1 Lévy Processes

Definition 1.1. A Lévy process with drift rate α, diffusion rate σ, and jump kernel ν is a stochastic process with distribution:

$X_0 = 0,$

$(*)$ $X_t = \alpha t + B_t + \int_0^t \int_{-\infty}^{\infty} xN(dx, dt)$

where B_t is Brownian motion and $N \sim PPP$ on $[0,\infty) \times \mathbb{R}$ with intensity measure $dt\, \nu(dx)$.

Fact 1.2. (Lévy-Khinchine) Any Markov process on $\mathbb{R}$ with $X_0 = 0$ and stationary independent increments is Lévy.

i.e. a markov process with

(a) distribution of increment $X_{t+h} - X_t$ only depends on h

(b) If $a < b \leq c < d$, $(X_d - X_c)$ independent of $(X_b - X_a)$ with

$$\int_{-\infty}^{\infty} \min(|x|, 1)\nu(dx) < \infty$$

$\big($ and $\nu([1,\infty)) + \nu((-\infty, 1]) < \infty$ $\big)$.

Note: $(*)$ makes sense if jumps are absolutely summable. i.e. let $N = \sum_i \delta_{(t_i,x_i)}$. Then the "jump component" of X is

$$J_t = \int_0^t \int_{-\infty}^{\infty} xN(dt, dx) = \sum_{i, t_i \leq t} x_i \qquad \text{where} \quad \sum_i |x_i| < \infty$$

Is $\sum_{|x_i|<1,\ t_i<t} |x_i| < \infty$?

$$\mathbb{E}\left[\sum_{|x_i|<1,\ t_i<t} |x_i|\right] = \mathbb{E}\left[\int_0^t \int_{-1}^{1} |x|N(dt, dx)\right]$$
$$= t \cdot \int_{-1}^{1} |x| \nu\, dx < \infty.$$

OR: If you have faster accumulation of jumps near zero,

$(**)$ $$X_t = \alpha t + \sigma B_t + \int_0^t \int_{-1}^{1} x\big[N(ds, dx) - dx\, \nu(dx)\big] + \int_0^t \int_{|x|>1} xN(ds, dx)$$

Need only $\displaystyle\int_{-\infty}^{\infty} \min(|x|^2, 1)\, \nu(dx) < \infty$

Properties:

(1) Stationary independent increments.

(2) X has generators

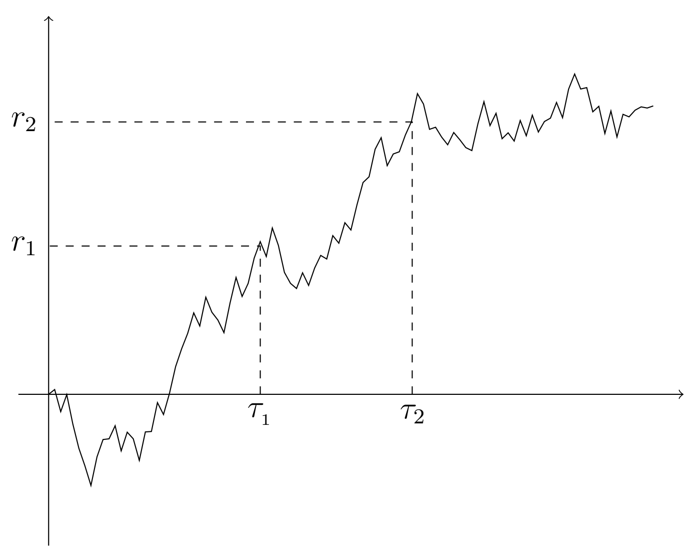
$$Gf(x) = xf'(x) + \frac{\sigma^2}{2} f''(x) + \int_{-\infty}^{\infty} \big[f(x+y) - f(x)\big]\nu(dy)$$

(3) Lévy - Keinchine fomula:

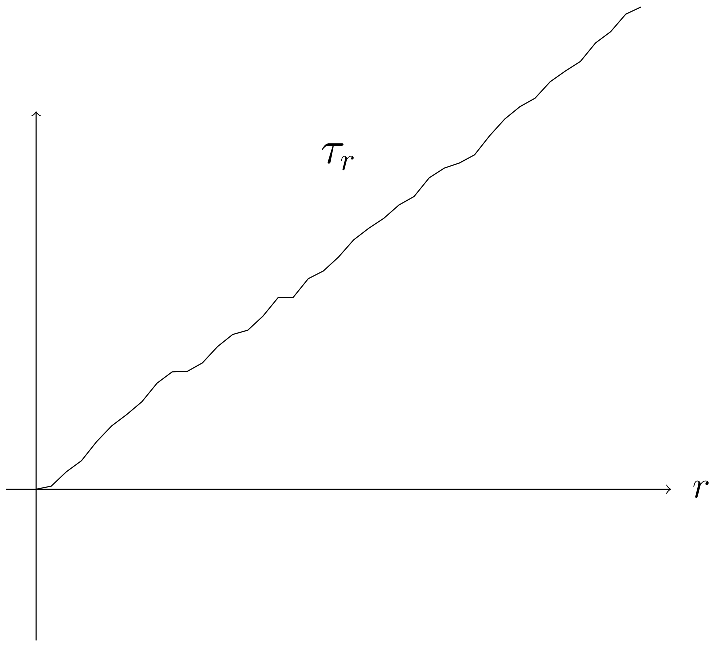
$$\begin{aligned} \mathbb{E}\big[e^{iuX_t}\big] &= e^{t\Psi(u)} \\ \Psi(u) &= e^{i\alpha u} - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty} \big[e^{iux} - 1\big]\nu(dx) \\ e^{t\Psi(u)} &= \int_{-\infty}^{\infty} e^{iux}\, \mathbb{P}^0\{X_t = x\}\, dx \end{aligned}$$

Example 1.3. "Stable Subordinators"

$(B_t)_{t\geq 0}$ $\qquad \tau_r = \inf\{t \geq 0 : B_t \geq r\}$



Claim: $(\tau_r)_{r\geq 0}$ is Lévy with $\alpha = 0$, $\sigma = 0$, and $N(dx) = x^{-3/2}dx$



"Subordinators" $\sim$ Lévy process $X_i \nearrow \beta$, $t \nearrow \infty$ with independent stationary increments.

2 First Hitting Times and Brownian Motion

Let $(B_t)_{t\geq 0}$ be Brownian motion,

$$\tau_x = \inf\big\{t \geq 0 : B_t \geq x\big\}, \quad \text{and} \quad M_t = \sup\big\{B_s : 0 \leq s \leq t\big\}.$$

Then $(\tau_x)_{x\geq 0}$ is a Markov process with stationary independent increments, i.e. a non decreasing Lévy process, aka subordinator.

Why is this Markov?
Define $\tilde{B}_t$ by $\big(\tilde{B}_t = B_{\tau_x+t} - x\big)_{t\geq 0} := \big(B_t\big)_{t\geq 0}$. Then

$$\begin{aligned}\tau_{x+y} - \tau_x &= \inf\big\{t \geq \tau_x : B_t = x + y\big\} \\ &= \inf\big\{t \geq 0 : \tilde{B}_t = y\big\} \\ &= \tau_y,\end{aligned}$$

which implies τ has stationary increments.

