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Petra Poklukar

On the real spectrum compactification of Teichmüller space

Master thesis

Supervisor: prof. dr. Marc Burger[1]
Co-supervisor: akad. prof. dr. Franc Forstnerič[2]

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[1]Department of Mathematics, ETH Zürich
[2]Faculty of Mathematics and Physics, University of Ljubljana

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Petra Poklukar

O realni spektralni kompaktifikaciji Teichmüllerjevega prostora

Magistrsko delo

Mentor: prof. dr. Marc Burger[1]
Somentor: akad. prof. dr. Franc Forstnerič[2]

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[1]Oddelek za matematiko, ETH Zürich
[2]Fakulteta za matematiko in fiziko, Univerza v Ljubljani

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Contents

Work description

In 1988 G. Brumfiel published a short paper "The real spectrum compactification of Teichmüller space" in which he described an approach to compactifying Teichmüller space using its realization V as a closed semi-algebraic subset of an appropriate affine space and applying techniques from real semi-algebraic geometry. More precisely, he introduced the real spectrum compactification $\mathrm{Spec}_{\mathrm{R}}(V)$ of V and showed that there is a natural continuous surjection of $\mathrm{Spec}_{\mathrm{R}}(V)$ to the Thurston compactification of Teichmüller space. The article of Brumfiel is written in a very concise way, and gives only indications on how to proceed in the proofs. The topic of the master thesis of Ms Poklukar is to provide detailed proofs of the statements in Brumfiel's paper and in particular provide the detailed background in semi-algebraic geometry necessary for the understanding of the yet largely unexploited original approach of Brumfiel.

prof. dr. Marc Burger

On the Real Spectrum Compactification of Teichmüller space

Abstract

In this thesis we consider the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ of Teichmüller space $\mathrm{T}_g(\Sigma)$ associated with a closed oriented surface Σ of genus $g \geq 2$. A geometric model of $\mathrm{T}_g(\Sigma)$ is presented after giving a brief overview of hyperbolic geometry and hyperbolic surfaces. We then focus on the real spectrum $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ of a commutative ring with identity A, and its subspace $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ consisting of closed points of $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$. We define the real spectrum compactification of a closed semi-algebraic subset W of a real algebraic set X as its closure in the space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, where $\mathcal{A}(X)$ denotes the affine coordinate ring of X. In order to apply this theory to Teichmüller space, we introduce its algebraic model defined as the space of conjugate classes of discrete faithful representations of the first fundamental group $\pi_1(\Sigma)$ in the group $\mathrm{SL}(2, \mathbb{R})$. Using the trace function, we embed $\mathrm{T}_g(\Sigma)$ into the real space $\mathbb{R}^M$ as a closed semi-algebraic set, which yields its real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$. We close this thesis by defining the well-known Thurston compactification of Teichmüller space $\hat{\mathrm{T}}_g(\Sigma)$, and comparing it with the obtained real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$. Finally, we prove that there exists a continuous surjection from $\widetilde{\mathrm{T}}_g(\Sigma)$ to $\hat{\mathrm{T}}_g(\Sigma)$.

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Keywords: hyperbolic surface, Teichmüller space, real spectrum of a ring, semi-algebraic set, real spectrum compactification, representation.

O realni spektralni kompaktifikaciji Teichmüllerjevega prostora

Povzetek

V magistrskem delu obravnavamo realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g(\Sigma)$ Teichmüllerjevega prostora $\mathrm{T}_g(\Sigma)$ prirejenega sklenjeni, orientirani ploskvi Σ roda $g \geq 2$. Na kratko povzamemo osnovne pojme s področja hiperbolične geometrije in hiperboličnih ploskev ter predstavimo geometričen model Teichmüllerjevega prostora. Nato se posvetimo realnemu spektru $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ komutativnega kolobarja z enico A in njegovemu podprostoru $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$, ki sestoji iz zaprtih točk prostora $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$. Definiramo realno spektralno kompaktifikacijo zaprte semi-algebraične podmnožice W realne algebraične množice X kot njeno zaprtje v prostoru $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, kjer $\mathcal{A}(X)$ označuje afini koordinatni kolobar množice X. Za uporabo te teorije na Teichmüllerjevemu prostoru najprej predstavimo njegov algebraični model, definiran kot prostor konjugiranih razredov diskretnih, zvestih reprezentacij prve fundamentalne grupe $\pi_1(\Sigma)$ v grupi $\mathrm{SL}(2, \mathbb{R})$. Vsaki taki reprezentaciji priredimo sled ustrezne matrike in s tem vložimo $\mathrm{T}_g(\Sigma)$ v realni prostor $\mathbb{R}^M$ kot zaprto semi-algebraično množico, kar porodi njegovo realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g(\Sigma)$. Nazadnje definiramo še Thurstonovo kompaktifikacijo Teichmüllerjevega prostora $\hat{\mathrm{T}}_g(\Sigma)$ in jo primerjamo z obravnavano realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g(\Sigma)$. Dokažemo, da obstaja zvezna surjekcija iz $\widetilde{\mathrm{T}}_g(\Sigma)$ v $\hat{\mathrm{T}}_g(\Sigma)$.

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Ključne besede: hiperbolična ploskev, Teichmüllerjev prostor, realni spekter kolobarja, semi-algebraična množica, realna spektralna kompaktifikacija, reprezentacija.

1. Introduction

The Teichmüller space of a closed connected orientable topological surface Σ of genus $g \geq 2$ was introduced in the 1930s by a German mathematician Oswald Teichmüller. He studied the so-called moduli space of equivalence classes of Riemann surfaces, where two such surfaces are considered equivalent if there exists a holomorphic homeomorphism between them. By introducing a new relation on the moduli space, he obtained a simpler space known as the Teichmüller space T_g associated with the surface Σ. It can be defined as the set of equivalence classes of hyperbolic surfaces marked by Σ, that is, hyperbolic surfaces diffeomorphic to Σ. Equivalently, it can be considered as the quotient space of hyperbolic metrics on Σ by the group $\mathrm{Diff}_0^+(\Sigma)$ of orientation-preserving diffeomorphisms of Σ isotopic to the identity id_Σ. Equipped with the Fenchel-Nielsen twist-length coordinates, Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$. Using the theory of L. Ahlfors and L. Bers, it can be endowed with a complex structure which makes it a complex manifold of complex dimension $3g-3$. Moreover, T_g carries several metrics, such as the Teichmüller metric induced by defining the distance between two conformal structures on Σ, the Weil-Petersson metric defined with the use of L^2-norm on the space of differentials at each marked surface of $\mathrm{T}_g(\Sigma)$, or the Thurston asymmetric metric which has convenient geometric properties although it does not satisfy the symmetry axiom. The Weil-Petersson metric also induces a symplectic structure on $\mathrm{T}_g(\Sigma)$, where the symplectic form can be defined using the Fenchel-Nielsen coordinates.

Teichmüller space can be embedded into function spaces in various ways. The closures of images of such embeddings give rise to the boundary structure of $\mathrm{T}_g(\Sigma)$ which turns out to be highly nontrivial even though Teichmüller space itself is topologically simply a ball of dimension $6g-6$. If the closure is also compact, the boundary structure determines a compactification of Teichmüller space. The best known is the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$ realized via an embedding of $\mathrm{T}_g(\Sigma)$ into the real projective space $\mathbb{P}^{\mathcal{S}}$, where $\mathcal{S}$ is a subset of $\pi_1(\Sigma)$ containing an element of each conjugate class represented by a simple closed curve on Σ. The Thurston compactification also has a nice geometric structure since it is topologically a closed disc of dimension $6g-6$. The Bers compactification is obtained by embedding $\mathrm{T}_g(\Sigma)$ into the Banach space of differentials on a given marked surface of $\mathrm{T}_g(\Sigma)$. Furthermore, every complete metric d on Teichmüller space determines a boundary defined as the set of equivalence classes of geodesic rays originating from a marked surface of $\mathrm{T}_g(\Sigma)$, where two rays $\gamma_1, \gamma_2 : [0,\infty) \to \mathrm{T}_g(\Sigma)$ are equivalent if the set $\{d(\gamma_1(t), \gamma_2(t)) \mid t \in [0,\infty)\}$ is bounded. This construction induces, for instance, the Teichmüller boundary and also the Weil-Petersson boundary if one takes the completion of the Weil-Petersson metric. However, all of the mentioned compactifications, except for the Thurston compactification, have a common disadvantage. The action of the mapping class group of the surface Σ, defined as the quotient $\mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0^+(\Sigma)$ of the group $\mathrm{Diff}^+(\Sigma)$ of orientation-preserving diffeomorphisms of Σ, on Teichmüller space does not extend continuously to the corresponding boundary. In this thesis, we present a new compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ of Teichmüller space introduced by G. W. Brumfiel [3], called the real spectrum compactification, for which the action of the mapping class group does extend continuously to its boundary. In fact, there exists a continuous surjection taking $\widetilde{\mathrm{T}}_g(\Sigma)$ onto the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$, which is in addition equivariant with respect to the mapping class group. Moreover, each element of the mapping class group can be shown to have a fixed point on $\widetilde{\mathrm{T}}_g(\Sigma)$, which implies the same conclusion for $\hat{\mathrm{T}}_g(\Sigma)$ without proving first that $\hat{\mathrm{T}}_g(\Sigma)$ is topologically a closed disc and applying the Brouwer fixed point theorem.

The real spectrum compactification is in fact defined for a closed semi-algebraic set W in the following way. Let W be a subset of a real algebraic set X. Then the real spectrum $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(X))$ of the affine coordinate ring $\mathcal{A}(X)$ of the set X is a topological space consisting of prime cones of $\mathcal{A}(X)$. It has a compact Hausdorff subspace $\mathrm{Spec}_{\mathrm{R}}^{\mathrm{m}}(\mathcal{A}(X))$, comprised of closed points of $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(X))$, which in addition contains X and hence W. Then the real spectrum compactification $\widetilde{W}$ of W is defined as the closure of W in $\mathrm{Spec}_{\mathrm{R}}^{\mathrm{m}}(\mathcal{A}(X))$. In order to apply this theory to Teichmüller space $\mathrm{T}_g(\Sigma)$, one needs to identify it with a closed semi-algebraic set. This is carried out using representations of the first fundamental group $\pi_1(\Sigma)$ in the group $\mathrm{PSL}(2,\mathbb{R})$, obtained in the following way. By the Uniformization theorem for Riemann surfaces, the surface Σ can be written as a quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ by a discrete torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}$. Since Γ can be identified with the group of covering transformations $\mathrm{Deck}_\pi(\mathbb{H})$ corresponding to the covering projection $\pi : \mathbb{H} \to \Sigma$, one obtains a representation $\rho : \pi_1(\Sigma) \longrightarrow \mathrm{Deck}_\pi(\mathbb{H}) \subset \mathrm{PSL}(2,\mathbb{R})$ of the first fundamental group $\pi_1(\Sigma)$ of Σ in the group $\mathrm{PSL}(2,\mathbb{R})$. In turns out that a conjugacy class of a discrete faithful representation of $\pi_1(\Sigma)$ in $\mathrm{PSL}(2,\mathbb{R})$ determines a point in the Teichmüller space $\mathrm{T}_g(\Sigma)$. Using the trace functions, this algebraic model of Teichmüller space can then be embedded into the real space $\mathbb{R}^M$ as a closed semi-algebraic set.

The thesis is organized as follows. In Section 3 we give a brief overview of the hyperbolic geometry necessary for further studies. We investigate isometries of the hyperbolic plane $\mathbb{H}$ and discuss hyperbolic surfaces. We also give a geometric model of Teichmüller space using the mentioned Fenchel-Nielsen coordinates. Section 4 is devoted to the real spectrum of a commutative ring A, where we also define the real spectrum compactification of a semi-algebraic set. In Section 5 we identify Teichmüller space with a closed semi-algebraic set and apply the theory of Section 4 to obtain its real spectrum compactification. Finally, in Section 6, we compare it with the Thurston compactification of Teichmüller space.

2. Preliminaries

2.1. **Real closed fields.** In this section we recall the most important properties of real closed fields. For the proofs we refer the reader to [2, Section 1].

Definition 2.1. Let F be a field endowed with a total order relation $\leq$. If the relation additionally satisfies

(i) $x \leq y \Rightarrow x + z \leq y + z$,
(ii) $0 \leq x,\ 0 \leq y \Rightarrow 0 \leq xy$,

for $x, y, z \in F$, we say it defines an *ordering* of the field F.

A field F equipped with an ordering $\leq$ is called an *ordered* field $(F, \leq)$.

Definition 2.2. A subset P of a field F satisfying

(i) $x \in P, y \in P \Rightarrow x + y \in P$,
(ii) $x \in P, y \in P \Rightarrow xy \in P$,
(iii) $x \in F \Rightarrow x^2 \in P$,

is said to be a *cone* of the field F. If, in addition,

(iv) $-1 \notin P$,

then P is called a *proper cone* of F.

Definition 2.3. A *positive cone* P of an ordered field $(F, \leq)$ is a cone of F consisting of all positive elements of F,

$$P = \{x \in F \mid x \geq 0\}.$$

For a subset P of a field F we denote by $-P = \{x \in F \mid -x \in P\}$, and by $\sum F^2 = \{\sum_{i=1}^n x_i^2 \mid x_i \in F\}$. With this notation, we obtain the following result.

Proposition 2.4. *The following holds for a field F.*

(i) If F is ordered, then the positive cone P of F is a proper cone for which $P \cup -P = F$.

(ii) If the equality $P \cup -P = F$ holds for a proper cone P of F, then the total order relation of F, given by $x \leq y \iff y - x \in P$, defines an ordering of F.

Theorem 2.5. *The following are equivalent for a field F.*

(i) There exists an ordering of F.

(ii) There exists a proper cone of F.

(iii) $-1 \notin \sum F^2$.

(iv) If for a collection of elements $x_1, \ldots, x_n \in F$ the sum $\sum_{i=1}^n x_i^2 = 0$, then $x_1 = \cdots x_n = 0$.

Definition 2.6. We say a field F is *real*, if it satisfies any of the equivalent properties of Theorem 2.5.

Let us recall that a field extension F_1 of a field F is called *algebraic* if all elements of F_1 are algebraic over F, i.e., each element of F_1 is a root of a polynomial with coefficients in F. We now define a real closed field and a real closure of a field.

Definition 2.7. A real field F with no nontrivial real algebraic extension is said to be a *real closed field*, i.e., there is no real field F_1, $F_1 \neq F$, such that $F_1 \supset F$ extends F algebraically and the inclusion $F \hookrightarrow F_1$ preserves the ordering of F.

Definition 2.8. A *real closure* of an ordered field $(F, \leq)$ is an algebraic extension R of F such that R is a real closed field whose ordering extends the ordering of F, i.e., the map $F \hookrightarrow R$ preserves the ordering.

In fact, every ordered field has a real closure which is unique in the following sense.

Theorem 2.9. *Every ordered field $(F, \leq)$ has a real closure which is unique up to a unique isomorphism of fields identical on F, i.e., if R and R' are two real closures of F, then there exists a unique isomorphism $\Phi : R \longrightarrow R'$ such that $\Phi|_F$ is the identity* id_F.

Definition 2.10. A *real* ideal I of a commutative ring A is an ideal such that for every collection $a_1, \ldots, a_p$ of elements of A

$$a_1^2 + \cdots + a_p^2 \in I \Rightarrow a_i \in I \text{ for all } i = 1, \ldots, p.$$

An immediate corollary of Theorem 2.5 is the following lemma.

Lemma 2.11. *A prime ideal I of a commutative ring A is real if and only if the field of fractions of A/I is real.*

2.2. **Semi-algebraic sets.** Throughout this section let R denote a real closed field and $R[x_1, \ldots, x_n]$ the ring of polynomials in variables $x_1, \ldots, x_n$ over R.

Definition 2.12. An *algebraic subset* X of R^n is a set of the form

$$\mathcal{Z}(B) = \{ \ x \in R^n \mid \forall f \in B \ f(x) = 0\}$$

for some subset $B \subset R[x_1, \ldots, x_n]$.

Let X be a subset of R^n. We denote by $\mathcal{I}(X)$ the ideal of $R[x_1, \ldots, x_n]$ consisting of polynomials that vanish on X:

$$\mathcal{I}(X) = \{f \in R[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall x \in X\}.$$

The set of equations determining an algebraic subset $X \subset R^n$ can be reduced to a single equation.

Proposition 2.13. *Every algebraic subset $X \subset R^n$ can be written as $X = \mathcal{Z}(f)$ for some polynomial $f \in R[x_1, \ldots, x_n]$.*

Proof. Let B be a subset of $R[x_1, \ldots, x_n]$ such that $X = \mathcal{Z}(B)$. Recall that the ring $R[x_1, \ldots, x_n]$ is Noetherian, and hence every ideal in $R[x_1, \ldots, x_n]$ is finitely generated. Let the polynomials $f_1, \ldots, f_m$ generate the ideal $\mathcal{I}(X)$. Define $f = f_1^2 + \cdots + f_m^2$. Then clearly $\mathcal{Z}(B) = X = \mathcal{Z}(f)$. $\square$

In the later sections we will be dealing with so-called semi-algebraic subsets of R^n. These are simply subsets of algebraic sets, which can be described by finitely many polynomial equalities and inequalities.

Definition 2.14. Let $\{f_i\}_{i=1,\ldots,l}$ and $\{g_j\}_{j=1,\ldots,m}$ be finite collections of polynomials in $R[x_1, \ldots, x_n]$. A *semi-algebraic subset* of R^n is a finite union of sets of the form

$$\{x \in R^n \mid f_1(x) > 0, \ldots f_l(x) > 0, g_1(x) = 0, \ldots, g_m(x) = 0\}.$$

To avoid introducing too much notation, we will shortly write that a semi-algebraic subset of R^n is a finite union of sets of the form $\{f_i > 0, g_j = 0\}$, where $\{f_i\}$ and $\{g_j\}$ are finite collections of polynomials as in Definition 2.14. Note that a finite union and a finite intersection of semi-algebraic sets is again a semi-algebraic set. The same conclusion holds for complements of semi-algebraic sets since

$$\begin{aligned}\{f_i > 0, g_j = 0\}^{\mathsf{c}} &= (\{f_i > 0\} \cap \{g_j = 0\})^{\mathsf{c}} \\ &= (\{-f_i > 0\} \cup \{f_i = 0\}) \cup (\{g_j > 0\} \cup \{-g_j > 0\}),\end{aligned}$$

which is a semi-algebraic set by definition.

3. Hyperbolic geometry

This section investigates hyperbolic manifolds and gives a geometric definition of Teichmüller space of a closed orientable surface of genus $g \geq 2$. We begin by studying isometries of the upper-half plane model $\mathbb{H}$ of the hyperbolic plane, and show that the group of orientation-preserving isometries of $\mathbb{H}$ can be identified with the projective special linear group of 2×2 matrices denoted by $\mathrm{PSL}(2, \mathbb{R})$. This result is particularly useful when studying orientable hyperbolic surfaces as it turns out that the covering transformations of the universal cover of a hyperbolic surface can be identified with a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Such an identification yields a homomorphism of the fundamental group of the hyperbolic surface into the group $\mathrm{PSL}(2, \mathbb{R})$. In later sections we will see that these homomorphisms give rise to an algebraic model of

Teichmüller space used to study its real spectrum compactification. However, this section focuses only on geometric model of Teichmüller space. We shall define it in two equivalent ways and show that it is homeomorphic to $\mathbb{R}^{6g-6}$, where $g \geq 2$ denotes the genus of the corresponding surface.

3.1. **Hyperbolic plane.** There exists several different models of the hyperbolic plane such as the Poincaré disc model, the upper half-plane model, or the hyperboloid model. However, we will only be dealing with the upper half-plane model $\mathbb{H}$ defined in this section. We will show that it can be equipped with a metric of constant Gaussian curvature -1, also called a hyperbolic metric, which gives rise to hyperbolic surfaces discussed in Section 3.2. Afterwards, we will investigate the orientation-preserving isometries of $\mathbb{H}$, [10], and we will show that they can be divided into elliptic, parabolic, and hyperbolic isometries. For the purposes of our further study, we will particularly focus on the hyperbolic case.

Definition 3.1. The open set

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

is called the *Poincaré upper half-plane model* of the hyperbolic plane.

We shall see that geodesics with respect to the metric ds^2 are exactly portions of straight lines perpendicular to the real axis $\mathbb{R}$ that are contained in $\mathbb{H}$, and Euclidean half-circles centred at $\mathbb{R}$. As in Euclidean geometry, any two distinct points can be joined by a unique geodesic segment. However, the geometry in the hyperbolic plane is not Euclidean since the fifth parallel postulate does not hold in $\mathbb{H}$. Namely, given a straight line L in $\mathbb{H}$ and a point z not on it, there is more than one geodesic passing through z and not intersecting L (see Figure 1).

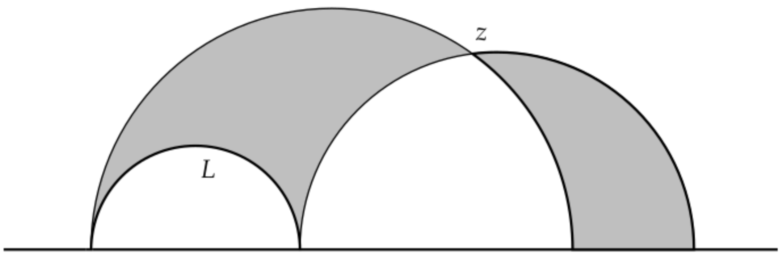


FIGURE 1. Two geodesics in $\mathbb{H}$ passing through the point z and not intersecting the geodesic L.

Let us calculate the Gaussian curvature K of the metric ds^2. Recall that Gaussian curvature K of a surface Σ in $\mathbb{R}^3$ can be expressed as the ratio of determinants

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}},$$

where II and I denote the second and the first fundamental form of Σ, respectively. Let $r(x, y)$ be a parametrization of $\mathbb{H}$. Then the first fundamental form I is given by

$$\begin{aligned} ds^2 &= Edx^2 + Fdxdy + Gdy^2 \\ &= r_{xx}dx^2 + r_{xy}dxdy + r_{yy}dy^2, \end{aligned}$$

which is in our case equal to

$$ds^2 = \frac{1}{y^2}dx^2 + \frac{1}{y^2}dy^2.$$

Since $F = r_{xy} = 0$, Gaussian curvature K of $\mathbb{H}$ can be expressed by

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial y}\frac{E_y}{\sqrt{EG}} + \frac{\partial}{\partial x}\frac{G_x}{\sqrt{EG}}\right).$$

For $\frac{1}{\sqrt{EG}} = y^2$ we compute

$$\begin{aligned} K &= -\frac{y^2}{2}\left(\frac{\partial}{\partial y}\frac{-2}{y} + 0\right) \\ &= -\frac{y^2}{2}\left(\frac{2}{y^2} + 0\right) \\ &= -1. \end{aligned}$$

As we will see in later sections, the metric ds^2 is in fact a Riemannian metric of constant Gaussian curvature -1, also called a *hyperbolic metric*.

We define the length of a differentiable curve $\gamma : [0,1] \to \mathbb{H}$, $\gamma(t) = x(t) + iy(t)$, by

$$\begin{aligned} L(\gamma) &:= \int_\gamma ds \\ &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}dt. \end{aligned}$$

Thus, the hyperbolic speed of the curve γ is the ratio between the Euclidean speed and its height above the real axis $\mathbb{R}$. A particle on the curve γ approaching $\mathbb{R}$ is therefore travelling faster in the hyperbolic metric than in the Euclidean metric.

Definition 3.2. Let $z, w \in \mathbb{H}$. The *hyperbolic distance* $\rho : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$ between z and w is given by

$$\rho(z, w) := \inf_{\gamma \in \Gamma} L(\gamma),$$

where Γ is the set of all differentiable curves γ connecting z and w.

It is not difficult to show that the map ρ is a distance map, i.e., it is nonnegative, symmetric and it satisfies the triangle inequality.

Although there are many interesting topics related to hyperbolic geometry, we are mainly interested in isometries of the hyperbolic plane $\mathbb{H}$. We will first classify them depending on whether or not they preserve the orientation. Afterwards, we will further classify the orientation-preserving isometries since they are important when studying universal covers of orientable hyperbolic surfaces, which we shall investigate in Section 3.2.

The group

$$\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1 \right\}$$

of 2×2 real matrices with determinant 1 acts on $\mathbb{H}$ by Möbius transformations in the following way. For every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ we obtain a map

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ z &\longmapsto \varphi_A(z) = \frac{az+b}{cz+d}. \end{aligned} \tag{1}$$

Maps of the form (1) are called *real Möbius transformations*. It is important to keep in mind that their coefficients a,b,c,d are real and satisfy $ad-bc=1$.

Proposition 3.3. $\varphi_A(\mathbb{H}) = \mathbb{H}$ *for any real Möbius transformation* φ_A.

Proof. We calculate the imaginary part of the image $w = \varphi_A(z)$.

$$w = \varphi_A(z) = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2},$$

$$\mathrm{Im}(w) = \frac{w-\bar{w}}{2i} = \frac{(ad-bc)(z-\bar{z})}{2i|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}. \tag{2}$$

Since $\mathrm{Im}(z) > 0$, we obtain $\mathrm{Im}(w) > 0$ and the proposition is proved. □

Let $\mathrm{Aut}(\mathbb{H})$ be the group of biholomorphic automorphisms of the hyperbolic plane $\mathbb{H}$. It is not difficult to see that it consists of real Möbius transformations (1). This follows readily from the well-known fact in complex analysis that the elements of the group $\mathrm{Aut}(\mathbb{D})$ of biholomorphic automorphisms of the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ are exactly transformations of the form $z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}}$ for which $a,b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. More precisely, conjugating any element $\gamma \in \mathrm{Aut}(\mathbb{D})$ with the biholomorphic map $T(z) = \frac{z-i}{-iz+1}$ of $\mathbb{H}$ to $\mathbb{D}$, we obtain a biholomorphism $T^{-1} \circ \gamma \circ T \in \mathrm{Aut}(\mathbb{H})$. Hence, $\mathrm{Aut}(\mathbb{H}) = T^{-1}\,\mathrm{Aut}(\mathbb{D})T$. Then a matrix calculation shows that

$$\frac{1}{4}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

and every biholomorphism of $\mathbb{H}$ is a real Möbius transformation.

Hence, we have obtained an epimorphism $\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{H})$ whose kernel equals $\{\pm \mathrm{I}\}$, where I is the 2×2 identity matrix. By the First Isomorphism Theorem,

$$\mathrm{Aut}(\mathbb{H}) \cong \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{I}\}.$$

The group $\mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{I}\}$ is called the *real projective special linear group* of 2×2 matrices, and it is denoted by $\mathrm{PSL}(2,\mathbb{R})$. We will see that it corresponds to orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$.

Definition 3.4. A bijective map $F : \mathbb{H} \longrightarrow \mathbb{H}$ is an *isometry* if it preserves the hyperbolic distances

$$\rho(z,w) = \rho(F(z), F(w)),$$

for all $z, w \in \mathbb{H}$. Isometries of $\mathbb{H}$ form a group under composition, denoted by $\mathrm{Isom}(\mathbb{H})$.

Theorem 3.5. *A real Möbius transformations* $\varphi_A : \mathbb{H} \longrightarrow \mathbb{H}$ *is an isometry. In particular,* $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$.

Proof. Let $\varphi = \varphi_A$ be a Möbius transformation corresponding to a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\gamma : [0,1] \longrightarrow \mathbb{H}$ be a differentiable curve given by $z(t) = (x(t), y(t))$. We shall prove that $L(\gamma) = L(\varphi(\gamma))$. Let us write

$$w = \varphi(z) = \frac{az+b}{cz+d},$$

and

$$w(t) = \varphi(z(t)) = u(t) + iv(t).$$

We calculate

$$\frac{dw}{dz}(t) = \frac{a(c \cdot z(t) + d) - (a \cdot z(t) + b)c}{(c \cdot z(t) + d)^2} = \frac{ad - bc}{(c \cdot z(t) + d)^2} = \frac{1}{(c \cdot z(t) + d)^2}.$$

Then equation (2) implies that

$$y(t) = \mathrm{Im}(z(t)) = \mathrm{Im}(w(t)) \cdot |c \cdot z(t) + d|^2 = v(t) \cdot |c \cdot z(t) + d|^2$$

and hence,

$$\left|\frac{dw}{dz}(t)\right| = \frac{\mathrm{Im}(w(t))}{\mathrm{Im}(z(t))} = \frac{v(t)}{y(t)}.$$

Therefore, we obtain

$$L(\varphi(\gamma)) = \int_0^1 \frac{\left|\frac{dw}{dt}(t)\right|}{v(t)} dt = \int_0^1 \frac{\left|\frac{dw}{dz}\frac{dz}{dt}(t)\right|}{v(t)} dt = \int_0^1 \frac{\left|\frac{dz}{dt}(t)\right|}{y(t)} dt = L(\gamma),$$

which proves the theorem. □

In order to classify the isometries of $\mathbb{H}$, we now study its geodesics. By estimating the hyperbolic length of an arbitrary path passing through two distinct points on a positive imaginary axis, we see that the straight rays orthogonal to $\mathbb{R}$ are geodesic. For an arbitrary pair of points $z, w \in \mathbb{H}$, we denote by L the unique Euclidean circle or straight line orthogonal to $\mathbb{R}$ joining those points. Assume L intersects $\mathbb{R}$ at some finite point α. Then the Möbius transformation of the form $z \longrightarrow \frac{\beta z - (1+\alpha\beta)}{z-\alpha}$ maps L to the positive imaginary axis for an appropriate value of β. Since such a transformation is an isometry and the straight lines are geodesics, we conclude that the segment of L joining z and w is the unique geodesic segment passing through those points. For a detailed proof we refer the reader to [11, Theorem 1.2.1].

The following theorem provides a classification of the isometries of the hyperbolic plane according to their orientation, [11, Theorem 1.3.1].

Theorem 3.6. *The group of isometries* $\mathrm{Isom}(\mathbb{H})$ *of the hyperbolic plane* $\mathbb{H}$ *is generated by real Möbius transformations and the transformation* $z \longmapsto -\bar{z}$.

Proof. Let us first observe that an isometry φ of $\mathbb{H}$ maps geodesics to geodesics. Let z, w be two distinct points in $\mathbb{H}$ and let $[z, w]$ denote the unique geodesic segment connecting them. Then

$$\rho(z, w) = \rho(z, \xi) + \rho(\xi, w) \text{ if and only if } \xi \in [z, w].$$

Since φ is an isometry, we have that $\varphi(\xi) \in [\varphi(z), \varphi(w)]$, which means that the segment $[z, w]$ is mapped to the segment $[\varphi(z), \varphi(w)]$. Hence, we can conclude that φ maps geodesics to geodesics.

