

# Rapid speciation despite conservation of phenotype

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## Abstract

We introduce an analytical theory to study the evolution of biological systems, such as gene regulatory networks. The evolutionary conservation of phenotype under selective and environmental stasis does not necessitate conservation of the underlying mechanism, as distinct molecular pathways can realize identical phenotypes. Here we give an exact expression for the set of all linear mechanisms with identical phenotypes, and expect evolution under neutrality to explore this set. We employ a quantitative genetic approach to model evolution under neutrality as a random process over the set of all phenotype-invariant mechanisms: only mutational tweaks to the pathway that leave the phenotype invariant are optimally fit. We show that there is never a unique linear system architecture for any phenotype and that the evolutionary exploration of these distinct and mutationally connected mechanisms can lead to the rapid accumulation of hybrid incompatibilities between allopatric populations and thus lead to the rapid formation of new species – in fewer generations than there are breeding individuals in a population.

Additional ideas to consider adding:

- hybrid vigor (need to do calculation)
- discuss linearity, linearization, and canalization in introduction

*Note: in the L<sup>A</sup>T<sub>E</sub>X source I'm putting in semantic linebreaks, so it's easy to edit and move around phrases and ideas.*

*Need to come up with a consistent term for “the  $A_{ij}$ ”s. – “regulatory coefficients”? “genotype”?*

## Introduction

Bridging the gulf between an organism's genome and phenotype is a poorly understood and complex molecular machinery. Progress in a suite of biological subdisciplines is stalled by our general lack of understanding of this molecular machinery: with respect to both its function and evolution. . There does exist a growing body of data on the evolutionary histories and molecular characterizations of particular gene regulatory networks [Jaeger, 2011, Davidson and Erwin, 2006, Israel et al., 2016], as well as thoughtful verbal and conceptual models [True and Haag, 2001, Pavlicev and Wagner, 2012, Weiss and Fullerton, 2000, Edelman and Gally, 2001]. However, verbal theories are often insufficient, if not downright misleading [Servedio et al., 2014]. This is especially pertinent given the staggering complexity and scope of contemporary research programs. This outlook necessitates the advancement of conceptual frameworks of such precision, only mathematics will suffice, as models allow the development of concrete numerical predictions.

The molecular machinery, interacting with the environment, and bridging genotype to phenotype can be mathematically described as a dynamical system – or a system of differential equations [Jaeger et al., 2015]. Movement in this direction is ongoing, as researchers have begun to study the evolution of both abstract [Wagner, 1994, 1996, Siegal and Bergman, 2002, Bergman and Siegal, 2003, Draghi and Whitlock, 2015] and empirically inspired computational and mathematical models of gene regulatory networks [Mjolsness et al., 1991,

Saying “stalled” reflects negatively on other fields - rephrase?

I don't think it reflects negatively. Even if it did, as long as it's a true statement it should be fine.

I don't think the HW example will speak to our audience. Also, better to say why math is good rather than why not-math is bad.

deleted HW. I don't think we need to write an anodyne – the current sentence is true enough.

probably some more recent Wagner papers in this line as well?

Jaeger et al., 2004, Kozlov et al., 2012, 2015, 2014, Crombach et al., 2016, Wotton et al., 2015, Chertkova et al., 2017]. If we allow the reasonable assumption that the genotype-phenotype map can be represented as a system of differential equations, we can immediately discuss its evolution and function in a much more mechanistic, yet general, manner.

In some fields that seek to fit parametric models to experimental data, such as control theory, chemical engineering, and statistics, it is well known that mathematical models can fundamentally be *unidentifiable* and/or *indistinguishable* – meaning that there can be uncertainty about an inferred model’s parameters or even its claims about causal structure, even with access to complete and perfect data [Bellman and Åström, 1970, Grewal and Glover, 1976, Walter et al., 1984]. Models with different parameter schemes, or even different mechanics can be equally accurate, but still not *actually* agree with what is being modelled. In control theory, where electrical circuits and mechanical systems are often the focus, it is understood that there can be an infinite number of “realizations”, or ways to reverse engineer the dynamics of a black box, even if all possible input and output experiments on the black box are performed [Kalman, 1963, Anderson et al., 1966, Zadeh and Deoser, 1976]. In chemical engineering, those who study chemical reaction networks sometimes refer to the fundamental unidentifiability of these networks as “the fundamental dogma of chemical kinetics” [Craciun and Pantea, 2008]. In computer science, this is framed as the relationship among processes that simulate one another [Van der Schaft, 2004]. Although this may frustrate the occasional engineer or scientist, viewed from another angle, the concepts of unidentifiability and indistinguishability can provide a starting point for thinking about externally equivalent systems – systems that evolution can explore, so long as the parameters and structures can be realized biologically. In fact, evolutionary biologists who study convergent versus parallel evolution, homology, and analogy are very familiar with such functional symmetries; macroscopically identical phenotypes in even very closely related species can in fact be divergent at the molecular and sequence level [True and Haag, 2001, Tsong et al., 2006, Hare et al., 2008, ?, Stergachis et al., 2014, Taylor et al., 2016, Matsui et al., 2015].

In this paper we outline a theoretical framework to study the evolution of biological systems. Presently, we focus solely on a neutral scenario, that is where phenotype is conserved over evolutionary time. however, this framework could be applied to a wider set of evolutionary scenarios.

We present an analytical description of the set of all linear biological systems with identical phenotypes – that is we describe the set of all gene network architectures that yield identical phenotypes, and show that all linear biological systems can, in principal, can undergo systems drift. In the neutral case, this set describes a manifold that evolution explores leaving phenotype invariant with respect to mutation, and predicts that if two populations become reproductively isolated, hybrid incompatibility can occur, despite the absence of adaptation, directional selection, or environmental change. Speciation typically occurs on timescales approximately on the order of  $N_e$  generations, where  $N_e$  is the effective population size.

## Gene networks as linear dynamical systems

**Systems theory** Organisms’ phenotypes are constructed by gene by gene and gene by environment interactions. Here we simply define the *phenotype* to be the temporal molecular dynamics directly under natural selection. The *what*, *when*, and *how much*, of an organism’s molecules that are physiologically or otherwise relevant to survival.

Thus an organism’s phenotype  $\phi(t)$  – a vector of molecular concentrations at time  $t$  – is determined both by the structure and organization of a biological system (*e.g.* a gene

here you are jumping to evolution – haven’t said yet why evolution “can explore” these – needs to say explicitly what we mean by neutral here

rather than “we expect” maybe say that the framework could be applied to non-neutral evolution

fixed

here’s the definition of neutral, but it’s not quite right?

how’s that?

We need something here saying what we mean by “gene network architecture”. Also, I don’t think we claim anything for “all biological systems”.

added the word “linear.”

Below we mix general language with language specific to GRNs, like switching between “internal state” and “transcription factor concentration. Rather than keep making comments like “(or some other system)”, we could say up top that this applies to other situations, but to make it concrete we’ll talk about regulatory networks.

regulatory network), given by the triple  $(A, B, C)$ , and by its environment  $u(t)$ .

