LECTURE NOTES

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Note about previous example If we start with Brownian motion and incorporate exponential decay, a more general central limit theorem applies (for more information, look up Lindeberg-Feller condition).

DEFINITION. Random variables $Z_1, \ldots, Z_n$ (which we can think of as a random vector of length n, $(Z_1, \ldots, Z_n)$) are *jointly Gaussian* if for any $a_1, \ldots, a_n \in \mathbb{R}$, we have

$$\sum_{k=1}^{n} a_k Z_k \sim N(m, \sigma^2)$$

for some m, σ^2. That is, these random variables are jointly Gaussian if and only if any linear combination of them is univariate Gaussian.

Definition/Notation Suppose that the random variables $(Z_1, \ldots, Z_n)$ are jointly Gaussian. Let $Z = (Z_1, \ldots, Z_n)$. Define a vector of means

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix},$$

where μ_i is the mean of Z_i. Define the $n \times n$ covariance matrix $\Sigma = \{\Sigma_{i,j}\}$, where $\Sigma_{i,j} = \operatorname{cov}[Z_i, Z_j]$. Then we write $Z \sim N(\mu, \Sigma)$.

Now, suppose that we have Z, μ, and Σ as in the above definition, and let $a_1, \ldots, a_n \in \mathbb{R}$. Let

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Then we know that $\displaystyle\sum_{k=1}^{n} a_k Z_k \sim N(m, \sigma^2)$ for some m, σ^2. A small amount of algebraic manipulation yields the following equations:

$$m = \sum_{k=1}^{n} a_k \mu_k = a^T \mu$$

and

$$\sigma^2 = \sum_{1 \le i,j \le n} a_i \Sigma_{i,j} a_j = a^T \Sigma a \,.$$

Example We can have random variables that are marginally Gaussian (that is, each of them have Gaussian distributions themselves) but that are not jointly Gaussian. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent random variables. Let $Z = \operatorname{sign}(X)|Y|$. So X and Z are "almost"

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independent, but have the same sign as each other. Then X and Z both have Gaussian distributions, but X and Z are not jointly Gaussian.

DEFINITION. A *Gaussian process* on an index set T is a collection of random variables $\{X_t\}_{t\in T}$ such that for any $n \in \mathbb{N}$, for any $(t_1, \ldots, t_n) \in T^n$, $(X_{t_1}, \ldots, X_{t_n})$ is jointly Gaussian. It is *centered* if $\mathbb{E}[X_t] = 0$ for all t.

Example Let $Z \sim N(\mu, \Sigma)$, where $Z = (Z_1, \ldots, Z_n)$. Then Z is a Gaussian process on the index set $\{1, 2, \ldots, n\}$.

Example Suppose that $\{B_t\}_{t\geq 0}$ is a Brownian motion, then it is a Gaussian process on $[0, \infty)$.

Facts About Jointly Gaussian Variables and Gaussian Processes

(i) If $Z \in \mathbb{R}^n$ and $Z \sim N(\mu, \Sigma)$ (that is, Z is an n-dimensional multivariate Gaussian), then

$$\mathbb{E}[f(Z)] = \int_{\mathbb{R}^n} f(x) \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(\frac{-(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}\right) dx.$$

(ii) The distribution of a Gaussian process is determined by its mean, $\mu(t) = \mathbb{E}[X_t]$ and covariance (also called covariance kernel) $\sigma^2(s,t) = \operatorname{cov}[X_s, X_t]$.

(iii) Covariance kernels are positive semidefinite. Given a Gaussian process on T, where T is a measure space, we can use this to define an inner product on some subset of the set of functions $T \to \mathbb{R}$ (I don't know what the subset is that we need to take in order for this definition to make sense). Given $f, g \colon T \to \mathbb{R}$, define

$$\langle f, g \rangle_\sigma \coloneqq \sum_{s\in T} \sum_{t\in T} f(s) g(t) \sigma^2(s,t)$$

and $\|f\|_\sigma^2 = \langle f, f \rangle_\sigma \geq 0$.

