TWIST REPRESENTATION ZETA FUNCTIONS: COMPUTATIONS AND THE SCALING PHENOMENON

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Abstract. We study the twist representation zeta function of finitely generated, torsion-free nilpotent groups.
We provide the method required to compute the twist zeta function, by means of the Kirillov orbit method and $\mathfrak{p}$-adic integration.
We provide a fully worked example of nilpotency class 3, for both methods.
We attempt to extend known results of the k-fold central products effect on commutator matrices, by specialising to the 2-fold central product.
By counterexample, we prove that in general, in contrast to the $\mathcal{B}$ matrix, there is no $\mathbb{Z}$-Lie lattice which gives a scaled $\mathcal{A}$ matrix.
Again by counterexample we prove that for nilpotency class greater than 2 there need not be a $\mathbb{Z}$-Lie lattice which has a scaled $\mathcal{R}$ matrix.
We then show that beyond nilpotency class 2, the k-fold central product does not induce scaling in the twist zeta function.

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Contents

1. Introduction and statement of main results

We begin with an overview of key definitions and ideas that will appear frequently throughout this paper. We also provide some context and motivation within the broader scope of research.

1.1. Background and motivation.

1.1.1. *Representations and their growth.* Let G be a group, and V a vector space over the field k. A representation of G on V is a group homomorphism, $\rho : G \to \mathrm{GL}(V)$. We say a representation is irreducible if there are no non-trivial invariant subspaces. A natural question which arises in the study of representations of groups is how many inequivalent representations exist of a group, for a given dimension. Initially, we approach this question through Maschke's theorem, which is a classical and powerful result in representation theory. Maschke's theorem states that if G is a finite group and the characteristic of k does not divide the order of the group G, then every representation of G can be expressed as the direct sum of irreducible representations. Another classical result in representation theory, Schur's orthogonality relations, tell us that if G is finite, then it has only finitely many irreducible representations. As a result, when G is finite it suffices to study the finite number of irreducible representations. On the other hand, if G is not a finite group, it is much more difficult to classify representations. As such, in this paper we focus our attention to a specific family of groups with nice properties, which will allow us to enumerate their representations.

Definition 1.1. Let G be a group. For $n \in \mathbb{N}$, define

$$r_n(G) \;=\; \left|\left\{ \begin{array}{l} \text{isomorphism classes of } n\text{-dimensional, irreducible} \\ \text{complex representations of } G \end{array} \right\}\right|.$$

A group is said to be *rigid* if $r_n(G)$ is finite for all $n \in \mathbb{N}$.

1.1.2. *Twist equivalence.* Two irreducible representations, ρ and σ, of the group G, are said to be twist equivalent if there exists a 1-dimensional representation, χ, of G such that $\rho \cong \chi \otimes \sigma$. If G is a topological group, we also require that χ be a continuous representation of G. It is straightfoward to check that twist equivalence is an equivalence relation on the set of irreducible representations of G. We call the equivalence classes under this equivalence relation twist-isoclasses.

Definition 1.2. Let G be a group. For $n \in \mathbb{N}$, define

$$\tilde{r}_n(G) \;=\; \left|\left\{ \begin{array}{c} \text{twist iso-classes of } n\text{-dimensional, irreducible} \\ \text{complex representations of } G \end{array} \right\}\right|.$$

A group is said to be *twist rigid* if $\tilde{r}_n(G)$ is finite for all $n \in \mathbb{N}$.

1.1.3. *$\mathcal{T}$-groups.* Our main groups of interest throughout this paper will be those that are finitely generated, torsion-free and nilpotent, called $\mathcal{T}$-groups. $\mathcal{T}$-groups are not rigid as they have infinitely many representations of dimension one. However, it has been proven that given a $\mathcal{T}$-group, G, and $n \in \mathbb{N}$, there exists a finite quotient $G(n)$ of G such that every n-dimensional representation of G is twist-equivalent to one that factors through $G(n)$, [14, Theorem 6.6]. This means that $\mathcal{T}$-groups are twist rigid which will allow us to study their representation growth, up to twist iso-classes.

1.1.4. *$\mathcal{O}$-Lie lattices and group schemes.* Let $\mathcal{O}$ be the ring of integers for the number field K.

Definition 1.3. An $\mathcal{O}$-Lie lattice is a finitely generated, free $\mathcal{O}$-module, Λ, together with a bilinear map

$$[\cdot,\cdot] : \Lambda \times \Lambda \to \Lambda,$$

such that the bracket is alternating, bilinear and satisfies the Jacobi identity.