If now I denotes the positive imaginary axis, then, by the above conclusion, $\varphi(I)$ is again a geodesic. We have seen that there exists an isometry $g \in \mathrm{PSL}(2, \mathbb{R})$ which maps $\varphi(I)$ to I. Since such a Möbius transformation g is uniquely determined by the values of three points, we may assume that $g(\varphi(0)) = 0, g(\varphi(i)) = i$, and $g(\varphi(\infty)) = \infty$. Therefore $g\varphi$ is an isometry that fixes all points of I. In the sequel, we shall use the following formula, [11, Theorem 1.2.6]

$$\sinh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|}{2(\operatorname{Im}(z)\operatorname{Im}(w))^{\frac{1}{2}}}, \tag{3}$$

which holds for every pair $w, z \in \mathbb{H}$. Let us write

$$\begin{aligned} z &= x + iy \in \mathbb{H} \\ g\varphi(z) &= u + iv \in \mathbb{H}. \end{aligned}$$

Then

$$\rho(z, it) = \rho(g\varphi(z), g\varphi(it)) = \rho(u + iv, it)$$

holds for all $t > 0$. Applying (3), we obtain that

$$\begin{aligned} \sinh(\frac{1}{2}\rho(z,it)) &= \sinh(\frac{1}{2}\rho(u+iv,it)) \iff \\ \frac{|x+i(y-t)|}{2(yt)^{\frac{1}{2}}} &= \frac{|u+i(v-t)|}{2(vt)^{\frac{1}{2}}} \iff \\ (x^2+(y-t)^2)v &= (u^2+(v-t)^2)y. \end{aligned}$$

If we divide both sides by t^2 and take the limit as $t \to \infty$, we obtain $v = y$, and hence $x^2 = u^2$. Therefore, the isometry $g\varphi$ is of the form

$$g\varphi(x + iy) = \pm x + iy,$$

or equivalently,

$$g\varphi(z) = z \text{ or } -\bar{z}.$$

If $g\varphi = \mathrm{id}$, then $\varphi = g^{-1}$ and hence a real Möbius transformation (1). If $g\varphi(z) = -\bar{z} = \frac{-|z|^2}{z}$, then a short computation shows that

$$\varphi(z) = \frac{bz + a|z|^2}{dz + b|z|^2} = \frac{a\bar{z} + b}{c\bar{z} + d},$$

where $ad - bc = -1$. This classifies the isometries of $\mathbb{H}$. □

The orientation of an isometry φ is determined by the sign of the determinant of the associated matrix. Therefore, we define the transformations of $\mathrm{PSL}(2,\mathbb{R})$ to be *orientation-preserving*, whereas the isometries of the form $g\varphi(z) = -\bar{z}$ are said to be *orientation-reversing*. The group of all orientation-preserving isometries is denoted by $\mathrm{Isom}^+(\mathbb{H})$, and it is, as we have seen, exactly the group of holomorphic automorphisms $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$.

We shall now take a closer look at the isometries given by $\mathrm{PSL}(2,\mathbb{R})$ and investigate their classification by the absolute value of the trace of the associated matrix.

Assume an isometry $\varphi_A \in \mathrm{Isom}^+(\mathbb{H})$ corresponds to a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote the absolute value of the trace of A by $\mathrm{Tr}(A) = |a + d|$.

Definition 3.7. Let φ_A be an orientation-preserving isometry of $\mathbb{H}$ and let $A \in \mathrm{PSL}(2,\mathbb{R})$ be the corresponding matrix. Then φ is said to be

(i) *elliptic*, if $\mathrm{Tr}(A) < 2$,
(ii) *parabolic*, if $\mathrm{Tr}(A) = 2$,
(iii) *hyperbolic*, if $\mathrm{Tr}(A) > 2$.

The same classification is obtained when studying the fixed points of an isometry φ_A which are exactly the solutions of the equation

$$cz^2 + (d-a)z - b = 0.$$

Observe that by the Brouwer fixed-point theorem φ_A has at least one fixed point on the closure of $\mathbb{H}$ which is the Riemann sphere $\hat{\mathbb{C}}$.

Assume φ_A is not equal to $\pm\mathrm{I}$. We first consider the case when $c = 0$ which implies that the fixed points of φ_A are ∞ and $\frac{b}{d-a}$. Since $ad - bc = ad = 1$, either $a = d$ and hence $a^2 = 1$, or $a \neq d$ and $d = a^{-1}$. In the first case, ∞ is the only fixed point and $\mathrm{Tr}(A) = 2$, whereas in the second case the fixed points are ∞, $\frac{b}{a^{-1}-a}$ and $\mathrm{Tr}(A) > 2$.

If $c \neq 0$, then roots of this equation are given by the standard formula

$$z_1, z_2 = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}.$$

Three cases can occur depending on whether the discriminant is greater, equal to or less than zero. Using the equality $ad - bc = 1$, we see that

$$(d-a)^2 + 4bc = (a+d)^2 - 4 = \mathrm{tr}(A)^2 - 4.$$

Hence, we conclude the following.

(i) φ_A fixes a point in $\mathbb{H}$ if and only if $\mathrm{Tr}(A) < 2$, i.e., φ_A is elliptic.
(ii) φ_A fixes a single point in $\mathbb{R} \cup \{\infty\}$ if and only if $\mathrm{Tr}(A) = 2$, i.e., φ_A is parabolic.
(iii) φ_A fixes two points in $\mathbb{R} \cup \{\infty\}$ if and only if $\mathrm{Tr}(A) > 2$, i.e., φ_A is hyperbolic.

In fact, we have another equivalent characterization of orientation-preserving isometries of $\mathbb{H}$.

Proposition 3.8. *The following holds for an isometry $\varphi_A \in \mathrm{Isom}^+(\mathbb{H})$.*

(i) φ_A is elliptic if and only if it is conjugate to a rotation $z \longmapsto e^{i\phi}z$.
(ii) φ_A is parabolic if and only if it is conjugate to a translation $z \longmapsto z + b$, $b \in \mathbb{R}$.
(iii) φ_A is hyperbolic if and only if it is conjugate to a dilation $z \longmapsto kz$, $k > 0$.

In later sections we will be dealing with discrete faithful representations of the fundamental group of a hyperbolic surface. Since such representations give rise to hyperbolic transformations, we shall prove this proposition only for the hyperbolic case. For the remaining two cases we refer to [19, Sections 10, 11].

Proof of Proposition 3.8. (iii) Observe first that a transformation φ_A fixing 0 and ∞ is necessarily a dilation as this implies that $c = b = 0$, and hence φ_A is of the form $z \longmapsto a^2 z$, $a > 0$ and $a \neq 1$. Conversely, if φ_A is a dilation, then 0 and ∞ are its fixed points.

Suppose now that φ_A is conjugate to a dilation. Since conjugate transformations have the same number of fixed points, then φ_A has exactly two fixed points in $\mathbb{R}$ by the above observation. Hence, it is hyperbolic.

Conversely, suppose φ_A is hyperbolic with fixed points z_1, z_2. Assume first that $z_1 = \infty$ and set $g(z) = z - z_2$. Then $g\varphi_A g^{-1}$ is clearly conjugate to φ_A. A short computation shows that the fixed points of $g\varphi_A g^{-1}$ are 0 and ∞, which means it is a dilation. Thus, φ_A is conjugate to a dilation. Let now $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$ and set $g(z) = \frac{z-z_2}{z-z_1}$. By dividing the coefficients with $\sqrt{-z_1 + z_2} > 0$, we may assume $g \in \mathrm{SL}(2, \mathbb{R})$. Since $g(z_1) = \infty$ and $g(z_2) = 0$, the fixed points of $g\varphi_A g^{-1}$ are exactly 0 and ∞. Therefore, φ_A is conjugate to a dilation. $\square$

Note that a dilation $D : z \longmapsto kz, k > 0$ corresponds to a matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, or equivalently, to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $k = \lambda^2$. The fixed points $r_D = 0$ and $a_D = \infty$ of D are called the *repelling fixed point* and the *attractive fixed point*, respectively. The positive imaginary axis I in $\mathbb{H}$ is then a geodesic joining r_D and a_D. Observe that D acts on I as a translation since

$$\rho(iy, \lambda^2 iy) = \log(\lambda^2).$$

Let $\varphi \in \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolic transformation of the form $\varphi = g \circ D \circ g^{-1}$ for an isometry $g \in \mathrm{PSL}(2, \mathbb{R})$ and a dilation $D : z \longmapsto \lambda^2 z$. Then $g(I)$ is a geodesic since g is an isometry, and it is joining the fixed points r_φ and a_φ as a semi-circle orthogonal to $\mathbb{R}$. Moreover, $g(I)$ is invariant under φ since $\varphi(g(I)) = g(D(I)) = g(I)$, and φ acts on it as a translation.

Definition 3.9. Let $\varphi \in \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolic transformation conjugated to a dilation D by an isometry $g \in \mathrm{PSL}(2, \mathbb{R})$. The geodesic $g(I)$ in $\mathbb{H}$, where I denotes the positive imaginary axis, joining the two fixed points r_φ and a_φ of φ is called the *axis* of φ.

3.2. **Hyperbolic surfaces.** This section is devoted to hyperbolic surfaces defined as topological surfaces endowed with hyperbolic atlases. These are in fact Riemann surfaces equipped with a hyperbolic metric, that is, a Riemannian metric of constant negative Gaussian curvature. It follows from the Uniformization theorem that the universal cover of a hyperbolic surface Σ is biholomorphic to the hyperbolic plane $\mathbb{H}$. If $\pi : \mathbb{H} \longrightarrow \Sigma$ denotes the universal cover of Σ, then the group $\mathrm{Deck}_\pi(\mathbb{H})$ of covering transformations can be identified with a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H})$. In this section, we provide a detailed investigation of these facts. We will also show that in this case all covering transformations are hyperbolic, which in particular implies that every free-homotopy class of a simple closed hyperbolic curve on Σ contains a unique simple closed geodesic. This result is important in later sections when studying the Thurston compactification of Teichmüller space.

By a surface we will always mean a smooth closed connected orientable 2-manifold.

Definition 3.10. Let Σ be a surface. A *hyperbolic* atlas on Σ is an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ satisfying the following properties.

(i) $\varphi_\alpha(U_\alpha) \subset \mathbb{H}$ for all $\alpha \in A$.
(ii) For every two charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{A}$ for which $U_\alpha \cap U_\beta \neq \emptyset$ there exists an orientation-preserving isometry $m \in \mathrm{Isom}^+(\mathbb{H})$ such that the *transition map* $\varphi_\beta \circ \varphi_\alpha^{-1}$ coincides with m on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Let us show that every hyperbolic atlas $\mathcal{A}$ on a surface Σ induces a Riemannian metric on Σ defined below.

Definition 3.11. Let M be a smooth connected surface. A *Riemannian metric* on M is a family of inner products $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ on the tangent space T_pM at each point $p \in M$ such that the map $p \longmapsto g_p(X, Y)$ is smooth for every two tangent vectors $X, Y \in T_pM$.

Definition 3.12. Let M be a smooth connected surface. A *hyperbolic metric* on Σ is a Riemannian metric g with negative Gaussian curvature.

We first observe that the metric ds^2 on $\mathbb{H}$, defined in Section 3.1, is a hyperbolic metric on $\mathbb{H}$ induced by the following inner product on the tangent space $T_z\mathbb{H}$ at a point $z \in \mathbb{H}$. Given two tangent vectors $\xi_1, \xi_2 \in T_z\mathbb{H} \approx \mathbb{R}^2$, $\xi_i = a_i + ib_i$, we define

$$g_z(\xi_1, \xi_2) = \frac{\langle \xi_1, \xi_2 \rangle_{\mathbb{R}^2}}{\mathrm{Im}(z)^2},$$

where $\langle \xi_1, \xi_2 \rangle_{\mathbb{R}^2} = a_1a_2 + b_1b_2$ denotes the usual inner product on $\mathbb{R}^2$.

Let (U, φ) be a coordinate chart of $\mathcal{A}$ at a point $p \in \Sigma$, and let $X, Y \in T_p(\Sigma)$ be two tangent vectors at p. If we adopt the definition of tangent vectors via equivalence classes of differentiable curves on surfaces, we can write $X = [\alpha], Y = [\beta]$ for two differentiable curves $\alpha, \beta : (-\epsilon, \epsilon) \to \Sigma$ such that $\alpha(0) = p = \beta(0)$. Let g be the above defined Riemannian metric on $\mathbb{H}$. Then a Riemannian metric h on Σ is given by

$$h_p(X, Y) = g_{\varphi(p)}((\varphi \circ \alpha)'(0), (\varphi \circ \beta)'(0)).$$

It is independent of the choice of a coordinate chart because the transition maps of the hyperbolic atlas $\mathcal{A}$ are isometries by definition. Since the Gaussian curvature of g is -1, we conclude that h is a hyperbolic metric on Σ induced by $\mathcal{A}$.

Conversely, since a hyperbolic metric on Σ is a Riemannian metric with negative Gaussian curvature, we conclude that every such metric is obtained from a hyperbolic atlas on Σ.

Definition 3.13. A *hyperbolic structure* on a surface Σ is a maximal hyperbolic atlas on Σ. Such a structure is said to be *complete* if the induced metric on the surface Σ is complete. A *hyperbolic surface* Σ is a surface endowed with a complete hyperbolic structure.

Let Σ be a hyperbolic surface. If $\mathcal{A}$ denotes a hyperbolic atlas on Σ, it can be considered as a complex atlas whose charts are complex maps and the transition maps are holomorphic. Hence Σ is a complex surface of complex dimension 1, called a Riemann surface, which is additionally endowed with a hyperbolic metric. Let $\widetilde{\Sigma}$ be the universal cover of Σ and $\pi : \widetilde{\Sigma} \to \Sigma$ the corresponding covering projection. The covering space $\widetilde{\Sigma}$ can be endowed with a hyperbolic structure by pulling back the hyperbolic structure on Σ along the map π. Since $\widetilde{\Sigma}$ is a simply connected Riemann surface, it is conformally equivalent to either the upper half-plane $\mathbb{H}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\hat{\mathbb{C}}$ by Uniformization theorem. As $\mathbb{C}$ and $\hat{\mathbb{C}}$ cannot be endowed with a hyperbolic metric, we conclude that $\widetilde{\Sigma}$ is conformally equivalent to $\mathbb{H}$. We shall therefore assume $\mathbb{H}$ is the universal cover of Σ and write $\pi : \mathbb{H} \longrightarrow \Sigma$ for the associated covering projection.

We will now study the group of covering transformations

$$\mathrm{Deck}_\pi(\mathbb{H}) := \{T_\gamma \in \mathrm{Homeo}(\mathbb{H}) \mid \pi \circ T_\gamma = \pi\}.$$

Since $\pi : \mathbb{H} \longrightarrow \Sigma$ is the universal cover of Σ, it is isomorphic to its fundamental group $\pi_1(\Sigma, p_0)$, where $p_0 \in \Sigma$ is a fixed base point. Let us recall how do we obtain this identification, [17]. Let $\widetilde{p}_0 \in \pi^{-1}(p_0) \subset \mathbb{H}$ be a lift of the basepoint p_0, and let $\widetilde{\alpha}$ be a lift of $[\alpha] \in \pi_1(\Sigma, p_0)$ with the starting point $\widetilde{p}_0$. We define an action of $\pi_1(\Sigma, p_0)$ on $\pi^{-1}(p_0)$ by

$$\begin{aligned} \pi^{-1}(p_0) \times \pi_1(\Sigma, p_0) &\longrightarrow \pi^{-1}(p_0) \\ (\widetilde{p_0}, \alpha) &\longmapsto \widetilde{\alpha}(1). \end{aligned}$$

Since $\mathbb{H}$ is simply connected this action is transitive, and hence for any two lifts $\widetilde{p_0}, \widetilde{p_1} \in \pi^{-1}(p_0)$ there exists and element $[\alpha] \in \pi_1(\Sigma, p_0)$ such that $\widetilde{p_0} \cdot [\alpha] = \widetilde{p_1}$. Therefore, we obtain a short exact sequence

$$0 \longrightarrow \pi_1(\mathbb{H}, \widetilde{p_0}) \xrightarrow{\pi_\#} \pi_1(\Sigma, p_0) \longrightarrow \mathrm{Deck}_\pi(\mathbb{H}) \longrightarrow 0$$

Since $\pi_1(\mathbb{H}, \widetilde{p_0}) = \{1\}$, we obtain an isomorphism $\pi_1(\Sigma, p_0) \cong \mathrm{Deck}_\pi(\mathbb{H})$ and therefore a homomorphism $\pi_1(\Sigma, p_0) \longrightarrow \mathrm{Deck}_\pi(\mathbb{H}) \subset \mathrm{PSL}(2, \mathbb{R})$.

The group $\mathrm{Deck}_\pi(\mathbb{H})$ acts on the universal cover $\mathbb{H}$ in the obvious way, and the action satisfies the following property, called the *covering space property*: for every $z \in \mathbb{H}$ there exists a neighbourhood U such that $U \cap T(U) \neq \emptyset$ only if $T = \mathrm{id}_{\mathbb{H}}$. Such an action will be called a *covering space action*, [8, Page 72]. Let $\widetilde{p} \in \mathbb{H}$ and $\pi(\widetilde{p}) = p \in \Sigma$. The orbit of the point $\widetilde{p}$ by this action

$$\begin{aligned} \mathrm{Deck}_\pi(\mathbb{H})\widetilde{p} &:= \{T_\gamma(\widetilde{p}) \in \mathbb{H} \mid T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})\} \\ &= \pi^{-1}(p) \end{aligned}$$

is exactly the fiber of the point $p \in \Sigma$. Therefore, the orbit space equals

$$\mathbb{H}/\,\mathrm{Deck}_\pi(\mathbb{H}) = \Sigma.$$

We will prove that the group of covering transformation can be identified with a discrete subgroup of the group of orientation-preserving isometries of $\mathbb{H}$. Such subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are called *Fuchsian groups*.

Definition 3.14. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$, or equivalently, a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H})$.

Let us first prove that covering transformations are isometries. Clearly, if g is a Riemannian metric on Σ then the pullback metric $\pi^*(g)$ via the covering map π gives a Riemannian metric on $\mathbb{H}$ for which π is a local isometry. For $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$ we have that $T_\gamma^*(\pi^*(g)) = (\pi \circ T_\gamma)^*(g) = \pi^*(g)$, and therefore T_γ is a local isometry. Since it is also a homeomorphism, we conclude that T_γ is an isometry.

We have seen that the group $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Since an orientation-preserving homeomorphism of $\mathbb{H}$ which is also an isometry is a biholomorphism, the deck group $\mathrm{Deck}_\pi(\Sigma)$ can be identified with a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$.

It remains to prove that Γ is also discrete. Recall that a group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is discrete if the subspace topology of Γ in $\mathrm{PSL}(2, \mathbb{R})$ is the discrete topology. We will prove that the covering space action of $\mathrm{Deck}_\pi(\mathbb{H})$ on the hyperbolic space $\mathbb{H}$ implies the discreteness of $\mathrm{Deck}_\pi(\mathbb{H})$.

Theorem 3.15. *If a subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acts on the hyperbolic plane $\mathbb{H}$ and the action satisfies the covering space property, then Γ is discrete.*

Proof. Suppose the action of Γ on $\mathbb{H}$ satisfies the covering space property but Γ is not a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Note that the covering space action is also free, i.e., if there is an element $z \in \mathbb{H}$ such that $T(z) = z$, then $T = \mathrm{id}$. Let z be any element of $\mathbb{H}$. By the previous observation, $T(z) \neq z$ for all $T \in \Gamma \setminus \{\mathrm{id}\}$. Since Γ is not a discrete group, there exists an element $T \in \Gamma$ such that there is no open set U for which $U \cap \Gamma = \{T\}$. Let $L : \Gamma \longrightarrow \Gamma$ be the left multiplication by T^{-1}. Since L is a homeomorphism, $L(U) \cap \Gamma \neq \{\mathrm{id}\}$ for every open set $L(U)$ containing the identity element. Hence, there exists a sequence of distinct transformations T_n, $T_n \neq \mathrm{id}$, which converges to id. Then $T_n(z) \to z$ as $n \to \infty$, and $T_n(z) \neq z$ for all n since the transformations T_n have no fixed points. Let U be a neighbourhood of z. Then $U \cap T_n(U) \neq \emptyset$ for n large enough but $T_n \neq \mathrm{id}$, which contradicts the covering space property. □

This theorem completes the proof that the group of covering transformations $\mathrm{Deck}_\pi(\mathbb{H})$ of a connected oriented hyperbolic surface Σ can be identified with a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$. Note that Γ is necessarily torsion-free, i.e., it does not contain nonidentity elements of finite order, since the fundamental group $\pi_1(\Sigma)$ is torsion-free.

It is important to note that in our case all nonidentity covering transformations are hyperbolic, which we prove next with the use of the following lemma.

Lemma 3.16. *Let Σ be a closed connected oriented hyperbolic surface and let $\pi : \mathbb{H} \to \Sigma$ be its universal cover. Then there exists a positive real number $\delta > 0$ such that $\rho(z, T_\gamma z) \geq \delta$ for all $z \in \mathbb{H}$ and all transformations $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$.*

Proof. We identify first $\mathrm{Deck}_\pi(\mathbb{H})$ with a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$. Let z be an arbitrary point in $\mathbb{H}$ and set

$$r(z) = \frac{1}{2}\rho(z, \Gamma z \setminus \{z\}) = \frac{1}{2}\inf\{\rho(z, T_\gamma z) \mid T_\gamma \in \Gamma \setminus \{\mathrm{id}\}\}.$$

Let $B(z,r)$ denotes the open ball centred at z with radius r. Observe that the open unit balls

$$\{B(T_\gamma z, r(z)) \mid T_\gamma \in \Gamma\}$$

are disjoint since Γ acts without fixed points on $\mathbb{H}$. The quotient $\mathbb{H}/\Gamma = \Sigma$ is a compact space with an open cover

$$\{B(\pi(w), r(w)) \mid w \in \mathbb{H}\}.$$

By the Lebesgue's number lemma, there exists a positive real number $\delta = 2l > 0$ such that every subset of Σ having the diameter less than $2l$ is contained in a subset of the open covering. Thus, $B(\pi(z), l)$ is contained in $B(\pi(w), r(w))$ for some $w \in \mathbb{H}$, which implies that $B(z,l)$ is contained in the union $\cup_{T_\gamma \in \Gamma} B(T_\gamma w, r(w))$. In fact, there exists $T_\gamma \in \Gamma$ such that $B(z,l)$ is contained in $B(T_\gamma w, r(w))$ since $B(z,l)$ is connected. Without loss of generality, we may assume that $T_\gamma = \mathrm{id}$ as we can replace the initial w by $T_\gamma^{-1} w$. Let T_η be an arbitrary nonidentity element of Γ. It is clearly not elliptic since Γ acts without fixed points on $\mathbb{H}$. Then the unit balls $B(w, r(w))$ and $B(T_\eta w, r(w))$ are disjoint, and hence $B(z,l)$ and $B(T_\eta z, l)$ are disjoint. This implies that $\rho(z, T_\eta z) \geq 2l = \delta$ for all $z \in \mathbb{H}$ and all nonidentity transformations $T_\eta \in \Gamma$. □

Theorem 3.17. *Let Σ be a closed connected oriented hyperbolic surface and let $\pi : \mathbb{H} \to \Sigma$ be its universal cover. Then all nontrivial elements of $\mathrm{Deck}_\pi(\mathbb{H})$ are hyperbolic.*

Proof. We have already seen that a transformation $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$ cannot be elliptic since the group of covering transformations acts without fixed points on $\mathbb{H}$. Assume T_γ is parabolic given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We may assume it fixes ∞ by Proposition 3.8, which implies that the parameter c is 0. Moreover, the parameters a and d are necessarily equal, otherwise T_γ would be hyperbolic. Therefore, T_γ is of the form $T_\gamma(z) = z + \frac{b}{a}$. We will use the following equality, [11, Theorem 1.2.6],

$$\cosh(\rho(z,w)) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)},$$

which holds for every $z, w \in \mathbb{H}$. Let us consider the sequence $z_n = ni$ of elements in $\mathbb{H}$. We compute

$$\begin{aligned}\cosh(\rho(z_n, T_\gamma(z_n))) &= 1 + \frac{|ni - ni - \frac{b}{a}|^2}{2n^2} \\ &= 1 + \frac{b^2}{2a^2n^2}.\end{aligned}$$

Then the limit

$$\lim_{n\to\infty} \cosh(\rho(z_n, T_\gamma(z_n))) = 1,$$

and thus

$$\lim_{n\to\infty} \rho(z_n, T_\gamma(z_n)) = 0,$$

which is a contradiction with Lemma 3.16. Hence every transformation $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$ is hyperbolic. ☐

Using the fact that all covering transformations are hyperbolic, we close this section by showing that every free-homotopy class of a simple closed hyperbolic curve on Σ contains a unique simple closed geodesic. The hyperbolic lengths of such geodesics give rise to the Thurston compactification of Teichmüller space discussed in Section 6.2.

Definition 3.18. Let $\alpha : \mathbb{R} \longrightarrow \Sigma$ be a geodesic line on Σ, that is, a locally distance preserving function. A *period* of α is a positive real number $p > 0$ such that $\alpha(t+p) = \alpha(t)$ for all $t \in \mathbb{R}$. If a geodesic line has a period, it is called a *periodic geodesic line.*

Definition 3.19. The image of a periodic geodesic line $\alpha : \mathbb{R} \to \Sigma$ is called a *closed geodesic* of the surface Σ.

Theorem 3.20. *Let Σ be a closed connected oriented hyperbolic surface and let $\pi : \mathbb{H} \longrightarrow \Sigma$ be its universal cover. Then the following holds.*

(i) The axis of every transformation $T \in \mathrm{Deck}_\pi(\mathbb{H})$ projects onto a closed geodesic of the surface Σ under the projection π.

(ii) If C is a closed geodesic of Σ, then there exists a transformation $T \in \mathrm{Deck}_\pi(\mathbb{H})$ whose axis projects onto C under the projection π.

Proof. (i) Suppose $T \in \mathrm{Deck}_\pi(\mathbb{H})$ is a transformation whose axis L in $\mathbb{H}$ is parametrized by a geodesic line $\widetilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{H}$, that is, $\widetilde{\alpha}(\mathbb{R}) = L$. Write $T = gDg^{-1}$ for a dilation $D(z) = \lambda^2 z$ and a transformation $g \in \mathrm{PSL}(2,\mathbb{R})$, and let $\widetilde{\beta} : \mathbb{R} \longrightarrow \mathbb{H}$ be a parametrization of the imaginary axis I given by $\widetilde{\beta}(t) = ie^t$. We know that D acts on I as a translation by a distance p. We compute

$$ie^{t+p} = \widetilde{\beta}(t+p) = D(\widetilde{\beta}(t)) = i\lambda^2 e^t,$$

and hence $p = \log(\lambda^2)$. Since $g(I) = L$, the transformation T acts on L as a translation by the distance $p = \rho(\widetilde{\alpha}(0), T\widetilde{\alpha}(0)) = \log(\lambda^2)$. This implies that $\alpha = \pi\widetilde{\alpha} : \mathbb{R} \longrightarrow \Sigma$ is a periodic geodesic line with the period p since π is a local isometry. Hence, $\alpha(\mathbb{R}) = C$ is a closed geodesic on Σ. In fact, the geodesic $C = \alpha(\mathbb{R}) = \pi\widetilde{\alpha}(\mathbb{R}) = \pi(L)$ is the image of the axis L under π.

(ii) Assume $\alpha : \mathbb{R} \longrightarrow \Sigma$ is a periodic geodesic line whose image $\alpha(\mathbb{R}) = C$ is the closed geodesic C. Let p be the smallest period of α, [15, Theorem 9.6.1]. Let $\widetilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{H}$ be a lift of α, i.e., $\pi\widetilde{\alpha} = \alpha$. Since π is a local isometry, the image $\widetilde{\alpha}(\mathbb{R}) = L$ is a geodesic in $\mathbb{H}$. As $\pi\widetilde{\alpha}(p) = \pi\widetilde{\alpha}(0)$, there exists a transformation $T \in \mathrm{Deck}_\pi(\mathbb{H})$ such that $\widetilde{\alpha}(p) = T\widetilde{\alpha}(0)$. Consider the map

$$T\widetilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{H},$$

which is also a lift of α as $\pi T\widetilde{\alpha} = \pi\widetilde{\alpha} = \alpha$. It agrees with the map $\hat{\alpha} : \mathbb{R} \longrightarrow \mathbb{H}$, given by $\hat{\alpha}(t) = \widetilde{\alpha}(t+p)$, at $t = 0$ since $T\widetilde{\alpha}(0) = \widetilde{\alpha}(p)$. As $\hat{\alpha}$ also lifts α, $\hat{\alpha} = T\widetilde{\alpha}$ by the unique lifting property. Then $T(L) = T\widetilde{\alpha}(\mathbb{R}) = \hat{\alpha}(\mathbb{R}) = L$, and hence L is the axis of T. □

Let $\gamma : [0,1] \to \Sigma$ be a closed curve of Σ. Since for any lift $\widetilde{\gamma} : [0,1] \longrightarrow \mathbb{H}$ the transformation $T \in \mathrm{Deck}_\pi(\mathbb{H})$ for which $\widetilde{\gamma}(1) = T\widetilde{\gamma}(0)$ is hyperbolic, we say that the curve γ is *hyperbolic*.

Definition 3.21. Let $\alpha, \beta : [0,1]_t \longrightarrow \Sigma$ be two closed hyperbolic curves of Σ. Then α and β are said to be *freely homotopic* if there exists a homotopy $H : [0,1]_t \times [0,1]_s \longrightarrow \Sigma$ from α to β such that $H(0,s) = H(1,s)$ for all $s \in [0,1]$.

Theorem 3.22. *Let $\gamma : [0,1] \longrightarrow \Sigma$ be a closed hyperbolic curve in Σ. There exists a periodic geodesic line $\alpha : [0,1] \longrightarrow \mathbb{R}$ with a unique period p such that γ is freely homotopic to the closed curve $\alpha_p : [0,1] \longrightarrow \Sigma$ given by $\alpha_p(t) = \alpha(pt)$. Moreover, the geodesic line α is unique up to a composition with a translation in $\mathbb{R}$.*

Proof. Let $\widetilde{\gamma} : [0,1] \longrightarrow \mathbb{H}$ be a lift of γ, and let $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$ be a transformation such that $\widetilde{\gamma}(1) = T_\gamma\widetilde{\gamma}(0)$. By Theorem 3.17, T_γ is hyperbolic and hence has an axis L in $\mathbb{H}$. Assume L is parametrized by a geodesic line $\widetilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{H}$ such that the parametrization agrees with the direction in which T_γ translates L. Since T_γ acts on L as a translation, there exists $p > 0$ such that

$$T_\gamma\widetilde{\alpha}(t) = \widetilde{\alpha}(t+p).$$

Moreover, $\alpha = \pi \circ \widetilde{\alpha}$ is a geodesic line and if we apply the covering projection π to the above equality, we obtain that

$$\alpha(t) = \pi\widetilde{\alpha}(t) = \pi T_\gamma\widetilde{\alpha}(t) = \pi\widetilde{\alpha}(t+p) = \alpha(t+p).$$

Thus, p is a period for $\alpha : [0,1] \longrightarrow \Sigma$. Let $\widetilde{\alpha}_p : [0,1] \longrightarrow \mathbb{H}$ be given by $\widetilde{\alpha}_p(t) = \widetilde{\alpha}(tp)$. Then the map

$$\begin{aligned} \widetilde{H} : [0,1] \times [0,1] &\longrightarrow \mathbb{H} \\ (t,s) &\longmapsto (1-s)\widetilde{\gamma}(t) + s\widetilde{\alpha}_p(t) \end{aligned}$$

is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\alpha}_p$. Let us prove that $H = \pi\widetilde{H}$ is a homotopy from γ to α_p. We first compute that

$$\begin{aligned} T_\gamma\widetilde{H}(0,s) &= (1-s)T_\gamma\widetilde{\gamma}(0) + sT_\gamma\widetilde{\alpha}_p(0) \\ &= (1-s)\widetilde{\gamma}(1) + s\widetilde{\alpha}(p) \\ &= (1-s)\widetilde{\gamma}(1) + s\widetilde{\alpha}_p(1) \\ &= \widetilde{H}(1,s). \end{aligned}$$

Therefore, $H(0,s) = \pi\widetilde{H}(0,s) = \pi T_\gamma\widetilde{H}(0,s) = \pi\widetilde{H}(1,s) = H(1,s)$, and γ is freely homotopic to α_p. We shall not prove the uniqueness of α. For this we refer to [15, Theorem 9.6.4]. $\square$

A bit more work is required to prove a similar result for simple closed hyperbolic curves as stated in the following theorem, [15, Theorem 9.6.5].