Such a biological system  $\mathcal{S}$  is given by,

$$\mathcal{S} := \begin{cases} \dot{\kappa}(t) &= A\kappa(t) + Bu(t) \\ \phi(t) &= C\kappa(t) \end{cases} \quad (1)$$

Generally  $A$  is any real  $n \times n$  matrix,  $B$  any  $n \times \ell$ , and  $C$  any  $\ell \times n$  dimensional matrix. Although many different biological systems can be modeled with this approach, for clarity, we focus on gene regulatory networks. Each  $i$ th row of  $A$  describes the *cis*-regulatory module for gene  $i$ , and each  $j$ th entry, the specific regulatory influence of gene  $j$  on gene  $i$ . As such,  $A_{ij}$  is the magnitude at which transcription factor  $\kappa_j(t)$  regulates transcription factor  $\kappa_i(t)$ , and if  $A_{ij} > 0$ , we say that  $\kappa_j$  upregulates  $\kappa_i$ . If  $A_{ij} < 0$ , we say that  $\kappa_j$  down-regulates  $\kappa_i$ .

The form of  $B$  determines precisely how the environment influences the organism – that is  $B$  filters and translates the input to the system.  $C$  filters and translates the dynamics of the system and precisely determines the output, that is, what is visible to selection. *E.g.* for a metabolic system,  $C_{ij}$  is the amount the  $j$ th metabolite  $u_j$  affects the production of the  $i$ th enzyme.

Furthermore, whereas the phenotype is a subset of molecules *visible* to selection, the *kryptotype* includes the molecular dynamics *hidden* from selection. That is  $\kappa(t)$  is a vector of the system's molecular concentrations at time  $t$ .

Finally, we can write the phenotype as a convolution of the system organization and the environment,

$$\phi(t) = Ce^{At}\kappa(0) + \int_0^t Ce^{A(t-s)}Bu(s)ds, \quad (2)$$

where we refer to  $h(t) := Ce^{At}B$  as the system's *impulse response*.

**Example 1** (Oscillating gene network: cell cycle control). *Cellular division is governed by many different processes, however it is thought that its rhythm is partially controlled by oscillating gene transcription [Orlando et al., 2008]. Here we consider a simplified model of oscillating gene transcription.*

*Suppose gene-2 up-regulates the transcription of gene-1 and that gene-1 down-regulates gene-2 with equal magnitudes, whose concentrations are given by  $\kappa_1(t)$  and  $\kappa_2(t)$ . Furthermore, suppose that only the dynamics of gene-1 are consequential to the cell cycle (perhaps the amount of gene-1 activates another downstream gene network). Lastly suppose that the production of both genes is equally stimulated by an impulse of a molecule present immediately after division.*

*If the rate each of these genes is expressed is a linear function of their concentrations, the dynamics of the system is given by*

$$\begin{aligned} \dot{\kappa}_1(t) &= \kappa_2(t) + u(t) \\ \dot{\kappa}_2(t) &= -\kappa_1(t) + u(t) \\ \phi(t) &= \kappa_1(t). \end{aligned}$$

*Equivalently, in matrix form the system regulatory coefficients are given as,  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$*

*Since the input is simply an impulse, its phenotype is equivalent to its impulse response,*

$$\phi(t) = h(t) = \sin t + \cos t.$$

This "Thus" needs expanding, e.g., "the solution to this equation is unique and given by" with a reference.

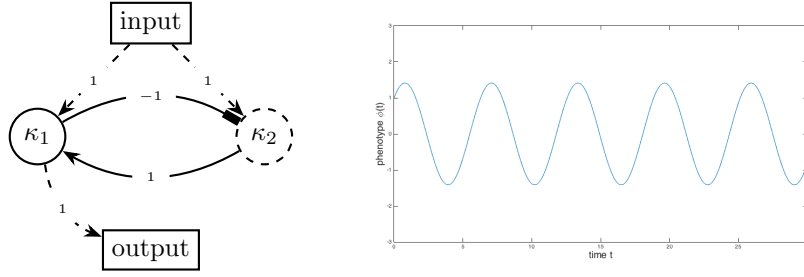


Figure 1: (Left) Graphical representation of the cell cycle control gene network, and (right) plot of the phenotype  $\phi(t)$  against time  $t$ .

We return to the evolution of such a system below.

**Phenotypically equivalent gene networks** Despite a symmetry in functionality or phenotype systems can often differ, sometimes substantially, at the molecular level. How many different mechanisms have the same function?

Gene regulatory networks with identical phenotypes do not necessarily have identical kryptotypes. Any linear and minimal gene network – minimal, informally meaning that the system’s phenotype is achieved with the fewest possible genes ( $n = \min$ , see A) – has an identical impulse response and thus phenotype given identical inputs  $u(t)$ , up to a change of coordinates.

$$\begin{aligned} h(t) &= Ce^{At}B = CV^{-1}Ve^{At}V^{-1}VB \\ &= CV^{-1}e^{VAV^{-1}t}VB = \bar{C}e^{\bar{A}t}\bar{B} \end{aligned} \quad (3)$$

Two gene networks,  $\mathcal{S} = (A, B, C)$ , and  $\bar{\mathcal{S}} = (\bar{A} = VAV^{-1}, \bar{B} = VB, \bar{C} = CV^{-1})$ , have the same phenotype, for all possible inputs, if they are related by a change of coordinates.

Therefore the set of all phenotypically identical (and minimal) systems can be parameterized as,  $\mathcal{S}_{\min}(V) = (VAV^{-1}, VB, CV)$ , where  $V$  is any invertible matrix and its elements are free parameters.

Therefore varying  $V$  can tweak the relationships between genes within a regulatory network, yet preserve the phenotype. Some gene networks, however, can grow or shrink, perhaps following gene duplications and deletions, and also still preserve their phenotypes. These changes cannot be described by  $\mathcal{S}(V)$ , but are encapsulated by 4, below.

More generally, we denote by  $\mathcal{A}_n(\mathcal{S}_0)$  the set of all  $n$ -dimensional systems equivalent to  $\mathcal{S}_0$ :

$$\begin{aligned} \mathcal{A}_n(\mathcal{S}_0) &= \{(A, B, C) : Ce^{At}B = C_0e^{A_0t}B_0 \text{ for } t \geq 0\} \\ &= \{(A, B, C) : CA^rB = C_0A_0^rB_0 \text{ for } 1 \leq r \leq n-1\}. \end{aligned} \quad (4)$$

Equivalence of the two characterizations follows from the Cayley-Hamilton theorem. Usually, the dimension  $n$  and the reference system  $\mathcal{S}_0$  is implicit and we write only  $\mathcal{A}$ . Further, we refer to the set of all linear systems with impulse response  $h(t)$ , regardless of dimension, to be  $\mathcal{K}$ ,

$$\mathcal{K} := \bigcup_{n=\min}^{\infty} \mathcal{A}_n(\mathcal{S}_0) \quad (5)$$

Two systems, even if in different dimensions, will have identical phenotypes if they are in  $\mathcal{K}$ . Further, there is no unique triple  $(A, B, C)$  for the mapping  $h : u \mapsto \phi$ . This implies that the gene regulatory network architecture per phenotype/environment pair is never unique – that is under these assumptions, all gene regulatory networks can drift/rewire, and still conserve

”has an identical phenotype” is not what you mean to say here

edited – does that work?

is this the place to say ”and only if, in the minimal dimension”?

