

LECTURE NOTES

MATT

Recall a realization of the basic noise for Gaussian processes looked like that in Figure ???. Now, arrows are either muted or (rarely) point up. See Figure ???.

§ 1 | Motivation

Suppose in some space X we lay down a large number of LED lights, each with their own battery, with density given by a σ -finite measure μ . We do this in a way so that, for each region $A \subset X$,

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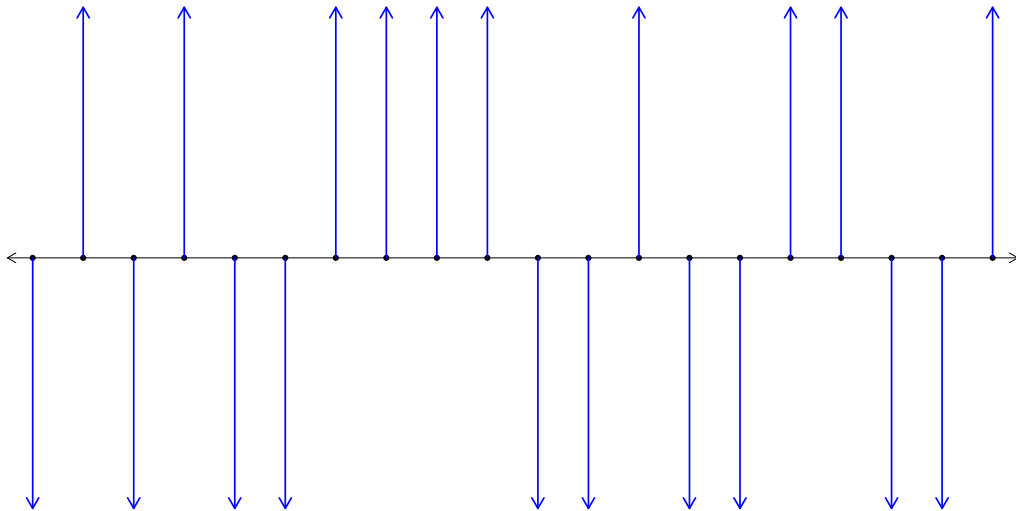


FIGURE 1. A realization of the basic noise used to construct a Gaussian process.

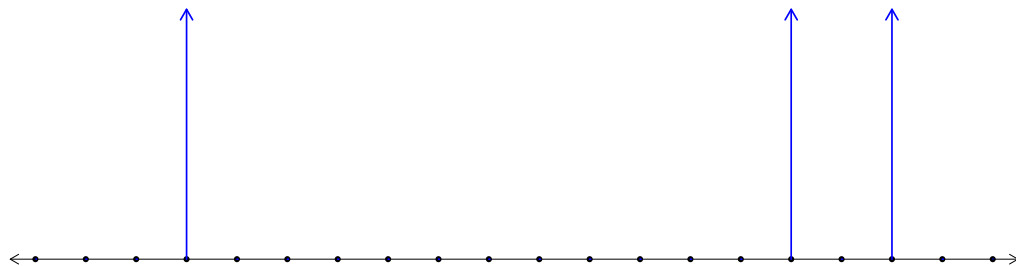


FIGURE 2. A realization of the basic noise used to construct a Poisson process.

we put down about $M\mu(A)$ lights in that region, where M is some large number. Independently we turn on each light with probability M^{-1} , and leave off otherwise.

We would like to answer the following question: how many lights in A are on? To that end, let $N(A)$ denote the number of lights on in A and compute

$$(1) \quad \mathbb{E}[N(A)] = \mathbb{E} \left[\sum_{\text{lights in } A} \mathbb{1}_{\{\text{light on}\}} \right] = \sum_{\text{lights in } A} \mathbb{P}\{\text{light is on}\} = M\mu(A) \left(\frac{1}{M} \right) = \mu(A).$$

Thus μ gives the expected density for the set of lights that are on in A . By construction, we know $N(A) \sim \text{Binom}(M\mu(A), M^{-1})$, and hence the distribution of $N(A)$ is approximately $\text{Pois}(\mu(A))$. To see this, put $L = M\mu(A)$ and observe,

$$(2) \quad \mathbb{P}\{N(A) = n\} = \binom{L}{n} \left(\frac{1}{M} \right)^n \left(1 - \frac{1}{M} \right)^{L-n}$$

$$(3) \quad = \frac{L(L-1) \cdots (L-n+1)}{n! M^n} \left(1 - \frac{1}{M} \right)^{L-n}$$

$$(4) \quad \simeq \frac{1}{n!} \left(\frac{L}{M} \right)^n \exp \left(-\frac{L}{M} \right) + \mathcal{O} \left(\frac{1}{M} \right)$$

$$(5) \quad \simeq \frac{\mu(A)^n}{n!} e^{-\mu(A)}$$

This motivates the following definition.