Example For Brownian motion, $\mathbb{E}[B_t] = 0$, $\operatorname{cov}[B_s, B_t] = \sigma^2(s,t) = \min(s,t)$. So, for $f, g \colon [0,\infty) \to \mathbb{R}$,

$$\langle f, g \rangle_\sigma = \int_0^\infty \int_0^\infty f(s) g(t) \min(s,t)\, ds\, dt\,.$$

(iv) If you have a linear space V, with a symmetric positive definite inner product $\langle \cdot, \cdot \rangle$, and a countable orthonormal basis $\{\varphi_k\}_{k=1}^\infty$, then you can define a centered isomorphic Gaussian process, ie. a Gaussian process on V such that $\mathbb{E}[X_t] = 0$ for all $t \in V$ and $\operatorname{cov}[X_s, X_t] = \langle s, t \rangle$. That is, you can construct random variables and index them using V in such a way that the covariance of two random variables is equal to the inner product of their indices.

This can be done in the following way. Let $\{Z_k\}_{k=1}^\infty$ be independent, identically distributed (iid) variables each with distribution $N(0,1)$. For $t \in V$, which we can write as $t = \sum_{k=1}^\infty \langle t, \phi_k \rangle \phi_k$, define $X_t = \sum_{k=1}^\infty \langle t, \phi_k \rangle Z_k \in \mathbb{R}$. Because Z_k are independent, and each have variance 1,

$$\operatorname{cov}[Z_i, Z_j] = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and so

$$\begin{aligned}\operatorname{cov}[X_s, X_t] &= \operatorname{cov}\left[\sum_k \langle s, \phi_k\rangle Z_k, \sum_j \langle t, \phi_j\rangle Z_j\right] \\ &= \sum_{j,k} \langle s, \phi_k\rangle \langle t, \phi_j\rangle \operatorname{cov}[Z_k, Z_j] \\ &= \sum_j \langle s, \phi_j\rangle \langle t, \phi_j\rangle \\ &= \langle s, t\rangle .\end{aligned}$$

Example Suppose we wanted to construct a Gaussian process with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix} .$$

That is, we want to construct random variables X_{v_1}, X_{v_2}, X_{v_3}, where $v_1, v_2, v_3 \in \mathbb{R}^3$, such that for any pair (X_{v_i}, X_{v_j}), we have $\operatorname{cov}[X_{v_i}, X_{v_j}] = \langle v_i, v_j\rangle = \Sigma_{i,j}$, where $\Sigma_{i,j}$ is the entry in the ith row and jth column of our covariance matrix.

If we start with three independent random variables, each with distribution $N(0,1)$, we can use the process above to construct $X_{v_1}, X_{v_2}, X_{v_3}$. In order to do this, however, we first need to determine what v_1, v_2, and v_3 are. To do this, we will use the fact that we need $\langle v_i, v_j\rangle = \Sigma_{i,j}$.

(I wasn't able to write down anything coherent about the discussion about how to do this/why it exists, so this is a gap.)

Once we have v_1, v_2, v_3, then we can start with $Z_{v_1}, Z_{v_2}, Z_{v_3}$, independent and each with distribution $N(0,1)$, we can construct $X_{v_1}, X_{v_2}, X_{v_3}$ using the process described above.

(v) Linear transformations of Gaussians are Gaussian: if $Z = (Z_1, \ldots, Z_n) \sim N(\mu, \Sigma)$ and $A \in \mathbb{R}^{k\times n}$, then $AZ \sim N(A\mu, A\Sigma A^T)$.

Example: Let $\Sigma = kk^T$ be the Cholesky decomposition of Σ, and let $Z = (Z_1, \ldots, Z_n)$ independent, each with distribution $N(0,1)$. Then if we let $(X_1, X_2, X_3) = X = kZ$, then $X \sim N(0, kIk^T = \Sigma)$.

If X, Y are jointly Gaussian, then X and Y are independent if and only if $\operatorname{cov}[X, Y] = 0$.