To make sense of this we need to say how  $C$  and  $B$  change with  $n$ .

what do you like better for  $\mathcal{K}$ ?  $\mathcal{A}$ ,  $\mathcal{A}_{\infty}$ , or  $\mathcal{K}_{h(t)}$ , or something else – maybe a modification of  $\mathcal{S}$ ? Using a script of “A” was more intuitive when we were leaving  $B$  and  $C$  invariant.

the phenotype. This set can be precisely defined and completely parameterized using the *Kalman decomposition* (Appendix A).

The Kalman decomposition decomposes a system into *reachable* and *observable* components – that is coefficients that respond to an input and coefficients that produce an observable output. A minimal system’s components are both reachable and observable. In this form, it is possible to add components that respond to an input, but do not influence the output, or components that, in principle, influence the output, but do not respond to any inputs. Any system with such components is, by definition, non-minimal, and any system can be modified to include any finite number of extra unreachable and/or unobservable components, without modifying its impulse response. Decomposing the system allows for the easy manipulation of “superfluous” (but not necessarily expendable) components. After unreachable/unobservable coefficients are added, changing coordinates “stirs” the system coefficients such that the previously superfluous coefficients can become essential to system function. Whether or not evolution constrains biological systems to be minimal or nearly minimal is an open question.

A large, adapted population will move around in  $\mathcal{K}$  under environmental and selective stasis. In the minimal dimension neutral evolution proceeds as a stochastic coordinate transformation on the elements in  $\mathcal{S}_{\min}$ . Even if there is conservation of phenotype through evolutionary time, the molecular mechanism underpinning the phenotype is not constrained and likely will not be conserved.

**Example 2** (All Phenotypically Equivalent Cell Cycle Control Networks). *The set of all two-gene regulatory networks phenotypically equivalent to the cell cycle control network in 1, where only  $A$  can vary, is given by  $\mathcal{S}(\tau) = (A(\tau), B, C)$ , such that,*

$$A(\tau) = \frac{1}{\tau - 1} \begin{bmatrix} \tau & -1 \\ 2\tau(\tau - 1) + 1 & -\tau \end{bmatrix} \text{ for } \tau \neq 1$$

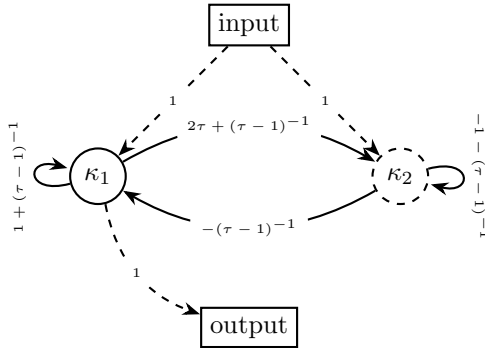


Figure 2:  $\mathcal{S}(\tau)$ , the set of all phenotype-equivalent cell cycle control networks  $\mathcal{S}(\tau) = \{\mathcal{A}_2 : B = [1 \ 1]^T, C = [1 \ 0]\}$ .

*Despite the phenotypic equivalence of all instantiations of  $\mathcal{S}(\tau)$ , the kryptotypes, vary as a function of  $\tau$ .*

should the sentence of evolution of nearly minimal be in the paper? Or should we save this for our next paper?

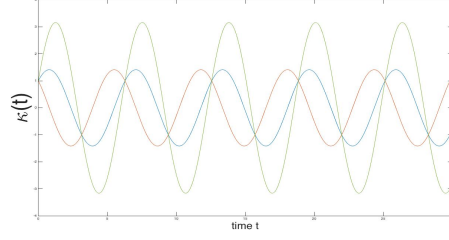


Figure 3: Gene-1 dynamics (blue) for both systems  $\mathcal{S}(0)$  and  $\mathcal{S}(2)$  are identical, however,  $\mathcal{S}(0)$  gene-2 dynamics (orange) differ from  $\mathcal{S}(2)$  (green). Both gene-1 dynamics are given by  $\kappa_1 = \sin t + \cos t$ , and gene-2 by  $\kappa_2 = \cos t - \sin t$  ( $\mathcal{S}(0)$ ) and  $\kappa'_2 = \cos t + 3 \sin t$  ( $\mathcal{S}(2)$ ).

**Example 3** (Metabolic network). Consider an organism that can metabolize two different sugars (present at logarithmic concentrations  $u_1$  and  $u_2$ ), with two enzymes (at log concentrations  $\phi_1$  and  $\phi_2$ ). Further suppose that the second sugar is preferred, and that depending on  $u_1$  and  $u_2$  the organism will synthesize an appropriate  $\phi_1$  and  $\phi_2$ . Furthermore, assume this system contains at least two transcription factors, whose log concentrations are  $\kappa_1, \kappa_2, \dots, \kappa_n$ . Minimally such a system may have the architecture,  $\mathcal{S}_{\min}(U) = (U \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} U^{-1}, U, U^{-1})$ . We can find alternative equivalent systems by changing coordinates ( $U \rightarrow U'$ ) or, more generally, by applying the Kalman decomposition A. To illustrate that phenotypic invariance does not require dimensional invariance, we apply the Kalman decomposition to  $\mathcal{S}_{\min}$  to find  $\mathcal{S}(D_{1-3}, V) = (V \begin{bmatrix} D_1 & D_2 \\ 0 & A \end{bmatrix} V^{-1}, V \begin{bmatrix} D_3 \\ 0 \end{bmatrix}, [0 \ C] V^{-1})$ , both of which are in  $\mathcal{K}$  ( $V$  can be any 4-dimensional and invertible matrix,  $D_{1-3} \in \mathbb{R}^2$ , and  $(A, B, C) \in \mathcal{S}_{\min}$ ).

**Sexual recombination and hybrid incompatibility** Phenotypically identical yet kryptotypically variant gene networks, when brought together and/or recombined during sexual reproduction, can lead to phenotypic inviability.

For instance, the diploid  $\{\mathcal{S}, \mathcal{S}'\}$  has system architecture  $\mathcal{S}_{\text{diploid}} = (\mathcal{S} + \mathcal{S}')/2$ , the average of its two haploid systems. During meiosis a diploid recombines its two genomes to form a new one (a gamete):  $(\mathcal{S} \times \mathcal{S}') \rightarrow \mathcal{S}''$ ; by randomly swapping matrix rows between two systems (Appendix B). Then two gametes join to form an offspring (e.g.  $\{\mathcal{S}'', \mathcal{S}\}$ ).

Even if two systems yield the same phenotype, their diploid and recombinant systems may not. This is because  $\mathcal{K}$  is not required to be closed under averaging or recombination.

If sexual recombination among systems drawn from  $\mathcal{K}$  yields systems with divergent phenotypes, populations containing significant diversity in  $\mathcal{K}$  can carry genetic load, and isolated populations may fail to produce hybrids with viable phenotypes (Figure 4).

