LECTURE NOTES

HELEN

§ 1 | Examples

1.1 Example Motivating example: Moments of the Poisson Let $(N_t)_{t\geq 0}$ be a Poisson counting process where $N \sim \text{PPP}(\text{constant rate } 1)$ on $[0,\infty)$. The fact that $(N_t)_{t\geq 0}$ is a Poisson counting process means that $N_t \in \mathbb{Z}$ and N_t "jumps by 1 unit at rate 1", ie. $N_t = N([0,t]) + N_0$. Note that writing $N \sim \text{PPP}(\text{constant rate } 1)$ is the same as writing $N \sim \text{PPP}(\lambda)$, where λ is the Lebesgue measure.

We want to answer the question: What is $\mathbb{E}[N_t^k]$? First we define

$$\mathbb{E}^x[f(N_t)] \coloneqq \mathbb{E}[f(N_t)|N_0 = x]\,.$$

We see that

$$\begin{aligned}
\mathbb{E}^x[f(N_t)] = f(x) + \int_0^t \frac{d}{ds}\mathbb{E}^x[f(N_s)]ds &= f(x) + \int_0^t \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}^x[f(N_{s+\epsilon}) - f(N_s)]\,ds \\
&= f(x) + \int_0^t \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}^x[\mathbb{E}^s[f(N_{s+\epsilon}) - f(N_s)|N_s]]\,ds
\end{aligned}$$

This motivates the following definition.

Definition. Let

$$(Gf)(y) \coloneqq \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}^y[f(N_\epsilon) - f(y)]\,.$$

The function G is called the **generator** for N_t.

Since $N_\epsilon - N_0 \sim \text{Poisson}(\epsilon)$,

$$\begin{aligned}
Gf(x) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}^x[f(N_\epsilon) - f(x)] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[f(N_\epsilon) - f(x)|N_0 = x] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\mathbb{P}\{N_\epsilon - N_0 = 0\}(f(x) - f(x)) + \mathbb{P}\{N_\epsilon - N_0 = 1\}(f(x+1) - f(x)) \\
&\quad + \mathbb{P}\{N_\epsilon - N_0 \geq 2\}\mathbb{E}[f(N_\epsilon) - f(x)|N_\epsilon - N_0 \geq 2]) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(e^{-\epsilon}\cdot\epsilon\cdot(f(x+1) - f(x)) + \sum_{n\geq 2}\frac{e^{-\epsilon}\epsilon^n}{n!}\mathbb{E}[f(N_\epsilon) - f(x)|N_\epsilon - N_0 \geq 2]\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(e^{-\epsilon}\cdot\epsilon\cdot(f(x+1) - f(x)) + \mathcal{O}(\epsilon^2)\right) \\
&= f(x+1) - f(x)
\end{aligned}$$

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So $Gf(x) = f(x+1) - f(x)$ is the generator for the constant rate 1 Poisson process.

By the Markov property,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}^x[\mathbb{E}^s[f(N_{s+\epsilon}) - f(N_s) | N_s]] = \mathbb{E}^x[Gf(N_s)] .$$

It follows from this and (1.1) that

$$\mathbb{E}^x[f(N_t)] = f(x) + \int_0^t \mathbb{E}^x[f(N_s + 1) - f(N_s)]\, ds$$

Next we need the following notation. Define $(x)_m := x(x-1)\cdots(x-m+1)$ and observe that

$$\begin{aligned}(x+1)_m - (x)_m &= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+1) \\ &= ((x+1) - (x-m+1))(x)_{m-1} \\ &= m(x)_{m-1} .\end{aligned}$$

Now by (1.1),

$$\mathbb{E}^0[(N_t)_m] = 0 + \int_0^t \mathbb{E}^0[(N_s+1)_m - (N_s)_m]\, ds = 0 + \int_0^t m\mathbb{E}^0[(N_s)_{m-1}]\, ds$$

and $\mathbb{E}^0[(N_t)_0] = 1$.

Let

$$g_m(t) := \int_0^t m\mathbb{E}^0[(N_s)_{m-1}]\, ds$$

and let $g_0(t) = 1$. Then $g_m(t) = \int_0^t m g_{m-1}(s)\, ds$, so

$$g_1(t) = \mathbb{E}^0[N_t] = \int_0^t 1\, ds = t$$

$$g_2(t) = \mathbb{E}^0[N_t(N_t - 1)] = \int_0^t 2s\, ds = t^2$$

and by induction,

$$\mathbb{E}^0[(N_t)_m] = g_m(t) = t^m$$

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1.2 Example What is the generator for Brownian motion? Let $(B_t)_{t\geq 0}$ be a Brownian motion, ie. $B_{t+s} - B_t \sim \mathrm{Normal}(0, s)$. Then

$$\begin{aligned}\mathbb{E}^x[f(B_\epsilon)] &\approx \mathbb{E}^x[f(x) + f'(x)(B_\epsilon - x) + \frac{1}{2} f''(x)(B_\epsilon - x)^2 + \cdots] \\ &= f(x) + f'(x)\cdot 0 + \frac{1}{2} f''(x) \cdot \epsilon + \mathcal{O}(\epsilon^{3/2})\end{aligned}$$

So

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}^x[f(B_\epsilon) - f(x)] = \frac{1}{2} f''(x)$$

and thus $Gf(x) = \dfrac{1}{2} f''(x)$ is the generator for standard Brownian motion. •

1.3 Example How can we make N_t into Brownian motion? We know that:

(1) By the Central Limit Theorem, adding up independent noise, centering and scaling gets you the Gaussian distribution.

(2) N_t is the number of points in a PPP on $[0,t]$, and so is a sum of a bunch of independent things.
(3) Brownian motion started at zero has $\mathbb{E}[B_t]=0$, $\mathbb{E}[B_t^2]=t$.

N_t does not have enough noise in each interval to be Brownian. So consider N_{Mt}. Since $\mathbb{E}[N_{Mt}] = Mt$, we can subtract Mt, and then divide by $\sqrt{M}$ to get the correct variance. Thus we define

$$X_t^{(M)} = \frac{N_{Mt} - Mt}{\sqrt{M}}$$

Let G_M denote the generator of $X_t^{(M)}$. Then

$$G_M f(x) = M\left(f\left(x + \frac{1}{\sqrt{M}}\right) - f(x)\right) - \frac{1}{\sqrt{M}} f'(x)\,.$$

This is discussed in more detail on Day 15. As $M \to \infty$, $G_M f(x) \to \frac{1}{2} f''(x)$. Therefore,

$$(X^{(M)})_{t\geq 0} \xrightarrow[M\to\infty]{d} (B_t)_{t\geq 0}$$

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