Rapid speciation despite conservation of phenotype

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Abstract

Even if a species' phenotype remains unchanged over evolutionary time, the underlying mechanism may have changed, as distinct molecular pathways can realize identical phenotypes. Here we use quantitative genetics and linear system theory to study how a gene network underlying a conserved phenotype evolves, as the genetic drift of small mutational tweaks to these molecular pathways cause a population to explore the set of mechanisms with identical phenotypes. In this setting we treat an organism as a "black box" for which the environment provides input and the phenotype is the output, and there exists an exact characterization of the set of all mechanisms that give the same input–output relationship. Within this framework, we show that there is never a unique network architecture for any phenotype and that the evolutionary exploration of these distinct and mutationally connected mechanisms can lead to the reproductive incompatibility of idependently evolving populations. This evolutionary exploration, which we refer to as system drift, proceeds at a rate proportional to the effective population size (N_e) and amount of intrapopulation genetic variation. We estimate that this process can lead to the formation of new species, perhaps in as few as $0.1N_e$ generations. This model also naturally yields, what appears to be, a distinct explanation for Haldane's rule, or why heterogametic hybrids tend to be disrupted more often than homogametes during the early stages of speciation.

Introduction

A complex molecular machinery translates an organism's genome into the characteristics on which natural selection acts, the phenotype. It is an overarching goal of many biological subdisciplines to attain a general understanding of the function and evolution of this molecular machinery. For example, there is a growing body of data on the evolutionary histories and molecular characterizations of particular gene regulatory networks [Jaeger, 2011, Davidson and Erwin, 2006, Israel et al., 2016], as well as thoughtful verbal and conceptual models [True and Haag, 2001, Pavlicev and Wagner, 2012, Weiss and Fullerton, 2000, Edelman and Gally, 2001]. Mathematical models of both particular regulatory networks and the evolution of such systems in general can provide guidance where intuition fails, and thus has the potential to discover general principles in the organization of biological systems as well as provide concrete numerical predictions [Servedio et al., 2014].

The dynamics of the molecular machinery and its interactions with the environment can be mathematically described as a dynamical system [Jaeger et al., 2015]. Movement in this direction is ongoing, as researchers have begun to study the evolution of both abstract [Wagner, 1994, 1996, Siegal and Bergman, 2002, Bergman and Siegal, 2003, Draghi and Whitlock, 2015] and empirically inspired computational and mathematical models of gene regulatory networks, [e.g. Mjolsness et al., 1991, Jaeger et al., 2004, Kozlov et al., 2012, 2015, 2014, Crombach et al., 2016, Wotton et al., 2015, Chertkova et al., 2017]. It is well known that in many contexts mathematical models can fundamentally be *nonidentifiable* and/or *indistinguishable* – meaning that there can be uncertainty about an inferred model's parameters or even its claims about causal structure, despite access to complete and perfect data [Bellman and Åström, 1970, Grewal and Glover, 1976, Walter et al., 1984]. Models with different parameter schemes, or even different mechanics can make equally accurate predictions, but still not actually reflect the internal dynamics of the system being modelled. In control theory, where electrical circuits and mechanical systems are often the focus, it is understood that there can be an infinite number of "realizations", or ways to reverse engineer the dynamics of a "black box", even if all possible input and output experiments on the "black box" are performed [Kalman, 1963, Anderson et al., 1966, Zadeh and Deoser, 1976]. The fundamental nonidentifiability of chemical reaction networks is sometimes referred to as "the fundamental dogma of chemical kinetics" [Craciun and Pantea, 2008]. In computer science, this is framed as the relationship among processes that simulate one another

[Van der Schaft, 2004]. Finally, the field of *inverse problems* studies those cases where, even if a one-to-one mapping between model and behavior is possible in theory, even tiny amounts of noise can make inference problems nonidentifiable in practice *cite?*.

Although nonidentifiability may frustrate the occasional engineer or scientist, viewed from another angle, this concept can provide a starting point for thinking about externally equivalent systems – systems that evolution can explore, so long as the parameters and structures can be realized biologically. These functional symmetries manifest in convergent and parallel evolution, as well as *developmental system drift*: the observation that macroscopically identical phenotypes in even very closely related species can in fact be divergent at the molecular and sequence level [True and Haag, 2001, Tanay et al., 2005, Tsong et al., 2006, Hare et al., 2008, **?**, Matsui et al., 2015, Dalal et al., 2016, Dalal and Johnson, 2017].

In this paper we outline a theoretical framework to study the evolution of biological systems, such as gene regulatory networks. We study the evolution of an optimally adapted population subject to stabilizing selection for phenotype – that is the population evolves under constant selective and environmental pressures. Even if the phenotype remains stable over evolutionary time, the underlying mechanism might not remain so, as many distinct (and mutationally connected) molecular pathways can realize identical phenotypes. Applying results from system theory, we present an analytical description of the set of all linear gene network architectures that yield identical phenotypes, and show that any biological system, in principal, can undergo system drift, as there is never a unique system architecture per phenotype. Even under stabilizing selection, a population can explore the set of all possible phenotypically equivalent gene networks. Phenotypically equivalent gene networks are not necessarily compatible with one another, as such system drift may result in the reproductive incompatibility between populations isolated for a sufficiently long period of time, even in the absence of any sort of adaptative, selective, or environmental change. Finally, in some cases we estimate that reproductive incompatibility due to system drift can manifest on timescales on the order of N_e generations.

analogous to those in the Dobzhansky-Muller model.

REFERENCE: Barton's paper (PETER).

Evolutionary system theory

Here we construct as abstract model to study biological systems, such as gene regulatory networks. The behavior (i.e. temporal dynamics) of such a system is determined by a collection of n coregulating moleculues – such as transcription factors, as well as external or environmental inputs. We write $\kappa(t)$ for the vector of n molecular concentrations at time t. The m "inputs" determined exogenously to the system are denoted $u(t)$, and the ℓ "outputs" are denoted $\phi(t)$. The output is merely a linear function of the internal state: $\phi_i(t) = \sum_j C_{ij}\kappa_i(t)$ for some matrix C. Since ϕ is what natural selection acts on, we refer to it as the *phenotype* (meaning the "visible" aspects of the organism), and in contrast refer to κ as the *kryptotype*, as it is "hidden" from direct selection. Although ϕ may depend on all entries of κ, it is usually of lower dimension than κ, and we tend to think of it as the subset of molecules relevant for survival. The dynamics are determined by the matrix of regulatory coefficients, A; a time-varying vector of inputs $u(t)$, and a matrix B that encodes the effect of each entry of u on the elements of the kryptotype. The rate at which the i^{th} concentration changes is a weighted sum of the concentrations of the other concentrations as well as the input:

$$\begin{aligned} \dot{\kappa}(t) &= A\kappa(t) + Bu(t) \\ \phi(t) &= C\kappa(t). \end{aligned} \tag{1}$$

Furthermore, we always assume that $\kappa(0) = 0$, so that the kryptotype measures deviations from initial concentrations. Here A can be any $n \times n$ matrix, B an $n \times m$, and C any $\ell \times n$ dimensional matrix, with usually ℓ and m less than n. We think of the system as the triple (A, B, C), which translates (time-varying) m-dimensional input $u(t)$ into the ℓ-dimensional output $\phi(t)$. Under quite general assumptions, we can write

the phenotype as

$$\phi(t) = Ce^{At}\kappa(0) + \int_0^t Ce^{A(t-s)}Bu(s)ds, \tag{2}$$

which is a convolution of the input $u(t)$ with the system's *impulse response*, which we denote as $h(t) := Ce^{At}B$.

Although many different biological systems can be modeled with this approach, for clarity, we focus on gene regulatory networks. In this interpretation, A_{ij} determines how the j^{th} transcription factor regulates the i^{th} transcription factor. If $A_{ij} > 0$, then κ_j upregulates κ_i, while if $A_{ij} < 0$, then κ_j downregulates κ_i. The i^{th} row of A is therefore determined by genetic features such as the strength of j-binding sites in the promoter of gene i, factors affecting chromatin accessibility near gene i, or basal transcription machinery activity. The form of B determines how the environment influences transcription factor expression levels, and C might be the rate of production of a downstream enzyme (although other arrangements could be made).

Here we have assumed that the system is linear, and begins from the "zero" state ($\kappa(0) = 0$). Of course, neither of these are necessarily true for real systems, but the dynamics of most nonlinear systems can be approximated locally by a linear systems near most points. Furthermore, the ease of analyzing linear systems makes this an attractive place to start. To demonstrate this approach, we apply it to construct a simple gene network in Example 1 below.

Example 1 (An oscillator)**.** *For illustration, we consider an extremely simplified model of oscillating gene transcription, as for instance is found in cell cycle control or the circadian rhythm. Suppose there are two genes, whose transcript concentrations are given by $\kappa_1(t)$ and $\kappa_2(t)$, and that gene-2 upregulates gene-1 and that gene-1 downregulates gene-2 with equal strength. Furthermore, suppose that only the dynamics of gene-1 are consequential to the oscillator (perhaps the amount of gene-1 activates another downstream gene network). Lastly suppose that the production of both genes is equally upregulated by an exogenous signal. The dynamics of the system are described by*

$$\begin{aligned}\dot{\kappa_1}(t) &= \kappa_2(t) + u(t)\\ \dot{\kappa_2}(t) &= -\kappa_1(t) + u(t)\\ \phi(t) &= \kappa_1(t).\end{aligned}$$

In matrix form the system regulatory coefficients are given as, $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, and $C = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$.

Suppose the input is an impulse at time zero (a delta function), and so its phenotype is equal to its impulse response,

$$\phi(t) = h(t) = \sin t + \cos t.$$

The system and its dynamics are referred to in Figure 1. We return to the evolution of such a system below.

