Project 4 - Detection theory

INF4480

Petter Andre Kristiansen

March 18, 2018

University of Oslo, Department of Informatics

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A CFAR detector

Exercise 1 A - What is the PDF $f_\theta(x)$ of the signal + noise?

- $H_0 : x = n$
- $H_1 : x = \theta + n$
- $f_n(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x)^2}{2\sigma^2}}$ for $H_0 : \theta = 0$
- $f_\theta(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$ for H_1

Exercise 1 B - How does P_{FA} depend on θ and σ^2

$$P_{FA} = Pr(D_1|H_0) = \int_{x_c}^{\infty} f_n dx \tag{1}$$

- Increasing θ results in a increase in the distance between f_n and f_θ
- Reducing the variance σ^2, results in narrower probability distribution functions which will also increase the distance
- Increasing the variance, makes the PDFs potentially overlapping more, depending on θ
- No overlap, means no false alarm

Exercise 1 B - How does P_{FA} depend on θ and σ^2

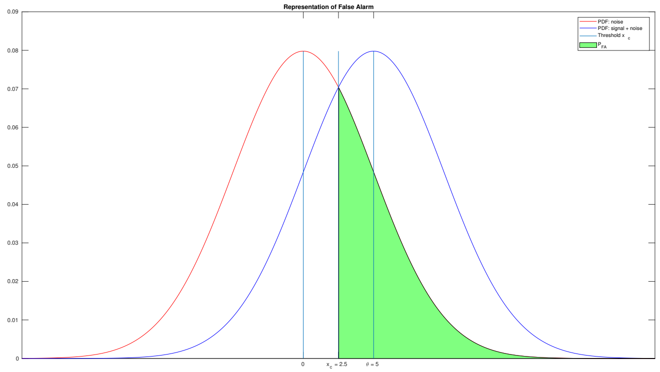

Figure 1: False alarm

Exercise 1 B - How does P_{FA} depend on θ and σ^2

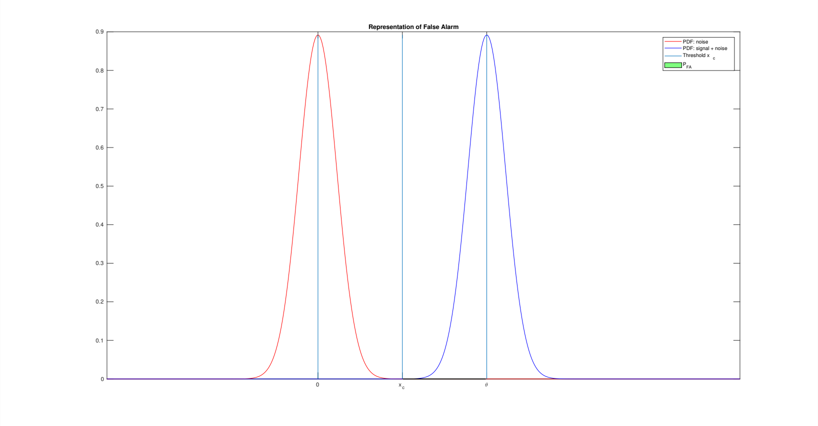

Figure 2: False alarm, smaller variance

Exercise 1 B - How does P_{FA} depend on θ and σ^2

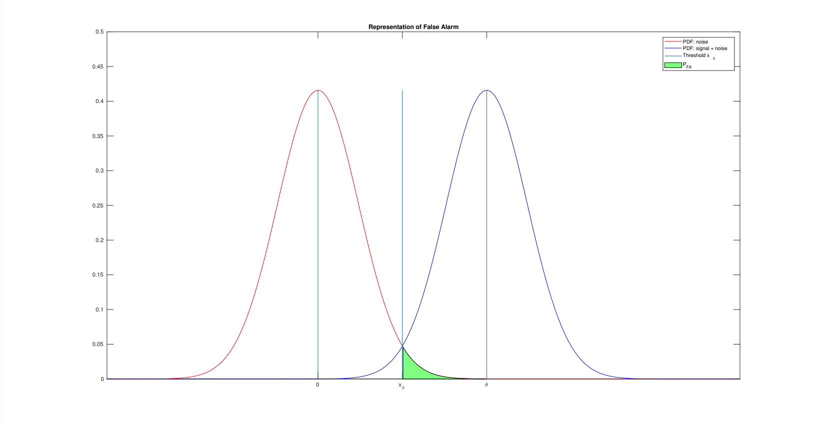

Figure 3: False alarm, larger θ

Exercise 1 C - Designing a CFAR detector

$H_0 : x = n$
$H_1 : x = 5 + n$

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} \cdot e^{-\frac{(x-5)^2}{2\sigma_1^2}}}{\frac{1}{\sqrt{2\pi\sigma_0^2}} \cdot e^{-\frac{(x)^2}{2\sigma_0^2}}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \tag{2}$$

$$\Lambda(x) = e^{-\frac{(x-5)^2}{2\sigma^2}} e^{\frac{(x)^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \tag{3}$$

Exercise 1 C - Designing a CFAR detector

$$\Lambda(x) = e^{\frac{-x^2+10x-25+x^2}{2\cdot 5^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \tag{4}$$

