This will apply to SDE of the following form:

$$dX_i(t) = \sum_k B_{ik} X_k(t) dt + \sum_{j=1}^{r} \sum_k G_{jik} X_k(t) dW_j(t) \tag{1}$$

where $\{W_i(t); 1 \leq i \leq r\}$ are independent standard Brownian motions.

The algorithm is as follows:

- Let $\bar{X}^h_p(x)$ be an approximation to the distribution of $X(h)$ given that $X(0) = x$, for $p \geq 1$.
- Construct a discrete-time process (R_p, Y_p) as follows: start with $R_0 = 1$ and $Y_0 = 1/n$, and given (R_p, Y_p),
- let $R_{p+1} = X^h_{p+1}(Y_p)$ and $Y_{p+1} = X^h_{p+1}(Y_p)/R_{p+1}$.
- Run for T/h steps.
- The estimate is $\lambda_h = \frac{1}{T} \sum_{p=1}^{T/h} \log R_p$.

Here is the information about the approximations. In the paper, they present the above SDE in the Stratonovich sense, but in the Appendix, they give the following information about Ito-sense SDE. Consider the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \tag{2}$$

where $X(t)$ and $b(x)$ are n–dimensional, and $\sigma(x) = \sigma^i_j(x)$ is an $n \times r$ matrix, for each x. Denote by σ_j the jth column of σ, and for a vector $y(x)$ let $Dy(x)$ be the matrix with $Dy(x)_{ij} = \partial_j y_i(x)$. Let U^j_p be (something like) iid standard Gaussians, let ξ^{jk}_p be iid with $\mathbb{P}\{\xi = +1\} = \mathbb{P}\{\xi = -1\} = \frac{1}{2}$, for $j \leq k$, and define

$$S^{jk}_p = \begin{cases} \frac{1}{2}\left(U^j_p U^k_p + \xi^{jk}_p\right) & \text{if } j < k \\ \frac{1}{2}\left(U^j_p U^k_p - \xi^{kj}_p\right) & \text{if } k > j \\ \frac{1}{2}\left((U^j_p)^2 - 1\right) & \text{if } k = j. \end{cases} \tag{3}$$

Then the *Euler scheme* is defined, for a given granularity h, at time step $p+1$, by

$$\bar{X}^h_{p+1} = \bar{X}^h_p + \sqrt{h} \sum_{j=1}^{r} \sigma_j(\bar{X}^h_p) U^j_{p+1} + hb(\bar{X}^h_p). \tag{4}$$

The *Mil'shteĭn scheme* is

$$\bar{X}^h_{p+1} = \bar{X}^h_p + \sqrt{h} \sum_{j=1}^{r} \sigma_j(\bar{X}^h_p) U^j_{p+1} + hb(\bar{X}^h_p) + h \sum_{j,k=1}^{r} D\sigma_j(\bar{X}^h_p)\sigma_k(\bar{X}^h_p) S^{kj}_{p+1}, \tag{5}$$

and the second-order scheme I'm calling the *Talay scheme* is

$$\begin{aligned} \bar{X}^h_{p+1} = \bar{X}^h_p &+ \sqrt{h} \sum_{j=1}^{r} \sigma_j(\bar{X}^h_p) U^j_{p+1} + hb(\bar{X}^h_p) + h \sum_{j,k=1}^{r} D\sigma_j(\bar{X}^h_p)\sigma_k(\bar{X}^h_p) S^{kj}_{p+1} \\ &+ h^{3/2} \frac{1}{2} \sum_{j=1}^{r} \left(Db(\bar{X}^h_p)\sigma_j(\bar{X}^h_p) + D\sigma_j(\bar{X}^h_p) b(\bar{X}^h_p)\right) U^j_{p+1} \\ &+ h^2 \left(\sum_{i=1}^{n} b_i(\bar{X}^h_p) \partial_i b(\bar{X}^h_p) + \frac{1}{2} \sum_{i,j=1}^{n} \left(\sigma(\bar{X}^h_p)\sigma(\bar{X}^h_p)^T\right)_{ij} \partial_i \partial_j b(\bar{X}^h_p) \right). \end{aligned} \tag{6}$$

In our linear case, $b(x) = Bx$ and $\sigma^i_j(x) = \sum_k G_{jik}x_k$, so that $\partial_j b_i(x) = B_{ij}$, $\partial_k \sigma^i_j(x) = G_{jik}$, and $(\sigma\sigma^T)_{ij} = \sum_{k\ell m} G_{\ell ik}G_{\ell jm}x_k x_m$, so the Euler scheme is

$$\bar{X}^h_{p+1} = \left(I + \sqrt{h}\sum_{j=1}^{r} G_j U^j_{p+1} + hB \right) \bar{X}^h_p, \tag{7}$$

the Mil'shteĭn is

$$\bar{X}^h_{p+1} = \left(I + \sqrt{h}\sum_{j=1}^{r} G_j U^j_{p+1} + hB + h\sum_{j,k=1}^{r}\sum_{\ell=1}^{n} S^{kj}_{p+1}G_j G_k \right) \bar{X}^h_p, \tag{8}$$

and the Talay is

$$\begin{aligned} \bar{X}^h_{p+1} = \Bigg(& I + \sqrt{h}\sum_{j=1}^{r} G_j U^j_{p+1} + hB + h\sum_{j,k=1}^{r} S^{kj}_{p+1}G_j G_k \\ & + h^{3/2}\frac{1}{2}\sum_{j,k=1}^{r}(BG_j + G_j B) + h^2\frac{1}{2}B^2 \Bigg) \bar{X}^h_p, \end{aligned} \tag{9}$$

where e.g. $G_j G_k X$ is a matrix product, $(G_j G_k X)_i = \sum_{\ell,m=1}^{n} G_{ji\ell}G_{k\ell m}X_m$.

Our example,

$$dX_t = \left(\mathrm{diag}(\mu) + D^T\right) X_t dt + \mathrm{diag}(X_t)\Gamma^T dB_t \tag{10}$$

has

$$Gjik = \delta_{ik}\Gamma_{jk} \tag{11}$$

$$B = \mathrm{diag}(\mu) + D^T. \tag{12}$$