LECTURE NOTES

FILL STALEY

Recall the following definitions:

- Covariance: $\mathrm{cov}[X,Y] = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- Variance: $\mathrm{var}[X] = \mathrm{cov}[X,X]$

Covariance is bilinear, so [†]

$$\mathrm{cov}[aX+bY,Z] = a\,\mathrm{cov}[X,Z] + b\,\mathrm{cov}[Y,Z]\,,$$

and in particular,

$$\mathrm{var}[aX] = a^2\,\mathrm{var}[X]\,.$$

Motivation: additive noise.

Let

$$X_k = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

be independent, for $k \in \mathbb{Z}$. A basic thing we might want to do with these values is add them up. So, let $S_{k,n}$ be the sum of the n adjacent values starting with the kth value; that is,

$$S_{k,n} = \sum_{j=k}^{k+n-1} X_j\,.$$

Note that $\mathbb{E}[X_k] = 0$ and $\mathrm{var}[X_k] = 1$, and so $\mathbb{E}[S_{k,n}] = 0$, $\mathrm{var}[S_{k,n}] = n$.

Recall the Central Limit Theorem, which essentially says "(well-enough behaved) additive noise makes Gaussian distributions". That is, adding up a bunch of small things that make the same-size contribution and rescaling yields basically a Gaussian distribution.

In our case, this says that

$$\frac{1}{\sqrt{n}} S_{k,n} \xrightarrow[d]{n\to\infty} N(0,1)\,.$$

ie. for any $a < b$,

$$\mathbb{P}\left\{a \le \frac{1}{\sqrt{n}} S_{k,n} \le b\right\} \xrightarrow{n\to\infty} \int_a^b \frac{1}{2\pi e^{-x^2/2}}\,dx$$

and, for "any" f,

$$\mathbb{E}\left[f\left(\frac{1}{\sqrt{n}} S_{k,n}\right)\right] \xrightarrow{n\to\infty} \mathbb{E}[f(Z)] = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx\,, \tag{1}$$

where $Z \sim N(0,1)$.

Facts About Gaussian Processes

Say $X \sim N(\mu, \sigma^2)$, ie.

$$\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(X) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(X-\mu)^2/2\sigma^2}\,dx\,.$$