Theorem 3.23. *Let $\gamma : [0,1] \longrightarrow \Sigma$ be a simple closed hyperbolic curve in Σ. There exists a periodic geodesic line $\alpha : [0,1] \longrightarrow \mathbb{R}$ with a unique period p such that γ is freely homotopic to the closed curve $\alpha_p : [0,1] \longrightarrow \Sigma$ given by $\alpha_p(t) = \alpha(pt)$. Moreover, the geodesic line α is unique up to a composition with a translation in $\mathbb{R}$, and the curve α_p is simple.*

Therefore, in every nontrivial free-homotopy class $[\gamma] \in \pi_1(\Sigma)$ there exists a unique simple closed geodesic $\alpha_p \in [\gamma]$. Let us compute its length. From the proof of Theorem 3.22 we have that the geodesic $\alpha_p : [0,1] \longrightarrow \Sigma$ is given by $\alpha_p(t) = \alpha(pt)$ for a geodesic line $\alpha : \mathbb{R} \longrightarrow \Sigma$ with a period p. Furthermore, $\alpha = \pi\widetilde{\alpha}$ is the image of $\widetilde{\alpha}$ under the covering projection π, where $\widetilde{\alpha}$ is the parametrization of the axis of the hyperbolic transformation $T_\gamma \in \mathrm{Deck}_\pi(\mathbb{H})$ for which $T_\gamma\widetilde{\gamma}(0) = \widetilde{\gamma}(1)$. Since T_γ is hyperbolic by Theorem 3.17, it can be written as $T_\gamma = gDg^{-1}$ for a dilation $D(z) = \lambda^2 z$ and a transformation $g \in \mathrm{PSL}(2,\mathbb{R})$. Then

$$\begin{aligned} \rho(\alpha_p(0), \alpha_p(1)) &= \rho(\alpha(0), \alpha(p)) \\ &= \rho(\pi\widetilde{\alpha}(0), \pi\widetilde{\alpha}(p)) \\ &= \rho(\widetilde{\alpha}(0), \widetilde{\alpha}(p+0)) \\ &= \rho(\widetilde{\alpha}(0), T_\gamma\widetilde{\alpha}(0)), \end{aligned}$$

since π is a local isometry. If we write $g(iy) = \widetilde{\alpha}(0)$ for some $y > 0$, then we obtain that

$$\begin{aligned} \rho(\widetilde{\alpha}(0), T_\gamma\widetilde{\alpha}(0)) &= \rho(g(iy), T_\gamma g(iy)) \\ &= \rho(g(iy), gD(iy)) \\ &= \rho(iy, D(iy)) \\ &= \log(\lambda^2), \end{aligned}$$

as g is an isometry.

3.3. **Teichmüller space.** This section is devoted to a geometric description of Teichmüller space of a closed hyperbolic surface of genus $g \geq 2$. We will define it in two equivalent ways, as a set of equivalence classes of marked hyperbolic surfaces and as a set of isotopy classes of hyperbolic metrics on a closed surface of genus $g \geq 2$. We will equip it with the Fenchel-Nielsen coordinates and use them to define a homeomorphism of Teichmüller space into $\mathbb{R}^{6g-6}$.

Let Σ be a fixed closed hyperbolic surface of genus $g \geq 2$.

Definition 3.24. Let S be a closed hyperbolic surface of genus $g \geq 2$ and $f : \Sigma \to S$ be a diffeomorphism. We call the pair (S, f) a *marked hyperbolic surface* of genus g, and $f : \Sigma \to S$ the *marking diffeomorphism.*

We define an equivalence relation $\sim$ on the set of hyperbolic surfaces marked by Σ in the following way. Two marked hyperbolic surfaces (S, f) and (S', f') are equivalent if and only if there exists an isometry $m : S \longrightarrow S'$ such that the map $m \circ f$ is isotopic to f', i.e., the following diagram commutes up to isotopy.

$$\begin{tikzcd} & \Sigma \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ S \arrow[rr, dotted, "m"] & & S' \end{tikzcd}$$

Recall that an *isotopy* between two diffeomorphisms $f_0, f_1 : \Sigma \to S$ is a continuous map $H : \Sigma \times [0,1] \to S$ such that $H(p, 0) = f_0(p)$ and $H(p, 1) = f_1(p)$ for every $p \in \Sigma$. Moreover, the map $H_t = H(\cdot, t)$ is an embedding for every $t \in [0, 1]$, i.e, a diffeomorphism such that $H_t(\Sigma)$ is a submanifold of S.

Definition 3.25. The *Teichmüller space* of the surface Σ

$$\mathrm{T}_g(\Sigma) := \{(S, f)\}/\sim$$

is the set of equivalence classes of hyperbolic surfaces marked by the surface Σ.

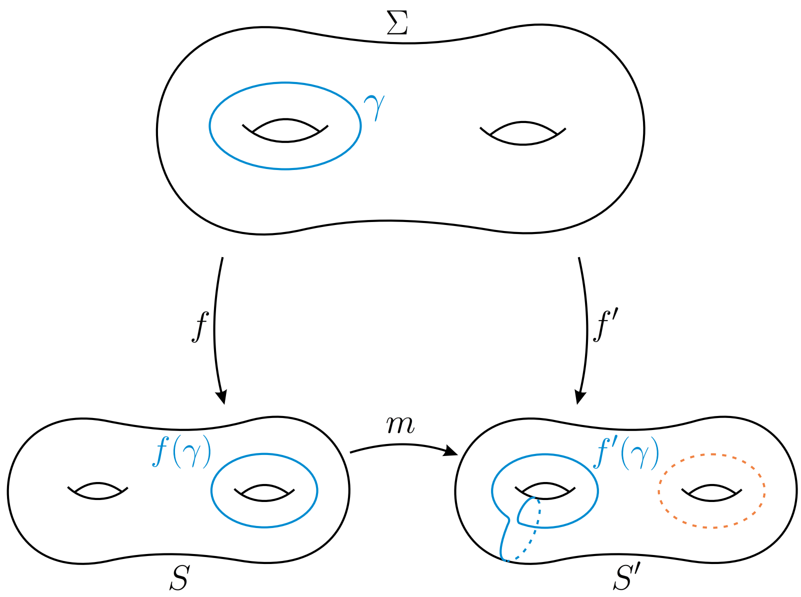


FIGURE 2. Two marked surfaces (S, f) and (S', f') representing different points in the Teichmüller space $\mathrm{T}_g(\Sigma)$.

In Figure 2, the map m is a translation in $\mathbb{R}^3$ and hence an isometry between the surfaces S and S'. However, the marked surfaces (S, f) and (S', f') represent different points in Teichmüller space $\mathrm{T}_g(\Sigma)$ since the curve $f'(\gamma)$ is clearly not isotopic to the curve $m(f(\gamma))$ denoted with the dashed line.

Let $\mathrm{Hyp}(\Sigma)$ be the set of *hyperbolic metrics* on Σ, i.e., complete Riemannian metrics such that every point $p \in \Sigma$ has a neighbourhood isometric to an open ball $\mathbb{H}$. Every marking (S, f) induces a hyperbolic metric on Σ via the pullback along the diffeomorphism $f : \Sigma \longrightarrow S$. We define an equivalence relation on the set $\mathrm{Hyp}(\Sigma)$ by identifying two metrics $h_1, h_2 \in \mathrm{Hyp}(\Sigma)$ if there exists an orientation-preserving diffeomorphism φ of Σ isotopic to the identity such that $h_1 = \varphi^*(h_2)$, where φ^* denotes the pullback along φ. This gives rise to another equivalent description of Teichmüller space of Σ.

Definition 3.26. The *Teichmüller space* of the surface Σ is the set of isotopy classes of hyperbolic metrics on Σ

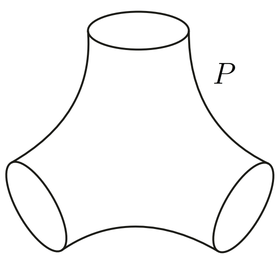
$$\mathrm{T}_g(\Sigma) := \mathrm{Hyp}(\Sigma) / \mathrm{Diff}_0^+(\Sigma),$$

where $\mathrm{Diff}_0^+(\Sigma)$ denotes the group of orientation-preserving diffeomorphisms of Σ isotopic to the identity.

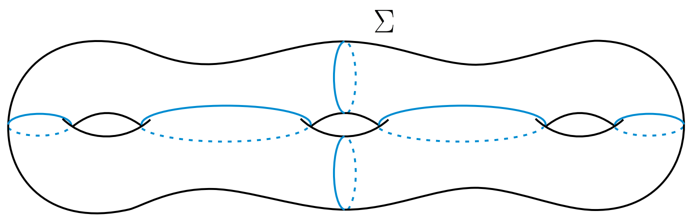
Since we have seen that every hyperbolic metric is obtained from a hyperbolic atlas, we could replace the set $\mathrm{Hyp}(\Sigma)$ in Definition 3.26 with the set of hyperbolic structures on Σ.

An important result states that Teichmüller space $\mathrm{T}_g(\Sigma)$ of a closed oriented hyperbolic surface Σ of genus $g \geq 2$ is homeomorphic to $\mathbb{R}^{6g-6}$. In the remainder of the section we shall describe the $6g-6$ parameters associated to $\mathrm{T}_g(\Sigma)$, also called Fenchel-Nielsen coordinates for $\mathrm{T}_g(\Sigma)$, which yield the mentioned homeomorphism. The construction of Fenchel-Nielsen coordinates is based on the so-called pants decomposition of the surface Σ which we depict in the following. We shall adopt an informal point of view and not give any proofs of the presented facts. These are discussed in several books such as [15, Section 9], [9, Section 3.2] and [18, Section 4.6].

We begin by decomposing a surface Σ into the so-called pairs of pants. A *pair of pants* is defined as the space P homeomorphic to the sphere S^2 with removed three disjoint open discs as depicted in Figure (3.a). A *pants decomposition* of a closed surface Σ is a collection of disjoint simple closed curves on Σ such that cutting Σ along these curves yields a disjoint union of pairs of pants, see Figure (3.b). Equivalently, a pants decomposition is a maximal collection of pairwise disjoint essential simple closed curves in Σ which are pairwise nonisotopic. Recall that a curve $\gamma \in \Sigma$ is essential if it is not contractible, that is, its fundamental class $[\gamma] \in \pi_1(\Sigma)$ is nontrivial.



(3.a) A pair of pants.



(3.b) Pants decomposition of a closed surface of genus 3.

It is a known fact that the pants decomposition of a closed orientable surface Σ of genus $g \geq 2$ consists of $3g-3$ disjoint simple closed curves on Σ. Moreover, cutting along these curves yields $2g-2$ pairs of pants, see Figure (3.b). When Σ is a closed surface endowed with a hyperbolic structure, each nontrivial homotopy class in Σ contains a unique simple closed geodesic by Theorem 3.23. Therefore, the curves in the pants decomposition of Σ can be represented by geodesics.

Let P be a pair of pants contained in Σ with geodesic boundary. A *seam* in P is a geodesic curve joining two different boundary circles and being perpendicular to them. We shall use the fact that any two boundary components of P can be joined by a unique seam, and all three seams are mutually disjoint, [15, Theorem 9.7.2]. Assume now a, b, c are lengths of the boundary geodesics of P. Cutting P along its seams yields two right-angled hexagons H_1, H_2 in $\mathbb{H}$. It is a fact that for any x, y, z real positive numbers there exists a right-angled hexagon in $\mathbb{H}$, unique up to isometry, with alternate sides of length x, y, z. This implies that H_1, H_2 are isometric and their nonseam alternating sides have lengths $a/2, b/2, c/2$ since they sum up to a, b, c in P. These lengths determine H_1 and H_2 up to isometry. Therefore, P is determined, up to isometry, by the lengths a, b, c. We refer the reader to [15, Section 9.7] for detailed proofs of these facts.

We shall now introduce the Fenchel-Nielsen coordinates on $\mathrm{T}_g(\Sigma)$ for a closed oriented hyperbolic surface Σ. We first choose a coordinate system of curves on Σ consisting of

(i) a collection of $3g-3$ homotopically distinct disjoint essential simple closed curves $\alpha_1, \dots, \alpha_{3g-3}$ corresponding to the pants decomposition of Σ, and
(ii) a collection of $g+1$ homotopically distinct disjoint essential simple closed curves $\beta_1, \dots, \beta_{g+1}$ such that the intersection of $\cup\beta_i$ with each pair of pants P yields three disjoint arcs connecting boundary components of P pairwise.

Without loss of generality, we assume that the curves $\alpha_1, \dots, \alpha_{3g-3}$ are geodesics. Let us denote their lengths by $l_1, \dots, l_{3g-3}$, respectively, which we call the *length parameters*. Cutting Σ along α_i yields $2g-2$ pairs of pants whose isometry types are determined by the lengths parameters l_i. We will determine the isometry type of Σ from that of the pairs of pants. To this end, we introduce the twist parameters which measure how much we rotate each two boundary circles of the pairs of pants before gluing them together.

Assume P is a pair of pants with boundary components α_1 and α_2 and let β_i be an essential simple closed curve intersecting α_1 and α_2. Denote by $\gamma_i = \beta_i \cap P$ the restriction of β_i to P, which is an arc connecting the boundary components α_1 and α_2 by construction, see Figure 4 on the left. Let N_1, N_2 be annular neighbourhoods of α_1 and α_2, respectively. We isotope the interior of P so that γ_i coincides with the unique seam δ of P joining α_1 and α_2 everywhere in P except in $N_1 \cup N_2$, see Figure 4. This isotopy fixes the endpoints of γ_i. We can further isotope the arc γ_i in the neighbourhoods N_1 and N_2, so that it becomes a spiral as depicted in Figure 4 on the right. This yields two real numbers measuring the amount of twisting as one goes from the middle of P, i.e., from $P \setminus (N_1 \cup N_2)$, to both endpoints $\gamma_i \cap \alpha_i$. We called them the *twisting numbers* of γ_i at α_i, and their sign depends on the orientation of α_i.

Suppose P, P' are the pairs of pants with the common boundary component α_i, and let β_i intersect α_i. Let $\gamma = \beta_i \cap P$ and $\gamma' = \beta_i \cap P'$ the arcs in P, P' corresponding to β_i. Denote the twisting numbers of γ and γ' at α_i by t and t', respectively. Then the *twist parameter* T at α_i is defined as

$$T(\alpha_i) = t - t' \in \mathbb{R}.$$

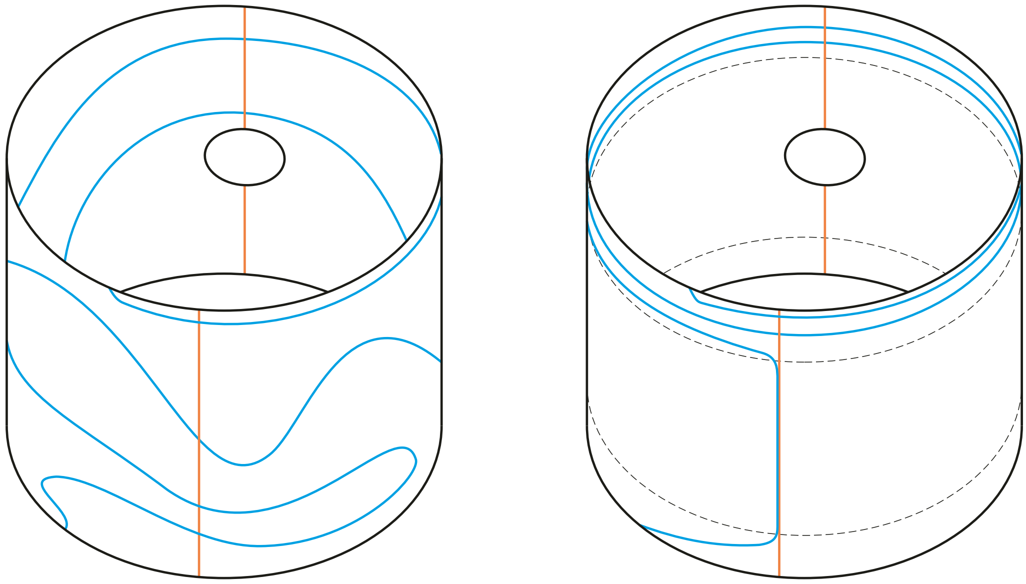


FIGURE 4. A simple arc γ_i (blue line in the left figure) joining two boundary components α_1 and α_2 of a hyperbolic pair of pants P can be modified along an isotopy on the interior of P, so that it agrees with the unique seam δ depicted as a straight orange line. In the annular neighbourhoods of α_1 and α_2 (dashed lines in the right figure) it can be further isotoped to become spiral as depicted in the right figure.

Observe that changing one of the twist parameters by 2π yields a new hyperbolic metric which is isometric to the first one, but there is no isometry isotopic to the identity between them. Hence, the twist parameters are not defined modulo 2π.

Since there are two choices of seams for a boundary component α_i, we need to prove that the twist parameter T is independent of this choice. To this end, assume the boundary components of the pair of pants depicted in Figure 4 on the right are glued to each other. Let us denote the common boundary component by α_i, and let us consider the universal cover of the annular neighbourhood N_i of α_i obtained after gluing, see Figure 5.

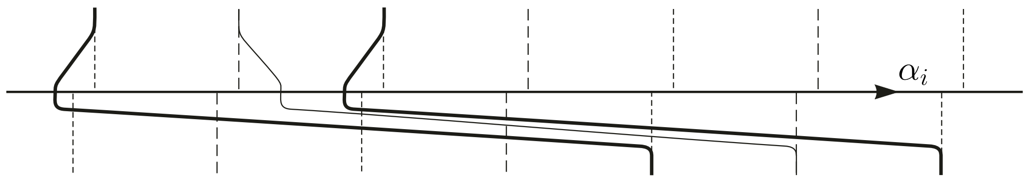


FIGURE 5. The unfolded annular neighbourhood N_i of α_i obtained after gluing.

The dashed lines represent the two seams δ, δ' of each copy of α_i. The thick curve comes from the curve β_i shown in Figure 4 on the right. Now drag β_i on each side of α_i so that it matches with the other pair of seams, but keep the endpoints of β_i on α_i fixed. If we drag in the opposite direction according to the orientation of α_i, we obtain the thin curve in Figure 5. As the two seams δ, δ' intersect α_i at the diametrically opposite points, we see that the thick and the thin curve yield the same twist parameter.

The length parameters and the twist parameters determine the homeomorphism of Teichmüller space $\mathrm{T}_g(\Sigma)$ onto $\mathbb{R}^{6g-6}$.

Theorem 3.27. *Let Σ be a closed oriented hyperbolic surface Σ of genus $g \geq 2$. Then the map*

$$\begin{aligned} \mathrm{T}_g(\Sigma) &\longrightarrow \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ [(S,f)] &\longmapsto (l(\alpha_1), \dots, l(\alpha_{3g-3}), T(\alpha_1), \dots, T(\alpha_{3g-3})) \end{aligned}$$

is a homeomorphism.

The proof of this important theorem can be found in [18, Theorem 4.6.23] or in [5, Theorem 10.6].

4. Real spectrum

In this section we study the set of prime cones of a commutative ring A called the real spectrum of A and denoted by $\mathrm{Spec_R(A)}$. Endowed with an appropriate topology, it becomes a compact topological space. However, the importance of the real spectrum is in its subspace, $\mathrm{Spec_R^m(A)}$, consisting of closed points of $\mathrm{Spec_R(A)}$. This subspace is particularly essential when studying the compactification of a semi-algebraic subset W of a real algebraic set X. Namely, it turns out that $W \subseteq X$ can be viewed as a subset of $\mathrm{Spec_R^m}(\mathcal{A}(\mathrm{X}))$, where $\mathcal{A}(X)$ denotes the affine coordinate ring of X. Moreover, $\mathrm{Spec_R^m}(\mathcal{A}(\mathrm{X}))$ is a compact Hausdorff topological space containing W as an open and dense subset. Then the real spectrum compactification of W is defined as the closure of W in the space $\mathrm{Spec_R^m}(\mathcal{A}(\mathrm{X}))$ of closed points of the real spectrum $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$. This section provides a detailed investigation of all these facts summarized by [2, Sections 7.1 and 7.2] unless stated otherwise.

We will always denote by A a commutative ring with identity.

4.1. **Prime cones.** We introduce the notion of a prime cone P of a ring A, and define its support $\mathrm{supp}(P)$ which turns out to be a real prime ideal of A. This enables us to study the field of fractions $k(\mathrm{supp}(P))$ of the integral domain $A/\mathrm{supp}(P)$. Given a prime cone P of A, we show that it is actually a preimage of a positive cone of an ordering of $k(\mathrm{supp}(P))$ under the canonical homomorphism taking A onto $k(\mathrm{supp}(P))$. Using this result, we give a more general condition on when a subset of A is also its prime cone.

Definition 4.1. A *cone* P of a commutative ring A is a subset of A satisfying the following conditions.

(i) If $f \in P$ and $g \in P$, then $f + g \in P$.
(ii) If $f \in P$ and $g \in P$, then $fg \in P$.
(iii) If $f \in A$, then $f^2 \in P$.

If, in addition,

(iv) $-1 \notin P$,

then P is called a *proper* cone of A.

We are interested in proper cones that satisfy an additional property, called prime cones, since they present points of the real spectrum of a ring A discussed in Section 4.2.

Definition 4.2. Let P be a proper cone of A. If the condition

$$fg \in P \Rightarrow f \in P \text{ or } -g \in P,$$

holds for all $f, g \in A$, then P is called a *prime cone* of A.

We introduce the notation $-P := \{f \in A \mid -f \in P\}$ for a subset $P \subset A$. In case P is a prime cone of A, two important relations can be obtained from the sets P and $-P$.

Proposition 4.3. *The following holds for a prime cone P of A.*

(i) $P \cup -P = A$, and

(ii) the set $\operatorname{supp}(P) := P \cap -P$*, called the* support *of P, is a prime ideal of A.*

Proof. (i) We know that $f^2 \in P$ for all $f \in A$ since P is a cone. Thus, $f \in A$ or $-f \in A$ by the definition of a prime cone.

(ii) Condition (i) of Definition 4.1 implies that $\operatorname{supp}(P)$ is an additive subgroup of P. Let us now prove that it is an ideal of A. Suppose $f \in \operatorname{supp}(P)$ and $g \in A$. We consider two cases, depending on whether or not $g \in P$.

- If $g \in P$, then $fg \in P$ and $(-f)g \in P$ because P is a cone. Since the condition $(-f)g \in P$ is equivalent to $fg \in -P$, $fg \in \operatorname{supp}(P)$.
- In case $g \notin P$, then, by (i), $g \in -P$. Following the same conclusion as in the previous case, $fg \in \operatorname{supp}(P)$.

Hence, $\operatorname{supp}(P)$ is an ideal. It remains to prove that it is a prime ideal. Suppose $fg \in \operatorname{supp}(P)$ but $f \notin \operatorname{supp}(P)$.

- In case $f \notin P$, then $-g \in P$ by definition of a prime cone. Since $fg \in -P$, or equivalently, $f(-g) \in P$ which is a prime cone, we have that $g \in P$.
- If $f \notin -P$, then $g \in P$ because, by assumption, $gf \in P$. Since $(-g)f \in P$ we have that $-g \in P$.

In both cases, $g \in \operatorname{supp}(P)$, which proves the proposition. □

Example 4.4. (i) Let us prove that prime cones of a field F are exactly positive cones of orderings of F.

($\Leftarrow$) Let $P = \{x \in F \mid x \geq 0\}$ be a positive cone of an ordering of F. By Proposition 2.4, the set P is a proper cone satisfying $P \cup -P = F$. It remains to prove that if $xy \in P$ then either $x \in P$ or $-y \in P$. Suppose $x \notin P$. Then $xy \geq 0$ and $x < 0$. Thus, $x^2y \leq 0$ since F is an ordered field. Assume now that $-y \notin P$, or equivalently, $y > 0$. Then $(xy)^2 \leq 0$, which is a contradiction since P is a cone of F.

($\Rightarrow$) Let P be a prime cone of F. By Proposition 2.4, the field F is ordered by $x \leq y \Leftrightarrow y - x \in P$. This means that $y \in P$ if and only if $y \geq 0$.

(ii) Suppose $F : A \longrightarrow B$ is a ring homomorphism and P a prime cone of B. Let us prove that $F^{-1}(P)$ is a prime cone of A. The preimage $F^{-1}(P)$ is clearly a cone which is also proper since a ring homomorphism sends the identity of A to the identity of B. Assume now that $xy \in F^{-1}(P)$ and $x \notin F^{-1}(P)$. Since P is a prime cone of B, $-F(y) \in P$ and thus, $F^{-1}(P)$ is a prime cone of A.

Definition 4.5. Let P be a prime cone of A. The field of fractions of $A/\operatorname{supp}(P)$ is called the *residue field of A at* $\operatorname{supp}(P)$, and denoted by $k(\operatorname{supp}(P))$.

Lemma 4.6. *Let P be a prime cone of A and $\varphi : A \longrightarrow k(\operatorname{supp}(P))$ the canonical homomorphism of the ring A onto the field of fractions of $A/\operatorname{supp}(P)$. Then the* *positive cone of an ordering of* $k(\mathrm{supp}(P))$ *is given by the set*

$$\overline{P} = \{\overline{f}/\overline{g} \in k(\mathrm{supp}(P)) \mid fg \in P\}.$$

Moreover, the preimage $\varphi^{-1}(\overline{P})$ *is the prime cone* P.

Proof. The fact that P is a prime cone implies that $\overline{P}$ is a cone satisfying $\overline{P} \cup -\overline{P} = k(\mathrm{supp}(P))$. If we prove that $\overline{P}$ is also proper, then Proposition 2.4 implies that $\overline{P}$ is the positive cone of an ordering of $k(\mathrm{supp}(P))$. It is sufficient to prove that $\varphi^{-1}(\overline{P}) = P$, since then $\overline{P}$ is a proper cone. We denote the equivalence class of an element $f \in A$ in the residue field $k(\mathrm{supp}(P))$ by $\overline{f} = \varphi(f)$.

($\Rightarrow$) We shall prove that $\varphi^{-1}(x)$ is an element of P for any $x \in \overline{P}$. By definition, $x = \overline{g}/\overline{h}$ and $gh \in P$. Since $\overline{h} \neq 0$ in $k(\mathrm{supp}(P))$, we have that $h \notin \mathrm{supp}(P)$. We may assume $h \notin -P$. Then $g \in P$ since gh is an element of P which is a prime cone of A. Let us denote the preimage $\varphi^{-1}(x)$ in A by f. Then we can write $fh = g + k$ for some $k \in \mathrm{supp}(P)$ since $\varphi(f) = \varphi(\frac{g}{h} + \frac{k}{h}) = x$. This implies that $fh \in P$, and since $-h \notin P$, we conclude that $f = \varphi^{-1}(x) \in P$.

($\Leftarrow$) Clearly, $\varphi^{-1}(\overline{f}) \in P$ for any $f \in P$. □

Corollary 4.7. *The support* $\mathrm{supp}(P)$ *of a prime cone* P *is a real prime ideal.*

Proof. We already know from Proposition 4.3 that $\mathrm{supp}(P)$ is a prime ideal. Let $f_1, \dots, f_p$ be a sequence of elements of A such that $f_1^2 + \cdots + f_p^2 \in \mathrm{supp}(P)$. Then $\pm(f_1^2 + \cdots + f_p^2) \in P = \varphi^{-1}(\overline{P})$, or equivalently, $\pm(\varphi(f_1)^2 + \cdots + \varphi(f_p)^2) \in \overline{P}$. As $\overline{P}$ is a positive cone of an ordering of $k(\mathrm{supp}(P))$, we conclude that $\varphi(f_i) = 0$ in $k(\mathrm{supp}(P))$ for every i. Hence, $f_i \in \mathrm{supp}(P)$ for every $i = 1, \dots, p$. □

Proposition 4.8. *For a subset* P *of a commutative ring* A *the following conditions are equivalent.*

(i) The set P *is a prime cone of* A.

(ii) There exists an ordered field F *and a homomorphism* $\alpha : A \longrightarrow F$ *such that* $P = \{f \in A : \alpha(f) \geq 0\}$.

Proof. (i) $\Rightarrow$ (ii) Let P be a prime cone of A. Define F to be the residue field $k(\mathrm{supp}(P))$, and let $\alpha : A \longrightarrow k(\mathrm{supp}(P))$ be the canonical homomorphism. Applying Lemma 4.6, P is the inverse image of the positive cone of an ordering of $k(\mathrm{supp}(P))$.

(ii) $\Rightarrow$ (i) Assume there is an ordered field F and a homomorphism $\alpha : A \longrightarrow F$. By Example 4.4 (i), the positive cone $\overline{P} = \{x \in F \mid x \geq 0\}$ of an ordering of F is also a prime cone of F. Then Example 4.4 (ii) implies that the inverse image $\alpha^{-1}(\overline{P}) = \{f \in A \mid \alpha(f) \geq 0\}$ is a prime cone of A. □

4.2. **The real spectrum of a ring.** In this section, we investigate the real spectrum of the commutative ring A which we denote by $\mathrm{Spec_R(A)}$. We endow it with the so-called real spectrum topology with respect to which it is a compact space. It turns out that especially important is the subspace of closed points of $\mathrm{Spec_R(A)}$, denoted by $\mathrm{Spec_R^m(A)}$, because it plays an important role when studying the real spectrum compactification of a semi-algebraic set discussed in Section 4.3. We prove that $\mathrm{Spec_R^m(A)}$ is a compact Hausdorff topological space, and we show that there exists a continuous retraction from $\mathrm{Spec_R(A)}$ to $\mathrm{Spec_R^m(A)}$. Finally, we consider different characterizations of points of $\mathrm{Spec_R^m(A)}$.

Let us recall that we consider A to be a commutative ring with identity, and let α be a prime cone of A. In Section 4.1 we introduced the notation $k(\operatorname{supp}(\alpha))$ for the field of fractions of $A/\operatorname{supp}(\alpha)$. We again write f for an element of A, and $\overline{f}$ for the image of f in $k(\operatorname{supp}(\alpha))$ under the canonical homomorphism $A \longrightarrow k(\operatorname{supp}(\alpha))$. Recall that $k(\operatorname{supp}(\alpha))$ is a real field by Lemma 2.11 and Corollary 4.7, with an ordering $\leq_\alpha$ induced by each prime cone α of A given by $0 \leq_\alpha \overline{f} \iff f \in \alpha$. Since every ordered field F has a real closure which uniquely extends the ordering of F by Theorem 2.9, we denote the real closure of the residue field $k(\operatorname{supp}(\alpha))$ by $F(\alpha)$. The assertion of the following lemma is essential for classification of closed points of the real spectrum of A.

Lemma 4.9. *Let $\alpha \in \operatorname{Spec_R(A)}$ be a prime cone of A. Then the real closure $F(\alpha)$ of the field of fractions $k(\operatorname{supp}(\alpha))$ is Archimedean over $k(\operatorname{supp}(\alpha))$, i.e., every element $x \in F(\alpha)$ is bounded by some element of $k(\operatorname{supp}(\alpha))$.*

Proof. Since $F(\alpha)$ is an algebraic extension of $k(\operatorname{supp}(\alpha))$ by definition, every element $x \in F(\alpha)$ is a root of a polynomial with coefficients over $k(\operatorname{supp}(\alpha))$. Suppose $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, where $a_0, \ldots, a_{n-1} \in k(\operatorname{supp}(\alpha))$. We shall prove that

$$|x| \leq 1 + |a_{n-1}| + \cdots + |a_0|.$$

In case $|x| \leq 1$, the assertion is obvious. If $|x| > 1$, we have that

$$|x|^n \leq |a_{n-1}||x|^{n-1} + |a_{n-2}||x|^{n-2} \cdots + |a_0|.$$

Then dividing by $|x|^{n-1}$ yields

$$\begin{aligned} |x| &\leq |a_{n-1}| + |a_{n-2}||x|^{-1} \cdots + |a_0||x|^{-(n-1)} \\ &< |a_{n-1}| + |a_{n-2}| \cdots + |a_0|, \end{aligned}$$

since $|x|^{-1} < 1$. Thus, every element $x \in F(\alpha)$ is bounded by an element of $k(\operatorname{supp}(\alpha))$. □

Let us change the notation and denote the image of an element $f \in A$ under the canonical homomorphism $A \longrightarrow F(\alpha)$ by $f(\alpha)$. Since $F(\alpha)$ is a totally ordered field, exactly one of the following statements holds for an element $f \in A$: $f(\alpha) > 0$, $f(\alpha) = 0$ or $f(\alpha) < 0$. By definition of the induced ordering of $F(\alpha)$, these statements are equivalent to

$$\begin{aligned} f(\alpha) > 0 &\iff f \in \alpha \setminus (-\alpha) \\ f(\alpha) = 0 &\iff f \in \operatorname{supp}(\alpha) \\ f(\alpha) < 0 &\iff -f \in \alpha \setminus (-\alpha) \iff f \in (-\alpha) \setminus \alpha. \end{aligned}$$

Note that by definition

$$f(\alpha) \geq 0 \iff f \in \alpha.$$

Definition 4.10. The *real spectrum* $\operatorname{Spec_R(A)}$ *of a ring* A is a topological space consisting of prime cones of the ring A,

$$\operatorname{Spec_R(A)} = \{\alpha \mid \alpha \text{ prime cone of } A\},$$

endowed with the *real spectrum topology* given by the subbasis of open sets

$$\widetilde{U}(f) = \{\alpha \in \operatorname{Spec_R(A)} \mid f(\alpha) > 0\},$$

where $f \in A$.