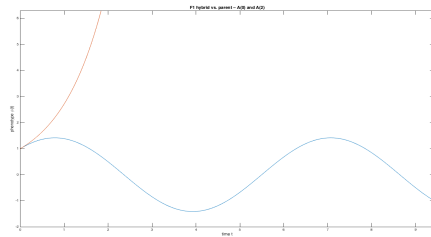


Figure 4: **Hybrid phenotypic breakdown**  $F_1$  hybrid (orange) and parental (blue) phenotypic cell-cycle control (oscillator) dynamics for an  $\mathcal{S}(0)$  by  $\mathcal{S}(2)$  cross from Example 2. The  $F_1$  hybrid gene expression fails to oscillate and instead grows exponentially ( $\phi_{F_0} = \sin t + \cos t$ , however  $\phi_{F_1} = e^t$ ).

**Example 4** (Hybrid Incompatibility in an Oscillating Gene Network). *Here we compare the phenotypes for  $F_2$  hybrids formed by crossing oscillators with different values for  $\tau$  from 2,  $\mathcal{S}(2)$  with  $\mathcal{S}(2.01)$ ,  $\mathcal{S}(2.1)$ , and  $\mathcal{S}(2.5)$ . Systems have identical phenotypes, however some of the hybrids exhibit markedly different dynamics.*

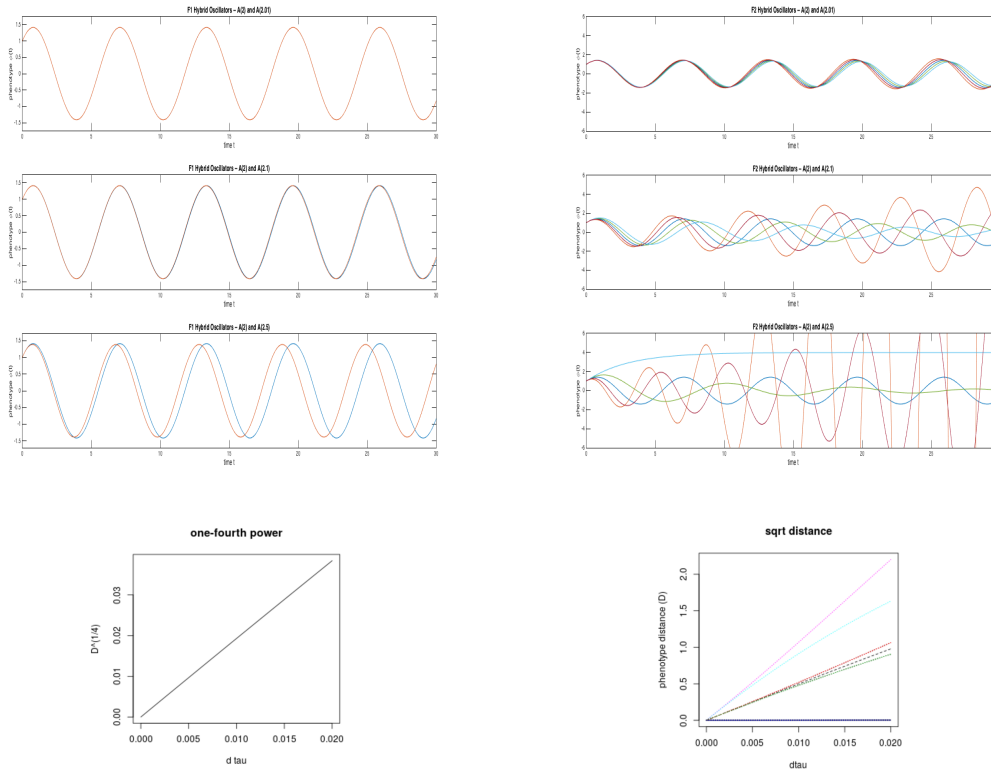


Figure 5:  $F_1$  (left) and  $F_2$  (right) hybrids crossing  $\mathcal{S}(2)$  with  $\mathcal{S}(2.01)$  (top),  $\mathcal{S}(2.1)$  (middle), and  $\mathcal{S}(2.5)$  (bottom).  $F_2$  hybrids display more phenotypic divergence than  $F_1$ s, on average. Further, some  $F_2$ s completely fail to oscillate, as seen in an  $\mathcal{S}(2.5)$   $F_2$  (light blue). *make axis labels bigger*

## Systems drift and the accumulation of incompatibilities

Thus far we have shown that many distinct molecular mechanisms can realize identical phenotypes and that these mechanisms may fail to produce viable hybrids. Here we discuss the rate at which evolution shifts molecular mechanisms and ask whether this process is fast enough to be a significant driver of speciation. To do so, we explore a general quantitative genetic model in which a population drifts stochastically near a set of equivalent and optimal systems. We work with a population of effective size  $N_e$ .

At any given time, there will be a range of regulatory coefficients present in the population due to segregating genetic variants. Over many generations, even if selective pressures do not change, this range of networks will shift as recombination, mutation, and demographic noise create new alleles and shift allele frequencies.

Suppose that a set  $x$  of coefficients that determine a system (this is  $\mathcal{S}$  above), produce a phenotype  $\Phi(x)$  (the time course of  $\phi(t)$ ). There is an optimal phenotype  $\Phi_0$ , and a set  $\mathcal{X}$  of “optimal” coefficients that produce this phenotype. Fitness depends on distance to the

commented out questions here b/c we don't answer them directly – what do you think?



optimal phenotype – we will write the “distance” between phenotypes  $\phi$  and  $\psi$  as  $d(\phi, \psi)$ , measured so that the fitness of an organism with coefficients  $x$  is  $\mathcal{F}(x) = \exp(-d(\Phi(x), \Phi_0)^2)$ . We will assume that the map  $\Phi$  is smooth and that the optimal set  $\mathcal{X}$  is locally isomorphic to  $\mathbb{R}^m$ .

**System drift** If the regulatory coefficient population variation has standard deviation  $\sigma$ , since subsequent generations resample from this diversity, the population mean coefficient will move a random distance of size  $\sigma/\sqrt{N_e}$  per generation, simply because this is the standard deviation of the mean of a random sample [?]. Selection will tend to restrain this motion, but mean movement along the optimal set  $\mathcal{X}$  is unconstrained. The amount of variance in particular directions in coefficient space depend on constraints imposed by selection and the covariance between genetic variation between different coefficients (the  $G$  matrix [?]).

We denote  $\sigma_N$  and  $\sigma_S$ , the standard deviations of coefficient variation along and perpendicular to  $\mathcal{X}$  respectively, and  $\gamma$ , the scale on which phenotype changes moving away from  $\mathcal{X}$ . Concretely,  $\gamma$  is the inverse of the derivative of  $d(\Phi(x + uz), \Phi(x))$  with respect to  $u$  for  $x \in \mathcal{X}$  and  $z$  perpendicular to the tangent space at  $x$ . With these parameters, a typical individual will have a fitness of around  $\exp(-(\sigma_S/\gamma)^2)$ .