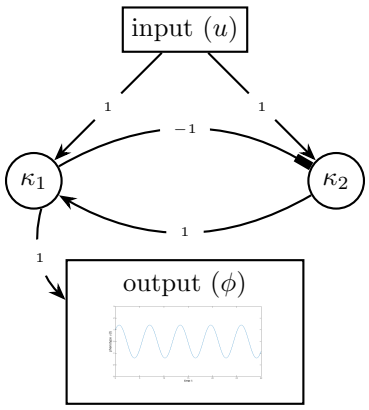

Figure 1: Diagram of the gene network in Example 1 .

Equivalent gene networks

As reviewed above, some systems with identical phenotypes are known to differ, sometimes substantially, at the molecular level; systems with identical phenotypes do not necessarily have identical kryptotypes. How many different mechanisms perform the same function?

Two systems are equivalent if they produce the same phenotype given the same input, i.e., have the same input–output relationship. We say that the systems defined by (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are **phenotypically equivalent** if their impulse response functions are the same: $h(t) = \bar{h}(t)$ for all $t \geq 0$. This implies that for any acceptable input $u(t)$, if $(\kappa_u(t), \phi_u(t))$ and $(\bar{\kappa}_u(t), \bar{\phi}_u(t))$ are the solutions to equation (1) of these two systems, respectively, then

$$\phi_u(t) = \bar{\phi}_u(t) \qquad \text{for all } t \geq 0.$$

One way to find other systems phenotypically equivalent to a given one is by change of coordinates: if V is an invertible matrix, then the systems (A, B, C) and (VAV^{-1}, VB, CV^{-1}) are phenotypically equivalent because their impulse response functions are equal:

$$\begin{aligned} h(t) &= Ce^{At}B = CV^{-1}Ve^{At}V^{-1}VB \\ &= CV^{-1}e^{VAV^{-1}t}VB = \bar{C}e^{\bar{A}t}\bar{B} = \bar{h}(t). \end{aligned} \tag{3}$$

However, not all phenotypically equivalent systems are of this form: systems can have identical impulse responses without being coordinate changes of each other. In fact, systems with identical impulse responses can involve interactions between different numbers of molecules, and thus have kryptotypes in different dimensions altogether.

This implies that most systems have at least n^2 degrees of freedom, where recall n is the number of components of the kryptotype vector. This is because for an arbitrary $n \times n$ matrix Z, taking V to be the identity matrix plus a small perturbation in the direction of Z above implies that moving A in the direction of $ZA - AZ$ while also moving B in the direction of ZB and C in the direction of $-CZ$ will leave the phenotype unchanged. If A is invertible, for instance, then any such Z will result in a different system.

It turns out that in general, there are more degrees of freedom, except if the system is *minimal* – meaning, informally, that it uses the smallest possible number of components to achieve the desired dynamics. Results in system theory show that any system can be realized in a particular minimal dimension (the dimension of the kryptotype, $n_{\min}$), and that any two phenotypically equivalent systems of dimension $n_{\min}$ are related by a change of coordinates.

Some gene networks, however, can grow or shrink, perhaps following gene duplications and deletions, and also still preserve their phenotypes. More generally, even if the system is not minimal, results from systems theory explicitly describe the set of all phenotypically equivalent systems. We refer to $\mathcal{N}(A_0, B_0, C_0)$ as the set of all systems phenotypically equivalent to the system defined by (A_0, B_0, C_0). Concretely, this is

$$\mathcal{N}(A_0, B_0, C_0) = \left\{(A, B, C) : Ce^{At}B = C_0e^{A_0t}B_0 \text{ for } t \geq 0\right\}. \tag{4}$$

These systems need not have the same kryptotypic dimension n, but must have the same input and output dimensions (ℓ and m, respectively).

The Kalman decomposition, which we now describe informally, elegantly characterizes this set [Kalman, 1963, Kalman et al., 1969, Anderson et al., 1966]. To motivate this, first note that the input $u(t)$ only directly pushes the system in certain directions (those lying in the span of the columns of B). As a result, different combinations of input can move the system in any direction that lies in what is known as the *reachable subspace*. Analogously, we can only observe motion of the system in certain directions (those lying in the span of the columns of C), and so can only infer motion in what is known as the *observable subspace*. The Kalman decomposition then classifies each direction in kryptotype space as either reachable or unreachable, and as either observable or unobservable. Only the components that are both reachable and observable determine the system's phenotype – that is, components that respond to an input and components that produce an observable output.

Concretely, the **Kalman decomposition** of a system (A, B, C) gives a change of basis P such that the transformed system (PAP^{-1}, PB, CP^{-1}) has the following form:

$$PAP^{-1} = \begin{bmatrix} A_{r\bar{o}} & A_{r\bar{o},ro} & A_{r\bar{o},\bar{r}\bar{o}} & A_{r\bar{o},\bar{r}o} \\ 0 & A_{ro} & 0 & A_{ro,\bar{r}o} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{\bar{r}\bar{o},\bar{r}o} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix},$$

and

$$PB = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix} \qquad (CP^{-1})^T = \begin{bmatrix} 0 \\ C_{ro}^T \\ 0 \\ C_{\bar{r}o}^T \end{bmatrix}.$$

The impulse response of the system is given by

$$h(t) = C_{ro} e^{A_{ro} t} B_{ro},$$

and therefore, the system is phenotypically equivalent to the *minimal* system (A_{ro}, B_{ro}, C_{ro}). (Here the subscript ro refers to the both *reachable and observable* subspace, while $\bar{r}\bar{o}$ refers to the *unreachable and unobservable* subspace, and similarly for $\bar{r}o$ and $r\bar{o}$.)

Any two minimal systems are related by a change of coordinates, and so the minimal subsystems obtained by the Kalman decomposition are unique up to a change of coordinates. In particular, this implies that there is no equivalent system with a smaller number of kryptotypic dimensions than the dimension of the minimal system. It is also remarkable to note that the gene regulatory network architecture to achieve a given input–output map is never unique – both the change of basis used to obtain the decomposition and, once in this form, all submatrices other than A_{ro}, B_{ro}, and C_{ro} can be changed without affecting the phenotype, and so represent degrees of freedom.

Note on implementation: The *reachable subspace*, which we denote by $\mathcal{R}$, is defined to be the closure of span(B) under applying A, and the *unobservable subspace*, denoted $\bar{\mathcal{O}}$, is the largest A-invariant subspace contained in the null space of C. The four subspaces, $r\bar{o}$, ro, $\bar{r}\bar{o}$, and nro are defined from these by intersections and orthogonal complements.

For the remainder of the paper, we interpret $\mathcal{N}$ as the phenotypically neutral landscape, wherein a large population will drift under environmental and selective stasis. Even if the phenotype is constrained and remains constant through evolutionary time, the molecular mechanism underpinning it is not constrained and likely will not be conserved.

Finally, note that if B and C are held constant – i.e., if the relationships between environment, kryptotype, and phenotype do not change – there are *still* usually degrees of freedom. These correspond to distinct genetic networks that perform indistinguishable functions. The following example 2 gives the set of minimal systems equivalent to the oscillator of Example 1, that all share common B and C matrices. The oscillator can also be equivalently realized by a three-gene (or larger) network, and will have even more evolutionary degrees of freedom available, as in e.g., Figure 3.

Example 2 (All Phenotypically Equivalent Oscillators)**.** *The oscillator of example 1 is minimal, and so any equivalent system is a change of coordinates by an invertible matrix V. If we further require B and C to be invariant then we need $VB = B$ and $CV = C$. Solving these equations, we find that a one-parameter family $(A(\tau), B, C)$ describes the set of all two-gene systems phenotypically equivalent to the oscillator, where*

$$A(\tau) = \frac{1}{\tau - 1} \begin{bmatrix} \tau & -1 \\ 2\tau(\tau - 1) + 1 & -\tau \end{bmatrix} \text{ for } \tau \neq 1.$$

The resulting set of systems, and their dynamics, are depicted in Figure 2

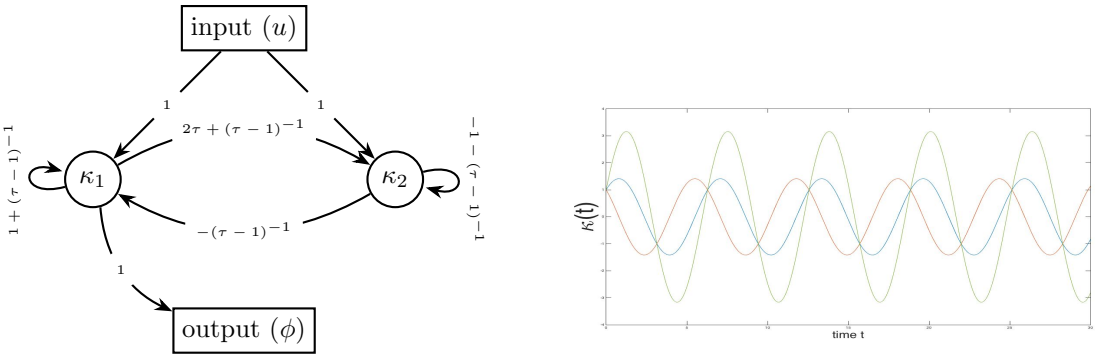

Figure 2: (Left) $A(\tau)$, the set of all phenotype-equivalent cell cycle control networks. (Right) Gene-1 dynamics (blue) for both systems $A(0)$ and $A(2)$ are identical, however, $A(0)$ gene-2 dynamics (orange) differ from $A(2)$ (green).