Then

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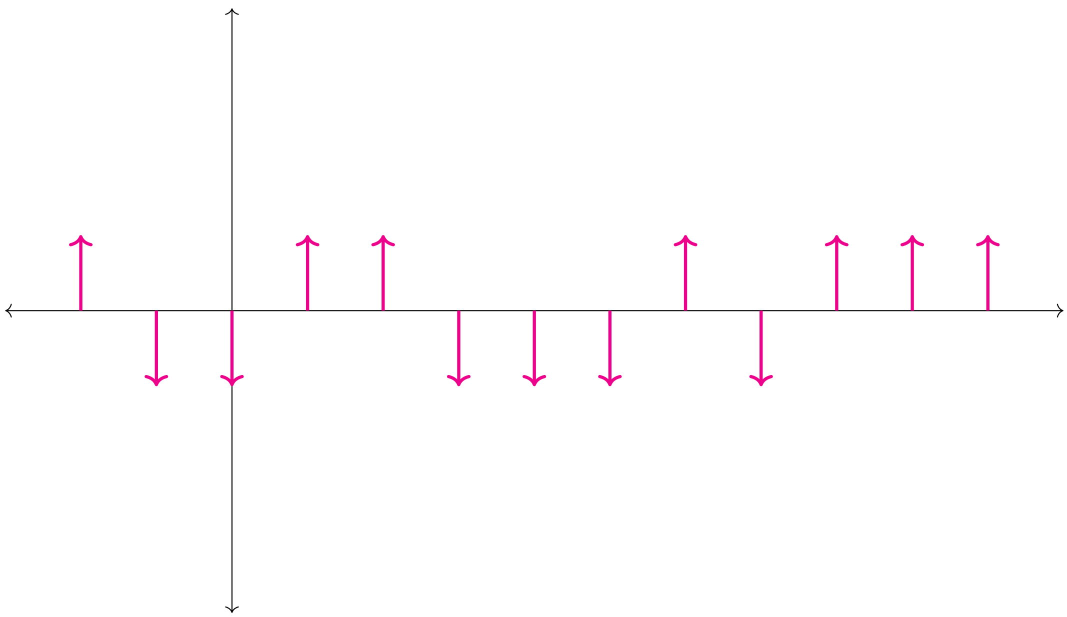
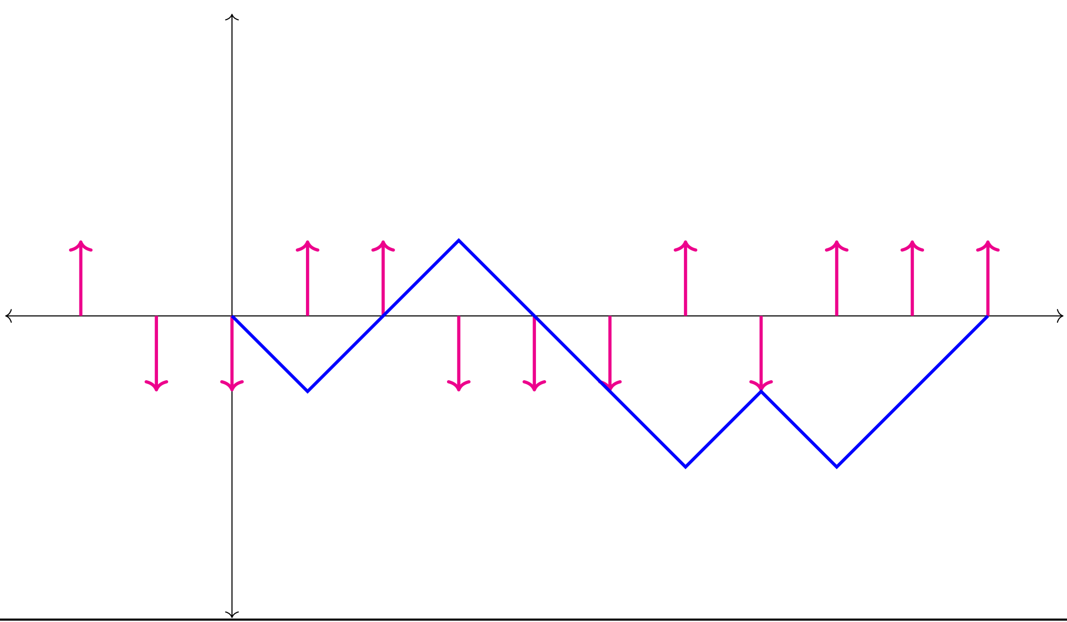
† Test.

(1) (scaling) if $a \in \mathbb{R}$, then $aX \sim N(a\mu, a^2\sigma^2)$; and
(2) (linearity) if $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

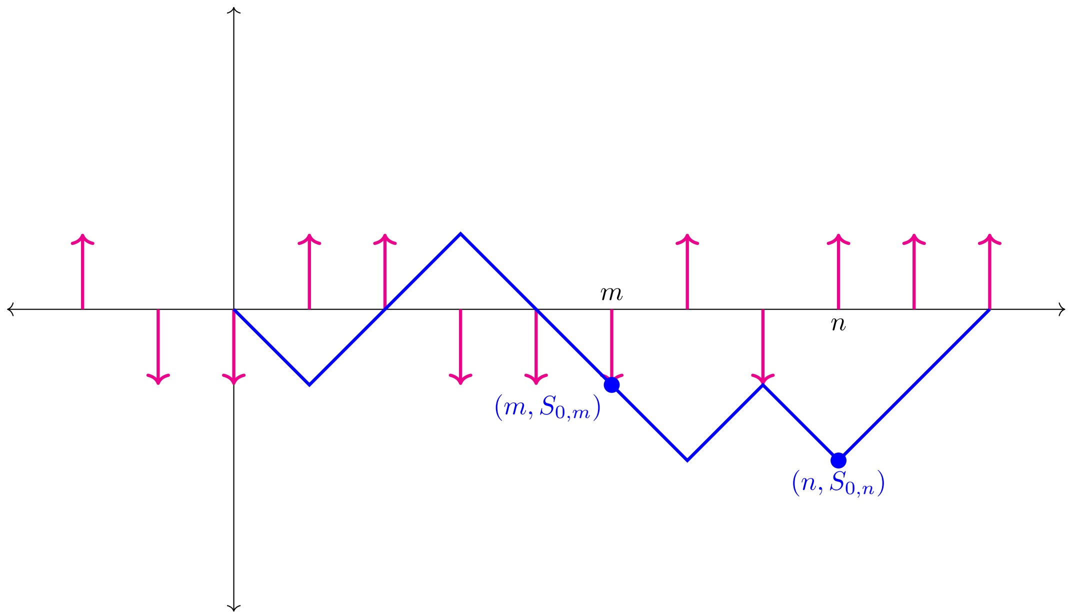
Let X_k and $S_{k,n}$ be defined as above. We can visualize these variables in the following way. We can visualize $\{X_k\}$ with the following picture:



