil. liquid-vapour
C_{nb}		Constant dependent on the heat plate material and liquid type (TABLES)

Calculation of burning temperature (film boiling):

- Lienhard and Dhira equation:

KRIT ≈ CRIT ≈ critical

A: surface of heating plate

$$(\alpha \Delta T_e)_{\mathrm{KRIT}} = q_{\mathrm{KRIT}} = \frac{\dot{Q}_{\mathrm{KRIT}}}{A}$$

$$q_{\mathrm{KRIT}} = C_{\mathrm{KRIT}} \Delta h_S \rho_v \left[\frac{\sigma g \left(\rho_l - \rho_v \right)}{\rho_v^2} \right]^{\frac{1}{4}}$$

Constant dependent on geometry of described system (TABLES).

Table 7-1: Values of the coefficient C_{nb} in Eq. (7.3) for various surface/fluid combinations, from Rohsenow (1952), Collier and Thome (1994), Vachon et al. (1968).

Fluid	Surface	C_{nb}
water	polished copper	0.0127
	lapped copper	0.0147
	scored copper	0.0068
	ground & polished stainless steel	0.0080
	teflon-pitted stainless steel	0.0058
	chemically etched stainless steel	0.0133
	mechanically polished stainless steel	0.0132
	brass	0.0060
	nickel	0.006
	platinum	0.0130
n-pentane	polished copper	0.0154
	polished nickel	0.0127
	lapped copper	0.0049
	emery-rubbed copper	0.0074
carbon tetrachloride	polished copper	0.0070
benzene	chromium	0.0101
ethyl alcohol	chromium	0.0027
isopropyl alcohol	copper	0.0023
n-butyl alcohol	copper	0.0030

http://fchart.com/ees/heat_transfer_library/boiling/hs2010.htm

Example

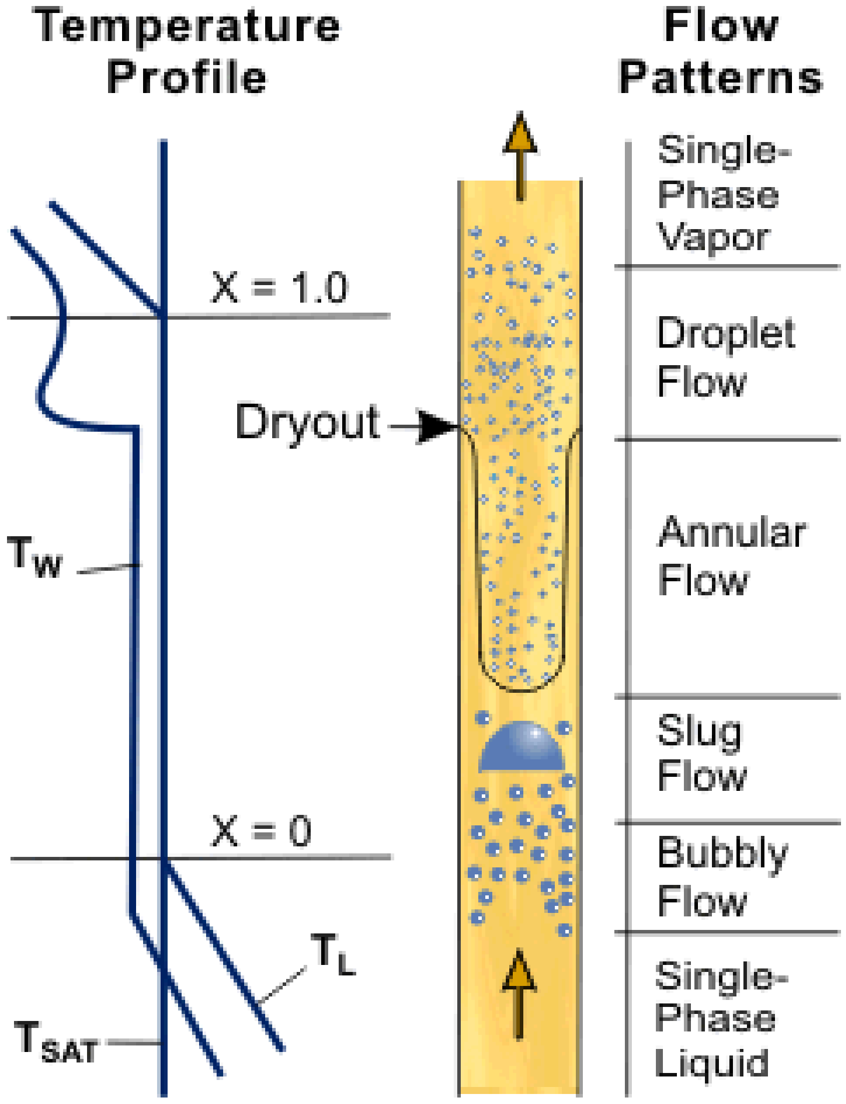


Figure 1

Regimes of heat transfer and two-phase flow in a heated channel

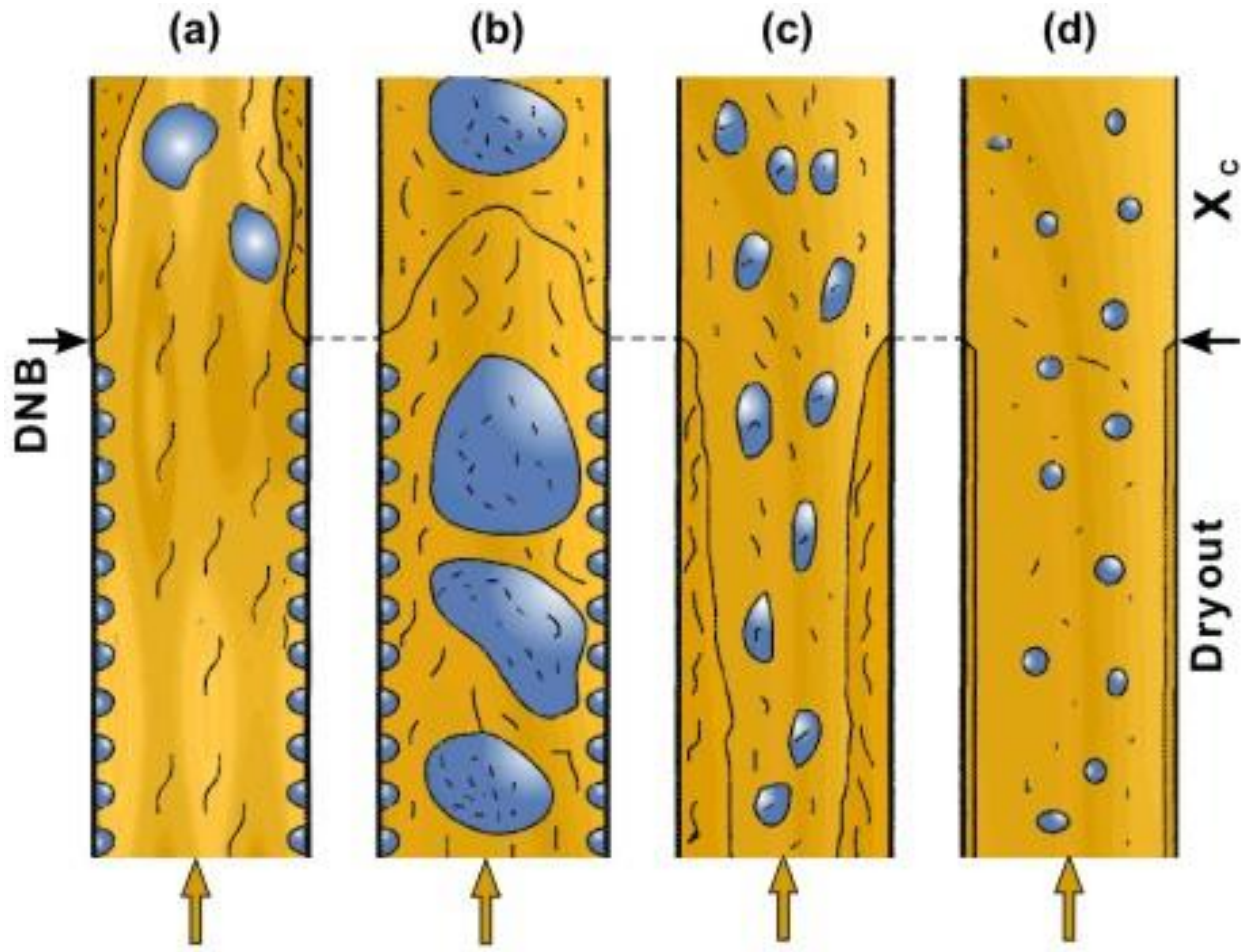


Figure 2

http://www.thermopedia.com/content/605/

UNIVERSITY OF CHEMISTRY AND TECHNOLOGY, PRAGUE
Faculty of Chemical Engineering
Department of Chemical Engineering

Lecture 11

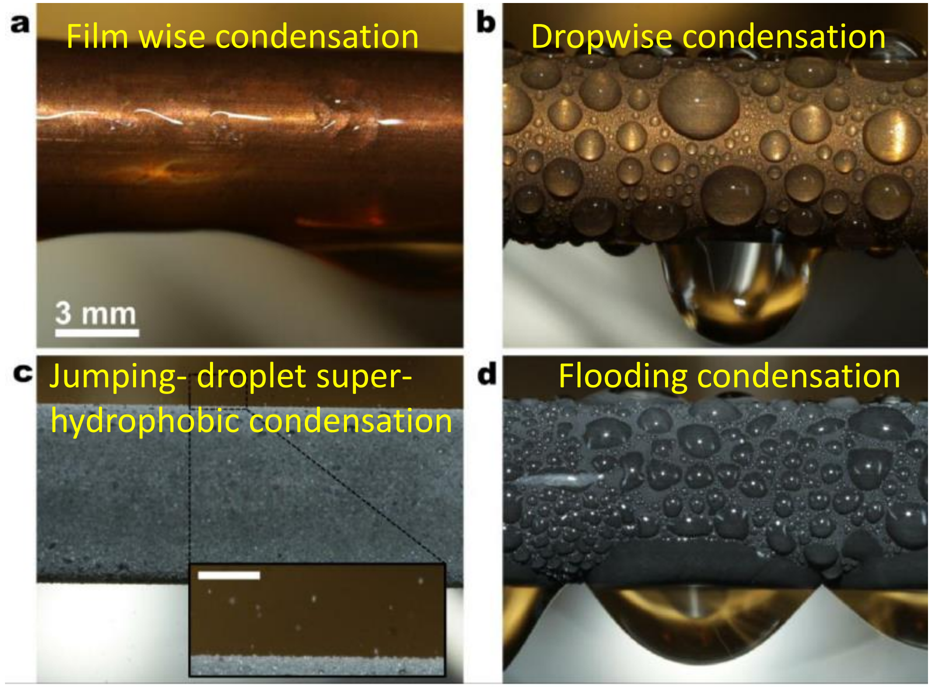
Heat transfer in condensation

Heat transfer in condensation

- **Condensation** occurs whenever a vapour comes into contact with a surface at a temperature lower than the saturation temperature corresponding to its vapour pressure.

1) **Drop condensation** – wanted, difficult to achieve, non-wetting surface, high heat transfer coefficient
2) **Film condensation** – usual, wetting suface

- The nature of condensation depends upon whether the liquid thus formed wets or does not wet the solid surface.
- If the liquid wets the surface, the condensate flows on the surface in the form of a lm and the process is called **film condensation**.
- If on the other hand, the liquid does not wet the solid surface, the condensate collects in the form of droplets, which either grow in size or coalesce with neighbouring droplets and eventually roll of the surface under the influence of gravity. This process is called **drop condensation**.