**Hybridization** The means of two allopatric populations separated for  $T$  generations will be a distance of order  $2\sigma_N\sqrt{T/N_e}$  apart along  $\mathcal{X}$ . A population of  $F_1$  hybrids has one haploid genome from each, whose coefficients are averaged, and so will have mean system coefficients at the midpoint between their means, and variance equal to  $\sigma^2$ . Each  $F_2$  hybrid will be homozygous for one parental allele on average at half of the loci in the genome, so the distribution of  $F_2$ s will have mean at the average of the two populations, as before, but variance equal to  $\sigma^2 + z^2/2$ , where  $z$  is the distance between the parental populations. These are depicted in figure ??.

*improve figure by putting labels on from the following* Suppose that two populations have drifted independently to differ by  $z$ , and that  $z$  is of the same order as  $\sigma$  but is smaller than  $\gamma$ . The mean  $F_1$  is the average of the parental means, and since the first-order terms in the Taylor series vanish, has phenotype differing from the optimum by a distance of order  $\|z\|^2$  (see appendix D). The mean  $F_2$  is the same, but the standard deviation is of order  $z$ , so that up to lower order terms, while the typical fitness of an individual in the original population is  $\mathcal{F}_0 = \exp(-(\sigma_S/\gamma)^2)$ ; of an  $F_1$  is  $\mathcal{F}_1/\mathcal{F}_0 = \exp(-(c_1\sigma_N^2 T/N_e)^2)$ ; and of an  $F_2$  is  $\mathcal{F}_2/\mathcal{F}_0 = \exp(-T/(N_e\gamma^2))$ .

If we prefer not to make any assumptions about the shape of the fitness function, we can more conservatively state that hybrid phenotypes diverge from their optimal parents at rates  $d(\Phi_{F_1}, \Phi_0) = \alpha_1\sigma_N^2 T/N_e$  and  $d(\Phi_{F_2}, \Phi_0) = \alpha_2\sigma_N\sqrt{T/N_e}$ , where  $\alpha_1$  and  $\alpha_2$  are constants. Whether or not hybrid phenotypic divergence is associated with a high fitness cost,  $F_2$  phenotypes should diverge much faster than  $F_1$ s, with a rate proportional to the square root of  $T/N_e$ , versus linearly.

**Speciation rates under neutrality** For cases where system structures and  $\gamma$  can be inferred, specific speciation rates can be estimated. For other situations we offer a back of the envelope calculation.

Consider a population that is distributed on  $\mathcal{X}$  such that the magnitude of phenotypically divergent  $\sigma_S$  and neutral  $\sigma_N$  diversity are constrained by selection;  $\sigma_S$  directly and  $\sigma_N$  indirectly via recombination load. Two reproductively isolated populations will, on average, be a distance of  $2\sigma_N\sqrt{T/N_e}$ . If we assume that  $\sigma_N$  is constrained by a function of  $\sigma_S$ , neutral movement along  $\mathcal{X}$  pushes hybrids off of the manifold by  $\sigma_S$  per  $\sigma_N$ . Lastly, if fitness is Gaussian:  $\mathcal{F}_2 = \exp(-d(x, x')^2/\beta^2)$ , and we assume that the strength of selection is only

need to add  $\gamma^2$  here?

say that better

since it moves at rate  $\sigma\sqrt{T/N_e}$  isn't saying  $\sigma/\sqrt{N_e}$  per generation not that clear?

I put F1s and F2s here, and  $\mathcal{I}$ , but with less explanation than below. Merge somehow.

I think you mean  $\sigma^2$ ? I changed it.

Can you clarify why  $z^2/2$ ? Is this because it's averaging two distributions?

I think  $\alpha_1$  is your  $c_1$  and  $\alpha_2$  is a function of  $\gamma^{-1}$ ?



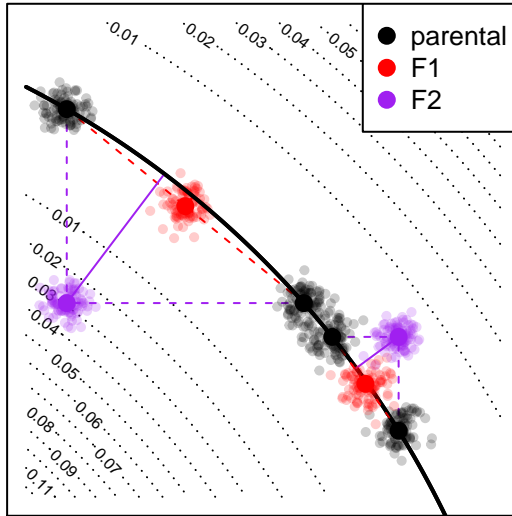


Figure 6: A conceptual figure of the fitness consequences of hybridization: axes represent system coefficients (i.e., entries of  $A$ ); the line of optimal system coefficients is down in black; dotted lines give phenotypic distances to the optimum. Two pairs of parental populations are shown in black, along the optimum; a hypothetical population of  $F_1$ s are shown for each in red, and the distribution of one type of  $F_2$  is shown in purple (other types of  $F_2$  are not shown). Solid lines depict the distance of the  $F_2$  to optimum. *Should show all types of  $F_2$ ? would be messy.*

several multiples of the segregating divergent diversity, such that  $\sigma_S = c\beta$ , we can simplify the expression estimating  $F_2$  hybrid fitness drops to be,

$$\exp\left(-c^2 \frac{T}{N_e}\right) \quad (6)$$

This suggests that, as an upper bound, if a population contains as much phenotypic diversity as allowed by selection, reproductively isolated populations can speciate rapidly – possibly on timescales shorter than  $N_e$  generations (It is likely that, on average,  $c \leq 2/3$ ).

## Discussion

The complexity of biological systems has limited our understanding of their function and evolution. Above we outline an approach, a first step, towards untangling this complexity in reference to function and evolution. This methodology borrows successfully applied tools from engineering and aims to synthesize these with the concepts and tools of molecular and evolutionary biology.

Theoretical models in evolution and population genetics often lack the molecular details of physiology or of the genotype-phenotype map. Here, we offer a tractable and simple model which includes these missing features. Further, we provide, in clear mathematical language, an analytical description of phenomena hitherto discussed verbally and conceptually (phenogenetic drift [Weiss and Fullerton, 2000], developmental systems drift [True and Haag, 2001], biological degeneracy [Edelman and Gally, 2001], *etc.*). The tractability and relative simplicity of this exposition enables the interested biologist to work out by hand, if desired, the

this is a claim we are the first to do something. best not to make these claims (and we aren't the first to make math models of these)

removed the word “only”

probably not desired

:p

dynamics of a genetic system, as well as perturbations to the system – an attribute not likely to be found in less tractable models and simulations.

We have suggested an interpretation of system identification: to see it as an evolutionarily neutral manifold, and not simply a computational nuisance. We have demonstrated a method to analytically determine the set of all phenotypically invariant gene networks; by a simple change of coordinates in the minimal configuration, or more generally by applying the Kalman decomposition in higher dimensions. This set is explored over evolutionary time when phenotype is conserved, and can lead to a diverse set of consequences, including the accumulation of Dobzhansky-Muller incompatibilities. We emphasize that these incompatibilities are a consequence of recombining different, yet functionally equivalent, mechanisms.

