LECTURE NOTES

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More rain: a hailstorm falls on our patio for 1 hour. The rate of hailstones having mass x grams is $\dfrac{1}{x}e^{-x}$ stones per hour (in total on the patio). Assume the hail is pure ice (water has density 1 gram per cubic cm).

(1) How many hailstones have mass greater than 0.01 grams fall?
(2) How many hailstones total fall?
(3) Pick a random stone with mass greater than 1 gram. What is the chance it weighs more than 2 grams?
(4) What is the total weight of all hails?
(5) Line up the hailstones side-by-side. What is the total length of hail?

1). Let $N([a,b]) = \#\{\text{hailstones with mass } a \leq x \leq b$. Then $N \sim PPP(\mu)$ on $(0,\infty)$ with $d\mu(x) = \dfrac{1}{x}e^{-x}dx$. Thus

$$N([0.01,\infty)) \sim \text{Pois}\left(\int_{0.01}^{\infty} \frac{1}{x}e^{-x}dx\right).$$

Define $E(x) = \int_x^\infty \frac{1}{y}e^{-y}dy$. Then $\mathbb{E}[N([0.01,\infty))] = E(0.01) = 4.03793$ (from an integral table) and $\text{var}[N([0.01,\infty))] = E(0.01) = 4.03793$.

2). How many hailstones total fall? $\mathbb{E}[N([\epsilon,\infty))] = E(\epsilon) \xrightarrow{\epsilon\searrow 0} \infty$ so $\mathbb{P}\{N(0,\infty)) = \infty\} = 1$. Picture: with the patio as the horizontal axis and x as the vertical axis, there's high density of low x values along the patio.

Property: *(conditional uniformity)* Let $N \sim PPP(\mu)$ on X and $A \subset X$ with $\mu(A) < \infty$. Conditioned on $N(A)$, the points of N that fall in A are i.i.d. with distribution proportional to μ; i.e. if $B \subset A$ then

$$\mathbb{P}\{N(B) = k | N(A) = n\} = \binom{n}{k}\left(\frac{\mu(B)}{\mu(A)}\right)^k \left(1 - \frac{\mu(B)}{\mu(A)}\right)^{n-k}.$$

3). Let $B = [2,\infty)$ and $A = [1,\infty)$. The mass of stones with mass greater than 1 gram has probability density $\dfrac{\frac{1}{x}e^{-x}}{E(1)}$ for $x \geq 1$. So

$$\mathbb{P}\{\text{stone} > 2g | \text{stone} > 1g\} = \frac{E(2)}{E(1)} = 0.2228992.$$

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4). Let $W = \int_0^\infty x\, dN(x) = \sum_i x_i$ = total wieght of all hailstones. We find the mean and variance of W.

$$\mathbb{E}[W] = \int_0^\infty x \frac{1}{x} e^{-x}\, dx = 1\,\text{gram}$$

since $\mathbb{E}[\int f\, dN] = \int f\, d\mu$.

Lemma: $\text{var}[\int f(x)\, dN(x)] = \int f(x)^2\, d\mu(x)$.

Proof: Let $C_j^{(\epsilon)}$ be a partition of X with all $C_j^{(\epsilon)}$ be "ϵ-small", and let $z_j^{(\epsilon)} \in C_j^{(\epsilon)}$ be the center of each $C_j^{(\epsilon)}$. Then

$$\int f(x)\, dN(x) = \sum_i f(x_i) \approx \sum_j N(C_j^{(\epsilon)}) f(z_j^{(\epsilon)})$$

so

$$\begin{aligned}
\text{var}\left[\int f(x)\, dN(x)\right] &\approx \text{var}\left[\sum_j N(C_j^{(\epsilon)}) f(z_j^{(\epsilon)})\right] \\
&= \sum \text{var}[N(C_j^{(\epsilon)}) f(z_j^{(\epsilon)})] \\
&= \sum f(z_j^{(\epsilon)})^2 \mu(C_j^{(\epsilon)}) \\
&\xrightarrow{\epsilon \searrow 0} \int f(x)^2 d\mu(x) \qquad \blacksquare
\end{aligned}$$

Thus $\text{var}[W] = \int_0^\infty x^2 \frac{1}{x} e^{-x}\, dx = 1$.

5). Since the density of water is 1 g/cm^3, $\mathbb{E}[\text{length}] = \int_0^\infty x^{1/3} \frac{1}{x} e^{-x}\, dx = \Gamma(\frac{1}{3})$.

Fact: if X and Y are random variables, $\mathbb{E}[e^{i\alpha X}] = \mathbb{E}[e^{i\alpha Y}]\ \forall \alpha \in \mathbb{R}$ if and only if $X \stackrel{d}{=} Y$ (i.e. $\mathbb{P}\{X \in A\} = \mathbb{P}\{Y \in A\}$ for all A, or $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ for all f).

The *characteristic function* for X is $\varphi_X(\alpha) = \mathbb{E}[e^{i\alpha X}]$; in other words, φ_X is the Fourier transform of the density function f_X of X.
Characteristic functions are convenient for calculating moments, but *not* probabilities.