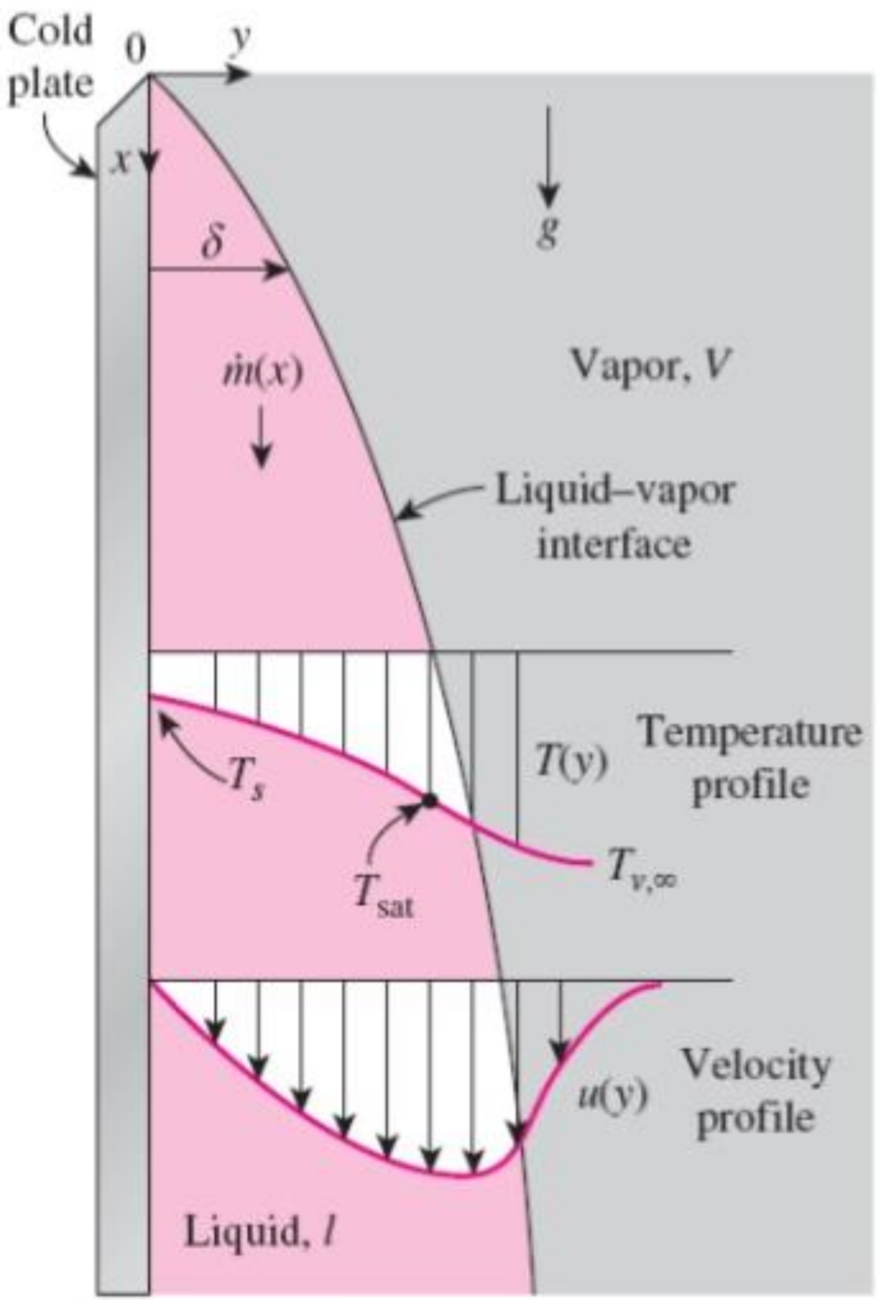
The most common and best understood case of condensation heat transfer is that of **film *condensation*** of a pure quiescent vapor on a solid surface.

- Application: electric power generation, process industries, refrigeration and air-conditioning…

https://www.researchgate.net/figure/Condensation-heat-transfer-modes-and-performance-Images-of-a-filmwise-condensation_fig1_259433420

Film condensation

- Liquid film starts forming at the top of the plate and flows downward under the influence of gravity.
- δ increases in the flow directions x
- Heat in the amount h_{lg} is released during condensation and is transferred through the film to the plate surface
- Ts must be below the saturation temperature for condensation.
- The temperature of the condensate is T_{sat} at the interface and decreases gradually to Ts at the wall.



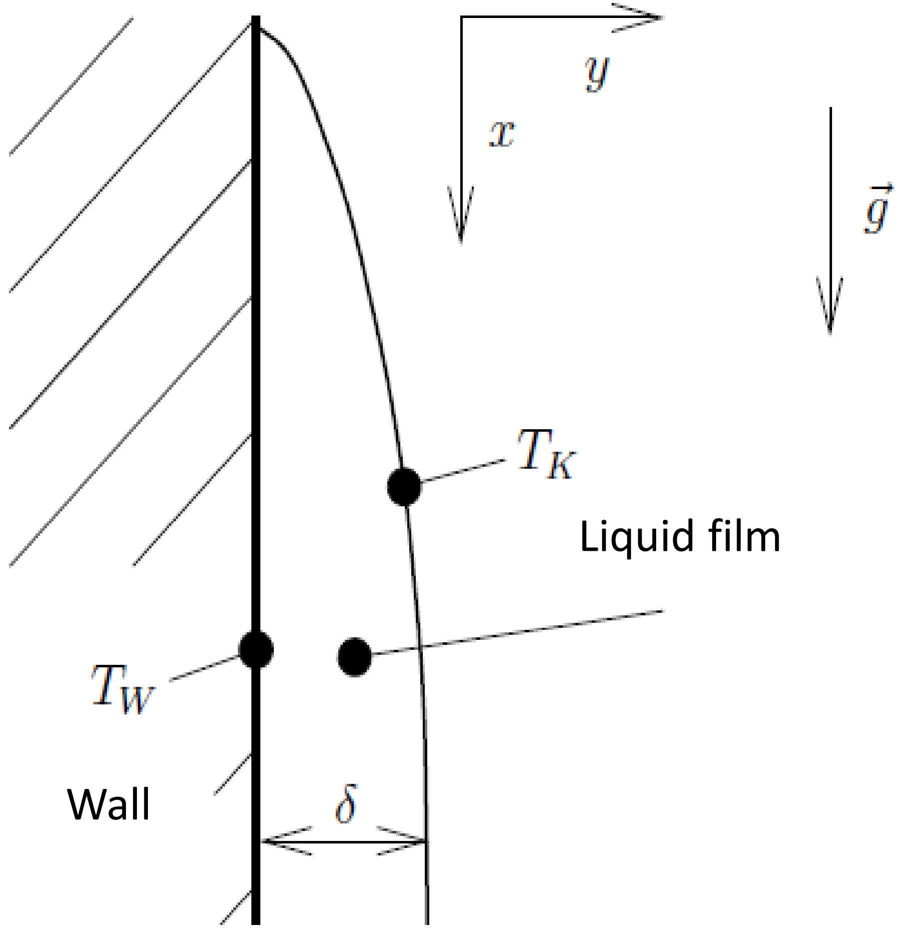
Film condensation

- Liquid film on the surface
- Film thickness δ
- Heat transfer over this film

$$\underbrace{q = \lambda}\frac{T_K - T_W}{\delta}$$

λ	$[\mathrm{W\ m^{-1}\ K^{-1}}]$	Liquid thermal conductivity (film)
T_K	$[\mathrm{K}]$	Temperature of condensation
T_W	$[\mathrm{K}]$	Temperature of heat exchange surface

$T_W < T_K$



- Heat flow: q = $\alpha(T_K - T_W)$
- Assumption: $\alpha \approx \frac{\lambda}{\delta}$ (α is heat transfer coefficient)
- δ (thickness of film) should be small → low wall (horizontal tubes)

NS equation in steady state:

- x-coordinate: $$\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho_l\,g+\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$
- y-coordinate: vapour (ρ_g) in y → ∞ is not moving
  - No film $$\frac{\partial p}{\partial x}=g\,\rho_g$$

$$\frac{\partial p}{\partial y}\doteq 0$$

- Assumption: pressure over the film is not changing very much
- Both equations combined: $$\underbrace{\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)}_{\text{Usually negligible}}=-\rho_g\,g+\rho_l\,g+\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$(\rho_g-\rho_l)g\doteq\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

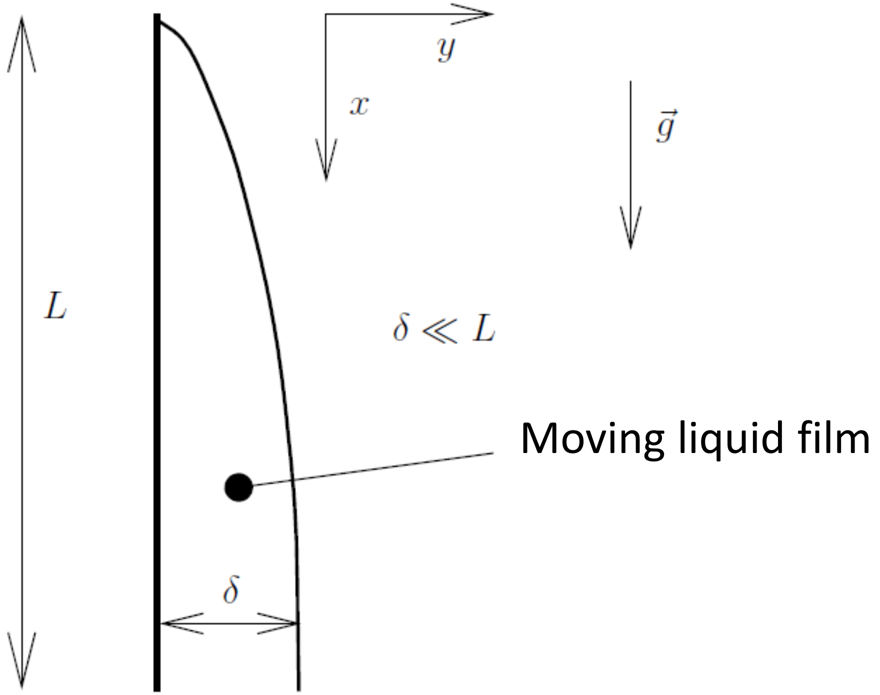
$$\underbrace{\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)}_{\text{Usually negligible}} = -\rho_g\, g + \rho_l\, g + \eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

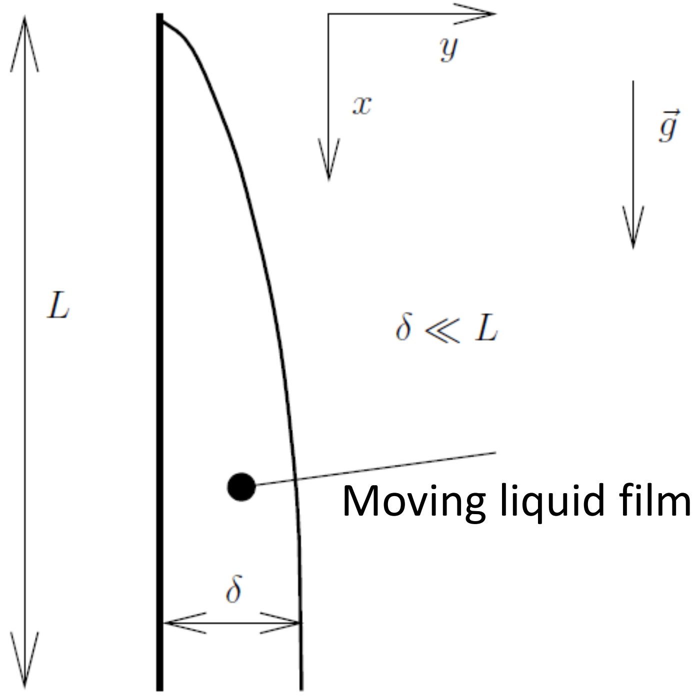
$$(\rho_g - \rho_l) g \doteq \eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\mathrm{Re} = \frac{U\delta\rho_l}{\eta} \ll 1$$

