1 Transfer Functions

2 GRN

Let $G^*(s) = \frac{(s-k)}{(s-\lambda_1)(s-\lambda_2)}$ be the optimal phenotype. This transfer function can be realized by a GRN composed of 2 or more genes. If k is equal to λ a one gene GRN can completely realize the dynamics.

The phenotype of an organism is, G(s) = $\begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The dynamics of the optimal GRN in the time domain (via the inverse Laplace transform) is,
$f^*(t) = \mathcal{L}^{-1}(G^*(s)) = \frac{e^{\lambda_1 t}(\lambda_1 - k) - e^{\lambda_2 t}(\lambda_2 - k)}{(\lambda_1 - \lambda_2)}$

These dynamics can be realized by an infinite number of $\geq$ 2 gene systems ($in\mathbb{R}$). The 2 dimensional realizations will all have the form below where x can be any real number. If the transfer function is of the form $G^*(s) = \frac{1}{(s-\lambda_1)}$, then $\lambda_2 = k$ can also be any real number.

$$\begin{bmatrix} x & \lambda_1 + \lambda_2 - k - x \\ \frac{-(x-\lambda_1)(x-\lambda_2)}{\lambda_1+\lambda_2-k-x} & \lambda_1 + \lambda_2 - x \end{bmatrix}$$

The fitness of an individual organism is computed as

$$\phi(f(t)) = e^{-\sqrt{\int_0^\infty (f^*(t) - f(t))^2 dt}} \tag{1}$$

Where as individual realization is $f(t) = \frac{e^{\gamma_1 t}(\gamma_1 - h) - e^{\gamma_2 t}(\gamma_2 - h)}{(\gamma_1 - \gamma_2)}$ If we look at the difference (instead of the squared difference) between optimal and actual, and assume that the real parts of the eigenvalues are ≤ 0, we get,

$$\varphi(f(t)) = e^{-\int_0^\infty (f^*(t) - f(t))dt} = e^{-\left| \frac{k}{\lambda_1 \lambda_2} - \frac{h}{\gamma_1 \gamma_2} \right|} \tag{2}$$

$$\frac{\partial(\det(A))}{\partial A_{ij}} = \lambda_1 \lambda_2 \mathrm{Tr} \left[A^{-1} \frac{\partial A}{\partial A_{ij}} \right] \tag{3}$$

And, $\gamma_1 \gamma_2 \approx \lambda_1 \lambda_2 + \mathrm{Tr} \left[\mathrm{adj}(A) \frac{\partial A}{\partial A_{ij}} \right] = \lambda_1 \lambda_2 + (\mathrm{adj}(A))^T_{ij}$

$$\nabla G = \begin{bmatrix} (s - a_{22})(s + a_{12} - a_{22}) \\ (s - a_{22})(s + a_{21} - a_{11}) \\ a_{12}(s + a_{12} - a_{22}) \\ a_{12}(s + a_{21} - a_{11}) \end{bmatrix} \times \left((s - \lambda_1)(s - \lambda_2) \right)^{-2} \tag{4}$$

$$
\nabla^2 G = \begin{bmatrix}
2(s-a_{22})^2(s+a_{12}-a_{22}) & (s-a_{22})(2a_{21}(s+a_{12}-a_{22})+(s-\lambda_1)(s-\lambda_2)) & 2a_{12}(s-a_{22})(s+a_{12}-a_{22}) & a_{12}(2a_{21}(s+a_{12}-a_{22})+(s-\lambda_1)(s-\lambda_2)) \\
\cdot & 2a_{21}(s-a_{22})(s+a_{21}-a_{11})^2 & (s-a_{22})(2a_{12}(s+a_{21}-a_{11})+(s-\lambda_1)(s-\lambda_2)) & (s+a_{21}-a_{11})(2a_{12}a_{21}+(s-\lambda_1)(s-\lambda_2)) \\
\cdot & \cdot & 2(a_{12})^2(s+a_{12}-a_{22}) & a_{12}(2a_{12}(s+a_{21}-a_{11})+(s-\lambda_1)(s-\lambda_2)) \\
\cdot & \cdot & \cdot & 2a_{12}(s-a_{11})(s+a_{21}-a_{11})
\end{bmatrix}
\times \big((s-\lambda_1)(s-\lambda_2)\big)^{-3} \tag{5}
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