Suppose that $(X_t)_{t\geq 0}$ is a continuous-time Markov process with generator matrix G, i.e. $G_{ij} > 0$ if $i \neq j$ and rows of G sum to zero. Let τ be an independent Gamma distributed time with shape parameter α and scale parameter β. We would like to compute the probabilities

$$Q(x, y; \alpha, \beta) := \mathbb{P}\{X_\tau = y \mid X_0 = x\}$$

in an efficient manner, focusing on the case when G is large and sparse.

Consider the corresponding "jump chain", which represents X as $X_t = M_{N(\lambda t)}$, where $N(t)$ is a unit Poisson process and M is a discrete-time Markov chain. Here λ is the maximum transition rate $\lambda = \max_i\{-G_{ii}\}$, and the transition matrix for M is given by

$$P_{xy} := \mathbb{P}\{M_{k+1} = y \mid M_k = x\} \tag{1}$$

$$= \begin{cases} G_{xy}/\lambda & x \neq y \\ 1 + G_{xx}/\lambda & x = y \end{cases}. \tag{2}$$

$N(\lambda\tau)$ has a negative binomial distribution, with parameters α and $\lambda/(\lambda+\beta)$:

$$\mathbb{P}\{N(\lambda\tau) = n\} = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} e^{-\lambda t} \frac{(\lambda t)^n}{n!} dt \tag{3}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha) n!} \frac{\lambda^n}{(\lambda+\beta)^{n+\alpha}} \int_0^\infty t^{n+\alpha-1} e^{-t} dt \tag{4}$$

$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} \frac{\beta^\alpha \lambda^n}{(\lambda+\beta)^{n+\alpha}} \qquad \text{for } n \geq 0. \tag{5}$$

It follows that

$$Q(x, y; \alpha, \beta) = \sum_{n\geq 0} \mathbb{P}\{N(\lambda\tau) = n\} P^n. \tag{6}$$

Note that

$$\frac{\mathbb{P}\{N(\lambda\tau) = n+1\}}{\mathbb{P}\{N(\lambda\tau) = n\}} = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha)\Gamma(n+2)} \frac{\lambda}{\lambda+\beta} \tag{7}$$

$$= \frac{n+\alpha}{n+1} \frac{\lambda}{\lambda+\beta}, \tag{8}$$

which is helpful for computing this. The error in approximating this by a finite sum up to n_0 is bounded by $\mathbb{P}\{N(\lambda\tau) > n_0\}$.

1 More calculations

Suppose that T is Exponential(β) and S is Exponential($\beta + u$), and that $N(t)$ is Poisson(λt). By Poisson thinning,

$$S \stackrel{d}{=} T + \sum_{k=0}^{M} T_k, \tag{9}$$

where T_k are independent copies of T and M is Geometric$(\beta/(\beta+u))$. Therefore,

$$N(S) \stackrel{d}{=} N(T) + N', \tag{10}$$

where N' is NegativeBinomial$(M,\lambda/(\lambda+\beta))$.

What is a Geometric mixture of Negative Binomials? Well, if

$$\mathbb{P}\{M = m\} = p(1-p)^m \tag{11}$$

$$\text{and } \mathbb{P}\{N' = n \mid M = m\} = \frac{\Gamma(n+m)}{\Gamma(m)n!} q^m (1-q)^n, \tag{12}$$

then

$$\begin{aligned}
\mathbb{P}\{N' = 0\} &= \sum_{m \geq 0} p(1-p)^m q^m && (13)\\
&= \frac{p}{1-q(1-p)}, \text{ and} && (14)\\
\mathbb{P}\{N' = n\} &= \sum_{m \geq 1} p(1-p)^m \frac{\Gamma(n+m)}{\Gamma(m)n!} q^m (1-q)^n && (15)\\
&= \frac{p(1-q)^n}{n!} \sum_{m \geq 1} q^m (1-p)^m \frac{\Gamma(n+m)}{\Gamma(m)} && (16)\\
&= \frac{p(1-q)^n}{n!} \frac{n! q(1-p)}{(1-q(1-p))^n} && (17)\\
&= (1 - \frac{p}{1-q(1-p)}) \frac{pq}{1-q} \left(\frac{1-q}{1-q(1-p)} \right)^n. && (18)
\end{aligned}$$

In other words, if we let $a = (1-p)(1-q)/(1-q(1-p))$ and $b = (1-q)/(1-q(1-p))$, then

$$\mathbb{P}\{N' = 0\} = 1 - a \tag{19}$$

$$\mathbb{P}\{N' = n\} = a(1-b)b^{n-1}, \tag{20}$$

i.e., if X be Bernoulli(a) and Y be Geometric(b) then $N' \stackrel{d}{=} X + Y$.

We want to apply this with $p = u/(\beta+u)$ and $q = \beta/(\lambda+\beta)$, which translates to

$$\begin{aligned}
a &= \frac{\lambda\beta}{u\lambda + u\beta + \lambda\beta} && (21)\\
&= \frac{\lambda/\beta}{(1+\lambda/\beta)(1+u/\beta) - 1} && (22)\\
b &= \frac{\lambda(\beta+u)}{\lambda(\beta+u) + \beta u} && (23)
\end{aligned}$$