δ Film thickness (10^{-5} m) Very low

U Velocity of moving film (10^{-3} m s^{-1}) Relatively low





$$\tilde{v}_x = \frac{v_x}{U} \qquad \tilde{x} = \frac{x}{L} \qquad \tilde{y} = \frac{y}{\delta}$$

$$(\rho_g - \rho_l) g = \frac{\eta U}{\delta^2} \left(\underbrace{\frac{\delta^2}{L^2}}_{\ll 1} \underbrace{\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}}_{\sim 1} + \underbrace{\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}}_{\sim 1} \right)$$

$$\frac{(\rho_g - \rho_l) g \, \delta^2}{\eta U} = \cancel{\frac{\delta^2}{L^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\underbrace{\frac{(\rho_g - \rho_l) g \, \delta^2}{\eta U}}_{\sim 1} = \underbrace{\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}}_{\sim 1}$$

Suitably chosen velocity: $\boxed{U = \frac{(\rho_l - \rho_g) g \, \delta^2}{\eta}}$ Then: $\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2} = -1$

$$\downarrow$$

$$\tilde{v}_x = -\frac{\tilde{y}^2}{2} + C_1 \tilde{y} + C_2$$

- C_1 and C_2 constants from boundary conditions: $\tilde{v}_x = -\frac{\tilde{y}^2}{2} + C_1\tilde{y} + C_2$

1) Wall: $(y = 0, \tilde{y} = 0) \Rightarrow \tilde{v}_x = 0$
2) Liquid-vapour interface: $(y = \delta, \tilde{y} = 1)$

$$\eta_l \left.\frac{\partial v_x}{\partial y}\right|_l = \eta_g \left.\frac{\partial v_x}{\partial y}\right|_g$$

- Viscosity tension between liquid and vapour is equal = momentum flow is constant in the phase interphase

$$\frac{\left.\frac{\partial v_x}{\partial y}\right|_l}{\left.\frac{\partial v_x}{\partial y}\right|_g} = \frac{\eta_g}{\eta_l} \rightarrow 0$$

- Vapour viscosity is much lower

Approximately valid: $\left.\frac{\partial v_x}{\partial y}\right|_l = 0 \Rightarrow \left.\frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right|_l = 0$

- Applied in the final solution:

$$y = 0 \qquad v_x = 0 \qquad\qquad \tilde{y} = 0 \qquad \tilde{v}_x = 0$$

$$y = \delta \qquad \frac{\mathrm{d}v_x}{\mathrm{d}y} = 0 \qquad\qquad \tilde{y} = 1 \qquad \frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}\tilde{y}} = 0$$

$$0 = C_2$$

$$\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}\tilde{y}} = 0 = -\tilde{y} + C_1$$

$$0 = -1 + C_1$$

$$C_1 = 1$$

$$\boxed{\tilde{v}_x(\tilde{y}) = -\frac{\tilde{y}^2}{2} + \tilde{y}}$$

$$\frac{v_x(y)}{U} = -\frac{y^2}{2\delta^2} + \frac{y}{\delta}$$

$$\boxed{v_x = U\left(-\frac{1}{2}\frac{y^2}{\delta^2} + \frac{y}{\delta}\right)}$$

Temperature profile in film condensation

Temperature profile in film condensation

- Assumptions:

1) Without heat source
2) Steady state

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Scaling:

$$\Theta = \frac{T - T_W}{T_K - T_W} \qquad \tilde{v}_x = \frac{v_x}{U} \qquad \tilde{v}_y = \frac{v_y}{U}$$

$$\tilde{x} = \frac{x}{L} \qquad \tilde{y} = \frac{y}{\delta} \qquad \boxed{\delta \ll L}$$

→ **FK equation**:

$$\frac{U\cancel{(T_K - T_W)}}{\cancel{\delta}} \left[\overset{\delta/\delta}{\frac{\delta}{L}} \frac{\partial \Theta}{\partial \tilde{x}} + \frac{\partial \Theta}{\partial \tilde{y}} \right] = \frac{a\cancel{(T_K - T_W)}}{\delta^{\cancel{2}}} \left[\overset{\delta^2/\delta^2}{\frac{\delta^2}{L^2}} \frac{\partial^2 \Theta}{\partial \tilde{x}^2} + \frac{\partial^2 \Theta}{\partial \tilde{y}^2} \right] \quad \Big| \frac{\delta}{a}$$

$$\underbrace{\frac{U\delta}{a} \left[\frac{\delta}{L} \frac{\partial \Theta}{\partial \tilde{x}} + \frac{\partial \Theta}{\partial \tilde{y}} \right]}_{\text{Negligible}} = \underbrace{\cancel{\frac{\delta^2}{L^2}}}_{\ll 1} \underbrace{\frac{\partial^2 \Theta}{\partial \tilde{x}^2}}_{\sim 1} + \underbrace{\frac{\partial^2 \Theta}{\partial \tilde{y}^2}}_{\sim 1} \qquad \underbrace{(\ll 1) \cdot (\sim 1)}_{\text{Negligible}}$$

$$\mathrm{Pe} = \frac{U\,\delta}{a} \ll 1$$

$$a = \frac{\lambda}{\rho\, c_p} = \frac{10^{-1}}{10^3 10^0} = 10^{-4}\ \mathrm{m^2\ s^{-1}} \quad \delta = 10^{-5}\ \mathrm{m} \quad U = 10^{-3}\ \mathrm{m\ s^{-1}} \ \Rightarrow \ \mathrm{Pe} \sim 10^{-4}$$

$$\frac{\partial^2 \Theta}{\partial \tilde{y}^2} = 0$$

Temperature profile in film condensation

→ **FK equation**:

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad \frac{\partial^2 \Theta}{\partial \tilde{y}^2} = 0$$

→ **Solution of FK equation → Linear temperature profile**

$$\Theta = C_1 \tilde{y} + C_2$$

Boundary conditions:

$$\begin{array}{llcll} y = 0 & T = T_W & \rightarrow & \tilde{y} = 0 & \Theta = 0 \\ y = \delta & T = T_K & \rightarrow & \tilde{y} = 1 & \Theta = 1 \end{array}$$

$$0 = 0 + C_2 \Rightarrow C_2 = 0$$

$$1 = C_1$$

$$\Theta = \tilde{y}$$

$$\frac{T - T_W}{T_K - T_W} = \frac{y}{\delta}$$

$$\boxed{T(y) = \frac{y}{\delta}(T_K - T_W) + T_W}$$

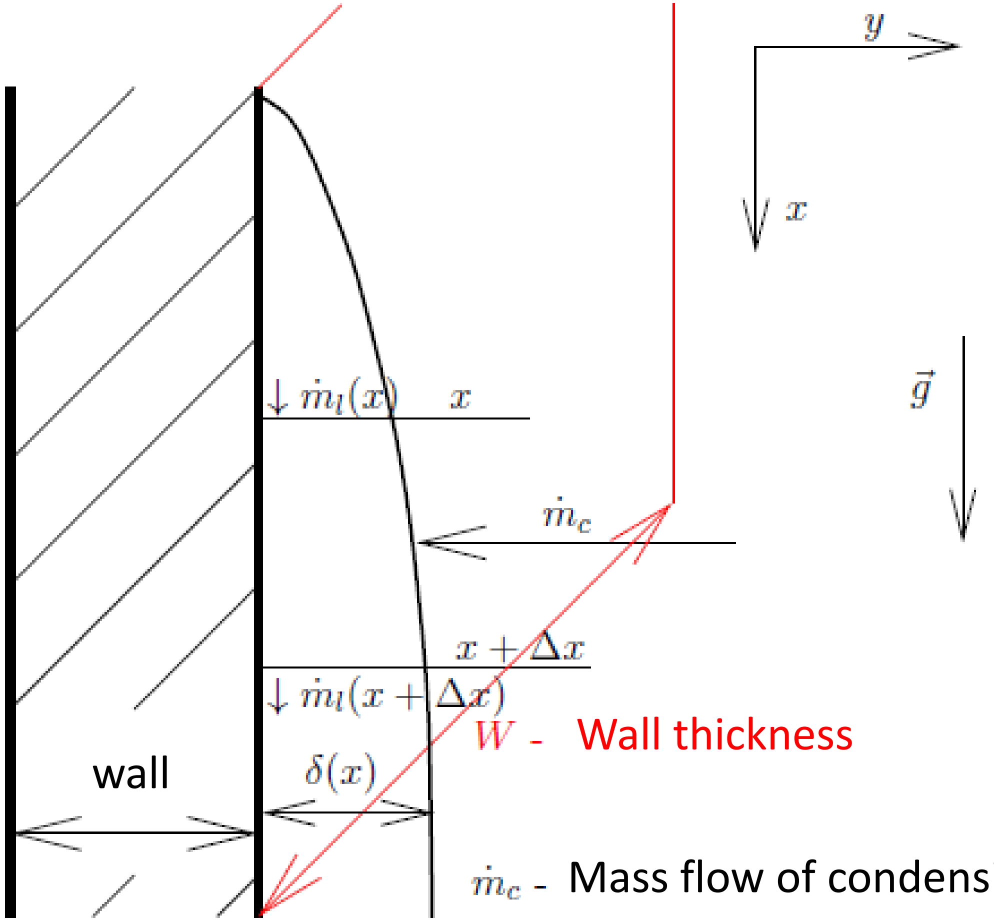
Estimation of film thickness – film condensation

