LECTURE NOTES

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0.1 Lemma Let $\tau_+ = \inf\{t \geq 0 \mid X_t \neq X_0\}$ be the time of the first jump. Suppose $\imath\{\tau_+ > 0\} > 0$. Then $\tau_+ \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$.

Proof. Let $F(t) = \mathbb{P}\{\tau_+ > t\}$ be the distribution function for τ_+. By the Markov property on X_t, we have

$$\mathbb{P}\{\tau_+ > t + s \mid \tau_+ > t\} = \mathbb{P}\{\tau_+ > s\} = F(s).$$

Unraveling the conditional probability on the left, we find

$$F(s) = \frac{\mathbb{P}\{\tau_+ > t + s \text{ and } \tau_+ > t\}}{\mathbb{P}\{\tau_+ > t\}} = \frac{F(t+s)}{F(t)}.$$

We then have that $F(t + s) = F(t)F(s)$, and the only continuous solution to this equation is $F(t) = e^{-\lambda t}$ for some $\lambda > 0$.

□

To solve our problem with splitting and dying cells, we model it with a continuous time Markov chain with state space $\{1, 2, \ldots, N\}$ and generator G given by

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \mu & -\mu - \lambda & \lambda & 0 & 0 & \ldots & 0 \\ 0 & 2\mu & -2\mu - 2\lambda & 2\lambda & 0 & \ldots & 0 \\ 0 & 0 & 3\mu & -3\mu - 3\lambda & 3\lambda & \ldots & 0 \\ \vdots & & & & & & \vdots \end{pmatrix}$$

Let $h(x) = \mathbb{P}^x\{\tau_N < \tau_0\}$ be the hitting probability. From the theorem on hitting probabilities, we know that

$$\begin{aligned} h(0) &= 0 \\ h(N) &= 1 \\ Gh(x) &= 0 \quad \forall x \neq 0, N. \end{aligned}$$

The last of these equations says

$$x\lambda(h(x+1) - h(x)) + x\mu(h(x-1) - h(x)) = 0$$

which rearranges into

$$h(x+1) - h(x) = \frac{\mu}{\lambda}(h(x) - h(x-1)).$$

Since $h(1) - h(0) = h(1)$ by our boundary condition we can use this as the seed for this recurrence relation. We get

$$h(x+1) - h(x) = \left(\frac{\mu}{\lambda}\right)^x h(1),$$

Date: 2 November 2018.

so

$$\begin{aligned} h(x) &= h(1) + \sum_{y=1}^{x-1} h(y+1) - h(y) \\ &= h(1)\left\{1 + \sum_{y=1}^{x-1}\left(\frac{\mu}{\lambda}\right)^y\right\} \\ &= h(1)\left(\frac{1-\left(\frac{\mu}{\lambda}\right)^x}{1-\frac{\mu}{\lambda}}\right). \end{aligned}$$

Next we use the other boundary condition:

$$1 = h(N) = h(1)\left(\frac{1-\left(\frac{\mu}{\lambda}\right)^N}{1-\frac{\mu}{\lambda}}\right)$$

which gives

$$h(1) = \left\{\frac{1-\left(\frac{\mu}{\lambda}\right)^N}{1-\frac{\mu}{\lambda}}\right\}^{-1},$$

and

$$h(x) = \frac{1-\left(\frac{\mu}{\lambda}\right)^x}{1-\left(\frac{\mu}{\lambda}\right)^N}.$$

This gives us the answer to our first question: what is the probability that a single mutation "grows?" This is answered by the value

$$h(1) = \frac{1-\left(\frac{\mu}{\lambda}\right)}{1-\left(\frac{\mu}{\lambda}\right)^N} \approx 1 - \frac{\mu}{\lambda} \text{ for large } N.$$

Stationary Distributions

0.2 Proposition Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain with generator G on state space $\mathcal{X}$. Suppose that $G^T\pi = 0$ for some probability distribution π on $\mathcal{X}$. Then

$$\mathbb{P}^\pi\{X_t = y\} = \pi(y).$$

The conditions on π above mean

$$\sum_{y\in\mathcal{X}} \pi(y) = 1, \quad \pi(y) \geq 0, \quad \sum_{y\in\mathcal{X}} \pi(y)G_{yx} = 0.$$

The notation $\mathbb{P}^\pi$ means

$$\shortmid^\pi\{X_t = y\} = \sum_{x\in\mathcal{X}} \pi(x)\shortmid^x\{X_t = y\}.$$

Such a distribution π is called a *stationary distribution.*

Proof. Let $f : \mathcal{X} \to \mathbb{R}$ be a function (thought of as a column vector) with $\sum_y \pi(y)f(y) < \infty$ (if we wished to generalize to a countable state space). Recall that

$$\mathbb{E}^x_{f(X_t)} = P_t f(x) = e^{tG} f(x).$$

Since $\pi G = 0$,

$$\pi P_t = \pi e^{tG} = \pi \sum_{n \geq 0} \frac{t^n}{n!} G^n = \pi.$$

So,

$$\mathbb{E}^{\pi}_{f(X_t)} = \pi P_t f = \pi f = \sum_x \pi(x) f(x).$$

The statement of the proposition now follows from taking f to be the indicator of y:

$$f(x) = \begin{cases} 1 & x = y \\ 0 & x \neq y. \end{cases}$$

$\square$

Fact. If π is unique, then $P_t \to \pi$ as $t \to \infty$. That is, for all f, x,

$$\mathbb{E}^{x}_{f(X_t)} \to \sum_y \pi(y) f(y) \quad \text{as } t \to \infty.$$

This can also be phrased as saying that X_t converges in distribution to π.