Lemma 1. *Partition of covariance For random variables X, Y, and Z, with $\mathbb{E}[X^2]$ and $\mathbb{E}[Y^2]$ finite,*

(a)
$$\mathrm{var}[X] = \mathbb{E}[\mathrm{var}[X|Z]] + \mathrm{var}[\mathbb{E}[X|Z]]$$

(b)
$$\mathrm{cov}[X, Y] = \mathbb{E}[\mathrm{cov}[X, Y|Z]] + \mathrm{cov}[\mathbb{E}[X|Z], \mathbb{E}[Y|Z]].$$

where $\mathrm{cov}[X, Y|Z] := \mathbb{E}[XY|Z] - \mathbb{E}[X|Z]\mathbb{E}[Y|Z]$.

Proof. This follows from the fact that $X, Y \mapsto \mathbb{E}[XY]$ is the inner product on $L^2(\mathbb{P})$ and the definition of conditional expectation with respect to Z as the orthogonal projection onto the subspace of Z-measurable random variables. Specifically, $\mathbb{E}[X]$ is the projection of X into the space of constant random variables (since $\mathbb{E}[(X - \mathbb{E}[X])c] = 0$ for any constant c); and $\mathbb{E}[X|Z] - \mathbb{E}[X]$ is the projection of X into the intersection of Z-measurable random variables and the orthogonal complement of constant random variables (since $\mathbb{E}[f(Z)\mathbb{E}[X|Z]] = \mathbb{E}[f(Z)X]$, by definition of conditional expectation). Therefore, $X = \mathbb{E}[X] + (\mathbb{E}[X|Z] - \mathbb{E}[X]) + (X - \mathbb{E}[X|Z])$ is the decomposition of X into three orthogonal subspaces; and we have a corresponding decomposition for Y, so by Pythagoras,

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[(\mathbb{E}[X|Z] - \mathbb{E}[X])(\mathbb{E}[Y|Z] - \mathbb{E}[Y])] + \mathbb{E}[(X - \mathbb{E}[X|Z])(Y - \mathbb{E}[Y|Z])] &(1)\\
&= \mathbb{E}[X]\mathbb{E}[Y] + \mathrm{cov}[\mathbb{E}[X|Z], \mathbb{E}[Y|Z]] + \mathbb{E}[\mathrm{cov}[X, Y|Z]], &(2)
\end{aligned}$$

since $\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[X]$ and likewise for Y and for XY. □

References