Let $(\Lambda, [\cdot,\cdot])$ be a nilpotent $\mathcal{O}$-Lie lattice of $\mathcal{O}$-rank h and nilpotency class c. Let $(x_1, \ldots, x_h)$ be an arbitrary $\mathcal{O}$-basis for Λ and let $\Lambda(R) := \Lambda \otimes_{\mathcal{O}} R$. Then $(x_1 \otimes 1, \ldots, x_h \otimes 1)$ is a R-basis for $\Lambda(R)$,[12, XVI, Proposition 2.3]. We can define a group structure on $\Lambda(R)$ by means of the Hausdorff series.
Provided that $\Lambda' \subseteq c!\Lambda$, for $x, y \in \Lambda(R)$ we define,

$$x * y = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[[x, y], y] + \cdots, \qquad x^{-1} = -x, [10].$$

The group $(\Lambda(R), *)$ is nilpotent of class c and with respect to the basis defined above and the group operations are given by polynomials over $\mathcal{O}$, which are independent of R. This defines a unipotent group scheme G_Λ over $\mathcal{O}$ which is isomorphic as a scheme to affine h-space over $\mathcal{O}$, representing the group functor,

$$R \longmapsto (\Lambda(R), *).$$

The group $G_\Lambda(\mathcal{O})$ is a $\mathcal{T}$-group of nilpotency class c, [17].

1.1.5. *Zeta Functions.* The study of zeta functions originates with the Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{s} \in \mathbb{C} \text{ a complex variable.}$$

This series can be reformulated as a product over the primes,

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{for } \mathrm{Re}(s) > 1,$$

called the Euler product.

A natural generalisation that follows is the Dedekind zeta function. Let K be an algebraic number field and let $\mathcal{O}$ be the ring of integers,

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} \frac{1}{\left(N_{K/\mathbb{Q}}(I)\right)^s},$$

where I is an ideal of $\mathcal{O}$.
There exists an analogous Euler product which allows us to consider the local properties of the Dedekind zeta function,

$$\zeta_K(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} \frac{1}{1 - \left(N_{K/\mathbb{Q}}(\mathfrak{p})\right)^{-s}}, \quad \text{for } \mathrm{Re}(s) > 1,$$

where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}$.

In the context of groups, Grunewald, Segal and Smith introduced zeta functions to study subgroup growth of finitely generated groups in [6]. Building upon this, the study of zeta functions encoding representation growth was first introduced by Larsen and Lubotzky in [13].

Definition 1.4. Let G be a group and s a complex variable.
The representation zeta function of G is

$$\zeta_G^{\mathrm{irr}}(s) := \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

The twist representation zeta function of G is

$$\zeta_G(s) := \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}.$$

Note that for $\mathcal{T}$-groups, the series $\sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}$ converges on a complex right half-plane, [17, Lemma 2.1].

In the context of $\mathcal{T}$-groups derived from $\mathcal{O}$-Lie lattices we get the twist zeta function,

$$\zeta_{G(\mathcal{O})}(s) \;=\; \sum_{n=1}^{\infty} \tilde{r}_n(G(\mathcal{O}))\, n^{-s}.$$

This admits an Euler product over the primes ideals of $\mathcal{O}$,

$$\zeta_{G(\mathcal{O})}(s) \;=\; \prod_{\mathfrak{p}} \zeta_{G(\mathcal{O}_\mathfrak{p})}(s).$$

where,

$$\zeta_{G(\mathcal{O}_\mathfrak{p})}(s) \;=\; \sum_{i=0}^{\infty} \tilde{r}_{p^i}\big(G(\mathcal{O}_\mathfrak{p})\big)\, p^{-is}, \text{[17, Proposition 2.2].}$$

1.1.6. *Techniques for computation.* There are several techniques one can use to compute zeta functions, of which we shall provide two in this paper. The first method is by means of the Kirillov orbit method, developed by Kirillov in [11]. The Kirillov orbit method provides a correspondence between irreducible representations of a Lie group and its co-adjoint orbits. In [7], Howe specializes the orbit method to $\mathfrak{p}$-adic Lie groups and hence $\mathcal{T}$-groups. The second method is via an Igusa-type integral, also called an Igusa zeta function. These are $\mathfrak{p}$-adic integrals associated with matrices of linear forms.