Clearly, the basis of the real spectrum topology consists of the sets $\widetilde{U}(f_1, \ldots, f_n) = \{\alpha \in \mathrm{Spec_R(A)} \mid f_1(\alpha) > 0, \ldots, f_n(\alpha) > 0\}$, where $f_1, \ldots, f_n$ is a finite collection of elements of A.

It turns out that points of $\mathrm{Spec_R(A)}$ can be interpreted in several equivalent ways and not only as prime cones of A.

Proposition 4.11. *The following are equivalent interpretations of a point of the real spectrum* $\mathrm{Spec_R(A)}$.

(i) A prime cone α.

(ii) A pair $(\eta, \leq)$ consisting of a prime ideal η of A and an ordering $\leq$ of the field of fractions $k(\eta)$ of A/η.

(iii) An equivalence class of homomorphisms $\alpha : A \longrightarrow F(\alpha)$, where $F(\alpha)$ is a real closed field, algebraic over the subring $\alpha(A) \subseteq F(\alpha)$. We define two homomorphisms $\alpha : A \longrightarrow F(\alpha)$ and $\alpha' : A \longrightarrow F(\alpha')$ to be equivalent if there exists an order preserving isomorphism $\varphi : F(\alpha) \longrightarrow F(\alpha')$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & F(\alpha) \arrow[d, "\varphi"] \\
& F(\alpha')
\end{tikzcd}$$

Proof. (i) $\Rightarrow$ (ii) Let α be a prime cone of A. By Corollary 4.7, the support $\mathrm{supp}(\alpha)$ is a real prime ideal which induces an ordering $\leq_\alpha$ of the real closure $F(\alpha)$ of the residue field $k(\mathrm{supp}(\alpha))$. Therefore, we take $(\eta, \leq) = (\mathrm{supp}(\alpha), \leq_\alpha)$.

(ii) $\Rightarrow$ (iii) Let η be a prime ideal of A and $\leq$ an ordering of the residue field $k(\eta)$. Define $F(\alpha)$ to be a real closure of $k(\eta)$ with respect to $\leq$, which exists by Theorem 2.9. Take α to be the canonical homomorphism $\alpha : A \longrightarrow k(\eta) \longrightarrow F(\alpha)$. Let F' be another real closure of $k(\eta)$ and $\alpha' : A \longrightarrow F'$ the canonical homomorphism. Then, by Theorem 2.9, there exists a unique order-preserving isomorphism $\Phi : F(\alpha) \longrightarrow F'$ such that $\Phi|_{k(\eta)} = \mathrm{id}_{k(\eta)}$. Therefore, $\Phi(\alpha(A)) = \alpha'(A)$.

(iii) $\Rightarrow$ (i) Given a real closed field $F(\alpha)$ and a homomorphism $\alpha : A \longrightarrow F(\alpha)$, the set $\{f \in A \mid f(\alpha) \geq 0\}$ is a prime cone of A by Proposition 4.8. $\square$

Example 4.12. Let us fix a real algebraic set $X \subseteq \mathbb{R}^n$, and denote its affine coordinate ring by $\mathcal{A}(X) = \mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X)$. We regard points of $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$ as homomorphisms $\mathcal{A}(X) \longrightarrow F(\cdot)$. Then for every point $x := (x_1, \ldots, x_n) \in X \subseteq \mathbb{R}^n$ we define a homomorphism

$$\begin{aligned} \alpha_x : \mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X) &\longrightarrow \mathbb{R} \\ F + \mathcal{I}(X) &\longmapsto F(x). \end{aligned}$$

Hence, every real algebraic set X is a subset of the real spectrum $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$. In addition, we compute that

$$X \cap \widetilde{U}(f) = \{x \in X \mid f(x) > 0 \in \mathbb{R}\}$$

which means that the subspace topology on X induced by the real spectrum topology coincides with the Euclidean topology on X, [3].

This example leads to the following result.

Proposition 4.13. *Let $X \subseteq \mathbb{R}^n$ be a real algebraic set. Then the map*

$$\begin{aligned} \alpha : X &\longrightarrow \mathrm{Spec_R}(\mathcal{A}(\mathrm{X})) \\ x &\longmapsto \alpha_x := \{f \in \mathcal{A}(X) \mid f(x) \geq 0\} \end{aligned}$$

is injective. Moreover, if X is endowed with the Euclidean topology, then $\alpha : X \longrightarrow \mathrm{Im}(\alpha)$ is a homeomorphism.

Proof. It can be readily seen that α_x is a prime cone of $\mathcal{A}(X)$. If $x \neq y$, then there exists a polynomial $f \in \mathcal{A}(X)$ such that $f(x) \geq 0$ but $f(y) < 0$. Hence, $\alpha_x \neq \alpha_y$ and α is injective. Let $\widetilde{U}(f_1, \dots, f_p)$ be a basic open subset of $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$. Then the preimage

$$\begin{aligned} \alpha^{-1}(\widetilde{U}(f_1, \dots, f_p)) &= \{x \in X \mid f_1(\alpha_x) > 0, \dots, f_p(\alpha_x) > 0\} \\ &= \{x \in X \mid f_1 \in \alpha_x \setminus (-\alpha_x), \dots, f_p \in \alpha_x \setminus (-\alpha_x)\} \\ &= \{x \in X \mid f_1(x) > 0, \dots, f_p(x) > 0\} \\ &= U(f_1, \dots, f_p), \end{aligned}$$

which is an open basic set of the Euclidean topology on X. This proves that α is an open continuous bijection onto its image, [2, Proposition 7.1.5]. $\square$

Our goal is to study the compactification of a closed semi-algebraic subset of a real algebraic set $X \subseteq \mathbb{R}^n$ which turns out to be a subspace of the space of closed points of the real spectrum $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$, denoted by $\mathrm{Spec_R^m}(\mathcal{A}(\mathrm{X}))$. To this end, we now define the space $\mathrm{Spec_R^m}(\mathrm{A})$ for an arbitrary commutative ring A, and we prove that it is compact and Hausdorff. We shall interpret the points of $\mathrm{Spec_R}(\mathrm{A})$ as prime cones of A.

First, we show that the real spectrum $\mathrm{Spec_R}(\mathrm{A})$ is a compact space in the real spectrum topology, [12, Section 4]. We start by identifying $\mathrm{Spec_R}(\mathrm{A})$ with the space of functions $Y := \{0,1\}^A$ from the ring A to the set $\{0,1\}$. We endow the space $\{0,1\}$ with the discrete topology, and the space Y with the product, or Tychonoff, topology. By definition, the basis of the product topology on Y is given by finite products of open sets in $\{0,1\}$. Thus, it consists of open sets of the form

$$D_{\varepsilon_1,\dots,\varepsilon_n}(f_1, \dots, f_n) = \{F \in Y \mid F(f_i) = \varepsilon_i \ \forall i = 1, \dots, n\},$$

where $f_1, \dots, f_n \in A$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$. We now define a map

$$\begin{aligned} \Psi : \mathrm{Spec_R}(\mathrm{A}) &\longrightarrow Y \\ \alpha &\longmapsto e_\alpha, \end{aligned}$$

where we identify a prime cone α with the characteristic function $e_\alpha : A \longrightarrow \{0,1\}$ of $\alpha \setminus (-\alpha)$, given by

$$e_\alpha(f) = \begin{cases} 1 & \text{if } f \in \alpha \setminus (-\alpha) \\ 0 & \text{if } f \notin \alpha \setminus (-\alpha) \end{cases}.$$

Thus, the real spectrum $\mathrm{Spec_R}(\mathrm{A})$ is identified with the set of functions $\mathrm{Im}(\Psi) = \{e_\alpha \mid \alpha \in \mathrm{Spec_R}(\mathrm{A})\}$ in Y. The pullback of the subspace topology of $\mathrm{Im}(\Psi)$ to the real spectrum $\mathrm{Spec_R}(\mathrm{A})$ via Ψ yields the so-called *Tychonoff topology* of $\mathrm{Spec_R}(\mathrm{A})$ which is finer than the real spectrum topology by the following lemma.

Lemma 4.14. *The real spectrum topology is weaker than the subspace topology on $\mathrm{Im}(\Psi)$ induced by the Tychonoff topology on Y.*

Proof. We show that every open set $\widetilde{U}(f_1, \ldots, f_n)$ is an open subset of $\mathrm{Im}(\Psi)$ with respect to the subspace topology induced by the Tychonoff topology of Y, [12]. An open subset of $\mathrm{Im}(\Psi)$ is of the form $D_{\varepsilon_1,\ldots,\varepsilon_n}(f_1, \ldots, f_n) \cap \mathrm{Im}(\Psi)$ since the family of sets $D_{\varepsilon_1,\ldots,\varepsilon_n}(f_1, \ldots, f_n)$ constitute a basis of the Tychonoff topology of Y. Let us choose $\varepsilon_1 = \cdots = \varepsilon_n = 1$. Then we have that

$$\begin{aligned} D_{1,\ldots,1}(f_1, \ldots, f_n) \cap \mathrm{Im}(\Psi) &= \{e_\alpha \in \mathrm{Im}(\Psi) \mid e_\alpha(f_i) = 1 \; \forall i = 1, \ldots, n\} \\ &= \{e_\alpha \in \mathrm{Im}(\Psi) \mid f_i \in \alpha \setminus (-\alpha) \; \forall i = 1, \ldots, n\} \\ &= \{e_\alpha \in \mathrm{Im}(\Psi) \mid f_i(\alpha) > 0\} \\ &= \widetilde{U}(f_1, \ldots, f_n), \end{aligned}$$

since a function $e_\alpha \in \mathrm{Im}(\Psi)$ corresponds to a prime ideal $\alpha \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$. □

Since our aim is to prove that $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ is compact in the real spectrum topology, we shall first prove that $\mathrm{Im}(\Psi)$ is closed in Y with respect to the product topology on Y. By Tychonoff theorem, the product space Y is compact and Hausdorff, which implies that $\mathrm{Im}(\Psi)$ with the subspace topology is a compact and Hausdorff subspace of Y. Then the real spectrum $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A}) \equiv \mathrm{Im}(\Psi)$ is compact and Hausdorff in the Tychonoff topology and hence, compact in the weaker, real spectrum topology.

Proposition 4.15. *The image* $\mathrm{Im}(\Psi)$ *is closed in the Tychonoff topology on* Y*.*

Proof. We shall prove that $Y \setminus \mathrm{Im}(\Psi)$ is open, [12, Proposition 4.2]. We know that for an element $F \in Y \setminus \mathrm{Im}(\Psi)$, the set $S := -F^{-1}(0) = \{f \in A \mid F(-f) = 0\}$ is not a prime cone of A since $F \notin \mathrm{Im}(\Psi)$. Hence, the set S does not satisfy at least one of the conditions of Definition 4.1. Assume it violates condition (ii). This means that there exist $f, g \in S$ such that $fg \notin S$, which is equivalent to $F(-f) = F(-g) = 0$ and $F(-fg) = 1$. Thus, F is contained in the open subset

$$F \in D_{0,0,1}(-f, -g, -fg) = \{H \in T \mid H(-f) = 0, H(-g) = 0, H(-fg) = 1\}.$$

If

$$D_{0,0,1}(-f, -g, -fg) \cap \mathrm{Im}(\Psi) \neq \emptyset,$$

there exists a characteristic function $e_\alpha \in D_{0,0,1}(-f, -g, -fg)$ for some prime cone α of A. The conditions $e_\alpha(-f) = e_\alpha(-g) = 0$ are equivalent to $f, g \in \alpha \setminus (-\alpha)$, and since α is a prime cone of A, we have that $fg \in \alpha \setminus (-\alpha) \iff e_\alpha(-fg) = 0$. This is a contradiction and thus, $D_{0,0,1}(-f, -g, -fg) \cap \mathrm{Im}(\Psi) = \emptyset$. Since we found an open set containing F and not intersecting $\mathrm{Im}(\Psi)$, the set $Y \setminus \mathrm{Im}(\Psi)$ is open. If the set S violates some other properties of a prime cone instead of (ii), we proceed in a similar way. □

We have proved the following theorem.

Theorem 4.16. *(i) The real spectrum* $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ *is a compact Hausdorff space in the Tychonoff topology.*

(ii) The real spectrum $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ *is a compact space in the real spectrum topology.*

□

Next, we shall see that the basis of the Tychonoff topology of $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ consists of so-called constructible subsets which are particularly important when investigating the compactification of a semi-algebraic set discussed in Section 4.3.

Definition 4.17. A *constructible subset* of $\mathrm{Spec_R(A)}$ is a finite union of a finite intersection of the sets $\widetilde{U}(f) = \{\alpha \in \mathrm{Spec_R(A)} \mid f(\alpha) > 0\}$ and $\widetilde{Z}(g) = \{\alpha \in \mathrm{Spec_R(A)} \mid g(\alpha) = 0\}$ for $f, g \in A$. Equivalently, a constructible set is a finite union of sets of the form

$$\{f_1 > 0, \dots, f_n > 0, g = 0\} := \{\alpha \in \mathrm{Spec_R(A)} \mid f_1(\alpha) > 0, \dots, f_n(\alpha) > 0, g(\alpha) = 0\}$$

Note that that a finite intersection $\bigcap_{i=1}^{m} \widetilde{Z}(g_i) = \bigcap_{i=1}^{m} \{g_i = 0\}$ can be reduced to a single set $\widetilde{Z}(g_1^2 + \dots + g_m^2) = \{g_1^2 + \dots + g_m^2 = 0\}$.

Proposition 4.18. *The basis of the Tychonoff topology of* $\mathrm{Spec_R(A)}$ *consists of the constructible subsets of* $\mathrm{Spec_R(A)}$.

Proof. The subsets $D_{\varepsilon_1,\dots,\varepsilon_n}(f_1, \dots, f_n) \cap \mathrm{Im}(\Psi)$ form a basis of the subspace topology of $\mathrm{Im}(\Psi)$. Thus, the basis of the Tychonoff topology of $\mathrm{Spec_R(A)}$ is given by the family of sets

$$\{\Psi^{-1}(U) \mid U \text{ is a basic open subset of } \mathrm{Im}(\Psi)\}.$$

Assume for simplicity that $\varepsilon_1 = \dots = \varepsilon_r = 0$ and $\varepsilon_{r+1} = \dots = \varepsilon_n = 1$ for some $r \in \{1, \dots, n\}$, and let $f_1, \dots, f_n \in A$. For a subset $U = D_{\varepsilon_1,\dots,\varepsilon_n}(f_1, \dots, f_n) \cap \mathrm{Im}(\Psi)$, we have that

$$\begin{aligned}
\Psi^{-1}(U) =& \{\alpha \in \mathrm{Spec_R(A)} \mid e_\alpha(f_i) = \varepsilon_i \ \forall i = 1, \dots, n\} \\
=& \{\alpha \in \mathrm{Spec_R(A)} \mid f_1, \dots, f_r \in \alpha \setminus (-\alpha) \text{ and } f_{r+1}, \dots, f_n \notin \alpha \setminus (-\alpha)\} \\
=& \{\alpha \in \mathrm{Spec_R(A)} \mid f_1(\alpha) > 0, \dots, f_r(\alpha) > 0\} \cap \\
& (\{\alpha \in \mathrm{Spec_R(A)} \mid -f_{r+1}(\alpha) > 0, \dots, -f_n(\alpha) > 0\} \cup \\
& \{\alpha \in \mathrm{Spec_R(A)} \mid f_{r+1}(\alpha) = 0, \dots, f_n(\alpha) = 0\}).
\end{aligned}$$

In short notation, we obtain that

$$\begin{aligned}
\Psi^{-1}(U) =& \{f_1 > 0, \dots, f_r > 0, -f_{r+1} > 0, \dots, -f_n > 0\} \cup \\
& \{f_1 > 0, \dots, f_r > 0, f_{r+1}^2 + \dots + f_n^2 = 0\},
\end{aligned}$$

which is a constructible set by definition. □

Constructible sets are important not only because they form a basis of the Tychonoff topology of $\mathrm{Spec_R(A)}$ but also because of the following convenient properties.

Proposition 4.19. *(i) A set is constructible if and only if it is closed and open in the Tychonoff topology of* $\mathrm{Spec_R(A)}$.

(ii) Every constructible set is compact in the real spectrum topology of $\mathrm{Spec_R(A)}$.

Proof. The proof of this proposition is taken from [1, Proposition 2.2].

(i) ($\Rightarrow$) We already know that a constructible set S is open. A short calculation shows that the complement of S is also a constructible set, which implies that S is closed.

($\Leftarrow$) Let S be open and closed in the Tychonoff topology. As an open set, it is a union of constructible sets by definition of a basis. By Theorem 4.16, it is also a closed subset of a compact space $\mathrm{Spec_R(A)}$, and hence compact. Thus, the union of constructible sets covering S is finite, and S is constructible.

(ii) By (i), a constructible set C is closed, and hence compact in the Tychonoff topology. Let $\mathcal{U} = \cup_i \widetilde{U}_i$ be an open cover of C consisting of basic open sets with respect to the real spectrum topology. By Lemma 4.14, every $\widetilde{U}_i$ can be written as

$$\widetilde{U}_i(f_1, \dots, f_n) = D_{1,\dots,1}(f_1, \dots, f_n) \cap \mathrm{Im}(\Psi),$$

which is a constructible set by Proposition 4.18. Hence, $\mathcal{U}$ is an open cover of C in the Tychonoff topology with respect to which C is compact. Therefore, there exists a finite subcover of $\mathcal{U}$, and C is compact in the real spectrum topology. □

Finally, we describe the subspace $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ of closed points of the real spectrum $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$, and we show that it is a compact Hausdorff space.

Definition 4.20. Let α, β be two points of $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$. The point β is said to be a *specialization* of α if it is in the closure $\overline{\{\alpha\}}$ of the singleton set $\{\alpha\}$ in the real spectrum topology, i.e., $\beta \in \overline{\{\alpha\}}$.

Proposition 4.21. *For $\alpha, \beta \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ the following are equivalent.*

(i) $\beta \in \overline{\{\alpha\}}$.
(ii) $\alpha \subset \beta$.
(iii) $f(\alpha) \geq 0 \Rightarrow f(\beta) \geq 0$ *for every* $f \in A$.
(iv) $f(\beta) > 0 \Rightarrow f(\alpha) > 0$ *for every* $f \in A$.

Proof. (i) ⇒ (ii) Suppose there exists $f \in \alpha \setminus \beta$. Thus $f(\alpha) \geq 0$ and $-f(\beta) > 0$ since $-f \in \beta \setminus (-\beta)$. The set $\widetilde{U}(-f) = \{\alpha \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A}) \mid -f(\alpha) > 0\}$ is an open neighbourhood of β which intersects $\{\alpha\}$ since $\beta \in \overline{\{\alpha\}}$ by assumption. This implies that $-f(\alpha) > 0$, or equivalently, $f(\alpha) < 0$, which contradicts the fact that $f(\alpha) \geq 0$. Hence $\alpha \subset \beta$, [1, Proposition 2.4].

(ii) ⇒ (iii) Clearly, if $f \in \alpha \subset \beta$, then both $f(\alpha) \geq 0$ and $f(\beta) \geq 0$.

(iii) ⇒ (iv) Suppose there exists an element $f \in A$ such that $f(\beta) > 0$ but $f(\alpha) \leq 0$. By assumption, $-f(\alpha) \geq 0$ implies that $-f(\beta) \geq 0$, which is a contradiction.

(iv) ⇒ (i) If we assume that $f(\beta) > 0$ implies $f(\alpha) > 0$ for every $f \in A$, then every open neighbourhood $\widetilde{U}(f_1, \dots, f_n)$ of β contains α. Therefore, $\beta \in \overline{\{\alpha\}}$. □

A subset relation between two prime cones of A yields the same relation between their supports.

Lemma 4.22. *For $\alpha, \beta \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ the following are equivalent.*

(i) $\alpha \subsetneq \beta$.
(ii) $\mathrm{supp}(\alpha) \subsetneq \mathrm{supp}(\beta)$.

Proof. (ii) ⇒ (i) Suppose there is an element $f \in \mathrm{supp}(\alpha) \setminus \mathrm{supp}(\beta)$. Then $f(\alpha) = 0$ and $f(\beta) \neq 0$. We may assume $f(\beta) > 0$. Since $\alpha \subset \beta$, then $f(\alpha) > 0$ by Proposition 4.21, which is a contradiction.

(i) ⇒ (ii) Assume there exists an element $f \in \alpha \setminus \beta$. Then $f(\alpha) \geq 0$ and $-f(\beta) > 0$ since $f \in (-\beta) \setminus \beta$. We consider two cases depending on whether or not $f \in -\alpha$.

- If $f \in \alpha \cap (-\alpha)$, then $f(\alpha) = 0$ implies that $f(\beta) = 0$ which is a contradiction as $f(\beta) < 0$.
- Suppose $f \in \alpha \setminus (-\alpha)$, or equivalently, $f(\alpha) > 0$. Let us prove that in this case $\beta \not\subset \alpha$. If β was a subset of α, then, by Proposition 4.21, $-f(\beta) > 0$ implies that $-f(\alpha) > 0$, which is a contradiction. Therefore, $\beta \not\subset \alpha$ and $\alpha \not\subset \beta$ but then also $\mathrm{supp}(\alpha) \not\subset \mathrm{supp}(\beta)$, which is a contradiction. □

Proposition 4.23. *(i) The specializations of a point $\alpha \in \mathrm{Spec_R(A)}$ form a totally ordered chain under inclusion, i.e., if $\alpha \subset \beta$ and $\alpha \subset \gamma$, then either $\beta \subset \gamma$ or $\gamma \subset \beta$.*

(ii) Every point $\alpha \in \mathrm{Spec_R(A)}$ is contained in a unique maximal specialization, i.e., there exists a unique closed point $\beta \in \mathrm{Spec_R^m(A)}$ such that $\alpha \subset \beta$.

Proof. (i) We shall prove the statement by contradiction, [2, Proposition 7.1.23]. Suppose $\beta \not\subset \gamma$ and $\gamma \not\subset \beta$. Then there exist $f \in \beta \setminus \gamma$ and $g \in \gamma \setminus \beta$. Since $A = \alpha \cup -\alpha$ by Proposition 4.3, then either $f-g \in \alpha \subset \gamma$ or $g-f \in \alpha \subset \beta$. In both cases we obtain a contradiction since either $f = g+(f-g) \in \gamma$ or $g = f+(g-f) \in \beta$.

(ii) By (i), all specializations $\beta_1, \beta_2, \dots$ of a point $\alpha \in \mathrm{Spec_R(A)}$ form a totally ordered chain under inclusion. Since all β_i are prime cones, it can be readily seen that the union $\cup_{i=1}\beta_i$ is a prime cone, which is also an upper bound for the chain of specializations. By Zorn's lemma, there exists a maximal specialization $\cup_{i=1}^{n}\beta_i$, [1, Proposition 2.6 (ii)]. ☐

From Propositions 4.21 and 4.23 (ii) we deduce that the subspace $\mathrm{Spec_R^m(A)}$ of closed points of $\mathrm{Spec_R(A)}$ consists precisely of maximal prime cones of A. In order to prove that it is a compact Hausdorff space, we consider the following lemma.

Lemma 4.24. *Let $\alpha, \beta \in \mathrm{Spec_R(A)}$. The following are equivalent.*

(i) $\alpha \not\subset \beta$ and $\beta \not\subset \alpha$.

(ii) $\alpha, \beta \in \mathrm{Spec_R(A)}$ can be separated, i.e. there exists $f \in A$ such that $\alpha \in \widetilde{U}(f)$ and $\beta \in \widetilde{U}(-f)$.

(iii) There exist two open disjoint sets U_α, U_β such that $\alpha \in U_\alpha$ and $\beta \in U_\beta$.

Proof. The proof of this lemma is taken from [12, Lemma 4.6].

(i) $\Rightarrow$ (ii) Let $f \in \alpha \setminus \beta$ and $g \in \beta \setminus \alpha$. We shall prove that $h := f - g \in A$ is the desired element. Recall that $A = \alpha \cup -\alpha = \beta \cup -\beta$ by Proposition 4.3. If $-h \in \alpha$, then $-h + f = g \in \alpha$, which is a contradiction. Thus $h \in \alpha \setminus -\alpha \Leftrightarrow \alpha \in \widetilde{U}(f)$. Similarly, if $h \in \beta$, then $h + g = f \in \beta$, which is a contradiction. Thus $-h \in \beta \setminus -\beta \Leftrightarrow \beta \in \widetilde{U}(-f)$.

(ii) $\Rightarrow$ (iii) Clearly, $U_\alpha := \widetilde{U}(f)$ and $U_\beta = \widetilde{U}(-f)$ are the desired open disjoint sets.

(iii) $\Rightarrow$ (i) Since the open neighbourhood U_α of α does not intersect $\{\beta\}$, we conclude that $\beta \not\subset \alpha$ by Proposition 4.21. Similarly, $\alpha \not\subset \beta$. ☐

Proposition 4.25. *The subspace $\mathrm{Spec_R^m(A)}$ of closed points of $\mathrm{Spec_R(A)}$ is a compact Hausdorff space.*

Proof. We have seen that two points $\alpha, \beta \in \mathrm{Spec_R^m(A)}$ correspond to maximal prime cones, i.e., $\alpha \not\subset \beta$ and $\beta \not\subset \alpha$. Then α and β can be separated by Lemma 4.24, and $\mathrm{Spec_R^m(A)}$ is a Hausdorff space. To prove compactness, suppose $\mathcal{U} := \cup_{i\in I}U_i$ is an open cover of $\mathrm{Spec_R^m(A)}$ and let $\alpha \in \mathrm{Spec_R(A)}$. By Lemma 4.23 (ii), there exists a unique maximal specialization $\beta \in \mathrm{Spec_R^m(A)}$ containing α, which is contained in an open set U_j of the cover $\mathcal{U}$. Since $\beta \in \overline{\{\alpha\}}$, we have that $U_j \cap \{\alpha\} \neq \emptyset$, or equivalently, $\alpha \in U_j$. Therefore $\mathcal{U}$ is also a cover of the real spectrum $\mathrm{Spec_R(A)}$. Since $\mathrm{Spec_R(A)}$ is compact, a finite subfamily of $\mathcal{U}$ already covers it and hence also $\mathrm{Spec_R^m(A)}$, [1, Proposition 2.7]. ☐

We shall see that the topology on $\mathrm{Spec_R^m(A)}$ is induced by a continuous retraction $r : \mathrm{Spec_R(A)} \longrightarrow \mathrm{Spec_R^m(A)}$. The assertion of the following lemma is used to prove the continuity of the map r.

Lemma 4.26. *Let $\alpha \in \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ and let S be a closed subset of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ with $\alpha \notin S$. Then α and S can be separated, i.e., there exist two open neighbourhoods $U_\alpha, U_S \subset \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ of α and S respectively, such that $U_\alpha \cap U_S = \emptyset$. In particular, there exist $f_1, \dots, f_n \in A$ such that $U_\alpha = \widetilde{U}(f_1, \dots, f_n)$ and $U_S = \widetilde{U}(-f_1) \cup \cdots \cup \widetilde{U}(-f_n)$.*

Proof. We imitate the proof that compact Hausdorff spaces are regular, [12, Lemma 4.9]. Since $\{\alpha\}$ and S are closed subsets of the compact Hausdorff space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$, they are also compact. Thus, α and S can be separated. To prove the second part of the lemma, choose an element $\beta \in S$. Since $\{\alpha\} \cap S = \emptyset$, we have that $\alpha \not\subset \beta$ and $\beta \not\subset \alpha$ which is, by Lemma 4.24, equivalent to $\alpha \in \widetilde{U}(f_\beta)$ and $\beta \in \widetilde{U}(-f_\beta)$ for some $f_\beta \in A$. Then the union $\cup_{\beta \in S} \widetilde{U}(-f_\beta)$ covers S which is compact. Therefore there exist a finite collection $f_1, \dots, f_n$ of elements in A such that $S \subset \cup_{i=1}^{n} \widetilde{U}(-f_i)$ and also $\alpha \in \cap \widetilde{U}(f_i)$. ☐

Proposition 4.27. *The map*

$$\begin{aligned} r : \mathrm{Spec}_{\mathrm{R}}(\mathrm{A}) &\longrightarrow \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A}) \\ \alpha &\longmapsto \textit{unique maximal specialization of } \alpha \end{aligned}$$

is a continuous retraction of $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ onto $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$.

Proof. Let $\alpha \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ and let β be its unique maximal specialization contained in an open set U of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$. To prove continuity, we have to find an open neighbourhood V of α in $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ such that $r(V) \subseteq U$, [12, Proposition 4.10]. The complement of U in $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$, denoted by S, is a closed subset of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ not containing β. By Lemma 4.26, β and S can be separated by two open disjoint sets U_β and U_S containing β and S, respectively. Since $\beta \in \overline{\{\alpha\}}$, we have that $\alpha \in U_\beta$. We shall prove that $r(U_\beta) \subseteq U$. Assume there exists $\alpha' \in U_\beta$ such that $r(\alpha') = \beta' \notin U$. Then $\beta' \in U^C = S \subset U_S$. As $\beta' \in \overline{\{\alpha'\}}$ implies that $\alpha' \in U_S$, we have that $\alpha' \in U_\beta \cap U_S = \emptyset$, which is a contradiction. Hence $r(U_\beta) \subseteq U$ and r is a continuous retraction. ☐

Proposition 4.28. *The space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ is endowed with the quotient topology induced by the map r.*

Proof. We shall prove that the topology of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$ is given by the family of closed sets $\{S \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A}) \mid r^{-1}(S) \text{ is closed in } \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})\}$, [12, Proposition 4.10]. If S is a closed subset of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$, then $r^{-1}(S)$ is closed since r is continuous. Assume now $r^{-1}(S)$ is closed in $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ for some subset S of $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathrm{A})$. Then $r^{-1}(S)$ is compact and hence $r(r^{-1}(S)) = S$ is compact since r is a continuous surjection. As S is a compact subset of a Hausdorff space, it is necessarily closed. ☐

Closed points of $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ can be characterized in another way when they are regarded as homomorphisms from the ring A to a real closed field F rather than as prime cones of the commutative ring A.