- Mass balance of elementary section dx
- Steady state

$$\begin{aligned} \dot{m}_l(x) + \dot{m}_c &= \dot{m}_l(x + \Delta x) \\ \cancel{\dot{m}_l(x)} + j_c W \cancel{\Delta x} &= \cancel{\dot{m}_l(x)} + \frac{d\dot{m}_l}{dx}\cancel{\Delta x} \\ \frac{d\dot{m}_l}{dx} &= j_c W \end{aligned}$$

j_c intensity flow of condensing vapour [kg m^{-2} s^{-1}]

$$\dot{m}_c = j_c \Delta x W$$



W - Wall thickness

$\dot{m}_c$ - Mass flow of condensing vapour

Estimation of film thickness – film condensation

Mass flow of liquid:

$$\dot{m}_l = \int_0^\delta v_x W \rho_l \,\mathrm{d}y$$

$$v_x = U\left(-\frac{1}{2}\frac{y^2}{\delta^2} + \frac{y}{\delta}\right)$$

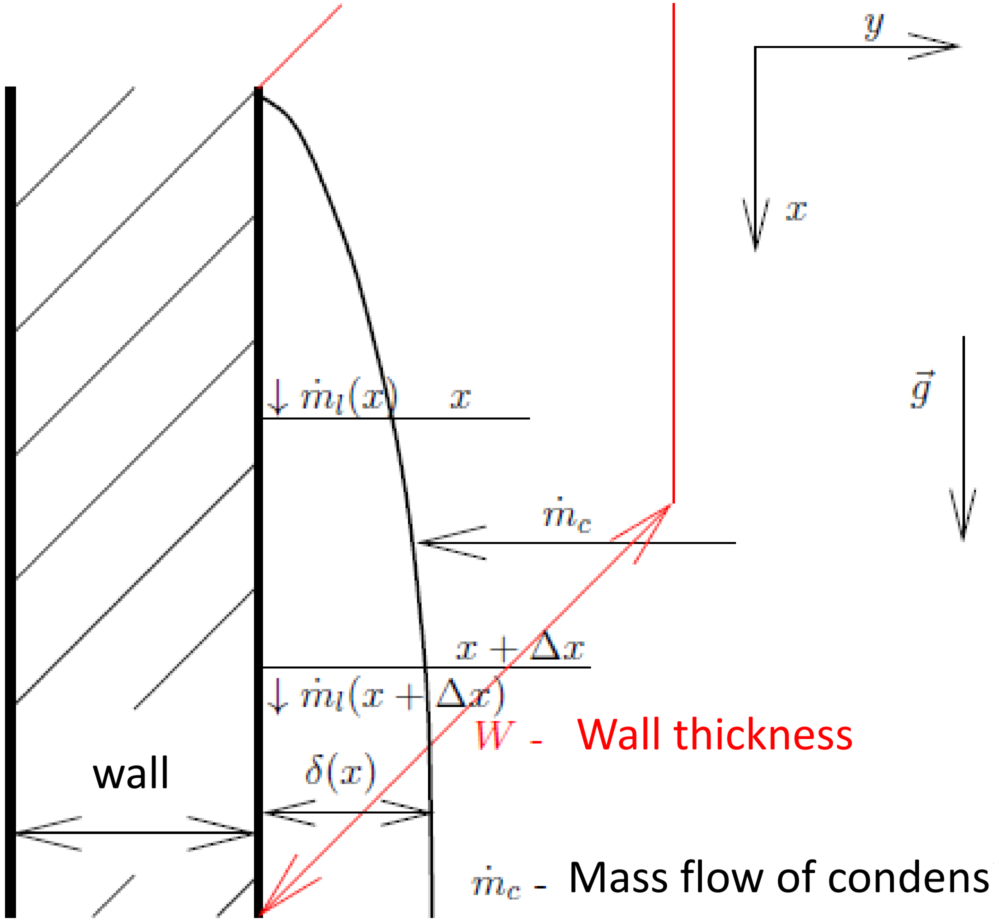
$$\dot{m}_l = W\rho_l U \int_0^\delta \left(-\frac{1}{2}\frac{y^2}{\delta^2} + \frac{y}{\delta}\right)\mathrm{d}y = W\rho_l U\left[-\frac{1}{6}\frac{y^3}{\delta^2} + \frac{y^2}{2\delta}\right]_0^\delta =$$

$$= W\rho_l U\left[-\frac{1}{6}\delta + \frac{1}{2}\delta\right] = \frac{1}{3}\delta W \rho_l U$$

$$\boxed{\dot{m}_l = \frac{1}{3}\frac{\rho_l W g(\rho_l - \rho_g)\delta^3}{\mu_l}}$$

$$j_c = \frac{1}{W}\frac{\mathrm{d}\dot{m}_l}{\mathrm{d}x} = \frac{1}{W}\frac{\mathrm{d}\dot{m}_l}{\mathrm{d}\delta}\frac{\mathrm{d}\delta}{\mathrm{d}x} = \frac{1}{\cancel{W}}\frac{\rho_l \cancel{W} g(\rho_l - \rho_g)\delta^2}{\mu_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}$$

$$\boxed{j_c = \frac{\rho_l g(\rho_l - \rho_g)\delta^2}{\mu_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}}$$



W - Wall thickness

$\dot{m}_c$ - Mass flow of condensing vapour

Enthalpy balance in elementary film section

- Mass flow of liquid:

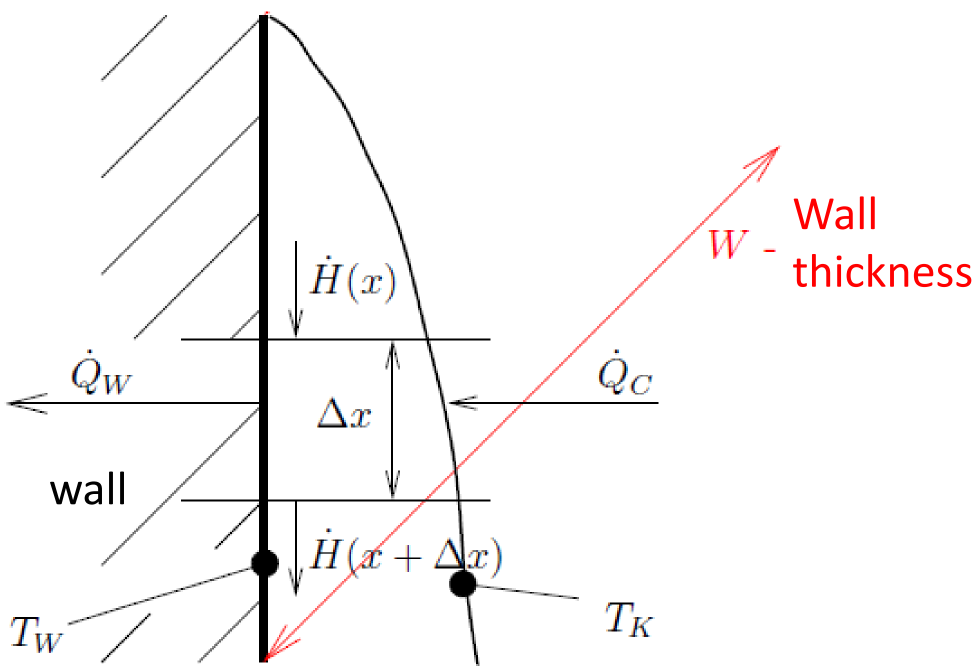


Figure: Energy flow in condensing film.

$$\dot{H}(x) + \dot{Q}_C = \dot{H}(x+\Delta x) + \dot{Q}_W$$

$\dot{Q}_C$ - Heat transferred by condensing vapour

$\dot{Q}_W$ - Heat removed by heat exchange surface

$$\dot{Q}_C = \dot{m}_C\, h_g = j_c\, \Delta x\, W\, h_g$$

Specific enthalpy of equilibrium vapour

$$\dot{Q}_W = q\, \Delta x\, W$$

$$q = -\lambda \frac{\mathrm{d}T}{\mathrm{d}y} \doteq \lambda \frac{T_K - T_W}{\delta}$$

Heat flow intensity through wall

$$\dot{Q}_W = \lambda \frac{T_K - T_W}{\delta}\, \Delta x\, W$$

Enthalpy balance in elementary film section

$$\cancel{\dot{H}(x)} + j_c \cancel{\Delta x} W h_g = \lambda \frac{T_K - T_W}{\delta} \cancel{\Delta x} W + \cancel{\dot{H}(x)} + \frac{\mathrm{d}\dot{H}(x)}{\mathrm{d}x} \cancel{\Delta x}$$

$$j_c W h_g = \lambda \frac{T_K - T_W}{\delta} W + \frac{\mathrm{d}\dot{H}(x)}{\mathrm{d}x}$$

- We must set a **reference state**: liquid phase + reference temperature T_{ref} is boiling point
- One component liquid/chemically pure liquid: boiling temperature = condensing temperature
- Then: $h_g = \Delta h_v$ = enthalpy of equilibrium of condensing vapour

$$j_c W \Delta h_v = \lambda \frac{T_K - T_W}{\delta} W + \frac{\mathrm{d}\dot{H}}{\mathrm{d}x}$$

- Integral form of enthalpy:

$$\dot{H} = \int_0^\delta W\, v_x\, \rho_l\, \langle c_{\mathrm{p}\,l} \rangle (T - T_{ref}) \mathrm{d}y$$

$\dot{V} = W\, v_x\, dy$	Volume flow of liquid
$\dot{m} = W\, v_x\, \rho_l$	Mass flow of liquid
$\langle c_{\mathrm{p}\,l} \rangle$	Mean heat capacity (without <>)
$h = \langle c_{\mathrm{p}\,l} \rangle (T - T_{ref})$	Specific enthalpy

Enthalpy balance in elementary film section

$$\begin{aligned} T_{ref} &= T_K \\ T &= \frac{y}{\delta}(T_K - T_W) + T_W \\ T - T_{ref} &= \frac{y}{\delta}(T_K - T_W) + T_W - T_K \quad = \quad (T_K - T_W)\left[\frac{y}{\delta} - 1\right] \end{aligned}$$

$$\dot{H} = \int_0^\delta W\,\rho_l\,c_{p\,l}(T_K - T_W)U\left[\frac{y}{\delta} - 1\right]\left[-\frac{1}{2}\frac{y^2}{\delta^2} + \frac{y}{\delta}\right]\mathrm{d}y =$$

$$= W\,\rho_l\,c_{p\,l}(T_K - T_W)U\int_0^\delta \left[-\frac{1}{2}\frac{y^3}{\delta^3} - \frac{y}{\delta} + \frac{3}{2}\frac{y^2}{\delta^2}\right]\mathrm{d}y =$$

$$= W\,\rho_l\,c_{p\,l}(T_K - T_W)U\left[-\frac{1}{8}\frac{y^4}{\delta^3} - \frac{y^2}{2\delta} + \frac{1}{2}\frac{y^3}{\delta^2}\right]_0^\delta =$$

$$= W\,\rho_l\,c_{p\,l}(T_K - T_W)U = \left[-\frac{1}{8}\delta - \frac{1}{2}\delta + \frac{1}{2}\delta\right] = W\,\rho_l\,c_{p\,l}(T_K - T_W)U\left[-\frac{1}{8}\delta\right]$$

$$\dot{H} = -\frac{1}{8}W\,\rho_l\,c_{p\,l}(T_K - T_W)U\delta = -\frac{1}{8}\frac{W\,\rho_l\,c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^3}{\eta_l}$$

$$\boxed{\frac{\mathrm{d}\dot{H}}{\mathrm{d}x} = \frac{\mathrm{d}\dot{H}}{\mathrm{d}\delta}\frac{\mathrm{d}\delta}{\mathrm{d}x} = -\frac{3}{8}\frac{W\,\rho_l\,c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}}$$

Now, we can apply formulas for j_c and (dH/dx)

Enthalpy balance in elementary film section

$$j_c W \Delta h_v = \lambda \frac{T_K - T_W}{\delta} W + \frac{\mathrm{d}\dot{H}}{\mathrm{d}x}$$

$$\frac{\rho_l\, g(\rho_l - \rho_g)\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}\cancel{W}\,\Delta h_v = \lambda \frac{T_K - T_W}{\delta}\cancel{W} - \frac{3}{8}\frac{\cancel{W}\rho_l c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}$$

- Calculation of δ(x) and α(x) done in MAPLE (time savings):

$$\alpha(x) = \frac{\lambda}{\delta_x} \qquad \overline{\alpha} = \frac{1}{L}\int_0^L \alpha(x)\mathrm{d}x$$

$$\frac{\mathrm{d}\dot{m}_l}{\mathrm{d}x} = j_c W$$

$$\dot{m}_l = \frac{1}{3}\frac{\rho_l\, W\, g(\rho_l - \rho_g)\delta^3(x)}{\eta_l}$$

$$\frac{\mathrm{d}\dot{m}_l}{\mathrm{d}x} = \frac{\rho_l\, W\, g(\rho_l - \rho_g)\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}$$

$$\boxed{j_c = \frac{\rho_l\, g(\rho_l - \rho_g)\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}}$$

$$\dot{H} = -\frac{1}{8}\frac{W\rho_l c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^3}{\eta_l}$$

$$\frac{\mathrm{d}\dot{H}}{\mathrm{d}x} = -\frac{3}{8}\frac{W\rho_l c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}$$