DEFINITION. Let μ be a σ -finite measure on some space X . A *Poisson Point Process* (PPP) on X with *mean measure* (or, *intensity*) μ is a random point measure N such that:

(a) For any Borel set $A \subset X$, we have $N(A) \in \mathbb{Z}_{\geq 0}$ and $N(A) \sim \text{Pois}(\mu(A))$, i.e.

$$(6) \quad \mathbb{P}\{N(A) = n\} = \frac{\mu(A)^n}{n!} e^{-\mu(A)}.$$

(b) If A and B are disjoint Borel subsets of X , then $N(A)$ and $N(B)$ are independent random variables.

Recall a point measure is just a measure whose mass is atomic. That is, if $\{x_i\} \subset X$ then a point measure is of the form

$$(7) \quad \mu = \sum_i a_i \delta_{x_i}$$

where δ_x is the unit point mass at x .

§ 2 | PPP Properties

It is sometimes useful to think of a PPP as a random collection of points. With this in mind, we list some important properties of $N \sim \text{PPP}(\mu)$ on some space X :

- *Enumeration:* It is always possible to enumerate the points of N , i.e. there is a random collection of points $\{x_i\} \subset X$ such that

$$(8) \quad N = \sum_i \delta_{x_i}.$$

- *Mean measure:* If $f: X \rightarrow \mathbb{R}$ then

$$(9) \quad \mathbb{E} \left[\int f(x) dN(x) \right] = \int f(x) d\mu(x).$$

Note: This is a more general property of point processes, as any point process has a mean measure. To see (9) holds without needing N to be a *Poisson* point process, let f be a simple function, i.e.

$$(10) \quad f(x) = \sum_{i=1}^n f_i \mathbf{1}_{A_i}(x), \quad \text{where} \quad X = \bigcup_i A_i, \quad A_i \cap A_j = \emptyset \text{ for } i \neq j.$$

Then we compute

$$(11) \quad \mathbb{E} \left[\int_X f(x) dN(x) \right] = \mathbb{E} \left[\sum_i f_i N(A_i) \right] = \sum_i f_i \mathbb{E} [N(A_i)] = \sum_i f_i \mu(A_i) = \int_X f(x) d\mu(x).$$

This can then be extended to arbitrary measurable functions through the standard limiting procedure.

- *Thinning:* Independently discard each point of N with probability $1 - p(x)$ for a point at $x \in X$. The result is a PPP(ν), where

$$(12) \quad \nu(A) = \int_A p(x) d\mu(x).$$

In other words, if $N = \sum_i \delta_{x_i}$ and $A_i = 1$ with probability $p(x_i)$ and $A_i = 0$ otherwise, then

$$(13) \quad \tilde{N} = \sum_i A_i \delta_{x_i} \sim \text{PPP}(\nu).$$

- *Additivity:* If $N_1 \sim \text{PPP}(\mu_1)$ and $N_2 \sim \text{PPP}(\mu_2)$ are independent on X , then $N_1 + N_2 \sim \text{PPP}(\mu_1 + \mu_2)$. In particular, if $\mathbb{P}\{X = n\} = \frac{\lambda^n}{n!} e^{-\lambda}$ and $\mathbb{P}\{Y = n\} = \frac{\nu^n}{n!} e^{-\nu}$ are independent, then

$$(14) \quad \mathbb{P}\{X + Y = n\} = \frac{(\lambda + \nu)^n}{n!} e^{-(\lambda + \nu)}.$$

- *Labeling:* For each point in a PPP, associate an independent label from a space Y according to some probability distribution ν . Let $N = \sum_i \delta_{x_i}$ for $\{x_i\} \subset X$ and let $G_1, G_2, \dots \in Y$ be iid with density ν . Then

$$(15) \quad \bar{N} := \sum_i \delta_{(x_i, G_i)} \sim \text{PPP}(\mu \times \nu)$$

on $X \times Y$.

§ 3 | Examples

Henceforth, let λ denote Lebesgue measure.

3.1 Example Let $N \sim \text{PPP}(\lambda)$ on $\mathbb{R}_{\geq 0}$, where λ is Lebesgue measure. As before, we think of the points of N as ‘lights’, here positioned on the positive reals.

- How far until the first light?
- Suppose each light is independently either red or green with probability $\frac{1}{2}$. How far until the first red light?

SOLUTION. Let $N = \sum_i \delta_{x_i}$ and put $T = \min\{x_i\}$. Using (??) we compute

$$(16) \quad \mathbb{P}\{T > t\} = \mathbb{P}\{N([0, t]) = 0\} = e^{-t}.$$

This solves part (a). For the colorblind readers, this also solves part (b).

Now let $\{\tilde{x}_i\} \subset \{x_i\}$ be the (random) set of red lights and define $\tilde{N} = \sum_i \delta_{\tilde{x}_i}$, the point process for the red lights from N . By the thinning property (??), $\tilde{N} \sim \text{PPP}(\frac{1}{2}\lambda)$. Similarly define $\tilde{T} = \min\{\tilde{x}_i\}$ and observe

$$(17) \quad \mathbb{P}\{\tilde{T} > t\} = \mathbb{P}\{\tilde{N}([0, t]) = 0\} = e^{-t/2},$$

thus (b) is solved. ◆
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3.2 Example Rain falls for 10 minutes on a large patio at a rate of $\nu = 5000$ drops per minute per square meter. Each drop splatters to a random radius R that has an Exponential distribution, with mean 1cm, independently of the other drops. Assume the drops are 1mm thick and the set of locations of the raindrops is a PPP.