Proposition 4.29. *Assume a point $\alpha \in \mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$ is regarded as a homomorphism $\alpha : A \longrightarrow F(\alpha)$. Then the following are equivalent.*

- *(i) The point α is closed.*
- *(ii) Every element $x \in F(\alpha)$ is bounded by an element of the subring $\alpha(A)$, i.e., $|x| < f(\alpha)$ for some $f \in A$.*

Proof. ($\Rightarrow$) Assume α is a closed point and suppose there exists an element $m \in F(\alpha)$ such that $m > a$ for all $a \in \alpha(A)$. Since the field $F(\alpha)$ is Archimedean over the field of fractions $k(\operatorname{supp}(\alpha))$ by Lemma 4.9, we may write $m = \frac{1}{b}$ for some $b \in \alpha(A)$, $b > 0$. Let us define

$$\begin{aligned} \mathcal{O}_{\frac{1}{b}} &:= \{x \in F(\alpha) \mid |x| < (1/b)^l \text{ for some } l \in \mathbb{Z}\}, \\ \mathcal{I}_{\frac{1}{b}} &:= \{x \in F(\alpha) \mid |x| < (1/b)^l \text{ for all } l \in \mathbb{Z}\}. \end{aligned}$$

It can be readily seen that $\mathcal{O}_{\frac{1}{b}}$ is a ring and $\mathcal{I}_{\frac{1}{b}}$ its ideal as any integer is bounded by m. By assumption, $\alpha(A) \subset \mathcal{O}_{\frac{1}{b}}$. Let us prove that $\alpha(A) \cap \mathcal{I}_{\frac{1}{b}} \neq (0)$. Since all elements of $\alpha(A)$ are bounded by the element $\frac{1}{b}$, we have that $b < \frac{1}{b}$. In particular, $b^n < \frac{1}{b}$ for all $n \in \mathbb{Z}$, or equivalently, $b < \left(\frac{1}{b}\right)^n$. Then $b \in \alpha(A) \cap \mathcal{I}_{\frac{1}{b}}$ and thus $\alpha(A) \cap \mathcal{I}_{\frac{1}{b}} \neq (0)$. We consider the following homomorphism

$$\psi : A \longrightarrow A/\operatorname{supp}(\alpha) \longrightarrow F(\alpha) \longrightarrow \mathcal{O}_{\frac{1}{b}}/\mathcal{I}_{\frac{1}{b}},$$

where $\mathcal{O}_{\frac{1}{b}}/\mathcal{I}_{\frac{1}{b}}$ is an ordered field. Therefore, the preimage $\mathfrak{p}$ of the ideal $\mathcal{I}_{\frac{1}{b}}$ under the map ψ is a real prime ideal containing $\operatorname{supp}(\alpha)$. By Proposition 4.8, the set $\beta := \{a \in A \mid \psi(a) \geq 0\}$ is a prime cone of A satisfying $\operatorname{supp}(\beta) = \mathfrak{p}$. Hence, $\operatorname{supp}(\alpha) \subsetneq \operatorname{supp}(\beta)$ which is equivalent to $\alpha \subsetneq \beta$ by Lemma 4.22. Since this implies that α is not a closed point, we obtain a contradiction.

($\Leftarrow$) Assume α is not closed, i.e., $\alpha \subsetneq \beta \subset A$. By Lemma 4.22, this implies that $\alpha \cap -\alpha \subsetneq \beta \cap -\beta \subset A$ and hence there exists an element $t \in \beta \cap -\beta \setminus \alpha \cap -\alpha$. We obtain the following commutative diagram

$$\begin{tikzcd} A \arrow[d, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\ A/\operatorname{supp}(\alpha) \arrow[r, "\bar{\varphi}"'] & A/\operatorname{supp}(\beta), \end{tikzcd}$$

i.e., $\bar{\varphi} \circ \varphi_\alpha = \varphi_\beta$. It is clear that $\varphi_\alpha(t) \neq 0$. Suppose $\varphi_\alpha(t) > 0$. By assumption, all elements of $F(\alpha)$ are bounded by the elements of the subring $\alpha(A)$, and therefore there exists $a \in \alpha(A)$ such that

$$\frac{1}{\varphi_\alpha(t)} < \varphi_\alpha(a) \iff 1 < \varphi_\alpha(t) \cdot \varphi_\alpha(a).$$

Applying the map $\bar{\varphi}$, we obtain

$$1 < \varphi_\beta(t) \cdot \varphi_\beta(a) = 0,$$

which is a contradiction since $F(\alpha)$ is an ordered field. $\square$

Example 4.30. Let $X \subseteq \mathbb{R}^n$ be a real algebraic set and $x \in X$. By Example 4.12, the point $x \in X$ can be regarded as a homomorphism $\alpha_x : \mathcal{A}(X) \longrightarrow \mathbb{R}$, and it can therefore be seen as a point in $\operatorname{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. Since such a homomorphism is surjective, we conclude that x is a closed point in $\operatorname{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ by Proposition 4.29, [3, Section 2].

4.3. **Compactification of semi-algebraic sets.** In this section we study the real spectrum compactification $\widetilde{W}$ of a closed semi-algebraic subset W of a real algebraic set $X \subseteq \mathbb{R}^n$, defined as the closure of W in the compact Hausdorff space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. We will see that $\widetilde{W}$ is closely related to constructible sets discussed in Section 4.2. Namely, we will associate the subset W to a unique constructible subset C_W of $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, and show that the real spectrum compactification $\widetilde{W}$ of W is then exactly the intersection of C_W with the space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$.

We already know that X is contained in $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ by Example 4.30. Let W be any subspace of X. The closure $\widetilde{W} := \mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W$ of W in $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ is a compact Hausdorff space since $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ is a compact Hausdorff space. Moreover, W is a dense subset of $\widetilde{W}$ because by the definition of the subspace topology

$$\mathrm{Cl}_{\widetilde{W}} W = \mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W \cap \widetilde{W} = \widetilde{W}.$$

Let us recall that for $f, g \in \mathcal{A}(X)$ Example 4.12 implies that

$$\widetilde{U}(f) \cap X = U(f) = \{x \in X \subset \mathbb{R}^n \mid f(x) > 0\}$$
$$\widetilde{Z}(g) \cap X = Z(g) = \{x \in X \subset \mathbb{R}^n \mid g(x) = 0\}.$$

We denote the collection of constructible subsets of $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ by $\mathscr{C}$ and the collection of semi-algebraic subsets of X by $\mathscr{S}$.

Proposition 4.31. *The map*

$$\begin{aligned} \tau : \mathscr{C} &\longrightarrow \mathscr{S} \\ C &\longmapsto X \cap C \end{aligned}$$

is bijective. Moreover, it preserves unions, intersections and inclusions.

The proof of Proposition 4.31 is based on the following important theorem whose proof can be found in [2, Theorem 4.1.2].

Theorem 4.32 (Artin-Lang)**.** *Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $\{f_i\}, \{g_j\}$ be finite collections of elements of $\mathcal{A}(X) = \mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X)$. Assume there exists an ordered field F and a homeomorphism $\alpha : \mathcal{A}(X) \longrightarrow F$ for which $f_i(\alpha) > 0$ and $g_j(\alpha) = 0$ for all i, j. Then there is a point $x \in X \subset \mathbb{R}^n$ such that $f_i(x) > 0$ and $g_j(x) = 0$ for all i, j.*

Proof of Proposition 4.31. A short computation implies that τ preserves unions, intersections and inclusions. We shall prove that it is bijective, [3, Section 3]. Given a semi-algebraic subset of the form $W = \{x \in X \mid f_1(x) > 0, \ldots, f_p(x) > 0, g(x) = 0\} \subset X$, we have seen above that $\widetilde{U}(f_1, \ldots, f_p) \cap \widetilde{Z}(g)$ is the corresponding constructible subset of $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, i.e., $X \cap (\widetilde{U}(f_1, \ldots, f_p) \cap \widetilde{Z}(g)) = W$. Thus, τ is surjective. To prove injectivity, it suffice to prove that $C \neq \emptyset$ implies $X \cap C \neq \emptyset$. If $C \neq \emptyset$, then it is a finite union of sets of the form $\{f_1 > 0, \ldots, f_p > 0, g = 0\}$. Since a point $\alpha \in C \subset \mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ can be regarded as a homomorphism $\alpha : \mathcal{A}(X) \longrightarrow F(\alpha)$ satisfying the conditions $f_1(\alpha) > 0, \ldots, f_p(\alpha) > 0, g(\alpha) = 0$, Artin-Lang Theorem 4.32 implies that α corresponds to a point $x \in X \subset \mathbb{R}^n$ such that $f_1(x) > 0, \ldots, f_p(x) > 0$ and $g(x) = 0$. By Example 4.12, every point x of the real algebraic set X can be regarded as a homomorphism $\alpha_x \in \mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. Hence, $\alpha_x \in X \cap C \neq \emptyset$. $\square$

Proposition 4.33. *Let C be a constructible set of* $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ *and $X \cap C$ the associated semi-algebraic subset of X. Then $X \cap C$ is dense in C.*

Proof. Let α be an arbitrary point in $C \subset \mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$ and let $\widetilde{U}(h_1, \dots, h_r)$ be a basic open neighbourhood of α in C. We shall prove that it intersects $X \cap C$. Recall that a constructible set C is of the form $C = \cup_{i=1}^n \left(\widetilde{U}(f_{1,i}, \dots, f_{p_i,i}) \cap \widetilde{Z}(g_i)\right)$. Then we compute

$$\begin{aligned} C \cap \widetilde{U}(h_1, \dots, h_r) &= \left[\cup_{i=1}^n \left(\widetilde{U}(f_1, \dots, f_{p_i}) \cap \widetilde{Z}(g_i)\right)\right] \cap \widetilde{U}(h_1, \dots, h_r) \\ &= \cup_{i=1}^n \left[\left(\widetilde{U}(f_{1,i}, \dots, f_{p_i,i}) \cap \widetilde{Z}(g_i)\right) \cap \widetilde{U}(h_1, \dots, h_r)\right] \\ &= \cup_{i=1}^n \left[\widetilde{U}(f_{1,i}, \dots, f_{p_i,i}, h_1, \dots, h_r) \cap \widetilde{Z}(g_i)\right]. \end{aligned}$$

Hence, the set $C \cap \widetilde{U}(h_1, \dots, h_r)$ is constructible and nonempty since $\alpha \in C \cap \widetilde{U}(h_1, \dots, h_r)$. Following the same conclusion as in the proof of Proposition 4.31, the condition $C \cap \widetilde{U}(h_1, \dots, h_r) \neq \emptyset$ implies that $X \cap C \cap \widetilde{U}(h_1, \dots, h_r) \neq \emptyset$. Since every open neighbourhood of $\alpha \in C$ intersects $X \cap C$, the set $X \cap C$ is dense in C, [3, Section 3]. $\square$

Theorem 4.34. *The map τ preserves open and closed sets, i.e., a semi-algebraic subset $W \subset X$ is open (resp. closed) in X if and only if the corresponding constructible set C_W is open (resp. closed) in the real spectrum topology of* $\mathrm{Spec_R}(\mathcal{A}(\mathrm{X}))$.

We shall first state the following important theorem, called Finiteness Theorem, since it is used in the proof of Theorem 4.34. Its proof can be found in [2, Theorem 2.7.2].

Theorem 4.35 (Finiteness Theorem). *Every open semi-algebraic subset $W \subset X$ can be written as a finite union of basic open sets*

$$W = \cup_{i=1}^n X \cap \widetilde{U}(f_{i,1}, \dots, f_{i,p_i}),$$

where $X \cap \widetilde{U}(f_{i,1}, \dots, f_{i,p_i}) = \{x \in X \mid f_{i,j} > 0, 1 \leq j \leq p_i\}$.

Proof of Theorem 4.34. Let W be a semi-algebraic subset of X and C_W the associated constructible set. It suffices to prove that the map τ preserves open sets. Namely, if $W \subset X$ is a closed semi-algebraic subset, then applying Finiteness Theorem 4.35 to the open set $X \setminus W$, we obtain that $X \setminus W = \cup_{i=1}^n X \cap \widetilde{U}(f_{i,1}, \dots, f_{i,p_i})$. Therefore, the set $W = \cup_{i=1}^n \{x \in X \mid -f_{1,i}(x) \geq 0, \dots, -f_{p_i,i}(x) \geq 0\}$ is a finite union of basic closed sets $X \cap \widetilde{W}(g_{j,i}) := \{x \in X \mid g_{1,i}(x) \geq 0, \dots, g_{p_i,i}(x) \geq 0\}$. Let us now prove the theorem for open sets.

($\Leftarrow$) Suppose W is an open subset of X. It can then be written as $W = \cup_{i=1}^n X \cap \widetilde{U}(f_{1,i}, \dots, f_{p_i,i})$ by Finiteness Theorem 4.35. Then Artin-Lang Theorem 4.32 implies that the set $C_W = \cup_{i=1}^n \widetilde{U}(f_{1,i}, \dots, f_{p_i,i})$ is the corresponding constructible set. Since it is a finite union of basic open sets $\widetilde{U}(f_{1,i}, \dots, f_{p_i,i})$, it is also open.

($\Rightarrow$) Assume C_W is an open constructible set and let $\mathcal{U} = \cup_i \widetilde{U}(f_{i,1}, \dots, f_{i,p_i})$ be an open cover of C_W consisting of basic open sets. By Proposition 4.19 (ii), the set C_W is compact in the real spectrum topology. Hence, there exists a finite subcover $\cup_{i=1}^n \widetilde{U}(f_{i,1}, \dots, f_{i,p_i})$ of $\mathcal{U}$ which already covers C_W. Therefore, the semi-algebraic set $W = \tau(C_W) = X \cap C_W = \cup_{i=1}^n X \cap \widetilde{U}(f_{i,1}, \dots, f_{i,p_i})$ is a finite union of basic open sets and hence open. $\square$

The properties of the map τ are crucial for characterization of the compactification $\widetilde{W}$ and its boundary $\widetilde{W} \setminus W$ which we study in the remainder of the section.

Proposition 4.36. *The closure* $\widetilde{W} = \mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W$ *of a closed semi-algebraic subset* $W \subset X$ *in the space* $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ *is of the form*

$$\widetilde{W} = C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})),$$

where C_W *is the constructible set associated to the semi-algebraic set* W*.*

Proof. Let $W \subset X$ be a closed semi-algebraic set. By Theorem 4.34, the corresponding constructible set C_W is closed in $\mathrm{Spec}_{\mathrm{R}}(\mathrm{A})$, and therefore the set $C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ is closed in $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. Applying the closure operator $\mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))}$ to

$$W \subset C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})) \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})),$$

we obtain

$$\mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W = \widetilde{W} \subset C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})) \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))\,.$$

Since the sets $\widetilde{W}$ and $C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ are both closed in $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, we conclude that $\widetilde{W}$ is closed in $C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. The subset relation

$$W \subset C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})) \subset C_W$$

implies that

$$\begin{aligned}\mathrm{Cl}_{C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W &= \mathrm{Cl}_{C_W} W \cap C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})) \\ &= C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))\end{aligned}$$

since W is dense in the associated constructible set C_W by Proposition 4.33. Next, we apply the closure operator $\mathrm{Cl}_{C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))}$ to the subsets

$$W \subset \widetilde{W} \subset C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})),$$

which yields

$$\mathrm{Cl}_{C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))} W = C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X})) \subset \widetilde{W} \subset C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))\,.$$

Hence, $\widetilde{W} = C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$. $\square$

Remark 4.37. Recall that a closed semi-algebraic set $W \subset X$ is a finite union a basic closed sets $W = \cup_{i=1}^{n} X \cap \overline{W}(g_{1,i}, \ldots, g_{p_i,i}) = \cup_{i=1}^{n} W_i$. The compactification of $\widetilde{W}$ is the union of $\cup_{i=1}^{n} \widetilde{W_i}$.

The set $\widetilde{W} \setminus W$ is called the *boundary of the real spectrum compactification* of W, denoted by $B(\widetilde{W})$. Points of the boundary $B(\widetilde{W})$ are called *boundary points* of the compactification $\widetilde{W}$, and are characterized in the following proposition.

Proposition 4.38. *Let* $W \subset X$ *be a closed semi-algebraic subset. Then a point* $\alpha : \mathcal{A}(X) = \mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X) \longrightarrow F(\alpha)$ *of* $\widetilde{W}$ *is a boundary point if and only if it satisfies the condition* $\sum x_i^2(\alpha) - r > 0$ *for all* $r \in \mathbb{R}$*. Equivalently,* α *is a boundary point if and only if there exists an element* $|x_i(\alpha)| \in F(\alpha)$ *which is infinitely large relative to* $\mathbb{R} \subset F(\alpha)$*, i.e.,* $|x_i(\alpha)| \geq r$ *for all* $r \in \mathbb{R}$*.*

Proof. Consider $\alpha \in \widetilde{W} = C_W \cap \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$ as a homomorphism $\alpha : \mathcal{A}(X) = \mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X) \longrightarrow F(\alpha)$. Assume that each $|x_i(\alpha)| \in F(\alpha)$ is bounded by a real number. Then every element of $\alpha(\mathcal{A}(X)) \subseteq F(\alpha)$ is bounded by a real number. Since α is a closed point, every element of the field $F(\alpha)$ is also bounded by an element of the subring $\alpha(\mathcal{A}(X))$ by Proposition 4.29. Therefore, for each $m \in F(\alpha)$ there exists and element $a \in \alpha(\mathcal{A}(X))$ such that

$$|m| \leq a,$$

but there also exists an element $r \in \mathbb{R}$ such that

$$|m| \leq a \leq r.$$

Hence, the ordered field $F(\alpha)$ is Archimedean. Since every Archimedean ordered field is isomorphic to a subfield of $\mathbb{R}$, [21], and $F(\alpha)$ contains the field of real numbers $\mathbb{R}$, we conclude that $F(\alpha) = \mathbb{R}$. Therefore, the point $\alpha : \mathcal{A}(X) \longrightarrow \mathbb{R}$ is also an element of X by Example 4.12. Thus, $\alpha \in X \cap C_W = W$. $\square$

Remark 4.39. Let $\widetilde{W} = \cup_{i=1}^{n} \widetilde{W_i}$ be the real spectrum compactification of W, where $W_i = X \cap \widetilde{W}(g_{1,i}, \ldots, g_{p_i,i})$. Clearly, a point $\alpha : \mathcal{A}(X) \longrightarrow F(\alpha)$ of $\widetilde{W_i}$ satisfies $g_{j,i}(\alpha) \geq 0$ for all $1 \leq j \leq p_i$. Proposition 4.38 implies that it also satisfies exactly one of the following properties:

(i) $F(\alpha) = \mathbb{R}$,

(ii) $F(\alpha)$ is non-Archimedean. Equivalently, there exists k, $1 \leq k \leq n$, such that the element $|x_k(\alpha)|$ is infinitely large relative to $\mathbb{R}$, i.e., $|x_k(\alpha)| > r$ for every $r \in \mathbb{R}$. Such points are exactly the boundary points of $\widetilde{W}$.

Corollary 4.40. *The set W is open in its real spectrum compactification $\widetilde{W}$.*

Proof. Let $\alpha : \mathcal{A}(X) \longrightarrow \mathbb{R}$ be an element of W. Since $\alpha \notin \widetilde{W} \setminus W$, for every $i = 1, \ldots, n$ there exists $r_i \in \mathbb{R}$ such that

$$|x_i(\alpha)| < r_i \text{ for all } x_i \in \mathbb{R}[x_1, \ldots, x_n].$$

Assume there exist indices k and l, $1 \leq l \leq k \leq n$ such that $x_i(\alpha) < 0$ for all $i = 1, \ldots, l$, $x_i(\alpha) = 0$ for all $i = l+1, \ldots, k$, and $x_i(\alpha) > 0$ for all $i = k+1, \ldots, n$. Clearly, if $x_i(\alpha) = 0$, then $(x_i + x_j)(\alpha) \neq 0$ for some $x_j(\alpha) \neq 0$. Assume, for example, $x_j(\alpha) > 0$. Let

$$\widetilde{U} := \widetilde{U}(-x_1, \ldots, -x_l, x_{l+1} + x_j, \ldots, x_k + x_j, x_{k+1}, \ldots, x_n)$$

be a neighbourhood of α and let $\beta \in \widetilde{U}$. We shall prove that all $|x_i(\beta)|$ are bounded by real numbers, which implies that $\beta \in W$. Assume there exists an index m such that $|x_m(\beta)| > r$ for all $r \in \mathbb{R}$. In all of the following cases we obtain a contradiction since $\beta \in \widetilde{U}$.

(i) If $m \in \{1, \ldots, l\}$, then, in particular, $x_m(\beta) > 0 \iff -x_m(\beta) < 0$.

(ii) If $m \in \{l, \ldots, k\}$, then, in particular, $-x_m(\beta) > x_j(\beta)$. This means that $(x_m + x_j)(\beta) < 0$.

(iii) If $m \in \{k+1, \ldots, n\}$, then, in particular, $-x_m(\beta) > 0 \iff x_m(\beta) < 0$.

Hence, all $|x_i(\beta)|$ are bounded and $\beta \in W$. This proves that W is an open subset of $\widetilde{W}$. $\square$

5. Teichmüller space as a semi-algebraic set

In Section 3.3 we have defined the Teichmüller space $\mathrm{T}_g(\Sigma)$ of a closed connected oriented hyperbolic surface Σ of genus $g \geq 2$ as the quotient space $\mathrm{Hyp}(\Sigma)/\operatorname{Diff}_0^+(\Sigma)$ of the set of complete hyperbolic metrics on Σ by the group of orientation-preserving diffeomorphisms of Σ isotopic to the identity id_Σ. The goal of this section is to identify $\mathrm{T}_g(\Sigma)$ with a closed semi-algebraic set and then apply the theory of Section

4.3 to obtain the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ of $\mathrm{T}_g(\Sigma)$. To this end, we first embed $\mathrm{T}_g(\Sigma)$ into the space of homomorphisms from its fundamental group $\pi_1(\Sigma)$ to the group $\mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}$, also called the space of representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}(2,\mathbb{R})$. This embedding is discussed in Section 5.1. However, we will see that Teichmüller space can also be seen as a subset of the space of representation of $\pi_1(\Sigma)$ in $\mathrm{SL}(2,\mathbb{R})$ rather than in $\mathrm{PSL}(2,\mathbb{R})$, which is a crucial observation for identification of $\mathrm{T}_g(\Sigma)$ with a real semi-algebraic subset discussed in Section 5.2.

5.1. **Teichmüller space of representations.** Let Σ be a closed connected oriented hyperbolic surface Σ of genus $g \geq 2$ and let $\pi_1(\Sigma, p_0)$ denote its first fundamental group with a base point $p_0 \in \Sigma$. We have seen in Section 3.2 that Σ can be written as a quotient $\mathbb{H}/\operatorname{Deck}_\pi(\mathbb{H})$ of the hyperbolic plane $\mathbb{H}$ by the group of covering transformations $\operatorname{Deck}_\pi(\mathbb{H})$, where $\pi : \mathbb{H} \longrightarrow \Sigma$ denotes the universal cover of Σ. Since the group $\operatorname{Deck}_\pi(\mathbb{H})$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ isomorphic to the fundamental group $\pi_1(\Sigma, p_0)$, we obtain a homomorphism $\pi_1(\Sigma, p_0) \to \operatorname{Deck}_\pi(\mathbb{H}) \subset \mathrm{PSL}(2,\mathbb{R})$ called a representation of $\pi_1(\Sigma, p_0)$ in $\mathrm{PSL}(2,\mathbb{R})$. In this section, we will show that every point of Teichmüller space $\mathrm{T}_g(\Sigma)$, defined in Section 3.3, can be associated to a conjugacy class of a discrete faithful representation of $\pi_1(\Sigma, p_0)$ in $\mathrm{PSL}(2,\mathbb{R})$.

Throughout this section Σ denotes a fixed closed connected oriented hyperbolic surface Σ of genus $g \geq 2$. Let $p_0 \in \Sigma$ be a fixed base point of its fundamental group $\pi_1(\Sigma, p_0)$, which we typically omit from notation when there is no confusion.

Definition 5.1. Let Γ be a group and F a topological group. A homomorphism $\Gamma \longrightarrow F$ is called a *representation* of Γ in F. The space of all such representations is denoted by $\operatorname{Hom}(\Gamma, F)$ and endowed with the topology of pointwise convergence.

Let us recall that the subbasis of the topology of point-wise convergence is given by the sets $S(\gamma, U) = \{\rho \in \operatorname{Hom}(\Gamma, F) \mid \rho(\gamma) \in U\}$ for a point $\gamma \in \Gamma$ and an open set $U \subset F$. Moreover, a sequence of homomorphisms $\{\rho_i : \Gamma \longrightarrow F\}_{i=1}^\infty$ converges to a homomorphism $\rho : \Gamma \longrightarrow F$ if and only if for all $\gamma \in \Gamma$ the sequence of images $\{\rho_i(\gamma)\}_{i=1}^\infty$ converges to $\rho(\gamma)$ in F.

In this section, we will be studying representations satisfying additional properties.

Definition 5.2. A representation $\rho \in \operatorname{Hom}(\Gamma, F)$ is said to be

(i) *faithful* if it is injective, and
(ii) *discrete* if the image $\rho(\Gamma)$ is discrete in F.

From now on we additionally assume that for a discrete faithful representation ρ of Γ in F, the quotient $F/\rho(\Gamma)$ is compact. Let us denote the space of all such representations by $\operatorname{DHom}(\Gamma, F)$. Let $\operatorname{Aut}(F)$ be the group of automorphisms of F and $\operatorname{Inn}(F)$ its subgroup of inner automorphisms. Recall that an inner automorphism of F is of the form $x \longmapsto axa^{-1}$ for an element $a \in F$ and all $x \in F$. The group $\operatorname{Aut}(F)$ acts on $\operatorname{Hom}(\Gamma, F)$ in the following way. For every $f \in \operatorname{Aut}(F)$ we obtain a map

$$\begin{aligned} \operatorname{Hom}(\Gamma, F) &\longrightarrow \operatorname{Hom}(\Gamma, F) \\ \rho &\longmapsto f \circ \rho. \end{aligned}$$

We introduce the notations

$$\mathcal{T}(\Gamma, F) = \operatorname{Hom}(\Gamma, F)/\operatorname{Inn}(F)$$

for the orbit space of the induced action of $\operatorname{Inn}(F)$ on $\operatorname{Hom}(\Gamma, F)$, and

$$\mathcal{DT}(\Gamma, F) = \operatorname{DHom}(\Gamma, F)/\operatorname{Inn}(F)$$

for the orbit space of the induced action of $\operatorname{Inn}(F)$ on $\operatorname{DHom}(\Gamma, F)$. In fact, $\mathcal{DT}(\Gamma, F)$ is the space of conjugate classes of discrete faithful representations of Γ in F, and it is exactly the image of $\operatorname{DHom}(\Gamma, F)$ under the projection $\operatorname{Hom} \longrightarrow \mathcal{T}$. If we endow $\mathcal{T}(\Gamma, F)$ with the quotient topology, then this projection is open since it is a quotient projection under the action of the group $\operatorname{Inn}(F)$.

In our case, the group Γ is the first fundamental group $\pi_1(\Sigma)$ of the surface Σ, and the topological group F is the group $\mathrm{PSL}(2, \mathbb{R})$. Our goal for the remainder of this section is to prove that there exists an injective map

$$\mathrm{T}_g(\Sigma) \longrightarrow \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R})),$$

taking Teichmüller space $\mathrm{T}_g(\Sigma) = \operatorname{Hyp}(\Sigma)/\operatorname{Diff}_0^+(\Sigma)$, defined in Section 3.3, onto the space of conjugate classes of discrete faithful representations of $\pi_1(\Sigma)$ in the group $\mathrm{PSL}(2, \mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}$. The construction of the map follows the one in [4, Section 2.1].

Let us consider the group $\operatorname{Diff}(\Sigma)$ of diffeomorphisms of the surface Σ and its normal subgroup $\operatorname{Diff}^+(\Sigma)$ consisting of orientation-preserving diffeomorphisms. Recall that the action of both $\operatorname{Diff}(\Sigma)$ and $\operatorname{Diff}^+(\Sigma)$ on the set $\operatorname{Hyp}(\Sigma)$ of complete hyperbolic metrics on Σ is defined via the pullback

$$\begin{aligned} \operatorname{Diff}^+(\Sigma) \times \operatorname{Hyp}(\Sigma) &\longrightarrow \operatorname{Hyp}(\Sigma) \\ (f, g) &\longmapsto f^*(g) \end{aligned}$$

Let $\widetilde{\Sigma}$ be the universal cover of Σ with a basepoint $\widetilde{p}_0 \in \widetilde{\Sigma}$. Since $\widetilde{\Sigma}$ is isometric to $\mathbb{H}$, for every $h \in \operatorname{Hyp}(\widetilde{\Sigma})$ there exists an orientation-preserving isometry

$$f_h : (\widetilde{\Sigma}, \widetilde{p}_0) \longrightarrow (\mathbb{H}, i),$$

which is not canonical. However, once we fix a base tangent vector $v \in T_{\widetilde{p}_0}\widetilde{\Sigma}$ such that the image $df_h(v) = \mathbb{R}^+ e$ for $e = 1 \in \mathbb{C}$, the isometry f_h is uniquely determined. By definition of a pullback of a metric, any diffeomorphism $\varphi \in \operatorname{Diff}^+(\widetilde{\Sigma})$ is an orientation-preserving isometry between the hyperbolic metrics $\varphi^*(h)$ and h, for any $h \in \operatorname{Hyp}(\widetilde{\Sigma})$. Thus, we obtain a map

$$c : \operatorname{Diff}^+(\widetilde{\Sigma}) \times \operatorname{Hyp}(\widetilde{\Sigma}) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$$

defined by

$$c(\varphi, h) := f_h \circ \varphi \circ f^{-1}_{\varphi^*(h)},$$

which is an element of $\mathrm{PSL}(2, \mathbb{R})$ since it is a composition of orientation-preserving isometries and hence an orientation-preserving isometry of $\mathbb{H}$. In addition, the map c satisfies the following *cocycle relation*

$$\begin{aligned} c(\varphi_1\varphi_2, h) &= f_h \circ \varphi_1\varphi_2 \circ f^{-1}_{(\varphi_1\varphi_2)^*(h)} \\ &= f_h \circ \varphi_1 \circ f^{-1}_{\varphi_1^*(h)} \circ f_{\varphi_1^*(h)} \circ \varphi_2 \circ f^{-1}_{\varphi_2^*(\varphi_1^*(h))} \\ &= c(\varphi_1, h)c(\varphi_2, \varphi_1^*(h)). \end{aligned}$$

Let us denote the covering projection from $\widetilde{\Sigma}$ to Σ by $\pi : \widetilde{\Sigma} \longrightarrow \Sigma$. We have seen that $\pi_1(\Sigma, p_0)$ is isomorphic to the group of covering transformations

$$\mathrm{Deck}_\pi(\widetilde{\Sigma}) = \{T_\gamma : \gamma \in \pi_1(\Sigma, p_0)\} < \mathrm{Diff}^+(\widetilde{\Sigma}).$$

Let $\mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}}$ denote the set of $\mathrm{Deck}_\pi(\widetilde{\Sigma})$-invariant elements in $\mathrm{Hyp}(\widetilde{\Sigma})$,

$$\mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}} := \{h \in \mathrm{Hyp}(\widetilde{\Sigma}) : T_\gamma^*(h) = h \text{ for every } T_\gamma \in \mathrm{Deck}_\pi(\widetilde{\Sigma})\}.$$

Consider the map

$$\begin{aligned} \mathrm{Hyp}(\Sigma) &\longrightarrow \mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}} \\ g &\longmapsto \pi^*(g), \end{aligned}$$

where π^* denotes the pullback along the covering map π. Clearly, $\pi^*(g)$ is a $\mathrm{Deck}_\pi(\widetilde{\Sigma})$-invariant metric on $\widetilde{\Sigma}$ since $(T_\gamma)^*\pi^*(g) = (\pi \circ T_\gamma)^*(g) = \pi^*(g)$. Using the definitions, one readily sees that this map is bijective. Then for every $h \in \mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}}$ we obtain a map

$$\begin{aligned} \rho_h : \pi_1(\Sigma) &\longrightarrow \mathrm{PSL}(2, \mathbb{R}) \\ \gamma &\longmapsto c(T_\gamma, h) = f_h \circ T_\gamma \circ f_h^{-1}. \end{aligned}$$

In fact, the cocycle identity implies that ρ_h is a homomorphism since

$$\begin{aligned} \rho_h(\gamma_1 \cdot \gamma_2) &= c(T_{\gamma_1} \circ T_{\gamma_2}, h) \\ &= c(T_{\gamma_1}, h) c(T_{\gamma_2}, (T_{\gamma_1})^*(h)) \\ &= c(T_{\gamma_1}, h) c(T_{\gamma_2}, h) \\ &= \rho_h(\gamma_1) \cdot \rho_h(\gamma_2). \end{aligned}$$

With respect to the map ρ_h, the isometry $f_h : \widetilde{\Sigma} \to \mathbb{H}$ is a $\mathrm{Deck}_\pi(\widetilde{\Sigma})$-equivariant map as the equality $f_h(T_\gamma \widetilde{p}) = \rho_h(\gamma) f_h(\widetilde{p})$ holds by definition. Since the map $\rho_h \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))$ for every $h \in \mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}}$, we can define a map

$$\begin{aligned} \theta : \mathrm{Hyp}(\Sigma) &\longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R})) \\ g &\longmapsto \rho_{\pi^*(g)}. \end{aligned}$$

In the remainder of this section we shall prove that this map descends to a map $\delta : \mathrm{Hyp}(\Sigma)/\,\mathrm{Diff}_0^+(\Sigma) \longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))/\,\mathrm{Inn}(\mathrm{PSL}(2, \mathbb{R}))$, where $\mathrm{Diff}_0^+(\Sigma)$ denotes the group of orientation-preserving diffeomorphisms of Σ isotopic to id_Σ.