$$ln\Lambda(x) = lne^{\frac{10x-25}{50}} \underset{H_0}{\overset{H_1}{\gtrless}} ln\lambda \tag{5}$$

$$ln\Lambda(x) = \frac{10x-25}{50} \underset{H_0}{\overset{H_1}{\gtrless}} ln\lambda \tag{6}$$

Exercise 1 C - Designing a CFAR detector

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{50 \cdot ln\lambda + 25}{10} = \gamma \tag{7}$$

$$P_{FA} = \int_{\gamma}^{\infty} f_n dx = 0.1 \tag{8}$$

Numerical in matlab using cumsum up until the area of P_{CR} reaches 0.9, γ results in 6.40776. Or we can use the Qinv.m code to calculate γ.

Exercise 1 C - Designing a CFAR detector

```
function y = qinv(x,mu,sigma)
% function y = Qinv(x,mu,sigma)
% The function calculates Qinverse(x), the inverse Gaussian
    tail probability.
% Input parameters
%       x         : The value of the Gaussian tail probability
%       Optional:
%         mu      : mean value of Gaussian distribution.
    Default: 0a
%         sigma   : standard deviation of Gaussian
    distribution. Default: 1
%
% Output          : Qinv
%
% AA, 4/3-2014    : First version
%
if nargin<3
    mu = 0;
    sigma = 1;
end
y = mu + sqrt(2)*sigma*erfinv(1-2*x);
```

Exercise 1 C - Designing a CFAR detector

P_D is then calculated with:

$$P_D = \int_{\gamma}^{\infty} f_\theta dx = \int_{6.40776}^{\infty} f_\theta dx = 1 - P_M = 1 - \int_{-\infty}^{6.40776} f_\theta dx \qquad (9)$$

Solving this numerically in matlab with a cumsum, results in:

$$P_D = 1 - 0.6108 = 0.3892 \qquad (10)$$

Exercise 1 C - Designing a CFAR detector

```matlab
function [PM, PD] = calc_pd(f, xc)
    %   Inputs:
    %           x   - Timeseries
    %           N   - Amount of samples in the timeseries
    %           fs  - PDF of signal + noise (theta)
    %           xc - Threshold limit (gamma)
    %   Outputs:
    %           PM  - Probability of miss
    %           PD  - Probability of choosing correct
    %   Petter Andre Kristiansen

    PD = integral(f, xc, inf);
    PM = 1 - PD;
end
```