W: Should be removed

$$j_c W \Delta h_v = \lambda \frac{T_K - T_W}{\delta} W + \frac{\mathrm{d}\dot{H}}{\mathrm{d}x}$$

$$j_c \Delta h_v = \frac{\rho_l g(\rho_l - \rho_g)\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}\Delta h_v \qquad A = \frac{\rho_l g(\rho_l - \rho_g)}{\eta_l}\Delta h_v$$

$$j_c \Delta h_v = \lambda \frac{T_K - T_W}{\delta} - \frac{3}{8}\frac{\cancel{W}\rho_l c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g\delta^2}{\eta_l}\frac{\mathrm{d}\delta}{\mathrm{d}x}$$

$$B = \lambda(T_K - T_W) \qquad C = -\frac{3}{8}\frac{\cancel{W}\rho_l c_{p\,l}(T_K - T_W)(\rho_l - \rho_g)g}{\eta_l}$$

Enthalpy balance in elementary film section

- Calculation of δ(x) and α(x) done in MAPLE (time savings):

$$
\begin{aligned}
A\delta^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} &= \frac{B}{\delta} + C\delta^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} \\
(A - C)\,\delta^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} &= \frac{B}{\delta} \\
\delta^3 \mathrm{d}\delta &= \frac{B}{A - C}\,\mathrm{d}x \\
\int_0^{\delta_1} \delta^3\,\mathrm{d}\delta &= \int_0^{x_1} D\,\mathrm{d}x \\
\frac{\delta_1^4}{4} &= D\,x_1 \\
\delta &= (4D\,x_1)^{\frac{1}{4}} \\
\alpha &= \frac{\lambda}{\delta} \sim x^{-\frac{1}{4}}
\end{aligned}
$$

$$D = \frac{B}{A - C}$$

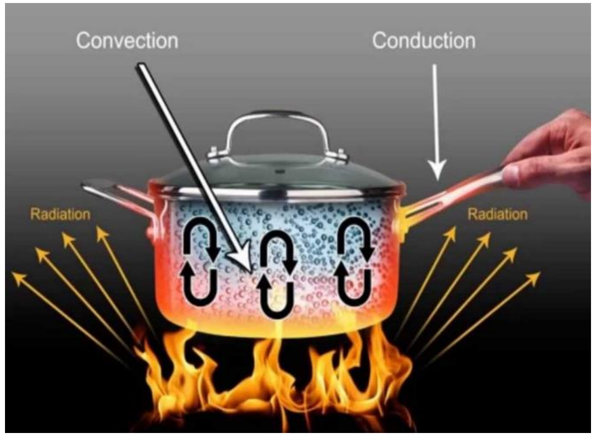
Lecture 12

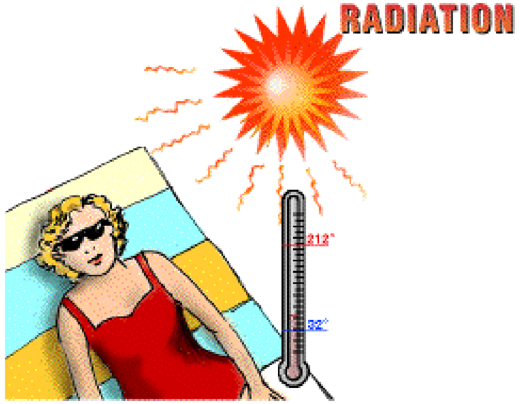
Heat transfer by radiation

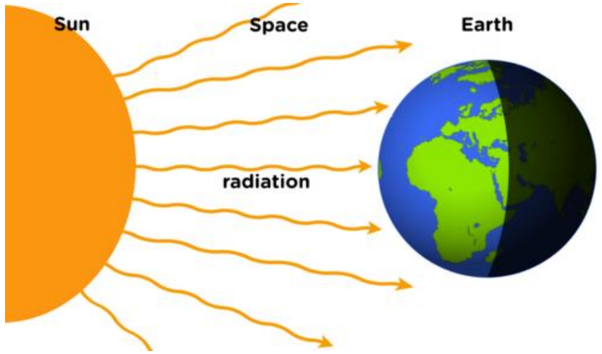
Heat transfer by radiation

- **Radiation** is the energy emitted from a surface as particles or waves → is transfer of heat by electromagnetic waves

  - It is different from conduction and convection as it requires no matter or medium to be present.
  - The radiative energy will pass perfectly through vacuum as well as clear air.
  - Conduction and convection depend on temperature differences to approximately the first power
  - the heat transfer by radiation depends on the differences of the individual body surface temperatures to the fourth power.
  - Therefore the radiation mode of heat transfer dominates over convection at high temperature levels as in fires.
  - Radiation depends on the temperature and surface properties of radiating object
  - Transfer of energy is even between to objects with similar temperature (their energy level is not changing)







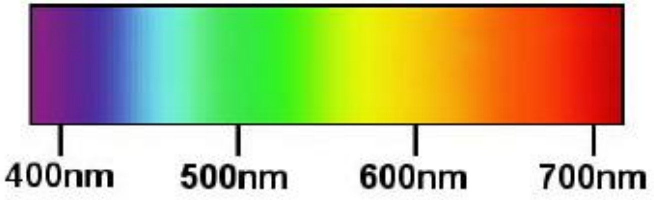
Electromagnetic spectrum

- Wavelength of light:

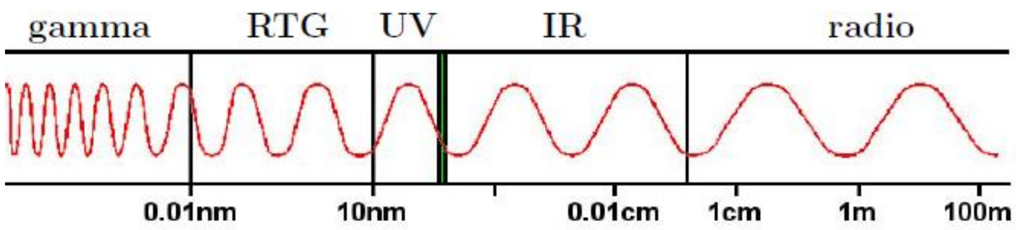
$$\lambda = \frac{c}{f} \quad \begin{array}{lll} f \text{ -} & \text{frequency} & [\mathrm{s}^{-1}] \\ c \text{ -} & \text{speed of light} & [\mathrm{m\ s}^{-1}] \end{array}$$

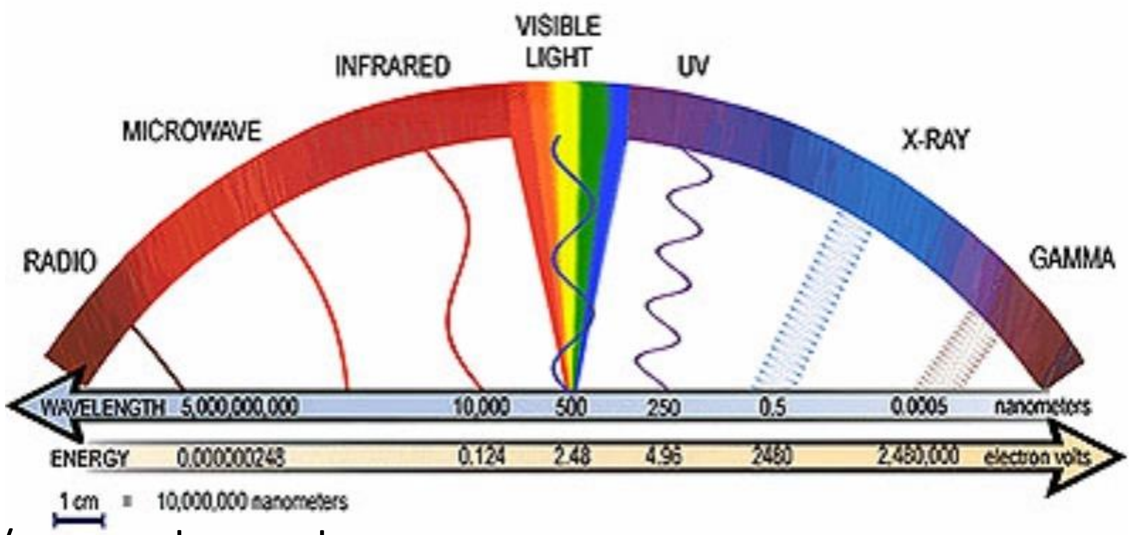
- Only small area of radiaton is generated by hot surfaces: 0.2 µm – 1 000 µm
- This interval contains ultraviolet (UV), visible (VIS) and infrared (IR) area

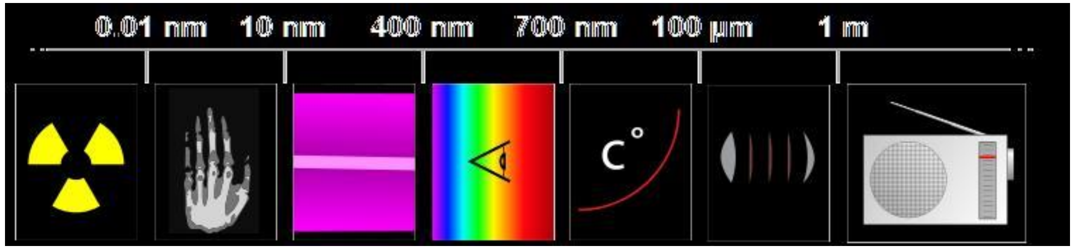
VIS area



↑wavelength ↓energy







http://www.auburn.edu

Black-body radiaton

Black-body radiaton Radiation heat transfer can be described by reference to the **'black body'**

Black body – Hypothetical body that absorbs all incoming radiation (all wavelengths and angles of incidence)

- Such bodies do not reflect light, and therefore appear black if their temperatures are low enough so as not to be self-luminous.
- All black bodies heated to a given temperature emit thermal radiation.
- emits equally in all directions (ideal diffuse emitter)
- emits maximum energy for particular temperature and wavelength
- emits various wavelengths for particular temperature

- **Maximum radiated energy** for temperature T is for wavelength λ_{max} :

Wien's displacement law	$\lambda_{max} T \doteq 2898\ \mu\text{m K}$

- **Final intensity flow of energy = Radiation energy per unit time** = for all wavelengths:

  It is proportional to the fourth power of the absolute temperature

  It is expressed by **Stefan-Boltzmann Law**

$$q_r = \sigma T^4 \ [\text{W m}^{-2}]$$

σ - Stefan-Boltzmann constant

$\sigma = 5{,}67 \cdot 10^{-8}\ [\text{W m}^{-2}\ \text{K}^{-4}]$

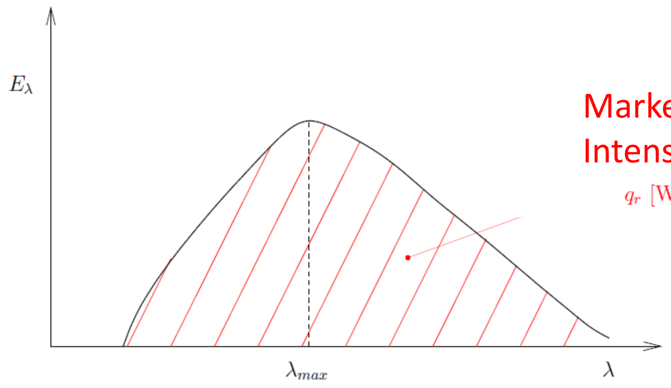
- This equation does not express energy flow for individuals wavelengths

Black-body radiaton

→ Planck's law:

Radiation energy per unit time: Emission power: Intensity flow of energy referring to individual wavelength = spectral density

Planck's law $$E_\lambda = \frac{C_1}{\lambda^5 \left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right]} \quad [\mathrm{W\ m^{-2}\ \mu m}]$$



Marked area =
Intensity flow of energy (total black body emissive power)
q_r [W m^{-2}]

FIG 2

Spectral distribution of the radiation emitted by the blackbody as a function of wavelength.