Let $\mathcal{N}^+$ denote the normalizer of $\mathrm{Deck}_\pi(\widetilde{\Sigma})$ in $\mathrm{Diff}^+(\widetilde{\Sigma})$,

$$\mathcal{N}^+ = \{\varphi \in \mathrm{Diff}^+(\widetilde{\Sigma}) \mid \varphi \,\mathrm{Deck}_\pi(\widetilde{\Sigma}) \varphi^{-1} = \mathrm{Deck}_\pi(\widetilde{\Sigma})\}.$$

Consider the following diagram

$$\begin{tikzcd} \{1\} \arrow[r] & \mathrm{Deck}_\pi(\widetilde{\Sigma}) \arrow[r, hook, "i"] & \mathcal{N}^+ \arrow[r, "\lambda"] \arrow[d, "a"] & \mathrm{Diff}^+(\Sigma) \arrow[r] & \{1\} \\ & & \mathrm{Aut}(\mathrm{Deck}_\pi(\widetilde{\Sigma})). & & \end{tikzcd}$$

The map i is the inclusion of $\mathrm{Deck}_\pi(\widetilde{\Sigma})$ into $\mathcal{N}^+$ and therefore injective. In order to define the map λ, let us first observe that a map $\varphi \in \mathcal{N}^+$ only permutes the fibers of the covering projection π because covering transformations T_γ preserve them. Hence, the map λ is defined to associate every $\varphi \in \mathcal{N}^+$ with the diffeomorphism

$\pi \circ \varphi \circ \pi^{-1} \in \mathrm{Diff}^+(\Sigma)$. Let us prove it is well-defined. Let $p \in \Sigma$ and let $\widetilde{p}_1, \widetilde{p}_2 \in \pi^{-1}(p)$ be its two lifts. Since $\widetilde{\Sigma}$ is connected, there exists $T_\gamma \in \mathrm{Deck}_\pi(\widetilde{\Sigma})$ such that $T_\gamma \widetilde{p}_1 = \widetilde{p}_2$. Then

$$\varphi(\widetilde{p}_2) = (\varphi T_\gamma)\widetilde{p}_1 = (T_{\gamma'}\varphi)\widetilde{p}_1$$

since $\varphi \in \mathcal{N}^+$. This implies that $\varphi(\widetilde{p}_2)$ is an element of the orbit $\mathrm{Deck}_\pi(\widetilde{\Sigma})\varphi(\widetilde{p}_1)$ and hence $\pi(\varphi(\widetilde{p}_1)) = \pi(\varphi(\widetilde{p}_2))$. Moreover, the map λ is surjective. Namely, as the universal cover $\widetilde{\Sigma}$ is path-connected and locally path-connected, every diffeomorphism $\psi \in \mathrm{Diff}^+(\Sigma)$ can be lifted to a diffeomorphism $\varphi : \widetilde{\Sigma} \to \widetilde{\Sigma}$, i.e., $\pi \circ \varphi = \psi \circ \pi$. This equality implies that $\pi \circ \varphi(\widetilde{p}) = \pi \circ \varphi(T_\gamma \widetilde{p})$ for every $\widetilde{p} \in \widetilde{\Sigma}$ and $T_\gamma \in \mathrm{Deck}_\pi(\widetilde{\Sigma})$. Then $\varphi T_\gamma(\widetilde{p})$ is an element of the orbit $\mathrm{Deck}_\pi(\widetilde{\Sigma})\varphi(\widetilde{p})$, which implies that $\varphi \in \mathcal{N}^+$. Therefore, the line in the above diagram is exact.

The map a is defined as

$$\begin{aligned} a : \mathcal{N}^+ &\longrightarrow \mathrm{Aut}(\mathrm{Deck}_\pi(\widetilde{\Sigma})) \\ \varphi &\longmapsto (a_\varphi : T_\gamma \longmapsto \varphi T_\gamma \varphi^{-1}). \end{aligned}$$

Observe that there exists an element $a_\varphi(\gamma) \in \pi_1(\Sigma)$ such that $\varphi T_\gamma \varphi^{-1} = T_{a_\varphi(\gamma)} \in \mathrm{Deck}_\pi(\widetilde{\Sigma})$ since $\varphi \in \mathcal{N}^+$. A straightforward calculation also show that a_φ is an isomorphism.

Lemma 5.3. *The equality*

$$\rho_h(a_\varphi(\gamma)) = c(\varphi, h)\rho_{\varphi^*(h)}(\gamma)c(\varphi, h)^{-1}$$

holds for every $\varphi \in \mathcal{N}^+$, $h \in \mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}}$ and $\gamma \in \pi_1(\Sigma)$.

Proof. We first observe that

$$c(\varphi, h)^{-1} = c(\varphi^{-1}, \varphi^*(h)).$$

Secondly, using the fact that $\varphi \in \mathcal{N}^+$, we obtain the following equality

$$(\varphi T_\gamma)^*(h) = (T_{\gamma'}\varphi)^*(h) = \varphi^*(T_{\gamma'})^*(h) = \varphi^*(h)$$

since $h \in \mathrm{Hyp}(\widetilde{\Sigma})^{\mathrm{Deck}}$. These observations and the definition of the map ρ_h imply that

$$\begin{aligned} c(\varphi, h)\rho_{\varphi^*(h)}(\gamma)c(\varphi, h)^{-1} &= c(\varphi, h)c(T_\gamma, \varphi^*(h))c(\varphi^{-1}, \varphi^*(h)) \\ &= c(\varphi T_\gamma, h)c(\varphi^{-1}, \varphi^*(h)) \\ &= c(\varphi T_\gamma, h)c(\varphi^{-1}, (\varphi T_\gamma)^*(h)) \\ &= c(\varphi T_\gamma \varphi^{-1}, h) \\ &= c(a_\varphi(T_\gamma), h) \\ &= \rho_h(a_\varphi(\gamma)), \end{aligned}$$

which is the desired equality. □

Let $\mathcal{N}_i^+$ be the subgroup of $\mathcal{N}^+$ consisting of diffeomorphisms $\varphi \in \mathcal{N}^+$ for which $a_\varphi \in \mathrm{Inn}(\mathrm{Deck}_\pi(\widetilde{\Sigma}))$. The following lemma states that the map λ yields an identification of $\mathcal{N}_i^+$ with diffeomorphisms of Σ isotopic to the identity, [4, Proposition 2.2].

Lemma 5.4. *Let $\mathrm{Diff}_0^+(\Sigma)$ be the subgroup of $\mathrm{Diff}^+(\Sigma)$ consisting of diffeomorphisms of Σ isotopic to the identity. Then we obtain the isomorphism*

$$\mathrm{Diff}_0^+(\Sigma) \cong \mathcal{N}_i^+ / \mathrm{Deck}_\pi(\widetilde{\Sigma}).$$

Let us now consider again the map

$$\begin{aligned}\theta : \mathrm{Hyp}(\Sigma) &\longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))\\ g &\longmapsto \rho_{\pi^*(g)}.\end{aligned}$$

If we identify representations of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ which are conjugated in $\mathrm{PSL}(2,\mathbb{R})$, we obtain the map

$$\begin{aligned}\delta : \mathrm{Hyp}(\Sigma) &\longrightarrow \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))\\ g &\longmapsto [\rho_{\pi^*(g)}].\end{aligned}$$

Let us prove that δ is invariant under the action of the group $\mathrm{Diff}_0^+(S)$, i.e., $[\rho_{\pi^*(g)}] = [\rho_{\pi^*(f^*(g))}]$ for a diffeomorphism $f \in \mathrm{Diff}_0^+(\Sigma)$. By Lemma 5.4, we can associate f to a diffeomorphism $\varphi_f \in \mathcal{N}_i^+/\operatorname{Deck}_\pi(\widetilde{\Sigma})$ for which $f \circ \pi = \pi \circ \varphi_f$ and a_{φ_f} is an inner automorphism of $\pi_1(\Sigma)$. Therefore, it suffices to prove that $[\rho_{\pi^*(g)}] = [\rho_{(\pi\circ\varphi_f)^*(g))}]$. Applying Lemma 5.3 and using the fact that the map $\rho_{\pi^*(g)}$ is a homomorphism, we get that δ is invariant under the action of the group $\mathrm{Diff}_0^+(S)$. Hence, it yields the map

$$\delta' : \mathrm{Hyp}(\Sigma)/\mathrm{Diff}_0^+(\Sigma) \longrightarrow \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R})),$$

or equivalently,

$$\begin{aligned}\delta' : \mathrm{T}_g(\Sigma) &\longrightarrow \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))\\ [g] &\longmapsto [\rho_{\pi^*(g)}].\end{aligned}$$

Proposition 5.5. *Let Σ be closed connected oriented hyperbolic surface of genus $g \geq 2$. Then the map δ' is injective.*

Proof. Assume $g_1, g_2 \in \mathrm{Hyp}(\Sigma)$ are metrics for which the maps $\rho_{\pi^*(g_1)}$ and $\rho_{\pi^*(g_2)}$ are conjugated in $\mathrm{PSL}(2,\mathbb{R})$. Write $\rho_{\pi^*(g_1)}(\gamma) = \zeta \cdot \rho_{\pi^*(g_2)}(\gamma) \cdot \zeta^{-1}$ for some element $\zeta \in \mathrm{PSL}(2,\mathbb{R})$. By definition, this is equivalent to

$$T_\gamma = (f^{-1}_{\pi^*(g_1)} \circ \zeta \circ f_{\pi^*(g_2)}) \circ T_\gamma \circ (f^{-1}_{\pi^*(g_1)} \circ \zeta \circ f_{\pi^*(g_2)})^{-1}.$$

Hence, the map $\psi := f^{-1}_{\pi^*(g_1)} \circ \zeta \circ f_{\pi^*(g_2)}$ is an orientation-preserving isometry of $\widetilde{\Sigma}$, which is clearly also $\operatorname{Deck}_\pi(\widetilde{\Sigma})$-equivariant. Moreover, it sends $\pi^*(g_2)$ to $\pi^*(g_1)$. Since $\widetilde{\Sigma}$ admits a hyperbolic structure, for every point $\widetilde{p} \in \widetilde{\Sigma}$ there exists a unique geodesic joining $\psi(\widetilde{p})$ and $\widetilde{p}$. We now construct a geodesic homotopy $\psi_t : \widetilde{\Sigma} \times I \to \widetilde{\Sigma}$ from ψ to $\mathrm{id}_{\widetilde{\Sigma}}$ with the property that the equality in hyperbolic distances $\rho(\psi_t(\widetilde{p}), \widetilde{p}) = t \cdot \rho(\psi(\widetilde{p}), \widetilde{p})$ holds for every $t \in I$. Since this homotopy is $\operatorname{Deck}_\pi(\widetilde{\Sigma})$-equivariant, it yields a homotopy between $\lambda(\psi)$ and id_Σ. Therefore, the map $\lambda(\psi)$ is an orientation-preserving diffeomorphism of Σ homotopic to the identity, which in addition maps g_2 to g_1. □

Observe that the map δ' is not surjective. Let $\rho \in \mathrm{Im}(\delta') \subset \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ and let τ be an orientation-reversing isometry of $\mathbb{H}$. Since τ normalizes the group $\mathrm{PSL}(2,\mathbb{R})$, we can define a representation

$$\begin{aligned}\rho' : \pi_1(\Sigma) &\longrightarrow \mathrm{PSL}(2,\mathbb{R})\\ \gamma &\longmapsto \tau\rho(\gamma)\tau^{-1}.\end{aligned}$$

Clearly, $\rho' \in \mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$. If we write $\rho(\gamma) = f_{\pi^*(g_0)} T_\gamma f^{-1}_{\pi^*(g_0)}$ for some $[g_0] \in \mathrm{T}_g(\Sigma)$, then $\rho' \notin \mathrm{Im}(\delta')$ since $\tau f_{\pi^*(g_0)}$ is an orientation-reversing isometry.

The image $\mathrm{Im}(\delta')$ of the map δ' will be denoted by $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$.

5.2. **Embedding of Teichmüller space into $\mathbb{R}^M$.** We have seen that Teichmüller space $\mathrm{T}_g(\Sigma)$ of a closed connected oriented hyperbolic surface Σ of genus $g \geq 2$ can be seen as a subspace of the space $\mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))$. In this section, we use this identification to embed $\mathrm{T}_g(\Sigma)$ into $\mathbb{R}^M$ for some M large enough. The embedding is given by traces of matrices $\rho(\gamma)$, where ρ is a representation and γ an element of $\pi_1(\Sigma)$. However, in order to realize this embedding, it is more convenient to study representations of $\pi_1(\Sigma)$ in the group $\mathrm{SL}(2, \mathbb{R})$ rather than in $\mathrm{PSL}(2, \mathbb{R})$ as in Section 5.1. Therefore, we investigate the space of representations $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$, or more precisely, the connected components of the space $\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$. The outline and the proofs of this section are taken from [16, Section 4.11] unless states otherwise.

Our first goal is to prove that there exists a surjection taking $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$ onto the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))$. To this end, we study lifts of representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}(2, \mathbb{R})$. We shall see that every representation $\rho : \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ actually lifts to a representation $\widetilde{\rho} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbb{R})$.

Definition 5.6. Let G be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A subgroup $\widetilde{G} \subset \mathrm{SL}(2, \mathbb{R})$ is a *lift* of G to $\mathrm{SL}(2, \mathbb{R})$ if the restriction of the quotient projection $P : \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ to $\widetilde{G}$ is an isomorphism from $\widetilde{G}$ to G.

Similarly, we can define a lift of a representation.

Definition 5.7. Let $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))$ be a representation and let $P : \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ be the quotient projection. A representation $\widetilde{\rho} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbb{R})$ is a *lift* of ρ to $\mathrm{SL}(2, \mathbb{R})$ if $P \circ \widetilde{\rho} = \rho$.

It is clear that the fundamental group $\pi_1(\Sigma)$ of a compact oriented surface Σ of genus $g \geq 2$ is torsion-free, that is, it does not contain nonidentity elements of finite order. Since any discrete torsion-free subgroup of $\mathrm{PSL}(2, \mathbb{R})$ can be lifted to $\mathrm{SL}(2, \mathbb{R})$, [16, Theorem 3.22.1], we conclude that any representation $\rho : \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ lifts to a representation $\widetilde{\rho} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbb{R})$. This is true since its image $\rho(\pi_1(\Sigma)) \subset \mathrm{PSL}(2, \mathbb{R})$ is a discrete group and hence has a lift $\widetilde{G}$ in $\mathrm{SL}(2, \mathbb{R})$. Therefore, ρ can be viewed as a representation of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$. The construction of such a lifting is described in [16, Section 3.22]. A more general result states that a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ lifts to $\mathrm{SL}(2, \mathbb{R})$ if it does not contain any nontrivial element of order 2.

Conversely, since $\pi_1(\Sigma)$ is torsion-free, a faithful representation $\widetilde{\rho}$ of $\pi_1(\Sigma)$ in $\mathrm{SL}(2, \mathbb{R})$ necessarily projects to a faithful representation $\rho = P \circ \widetilde{\rho}$ in $\mathrm{PSL}(2, \mathbb{R})$. Injectivity of the obtained representation follows from the fact that the image $\widetilde{\rho}(\pi_1(\Sigma))$ does not contain $-\mathrm{I}$ which is the nontrivial element of $\ker(P) = \{\pm \mathrm{I}\}$. More precisely,

$$\begin{aligned} \ker(P \circ \widetilde{\rho}) &= \{\gamma \in \pi_1(\Sigma) \mid P(\widetilde{\rho}(\gamma)) = \mathrm{I}\} \\ &= \{\gamma \in \pi_1(\Sigma) \mid \widetilde{\rho}(\gamma) = \pm \mathrm{I}\}. \end{aligned}$$

Assume there exists a nontrivial element $\gamma \in \pi_1(\Sigma)$ such that $\widetilde{\rho}(\gamma) = -\mathrm{I}$. Then $\widetilde{\rho}(\gamma^2) = \mathrm{I}$ implies that $\gamma^2 \in \ker(\widetilde{\rho}) = \{1\}$. Hence $\gamma^2 = \mathrm{id}$ which contradicts the fact that the fundamental group is torsion-free.

Therefore, we have proved that the map

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R})) \longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))$$

is an open continuous surjection. By the above conclusion, discrete and faithful representations $\rho \in \mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ are preserved under this map and the space $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is mapped to $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$.

Our next step is to prove that $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ has finitely many connected components which are in bijective correspondence with the connected components of $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. To this end, let us first prove that $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is a real algebraic set, [7, Section 1]. Since Σ is a closed oriented surface of genus $g \geq 2$, its fundamental group admits a representation

$$\pi_1(\Sigma) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle,$$

where $[\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ denotes the commutator of α_i and β_i. A representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is therefore completely determined by its values on the generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$. Since the images $A_i = \rho(\alpha_i), B_i = \rho(\beta_i) \in \mathrm{SL}(2,\mathbb{R})$ of these generators satisfy the relation

$$\prod_{i=1}^{g} [A_i, B_i] = 1, \tag{4}$$

we conclude that the subspace of $2g$-tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying (4) represents an embedding of the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ into the space $\mathrm{SL}(2,\mathbb{R})^{2g}$. Moreover, since equation (4) is a matrix multiplication and hence a polynomial equation in variables $A_1, B_1, \dots, A_g, B_g$, the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is a real algebraic set.

In [20], H. Whitney proved that any real algebraic set has finitely many connected components. Using his result and the fact that the set $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is real algebraic, we conclude that the quotient space $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ has finitely many components. Let us prove that these components can be mapped bijectively to the components of the space $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$.

Lemma 5.8. *The connected components of* $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ *are in bijective correspondence with the connected components of* $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$.

Proof. Since the projection $\pi : \mathrm{Hom} \longrightarrow \mathcal{T}$ is continuous, the image of a connected component of $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ under π is connected. Therefore it remains to prove that the preimage $\pi^{-1}(C)$ of a connected component $C \subset \mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is connected. Suppose there exists a separation of $\pi^{-1}(C)$, that is, two nonempty disjoint open sets A and B such that $A \sqcup B = \pi^{-1}(C)$. We may assume $\pi(A) \cap \pi(B) \neq \emptyset$, otherwise these sets would separate the connected component C. Let $\rho \in A$ and $\sigma \in B$ be two representations that map to the same representation in $\pi(A) \cap \pi(B)$. Consider the orbit $\mathrm{SL}(2,\mathbb{R})\rho$ of the representation ρ. On one hand, it is connected since it is the image of the connected space $\mathrm{SL}(2,\mathbb{R})$ under the map $\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SL}(2,\mathbb{R})\rho$. On the other hand, the sets $\mathrm{SL}(2,\mathbb{R})\rho \cap A$ and $\mathrm{SL}(2,\mathbb{R})\rho \cap B$ are open and nonempty since $\pi(\rho) = \pi(\sigma)$, or equivalently, $\sigma \in \mathrm{SL}(2,\mathbb{R})\rho$ by assumption. Hence, they separate $\mathrm{SL}(2,\mathbb{R})\rho$, which is a contradiction. Therefore, $\pi^{-1}(C)$ is connected and we obtain a bijection between the connected components of $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ and that of $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. □

Let us denote by $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})$ the part of $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ projecting to $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Im}(\delta')$ which is the part of $\mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ corresponding to hyperbolic structures on Σ that agree with its orientation.

In order to embed Teichmüller space $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ into $\mathbb{R}^M$ as a semi-algebraic set, we need prove that $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is a union of components of $\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. The proof is based on the following theorem, [13, Proposition 3.1.6] or [16, Theorem 4.11.1].

Theorem 5.9. *Connected components of* $\mathrm{DHom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ *correspond to connected components of* $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$.

Let us recall that $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ can be embedded into the space $\mathrm{SL}(2,\mathbb{R})^{2g}$ which is locally connected. Thus, connected components of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ are closed and open. Now, using Theorem 5.9 and the fact that the projection $\mathrm{Hom} \longrightarrow \mathcal{T}$ is open, we conclude that the components of $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ correspond to components of $\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. This implies also that components of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ are components of $\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$.

Lemma 5.10. *Let* $\rho, \rho' \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ *be representations belonging to the same component of* $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ *and let* $[\alpha] \in \pi_1(\Sigma)$. *Then traces of matrices* $\rho([\alpha]), \rho'([\alpha]) \in \mathrm{SL}(2,\mathbb{R})$ *have the same sign.*

Proof. Suppose $\rho, \rho' \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ belong to the same component of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ and let $\alpha \in \pi_1(\Sigma)$ be an element for which the traces $\rho(\alpha)$, $\rho'(\alpha) \in \mathrm{SL}(2,\mathbb{R})$ have opposite sign. Since ρ, ρ' are contained in the same component of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$, there exists a continuous map

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) \\ t &\longmapsto [\rho_t : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R})] \end{aligned}$$

joining them, i.e., $\rho_0 = \rho$ and $\rho_1 = \rho'$. Applying the trace function of a matrix, we obtain a continuous map

$$\begin{aligned} [0,1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \mathrm{tr}(\rho_t(\alpha)). \end{aligned}$$

By assumption, this map changes sign on $[0,1]$ and therefore there exists $s \in [0,1]$ such that $\mathrm{tr}(\rho_s(\alpha)) = 0$.

Let us first prove that representations $\rho_t \in \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ correspond to hyperbolic Möbius transformations. For every $t \in [0,1]$ we have that $\rho_t(\pi_1(\Sigma))$ is a discrete group of isometries of $\mathbb{H}$ such that the quotient $\mathbb{H}/\rho_t(\pi_1(\Sigma))$ is compact. Then, by Theorem 3.17, all nonidentity elements of $\rho_t(\pi_1(\Sigma))$ are hyperbolic. Hence, the matrices $\rho_t(\alpha)$ correspond to hyperbolic Möbius transformations, and they are conjugate in $\mathrm{SL}(2,\mathbb{R})$ to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some $|\lambda| > 1$ by Proposition 3.8 (iii). Thus, for all $t \in [0,1]$ the absolute value $|\,\mathrm{tr}(\rho_t(\alpha))| > 2$, which is a contradiction. $\square$

Let us remark that two representations $\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ project to the same point $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ if and only if $\rho_1(\gamma) = \xi(\gamma)\rho_2(\gamma)$ for a function $\xi : \pi_1(\Sigma) \longrightarrow \{\pm 1\}$ and all $\gamma \in \pi_1(\Sigma)$. This observation is crucial in the proof of the following theorem.

Theorem 5.11. *The space* $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ *has finitely many connected components. Moreover, each component is homeomorphic to the Teichmüller space of* Σ.

Proof. The space $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ projects to $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ by definition, and we have seen that the projection is open and continuous. We have also proved that $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$, and hence $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$, have finitely many components. It remains to prove that two representations belonging to different connected components of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ have distinct images under the projection

$$\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) \longrightarrow \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R})).$$

If we suppose that $\rho, \sigma \in \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ project to the same representation of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$, then there exists a map $\xi : \pi_1(\Sigma) \longrightarrow \{\pm 1\}$ such that $\rho(\gamma) = \xi(\gamma)\sigma(\gamma)$ holds for all $\gamma \in \pi_1(\Sigma)$. Let $\gamma \in \pi_1(\Sigma)$ such that $\xi(\gamma) = -1$. Such an element clearly exists since $\rho \neq \sigma$. Applying the trace function, we obtain that $\mathrm{tr}(\rho(\gamma)) = -\mathrm{tr}(\sigma(\gamma))$. Then, by Lemma 5.10, ρ and σ do not belong to the same component of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. $\square$

Using the above results, we now embed Teichmüller space into $\mathbb{R}^M$, for M large enough, as a connected component of a real algebraic set [16, Section 4.12].

Let $\gamma_1, \ldots, \gamma_m$ be any collection of generators of $\pi_1(\Sigma)$. For $1 \leq j \leq m$ we define

$$I_m = \{(\nu_1, \ldots, \nu_j) \mid 1 \leq \nu_1 < \cdots < \nu_j \leq m\}$$

as the set of ordered j-tuples of natural numbers. The map

$$\begin{aligned} \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) &\longrightarrow \mathbb{R}^M \\ \rho &\longmapsto (\mathrm{tr}(\rho(\gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_j})))_{(\nu_1,\ldots,\nu_j)\in I_m} \end{aligned}$$

induces the map

$$h : \mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) \longrightarrow \mathbb{R}^M$$

since the trace of matrices remains unchanged under conjugation. Let introduce the notation

$$A = h(\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))),$$

and let $\mathcal{I}(A)$ be the ideal of polynomials vanishing on A. For $f \in \mathcal{I}(A)$ the equation $f(h(\rho)) = 0$ holds for every $\rho \in \mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$, which is equivalent to $f((\mathrm{tr}(\rho(\gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_j})))_{(\nu_1,\ldots,\nu_j)\in I_m}) = 0$. Therefore these polynomials are exactly the relations between the values of trace functions on $\pi_1(\Sigma)$. Let us denote the set of zeros of $\mathcal{I}(A)$ by

$$X_g = \{x \in \mathbb{R}^M \mid f(x) = 0,\ \forall f \in \mathcal{I}(A)\},$$

or equivalently,

$$X_g = \mathcal{Z}(\mathcal{I}(A)) \subset \mathbb{R}^M$$

with the notations of Section 2.2. It is not difficult to see that $h(\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})))$ maps into X_g. Using more general results on trace functions, [16, Appendix B], one can show that we actually obtain an equality

$$A = h(\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))) = X_g,$$

which is a nontrivial result and we will thus not prove it here. Moreover, it follows from [14, Theorem 3.4] that the affine coordinate ring $\mathcal{A}(X_g) = \mathbb{R}[x_1, \ldots, x_M]/\mathcal{I}(A)$ is in fact generated by the images of the map h. More precisely, the generators of $\mathcal{A}(X_g)$ are given by the elements $\{\mathrm{tr}(\rho(\gamma_{\nu_q}\ldots\gamma_{\nu_j})) \mid (\nu_1, \ldots, \nu_j) \in I_m\}$.

If we denote by

$$DX_g = h(\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))),$$

we obtain the following important theorem.

Theorem 5.12. *The map*

$$h : \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) \longrightarrow DX_g$$

is a homeomorphism. Moreover, the set DX_g is a union of components of the set X_g.

Proof. The map h is clearly surjective. It is also injective since $h([\rho]) = h([\sigma])$ implies that $\mathrm{tr}(\rho(\alpha_{\nu_1}\alpha_{\nu_2}\cdots\alpha_{\nu_j})) = \mathrm{tr}(\sigma(\alpha_{\nu_1}\alpha_{\nu_2}\cdots\alpha_{\nu_j}))$ for all $(\nu_1,\ldots,\nu_j) \in I_m$. Hence $\rho = A\sigma A^{-1}$ for a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ and $[\rho] = [\sigma]$. The map h is also continuous as the trace map

$$\begin{aligned} \pi_1(\Sigma) &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \mathrm{tr}(\rho(\gamma)) \end{aligned}$$

is continuous for every $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. A nontrivial fact is that the map h is actually a homeomorphism, which we shall not prove here. Since homeomorphism preserves the connected components, and since $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is a finite union of components of $\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$, we conclude that the set DX_g is a finite union of components of X_g. □

By Theorem 5.11, each component of $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ is homeomorphic to Teichmüller space $\mathrm{T}_g(\Sigma)$, which implies the following result.

Theorem 5.13. *Each connected component of the space DX_g is homeomorphic to the Teichmüller space $\mathrm{T}_g(\Sigma)$ of the surface Σ.*

The set DX_g is a finite union of connected components of X_g which are closed and open since $X_g \subset \mathbb{R}^M$ is a locally connected space. Therefore, DX_g is a finite union of closed sets, and hence a closed semi-algebraic subset of $X_g \subset \mathbb{R}^M$. We have the following situation

$$\mathrm{T}_g(\Sigma) \subset DX_g \subset X_g \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(X_g)),$$

where $\mathcal{A}(X_g) = \mathbb{R}[x_1,\ldots,x_M]/\mathcal{I}(X_g)$ denotes the affine coordinate ring of the real algebraic set X_g. Applying the theory of Section 4.3, we now see that a component of the real spectrum compactification of $\widetilde{DX_g}$ characterizes *the real spectrum compactification* $\widetilde{\mathrm{T}}_g(\Sigma)$ *of Teichmüller space* $\mathrm{T}_g(\Sigma)$.

6. Comparison of the real spectrum compactification with the Thurston compactification of Teichmüller space

There exists several nonequivalent compactifications of Teichmüller space $\mathrm{T}_g(\Sigma)$ of a closed oriented hyperbolic surface Σ of genus $g \geq 2$, which differ in obtaining the points at infinity $B(\mathrm{T}_g(\Sigma))$. Many of them depend on the choice of a point in $\mathrm{T}_g(\Sigma)$ which is often inconvenient for further studies. However, Thurston's compactification does not have this disadvantage. For each point in Teichmüller space, Thurston used the lengths of simple closed geodesics on Σ to embed $\mathrm{T}_g(\Sigma)$ to the infinite dimensional real projective space $\mathbb{P}^{\mathcal{S}}$, where the set $\mathcal{S}$ is defined as a subset of $\pi_1(\Sigma)$, [6]. The closure of the image of this embedding in $\mathbb{P}^{\mathcal{S}}$ then defines the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$ of Teichmüller space $\mathrm{T}_g(\Sigma)$. Instead of taking the lengths of simple closed geodesics on Σ, one can also consider the closely related number $\log(|\,\mathrm{tr}(\rho(\gamma))|)$ for a loop $\gamma \in \pi_1(\Sigma)$ and a representation $\rho \in \mathrm{T}_g(\Sigma)$. It turns out that such a map extends continuously to the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ by the same formula. This section provides detailed study of these facts. Its outline as well as the proofs are taken directly from [3, Section 7]. In Subsection 6.1, we discuss the real projective space $\mathbb{P}^{\mathcal{S}}$ and we define a logarithm map from a closed semi-algebraic set W to $\mathbb{P}^{\mathcal{S}}$. We show that such a map extends continuously to the real spectrum compactification $\widetilde{W}$ when the set $\mathcal{S}$ satisfy certain properties. We use these rather general results in Subsection 6.2 to show that there exists a continuous surjection from the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ to the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$.

Throughout the entire section, Σ will denote a fixed closed oriented hyperbolic surface Σ of genus $g \geq 2$.