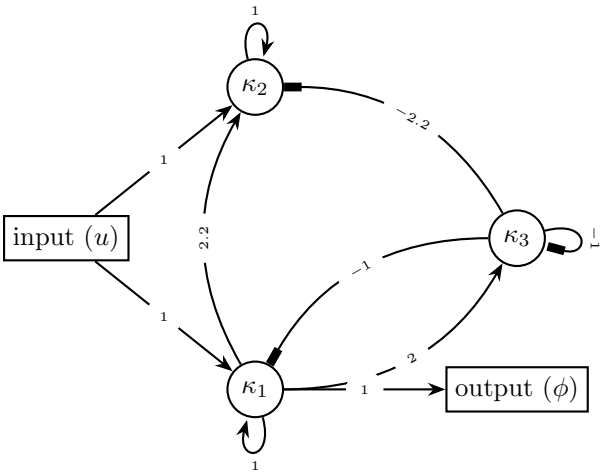

Figure 3: A possible non-minimal three-gene oscillator phenotypically equivalent to the systems in Examples 1 and 2.

Sexual reproduction and recombination Parents with phenotypically equivalent yet differently wired gene networks may produce offspring with dramatically different phenotypes. If the phenotypes are significantly divergent then the offspring may be inviable or otherwise dysfunctional, despite both parents being well adapted. If this is consistent for the entire population, we would consider them to be separate species, in accord with the biological species concept [Mayr, 2000].

Diploid organisms have two genomes, each of which encodes a set of system coefficients. We assume that a diploid which has inherited systems (A', B', C') and (A'', B'', C'') from each of its parents, has phenotype determined by the system that averages these two, $((A' + A'')/2, (B' + B'')/2, (C' + C'')/2)$.

Each genome an organism inherits is generated by meiosis, in which both of its diploid parents recombine their two genomes. We will assume that each coefficient (i.e., entry of A, B or C) is determined by a single nonrecombining locus, so that each coefficient in the system produced by meiosis is an independent random choice between the two parental coefficients. With these definitions, an F_1 offspring carries a system copy from each parent, and an F_2 is an offspring of two independently formed F_1s. If the parents are from distinct populations, these are simply first– and second–generation hybrids, respectively.

This is a simplification: since the i^{th} row of A summarizes how each gene regulates gene i, and hence

is determined by the promoter region of gene i, we would actually expect the elements of a row of A to tend to be inherited together. Similarly, we expect in practice heritable variation in each coefficient to be determined by more than one locus – but this may be a reasonable approximation.

Offspring formed from two phenotypically identical systems do not necessarily exhibit the same phenotype as both of its parents – in other words $\mathcal{N}$, the set of all systems phenotypically equivalent to a given one, is not, in general, closed under averaging or recombination. Next we discuss how this fact can contribute to hybrid incompatibility and genetic load. If sexual recombination among systems drawn from $\mathcal{N}$ yields systems with divergent phenotypes, populations containing significant diversity in $\mathcal{N}$ can carry genetic load, and isolated populations may fail to produce hybrids with viable phenotypes.

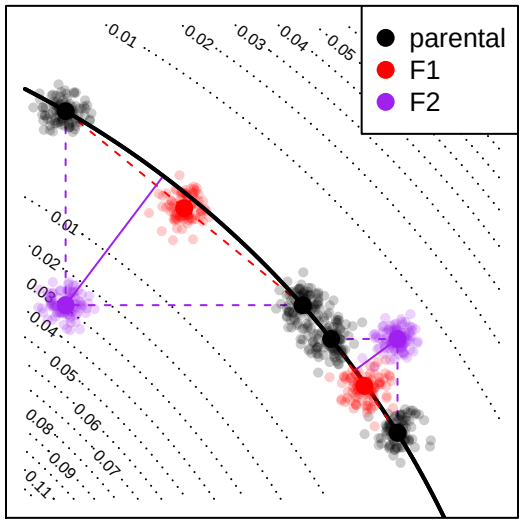

Figure 4: A conceptual figure of the fitness consequences of hybridization: axes represent system coefficients (i.e., entries of A); the line of optimal system coefficients is down in black; dotted lines give phenotypic distances to the optimum. Two pairs of parental populations are shown in black, along the optimum; a hypothetical population of F_1s are shown for each in red, and the distribution of one type of F_2 is shown in purple (other types of F_2 are not shown). Solid lines depict the distance of the F_2 to optimum.

Hybrid incompatibility Two parents with the optimal phenotype can produce offspring whose phenotype is suboptimal if the parents have different underlying systems. How quickly do hybrid phenotypes break down as genetic distance between parents increases? To quantify this, we will measure how far a system's phenotype is from optimal using a weighted difference between impulse response functions. Suppose that $\rho(t)$ is a nonnegative, smooth, square-integrable weighting function, suppose that $h_0(t)$ is the *optimal* impulse response function and define the "distance to optimum" of another impulse response function to be

$$D(h) = \left(\int_0^\infty \rho(t) \|h(t) - h_0(t)\|^2 dt \right)^{1/2}. \tag{5}$$

Consider reproduction between a parent with system (A, B, C) and another displaced by distance ϵ in the direction (X, Y, Z), i.e., having system $(A + \epsilon X, B + \epsilon Y, C + \epsilon Z)$. We assume both are "perfectly adapted" systems, i.e., having impulse response function $h_0(t)$, and their offspring has impulse response function $h_\epsilon(t)$. A Taylor expansion of $D(h_\epsilon)$ in ϵ is explicitly worked out in Appendix D, and shows that the phenotype of an F_1 hybrid between these two is at distance proportional to ϵ^2 from optimal, while F_2 hybrids are at distance proportional to ϵ. This is because an F_1 hybrid has one copy of each parental system, and therefore lies directly between the parental systems (see Figure 4) – the parents both lie in $\mathcal{N}$, which is the valley defined

by D, and so their midpoint only differs from optimal due to curvature of $\mathcal{N}$. In contrast, an F_2 hybrid may be homozygous for one parental type in some coefficients and homozygous for the other parental type in others; this means that each coefficient of an F_2 may be equal to either one of the parents, or intermediate between the two; this means that possible F_2 systems may be as far from the optimal set, $\mathcal{N}$, as the distance between the parents. The precise rate at which the phenotype of a hybrid diverges depends on the geometry – in Figure 4, this is depicted as the angle of the black line (the optimal set) with respect to the coordinates.

Example 3 (Hybrid Incompatibility in the Oscillator)**.** *Offspring of two equivalent systems from Example 2 can easily fail to oscillate. For instance, the F_1 offspring between homozygous parents at $\tau = 0$ and $\tau = 2$ has phenotype $\phi_{F_1}(t) = e^t$, rather than $\phi(t) = \sin t + \cos t$. However, the coefficients of these two parental systems differ substantially, probably more than would be observed between diverging populations. In figure 5 we compare the phenotypes for F_1 and F_2 hybrids between more similar parents, and see increasingly divergent phenotypes as the difference between the parental systems increases. (In this example, the coefficients of $A(\epsilon)$ differ from those of $A(0)$ by an average factor of $1 + \epsilon/2$; such small differences could plausibly be caused by changes to promoter sequences.) This divergence is quantified in Figure 6, which shows that mean distance to optimum phenotype of the F_1 and F_2 hybrid offspring between $A(0)$ and $A(\epsilon)$ increases with ϵ^2 and ϵ, respectively.*

The coefficients in A – i.e., the regulatory coefficients – differ between parents by only a few percent (around 0.5% for $\epsilon = -1/100$ and 5% for $\epsilon = -1/10$). This is well within the amount of regulatory coefficient variation we expect to find segregating within real populations (discussed further below). For these small values of ϵ, hybrid phenotypes remain relatively stable, consistent with the idea that natural selection will allow such intrapopulation variation. For larger values of $|\epsilon|$ (here $1/2$), hybrid phenotypes can become dysfunctional, and depending on context, manifest as hybrid inviability.