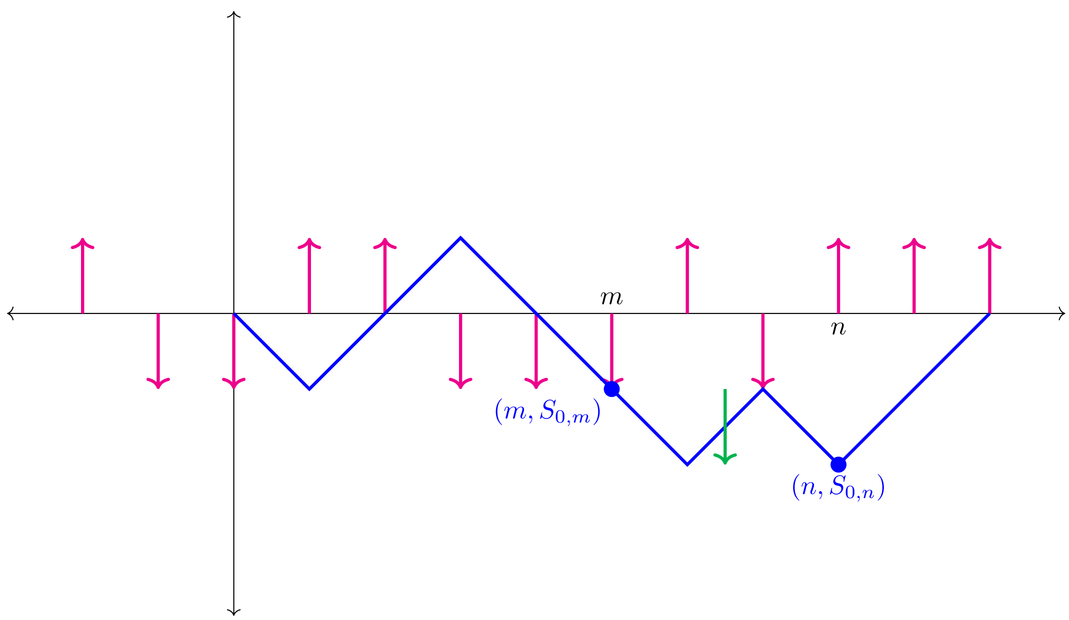
Where the location of the tip of each arrow is (k, X_k). We can visualize $\{S_{0,n}\}_{n=1,2,3,...}$ using the following picture:



Using this picture, if we suppose m and n are the values on the horizontal axis marked below, then $S_{0,m}$ and $S_{0,n}$ are the height of the points marked below.