6.1. **Non-Archimedean logarithms.** This is a preliminary section for a comparison of the Thurston compactification and the real spectrum compactification of Teichmüller space $\mathrm{T}_g(\Sigma)$ of the surface Σ. We will first define the real projective space $\mathbb{P}^{\mathcal{S}}$ for a set $\mathcal{S} \subset \mathcal{A}(X)$, where $\mathcal{A}(X)$ denotes the affine coordinate ring of a real algebraic set $X \subset \mathbb{R}^n$. Afterwards, we will take a closer look at a map $\theta : W \longrightarrow \mathbb{P}^{\mathcal{S}}$ taking a closed semi-algebraic set $W \subset X$ onto the real projective space $\mathbb{P}^{\mathcal{S}}$. The goal of this section is to prove that θ, realized via the logarithm, extends to a continuous map from the real spectrum compactification $\widetilde{W}$ to $\mathbb{P}^{\mathcal{S}}$. We shall see that the extension is given by the exact same formula as the initial map θ. However, since the boundary $B(\widetilde{W}) = \widetilde{W} \setminus W$ give rise to non-Archimedean fields and therefore infinitely large elements relative to $\mathbb{R}$, we shall first define the non-Archimedean logarithm and show it satisfies the usual properties. The extension of the logarithm will then yield the extension of θ.

Let W be a closed semi-algebraic subset of a real algebraic set X. We begin by defining the real projective space $\mathbb{P}^{\mathcal{S}}$. Let a set $\mathcal{S} \subset \mathcal{A}(X)$ be a family of elements in $\mathcal{A}(X)$ satisfying the following conditions.

(i) The set $\mathcal{S}$ contains all coordinate functions $x_i \in \mathcal{A}(X)$, $i = 1, \dots, n$.
(ii) $|f(x)| \geq 1$ for all $f \in \mathcal{S} \subset \mathcal{A}(X)$ and for all $x \in W \subset X$.
(iii) There exists at least one element $f \in \mathcal{S}$ such that $|f(x)| > 1$ for all $x \in W \subset X$.

We introduce an equivalence relation $\sim$ on the set $[0, \infty)^{\mathcal{S}} \setminus \{\vec{0}\}$, where $\vec{0}$ denotes the $\mathcal{S}$-tuple of zeros. We identify all positive scalar multiples of nonzero $\mathcal{S}$-tuples, i.e., $(a_f)_{f\in\mathcal{S}} \sim (b_f)_{f\in\mathcal{S}}$ if and only if there exists $\lambda > 0$ such that $(a_f)_{f\in\mathcal{S}} = \lambda(b_f)_{f\in\mathcal{S}}$. The obtained quotient space is called the *real projective space* and it is denoted by $\mathbb{P}^{\mathcal{S}} := ([0, \infty)^{\mathcal{S}} \setminus \{\vec{0}\})/\sim$.

Next, we consider the map

$$\begin{aligned} \theta : W &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ x &\longmapsto (\log_b(|f(x)|))_{f\in\mathcal{S}}. \end{aligned}$$

Let us prove it is well-defined. Choose an element $x \in W$ and let $b \in \mathbb{R}$, $b > 0$ be the base element of the logarithm. By definition, there exists an element $f \in \mathcal{S}$ such that $|f(x)| > 1$, and therefore the image $(\log_b(|f(x)|))_{f\in\mathcal{S}}$ of x is an element of $[0, \infty)^{\mathcal{S}} \setminus \{\vec{0}\}$. Moreover, the choice of the base element b of the logarithm is irrelevant since the change of base, given by the formula $\log_b(|f(x)|) = \log_b(b') \log_{b'}(|f(x)|)$, multiplies logarithms by a positive real number $\log_b(b')$. Then $\log_{b'}(|f(x)|)$ and

$\log_b(|f(x)|)$ define the same point in the quotient $\mathbb{P}^{\mathcal{S}}$, and we will hence omit the base element b from the notation $\log_b(|f(x)|)$.

Our goal is to extend the map $\theta : W \to \mathbb{P}^{\mathcal{S}}$ to the map $\widetilde{\theta} : \widetilde{W} \to \mathbb{P}^{\mathcal{S}}$ taking the real spectrum compactification $\widetilde{W}$ of the closed semi-algebraic set W onto $\mathbb{P}^{\mathcal{S}}$. Let us recall from Remark 4.39 that a point $\alpha \in W$ corresponds to a homomorphism $\alpha : \mathcal{A}(X) \to \mathbb{R}$, whereas a boundary point $\alpha \in \widetilde{W} \setminus W$ corresponds to a homomorphism $\alpha : \mathcal{A}(X) \to F(\alpha)$ for a non-Archimedean field $F(\alpha)$. In the later case, there exists an infinitely large element $|x_i(\alpha)| \in F(\alpha)$, $1 \le i \le n$, relative to $\mathbb{R} \subset F(\alpha)$ by Proposition 4.38. Therefore, we need to extend the logarithm to an arbitrary ordered field such that its base element can also be a big element defined below.

Definition 6.1. Let F be an ordered field. An element $b \in F$, $b > 0$, is a *big element* if for every $a \in F$ there exists an integer $m \ge 1$ such that $a < b^m$.

Definition 6.2. Let $b \in F$ be a big element and let F^+ denote the set of positive elements of an ordered field F. We define a *real valued logarithm* over F^+ as

$$\begin{aligned} \log_b : F^+ &\longrightarrow \mathbb{R} \\ a &\longmapsto \log_b(a), \end{aligned}$$

where $\log_b(a) \in \mathbb{R}$ is the Dedekind cut of $\mathbb{Q}$ given by

$$\frac{m'}{n} \le \log_b(a) \le \frac{m}{n} \tag{5}$$

if $b^{m'} \le a^n \le b^m$ for $n, m, m' \in \mathbb{Z}$ and $n > 0$.

Note that the equality in (5) is obtained if $\log_b(a) \in \mathbb{Q}$. The logarithm of Definition 6.2 satisfies the following usual properties.

Proposition 6.3. *Let b be a big element of an ordered field F. Then the following holds.*

- *(i)* $\log_b(b^m) = m$ *for all* $m \in \mathbb{Z}$.
- *(ii) If* $0 < a < a'$, *then* $\log_b(a) \le \log_b(a')$.
- *(iii) Let b' be another big element in F. Then* $\log_{b'}(b) > 0$ *and* $\log_{b'}(a) = \log_{b'}(b)\log_b(a)$.

Proof. (i) Since $b^m \le (b^m)^1 \le b^m$, we have that $m \le \log_b(b^m) \le m$ and hence $\log_b(b^m) = m$ for every $m \in \mathbb{Z}$.

(ii) As $0 < a < a'$, we have that $a^n < (a')^n$ for every $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be large enough, so that there exists an index $m \in \mathbb{Z}$ for which $a^{n_0} \le b^m \le (a')^{n_0}$. Let $m' \in \mathbb{Z}$ be such that $b^{m'} \le a^{n_0}$. Since b is a big element, there exists an index $p \in \mathbb{Z}$ such that $(a')^{n_0} \le b^p$. We obtain the following inequalities

$$b^{m'} \le a^{n_0} \le b^m \le (a')^{n_0} \le b^p,$$

which imply that $\frac{m'}{n_0} \le \log_b(a) \le \frac{m}{n_0} \le \log_b(a') \le \frac{p}{n_0}$.

(iii) Let $b^{m'} \le a^n \le b^m$ for some elements $m, m' \in \mathbb{Z}$ and $n \in \mathbb{N}$. Since b' is a big element, there exists an index $p \in \mathbb{Z}$ such that $b^m \le (b')^p$. Similarly, as b is a big element, we may assume $(b')^{p'} \le b^{m'}$ for some $p' \in \mathbb{Z}$. Then the following holds

$$(b')^{p'} \le b^{m'} \le a^n \le b^m \le (b')^p,$$

which imply that $\frac{p'}{m} \le \log_{b'}(b) \le \frac{p}{m}$ and also $\frac{p'}{m'} \le \log_{b'}(b) \le \frac{p}{m'}$. Since $\frac{m'}{n} \le \log_b(a) \le \frac{m}{n}$, we obtain that $\frac{p'}{n} \le \log_{b'}(b)\log_b(a) \le \frac{p}{n}$. As $\frac{p'}{n} \le \log_{b'}(a) \le \frac{p}{n}$, we conclude that $\log_{b'}(a) = \log_{b'}(b)\log_b(a)$. $\square$

Using the non-Archimedean logarithm of Definition 6.2, we now extend the map $\theta : W \to \mathbb{P}^{\mathcal{S}}$ to a continuous map $\widetilde{\theta} : \widetilde{W} \to \mathbb{P}^{\mathcal{S}}$ from the real spectrum compactification $\widetilde{W}$ of the closed semi-algebraic set W to the real projective space $\mathbb{P}^{\mathcal{S}}$. It turns out that the extension is given by the exact same formula.

Proposition 6.4. *The map*

$$\begin{aligned} \widetilde{\theta} : \widetilde{W} &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ \alpha &\longmapsto (\log(|f(\alpha)|))_{f\in\mathcal{S}} \end{aligned}$$

is a continuous extension of the map $\theta : W \longrightarrow \mathbb{P}^{\mathcal{S}}$, where $f(\alpha)$ is the image of $f \in \mathcal{S} \subset \mathcal{A}(X)$ under the homomorphism $\alpha : \mathcal{A}(X) \to F(\alpha)$.

Proof. We prove first prove $\widetilde{\theta}$ is well-defined. Recall that for $f, g \in \mathcal{A}(X)$ the following holds

$$\begin{aligned} \widetilde{U}(f) \cap X = U(f) = \{x \in X \subset \mathbb{R}^n \mid f(x) > 0\} \\ \widetilde{Z}(g) \cap X = Z(g) = \{x \in X \subset \mathbb{R}^n \mid g(x) = 0\}, \end{aligned}$$

which implies that

$$\widetilde{U}(|f|-1) \cap \widetilde{Z}(|f|-1) \cap X = U(|f|-1) \cap Z(|f|-1)$$

holds for $f \in \mathcal{A}(X)$. This means that if $|f(x)| \geq 1$ for all $x \in W$, then $|f(\alpha)| \geq 1$ for all $\alpha \in \widetilde{W}$, and hence $\log(|f(\alpha)|) \geq 0$ for all $\alpha \in \widetilde{W}$. The choice of the big base element is again irrelevant since, by Proposition 6.3 (iii), $\log_b(|f(\alpha)|)$ and $\log_{b'}(|f(\alpha)|)$ define the same point in $\mathbb{P}^{\mathcal{S}}$ for any two big elements $b, b' \in F(\alpha)$. Next, we prove that there exists an element $f \in \mathcal{S}$ such that $\log(|f(\alpha)|) \neq 0$ for every $\alpha \in \widetilde{W}$. It suffices to show that some element $|f(\alpha)| \in F(\alpha)$ is a big element in the non-Archimedean field $F(\alpha)$ since then $\log_{|f(\alpha)|} |f(\alpha)| = 1$ by Proposition 6.3 (i). This fact follows from Propositions 4.29 and 4.38. Namely, as $\alpha \in \widetilde{W}$ is closed, Proposition 4.29 implies that every element $x \in F(\alpha)$ is bounded by an element $f(\alpha)$ of the subring $\alpha(\mathcal{A}(X))$. Since $\mathcal{S}$ contains generators of $\mathcal{A}(X)$ and since there exists an integer $1 \leq k \leq n$ such that $|x_k(\alpha)| \in F(\alpha)$ is infinitely large relative to $\mathbb{R}$ by Proposition 4.38, the image $f(\alpha)$ is bounded by some power of $|x_k(\alpha)|$. Therefore, all elements of $F(\alpha)$ are bounded by some power of $|x_k(\alpha)|$ and $|x_k(\alpha)|$ is a big element. Thus, the map $\widetilde{\theta}$ is well-defined.

Next, we prove $\widetilde{\theta}$ is continuous. Let $r = 2 + \sum_{i=1}^n x_i^2 \in \mathcal{A}(X)$. Since $|x_k(\alpha)|$ is a big element for some k, $r(\alpha)$ is a big element. If we show that the map

$$\begin{aligned} \Theta : \widetilde{W} &\longrightarrow [0,\infty)^{\mathcal{S}} \setminus \{\vec{0}\} \\ \alpha &\longmapsto (\log(|f(\alpha)|)/\log(r(\alpha)))_{f\in\mathcal{S}}, \end{aligned}$$

is continuous, then the composition $\widetilde{\theta} = \pi \circ \Theta$, where $\pi : [0,\infty)^{\mathcal{S}} \setminus \{\vec{0}\} \to \mathbb{P}^{\mathcal{S}}$ is the quotient projection, is clearly continuous. Note that the quotient of logarithms is independent of the choice of the big base element. Namely, changing the base element only multiplies the logarithm by a positive constant by Proposition 6.3 (iii), which yields the same point in the projective space $\mathbb{P}^{\mathcal{S}}$. It suffices to show that every component of Θ is a continuous map, which means we need to prove that the map

$$\begin{aligned} \Theta_f : \widetilde{W} &\longrightarrow [0,\infty) \\ \alpha &\longmapsto \log(|f(\alpha)|)/\log(r(\alpha)). \end{aligned}$$

is continuous for an element $f \in \mathcal{S}$. For all $s \in [0, \infty)$ the intervals $[0, s)$ and (s, ∞) form a subbasis of the topology on $[0, \infty)$. Hence, we need to prove that $\Theta_f^{-1}((s, \infty))$ and $\Theta_f^{-1}([0, s))$ are open in $\widetilde{W}$ for every $s \in [0, \infty)$. Assume $r(\alpha)$ is the big base element of the logarithm. Observe that $\log_{r(\alpha)} r(\alpha) = 1$ by Proposition 6.3 (i). We first compute

$$\begin{aligned}
\Theta_f^{-1}([0, s]) &= \{\alpha \in \widetilde{W} \mid 0 \leq \log_{r(\alpha)} |f(\alpha)| \leq s, s \in \mathbb{R}_{\geq 0}\} \\
&= \{\alpha \in \widetilde{W} \mid 0 \leq \log_{r(\alpha)} |f(\alpha)| \leq \frac{m}{n}, \text{ for all } \frac{m}{n} \in \mathbb{Q}, \frac{m}{n} > s\} \\
&= \bigcap_{\frac{m}{n} > s, \frac{m}{n} \in \mathbb{Q}} \{\alpha \in \widetilde{W} \mid \log_{r(\alpha)} |f(\alpha)| \leq \frac{m}{n}\} \\
&= \bigcap_{\frac{m}{n} > s, \frac{m}{n} \in \mathbb{Q}} \{\alpha \in \widetilde{W} \mid |f(\alpha)|^n \leq r(\alpha)^m\}.
\end{aligned}$$

In the same way, we obtain

$$\Theta_f^{-1}([s, \infty)) = \bigcap_{\frac{m}{n} < s, \frac{m}{n} \in \mathbb{Q}} \{\alpha \in \widetilde{W} \mid r(\alpha)^m \leq |f(\alpha)|^n\}.$$

Let us prove that these sets are closed in $\widetilde{W}$. Each of the sets

$$\{\alpha \in \widetilde{W} \mid |f(\alpha)|^n \leq r(\alpha)^m\} = \{r^m - f^n \geq 0, r^m + f^n \geq 0\} \subset \widetilde{W},$$

is constructible and hence compact by Proposition 4.19 (ii). Therefore, the set $\Theta_f^{-1}([0, s])$ is compact because it is an intersection of compact sets. Since it is a compact space contained in the Hausdorff space $\widetilde{W} \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(\mathrm{X}))$, it is closed in $\widetilde{W}$. Similarly, we conclude that the set $\Theta_f^{-1}([s, \infty))$ is closed in $\widetilde{W}$.

As the sets $\Theta_f^{-1}((s, \infty))$ and $\Theta_f^{-1}([0, s))$ are complements of closed sets $\Theta_f^{-1}([0, s])$ and $\Theta_f^{-1}([s, \infty])$, respectively, they are open in $\widetilde{W}$, and the map Θ_f is continuous for every $f \in \mathcal{S}$. Therefore, the maps Θ and $\widetilde{\theta}$ are continuous. $\square$

6.2. **Comparison.** This section is devoted to a comparison of the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$ with the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ of Teichmüller space $\mathrm{T}_g(\Sigma)$. We will first study the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$ which can be defined as the closure of $\theta'(\mathrm{T}_g(\Sigma))$ in the real projective space $\mathbb{P}^{\mathcal{S}}$, where θ' is a logarithm map realized using lengths of simple closed geodesics on the hyperbolic surface Σ. In order to compare it with the real spectrum compactification, we shall consider a map $\theta : \mathrm{T}_g(\Sigma) \to \mathbb{P}^{\mathcal{S}}$ given by $\log(|\operatorname{tr}(\rho(\cdot)|)$ for a representation $\rho \in \mathrm{T}_g(\Sigma)$. Applying the theory of Section 6.1, we shall see that both maps θ and θ' extend to the maps from the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$, and that the extensions actually coincide at the boundary $B(\widetilde{\mathrm{T}}_g(\Sigma))$. The main goal of this section is to show that the extension of θ yields a surjective map taking the real spectrum compactification $\widetilde{\mathrm{T}}_g(\Sigma)$ onto the Thurston compactification $\hat{\mathrm{T}}_g(\Sigma)$.

Assume $\rho \in \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$ is the discrete faithful representation determining the hyperbolic structure on Σ. Let γ be a simple closed curve on Σ. By Theorem 3.17, the transformation $\rho(\gamma)$ is hyperbolic, and hence conjugate to a dilation $D : z \longmapsto \lambda^2 z$ for some $|\lambda| > 1$. Since conjugated representations determine the same point in $\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R}))$, we may assume $\rho(\gamma)$ is represented by the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. By Theorem 3.23, the free-homotopy class of the simple closed curve γ contains a unique simple closed geodesic whose length we denote by $l_\rho(\gamma)$. Recall from Section 3.2 that $l_\rho(\gamma)$ is equal to $2\log(\lambda)$.

Let us consider the relation between λ and the trace $\operatorname{tr}(\rho(\gamma))$ of $\rho(\gamma)$. Write $t = \operatorname{tr}(\rho(\gamma)) = \lambda + \lambda^{-1}$, or equivalently, $\lambda^2 - \lambda t + 1 = 0$. Solutions of this quadratic equation are given by

$$\lambda = \frac{t + \sqrt{t^2 - 4}}{2}, \lambda^{-1} = \frac{t - \sqrt{t^2-4}}{2}.$$

If λ and t are positive, then $t > 2$ and hence $\lambda > 1$. We obtain the following equalities

$$t = \lambda\left(1 + \frac{1}{\lambda^2}\right) \tag{6}$$

$$\lambda = t\left(\frac{1 + \sqrt{1 - \frac{4}{t^2}}}{2}\right), \tag{7}$$

which will be needed when defining the maps $\theta, \theta' : \mathrm{T}_g(\Sigma) \to \mathbb{P}^{\mathcal{S}}$.

Our goal is to define the Thurston compactification of Teichmüller space which turns out to be the closure of the image $\theta'(\mathrm{T}_g(\Sigma))$ for an appropriately defined map $\theta' : \mathrm{T}_g(\Sigma) \longrightarrow \mathbb{P}^{\mathcal{S}}$, where the set $\mathcal{S}$ satisfies the conditions of Section 6.1. For simplicity, we shall from now on omit Σ from the notation $\mathrm{T}_g(\Sigma)$. Recall from Section 5.2 that we have the following situation

$$\mathrm{T}_g \subset X_g \subset \mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(X_g)),$$

where T_g is a closed real semi-algebraic subset of the real algebraic set $X_g = h(\mathcal{T}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{R})))$. Therefore, the first condition on the set $\mathcal{S}$ is to contain the generators of the affine coordinate ring $\mathcal{A}(X_g)$ of the set X_g. However, we will define $\mathcal{S}$ as a subset of $\pi_1(\Sigma)$ and show that every element of so defined set $\mathcal{S}$ can be interpreted as a generator of $\mathcal{A}(X_g)$.

Let $[\mathbb{S}^1, \Sigma]$ be the set of homotopy classes of maps $\mathbb{S}^1 \longrightarrow \Sigma$ with no condition on base points. We refer to the fundamental group $\pi_1(\Sigma, p_0)$ as the set of homotopy classes of maps $\mathbb{S}^1 \longrightarrow \Sigma$, where the homotopies preserve the base point. By ignoring the base point, we obtain a natural map $\Psi : \pi(\Sigma, p_0) \longrightarrow [\mathbb{S}^1, \Sigma]$ which is surjective. Moreover, it yields a bijection from the set of conjugacy classes of $\pi_1(\Sigma, p_0)$ to $[\mathbb{S}^1, \Sigma]$. Recall that each free-homotopy class in $[\mathbb{S}^1, \Sigma]$ contains a unique simple closed geodesic by Theorem 3.23. We define the set $\mathcal{S}$ to be a subset of $\pi_1(\Sigma, p_0)$ containing an element of each conjugacy class of $\pi_1(\Sigma, p_0)$. For a fixed representation $\rho \in \mathrm{T}_g$ we then associate to each element $\gamma \in \mathcal{S}$ the trace $\operatorname{tr}(\rho(\gamma))$ which is, as we have seen in Section 5.2, a generator of the affine coordinate ring $\mathcal{A}(X_g)$. Then equation (7) implies that λ is also a generator of $\mathcal{A}(X_g)$.

Next, we need to show that $|f([\rho])| \geq 1$ for every $f \in \mathcal{S} \subset \mathcal{A}(X_g)$ and every $[\rho] \in \mathrm{T}_g$. Let ρ be a representative of the class $[\rho] \in \mathrm{T}_g$ and let $\gamma_1, \dots, \gamma_j$ be a collection of elements in $\mathcal{S}$. For $\epsilon_1, \dots, \epsilon_j \geq 1$ we compute

$$\begin{aligned} |\operatorname{tr}(\rho(\gamma_1))^{\epsilon_1} \operatorname{tr}(\rho(\gamma_2))^{\epsilon_2} \cdots \operatorname{tr}(\rho(\gamma_j))^{\epsilon_j}| &\geq |\operatorname{tr}(\rho(\gamma_1))|^{\epsilon_1} |\operatorname{tr}(\rho(\gamma_2))|^{\epsilon_2} \cdots |\operatorname{tr}(\rho(\gamma_j))|^{\epsilon_j} \\ &\geq 2^{\epsilon_1 + \cdots + \epsilon_j} \end{aligned}$$

since all $\rho(\gamma_k)$, $1 \leq k \leq j$, are hyperbolic transformations of $\mathbb{H}$. Hence, for every $f \in \mathcal{S} \subset \mathcal{A}(X_g)$ and every $[\rho] \in \mathrm{T}_g$ we have that $|f([\rho])| \geq 2$, and the set $\mathcal{S}$ satisfies all conditions of Section 6.1.

We now define the map θ' in the following way

$$\begin{aligned} \theta' : \mathrm{T}_g &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ [\rho] &\longmapsto \left(\frac{l_\rho(\gamma)}{2}\right)_{\gamma \in \mathcal{S}}. \end{aligned}$$

It turns our that θ' is an embedding, which we shall not prove here. For the proof, we refer to [6, Exposé 7].

Definition 6.5. Let Σ be a closed connected oriented surface Σ of genus $g \geq 2$. The *Thurston compactification* $\hat{\mathrm{T}}_g(\Sigma)$ of Teichmüller space $\mathrm{T}_g(\Sigma)$ is the closure $\overline{\theta'(\mathrm{T}_g(\Sigma)}$ of $\mathrm{T}_g(\Sigma)$ in the real projective space $\mathbb{P}^{\mathcal{S}}$.

Finally, we compare the Thurston compactification $\hat{\mathrm{T}}_g$ with the real spectrum compactification $\widetilde{\mathrm{T}}_g$. To this end, we define the map

$$\begin{aligned} \theta : \mathrm{T}_g &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ [\rho] &\longmapsto (\log(|\operatorname{tr}(\rho(\gamma))|))_{\gamma \in \mathcal{S}}. \end{aligned}$$

By Proposition 6.4, both θ and θ' extend to continuous maps $\widetilde{\theta}, \widetilde{\theta}' : \widetilde{\mathrm{T}}_g \longrightarrow \mathbb{P}^{\mathcal{S}}$ by the exact same formula. Let us prove that the extensions $\widetilde{\theta}$ and $\widetilde{\theta}'$ actually coincide on the boundary $B(\mathrm{T}_g) = \widetilde{\mathrm{T}}_g \setminus \mathrm{T}_g$. Let $[\rho] \in B(\mathrm{T}_g)$ be a fixed boundary point. We need to show that the equality

$$\frac{l_\rho(\gamma)}{2} = \log(|\operatorname{tr}(\rho(\gamma))|)$$

holds for every $\gamma \in \mathcal{S}$. Recall from Remark 4.39 that the boundary points are exactly homomorphisms $\mathcal{A}(X_g) \to F$ for a non-Archimedean field F. This implies that $|\operatorname{tr}(\rho(\gamma))| \in F$ and hence, by (7), $\lambda \in F$. As $1 < \lambda < t < 2\lambda$, Proposition 6.3 (ii) implies that

$$\log(\lambda) \leq \log(t) \leq \log(2\lambda) \leq C\log(\lambda) \in \mathbb{P}^{\mathcal{S}}$$

for a suitable positive constant $C \in (0, \infty)$. Since $\log(\lambda)$ and $C\log(\lambda)$ coincide in $\mathbb{P}^{\mathcal{S}}$, we conclude that $\log(t) = \log(\lambda) = \frac{l_\rho(\gamma)}{2}$. Thus, the maps θ and θ' coincide at the boundary $B(\mathrm{T}_g)$.

Proposition 6.6. *The map $\widetilde{\theta} : \widetilde{\mathrm{T}}_g \longrightarrow \mathbb{P}^{\mathcal{S}}$ yields a continuous surjection $\widetilde{\theta} : \widetilde{\mathrm{T}}_g \longrightarrow \hat{\mathrm{T}}_g$, extending the embedding $\theta : \mathrm{T}_g \longrightarrow \mathbb{P}^{\mathcal{S}}$.*

Proof. The map $\widetilde{\theta} : \widetilde{\mathrm{T}}_g \longrightarrow \hat{\mathrm{T}}_g$ is a continuous map from the compact space $\widetilde{\mathrm{T}}_g$ to the Hausdorff space $\hat{\mathrm{T}}_g = \mathrm{Cl}_{\mathbb{P}^{\mathcal{S}}}(\theta'(\mathrm{T}_g)) \subset \mathbb{P}^{\mathcal{S}}$. Therefore, $\widetilde{\theta}$ is closed. Since T_g is dense in $\widetilde{\mathrm{T}}_g$, the continuity of $\widetilde{\theta}$ implies that

$$\widetilde{\theta}(\widetilde{\mathrm{T}}_g) = \widetilde{\theta}(\mathrm{Cl}_{\widetilde{\mathrm{T}}_g}(\mathrm{T}_g)) \subset \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\widetilde{\theta}(\mathrm{T}_g)) = \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\theta'(\mathrm{T}_g))$$

since θ extends θ'. As $\widetilde{\theta}(\widetilde{\mathrm{T}}_g)$ is closed in $\hat{\mathrm{T}}_g$, we have that

$$\mathrm{Cl}_{\hat{\mathrm{T}}_g}(\theta'(\mathrm{T}_g)) = \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\widetilde{\theta}(\mathrm{T}_g)) \subset \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\widetilde{\theta}(\widetilde{\mathrm{T}}_g)) = \widetilde{\theta}(\widetilde{\mathrm{T}}_g).$$

Therefore, $\widetilde{\theta}(\widetilde{\mathrm{T}_g}) = \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\widetilde{\theta}(\mathrm{T}_g)) = \mathrm{Cl}_{\hat{\mathrm{T}}_g}(\theta'(\mathrm{T}_g)) = \hat{\mathrm{T}}_g$ as $\theta'(\mathrm{T}_g) \subset \hat{\mathrm{T}}_g \subset \mathbb{P}^{\mathcal{S}}$. Hence, $\widetilde{\theta}$ is surjective. $\square$

From Proposition 6.6, one can derive an important fact about the Thurston compactification which we only explain informally. A bit more work is required to see that the group $\mathrm{Out}(\pi_1(\Sigma))$ of outer automorphisms of the fundamental group $\pi_1(\Sigma)$ acts continuously on the real spectrum compactification $\widetilde{\mathrm{T}}_g$ as well as on the Thurston compactification $\hat{\mathrm{T}}_g$ of Teichmüller space. With respect to this action, the map $\widetilde{\theta}$ then becomes $\mathrm{Out}(\pi_1(\Sigma))$-equivariant. Using the Brouwer fixed point theorem and the fact that $\mathrm{Out}(\pi_1(\Sigma))$ coincides with the mapping class group of the surface Σ, one can prove that each element of $\mathrm{Out}(\pi_1(\Sigma))$ has at least one fixed point on $\widetilde{\mathrm{T}}_g \simeq \widetilde{\mathbb{R}}^{6g-6}$. Then the $\mathrm{Out}(\pi_1(\Sigma))$-equivariance of the map $\widetilde{\theta}$ implies that the same conclusion holds for the Thurston compactification $\hat{\mathrm{T}}_g$. Hence, we obtain the fixed point property for $\hat{\mathrm{T}}_g$ without proving first that $\hat{\mathrm{T}}_g$ is topologically a closed disc of dimensions $6g-6$ which is a nontrivial result.

7. Conclusion

The main objective of this thesis was to provide sufficient theoretical background to understand the real spectrum compactification of a closed semi-algebraic set, and to apply this theory to the Teichmüller space associated to a closed oriented surface Σ of genus $g \geq 2$. Such a surface can be endowed with a Riemannian metric of constant negative Gaussian curvature, called a hyperbolic metric, which gave rise to a geometric model of Teichmüller space T_g described as a set of isotopy classes of hyperbolic metrics on Σ. Using the Fenchel-Nielsen coordinates, we saw that T_g is homeomorphic to $\mathbb{R}^{6g-6}$. However, since we were interested in its real spectrum compactification, we also presented an algebraic model of Teichmüller space. Namely, we constructed an injective map taking T_g onto the space $\mathcal{DT}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ of conjugate classes of discrete faithful representations of the fundamental group $\pi_1(\Sigma)$ in the group $\mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$. This was the first step towards identifying Teichmüller space with a closed semi-algebraic set. Instead of working with representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}(2,\mathbb{R})$, we lifted them to representations in the group $\mathrm{SL}(2,\mathbb{R})$, and showed that T_g is homeomorphic to a connected component of the space $\mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. Associating to each such representation $\rho \in \mathcal{DT}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$ the trace of a matrix $\rho(\gamma)$ for an element $\gamma \in \pi_1(\Sigma)$, we embedded T_g into the real space $\mathbb{R}^M$. In fact, this embedding also identified Teichmüller space with a closed semi-algebraic subset of a real algebraic set X_g, which enabled us to study its real spectrum compactification constructed in the following way. We took a closer look at the set of prime cones $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(X_g))$ of the affine coordinate ring $\mathcal{A}(X_g) = \mathbb{R}[x_1, \ldots, x_M]/\mathcal{I}(X_g)$ associated to the set X_g. We proved that it is a topological space whose subspace $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(X_g))$, consisting of closed points of $\mathrm{Spec}_{\mathrm{R}}(\mathcal{A}(X_g))$, is a compact Hausdorff space containing the set X_g, and therefore also T_g. The real spectrum compactification of Teichmüller space was then defined as the closure $\widetilde{\mathrm{T}}_g = \mathrm{Cl}_{\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(X_g))} \mathrm{T}_g$ of T_g in the space $\mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(X_g))$. We closed the thesis by comparing it with the well-known Thurston compactification $\hat{\mathrm{T}}_g$ of Teichmüller space defined by the lengths of simple closed geodesics on Σ. We proved that there exists a continuous surjection from $\widetilde{\mathrm{T}}_g$ to $\hat{\mathrm{T}}_g$ which implies, with a bit more work, that an outer automorphism of $\pi_1(\Sigma)$ has at least one fixed point on $\hat{\mathrm{T}}_g$ without proving first that $\hat{\mathrm{T}}_g$ is topologically a closed disc of dimension $6g-6$. However, we only gave an idea of the later fact whose detailed explanation is given in [3, Section 7]. This paper by G. Brumfiel, was in fact our basis and it contains a further study on the real spectrum compactification of Teichmüller space.