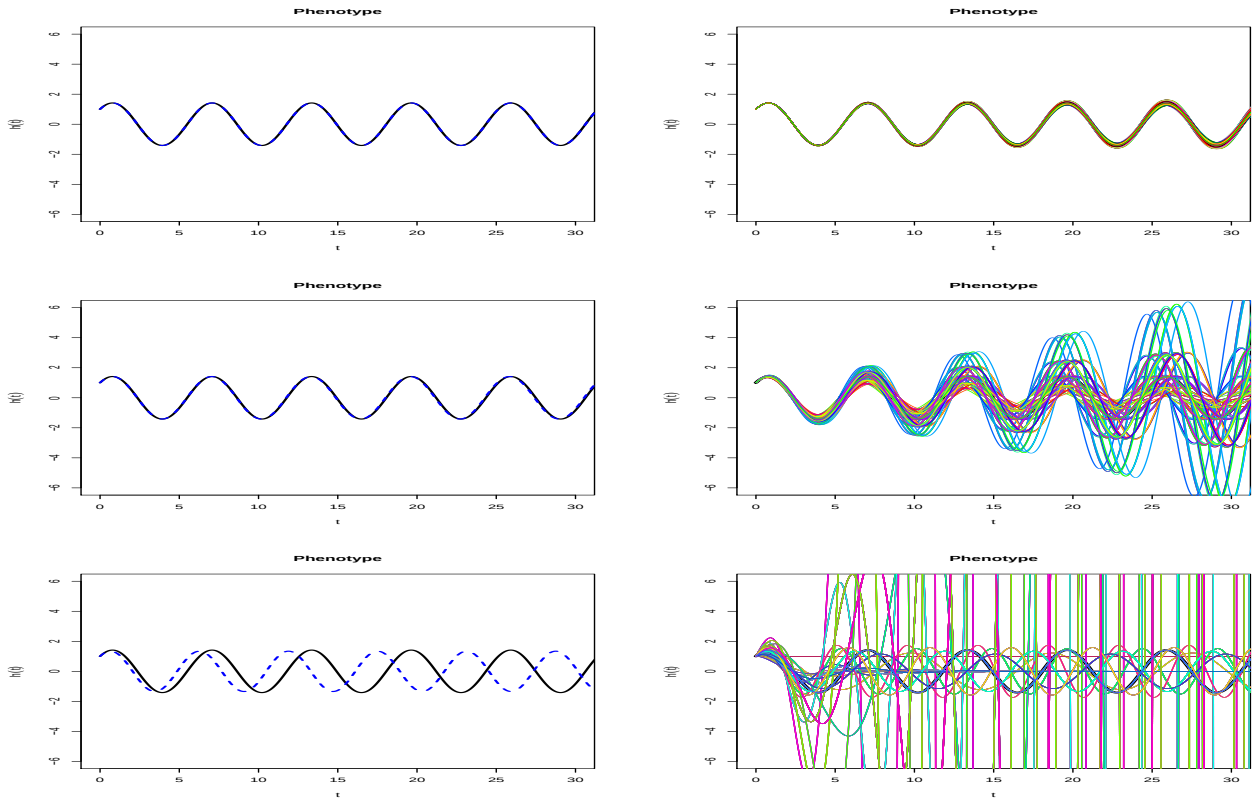

Figure 5: **(left)** Phenotypes of F_1 hybrids between an $A(0)$ parent and, top-to-bottom, an $A(-1/100)$, an $A(-1/10)$, and $A(-1/2)$ parent. Parental phenotypes ($\sin t + \cos t$) are shown in solid black, and hybrid phenotypes in dashed blue. **(right)** Phenotypes of all 256 possible F_2 hybrids between the same set of parents, with parental phenotype again in black. Different colored lines correspond to different F_2s; note that some completely fail to oscillate.

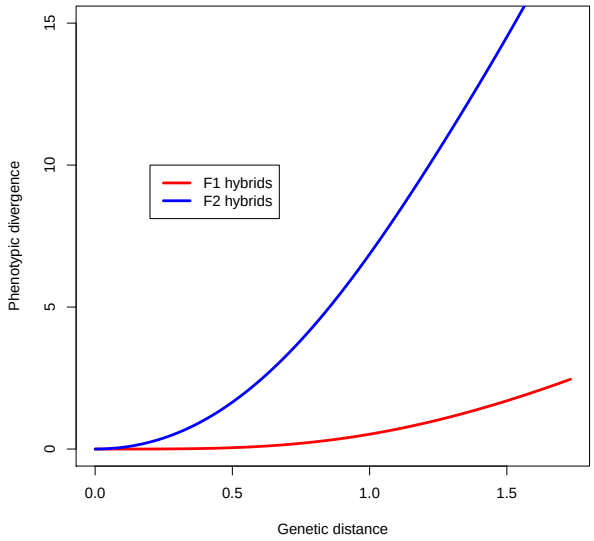

Figure 6: Phenotype distance to optimum, D, using $\rho(t) = \exp(-t/4\pi)$ for F_1 (red) and F_2 (blue) hybrids between an $A(0)$ and an $A(\epsilon)$ parent. Genetic distance is computed as the average coefficient difference between two systems.

Haldane's rule This model naturally predicts Haldane's rule, the observation that if only one hybrid sex is sterile or inviable it is likely the heterogametic sex (e.g., the male in XY sex determination systems) [Haldane, 1922]. For example, consider an XY species with a two-gene network where the first gene resides on an autosome and the second gene on the X chromosome. A male whose pair of haplotypes is $\left(\left[\begin{smallmatrix} A_1 & A_2 \\ \cdot & \cdot \end{smallmatrix}\right], \left[\begin{smallmatrix} A_1 & A_2 \\ X_1 & X_2 \end{smallmatrix}\right]\right)$ has phenotype determined by $A = \left[\begin{smallmatrix} A_1 & A_2 \\ X_1 & X_2 \end{smallmatrix}\right]$, thanks to *Drosophila*-like dosage compensation mechanism (the X is twice upregulated in heterogametes relative to homogametes), while a female homozygous for the haplotype $\left[\begin{smallmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{X}_1 & \bar{X}_2 \end{smallmatrix}\right]$, has phenotype determined by $A = \left[\begin{smallmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{X}_1 & \bar{X}_2 \end{smallmatrix}\right]$. An F_1 male offspring of these two will have its phenotype determined by $\left[\begin{smallmatrix} (A_1+\bar{A}_1)/2 & (A_2+\bar{A}_2)/2 \\ \bar{X}_1 & \bar{X}_2 \end{smallmatrix}\right]$. If both genes resided on the autosomes this system would only be possible in an F_2 cross. More generally, if the regulatory coefficients for a system are shared between the sex and one or more autosomal chromosomes, F_1 males are effectively equivalent to purely autosomal-system F_2s. As such, in this setting, Haldane's rule is mechanistically similar to *hybrid breakdown*, the observation that F_2s will often be less fit than F_1 hybrid crosses, in the early stages of speciation. This mechanism appears to be distinct from the "dominance", "faster-X", and "faster-male" theories [Orr, 1997, Coyne and Orr, 1998, Turelli and Orr, 1995] usually suggested to explain Haldane's rule.

System drift and the accumulation of incompatibilities

Thus far we have shown that many distinct molecular mechanisms can realize identical phenotypes and that these mechanisms may fail to produce viable hybrids. This begs the question: does evolution shift molecular mechanisms fast enough to be a significant driver of speciation? To approach this question, we explore a general quantitative genetic model in which a population drifts stochastically near a set of equivalent and optimal systems due to the action of recombination, mutation, and demographic noise. Although this is motivated by the results on linear systems above, the quantitative genetics calculations are more general, and only depend on the presence of genetic variation and a continuous set of phenotypically equivalent systems.

We will suppose that each organism's phenotype is determined by its vector of coefficients, denoted by $x = (x_1, x_2, \ldots, x_L)$, and that the corresponding fitness is determined by the distance of its phenotype to optimum. The optimum phenotype is unique, but is realized by many distinct x – those falling in the "optimal set" $\mathcal{N}$. The phenotypic distance to optimum of an organism with coefficients x is denoted $D(x)$. In the results above, $x = (A, B, C)$ and $D(x)$ is given by equation (5). determines a system (this is a system (A, B, C) above) with fitness determined by its phenotypic distance $D(x)$ from an optimum (this is analogous to $\mathcal{N}$ above). Concretely, we define the fitness at x to be $\exp(-D(x)^2)$. We will assume that in the region of interest, the map D is smooth and that we can locally approximate the optimal set $\mathcal{N}$ as a quadratic surface. As above, an individual's coefficients are given by averaging its parentally inherited coefficients and adding random noise due to segregation. Concretely, we use the *infinitesimal model* for reproduction [**?**] – the offspring of parents at x and x' will have coefficients $(x + x')/2 + \varepsilon$, where ε is a Gaussian displacement due to random assortment of parental alleles.

System drift We work with a randomly mating population of effective size N_e. If the regulatory coefficient population variation has standard deviation σ in a particular direction, since subsequent generations resample from this diversity, the population mean coefficient will move a random distance of size $\sigma/\sqrt{N_e}$ per generation, simply because this is the the standard deviation of the mean of a random sample [**?**]. Selection will tend to restrain this motion, but movement along the optimal set $\mathcal{N}$ is unconstrained, and so we expect the population mean to drift along the optimal set like a particle diffusing. The amount of variance in particular directions in coefficient space depends on constraints imposed by selection and correlations between the genetic variation underlying different coefficients (the G matrix [**?**]). It therefore seems reasonable to coarsely model the time evolution of population variation in regulatory coefficients as a "cloud" of width σ about the population mean, which moves as an unbiased Brownian motion through the set of network coefficients that give the optimal phenotype.

There will in general be different amounts of variation in different directions; to keep the discussion intuitive, we only discuss σ_N, the amount of variation in "neutral" directions (i.e., directions along $\mathcal{N}$), and σ_S, the amount of variation in "selected" directions (perpendicular to $\mathcal{N}$). The other relevant scale we denote by γ, which the scale on which distance to phenotypic optimum changes as x moves away from the optimal set, $\mathcal{N}$. Concretely, γ is $1/(\frac{d}{du}D(x + uz))$ with respect to u where x is optimal and z is a "selected" direction perpendicular to $\mathcal{N}$. With these parameters, a typical individual will have a fitness of around $\exp(-(\sigma_S/\gamma)^2)$. Of course, there are in general many possible neutral and selected directions; we take these values to be representative of the possible directions.

Hybridization The means of two allopatric populations each of effective size N_e separated for T generations will be a distance roughly of order $2\sigma_N\sqrt{T/N_e}$ apart along $\mathcal{X}$. (Consult figure 4 for a conceptual diagram.) A population of F_1 hybrids has one haploid genome from each, whose coefficients are averaged, and so will have mean system coefficients at the midpoint between their means. Each F_2 hybrid will be homozygous for one parental allele on average at half of the loci in the genome, so the distribution of F_2s will have mean at the average of the two populations, but will have higher variance. The variance of F_2s can be shown to increase linearly with the square of the distance between parental population means under models of both simple and polygenic traits. This is suggested by figure 4 and shown in Appendix A. *connect to Barton etc polygenic adaptation lit* Concretely, we expect the population of F_1s to have variance σ_S^2 in the selected direction (the same as within each parental population), but the population of F_2s will have variance of order $\sigma_S^2 + 4\omega\sigma_N^2 T/N_e$, where ω is a factor that depends on the genetic basis of the coefficients. In a model of p traits in which the optimal set $\mathcal{N}$ has dimension q, using the polygenic model of appendix A, $\omega = (p - q)/8$. If each trait is controlled by a single locus, as in figure 4, the value is similar.