Furthermore, using a quantitative genetic approach, we estimated that a genetically variable population will drift in neutral system space at a rate determined by its intra-population variation and its effective population size. Because mechanistically distinct yet phenotypically equivalent biological systems can fail to produce viable hybrids, we predict allopatric populations to accumulate genetic incompatibilities at a rate on the order of  $N_e$  if intrapopulation regulatory variation is constrained by selection. Additionally we see second-generation hybrid fitness plummet much faster than that of first-generation hybrids. This result is also consistent with Haldane’s rule; that if only one hybrid sex is inviable or sterile it is likely the heterogametic sex. The consistency comes from gene networks localized to the sex chromosomes functioning as an  $F_2$  hybrid cross within a diploid  $F_1$  heterogamete as there is only one sex chromosome.

Lastly, we show that hybrid gene networks break down as function of genetic distance, and may, in part, explain broad patterns of reproductive isolation among diverse phyla [Roux et al., 2016].

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check original Bateson/DM papers to see if this accords with those defns

I read through Orr’s review of those papers, which included excerpts. It seems like the DMI definition is very general and this accords.

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## Examples

### A Kalman Decomposition

**Definition 1** (Phenotypic equivalence of systems). *Let  $(\kappa(t), \phi(t))$  and  $(\bar{\kappa}(t), \bar{\phi}(t))$  be the solutions to (??) with coefficient matrices  $(A, B, C)$  and  $(\bar{A}, \bar{B}, \bar{C})$  respectively, and both  $\kappa(0)$  and  $\bar{\kappa}(0)$  are zero. The systems defined by  $(A, B, C)$  and  $(\bar{A}, \bar{B}, \bar{C})$  are **phenotypically equivalent** if*

$$\phi(t) = \bar{\phi}(t) \quad \text{for all } t \geq 0.$$

*Equivalently, this occurs if and only if*

$$h(t) = \bar{h}(t) \quad \text{for all } t \geq 0,$$

*where  $h$  and  $\bar{h}$  are the impulse responses of the two systems.*

One way to find other systems equivalent to a given one is by change of coordinates (“algebraic equivalence”): if  $V$  is an invertible matrix, then the systems  $(A, B, C)$  and  $(VAV^{-1}, VB, CV^{-1})$  have the same dynamics because their transfer functions are equal:

$$CV^{-1}(zI - VAV^{-1})^{-1}VB = CV^{-1}V(zI - A)^{-1}V^{-1}VB = C(zI - A)^{-1}B.$$

However, the converse is not necessarily true: systems can have identical transfer functions without being changes of coordinates of each other. In fact, systems with identical transfer functions can involve interactions between different numbers of molecular species.

The set of all systems phenotypically equivalent to a given system  $(A, B, C)$  is elegantly described using the Kalman decomposition, which also clarifies the system dynamics? tells us a lot about how it works? To motivate this, first note that the input  $u(t)$  only directly pushes the system in directions lying in the span of the columns of  $B$ . As a result, different combinations of input can move the system in any direction that lies in the *reachable subspace*, which we denote by  $\mathcal{R}$ , and is defined to be the closure of  $\text{span}(B)$  under applying  $A$  (or equivalently, the span of  $B, AB, A^2B, \dots, A^{n-1}B$ ). Analogously to this, we define the *observable subspace*,  $\mathcal{O}$ , to be the closure of  $\text{span}(C^T)$  under applying  $A$ . (Or:  $\bar{\mathcal{O}}$  is the largest  $A$ -invariant subspace contained in the null space of  $C$ ; and  $\mathcal{R}$  is the largest  $A$ -invariant subspace contained in the image of  $B$ .)

or something

If we define

1. The columns of  $P_{r\bar{o}}$  are an orthonormal basis for  $\mathcal{R} \cap \bar{\mathcal{O}}$ .
2. The columns of  $P_{ro}$  are an orthonormal basis of the complement of  $\mathcal{R} \cap \bar{\mathcal{O}}$  in  $\mathcal{R}$ .
3. The columns of  $P_{\bar{r}o}$  are an orthonormal basis of the complement of  $\mathcal{R} \cap \bar{\mathcal{O}}$  in  $\bar{\mathcal{O}}$ .
4. The columns of  $P_{\bar{r}\bar{o}}$  are an orthonormal basis of the remainder of  $\mathbb{R}^n$ .

If we then define

$$P = [ P_{r\bar{o}} \mid P_{ro} \mid P_{\bar{r}o} \mid P_{\bar{r}\bar{o}} ],$$

then

$$P^T P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & U & 0 \\ 0 & V & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Check this. Can we get  $U = V = 0$ ?

The following theorem can be found in SOME REFERENCE.

**Theorem 1** (Kalman decomposition). *For any system  $(A, B, C)$  with corresponding Kalman basis matrix  $P$ , the transformed system  $(PAP^{-1}, PB, CP^{-1})$  has the following form:*

$$\hat{A} = PAP^{-1} = \begin{bmatrix} A_{r\bar{o}} & A_{r\bar{o},ro} & A_{r\bar{o},\bar{r}\bar{o}} & A_{r\bar{o},\bar{r}o} \\ 0 & A_{ro} & 0 & A_{ro,\bar{r}o} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{\bar{r}\bar{o},\bar{r}o} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix},$$

and

$$\hat{B} = PB = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix},$$



and

$$\hat{C} = CP^{-1} = \begin{bmatrix} 0 & C_{ro} & C_{\bar{r}\bar{o}} & 0 \end{bmatrix}.$$

The transfer function of both systems is given by

$$H(z) = C_{ro}(zI - A_{ro})^{-1}B_{ro}.$$

In the latter case, we say that the system is *minimal* – there is no equivalent system with a smaller number of species. Note that this says that any two equivalent minimal systems are changes of basis of each other.

Since any system can be put into this form, and once in this form, its transfer function is determined only by  $C_{ro}$ ,  $A_{ro}$ , and  $B_{ro}$ , therefore, the set of all equivalent systems are parameterized by the dimension  $n$ , the choice of basis ( $P$ ), the remaining submatrices in  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  (which are unconstrained), and an invertible transformation of  $\text{span}(P_{ro})$ , which we call  $T_{ro}$ .

**Theorem 2** (Parameterization of equivalent systems). *Let  $(A, B, C)$  be a minimal system.*

- (a) *Every equivalent system is of the form given in Theorem 1, i.e., can be specified by choosing a dimension,  $n$ ; submatrices in  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  except for  $A_{ro} = A$ ,  $B_{ro} = B$ , and  $C_{ro} = C$ ; and choosing an invertible matrix  $P$ .*
- (b) *The parameterization is unique if  $P$  is furthermore chosen so that each  $P_x$  other than  $P_{ro}$  is a projection matrix, and that*

$$0 = P_x^T P_y$$

for all  $(x, y)$  except  $(ro, \bar{r}\bar{o})$ .

conjecture:

In some situations we may be interested in only “network rewiring”, where  $A$  changes while  $B$  and  $C$  do not. For instance, if all non-regulatory functions of each molecule are strongly constrained, then  $C$  cannot change. Likewise, if responses of each molecule to the external inputs are not changed by evolution, then  $B$  does not change.