2.1 Properties of This Process

Recall: Brownian Scaling: $\left(\overline{B}_s = \frac{1}{c} \cdot B_{c^2 s}\right)_{s \geq 0} = \left(B_s\right)_{s \geq 0}$. Then

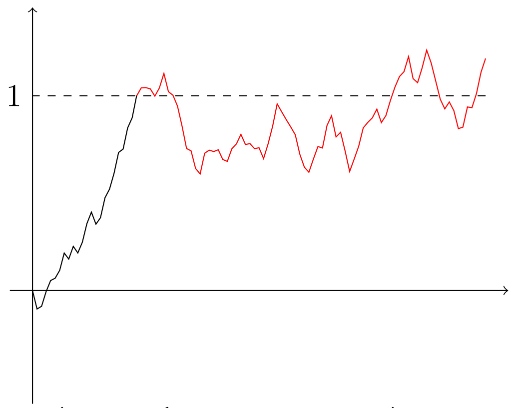
$$\begin{aligned}\tau_x &= \inf \left\{t \geq 0 : B_t \geq x\right\} \\ &= \inf \left\{t \geq 0 : \ \frac{1}{c} \cdot B_{c^2 t} \geq x\right\} \\ &= \frac{1}{c^2} \cdot \inf \left\{t \geq 0 : \ B_t \geq cx\right\} \\ &= \frac{1}{c^2} \cdot \tau_{cx}.\end{aligned}$$

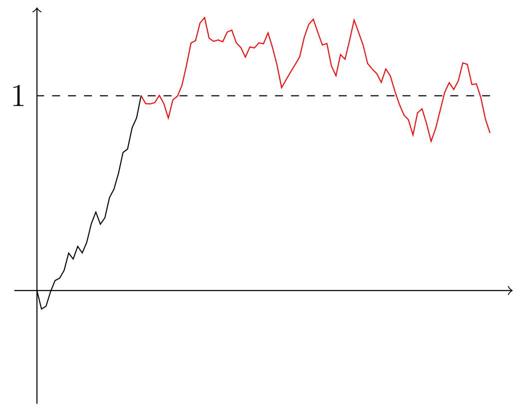
So $\tau_x = x^2 \tau_1$.

Lemma 2.1. Reflection Principle:

$$\mathbb{P}\{B_t < 1, M_t \geq 1\} = \mathbb{P}\{B_t \geq 1\}.$$

Proof. There is a bijection obtained by reflecting the $(t \geq \tau_1)$-portion of the process across the line $x = 1$:





So $\left(W_s = \begin{cases} B_s & s \leq \tau_1 \\ 2 - B_s & s \geq \tau_1 \end{cases}\right) = \left(B_s\right)_{s \geq 0}$.

□

$$
\begin{aligned}
\mathbb{P}\{\tau_1 < t\} &= 2\mathbb{P}\{B_t > 1\} \\
&= \frac{2}{\sqrt{2\pi t}} \int_1^\infty \exp\left(\frac{-x^2}{2t}\right) dx \\
&= 2\mathbb{P}\left\{B_1 > \frac{1}{\sqrt{t}}\right\} \\
&= \frac{2}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{t}}}^\infty \exp\left(\frac{-x^2}{2}\right) dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^t s^{-3/2} \cdot e^{-1/(2s)} ds \qquad \left(\text{letting } s = \tfrac{1}{x^2}\right)
\end{aligned}
$$

$$
\boxed{\mathbb{P}\{\tau_1 \in ds\} = \sqrt{\frac{2}{\pi}} \cdot s^{-3/2} \cdot e^{-1/(2s)} ds} \leftarrow \text{density}
$$

Now

$$
\begin{aligned}
\mathbb{P}\{\tau_x < t\} &= \mathbb{P}\{x^2 \tau_1 < t\} \\
&= \mathbb{P}\left\{\tau_1 < \frac{t}{x^2}\right\}.
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{P}\{\tau_x \in dt\} &= \frac{d}{dt}\left(\quad , \quad \right) \\
&= \frac{1}{x^2} \cdot \frac{d}{dt}\, \mathbb{P}\left\{\tau_1 < \frac{t}{x^2}\right\} \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{x^2} \cdot \left(\frac{t}{x^2}\right)^{-\frac{3}{2}} \cdot \exp\left(\frac{-x^2}{2t}\right).
\end{aligned}
$$

Note: must be a <u>jump</u> Lévy process (can't have drift component nor Brownian component).