What are the fitness consequences? A population of F_2s will begin to be substantially less fit than the parentals once they differ from the optimum by a distance of order γ, i.e., once $\sqrt{4\omega T/N_e} \approx \gamma/\sigma_N$. This implies that hybrid incompatibility among F_2s should appear on a time scale of $N_e(\gamma/\sigma_N)^2/(4\omega)$ generations. The F_1s will not suffer fitness consequences until the hybrid mean is further than γ from the optimum; as shown in appendix B (and suggested by figure 4), this deviation of the mean from optimum grows with the

square of the distance between the parental populations, and so we expect fitness costs in F_1s to appear on a time scale of N_e^2 generations.

For a more concrete prediction, suppose that the distribution among hybrids is Gaussian. A population whose trait distribution is Gaussian with mean μ and variance σ, has mean fitness

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{-\frac{x^2}{2\gamma^2}} dx = \sqrt{\frac{1}{1+\sigma^2/\gamma^2}} \exp\left\{-\frac{\mu^2}{\gamma^2}\left(\frac{1}{1+\sigma^2/\gamma^2}\right)\right\}. \tag{6}$$

This assumes a single trait, for simplicity; the multivariate case is done in appendix C. A population of F_2s will have, as above, variance $\sigma^2 = \sigma_S^2 + 4\omega\sigma_N^2 T/N_e$. The mean diverges with the square of the distance between the parentals with a speed depending on the local geometry of the optimal set (calculated in appendix D), so we set $\mu = c_\mu \gamma T/N_e$. The mean fitness in parental populations is as in equation 6 with $\mu = 0$ and $\sigma = \sigma_S$. This implies that if we define $\mathcal{F}_2(T)$ to be the mean relative fitness among F_2 hybrids between two populations separated by T generations, (i.e., the mean fitness divided by the mean fitness of the parents) then *figure out how dimensions come in here*

$$\mathcal{F}_2(T) = \left(1 + \frac{4\omega(\sigma_N/\gamma)^2}{(1+(\sigma_S/\gamma)^2)}\frac{T}{N_e}\right)^{-1/2} \exp\left\{-\left(c_\mu \frac{T}{N_e}\right)^2 \left(\frac{1}{1+(\sigma_S/\gamma)^2 + 4\omega(\sigma_N/\gamma)^2 T/N_e}\right)\right\}. \tag{7}$$

If each of the q selected directions acts independently, the drop in fitness will be $\mathcal{F}_2(T)^q$ (the expression for the correlated cases is given in Appendix C). We discuss the implications of this expression in the next section.

Speciation rates under neutrality Above, equation (7) shows how fast hybrids become inviable as the time that the parental populations are isolated increases; what does this tell us about speciation rates under neutrality? From equation (7) we can observe that time is always scaled in units of N_e generations, the population standard deviations are always scaled by γ, and the most important term is the rate of accumulation of segregation variance, $4\omega(\sigma_N/\gamma)^2$. All else being equal, this process will lead to speciation more quickly in smaller populations and in populations with more neutral genetic variation (larger σ_N). These parameters are related – larger populations generally have more genetic variation – but since these details depend on the situation, we leave these separate.

How does this prediction depend on the system size and constraint? If there are p trait dimensions, constrained in q dimensions, and if ω is proportional to $p-q$, then the rate that F_2 fitness drops is, roughly, $(1+4(p-q)CT/n_e)^{-q/2} \propto q(p-q)$, where C is a constant. This suggests that both degree of constraint and number of available neutral directions affect the speed of accumulation of incompatibilities. However, note that in real systems, it is likely that γ also depends on p and q. *revisit with sims*

Suppose in a large, genetically diverse population, the amount of heritable variation in the neutral and selected directions are roughly equal ($\sigma_N \approx \sigma_S$) but the overall amount of variation is (weakly) constrained by selection ($\sigma_N \approx \gamma$). If so, then the first term of equation (7) is $1/\sqrt{1+2\omega T/N_e} \approx 1 - \omega T/N_e$. If also $\omega = 1$, then, for instance, after $0.1N_e$ generations the average F_2 fitness has dropped by 10% relative to the parentals.

Consider instead a much smaller, isolated population whose genetic variation is primarily constrained by genetic drift, so that $\sigma_N \approx \sigma_S \ll \gamma$. Setting $a = (\sigma_N/\gamma)^2$ to be small, the fitness of F_2s is $\mathcal{F}_2 \leq 1/\sqrt{1+4\omega a T/N_e} \approx 1 - 2\omega a T/N_e$. Hybrid fitness seems to drop more slowly in this case in figure 7, but since time is scaled by N_e, so speciation may occur *faster* than in a large population. However, at least in some models [**?**], in small populations at mutation-drift equilibrium the amount of genetic variance (σ_N^2) is proportional to N_e, which would compensate for this difference, perhaps even predicting the rate of decrease of hybrid fitness to be *independent* of population size for small populations.

In the other direction, consider large metapopulations (or isolated "species complexes") among which heritable variation is strongly constrained by selection (i.e., there is substantial recombination load), so that $\sigma_S \approx \gamma$ but σ_N/γ is large. Then the fitness of F_2s is $\mathcal{F}_2 \leq 1/\sqrt{1+2\omega a T/N_e} \approx 1 - \omega a T/N_e$, and could be extremely rapid if a is large.

For instance, in a population of one million organisms that has 10 generations per year (a drosophilid species, perhaps) under the "large population" scenario of Figure 7A, system drift would lead to a substantial fitness drop of around 10% in F_2 hybrids in only 10,000 years. This drop may be enough to induce evolutionary reinforcement of reproductive isolation. If one thousand of these organisms is isolated (perhaps on an island, as in Figure 7B), then a similar drop could occur in around 120 years. On the other hand, if the population is one of several of similar size that have recently come into secondary contact after population re-expansion, the situation may be similar to that of Figure 7C with $N_e = 10^6$, the same drop could occur after 1,100 years. However, hyperdiverse populations of this type may not be stable on these time scales.

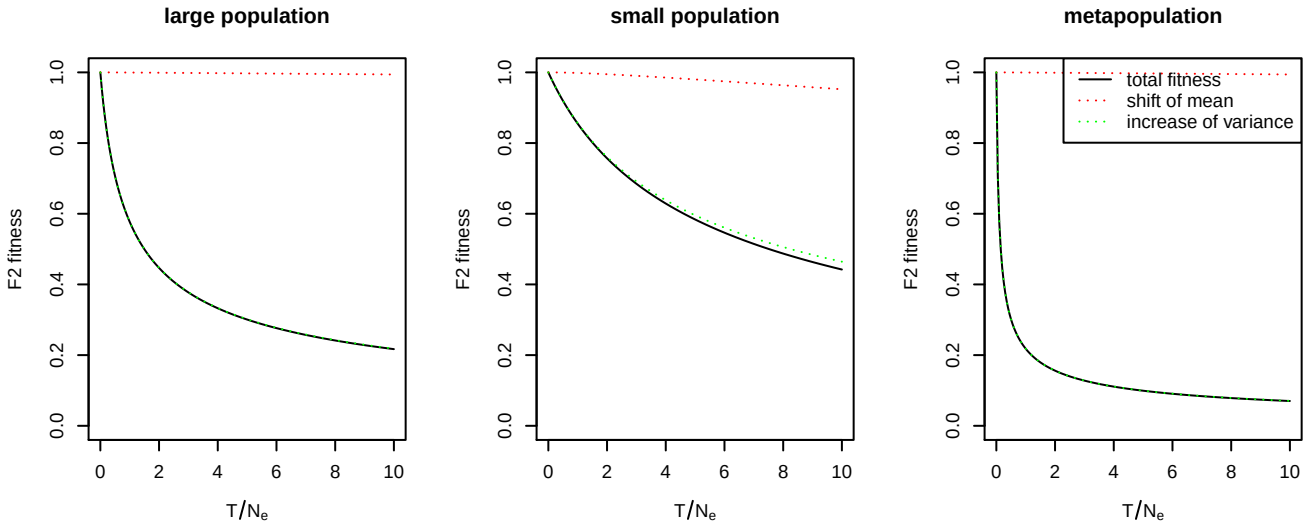

Figure 7: Mean drop if F_2 fitness relative to parental species, with $\omega = 1$ and **(A)** $\sigma_N^2 = \sigma_S^2 = \gamma^2$ **(B)** $\sigma_N^2 = \sigma_S^2 = 0.1\gamma^2$ **(C)** $0.1\sigma_N^2 = \sigma_S^2 = \gamma^2$. The total fitness is from equation (7), and is the product of the "shift of mean" and "increase of variance" terms (the exponential and the square root, respectively). Note that, as we assume, the contribution of the shift in mean is small relative to that of increased variance.

Genetic variation in empirical regulatory systems

What is known about the key quantity above, the amount of heritable variation in real regulatory networks? The coefficient A_{ij} from the system (1) measures how much the rate of net production of i changes per change in concentration of j. It is generally thought that regulatory sequence change contributes much more to inter- and intraspecific variation than does coding sequence change affecting molecular structure [**?**]. In the context of transcription factor networks this may be affected not only by the binding strength of molecule j to the promoter region of gene i but also the effects of other transcription factors (e.g., cooperativity) and local chromatin accessibility [Stefflova et al., 2013]. For this reason, the mutational target size for variation in A_{ij} may be much larger than the dozens of base pairs typically implicated in the handful of binding sites for transcription factor j of a typical promoter region, and single variants may affect many entries of $\mathcal{S}$ simultaneously.