1.2. Main results.

Theorem 1. *Fix a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and write $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ for the corresponding completion. Let q denote the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$.*
We consider the class 3 nilpotent $\mathbb{Z}$-Lie lattice,

$$\mathfrak{f}_{3,2} = \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y,\ [x_1, y] = z_1\ [x_2, y] = z_2 \rangle.$$

By means of the Poincaré series, we compute the twist representation zeta function,

$$\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}(s) = \frac{(1-q^{-s})^2}{(1-q^{1-s})(1-q^{2-s})}.$$

Theorem 2. *Given the $\mathcal{A}$ commutator matrix,*

$$\mathcal{A}(X_1, X_2, X_3, X_4) \;=\; \begin{pmatrix} -X_2 I_2 \\ X_1 I_2 \end{pmatrix} = \begin{pmatrix} -X_2 & 0 \\ 0 & -X_2 \\ X_1 & 0 \\ 0 & X_1 \end{pmatrix}.$$

There is no class 2 nilpotent $\mathbb{Z}$-Lie lattice we can construct which has this $\mathcal{A}$ commutator matrix.

Corollary A. *There is no analogous statement to Proposition 2.5 in [9] for the $\mathcal{A}$ commutator matrix.*

Theorem 3. *Let $\mathcal{R}_{\mathfrak{f}}$ be the $\mathcal{R}$ matrix for $\mathfrak{f}_{3,2}$ as defined above.*
Then given two copies of $\mathfrak{f}_{3,2}$,

$$\mathfrak{f}_1 := \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y,\ [x_1, y] = z_1, [x_2, y] = z_2 \rangle,$$

$$\mathfrak{f}_2 := \langle a_1, a_2, b, c_1, c_2 \mid [a_1, a_2] = b,\ [a_1, b] = c_1, [a_2, b] = c_2 \rangle.$$

We can identify their centres and their derived subgroups like so, $\mathfrak{g} :=$

$$\left\langle \begin{array}{c} x_1, x_2, y, z_1, z_2 \\ a_1, a_2, b, c_1, c_2 \end{array} \;\middle|\; \begin{array}{c} [x_1, x_2] = y = [a_1, a_2], \\ [x_1, y] = z_1 = [a_1, b] = c_1, [x_2, y] = z_2 = [a_2, b] = c_2 \end{array} \right\rangle.$$

This gives us the $\mathcal{R}$ commutator matrix,

$$\mathcal{R}_{\mathfrak{g}}(Y_1, Y_2, Y_3) \;=\; \mathcal{R}_{\mathfrak{f}}(Y_1, Y_2, Y_3) \otimes I_2,$$

where $\otimes$ refers to the Kronecker product.

Corollary B. *Let $\mathfrak{h}$ be a $\mathbb{Z}$-Lie lattice of nilpotency class greater than 2 and let $\mathcal{R}_{\mathfrak{h}}$ denote its $\mathcal{R}$ commutator matrix. In general, there does not exist a $\mathbb{Z}$-Lie lattice with $\mathcal{R}$ commutator matrix equal to $\mathcal{R}_{\mathfrak{h}} \otimes I_k$, where $1 < k \in \mathbb{N}$.*

Theorem 4. *Let $\mathfrak{h}$ be a nilpotent $\mathbb{Z}$-Lie lattice. The identity*

$$\zeta_{\mathfrak{h}(\mathfrak{o})}(ks) \;=\; \zeta_{\times_Z^k \mathfrak{h}(\mathfrak{o})}(s),$$

where $\times_Z^k$ denotes the k-fold central product, does not hold in general when $\mathfrak{h}$ has nilpotency class greater than 2.

1.3. **Notation.** The following list contains frequently used notation.

$\mathbb{N}$	$\{1, 2, \ldots,\}$
$\mathbb{N}_0$	$\{0, 1, 2, \ldots,\}$
$[n]$	$\{1, 2, \ldots, n\}$
I_n	$n \times n$ identity matrix
K	number field
$\mathcal{O}$	ring of integers of K
$\mathfrak{p}$	non-zero prime ideal of $\mathcal{O}$
$\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$	completion of $\mathcal{O}$ at $\mathfrak{p}$
q	cardinality of $\mathcal{O}/\mathfrak{p}$
$\mathfrak{o}^n$	n-fold Cartesian power $\mathfrak{o} \times \cdots \times \mathfrak{o}$
$\mathfrak{p}^m$	mth ideal power $\mathfrak{p} \cdots \mathfrak{p}$
$\iota(N)$	the isolator of N
$v_{\mathfrak{p}}$	$\mathfrak{p}$-adic valuation
$\lvert \cdot \rvert_{\mathfrak{p}}$	$\mathfrak{p}$-adic norm
$\lvert\lvert(z_i)_{i \in I}\rvert\rvert_{\mathfrak{p}}$	$\max(\lvert z_i\rvert_{\mathfrak{p}})_{i\in I}$, I a finite index set
M^*	$M \setminus \mathfrak{p}M$, M a module
$W(\mathfrak{o})$	$(\mathfrak{o}^d)^*$
$W_N(\mathfrak{o})$	$((\mathfrak{o}/\mathfrak{p}^N)^d)^*$
$\times_Z^k G$	k-fold central product of G

2. General Method

In this section, we present the methods to compute twist representation zeta functions, as seen in [17].