Blackbody provides an ideal limit to the emissive behavior of any surface.

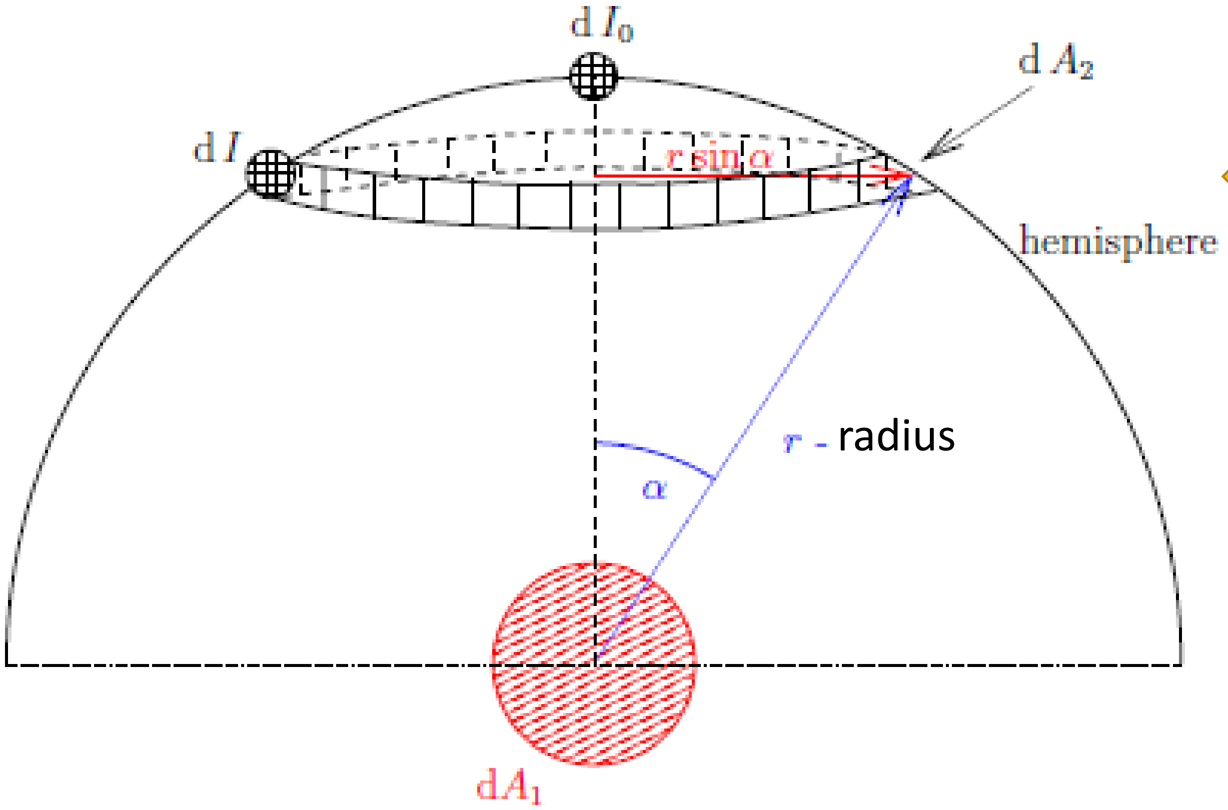
It is the absolute upper bound on the spectral emissive power that can be achieved by any real surface.

The distribution of blackbody spectral emissive power increases and shifts toward lower wavelengths as the temperature of the blackbody increases. Amount of the thermal radiation emitted by the blackbody is a strong function of temperature.

Black-body radiation

→ Then: Black body emissive power

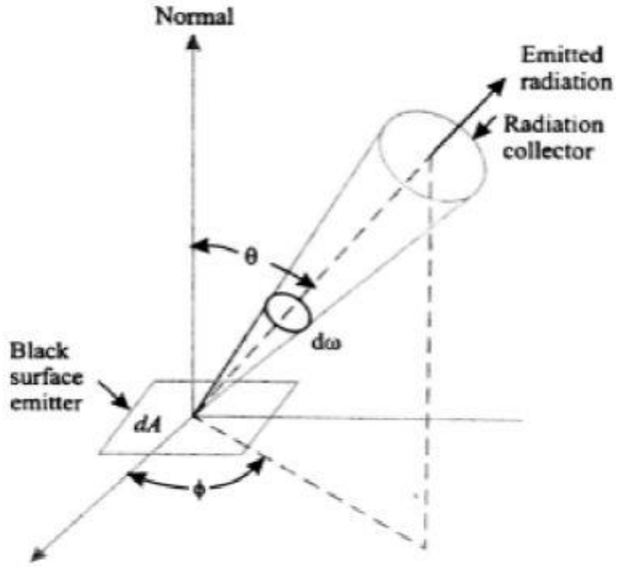
$$q_r = \sigma T^4 = \int_0^\infty E_\lambda \, d\lambda = \int_0^\infty \frac{C_1}{\lambda^5 \left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right]} d\lambda$$



Integrated over all possible wavelengths. Cant be integrated between Arbitrary wavelength limits (No analytical solution)

- Radiation exchanged between two black surfaces:

$$Intensity\ of\ radiation\ I = \frac{Amount\ of\ Energy}{Incoming\ surface\ area}$$



$$dI = dI_0 \cos \alpha$$

dI_0 - Energy from dA_1 incoming to surface perpendicularly above.

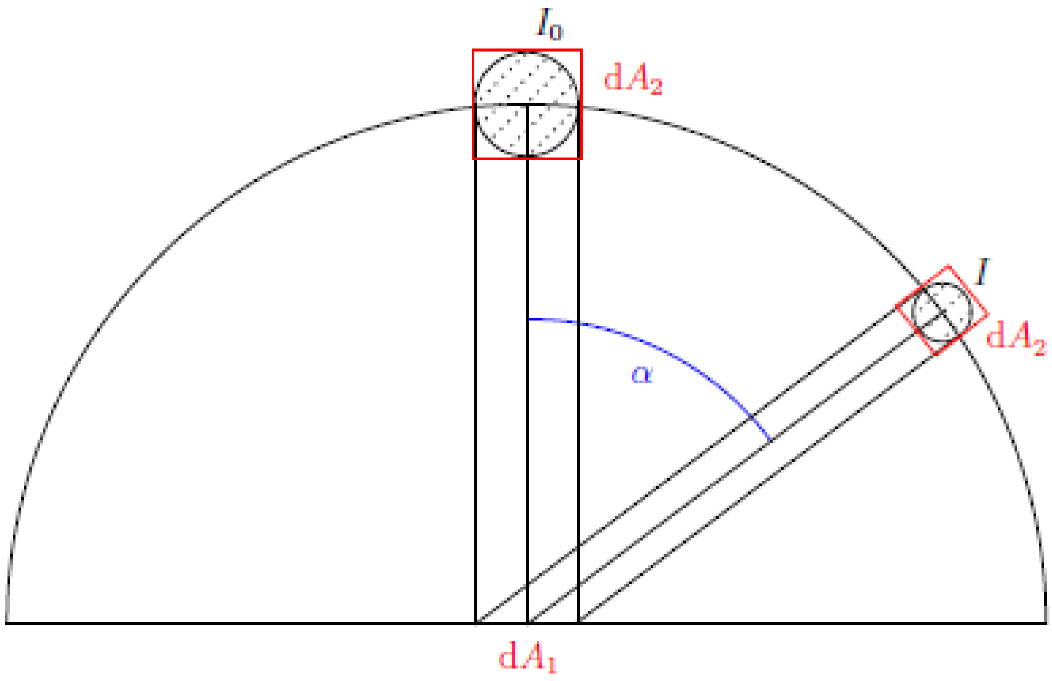
dI - Energy from dA_1 incoming to surface at an angle α. From that spot, surface dA_1 seems to be smaller.

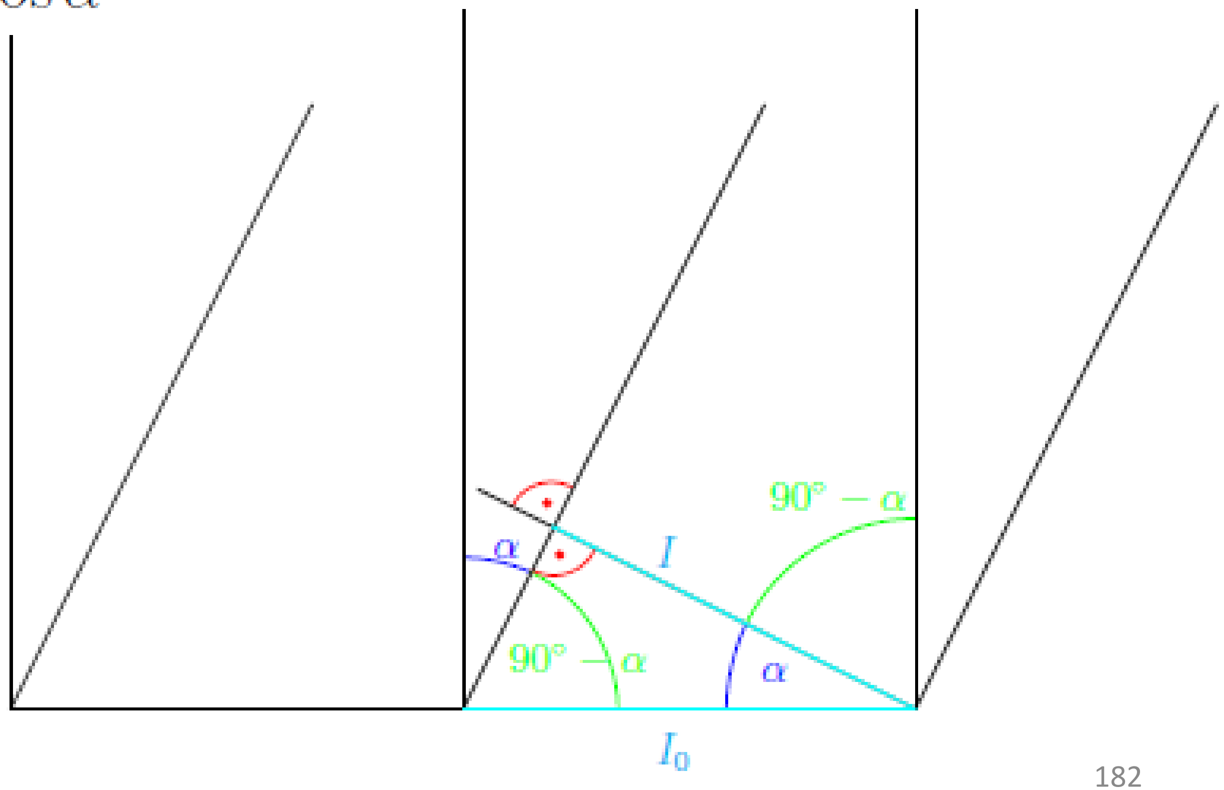
Lambert's cosine law

Radiant intensity from an ideal diffuse radiator (emitter) is directly proportional to the cosine of the angle between the direction of the incident light and the surface normal

$$\cos \alpha = \frac{I}{I_0}$$

$$I = I_0 \cos \alpha$$





Lambert's cosine law

- Energy received by surface A_2 from surface A_1:

$$\mathrm{d}q_{A2} = \mathrm{d}I\,\mathrm{d}A_2 = \mathrm{d}I_0 \cos\alpha\,\mathrm{d}A_2$$