Variation in binding site occupancy may overestimate variation in A, since it does not capture buffering effects (if for instance only one site of many needs be occupied for transcription to begin), and variation in expression levels measures changes in steady-state concentration (our κ_i) rather than the *rate* of change. Nonetheless, Kasowski et al. [2010] found differential occupancy in 7.5% of binding sites of a transcription factor (p65) between human individuals. Verlaan et al. [2009] showed that cis-regulatory variation accounts for around 2–6% of expression variation in human blood-derived primary cells, while [Lappalainen et al.,

2013] found that human population variation explained about 3% of expression variation. Taken together, this suggests that variation in the entries of $\mathcal{S}$ may be on the order of at least a few percent between individuals of a population – doubtless varying substantially between species and between genes.

The impact of regulatory variation (σ) above depends only on its magnitude relative to selection (γ), but it seems reasonable that these two quantities are of the same magnitude.

Discussion

Above we synthesize concepts and tools from molecular and evolutionary biology with tools from control theory to study the evolution of a mechanistic model of the molecular genotype-phenotype map under stabilizing selection. This model allows us to analytically describe the processes discussed verbally as phenogenetic drift [Weiss and Fullerton, 2000] and developmental system drift [True and Haag, 2001]. In this context, the Kalman decomposition [Kalman, 1963] implies that nearly all systems are nonidentifiable, and gives an analytical description of all phenotypically equivalent gene networks. System nonidentifiability, rather than being a computational nuisance, implies the existence of axes of genetic variation that are not constrained by selection. The independent movement of separated populations along these axes can lead to reduction in hybrid viability, and hence speciation, at a speed that depends on effective population size and amount of genetic variation. In this model, at biologically reasonable parameter values, system drift is a significant – and often rapid – driver of speciation. This may at first be surprising because hybrid inviability appears as a consequence of recombining different, yet functionally equivalent, mechanisms.

Consistent with empirical observation of *hybrid breakdown* [e.g. Plötner et al., 2017]), we see that the fitnesses of F_2 hybrids drop at a much faster rate than those of F_1s, since F_1 and F_2 phenotypes diverge linearly and at the square root of time relative to parentals, respectively. *refer to Turelli here* Another natural consequence of the model is Haldane's rule, that if only one F_1 hybrid sex is inviable or sterile it is likely to be the heterogametic sex. This occurs because if the genes underlying a regulatory network are distributed among both autosomes and the sex chromosome, then heterogametic F_1s show variation similar to that seen in F_2s.

Is there evidence that this is actually occurring? System drift and network rewiring has been inferred across the tree of life [Dalal and Johnson, 2017, Johnson, 2017], and there is often significant regulatory variation segregating within populations [**?**]. *cite sergey?* Transcription in hybrids between closely related species with conserved transcriptional patterns can also be divergent [Haerty and Singh, 2006, Maheshwari and Barbash, 2012, Coolon et al., 2014, Michalak and Noor, 2004]. Furthermore, in cryptic species complexes (e.g., sun skinks [Barley et al., 2013]), genetically distinct species may be nearly morphologically indistinguishable. *check the skinks have RI!! the paper just says highly "genetically distinct"; nothing is stated about RI revisit* Of course the basic assumptions of this model will be violated in practice (constant selective pressures, etc.), however, this model can function as a "neutral null" description of gene network evolution (a strategy advocated for in Lynch [2007], Fay and Wittkopp [2008], Koonin [2016]). If this model succeeds in describing actual system evolution, it can be inferred that the mechanism underlying species formation does not require the inclusion of adaptation or changes in selection to explain the emergence of hybrid incompatibility.

The origin of species not by means of natural selection? Although, as classically formulated, the Dobzhansky-Muller model of hybrid incompatibility is agnostic to the relative importance of neutral versus selective genetic substitutions [Coyne and Orr, 1998], and plausible mechanisms for the origin of Dobzhansky–Muller incompatibilities by neutral genetic drift have been proposed [**?**] and under stabilizing selection [Fierst and Hansen, 2009], previous authors have argued that neutral processes are likely too slow to be a significant driver of speciation [Nei et al., 1983, Seehausen et al., 2014]. Using simulations, Porter and Johnson [2002] demonstrated the accumulation of hybrid incompatibilities under directional, but not stabilizing selection, and Palmer and Feldman [2009] observed the appearence of incompatibilities in suboptimally adapted populations in a constant environment. This, in light of the few known incompatibilities, has lead some to conclude that hybrid incompatibility is typically a byproduct of positive selection [Orr et al., 2004, Schluter,

2009] or a conseuqence of genetic conflict [Presgraves, 2010, Crespi and Nosil, 2013]. In contrast, the model we develop here, suggests that even under strictly neutral conditions, system drift can lead to speciation, perhaps at a rate fast enough to play a role in species formation across the tree of life. Our results show that hybrids, under neutral processes, break down as a function of genetic distance, and despite the present model's myriad simplifications and assumptions, it may, in part, explain the observed broad patterns of reproductive isolation, such as the observed consistent relationship between molecular divergence and genetic isolation across diverse taxa and ecologies by Roux et al. [2016], and the suggestivley nonadaptive, clocklike speciation patterns observed by Hedges et al. [2015] (as diversification is not observed to be accelerated following mass extinctions). While the incompatibility of any one given network within an organism may be small, we note that organisms are made up of many different networks, and that the cumulative impact of system drift across all of these may be substantial. Further, even in populations experiencing directional selection, networks underlying conserved parts of the phenotype can still experience neutral drift and harbor incompatibilities.

Nonlinearity Of course, real regulatory networks are not linear dynamical systems. Most notably, physiological limits put upper bounds on expression levels, implying saturating response curves [**?**]. It remains to be seen how well these results carry over into real systems, but the fact that most nonlinear systems can be locally approximated by a linear one suggests qualitatively similar results. Furthermore, nonidentifiability (which implies the existence of neutral directions) is often found in practice in moderately complex models of biological systems. *need citation*

Peter: Quant gen discussion Maybe compare to Fierst and Hansen [2009]?
Do we talk about the G matrix and mutational correlation somewhere?
Note that islands or bottlenecks leave large σ_N but small N_e so may go fast.
What is Ne? In a metapopulation depends on migr rate: classic Turelli paper.
This has implications for genetic load also!!
hominids; neanderthal load.

Acknowledgements

We would like to thank Sergey Nuzhdin, Stevan Arnold, Erik Lundgren, and Hossein Asgharian for valuable discussion.

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A Genetic drift with a multivariate trait

For completeness, we provide a brief exposition of how a population evolves due to genetic drift with a quantitative genetics model, as in Lande [1981] or Hansen and Martins [1996]. These do not directly model underlying genetic basis, but developing a more accurate model is beyond the scope of this paper.

Suppose that the population is distributed in trait space as a Gaussian with covariance matrix Σ and mean μ, whose density we write as $f(\cdot; \Sigma, \mu)$. Selection has the effect of multiplying this density by the fitness function and renormalizing, so that if expected fitness of x is proportional to $\exp(-\|Lx\|^2/2)$, then the distribution post-selection has density at x proportional to $f(x; \Sigma, \mu)\exp(-\|Lx\|^2/2)$. By the computation below ("Completing the square"), the result is a Gaussian distribution with covariance matrix $(\Sigma^{-1}+L^TL)^{-1}$ and mean $(\Sigma^{-1} + L^TL)^{-1}\Sigma^{-1}\mu$.

After selection, we have reproduction: suppose this occurs as in the infinitesimal model [**?**], so that each offspring of parents with traits x and y is drawn independently from a Gaussian distribution with mean $(x + y)/2$ and covariance matrix R. Here, R is the contribution of "segregation variance". If $\widetilde{\Sigma} = (\Sigma^{-1} + L^TL)^{-1}$ is the covariance matrix of the parents post-selection, then the distribution of offspring will again be Gaussian, with mean equal to that of the parents and covariance matrix $\widetilde{\Sigma}/2 + R$.

In summary, a generation under this model modifies the mean (μ) and covariance matrix (Σ) of a population as follows:

$$\mu \mapsto \mu' = (\Sigma^{-1} + L^T L)^{-1}\Sigma^{-1}\mu$$
$$\Sigma \mapsto \Sigma' = \frac{1}{2}(\Sigma^{-1} + L^T L)^{-1} + R.$$

What measures are stable under this transformation? The condition $\mu = \mu'$ reduces to $\Sigma L^T L\mu = 0$; if we assume R and therefore Σ are of full rank, then this happens if and only if μ is in the null space of L, i.e., if μ lies in a neutral direction. The condition $\Sigma' = \Sigma$ can also be solved, at least numerically. After rearrangement, it reduces to $\Sigma L^T L\Sigma + (I/2 - RL^T L)\Sigma = R$. We can find a more explicit description if we assume that $x^T L^T Lx = \sum_{i=1}^k x_i^2$, i.e., that selection only cares about the first k coordinates, and then with no interactions between traits. If so, the condition $\Sigma' = \Sigma$ can be written in block form as

$$\begin{bmatrix} \Sigma_{11}^2 + (I/2 - R_{11})\Sigma_{11} & \Sigma_{11}\Sigma_{12} + (I/2 - R_{11})\Sigma_{12} \\ \Sigma_{12}^T\Sigma_{11} + \Sigma_{12}^T/2 - R_{12}^T\Sigma_{11} & \Sigma_{22} - R_{12}^T\Sigma_{12} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}.$$

The first equation, $\Sigma_{11}^2 + (I/2 - R_{11})\Sigma_{11} = R_{11}$, can be solved with the quadratic formula:

$$\Sigma_{11} = (R_{11} - I/2 + Q)/2$$

for any Q that commutes with R_{11} and is a solution to $Q^2 = (R_{11} - I/2)^2 + 4R_{11}$. Since we need Σ to be positive definite, we take the solution with positive eigenvalues. Given Σ_{11}, the remaining components are

$$\Sigma_{12} = (\Sigma_{11} + I/2 - R_{11})^{-1} R_{12}$$
$$\Sigma_{22} = R_{12}^T\Sigma_{12} + R_{22}.$$

Importantly, the mean μ does not affect either how the covariance matrix moves, or its stable shape.