Another way of saying it: pick the  $\mathcal{R}$  and  $\mathcal{O}$  subspaces, that must intersect in something of the minimal dimension; then let  $P$  be the appropriate basis?

## A.1 Neutral directions from the Kalman decomposition

The Kalman decomposition above says that any system  $(A, B, C)$  can be decomposed into  $(P, \hat{A}, \hat{B}, \hat{C})$  so that

$$A = P^{-1} \hat{A} P$$

$$B = P^{-1} \hat{B}$$

$$C = \hat{C} P,$$

and we know precisely how we can change these to preserve the transfer function:

1.  $P \rightarrow P + \epsilon Q$  as long as the result is still invertible,
2.  $\hat{A} \rightarrow \hat{A} + \epsilon X$  as long as  $X$  is zero in the correct places,
3.  $\hat{B} \rightarrow \hat{B} + \epsilon Y$  as long as  $Y$  is zero in the correct places,
4.  $\hat{C} \rightarrow \hat{C} + \epsilon Z$  as long as  $Z$  is zero in the correct places.

By taking  $\epsilon \rightarrow 0$ , these tell us the local directions we can move a system  $(A, B, C)$  in. All statements below are up to first order in  $\epsilon$ , omitting terms of order  $\epsilon^2$ .

First, since  $(P + \epsilon Q)^{-1} = P^{-1} + \epsilon P^{-1} Q P^{-1}$ , modifying  $P \rightarrow P + \epsilon Q$  changes

$$\begin{aligned} A &\rightarrow A + \epsilon P^{-1} \hat{A} Q - \epsilon P^{-1} Q P^{-1} \hat{A} P \\ &= A + \epsilon (A P^{-1} Q - P^{-1} Q A), \\ B &\rightarrow B - \epsilon P^{-1} Q B \\ C &\rightarrow C + \epsilon C P^{-1} Q. \end{aligned}$$

Since  $P$  is invertible and  $Q$  can be anything (if  $\epsilon$  is small enough), this allows changes in the direction of an arbitrary  $W$ :

$$\begin{aligned} A &\rightarrow A + \epsilon (A W - W A), \\ B &\rightarrow B - \epsilon W B \\ C &\rightarrow C + \epsilon C W. \end{aligned}$$

Then,  $\hat{A} \rightarrow A + \epsilon X$  does

$$A \rightarrow A + \epsilon P^{-1} X P$$

and  $\hat{B} \rightarrow B + \epsilon Y$  does

$$B \rightarrow B + \epsilon P^{-1} Y$$

and  $\hat{C} \rightarrow C + \epsilon Z$  does

$$C \rightarrow C + \epsilon Z P.$$

These degrees of freedom look like they depend on  $P$ , which is not unique, but for any two choices of  $P$  there are corresponding choices of  $X$  that give the same actual change in  $A$  (and likewise for  $Y$  and  $Z$ ).

Therefore, this gives us an upper bound on the number of degrees of freedom, in terms of the dimensions of the blocks in the Kalman decomposition ( $n_{ro}$  etc) and the dimensions of  $B$  and  $C$  ( $n_B$  and  $n_C$  respectively): namely, for  $W$ ,  $X$ ,  $Y$ , and  $Z$  respectively:

$$n^2 + (n_{r\bar{o}} + n_{ro} n_{\bar{r}o} + n_{\bar{r}\bar{o}}(n_{\bar{r}\bar{o}} + n_{\bar{r}o}) + n_{\bar{r}o}^2) + n_B n_{r\bar{o}} + n_C n_{\bar{r}\bar{o}}.$$

However, some of these may be redundant. For instance, changing  $P$  in the direction of a  $Q$  that satisfies both  $A P^{-1} Q = P^{-1} Q A$  and  $C P^{-1} Q = 0$  is equivalent to changing  $B$  by  $Y = Q B$ .

## B Meiotic recombination in linear systems

Recombination is performed by taking two analogous system components from  $\mathcal{S}$  and  $\mathcal{S}'$  and randomly swapping rows or columns. *E.g.* gamete systems  $(A'', B'', C'')$ , produced by the diploid  $\{\mathcal{S}, \mathcal{S}'\}$ , are formed by recombining (randomly swapping rows or columns) between two, possibly distinct, systems  $\mathcal{S} = (A, B, C)$  and  $\mathcal{S}' = (A', B', C')$  such that,

$$\mathcal{S}'' = \begin{pmatrix} A'' &= M A + (I - M) A', \\ B'' &= M B + (I - M) B', \\ C'' &= M C + C' (I - M) \end{pmatrix}$$

where  $M$  is a diagonal matrix where each diagonal element is a Bernoulli random variable ( $M_{ii} = 0$  or  $1$  with equal probability, and  $M_{ij} = 0$  if  $i \neq j$ ). If systems are different dimensions the smaller system elements can be augmented with 0s (*e.g.*  $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}$ ).

## C Genetic drift with a multivariate trait

For completeness, we provide a brief argument of how the population mean moves under genetic drift with a quantitative genetics model, as in [Lande \[1981\]](#) or [Hansen and Martins \[1996\]](#). These ignore details of the underlying genetic basis, but developing a more accurate model is beyond the scope of this paper.

**Completing the square** First note that

$$(x - y)^T A(x - y) = x^T A(x - 2y) + y^T A y,$$

and so

$$\begin{aligned} (x - y)^T A(x - y) + x^T B x &= x^T (A + B) (x - 2(A + B)^{-1} A y) + y^T A y \\ &= (x - (A + B)^{-1} A y)^T (A + B) (x - (A + B)^{-1} A y) + (\text{terms that don't depend on } x). \end{aligned}$$

Therefore, if  $f(x; \Sigma, y)$  is the density of a Gaussian with mean  $y$  and covariance matrix  $\Sigma$  then substituting  $A = \Sigma^{-1}$  and  $B = U^{-1}$  above,

$$\frac{f(x; \Sigma, y) f(x; U, 0)}{\int_x f(z; \Sigma, y) f(z; U, 0) dz} = f(x; (\Sigma^{-1} + U^{-1})^{-1}, (\Sigma^{-1} + U^{-1})^{-1} \Sigma^{-1} y).$$

Now suppose that the population is distributed in genotype space as a Gaussian with covariance matrix  $\Sigma$  and mean  $y$ . Selection has the effect of multiplying this density by the fitness function and renormalizing, so that if expected fitness of  $x$  is proportional to  $f(x; U, z)$  then the above argument shows that the next generation will be sampled from a Gaussian distribution with covariance matrix  $(\Sigma^{-1} + U^{-1})^{-1}$  and mean  $z + (\Sigma^{-1} + U^{-1})^{-1} \Sigma^{-1} (y - z)$ . Taking a sample of size  $N$  to construct the next generation will produce something close to this but with a slightly (stochastically) deviating mean. The next generation's mean is drawn from a Gaussian distribution with mean with covariance matrix  $(\Sigma^{-1} + U^{-1})^{-1}/N$  and mean  $z + (\Sigma^{-1} + U^{-1})^{-1} \Sigma^{-1} (y - z)$ .

