LECTURE NOTES

HELEN

§ 1 | Theory

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov process on a locally compact, separable metric space S, and define $C_0 \coloneqq C_0(S)$ to be the set of all continuous functions $f\colon S \to \mathbb{R}$ vanishing at infinity, ie, given $\epsilon > 0$, there is a compact $K \subset S$ such that $f(x) < \epsilon$ for all $x \in K$. Note that C_0 is a Banach space with the uniform norm: $\|f\|_\infty \coloneqq \sup_{x\in S}|f(x)|$. Assume $\mathbb{P}(\{X_t \in S\}) = 1$ for all t.

Definition. Define the transition semigroup $(P_t)_{t\geq 0}$ by $(P_t f)(x) \coloneqq \mathbb{E}^x[f(X_t)]$ for $f \in C_0$.

Note: The assumption

$$(X_t|X_0 = x) \xrightarrow[x\to y]{d} (X_t|X_0 = y)$$

implies $P_t\colon C_0 \to C_0$.

We have the following properties:

(1) $P_0 = I$ since $P_0 f(x) = \mathbb{E}^x[f(X_0)] = f(x)$.
(2) $P_s P_t = P_{s+t}$ since

$$P_s P_t f(x) = \mathbb{E}^x[P_t f(x_s)] = \mathbb{E}^x[\mathbb{E}^{x_s}[f(x_t)]] = \mathbb{E}^x[f(x_{t+s})] = P_{s+t} f(x)\,.$$

(3) If we assume that

$$(X_t|X_0 = x) \xrightarrow[t\to 0]{d} x$$

for each x then $P_t \to \mathrm{id}$ as $t \to 0$.

Definition. The generator of $(X_t)_{t\geq 0}$ and/or $(P_t)_{t\geq 0}$ is

$$G = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(P_\epsilon - \mathrm{id})$$

and

$$P_t = e^{tG} = \sum_{n\geq 0} \frac{t^n}{n!} G^n$$

if the above statements make sense.

Note: If $(X_t)_{t\geq 0}$ satisfies () and (2) it is said to be **Feller**. The generator of a Feller process uniquely determines its distribution. This is clear from () when it makes sense. To prove this claim in general, use resolvents.

The generator has the following properties:

(1) $G1 = 0$ since $(P_t - \mathrm{id})1(x) = \mathbb{E}^x[1 - 1] = 0$

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(2) π is a stationary measure for $(X_t)_{t\geq 0}$ if and only if

$$\int Gf(x)\,d\pi(x) = 0$$

for all f.

Next we consider some examples.

1.1 Example Let $Gf(x) = f'(x)$. Then

$$P_t f(x) = \sum_{n\geq 0} \frac{t^n}{n!} f^{(n)}(x) = \sum_{n\geq 0} \frac{t^n}{n!} f^{(n)}(x+t-t) = f(x+t)\,.$$

So $X_t = X_0 + t$. Therefore, (d/dx) corresponds to "deterministic flow at rate 1." •

1.2 Example Let $Gf(x) = \dfrac{1}{2} f''(x)$. Recall that this is the generator for Brownian motion. Then

$$P_t f(x) = \sum_{n\geq 0} \frac{2^{-n} t^n}{n!} f^{(2n)}(x)\,.$$

Denote by $\widehat{f}(\xi)$ the Fourier transform of f. Then

$$\widehat{P_t f}(x) = \sum_{n\geq 0} \frac{2^{-n} t^n}{n!} (-\xi^2)^n \hat{f}(\xi) = e^{-\frac{t}{2}\xi^2} \widehat{f}(\xi)\,.$$

Because $e^{-(t/2)\xi^2}$ is the Fourier transform of the Gaussian density with variance t, and the Fourier transform takes convolution to multiplication,

$$P_t f(x) = \sum_{n\geq 0} \frac{2^{-n} t^n}{n!} f^{(2n)}(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\dfrac{(y-x)^2}{2t}} f(y)dy\,.$$

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It turns out that if $Gf(x) = f^{(k)}(x)$ for $k > 2$, G is not the generator of a Feller process. One reason for this is that for $k > 2$ there is no way to write a discrete approximation to $f^{(k)}(x)$ as a sum over values of f with all coefficients except that of $f(x)$ positive. More generally, we can appeal to the following theorem.

1.3 Theorem (Hille-Yosida Theorem) Let A be a linear operator on $\mathcal{D} \subset C_0$. Then A has closure that is the generator of a Feller process if and only if

(i) $\mathcal{D}$ is dense in C_0;
(ii) the range of $\lambda - A$ is dense in C_0 for some $\lambda > 0$;
(iii) (Positive Maximum Principle) if $f(x) \leq f(x_0)$ for all $x \in S$ and $f(x_0) > 0$, then $Af(x_0) \leq 0$.

Note: If all three conditions in the above theorem hold, then $(\lambda - A)^{-1}$ exists for all $\lambda > 0$. To see this, suppose that $\lambda - A$ is not invertible on C_0 for some $\lambda > 0$. Then there exists $f \in C_0$ such that $Af = \lambda f$. Then $\mathbb{E}^x[f(X_t)] = e^{t\lambda} f(x)$. But $\mathbb{E}^x[f(X_t)]$ is bounded since $f \in C_0$, so we have reached a contradiction.

Note: If $A = \lim_{t \to 0} \frac{1}{t}(P_t - \mathrm{id})$, then third condition automatically holds. For if

$$Af(x_0) = \lim_{t \to 0} \frac{1}{t}(P_t - \mathrm{id})f(x_0) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}^{x_0}[f(x_t) - f(x_0)] \,.$$

Since $f(x_t) - f(x_0) \leq 0$ by assumption, $\mathbb{E}^{x_0}[f(x_t) - f(x_0)] \leq 0$, and $Af(x_0) \leq 0$.