Above we have described the *expected* motion of the mean and covariance.. However, random resampling will introduce noise. Suppose that a population of N individuals behaves approximately as described above. By the above, we may expect that the covariance matrix stays close to a constant value Σ, computed from R and L as above, so that we need only consider motion of the mean, μ. Since we take a sample of size N to construct the next generation, the next generation's mean is drawn from a Gaussian distribution with mean μ' and covariance matrix Σ/N. Defining $\Gamma = (I - (I + \Sigma L^T L)^{-1})$, this can be written as

$$\mu' - \mu = \Gamma\mu + \epsilon/\sqrt{N},$$

where ϵ is a multivariate Gaussian with mean zero and covariance matrix Σ. Let $\mu(k)$ denote the mean in the k^{th} generation, and suppose that μ differs from optimal by something of order $1/N$: if $\nu(t) = N\mu(tN)$, then the previous equation implies that as $N \to \infty$, in the limit ν solves the Itô equation

$$d\nu(t) = \Gamma\nu(t)dt + \Sigma^{1/2}dW(t),$$

where now $W(t)$ is a multivariate white noise. This has an explicit solution as a multivariate Ornstein-Uhlenbeck process:

$$\nu(t) = e^{-t\Gamma}\nu_0 + \int_0^t e^{-(t-s)\Gamma}\Sigma^{1/2}dW(s).$$

The asymptotic variance of this process in the direction z is

$$\lim_{t\to\infty} \text{Var}[\nu(t)\cdot z] = \int_0^\infty z^T e^{-s\Gamma}\Sigma e^{-s\Gamma}z ds, \tag{8}$$

which is infinite iff $\Gamma z = 0$, which occurs iff $Lz = 0$. In other words, population mean trait values lie away from the optimal set by a Gaussian displacement of order $1/N$ with a covariance matrix given by equation (8).

Now write what this means intuitively.

Completing the square First note that if A is symmetric,

$$(x-y)^T A(x-y) = x^T A\,(x-2y) + y^T Ay,$$

and so if B is also symmetric and $A+B$ is invertible,

$$\begin{aligned}(x-y)^T A(x-y) + x^T Bx &= x^T(A+B)\left(x - 2(A+B)^{-1}Ay\right) + y^T Ay \\ &= \left(x-(A+B)^{-1}Ay\right)^T (A+B)\left(x-(A+B)^{-1}Ay\right) \\ &\quad + y^T Ay - y^T A^T (A+B)^{-1}Ay.\end{aligned}$$

Therefore, by substituting $A=\Sigma^{-1}$ and $B=L^TL$,

$$\frac{f(x;\Sigma,y)\exp(-x^TL^TLx/2)}{\int f(z;\Sigma,y)\exp(-z^TL^TLz/2)dz} = f(x;(\Sigma^{-1}+L^TL)^{-1},(\Sigma^{-1}+L^TL)^{-1}\Sigma^{-1}y).$$

Evolution of segregation covariance

The description above does not completely describe how two diverging populations interact, since the amount of *segregation variance*, quantified by R, will not stay constant. To get an idea of how this might change, suppose that a multivariate trait is determined by L unlinked, biallelic loci, and that the i^{th} locus has two alleles with additive effects $\pm x_i$, so that begin homozygous for the + allele contributes $+2x_i$ to the trait. For simplicity, we will neglect the effects of selection. *fixup the below to be actually multivariate* If the + allele at locus i is at frequency p_i in a population, then the mean and genetic variance of the trait in a diploid population with random mating is *the mean should be* $\sum_i x_i(2p_i-1)$

$$\begin{aligned}m &= 2\sum_i p_i x_i \\ s^2 &= 2\sum_i p_i(1-p_i)x_i^2.\end{aligned}$$

Segregation variance between two parents depends on the loci at which either are heterozygous, and each locus contributes independently since alleles are additive. If the alleles are at Hardy-Weinberg proportions, then since segregation is a fair coin flip, a heterozygous locus contributes $x_i^2/4$ to the variance, and so the *mean* segregation variance, averaging across parents, is

$$R_0(p) = \frac{1}{2}\sum_i p_i(1-p_i)x_i^2.$$

On the other hand, if the second parent came from a distinct population with frequencies q_i (an F_1 hybrid), this would be

$$\begin{aligned}R_1(p,q) &= \frac{1}{4}\sum_i p_i^2(1-p_i)^2x_i^2 + \frac{1}{4}\sum_i q_i^2(1-q_i)^2x_i^2 \\ &= (R_0(p)+R_0(q))/2.\end{aligned}$$

If we assume that the populations are at equilibrium, $R_0(p)\approx R_0(q)$, and so $R_1(p,q)\approx R_0(p)$.

Now consider an F_2 hybrid, where both parents are F_1 and so each heterozygous at locus i with probability $p_i(1-q_i)+(1-p_i)q_i$. Then

$$R_2(p,q) = \frac{1}{4}\sum_i \left(p_i(1-q_i)+(1-p_i)q_i\right)x_i^2.$$

Suppose that the two populations are slightly drifted from each other, with frequency difference $p_i - q_i = 2\epsilon_i$. Then,

$$\begin{aligned} p(1-q) + p(1-q) &= (u+\epsilon)(1-u+\epsilon) + (u-\epsilon)(1-u-\epsilon) \\ &= 2u(1-u) + 2\epsilon^2. \end{aligned}$$

If the frequencies have evolved neutrally in unconnected, Wright-Fisher populations of effective size N for t generations from a common ancestor with allele frequency u, then ϵ has mean zero and variance roughly $u(1-u)t/N$. Still assuming the populations are at stationarity, so that R_0 is constant between the two, and taking the frequencies p_i as a proxy for the ancestral frequencies u_i, this implies that we expect

$$\begin{aligned} R_2 &\approx R_0 + \frac{1}{2}\sum_i p_i(1-p_i)x_i^2 t/N \\ &= \left(1 + \frac{t}{N}\right) R_0. \end{aligned}$$

On the other hand, the expected squared difference in trait *means* here is

$$4\sum_i p_i(1-p_i)x_i^2 t/N = 8R_0 t/N. \tag{9}$$

This implies that under this model, segregation variance in F_2s between two populations is roughly increased by a factor of 1/8 of the difference between their means.

B Away from the optimum

Let two points on $\mathcal{N}$ be x_1 and x_2, let $\bar{x} = (x_1 + x_2)/2$, and let $z = (x_2 - x_1)/2$. Then with ∇D and $\nabla^2 D$ the first and second derivatives of D, respectively, Taylor expanding about x_1 and x_2 finds that

$$\begin{aligned} D(\bar{x}) &= D(x_1) + \nabla D(x_1)\cdot z + \frac{1}{2} z^T \nabla^2 D(x_1) z + O(\|z\|^3) \\ &= D(x_2) - \nabla D(x_2)\cdot z + \frac{1}{2} z^T \nabla^2 D(x_2) z + O(\|z\|^3). \end{aligned}$$

Now, since $D(x_1) = D(x_2) = \nabla D(x_1) = \nabla D(x_2) = 0$ and

$$\begin{aligned} \nabla D(x_2) = \nabla D(x_1) + 2z^T \nabla^2 D(x_1) + O(\|z\|^2), \quad \text{and} \\ \nabla^2 D(x_2) = \nabla^2 D(x_1) + O(\|z\|), \end{aligned}$$

adding together the two equations above and dividing by two gets that

$$D(\bar{x}) = \Phi_0 - \frac{3}{2} z^T \nabla^2 D(x_1) z + O(\|z\|^3).$$

C Gaussian load

Suppose that a population has a Gaussian distribution in d-dimensional trait space with mean μ and covariance matrix Σ, and that fitness of an individual at x is $\exp(-\|Lx\|^2/2)$. Then, completing the square as

above with $A = \Sigma^{-1}$, $y = \mu$, and $B = L^T L$, and defining $Q = (\Sigma^{-1} + L^T L)^{-1}$,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}^n \det(\Sigma)^{1/2}} \int e^{-\frac{1}{2}x^T\Sigma^{-1}x} e^{-\frac{1}{2}x^T L^T L x} dx \\
&\quad = \frac{1}{\sqrt{2\pi}^n \det(\Sigma)^{1/2}} \int e^{-\frac{1}{2}(x-Q\Sigma^{-1}\mu)^T Q^{-1}(x-Q\Sigma^{-1}\mu)} dx \\
&\qquad \times e^{\mu^T(I-\Sigma^{-1}Q)\Sigma^{-1}\mu} \\
&\quad = \sqrt{\frac{\det(Q)}{\det(\Sigma)}} \exp\left\{\mu^T\left(I - \Sigma^{-1}Q\right)\Sigma^{-1}\mu\right\} \\
&\quad = \sqrt{\frac{1}{\det(\Sigma)\det(\Sigma^{-1} + L^T L)}} \exp\left\{\mu^T\left(I - (I + L^T L\Sigma)^{-1}\right)\Sigma^{-1}\mu\right\}
\end{aligned}$$

Now suppose that $\Sigma = \sigma^2 I$ and $L = I/\gamma$. Then,

$$\begin{aligned}
\sqrt{\frac{1}{\det(\Sigma)\det(\Sigma^{-1} + L^T L)}} &= \sqrt{\frac{1}{\sigma^{2d}(1/\sigma^2 + 1/\gamma^2)^d}} \\
&= \frac{1}{(1 + (\sigma/\gamma)^2)^{d/2}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\left(I - (I + L^T L\Sigma)^{-1}\right)\Sigma^{-1} &= \frac{1}{\sigma^2}\left(1 - (1 + (\sigma/\gamma)^2)^{-1}\right) I \\
&= \frac{1}{\gamma^2}\frac{1}{(1 + (\sigma/\gamma)^2)} I
\end{aligned}$$

D Differentiating the fitness function

Add refs to sensitivity analysis.