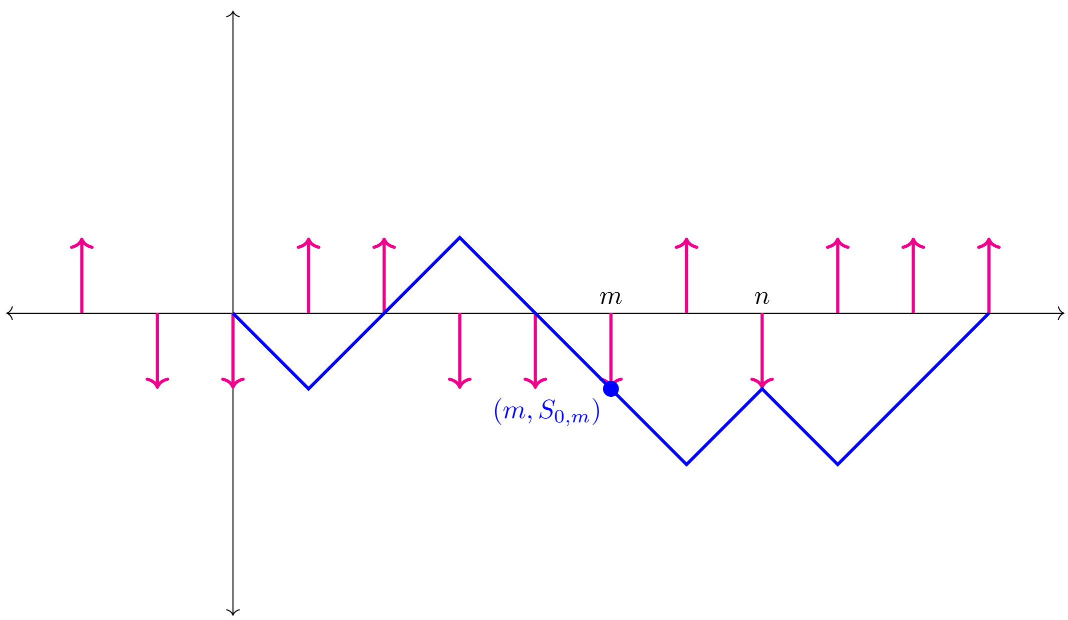
Additionally, $S_{m,n-m}$ is the signed distance indicated by the green arrow below, which gives the vertical distance from the first blue point to the second.



This gives a visualization of the fact that for $n \geq m$,

$$S_{0,n} = S_{0,m} + S_{m,n-m}.$$

Similarly, we can visualize $\{S_{-2,n}\}_{n=1,2,\ldots}$. The portion of the blue curve that's on the right half of the vertical axis is the same as in the pictures above, since $S_{-2,2} = 0$.



Observe that $\mathrm{var}[S_{0,n}] = n$, which can be used to show the more general statement that for $m \leq n$, $\mathrm{cov}[S_{0,m}, S_{0,n}] = m$. This comes from the following chain of equalities:

$$\begin{aligned}
\mathrm{cov}[S_{0,m}, S_{0,n}] &= \mathrm{cov}[S_{0,m}, S_{0,m} + S_{m,n-m}] \\
&= \mathrm{cov}[S_{0,m}, S_{0,m}] + \mathrm{cov}[S_{0,m}, S_{m,n-m}] && \text{since cov is bilinear} \\
&= m + 0 && \text{since } S_{0,m} \text{ and } S_{m,n-m} \text{ are independent} \\
&= m.
\end{aligned}$$

Add reference

Note that Equation 1holds in the more general case that $\mathbb{E}[Z] = 0$ and $\mathrm{var}[Z] = 1$.