Exercise 1 D - Plotting the ROC curves

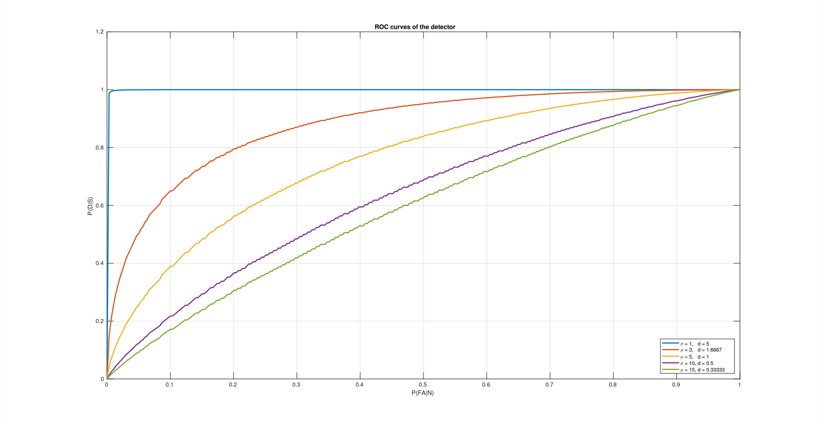

Figure 4: Representation of the ROC curves, unique pairs for each P_{FA}

Exercise 1 D - Plotting the ROC curves

- Lower σ means higher signal-to-noise-ratio(SNR)
- $d' = \frac{\theta}{\sigma}$
- This means, we get less "False alarms" with lower values of σ
- From the curves, $\sigma = 1$ gives us the best result faster

Exercise 1 E - Plotting $f_n(x)$ & $f_\theta(x)$

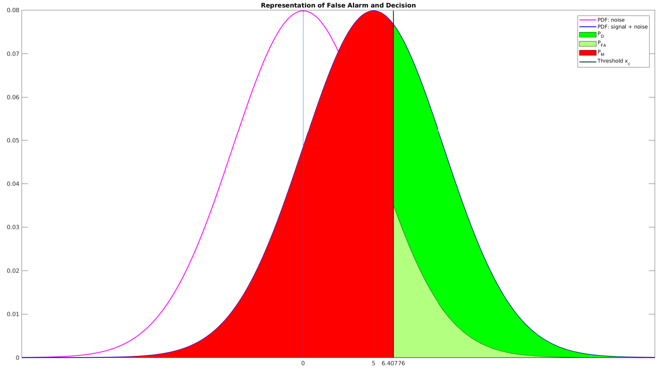

Figure 5: Representation of $f_n(x)$ & $f_\theta(x)$

Exercise 1 F - Improving the CFAR detector

- H_0' : $x_n = n_n$, for $n = 1, ..., N$.
- H_1' : $x_n = \theta + n_n$, for $n = 1, ..., N$.

The joint PDF of the noise is given by:

$$f_n(x) = \frac{1}{\sqrt{2\pi N\sigma^2}} \exp - \frac{x^2}{2N\sigma^2} \tag{11}$$

$$f_\theta(x) = \frac{1}{\sqrt{2\pi N\sigma^2}} \exp - \frac{(x - N\theta)^2}{2N\sigma^2} \tag{12}$$

Exercise 1 F - Improving the CFAR detector

```matlab
% Calculate PD for for all N, until PD > 0.9

close all; clear; clc;
sigma = 5;
fs_mu = 5;
lower = -10*sigma;  % lower limit
upper = 10*sigma;   % upper limit

samples = 10000;
PD = 0; N = 0; PFA = 0.01;

while 0.9 > PD
    N = N + 1;
    fn = @(x) (1/sqrt(2*pi*N*sigma.^2))*exp(-x.^2/(2*N*sigma.^2));
    fs = @(x) (1/sqrt(2*pi*N*sigma.^2))*exp(-(x - N*fs_mu).^2/(2*N*sigma.^2));
    x = linspace(N*lower,N*upper,samples);
    xc = cfar(x, fn, PFA);
    [~,PD] = calc_pd(fs, xc);
end