Lemma: Let $\psi(\alpha) = \log \varphi_X(\alpha)$ be the *cumulant generating function.* Then

$$\frac{d}{d\alpha}\psi(0) = i\,\mathbb{E}[X]$$
$$\frac{d^2}{d\alpha^2}\psi(0) = -\text{var}[X]$$

Proof: (1)

$$\begin{aligned}
\frac{d}{d\alpha}\mathbb{E}[e^{i\alpha X}] &= \mathbb{E}\left[\frac{d}{d\alpha} e^{i\alpha X}\right] \\
&= \mathbb{E}[iXe^{i\alpha X}]
\end{aligned}$$

so $\frac{d}{d\alpha}\psi(0) = \dfrac{\mathbb{E}[iXe^{i0X}]}{\mathbb{E}[e^{i0X}]} = i\mathbb{E}[X]$.

(2) Similarly:

$$\frac{d^2}{d\alpha^2}\psi(\alpha) = \frac{\mathbb{E}[-X^2e^{i\alpha X}] - \mathbb{E}[iXe^{i\alpha X}]^2}{\mathbb{E}[ie^{i\alpha X}]^2}$$
$$\frac{d^2}{d\alpha^2}\psi(0) = \mathbb{E}[X^2] = \mathbb{E}[X]^2 = \mathrm{var}[X]$$

as desired. ■

Characteristic functions: Let $N \sim PPP$ on X with intensity μ and let $f : X \to \mathbb{R}$. Then

$$\mathbb{E}\left[\exp\left(i\alpha \int f(x)\, dN(x)\right)\right] = \exp\left(\int_X (e^{i\alpha f(x)} - 1)\mu(dx)\right).$$

Lemma: If $Z \sim \mathrm{Pois}(\gamma)$, then

$$\mathbb{E}[e^{i\alpha Z}] = \sum_{n\geq 0} e^{i\alpha n}e^{-\gamma}\frac{\gamma^n}{n!} = e^{-\gamma}\sum_{n\geq 0}(\gamma e^{i\alpha})^n/n! = \exp(\gamma(e^{i\alpha} - 1))$$

Proof (theorem): Let f be piecewise constant $f(x) = f_i$ for $x \in A_i$ with $\sqcup_i A_i = X$. Then $\int f(x)\, dN(x) = \sum_i N(A_i)f_i$ (and the $N(A_i)$ are all independent) so

$$\begin{aligned}
\mathbb{E}[e^{i\alpha \int f\, dN}] &= \mathbb{E}\left[\prod_j e^{i\alpha N(A_j)f_j}\right] \\
&= \prod_j \mathbb{E}[e^{i\alpha N(A_j)f_j}] \\
&= \prod_j \exp(\mu(A_j)(e^{i\alpha f_j} - 1)) \\
&= \exp\left(\int_X (e^{i\alpha f(x)} - 1)\mu(dx)\right) \qquad ■
\end{aligned}$$

Corollary: $\mathbb{E}[\int f\, dN] = \int f\, d\mu$ and $\mathrm{var}[\int f\, dN] = \int f^2\, d\mu$.

Ex: **Cauchy process:** Let $N \sim PPP$ on $[0,\infty)\times(\mathbb{R}\backslash\{0\})$ with mean measure $\mu(dt, dx) = \dfrac{dt\, dx}{|x|^2}$. Picture: with t as the horizontal and x as the vertical axes, higher density of points near the t-axis.

Let $C_t = \int_0^t \int_{\mathbb{R}} x\, dN(s,x) =$ sum of x-coordinates of points in $[0,t] \times \mathbb{R}$.
Note:

$$\begin{aligned}
\mathbb{P}\{\text{no jumps in } [t, t+\epsilon) &= \mathbb{P}\{C_s = C_t : s \in [t, t+\epsilon)\} \\
&= \mathbb{P}\{N([t, t+\epsilon) \times \mathbb{R}) = 0\} \\
&= \exp\left(-\epsilon \int \frac{1}{|x|^2} dx\right) = 0
\end{aligned}$$

Also:

$$\mathbb{P}\{\text{no jumps bigger than } \delta \text{ in } [t, t+\epsilon)\} = \exp\left(-2\epsilon \int_\delta^\infty \frac{dx}{x^2}\right) = e^{-2\epsilon/\delta}.$$

What is the distribution of C_t?

$$
\begin{aligned}
\mathbb{E}[e^{i\alpha C_t}] &= \exp\left(\int_0^t \int_{-\infty}^{\infty} (e^{i\alpha X} - 1)\frac{1}{|x|^2} dx\, dt\right) \\
&= \exp(-t|\alpha|) \\
&= \int_{-\infty}^{\infty} e^{iz\alpha} \frac{dz}{\pi t(1 + (z/t)^2)}
\end{aligned}
$$

i.e. $C_t \sim \text{Cauchy}(t) = \text{probability density} \dfrac{1}{\pi t(1 + (z/t)^2)}$.

An interesting property: $C_n = C_1 + (C_2 - C_1) + ... + (C_n - C_{n-1}) = n$ i.i.d. $\sim C_1$. Thus $\frac{1}{n}C_n \overset{d}{=} C_1$ and $\text{var}[\frac{1}{n}C_n] = \frac{1}{n}\text{var}[C_1]$.