Brownian Motion

Let $S_{k,n}$ be defined as above and

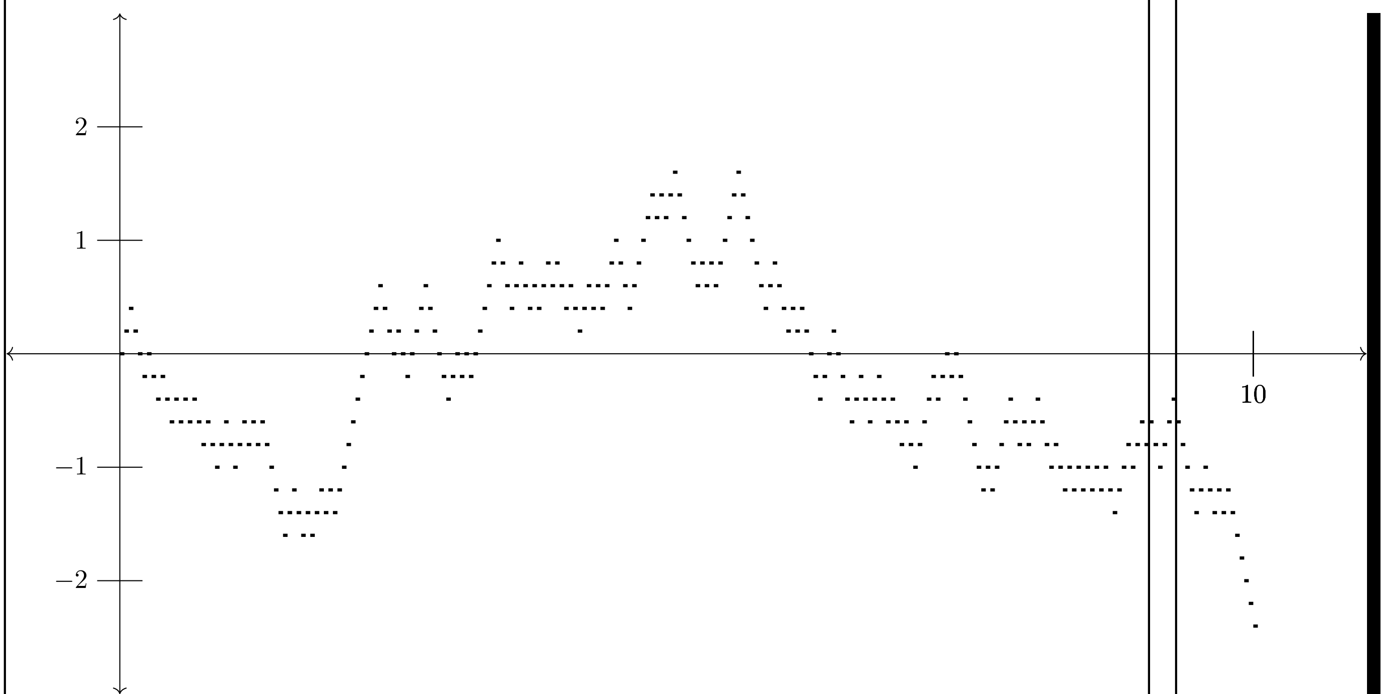
$$B_t^{(N)} = \frac{1}{\sqrt{N}} S_{0,\lfloor tN \rfloor}.$$

Then let

$$B_t = \lim_{N \to \infty} \frac{1}{\sqrt{N}} S_{0,\lfloor tN \rfloor}$$

Then $\{B_t\}$ is Brownian motion. We can visualize Brownian motion by considering the graphs yielded by the maps $t \mapsto B^t$ and $t \mapsto B_t^{(N)}$.

For example, if we let $N = 25$, the function $t \mapsto B_t^{(N)}$ yields the following graph for $t = 0$ to $t = 10$, for a particular sequence $\{X_k\}_{k \in \mathbb{Z}_{\geq 0}}$.



The Central Limit Theorem tells us that $B_t - B_s \sim N(0, t-s)$. Additionally,

$$\mathrm{var}[B_t] = \lim_{N\to\infty} \frac{\lfloor tN \rfloor}{N} = t$$

and

$$\mathrm{cov}[B_s, B_t] = s$$

for $s \leq t$.

Definition. A **standard Brownian motion** is a stochastic process $\{B_t\}_{t\geq 0}$ such that

(i) $B_0 = 0$
(ii) $B_t - B_s \sim N(0, t-s)$—that is, the variance of an increment is proportional to the time difference
(iii) $B_t - B_s$ is independent of $B_v - B_u$ for $u < v \leq s < t$

That is, Brownian motion is the stochastic process with independent Gaussian increments, ie. how it moves in an interval just depends on the length of that interval, not where it is.

Example Suppose a stream of energetic particles is absorbed by some object and the energy is slowly released from that object. Suppose at time t, the proportion of energy that remains in the object from a particle absorbed t time units ago is e^{-t}. Assuming that the object only absorbs energy at times $t = 0, 1, 2, \ldots$, let X_t be the amount of energy absorbed at time t. Let Z_n be the total energy contained in the object at time n. Then

$$Z_n = \sum_{k=0}^{\infty} e^{-k} X_{n-k} .$$

We can scale the x and y axes: let

$$Z_{\lfloor tN \rfloor} = \frac{1}{\sqrt{N}} \sum_{k=0}^{\infty} e^{-k/N} X_{\lfloor tN \rfloor - k}.$$

Then

$$Z_{\lfloor tN \rfloor} \xrightarrow{N \to \infty} Z_t .$$