Suppose that $\rho(t) \geq 0$ is a weighting function on $[0, \infty)$ so that fitness is a function of $L^2(\rho)$ distance of the impulse response from optimal. With $h_0(t) = C_0 e^{tA_0} B_0$ a representative of the optimal set:

$$\begin{aligned}
D(A, B, C) &:= \int_0^\infty \rho(t) \left|h_A(t) - h_0(t)\right|^2 dt \\
&:= \int_0^\infty \rho(t) \left|Ce^{At}B - C_0 e^{A_0 t} B_0\right|^2 dt \\
&= \int_0^\infty \rho(t) \operatorname{tr}\left\{\left(Ce^{At}B - C_0 e^{A_0 t} B_0\right)^T \left(Ce^{At}B - C_0 e^{A_0 t} B_0\right)\right\} dt \\
&= \int_0^\infty \rho(t) \operatorname{tr}\left\{\left(Ce^{At}B - C_0 e^{A_0 t} B_0\right) \left(Ce^{At}B - C_0 e^{A_0 t} B_0\right)^T\right\} dt.
\end{aligned} \tag{10}$$

How does this change as we perturb about (A_0, B_0, C_0)? First we differentiate with respect to A, keeping $B = B_0$ and $C = C_0$ fixed. Since

$$\frac{d}{du} e^{(A+uZ)t}|_{u=0} = \int_0^t e^{As} Z e^{A(t-s)} ds, \tag{11}$$

we have that

$$\begin{aligned}\frac{d}{du}D(A+uZ,B_0,C_0)|_{u=0} &= 2\int_0^\infty \rho(t)\,\mathrm{tr}\left\{C_0\left(\int_0^t e^{As}Ze^{A(t-s)}ds\right)B_0B_0^T\left(e^{At}-e^{A_0t}\right)^T C_0^T\right\}dt \\ &= 2\int_0^\infty \rho(t)\,\mathrm{tr}\left\{C_0\left(\int_0^t e^{As}Ze^{A(t-s)}ds\right)B_0\left(h_A(t)-h_0(t)\right)^T\right\}dt\end{aligned} \tag{12}$$

and, by differentiating this and supposing that A is on the optimal set, i.e., $h_A(t) = h_0(t)$, (so wolog $A = A_0$):

$$\begin{aligned}\mathcal{H}^{A,A}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv}D(A_0+uY+vZ,B_0,C_0)|_{u=v=0} \\ &= \int_0^\infty \rho(t)\,\mathrm{tr}\left\{C_0\left(\int_0^t e^{A_0s}Ye^{A_0(t-s)}ds\right)B_0B_0^T\left(\int_0^t e^{A_0s}Ze^{A_0(t-s)}ds\right)^T C_0^T\right\}dt.\end{aligned} \tag{13}$$

The function $\mathcal{H}$ will define a quadratic form. To illustrate the use of this, suppose that B and C are fixed. By defining Δ_{ij} to be the matrix with a 1 in the (i,j)th slot and 0 elsewhere, the coefficients of the quadratic form is

$$H_{ij,k\ell}(A) := \mathcal{H}(\Delta_{ij},\Delta_{k\ell}). \tag{14}$$

We could use this to compute the gradient of D, or to get the quadratic approximation to D near the optimal set. To do so, it'd be nice to have a way to compute the inner integral above. Suppose that we can diagonalize $A = U\Lambda U^{-1}$. Then

$$\int_0^t e^{As}Ze^{A(t-s)}ds = \int_0^t Ue^{\Lambda s}U^{-1}ZUe^{\Lambda(t-s)}U^{-1}ds \tag{15}$$

Now, notice that

$$\int_0^t e^{s\lambda_i}e^{(t-s)\lambda_j}ds = \frac{e^{t\lambda_i}-e^{t\lambda_j}}{\lambda_i-\lambda_j}. \tag{16}$$

Therefore, defining

$$X_{ij}(t,Z) = \left(U^{-1}ZU\right)_{ij}\frac{e^{t\lambda_i}-e^{t\lambda_j}}{\lambda_i-\lambda_j} \tag{17}$$

moving the U and U^{-1} outside the integral and integrating we get that

$$\int_0^t e^{As}Ze^{A(t-s)}ds = UX(t,Z)U^{-1}. \tag{18}$$

Following on from above, we see that if $Z = \Delta_{k\ell}$, then

$$X_{ij}^{k\ell}(t) = \frac{e^{t\lambda_i}-e^{t\lambda_j}}{\lambda_i-\lambda_j}(U^{-1})_{\cdot k}U_{\ell\cdot}, \tag{19}$$

where $U_{k\cdot}$ is the kth row of U, and so

$$H_{ij,k\ell}(A) = \int_0^\infty \rho(t)\,\mathrm{tr}\left\{CUX^{ij}(t)U^{-1}BB^T(U^{-1})^TX^{k\ell}(t)^TU^TC^T\right\}dt. \tag{20}$$

This implies that

$$D(A_0+\epsilon Z) \approx \epsilon^2\sum_{ijk\ell}H^{ij,k\ell}Z_{ij}Z_{k\ell} \tag{21}$$

and so

$$D(A_0+\epsilon Z) \approx \epsilon^2\sum_{ijk\ell}H^{ij,k\ell}Z_{ij}Z_{k\ell} \tag{22}$$

By section A, if we set $\Sigma = \sigma^2 I$ and $U = H$, then a population at $A_0 + Z$ experiences a restoring force of strength $(I + \sigma^2 H^{-1})^{-1} Z$ (treating Z as a vector and H as an operator on these). If σ^2 is small compared to H^{-1} then this is approximately $-\sigma^2 H^{-1} Z$. This suggests that the population mean follows an Ornstein-Uhlenbeck process, as described (in different terms) in Hansen and Martins [1996].

More generally, B and C may also change. To extend this we need the remaining second derivatives of D. First, in B:

$$\begin{aligned}
\mathcal{H}^{B,B}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv} D(A_0, B_0 + uY + vZ, C_0)|_{u=v=0} \\
&= \frac{1}{2}\int_0^\infty \rho(t)\,\mathrm{tr}\left\{ C_0 e^{tA_0} \frac{d}{du}\frac{d}{dv}(uY+vZ)(uY+vZ)^T|_{u=v=0} e^{tA_0^T} C_0^T \right\} dt \\
&= \frac{1}{2}\int_0^\infty \rho(t)\,\mathrm{tr}\left\{ C_0 e^{tA_0} \left(YZ^T + ZY^T\right) e^{tA_0^T} C_0^T \right\} dt.
\end{aligned} \tag{23}$$

Next, in C:

$$\begin{aligned}
\mathcal{H}^{B,B}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv} D(A_0, B_0, C_0 + uY + vZ)|_{u=v=0} \\
&= \frac{1}{2}\int_0^\infty \rho(t)\,\mathrm{tr}\left\{ B_0 e^{tA_0^T} \frac{d}{du}\frac{d}{dv}(uY+vZ)^T(uY+vZ)|_{u=v=0} e^{tA_0} B_0 \right\} dt \\
&= \frac{1}{2}\int_0^\infty \rho(t)\,\mathrm{tr}\left\{ B_0 e^{tA_0^T} \left(YZ^T + ZY^T\right) e^{tA_0} B_0 \right\} dt.
\end{aligned} \tag{24}$$

Now, the mixed derivatives in B and C:

$$\begin{aligned}
\mathcal{H}^{B,C}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv} D(A_0, B_0 + uY, C_0 + vZ)|_{u=v=0} \\
&= \int_0^\infty \rho(t)\,\mathrm{tr}\left\{ Y e^{tA_0^T} C_0^T Z e^{tA_0} B_0 \right\} dt.
\end{aligned} \tag{25}$$

In A and B

$$\begin{aligned}
\mathcal{H}^{A,B}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv} D(A_0 + uY, B_0 + vZ, C_0)|_{u=v=0} \\
&= \int_0^\infty \rho(t)\,\mathrm{tr}\left\{ C_0 \left(\int_0^t e^{sA_0} Y e^{(t-s)A_0} ds \right) B_0 Z^T e^{tA_0} C_0 \right\} dt,
\end{aligned} \tag{26}$$

and finally in A and C:

$$\begin{aligned}
\mathcal{H}^{A,C}(Y,Z) &:= \frac{1}{2}\frac{d}{du}\frac{d}{dv} D(A_0 + uY, B_0, C_0 + vZ)|_{u=v=0} \\
&= \int_0^\infty \rho(t)\,\mathrm{tr}\left\{ C_0 \left(\int_0^t e^{sA_0} Y e^{(t-s)A_0} ds \right) B_0 B_0 e^{tA_0} Z \right\} dt.
\end{aligned} \tag{27}$$