- (a) What is the mean and variance of the total amount of water falling on a square with area 1m^2 ?
- (b) A very small ant is running around the patio. See Figure ?? . What is the chance the ant gets hit?

SOLUTION. Let $N = \sum_i \delta_{(x_i, y_i)}$ where (x_i, y_i) is the center of the i th drop. Take $N \sim \text{PPP}(\nu\lambda)$ and let M denote the number of drops in $[0, 1]^2$, so that $M = N([0, 1]^2) \sim \text{Pois}(\nu)$. Then the total volume V is

$$(18) \quad V = \sum_{i=1}^M \frac{\pi}{10^3} R_i^2$$

where R_i is the radius of the i th drop. Note this is a sum of random variables where the number of terms is also a random variable. Thus we use Wald's equation (??) to obtain

$$(19) \quad \mathbb{E}[V] = \frac{\pi}{10^3} \mathbb{E}[M] \mathbb{E}[R_1^2] = \frac{\pi}{10^3} \cdot \nu \cdot \frac{2}{100^2} = \frac{2\pi}{10^7} \nu$$

The second step in (??) was obtained from the fact that an exponentially distributed random variable X with mean β^{-1} has higher moments given by

$$(20) \quad \mathbb{E}[X^n] = \frac{n!}{\beta^n}.$$

This is proved by an iterated application of integration by parts, and the result gives rise to

$$(21) \quad \text{var}[X^n] = \mathbb{E}[X^{2n}] - \mathbb{E}[X^n]^2 = \frac{(2n)! - (n!)^2}{\beta^{2n}}.$$

The $n = 2$ case will turn out to be useful when computing the variance of V .

Indeed, to compute the variance we utilize the variance decomposition formula. Observe,

$$\begin{aligned}
 (22) \quad \text{var}[V] &= \mathbb{E}[\text{var}[V \mid M]] + \text{var}[\mathbb{E}[V \mid M]] \\
 (23) \quad &= \mathbb{E}\left[M\left(\frac{\pi}{10^3}\right)^2 \text{var}(R^2)\right] + \text{var}\left[M\left(\frac{\pi}{10^3}\right) \mathbb{E}[R^2]\right] \\
 (24) \quad &= \nu\left(\frac{\pi}{10^3}\right)^2 \left(\frac{20}{100^4}\right) + \nu\left(\frac{\pi}{10^3}\right)^2 \left(\frac{2}{100^2}\right)^2 \\
 (25) \quad &= \left(\frac{\pi}{10^3}\right)^2 \left(\frac{24}{100^4}\right) \nu.
 \end{aligned}$$

This solves part (a).

Now, part (b) can be solved by way of the labeling property. Here, we use the radius R_i of the i th drop to label the point (x_i, y_i) . Recall the density of an Exponential random variable with mean 0.01 is $100 \exp(-100r) dr$. So we define a measure μ on $X := \mathbb{R}^2 \times [0, \infty)$ by

$$(26) \quad \mu(A) = \int_A 100\nu \exp(-100r) dx dy dr.$$

We think of X as the (closed) upper half plane in \mathbb{R}^3 where the third coordinate is a realization of R . By the labeling property (??), $\bar{N} := \sum_i \delta_{(x_i, y_i, R_i)} \sim \text{PPP}(\mu)$ on X . For the ant to remain dry, any drop with radius r must land outside the circle of radius r centered at the ant. Viewed from the space X , we want to integrate over the cone with its tip at the ant, whose horizontal cross-section at height r is a circle of radius r . From this we compute

$$(27) \quad \mathbb{P}\{\text{ant is dry}\} = \mathbb{P}\{\bar{N}(A) = 0\} = \exp(-\mu(A)) = \exp\left(-100\pi\nu \int_0^\infty r^2 e^{-100r} dr\right) = \exp\left(-\frac{\pi\nu}{5000}\right).$$

Plugging in the given value for ν yields $\mathbb{P}\{\text{ant is dry}\} = \exp(-\pi) \approx 0.0432$. The ant had better grab an umbrella! ◆

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3.1

Wald's Equation

The following is the statement of Wald's equation, taken from Wikipedia ^{*}.

3.3 Theorem (Wald's Equation) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of real-valued, independent and identically distributed random variables and let N be a nonnegative integer-valued random variable that is independent of the sequence $(X_n)_{n \in \mathbb{N}}$. Suppose that N and the X_n have finite expectations. Then

$$(28) \quad \mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}[N] \mathbb{E}[X_1].$$

^{*} The proof is also on Wikipedia.

FIGURE 3. A realization of the ant from Example ???. Looks like he had an umbrella after all.