- Surface $\mathrm{d}A_2$ = strip:

$$\mathrm{d}A_2 = \underbrace{2\pi r \sin\alpha}_{\text{Circumference of strip}}\ \underbrace{r\,\mathrm{d}\alpha}_{\text{Width of strip}}$$

$$\mathrm{d}q_{A2} = \mathrm{d}I_0\ 2\pi r^2 \sin\alpha\ \cos\alpha\ \mathrm{d}\alpha$$

- Total energy radiated by surface $\mathrm{d}A_1$ = energy incoming to the surface A_2:

$$\mathrm{d}E_{1\to 2} = \underbrace{W_1}\,\mathrm{d}A_1 = \int_{A_2} \mathrm{d}q_{A_2} = \int_0^{\pi/2} 2\pi r^2\ \mathrm{d}I_0 \sin\alpha \cos\alpha\ \mathrm{d}\alpha$$

Lambert's cosine law

- Radiation energy related to the surface:

$$\int \sin ax \ \cos ax \ \mathrm{d}x = \frac{\sin^2 ax}{2a}$$

$$a = 1$$

$$\int \sin x \ \cos x \ \mathrm{d}x = \frac{\sin^2 x}{2}$$

$$W_1 \, \mathrm{d}A_1 = 2\,\pi\,r^2\,\mathrm{d}I_0 \left[\frac{\sin^2 x}{2}\right]_0^{\frac{\pi}{2}} = 2\,\pi\,r^2\,\mathrm{d}I_0 \frac{1}{2} = \pi\,r^2\,\mathrm{d}I_0$$

- We can express dI_0:

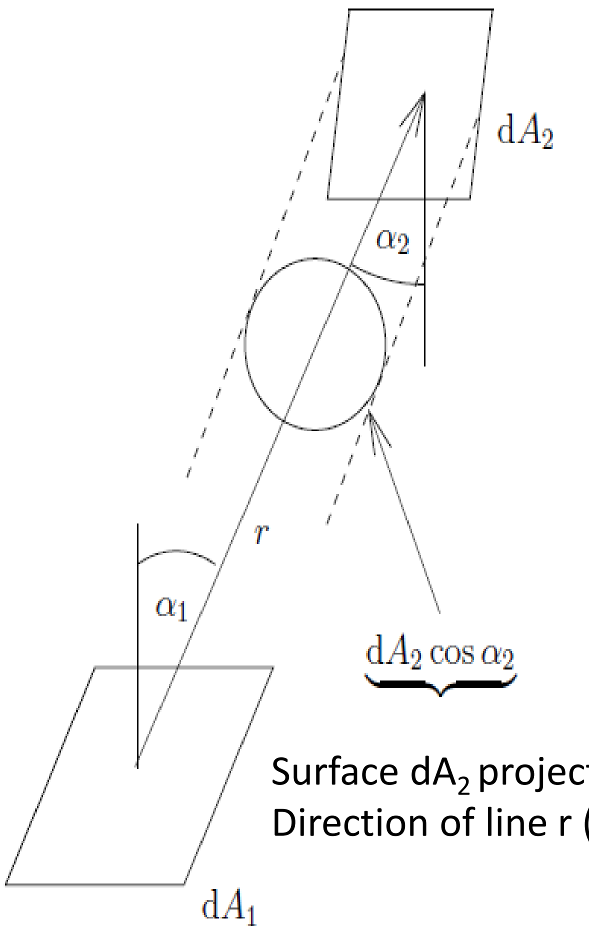
$$\left\{\begin{aligned} \mathrm{d}I_0 &= \frac{W_1 \mathrm{d}A_1}{\pi r^2} \\ \mathrm{d}I &= \cos\alpha \ \mathrm{d}I_0 \ , \end{aligned}\right.$$

$$\mathrm{d}I = \frac{W_1}{\pi r^2} \cos\alpha \ \mathrm{d}A_1$$

- Energy radiated from surface dA_1 to the surface dA_2 in the distance r

Lambert's cosine law

- Surface dA_2 is not necessarily perpendicular to the connecting line r:



Surface dA_2 projected to
Direction of line r (cosine law).

$$q_{\,\mathrm{d}A_1 \to \mathrm{d}A_2} = \underbrace{\mathrm{d}I_1 \;\; \cos\alpha_2 \;\; \mathrm{d}A_2}$$

$dI = dI_1 \cos\alpha_2$

dI_1: energy radiated by surface dA_1 over the surface dA_2

$\cos\alpha_2 dA_2$: surface dA_2 projected into perpendicular direction

$$q_{\,\mathrm{d}A_1 \to \mathrm{d}A_2} = \frac{W_1}{\pi\, r^2} \cos\alpha_1 \cos\alpha_2 \;\mathrm{d}A_1 \;\mathrm{d}A_2$$

Total energy radiated from A_1 to A_2:

$$q_{1\to 2} = W_1 \int_{A_1}\int_{A_2} \frac{\cos\alpha_1 \cos\alpha_2}{\pi\, r^2} \;\mathrm{d}A_1 \;\mathrm{d}A_2$$

$$\int_{A_1}\int_{A_2} \frac{\cos\alpha_1 \cos\alpha_2}{\pi\, r^2} \;\mathrm{d}A_1 \;\mathrm{d}A_2 = F_{12} A_1 = F_{21} A_2$$

View factor

Diffuse surfaces emit radiation uniformly in all directions. This behavior reduces the complexity associated with determining the net radiation exchange between black surfaces.

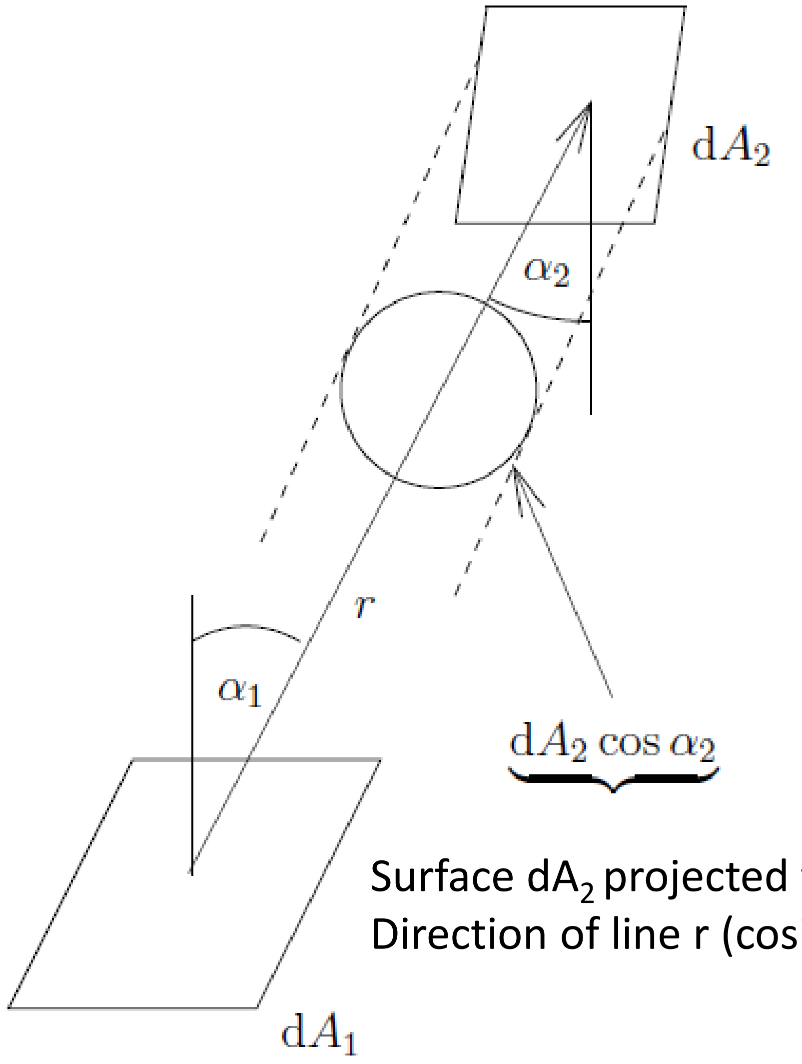
In the limit that the surfaces involved in a radiation problem are all diffuse emitters, the fraction of radiation emitted by one surface that hits another depends only on the relative geometric orientation of the two surfaces, and not on the characteristics of the surfaces themselves.

The geometric orientation is captured by the view factor.

The view factor is defined as the fraction of the total radiation that leaves surface 1 and goes directly (no intermediate surface) to surface 2.

$$F_{12} = \frac{energy\ emitted\ by\ surface\ 1\ coming\ to\ surface\ 2}{total\ energy\ emitted\ by\ surface\ 1}$$

Lambert's cosine law



Surface dA_2 projected to Direction of line r (cosine law).

Figure: Heat transfer by radiation between surfaces A_1 and A_2.

$$F_{12} = \frac{energy\ emitted\ by\ surface\ 1\ coming\ to\ surface\ 2}{total\ energy\ emitted\ by\ surface\ 1}$$

$$q_{1\to 2} = W_1 F_{12}\, A_1 = W_1 F_{21}\, A_2$$

$$W_1 = \sigma\, T_1^4$$

$$q_{1\to 2} = \sigma\, T_1^4 F_{12}\, A_1 = \sigma\, T_1^4 F_{21}\, A_2$$

Total energy transfered from A_2 to A_1:

$$q_{2\to 1} = W_2 F_{12}\, A_1 = W_2 F_{21}\, A_2$$

$$W_2 = \sigma\, T_2^4$$

$$q_{2\to 1} = \sigma\, T_2^4 F_{12}\, A_1 = \sigma\, T_2^4 F_{21}\, A_2$$

Total energy flow between 1 and 2:

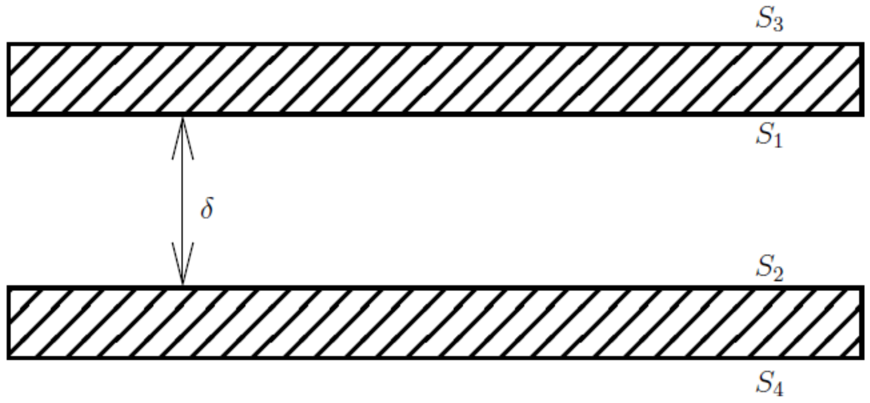
$$q_{12} = q_{1\to 2} - q_{2\to 1}$$

$$q_{12} = \sigma\, T_1^4 F_{12}\, A_1 - \sigma\, T_2^4 F_{12}\, A_1 = \sigma\, F_{12}\, A_1\, (T_1^4 - T_2^4)$$

