**Zero State Definition:**
If a system is in the zero-state then the zero-input will yield a null response.
**Zero-State Response Definition:**
The zero-state response of a system to $\delta(t-\xi)$ is,

$$\begin{aligned} \delta(t-\xi) &= \int_{t_0}^{t} f(t,\xi^{'})\delta(\xi^{'}-\xi)d\xi^{'} \\ &= f(t,\xi) \text{ for } t \geq \xi \geq t_0 \\ &= 0 \text{ for } \xi > t \geq t_0 \end{aligned} \tag{1}$$

Therefore the zero-state response of a system to input $u$ is,

$$A(u) = \int_{t_0}^{t} f(t,\xi)u(\xi)d\xi \quad t_0 \leq \xi \leq t \tag{2}$$

A system is zero-state time invariant if and only if its impulse response $f(t,\xi)$ is of the form $f(t-\xi)$.
**Proof:**
Let $f(t,\xi) = w(\tau,t)$
where $\tau \triangleq (t-\xi)$
The zero-state response of a system to $\delta(t-\xi)$ is $f(t,\xi)$
thus the zero-state response of the system to $\delta(t-(\xi+\lambda))$ where $\lambda$ is an arbitrary shift is $f(t,\xi+\lambda)$ or $w(\tau-\lambda,t)$
this implies $w(\tau-\lambda,t) = w(\tau-\lambda,t-\lambda)$ $\forall t \forall \tau \forall \lambda$
**Transfer Function Definition:**

$$G(s) \triangleq \int_{-\infty}^{\infty} e^{-st} f(t)dt \tag{3}$$

If $y$ is the zero-state response to $u$ then,

$$Y(s) = G(s)U(s) \tag{4}$$

The Transfer Function gives incomplete information about the zero-input response and fully characterizes the zero-state response.
The Transfer Function can also be viewed as the steady state response of a system to input $e^{st}$ divided by $e^{st}$: $G(s) \triangleq \frac{\text{steady state response of a system to } e^{st}}{e^{st}}$
Zero-State Equivalent Linear Dynamical Systems.

$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5}$$

$$\bar{\Sigma} \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} \end{cases} \tag{6}$$

**Definition:** two dynamical systems $\Sigma$ and $\bar{\Sigma}$ are algebraically equivalent if (where $\bar{x} = Px$),
(a) $\bar{A} = PAP^{-1}$
(b) $\bar{B} = PB$
(c) $\bar{C} = CP^{-1}$

Where $P$ is a nonsingular square matrix of rank $n$. If two systems are algebraically equivalent then they are also zero-state equivalent. Two systems that are zero-state equivalent are not necessarily algebraically equivalent and can be in different dimensions.

Definition: two dynamical systems are zero-state equivalent if they have the same transfer function $G(s)$.

The transfer function, $G(s) = C(sI - A)^{-1}B$, is the Laplace transform of the system's time domain dynamics.

$G(s) = \bar{G}(s) \iff$

$CA^iB = \bar{C}\bar{A}^i\bar{B}$ for all $i \geq 0$.

**Proof:**

$G(s) = C(sI - A)^{-1}B$

$= Cs^{-1}(I - s^{-1}A)^{-1}B$

$= Cs^{-1}\left(\sum_{i=0}^{\infty}(s^{-1}A)^i\right)B$

$= \sum_{i=0}^{\infty} CA^iBs^{i+1}$

$\sum_{i=0}^{\infty} CA^iBs^{-(i+1)} = \sum_{i=0}^{\infty} \bar{C}\bar{A}^i\bar{B}s^{-(i+1)}$

$\iff CA^iB = \bar{C}\bar{A}^i\bar{B}\ \forall i$

Note: the controllability matrix $\mathcal{C}$ is defined as:

$$\begin{bmatrix} B \ AB \ ... \ A^{n-1}B \end{bmatrix} \tag{7}$$

And the observability matrix $\mathcal{O}$ is defined as:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{8}$$