```

Exercise 1 G - Plotting the improving CFAR detector

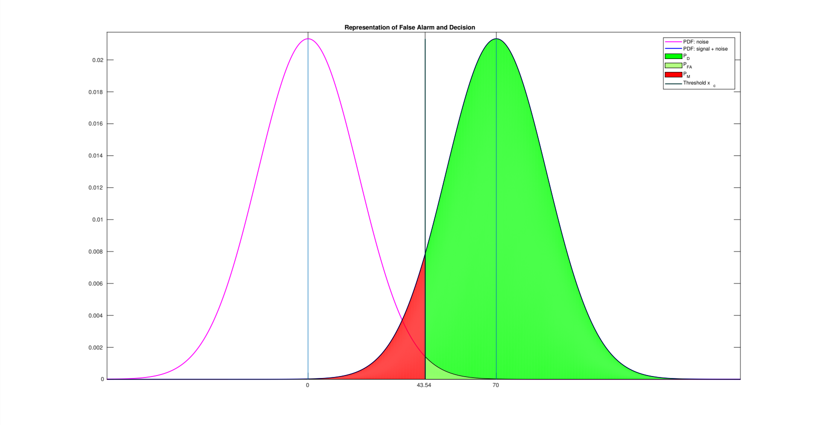

Figure 6: Representation of $f_n(x)$ & $f_\theta(x)$ after improvement, $\mathrm{N} = 14$

A maximum likelihood detector

Exercise 2 A - What does the detector look like

- $\theta = 1$
- $\sigma = \frac{3}{13}$

The ML detector is given by the difference between $f_\theta(x)$ and $f_n(x)$:

$$\frac{f_\theta(x)}{f_n(x)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{13}$$

Exercise 2 A - What does the detector look like

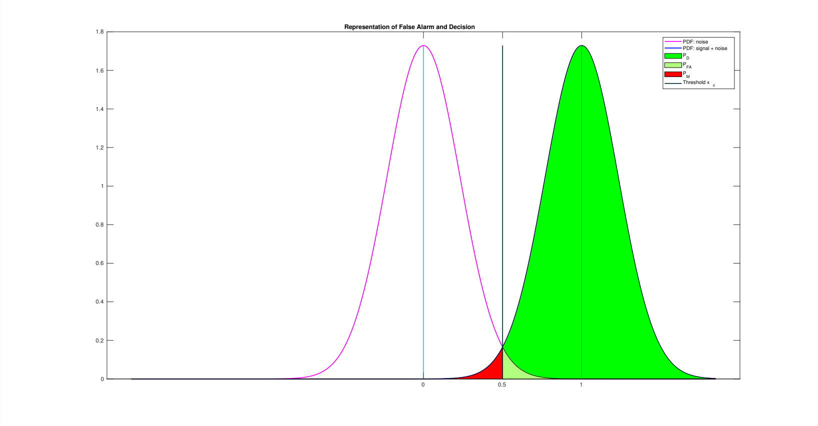

Figure 7: Representation of the detector

Exercise 2 A - How well will it perform with respect to P_{FA}, P_M, P_D

Resulting in a threshold $xc = (0 + \theta)/2 = 0.5$ and giving:

- $P_{FA} = P_M = 0.0151$
- $P_D = 0.9849$

Exercise 2 B - Constructing the test data

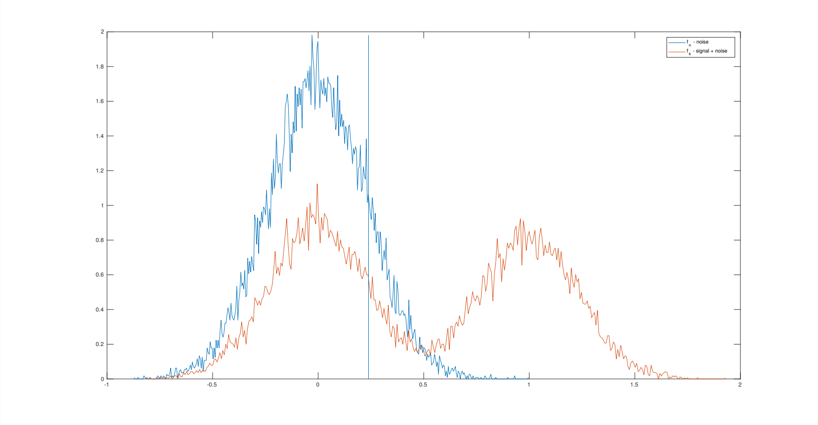

Figure 8: Representation of the detector

Exercise 2 B - Constructing the test data

- I had problems generating the signals properly.
- The signal generator looked like it was generating a uniform distribution
- This is probably why I got two peaks in the PDF
- These problems affected the other exercies c) and d)

Questions?