LECTURE NOTES

ISAAC

0.1 Theorem A [Markov] Levy process is (strictly) *stable* if

$$(X_t)_{t\geq 0} \overset{\mathrm{d}}{=} (\tfrac{X_{r_t}}{r^{1/p}})_{t\geq 0},$$

which happens iff [it is Levy and]

(1) $p = 2$ and X is BM.
(2) $0 < p < 2$ and X is pure jump † with $\nu(dx) \propto x^{-1-p}dx$

Proof. Let $X \sim \mathit{Levy}(b, \sigma, \nu)$ have drift, diffusion, and jump kernel coefficients. (i.e. with generator

$$G(f) = b\frac{d}{dx}f + \frac{\sigma^2}{2}\frac{d^2}{dx^2}f + \int \nu(dy)(f(x+y) - f(x)).)$$

Note, we will write $s = r^p$ in the following for notational simplicity. s is the coefficient 'r' from the theorem statement (i.e., $X_{r_t}/r^{1/p} = X_{s_t}/s^{1/p}$). Then, if we examine $X(r^p t)$, the generator associated with the drift is

$$b\tfrac{d}{dx} \to X_t = bt,$$

so $X_{r^p t} = br^p t$. Similarly, $\frac{\sigma^2}{2}\frac{d^2}{dx^2} \to X_t = \sigma B_t \sim \mathbb{N}(0, \sigma^2 t)$. So, $X_{r^p t} \sim \mathbb{N}(0, \sigma^2 r^p t) \overset{\mathrm{d}}{=} (\sigma r^{p/2})B_t$. And for the jumps, similar logic shows that $X(r^p t) \sim \mathit{Levy}(r^p b, r^{p/2}\sigma, r^p\nu)$. This is equal in distribution to

$$rX(t) \sim \mathit{Levy}(rb, r\sigma, \tilde{\nu})$$

where $\tilde{\nu}([a, b]) = \nu([\frac{a}{r}, \frac{b}{r}])$. If $(X(r^p t))_{t\geq 0} \overset{\mathrm{d}}{=} (rX(t))_{t\geq 0}$ then

(1) either $b = 0$ or $p = 1$
(2) either $\sigma = 0$ or $p = 2$
(3) either $\nu = 0$ or $r^p\nu = \tilde{\nu}$.

If $r^p\nu = \tilde{\nu}$ then $\nu([x, \infty)) = r^{-p}\nu([\frac{x}{r}, \infty))$ so setting $r = x$ implies that

$$\nu([x, \infty)) = x^{-p}\nu([1, \infty)) \implies \nu([x, \infty)) \propto x^{-p}.$$

This also holds for $\nu((-\infty, -x])$, and so if ν is absolutely continuous wrt Lebesgue measure then

$$\nu(dx) = \begin{cases} C_+ x^{-p-1}dx & x \geq 0 \\ C_- x^{-p-1}dx & x < 0 \end{cases}$$

for some $C_\pm \geq 0$. (this is what is meant when writing *pure jump* with the corresponding measure in the theorem statement. ∎

DEFINITION (Stable limits). Let $\{Y_i\}_i$ be a family of iid (nondegenerate) Random Variables in $\mathbb{R}$ and let $S_n = Y_1 + Y_2 + \cdots + Y_n$. Suppose $S_n/n^\alpha \xrightarrow{d} X$ converges in distribution for some

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distribution X. Then X is (strictly) stable, with index $p = 1/\alpha$ i.e., having the distribution of $X(1)$ from the previous theorem.

Proof. $\frac{S_{nk}}{(nk)^\alpha} \xrightarrow{d} X$ but also converges to $\frac{X_1+\cdots+X_k}{k^\alpha}$ where X_i are iid and $\sim X$, all of this because $S_{nk} \overset{\mathrm{d}}{=} S_n^{(1)} + \cdots + S_n^{(k)}$ are iid copies of S_n. That is, we see that the distribution X can be rescaled into a k-fold convolution of k copies of itself: $X \overset{\mathrm{d}}{=} \frac{X_1+\cdots+X_k}{k^\alpha}$.

Now let Z be the Poissonization of S, i.e. $Z_t = S_{N_t}$ where $N_t = PP(1)$ (poisson process) on $[0, \infty)$. Then, $Z \sim Levy(0, 0, \nu)$ where ν is the probability distribution of Y. and $S_n/n^\alpha \approx Z_n/n^\alpha$ since $N_n = n + \mathcal{O}(\sqrt{n})$, and $Z_n/n^\alpha \sim Levy(0, 0, \nu^{(n)})$ where $\nu^{(n)}([x, \infty)) = \nu([n^\alpha x, \infty))$

Let $Z_t^{(n)} = \frac{1}{n^\alpha} Z_{nt}$ The assumption $X_n = Y_1 + \cdots + Y_n$ implies that $Z_1^{(n)}$ converges in distribution to X, and so by the fact that $X \overset{\mathrm{d}}{=} \frac{X_1+\cdots+X_k}{k^\alpha}$, then X_t as a process is stable if the condition holds for time $t \neq 1$. but we could do the same argument to show that in fact the quoted assumption above implies $(Z_t^{(n)})_{t\geq 0} \xrightarrow{d} (U_t)_{t \geq 0}$ where U is a stable process whose increments are determined by X: $U_1 \overset{\mathrm{d}}{=} X$.

□

Note: $\tau_1 = \inf\{t \geq 0 : B_t = 1\} \sim Stable(\frac{1}{2})$ on $[0, \infty)$. So, $\tau_n \overset{\mathrm{d}}{=} \tau_1^{(n)} + \cdots + \tau_1^{(n)}$ decomposes into a sum of n iid copies of τ_1. Then

$$\tau_n/n^2 \overset{\mathrm{d}}{=} \frac{\tau_1^{(n)} + \cdots + \tau_1^{(n)}}{n^2} \overset{\mathrm{d}}{=} \tau_1,$$

(but also converges in distribution to τ_1 ($\xrightarrow{d} \tau_1$).)