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Uvod. Teichmüllerjev prostor T_g prirejen sklenjeni povezani orientabilni ploskvi Σ roda $g \geq 2$ je v štiridesetih letih prejšnjega stoletja vpeljal nemški matematik Oswald Teichmüller. Izpeljan je iz tako imenovanega modularnega prostora, ki sestoji iz ekvivalečnih razredov Riemannovih ploskev, kjer sta dve ploskvi ekvivalentni natanko tedaj, ko med njima obstaja holomorfen homeomorfizem. Oswald Teichmüller je na modularnem prostoru vpeljal novo ekvivalečno relacijo in tako dobil prostor, danes imenovan Teichmüllerjev prostor, ki je mnogo enostavnejši od prvotnega modularnega prostora. Definiramo ga lahko kot kvocientni prostor hiperboličnih metrik po grupi difeomorfizmov ploskve Σ, ki ohranjajo orientacijo in so hkrati izotopni identiteti id_Σ. Ekvivalentno ga lahko vidimo kot množico ekvivalenčnih razredov hiperboličnih ploskev zaznamovanih s Σ, to je, hiperboličnih ploskev difeomorfnih Σ. Tako definiran Teichmüllerjev prostor T_g do izotopije natančno parametrizira vse hiperbolične strukture na Σ in je, opremljen s Fenchel-Nielsenovimi koordinatami, homeomorfen prostoru $\mathbb{R}^{6g-6}$. Na njem lahko vpeljemo več metrik, kot so Teichmüllerjeva metrika, Thurstonova asimetrična metrika in Weil-Peterssonova metrika, slednja pa porodi tudi simplektično strukturo na T_g.

Teichmüllerjev prostor lahko vložimo v različne funkcijske prostore. Zaprtje slik takih vložitev porodi robno strukturo na T_g, ki je zanimiva zaradi svoje netrivialnosti. Ko je zaprtje tudi kompaktno, govorimo o kompaktifikaciji Teichmüllerjevega prostora. Med najbolj znanimi so na primer Teichmüllerjeva, Bersova, Weil-Peterssonova in Thurstonova kompaktifikacija. Primerjamo jih lahko na primer glede na to, ali se delovanje grupe razredov difeomorfizmov $\mathrm{MCG}(\Sigma)$ ploskve Σ na T_g zvezno razširi tudi na ustrezen rob. Pri tem je grupa $\mathrm{MCG}(\Sigma)$ definirana kot kvocient grupe difeomorfizmov Σ, ki ohranjajo orientacijo, po podgrupi tistih, ki so hkrati tudi izotopni identiteti id_Σ. Izkaže se, da je za vse razen za Thurstonovo kompaktifikacijo skupno dejstvo, da omenjena razširitev ne obstaja.

V magistrskem delu predstavimo novo kompaktifikacijo Teichmüllerjevega prostora, imenovano realna spektralna kompaktifikacija $\widetilde{\mathrm{T}}_g$, ki jo je v [3] vpeljal Gregory W. Brumfiel, za katero obstaja zvezna razširitev delovanja grupe $\mathrm{MCG}(\Sigma)$ na njen rob. Ta lastnost nam omogoča primerjavo s Thurstonovo kompaktifikacijo $\hat{\mathrm{T}}_g$, za katero se izkaže, da obstaja zvezna surjekcija iz $\widetilde{\mathrm{T}}_g$ v $\hat{\mathrm{T}}_g$.

Realna spektralna kompaktifikacija je definirana za poljubno zaprto realno semi-algebraično množico W na sledeč način. Naj bo X realna algebraična množica, ki vsebuje W, in $\mathcal{A}(X)$ njej prirejen afin koordinatni kolobar. Potem W lahko obravnavamo kot podmnožico topološkega prostora $\mathrm{Spec}_\mathrm{R}(\mathcal{A}(X))$, sestavljenega iz prastožcev kolobarja $\mathcal{A}(X)$. Še več, W je vsebovana v kompaktnem Hausdorffovem podprostoru $\mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathcal{A}(X))$ prostora $\mathrm{Spec}_\mathrm{R}(\mathcal{A}(X))$. Potem je njena realna spektralna kompaktifikacija definirana kot zaprtje W v $\mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathcal{A}(X))$.

Da bi realno spektralno kompaktificirali Teichmüllerjev prostor, ga moramo predstaviti kot zaprto realno semi-algebraično množico. Vsako točko v geometrijskemu modelu T_g najprej identificiramo z ekvivalečnim razredom diskretne zveste reprezentacije prve fundamentalne grupe $\pi_1(\Sigma)$ v grupi $\mathrm{PSL}(2, \mathbb{R})$. Vsakemu takemu ekvivalenčnemu razredu nato priredimo sled ustrezne matrike in s tem vložimo T_g v $\mathbb{R}^M$ kot zaprto semi-algebraično množico.

Hiperbolična geometrija. To poglavje je namenjeno kratkemu uvodu v hiperbolično geometrijo in hiperbolične ploskve. Posvetimo se predvsem izometrijam hiperbolične ravnine $\mathbb{H}$, ki ohranjajo orientacijo, saj nastopajo v definiciji hiperboličnega atlasa na ploskvi. Klasificiramo jih na eliptične, parabolične in hiperbolične izometrije. Nato nas zanimajo krovne translacije univerzalnega krova hiperbolične ploskve, saj porodijo algebraični model Teichmüllerjevega prostora. Pokažemo, da so krovne translacije hiperbolične izometrije $\mathbb{H}$, ki ohranjajo orientacijo. Na koncu predstavimo geometrični model Teichmüllerjevega prostora prirejen sklenjeni orientabilni hiperbolični ploskvi roda $g \geq 2$.

Hiperbolična ravnina je definirana kot odprta množica

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

opremljena z metriko

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

za katero pokažemo, da ima Gaussovo ukrivljenost enako -1. Dolžina $L(\gamma)$ diferenciabilne krivulje $\gamma : [0,1] \longrightarrow \mathbb{H}$, $\gamma(t) = x(t) + iy(t)$ je potem podana kot

$$L(\gamma) := \int_\gamma ds.$$

Naj bo Γ množica vseh diferenciabilnih krivulj skozi poljubni točki z in w v $\mathbb{H}$. Potem je preslikava $\rho : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$, definirana kot

$$\rho(z, w) = \inf_{\gamma \in \Gamma} L(\gamma),$$

hiperbolična razdalja med z in w. Preslikavam, ki ohranjajo hiperbolično razdaljo pravimo izometrije ravnine $\mathbb{H}$. Z operacijo kompozituma tvorijo grupo $\operatorname{Isom}(\mathbb{H})$. Posvetimo se njeni podgrupi $\operatorname{Isom}^+(\mathbb{H})$, sestavljeni iz izometrij, ki ohranjajo orientacijo. Dokažemo, da je generirana z Möbiusovimi transformacijami oblike $\varphi(z) = \frac{az+b}{cz+d}$, kjer so koeficienti $a, b, c, d \in \mathbb{R}$ in velja $ad - bc = 1$. Izkaže se, da je izomorfna grupi $\operatorname{Aut}(\mathbb{H})$ holomorfnih avtomorfizmov $\mathbb{H}$, katero pa lahko vidimo kot realno projektivno specialno linearno grupo 2×2 matrik $\operatorname{PSL}(2, \mathbb{R}) = \operatorname{SL}(2, \mathbb{R})/\{\pm \mathrm{I}\}$. Vsako izometrijo φ lahko torej klasificiramo glede na absolutno vrednost sledi prirejene matrike A v grupi $\operatorname{PSL}(2, \mathbb{R})$. V magistrskem delu predvsem obravnavamo tako imenovane hiperbolične izometrije, za katere je absolutna vrednost sledi strogo večja kot 2. Dokažemo, da ima vsaka taka izometrija natanko dve fiksni točki v $\mathbb{R} \cup \{\infty\}$, ter da je konjugirana dilataciji $z \longmapsto kz$, $k > 0$.

Hiperbolično ploskev definiramo kot gladko sklenjeno povezano orientabilno topološko ploskev opremljeno z maksimalnim hiperboličnim atlasom $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$, za katerega velja, da je $\varphi_\alpha(U_\alpha) \subset \mathbb{H}$ za vsak $\alpha \in A$, prehodna preslikava $\varphi_\alpha \circ \varphi_\beta^{-1}$ pa se za vsak par $\alpha, \beta \in A$ na množici $\varphi_\beta(U_\alpha \cap U_\beta)$ ujema z izometrijo $m \in \operatorname{Isom}^+(\mathbb{H})$. Dodatno zahtevamo tudi, da je metrika, porojena iz hiperboličnega atlasa, kompletna. Pokažemo, da je vsaka taka metrika v resnici Riemannova metrika z negativno Gaussovo ukrivljenostjo, imenovana hiperbolična metrika na ploskvi.

Vsak hiperboličen atlas na hiperbolični ploskvi Σ inducira kompleksen atlas, katerega koordinatne karte slikajo v $\mathbb{C}$, prehodne preslikave pa so holomorfne. Zato lahko Σ obravnavamo kot kompleksno ploskev, za katero pa velja uniformizacijski izrek. Odtod sklepamo, da je njen univerzalni krov $\widetilde{\Sigma}$ konformno ekvivalenten hiperbolični ravnini $\mathbb{H}$. Grupa krovnih translacij je torej enaka $\operatorname{Deck}_\pi(\mathbb{H}) := \{T_\gamma \in \operatorname{Homeo}(\mathbb{H}) \mid \pi \circ T_\gamma = \pi\}$, kjer $\pi : \mathbb{H} \to \Sigma$ označuje krovno projekcijo. Lahko je videti, da so krovne translacije izometrije $\mathbb{H}$, ki ohranjajo orientacijo. Iz delovanja grupe

$\mathrm{Deck}_\pi(\mathbb{H})$ na $\mathbb{H}$ pa sledi, da $\mathrm{Deck}_\pi(\mathbb{H})$ lahko identificiramo z diskretno podgrupo grupe $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H})$, imenovano fuzijska grupa. Z uporabo hiperbolične trigonometrije se izkaže tudi, da je vsaka krovna translacija hiperbolična, to je, ima natanko dve fiksni točki v $\mathbb{R} \cup \{\infty\}$.

Fiksirajmo sklenjeno hiperbolično ploskev Σ roda $g \geq 2$. Par (S, f), kjer je S sklenjena hiperbolična ploskev roda g, imenujemo hiperbolična ploskev zaznamovana s Σ, če obstaja difeomorfizem $f : \Sigma \to S$. Na množici hiperboličnih ploskev zaznamovanih s Σ vpeljemo naslednjo ekvivalenčno relacijo: ploskvi (S, f) in (S', f') sta ekvivalentni, če obstaja izometrija $m : S \to S'$, za katero je preslikava $m \circ f$ izotopna f'. Množico ekvivalenčnih razredov

$$\mathrm{T}_g := \{(S, f)\}/\sim$$

imenujemo Teichmüllerjev prostor ploskve Σ.

Vsaka hiperbolična ploskev (S, f) zaznamovana s Σ s povlekom nazaj f^* preko difeomorfizma f porodi hiperbolično metriko na ploskvi Σ. Če torej na množici $\mathrm{Hyp}(\Sigma)$ hiperboličnih metrik na Σ vpeljemo primerno ekvivalenčno relacijo, dobimo ekvivalentno definicijo Teichmüllerjevega prostora. Pri tem identificiramo dve metriki $h_1, h_2 \in \mathrm{Hyp}(\Sigma)$, če obstaja difeomorfizem $\varphi : \Sigma \to \Sigma$, ki ohranja orientacijo in je izotopen id_Σ, za katerega je $h_1 = \varphi^*(h_2)$. Potem je Teichmüllerjev prostor enak množici izotopnih razredov hiperboličnih metrik na Σ

$$\mathrm{T}_g := \mathrm{Hyp}(\Sigma)/\,\mathrm{Diff}_0^+(\Sigma),$$

kjer je $\mathrm{Diff}_0^+(\Sigma)$ grupa difeomorfizmov Σ, ki ohranjajo orientacijo in so izotopni id_Σ.

Vsako sklenjeno povezano orientirano hiperbolično ploskev Σ roda $g \geq 2$ opremimo s tako imenovanimi Fenchel-Nielsenovimi koordinatami, ki jih porodi dekompozicija Σ na pare hlač. Pokažemo, da parametri dolžine in zasuka, ki nastopajo v Fenchel-Nielsenovih koordinatah, določajo homeomorfizem

$$\mathrm{T}_g \longrightarrow \mathbb{R}^{6g-6}.$$

Realni spekter. V tem poglavju obravnavamo realni spekter $\mathrm{Spec_R(A)}$ komutativnega kolobarja z enico A in njegov podprostor $\mathrm{Spec_R^m(A)}$, sestavljen iz zaprtih točk prostora $\mathrm{Spec_R(A)}$. Definiramo tudi realno spektralno kompaktifikacijo zaprte realne semi-algebraične množice.

Najprej predstavimo pojem pravega stožca, ki je podan kot podmnožica α kolobarja A, za katero veljajo naslednji pogoji: (i) za poljubna dva elementa $f, g \in \alpha$ je $f + g \in \alpha$ in $fg \in \alpha$, (ii) za vsak $f \in A$ je $f^2 \in \alpha$, (iii) ter $-1 \notin \alpha$. Potem je realni spekter $\mathrm{Spec_R(A)}$ definiran kot množica

$$\mathrm{Spec_R(A)} := \{\alpha \mid \alpha \text{ prastožec kolobarja } A\},$$

kjer je prastožec α pravi stožec, ki dodatno zadošča pogoju

$$fg \in \alpha \Rightarrow f \in \alpha \text{ ali } -g \in \alpha.$$

Da bi dokazali, da je $\mathrm{Spec_R(A)}$ topološki prostor, si ogledamo tako imenovan nosilec $\mathrm{supp}(\alpha) := \alpha \cap -\alpha$ prastožca α, ki je praideal kolobarja A. Potem je kvocientni kolobar $A/\,\mathrm{supp}(\alpha)$ celostno polje, ki ga vložimo v obseg ulomkov $k(\mathrm{supp}(\alpha))$. Ogledamo si kanonični homomorfizem

$$\begin{aligned} A &\longrightarrow F(\alpha) \\ f &\longmapsto f(\alpha), \end{aligned}$$

kjer je $F(\alpha)$ realno zaprtje obsega $k(\mathrm{supp}(\alpha))$. Ker je $F(\alpha)$ urejen obseg, za element $f \in A$ velja natanko ena izmed izjav: $f(\alpha) < 0, f(\alpha) = 0$ ali $f(\alpha) > 0$. Realna spektralna topologija prostora $\mathrm{Spec}_\mathrm{R}(\mathrm{A})$ je potem podana s podbazo odprtih množic

$$\widetilde{U}(f) := \{\alpha \in \mathrm{Spec}_\mathrm{R}(\mathrm{A}) \mid f(\alpha) > 0\},$$

kjer je $f \in A$. Pokažemo, da je $\mathrm{Spec}_\mathrm{R}(\mathrm{A})$ glede na to topologijo kompakten topološki prostor.

Ker $\mathrm{Spec}_\mathrm{R}(\mathrm{A})$, opremljen z realno spektralno topologijo, ni T_1 prostor, to je, singeltoni niso zaprte množice, obravnavamo njegov podprostor $\mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathrm{A})$, sestavljen iz zaprtih točk prostora $\mathrm{Spec}_\mathrm{R}(\mathrm{A})$. Izkaže se, da so to natanko maksimalni prastožci kolobarja A. Dokažemo, da je $\mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathrm{A})$ kompakten Hausdorffov prostor.

Če vsak prastožec $\alpha \in \mathrm{Spec}_\mathrm{R}(\mathrm{A})$ ekvivalentno predstavimo kot ekvivalenčni razred homomorfizma $\alpha : A \to F(\alpha)$, dobimo naslednjo karakterizacijo zaprtih točk: točka $\alpha : A \to F(\alpha) \in \mathrm{Spec}_\mathrm{R}(\mathrm{A})$ je zaprta natanko tedaj, ko za vsak $x \in F(\alpha)$ obstaja $f \in A$, da je $|x| < f(\alpha)$. To je pomembno v primeru, ko za kolobar A vzamemo afin koordinatni kolobar $\mathcal{A}(X)$ realne algebraične množice X. Spomnimo se najprej, da je realna algebraična množica X podana kot množica točk $X := \{x \in \mathbb{R}^n \mid f(x) = 0\ \forall f \in B\}$, kjer je B podmnožica kolobarja polinomov $\mathbb{R}[x_1, \ldots, x_n]$. Afin koordinatni kolobar $\mathcal{A}(X)$ je potem enak kvocientnemu kolobarju $\mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}(X)$, kjer je $\mathcal{I}(X) := \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid f(x) = 0\ \forall x \in X\}$ ideal kolobarja $\mathbb{R}[x_1, \ldots, x_n]$. Če za vsako točko $x \in X$ definiramo homomorfizem

$$\begin{aligned} \mathcal{A}(X) &\longrightarrow \mathbb{R} \\ F + \mathcal{I}(X) &\longmapsto F(x), \end{aligned}$$

vidimo, da je $X \subset \mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathrm{A}) \subset \mathrm{Spec}_\mathrm{R}(\mathrm{A})$. To porodi definicijo realne spektralne kompaktifikacije $\widetilde{W}$ zaprte semi-algebraične podmnožice $W \subset X$ kot zaprtje

$$\widetilde{W} := \mathrm{Cl}_{\mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathcal{A}(\mathrm{X}))}\, W.$$

Pri tem je semi-algebraična množica W podana kot končna unija množic oblike $\{x \in \mathbb{R}^n \mid f_i(x) > 0, g_j(x) = 0\}$, kjer sta $\{f_i\}_{i=1}^l$ in $\{g_j\}_{j=1}^m$ končna nabora polinomov iz $\mathbb{R}[x_1, \ldots, x_n]$. Dokažemo, da je W odprta in gosta podmnožica v $\widetilde{W}$. Obravnavamo tudi robne točke množice W, to je, elemente množice $\widetilde{W} \setminus W$. Izkaže se, da je točka $x \in W \subset \mathrm{Spec}_\mathrm{R}^\mathrm{m}(\mathcal{A}(\mathrm{X}))$, ki jo razumemo kot homomorfizem $\alpha : \mathcal{A}(X) \to F(\alpha)$, robna točka natanko tedaj, ko obstaja $|x_i(\alpha)| \in F(\alpha)$, za katerega velja, da je $|x_i(\alpha)| > r$ za vsak $r \in \mathbb{R}$.

Teichmüllerjev prostor kot zaprta semi-algebraična množica. V tem poglavju najprej identificiramo Teichmüllerjev prostor s prostorom reprezentacij. To porodi algebraični model T_g, ki ga vložimo v $\mathbb{R}^M$, za dovolj velik M, kot zaprto realno semi-algebraično množico. Z uporabo teorije iz prejšnjega poglavja tako dobimo realno spektralno kompaktifikacijo Teichmüllerjevega prostora.

Homomorfizem $\rho : \Gamma \to F$ iz grupe Γ v topološko grupo F imenujemo reprezentacija Γ v F. Prostor vseh reprezentacij $\Gamma \to F$ označimo s $\mathrm{Hom}(\Gamma, F)$. Reprezentacija ρ je zvesta, če je injektivna, in diskretna, če je slika $\rho(\Gamma)$ diskretna v F. Prostor diskretnih zvestih reprezentacij Γ v F, za katere dodatno privzamemo, da je kvocient

$F/\rho(\Gamma)$ kompakten, označimo s $\operatorname{DHom}(\Gamma, F)$. Grupa $\operatorname{Inn}(F)$ notranjih avtomorfizmov F deluje na prostoru $\operatorname{Hom}(\Gamma, F)$ na naslednji način:

$$\begin{aligned}\operatorname{Inn}(F) \times \operatorname{Hom}(\Gamma, F) &\longrightarrow \operatorname{Hom}(\Gamma, F) \\ (f, \rho) &\longmapsto f \circ \rho.\end{aligned}$$

V magistrski nalogi se ukvarjamo s prostorom orbit

$$\mathcal{DT}(\Gamma, F) := \operatorname{DHom}(\Gamma, F)/\operatorname{Inn}(F),$$

kjer za grupo Γ vzamemo prvo fundamentalno grupo $\pi_1(\Sigma)$ sklenjene povezane orientabilne hiperbolične ploskve Σ, za topološko grupo F pa grupo $\operatorname{PSL}(2, \mathbb{R})$ izometrij $\mathbb{H}$, ki ohranjajo orientacijo. Spomnimo se, da je geometričen model Teichmüllerjevega prostora podan kot množica izotopnih razredov hiperboličnih metrik na Σ, to je,

$$\mathrm{T}_g := \operatorname{Hyp}(\Sigma)/\operatorname{Diff}_0^+(\Sigma),$$

kjer $\operatorname{Diff}_0^+(\Sigma)$ označuje grupo difeomorfizmov Σ, ki ohranjajo orientacijo in so izotopni identiteti id_Σ. Dokažemo naslednjo trditev:

Trditev 1. *Naj bo Σ sklenjena povezana orientabilna hiperbolična ploskev roda $g \geq 2$. Potem obstaja injektivna preslikava*

$$\delta' : \mathrm{T}_g \longrightarrow \mathcal{DT}(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R})).$$

Pokažemo, da δ' ni surjektivna, saj vsaka izometrija $\mathbb{H}$, ki obrne orientacijo, porodi diskretno zvesto reprezentacijo $\pi_1(\Sigma) \to \operatorname{PSL}(2, \mathbb{R})$, ki pa ni v sliki δ'. Slika $\operatorname{Im}(\delta')$ porodi algebraični model Teichmüllerjevega prostora T_g, ki ga označimo s $\mathcal{DT}_+(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R}))$. Da bi ga vložili v $\mathbb{R}^M$ kot zaprto realno semi-algebraično množico, si raje ogledamo prostor $\mathcal{DT}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))$ diskretnih zvestih reprezentacij $\pi_1(\Sigma)$ v grupi $\operatorname{SL}(2, \mathbb{R})$. Pokažemo, da vsako reprezentacijo $\rho \in \operatorname{Im}(\delta') = \mathcal{DT}_+(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R}))$ lahko dvignemo do reprezentacije $\rho : \pi_1(\Sigma) \to \operatorname{SL}(2, \mathbb{R})$. Natančneje, pokažemo, da obstaja zvezna odprta surjekcija

$$\operatorname{Hom}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R})) \longrightarrow \operatorname{Hom}(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R})),$$

ki slika prostor $\operatorname{DHom}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))$ v prostor $\operatorname{DHom}(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R}))$. Zato lahko ekvivalentno obravnavamo tisti del prostora $\mathcal{DT}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))$, ki se z zgornjo projekcijo slika v Teichmüllerjev prostor $\mathcal{DT}_+(\pi_1(\Sigma), \operatorname{PSL}(2, \mathbb{R}))$. Označimo ga s $\mathcal{DT}_+(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))$. Dokažemo naslednji izrek:

Izrek 1. *Prostor $\mathcal{DT}_+(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))$ ima končno mnogo komponent za povezanost. Vsaka od njih je homeomorfna Teichmüllerjevemu prostoru T_g.*

Vložitev Teichmüllerjevega prostora T_g v $\mathbb{R}^M$ definiramo na sledeč način. Izberemo poljuben nabor generatorjev $\gamma_1, \ldots, \gamma_m$ grupe $\pi_1(\Sigma)$ in označimo z

$$I_m := \{(\nu_1, \ldots, \nu_j) \mid 1 \leq \nu_1 \leq \cdots \leq \nu_j \leq m\}$$

množico urejenih j-teric naravnih števil, kjer je $1 \leq j \leq m$. Potem preslikava

$$\begin{aligned}\operatorname{Hom}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R})) &\longrightarrow \mathbb{R}^M \\ \rho &\longmapsto (\operatorname{tr}(\rho(\gamma_{\nu_1} \cdots \gamma_{\nu_j})))_{(\nu_1, \ldots, \nu_j) \in I_m},\end{aligned}$$

kjer tr označuje sled matrike, inducira preslikavo

$$h : \operatorname{Hom}(\pi_1(\Sigma), \operatorname{SL}(2, \mathbb{R}))/\operatorname{Inn}(\operatorname{SL}(2, \mathbb{R})) \longrightarrow \mathbb{R}^M,$$

saj je sled matrike invariantna glede na konjugiranje. Izkaže se, da je slika

$$X_g := h(\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))/\,\mathrm{Inn}(\mathrm{SL}(2,\mathbb{R})))$$

realna algebraična množica. Če označimo

$$DX_g := h(\mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))),$$

potem velja naslednji izrek:

Izrek 2. *Preslikava*

$$h : \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})) \longrightarrow DX_g$$

je homeomorfizem. Še več, množica DX_g je unija povezanih komponent realne algebraične množice X_g.

Potem izreka 1 in 2 inducirata želen rezultat:

Izrek 3. *Vsaka povezana komponenta prostora DX_g je homeomorfna Teichmüllerjevemu prostoru T_g ploskve Σ.*

Ker je DX_g zaprta realna semi-algebraična množica, z uporabo teorije prejšnjega poglavja dobimo, da je

$$\mathrm{T}_g \subset DX_g \subset X_g \subset \mathrm{Spec}^{\mathrm{m}}_{\mathrm{R}}(\mathcal{A}(X_g)),$$

kjer je $\mathcal{A}(X_g) = \mathbb{R}[x_1, \ldots, x_M]/\mathcal{I}(X_g)$ afin koordinatni kolobar realne algebraične množice X_g. Potem vsaka komponenta za povezanost realne sprektalne kompaktifikacije $\widetilde{DX_g}$ realizira realno spetralno kompaktifikacijo $\widetilde{\mathrm{T}}_g$ Teichmüllerjevega prostora T_g.

Primerjava realne spektralne kompaktifikacije Teichmüllerjevega prostora s Thurstono kompaktifikacijo. William P. Thurston je v [6] predstavil kompaktifikacijo Teichmüllerjevega prostora dobljeno kot zaprtje vložitve T_g v neskončno razsežen realni projektivni prostor $\mathbb{P}^{\mathcal{S}}$. Podrobneje jo definiramo v tem poglavju in jo primerjamo z obravnavano realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g$. Pokažemo, da obstaja zvezna surjektivna preslikava iz $\widetilde{\mathrm{T}}_g$ v Thurstonovo kompaktifikacijo $\hat{\mathrm{T}}_g$.

Z oznakami prejšnjega poglavja najprej definiramo realni projektivni prostor $\mathbb{P}^{\mathcal{S}}$. Naj bo $\mathcal{S}$ podmnožica afinega koordinatnega kolobarja $\mathcal{A}(X_g)$ realne algebraične množice X_g, ki zadošča naslednjim pogojem: (i) $\mathcal{S}$ vsebuje vse koordinatne preslikave $x_i \in \mathcal{A}(X_g)$, $i = 1, \ldots, n$; (ii) za vsak $f \in \mathcal{S}$ in vsak $x \in \mathrm{T}_g \subset X_g$ je $|f(x)| \geq 1$; (iii) obstaja $f \in \mathcal{S}$, da je $|f(x)| > 1$ za vsak $x \in \mathrm{T}_g$. Potem je

$$\mathbb{P}^{\mathcal{S}} := ([0,\infty)^{\mathcal{S}} \setminus \{\vec{0}\})/\sim,$$

kjer sta $\mathcal{S}$-terici $(a_f)_{f\in\mathcal{S}}, (b_f)_{f\in\mathcal{S}} \in [0,\infty)^{\mathcal{S}} \setminus \{\vec{0}\}$ ekvivalentni natanko tedaj, ko obstaja $\lambda > 0$, za katerega je $(a_f)_{f\in\mathcal{S}} = \lambda(b_f)_{f\in\mathcal{S}}$. Za množico $\mathcal{S}$ vzamemo podmnožico $\pi_1(\Sigma)$, ki vsebuje predstavnika vsakega konjugiranega razreda v $\pi_1(\Sigma)$ homotopnega enostavni sklenjeni krivulji na Σ, in pokažemo, da ustreza zgornjim pogojem.

Thurstonova kompaktifikacija je definirana kot zaprtje vložitve T_g v $\mathbb{P}^{\mathcal{S}}$. Z realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g$ jo primerjamo tako, da tudi $\widetilde{\mathrm{T}}_g$ zvezno preslikamo v $\mathbb{P}^{\mathcal{S}}$ in pokažemo, da ta preslikava porodi omenjeno surjekcijo. Zato si najprej v splošnem ogledamo preslikavo oblike

$$\begin{aligned} \theta : W &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ x &\longmapsto (\log_b(|f(x)|))_{f\in\mathcal{S}}, \end{aligned}$$

kjer je W zaprta semi-algebraična podmnožica realne algebraične množice X, in $b \in \mathbb{R}, b > 0$. Naš cilj je dokazati, da obstaja zvezna razširitev θ na $\widetilde{W}$. Spomnimo se, da je homomorfizem $\alpha : \mathcal{A}(X) \to F(\alpha) \in \widetilde{W}$ robna točka natanko tedaj, ko obstaja $|x_i(\alpha)| \in F(\alpha)$ za katerega je $|x_i(\alpha)| > r$ za vsak $r \in \mathbb{R}$. Da bi torej razširili θ na realno spektralno kompaktifikacijo $\widetilde{W}$, najprej razširimo logaritem $\log_b : F^+ \to \mathbb{R}$ na množico F^+ pozitivnih elementov urejenega obsega F. Nato pokažemo, da se θ z enakim predpisom zvezno razširi do preslikave $\widetilde{\theta} : \widetilde{W} \to \mathbb{P}^{\mathcal{S}}$.

Definiramo Thurstonovo kompaktifikacijo $\hat{\mathrm{T}}_g$ na sledeč način. Naj bo γ enostavna sklenjena krivulja na Σ in $\rho \in \mathcal{DT}_+(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R}))$. Potem je $\rho(\gamma)$ hiperbolična krovna translacija in zato konjugirana dilataciji $z \longmapsto \lambda^2 z$, $|\lambda| > 1$. Ker homotopski razred γ vsebuje enolično določeno enostavno sklenjeno geodetko, katere dolžina $l_\rho(\gamma)$ je enaka $2\log(\lambda)$, dobimo naslednjo preslikavo

$$\begin{aligned} \theta' : \mathrm{T}_g &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ [\rho] &\longmapsto \left(\frac{l_\rho(\gamma)}{2}\right)_{\gamma \in \mathcal{S}}. \end{aligned}$$

Potem je Thurstonova kompaktifikacija $\hat{\mathrm{T}}_g$ Teichmüllerjevega prostora definirana kot zaprtje $\overline{\theta'(\mathrm{T}_g)}$ prostora T_g v $\mathbb{P}^{\mathcal{S}}$. Da jo primerjamo z realno spektralno kompaktifikacijo $\widetilde{\mathrm{T}}_g$ definiramo preslikavo

$$\begin{aligned} \theta : \mathrm{T}_g &\longrightarrow \mathbb{P}^{\mathcal{S}} \\ [\rho] &\longmapsto (\log(|\operatorname{tr}(\rho(\gamma))|))_{\gamma \in \mathcal{S}}. \end{aligned}$$

Dokažemo, da se razširitvi $\widetilde{\theta}, \widetilde{\theta}' : \widetilde{\mathrm{T}}_g \to \mathbb{P}^{\mathcal{S}}$ ujemata na robu $\widetilde{\mathrm{T}}_g \setminus \mathrm{T}_g$, kar pa porodi naslednjo trditev:

Trditev 2. *Preslikava $\widetilde{\theta} : \widetilde{\mathrm{T}}_g \to \mathbb{P}^{\mathcal{S}}$ porodi zvezno surjekcijo $\widetilde{\theta} : \widetilde{\mathrm{T}}_g \to \hat{\mathrm{T}}_g$.*