Alternatively:

$$q_{12} = \sigma\, F_{21}\, A_2\, (T_1^4 - T_2^4)$$

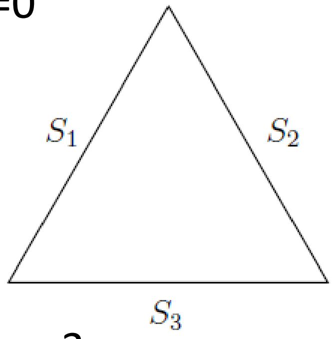
View factor



- Proportion of the radiation from surface 1 that hits surface 2
- For δ→0 all radiation from surface S_1 hit surface S_2

$$\lim_{\delta \to 0} F_{12} = 1$$

- Surface 3 and 4 „cannot see each other“ → $F_{34}=0$
- Closed system: $\sum_j F_{ij} = 1$



- For example: $F_{11} + F_{12} + F_{13} = 1$
- All radiation emitted by surface 1 hits surface 2 or 3
- Calculation of view factor is complicated
- Libraries for various geometries: http://www.thermalradiation.net/indexCat.html

Emissivity

- Black-body: 100% emissivity, emits maximum energy for particular temperature
- Other objects: lower emissivity

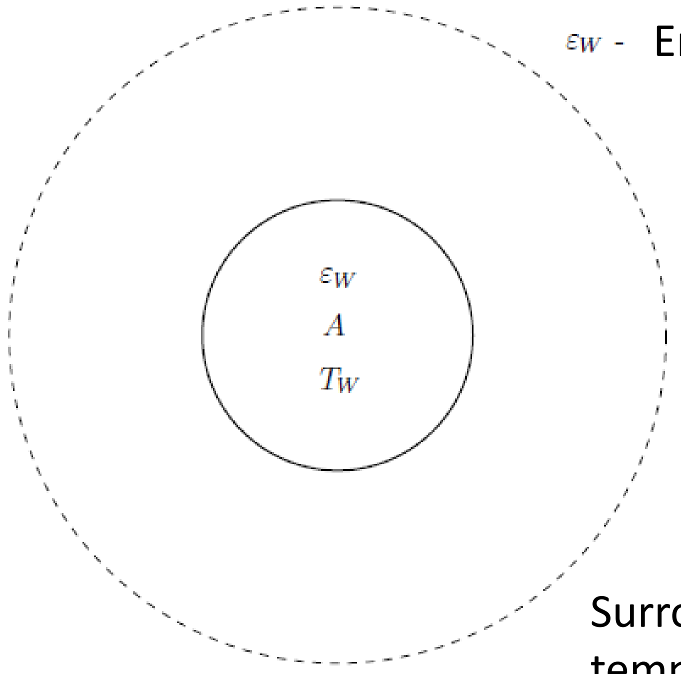
Emissivity $\rightarrow \quad \varepsilon = \dfrac{W}{W_b} = \begin{array}{l} \leftarrow \text{Energy emitted by other object} \\ \leftarrow \text{Energy emitted by black-body} \end{array}$

$$W = \varepsilon W_b = \varepsilon \sigma T^4$$

Usual emissivity values:

Polished metals	$\varepsilon = 0,02 - 0,10$
Passivated metals	$\varepsilon = 0,60 - 0,85$
Paper, wall	$\varepsilon = 0,65 - 0,95$
Dyes	$\varepsilon = 0,80 - 0,96$

Combined heat transfer



ε_W - Emissivity

Surroundings,
temperature T

- Heat transfer from surface A to the surroundings
- Intensity **flow of radiated heat**:

$$Q_{A\text{- Surround.}} = \sigma F_{A\to 0}\, A\, \varepsilon_W \left(T_W^4 - T^4\right)$$
$$q_r = \frac{Q_{A\to 0}}{A} = \sigma F_{A\to 0}\, \varepsilon_W \left(T_W^4 - T^4\right)$$

- All radiated heat strikes surroundings:

$$F_{A\to 0} = 1, \quad q_r = \sigma\, \varepsilon_W \left(T_W^4 - T^4\right)$$

- Intensity **flow of heat conduction**:

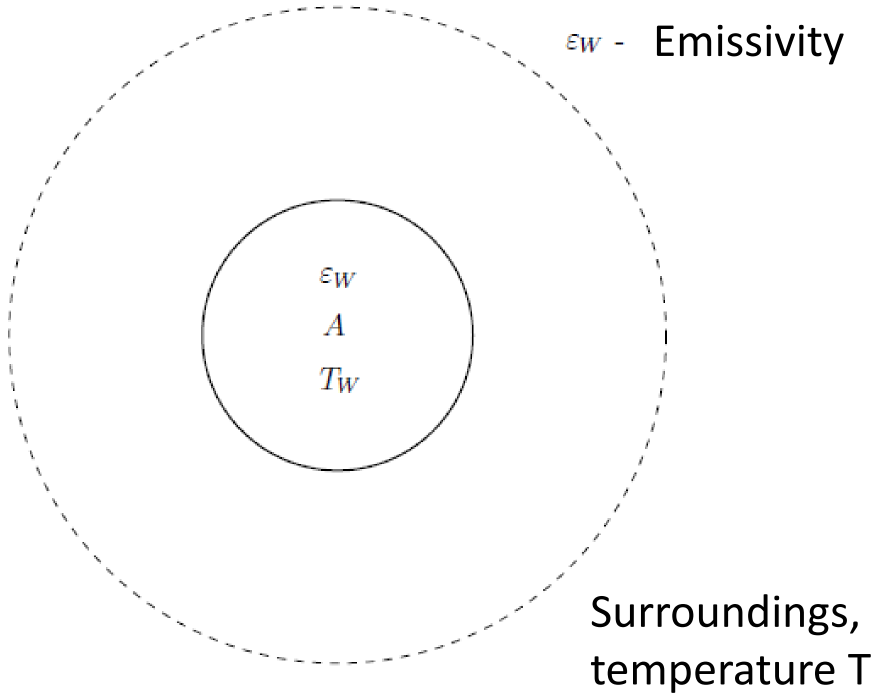
$$q_c = \alpha \left(T_W - T\right)$$

α heat transfer coefficient

- For very hot objects, we have to **sum up both contributions**:

$$q = q_r + q_c = \alpha \left(T_W - T\right) + \sigma\, \varepsilon_W \left(T_W^4 - T^4\right)$$

Combined heat transfer



- Radiation coefficient α_r:

$$q_r = \alpha_r \left(T_W - T\right)$$

$$q_r = \sigma\, \varepsilon_W \left(T_W^4 - T^4\right) = \sigma\, \varepsilon_W \left(T_W^2 - T^2\right)\left(T_W^2 + T^2\right) =$$
$$= \sigma\, \varepsilon_W \left(T_W^2 + T^2\right)\left(T_W + T\right)\left(T_W - T\right)$$

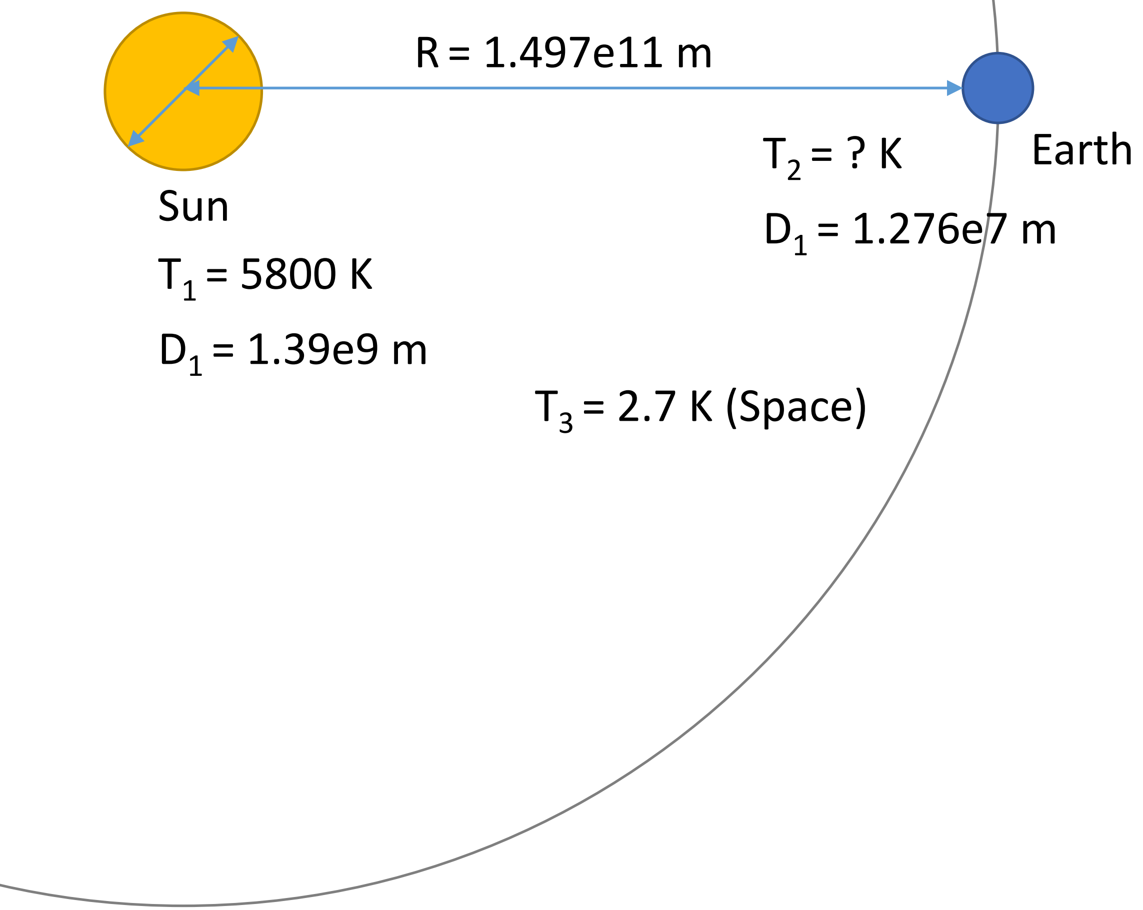
- If $T_W \approx T$, then:

$$q_r \doteq \underbrace{\sigma\, \varepsilon_W\, 2\, T_W^2\, 2\, T_W}_{\alpha_r} \left(T_W - T\right) \Rightarrow \boxed{\alpha_r = 4 T_W^3 \varepsilon_W \sigma}$$

$$\boxed{q = (\alpha + \alpha_r)(T_W - T)}$$

Exercise: Estimation of earth surface energy

- Energy from the Sun, from the Earth core
- Earth absorbs the same energy as emits (constant temperature)



$$F_{23} \to 1 \qquad A_1 = \pi D_1^2 \quad \text{(Sun)}$$

$$F_{21} > F_{12} \qquad A_2 = \pi D_2^2 \quad \text{(Earth)}$$

$$E_{12} = E_{21} + E_{23}$$

$$\delta \cdot (T_1^4 - T_2^4) \cdot F_{12} \cdot A_1 = \delta \cdot (T_2^4 - T_3^4) \cdot F_{23} \cdot A_2$$

$$F_{12} = \frac{incoming\ radiation}{emitted\ radiation} = \frac{surface\ of\ Sun\ cross\ section}{surface\ of\ big\ sphere\ around} = \frac{\frac{\pi D_2^2}{4}}{4\pi R^2} = 4.5e-10$$

$$T_2 \approx 278.9\ \text{K} \approx 6\ °\text{C}$$