Roughly, what is this doing? Suppose that the population mean differs from the optimum by  $\epsilon$ , that  $\Sigma = \sigma^2 I$  and  $U = I/\beta^2$  (so, stabilizing selection happening on a distance scale of  $\beta$ ). Then the population mean gets closer to the optimum on average, moving to  $\epsilon/(1 + \sigma^2 \beta^2)$  and adds noise of size  $(1/\beta)\sigma/\sqrt{N\sigma^2 + N1/\beta^2}$ . At equilibrium, these two movements will be of the same order, so that  $\epsilon$  is of order  $(\sigma/\sqrt{N})\sqrt{1 + \sigma^2 \beta^2}$ .

## D Away from the optimum

Let two points on  $\mathcal{X}$  be  $x_1$  and  $x_2$ , let  $\bar{x} = (x_1 + x_2)/2$ , and let  $z = (x_2 - x_1)/2$ . Then with  $D\Phi$  and  $D^2\Phi$  the first and second derivatives of  $\Phi$ , respectively, then Taylor expanding about  $x_1$  and  $x_2$  finds that

$$\begin{aligned} \Phi(\bar{x}) &= \Phi(x_1) + D\Phi(x_1) \cdot z + \frac{1}{2} z^T D^2\Phi(x_1) z + O(\|z\|^3) \\ &= \Phi(x_2) - D\Phi(x_2) \cdot z + \frac{1}{2} z^T D^2\Phi(x_2) z + O(\|z\|^3). \end{aligned}$$

Now, since  $\Phi(x_1) = \Phi(x_2) = \Phi_0$  and

$$\begin{aligned} D\Phi(x_2) &= D\Phi(x_1) + 2z^T D^2\Phi(x_1) + O(\|z\|^2), \quad \text{and} \\ D^2\Phi(x_2) &= D^2\Phi(x_1) + O(\|z\|), \quad \text{and} \end{aligned}$$

adding together the two equations above and dividing by two gets that

$$\Phi(\bar{x}) = \Phi_0 - \frac{3}{2} z^T D^2 \Phi(x_1) z + O(\|z\|^3).$$

## E Differentiating the fitness function

Suppose that  $\rho(t) \geq 0$  is a weighting function on  $[0, \infty)$  so that fitness is a function of  $L^2(\rho)$  distance of the impulse response from optimal. With  $A_0$  a representative of the optimal set:

$$\begin{aligned} D(A) &:= \int_0^\infty \rho(t) |h_A(t) - h_{A_0}(t)|^2 dt \\ &:= \int_0^\infty \rho(t) |C e^{At} B - C e^{A_0 t} B|^2 dt \\ &= \int_0^\infty \rho(t) |C (e^{At} - e^{A_0 t}) B|^2 dt \\ &= \int_0^\infty \rho(t) C (e^{At} - e^{A_0 t}) B B^T (e^{At} - e^{A_0 t})^T C^T dt \end{aligned} \tag{7}$$

How does this change with  $A$ ? Since

$$\frac{d}{du} e^{(A+uZ)t} \Big|_{u=0} = \int_0^t e^{As} Z e^{A(t-s)} ds, \tag{8}$$

we have that

$$\begin{aligned} \frac{d}{du} D(A+uZ) \Big|_{u=0} &= 2 \int_0^\infty \rho(t) C \left( \int_0^t e^{As} Z e^{A(t-s)} ds \right) B B^T (e^{At} - e^{A_0 t})^T C^T dt \\ &= 2 \int_0^\infty \rho(t) C \left( \int_0^t e^{As} Z e^{A(t-s)} ds \right) B (h_A(t) - h_{A_0}(t))^T dt \end{aligned} \tag{9}$$

and, by differentiating this and supposing that  $A$  is on the optimal set, i.e.,  $h_A(t) = h_{A_0}(t)$ , (so wolog  $A = A_0$ ):

$$\begin{aligned} \mathcal{H}(Y, Z) &:= \frac{1}{2} \frac{d}{du} \frac{d}{dv} D(A_0 + uY + vZ) \Big|_{u=v=0} \\ &= \int_0^\infty \rho(t) C \left( \int_0^t e^{A_0 s} Y e^{A_0(t-s)} ds \right) B B^T \left( \int_0^t e^{A_0 s} Z e^{A_0(t-s)} ds \right)^T C^T dt. \end{aligned} \tag{10}$$

Here  $\mathcal{H}$  is the quadratic form underlying the Hamiltonian. By defining  $\Delta_{ij}$  to be the matrix with a 1 in the  $(i, j)$ th slot and 0 elsewhere, the coefficients of the quadratic form is

$$H_{ij, k\ell}(A) := \mathcal{H}(\Delta_{ij}, \Delta_{k\ell}). \tag{11}$$

We could use this to compute the gradient of  $D$ , or to get the quadratic approximation to  $D$  near the optimal set. To do so, it'd be nice to have a way to compute the inner integral above. Suppose that we can diagonalize  $A = U \Lambda U^{-1}$ . Then

$$\int_0^t e^{As} Z e^{A(t-s)} ds = \int_0^t U e^{\Lambda s} U^{-1} Z U e^{\Lambda(t-s)} U^{-1} ds \tag{12}$$

Now, notice that

$$\int_0^t e^{s\lambda_i} e^{(t-s)\lambda_j} ds = \frac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j}. \tag{13}$$

Therefore, defining

$$X_{ij}(t, Z) = (U^{-1}ZU)_{ij} \frac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j} \quad (14)$$

moving the  $U$  and  $U^{-1}$  outside the integral and integrating we get that

$$\int_0^t e^{As} Z e^{A(t-s)} ds = UX(t, Z)U^{-1}. \quad (15)$$

Following on from above, we see that if  $Z = \Delta_{k\ell}$ , then

$$X_{ij}^{k\ell}(t) = \frac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j} (U^{-1})_{\cdot k} U_{\ell}, \quad (16)$$

where  $U_{k\cdot}$  is the  $k$ th row of  $U$ , and so

$$H_{ij,k\ell}(A) = \int_0^\infty \rho(t) C U X^{ij}(t) U^{-1} B B^T (U^{-1})^T X^{k\ell}(t)^T U^T C^T dt. \quad (17)$$

This implies that

$$D(A_0 + \epsilon Z) \approx \epsilon^2 \sum_{ijkl} H^{ij,k\ell} Z_{ij} Z_{k\ell} \quad (18)$$

and so

$$D(A_0 + \epsilon Z) \approx \epsilon^2 \sum_{ijkl} H^{ij,k\ell} Z_{ij} Z_{k\ell} \quad (19)$$

By section [C](#), if we set  $\Sigma = \sigma^2 I$  and  $U = H$ , then a population at  $A_0 + Z$  experiences a restoring force of strength  $(I + \sigma^2 H^{-1})^{-1} Z$  (treating  $Z$  as a vector and  $H$  as an operator on these). If  $\sigma^2$  is small compared to  $H^{-1}$  then this is approximately  $-\sigma^2 H^{-1} Z$ . This suggests that the population mean follows an Ornstein-Uhlenbeck process, as described (in different terms) in [Hansen and Martins \[1996\]](#).