Let Λ be a nilpotent $\mathcal{O}$-Lie lattice such that $\Lambda' \subseteq c!\Lambda$ and $\mathbf{G}_\Lambda$ the unipotent group scheme over $\mathcal{O}$ associated to Λ.

2.1. **Kirillov orbit method.** We begin by fixing a non-trivial prime ideal $\mathfrak{p}$ and set $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$. Define the $\mathfrak{o}$-Lie lattice $\mathfrak{g} := \Lambda \otimes_{\mathcal{O}} \mathfrak{o}$, let $\mathfrak{z}$ denote its centre and write $\widehat{\mathfrak{g}}$ for its Pontryagin dual. For all $\psi \in \widehat{\mathfrak{g}}$ there exists an associated alternating bi-additive form B_ψ. If p is odd or $p = 2$ and $c \geq 4$, we have

$$\zeta_{G(\mathfrak{o})}(s) = \sum_{\psi \in \widehat{\mathfrak{g}'}} |\mathfrak{g} : \mathrm{Rad}(B_\psi)|^{-s/2} |\mathfrak{g} : \mathfrak{g}_{\psi,2}|^{-1},$$
$$\mathfrak{g}_{\psi,2} = \{x \in \mathfrak{g} \mid \psi([x, \mathfrak{g}']) = 1\} [17, \text{Theorem } 2.6].$$

2.2. **Commutator Matrices.** In order to compute the generating function required for the orbit method, we need to find its terms explicitly.

Definition 2.1. Let M be a module and N a submodule of M. The isolator of N in M is the smallest submodule $L \leq M$ containing N such that M/L is torsion-free, denoted by $\iota(N)$. We say that N is isolated in M if $\iota(N) = N$.

First note that $\mathfrak{z}$ is isolated, hence $\mathfrak{z} = \iota(\mathfrak{z})$, [17, Lemma 2.5].

Let,

$$h = \mathrm{rank}_{\mathfrak{o}}(\mathfrak{g}), \quad d = \mathrm{rank}_{\mathfrak{o}}(\mathfrak{g}'),$$
$$k = \mathrm{rank}_{\mathfrak{o}}(\iota(\mathfrak{g}')/\iota(\mathfrak{g}' \cap \mathfrak{z})) = \mathrm{rank}_{\mathfrak{o}}(\iota(\mathfrak{g}' + \mathfrak{z})/\mathfrak{z}),$$
$$r - k = \mathrm{rank}_{\mathfrak{o}}(\mathfrak{g}/\iota(\mathfrak{g}' + \mathfrak{z})), \quad r = \mathrm{rank}_{\mathfrak{o}}(\mathfrak{g}/\mathfrak{z}).$$

Let $\overline{} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{g}'$ be the canonical surjection map and $\overline{x}$ the image of x under this map. Let π be the uniformiser of $\mathfrak{o}$ and choose an $\mathfrak{o}$-basis for $\mathfrak{g}$,

$$\boldsymbol{e} = (e_1, \ldots, e_h), \ \mathfrak{z} = \langle e_{r+1}, \ldots, e_h \rangle_{\mathfrak{o}}$$

and natural numbers $b_1, \ldots, b_d$ such that,

$$\overline{\mathfrak{g}' + \mathfrak{z}} = \langle \overline{\pi^{b_1} e_{r-k+1}}, \ldots, \overline{\pi^{b_k} e_r} \rangle_{\mathfrak{o}}, \quad \iota(\overline{\mathfrak{g}' + \mathfrak{z}}) = \langle \overline{e_{r-k+1}}, \ldots, \overline{e_r} \rangle_{\mathfrak{o}}$$
$$\mathfrak{g}' \cap \mathfrak{z} = \langle \pi^{b_{k+1}} e_{r+1}, \ldots, \pi^{b_d} e_{r-k+d} \rangle_{\mathfrak{o}}, \quad \iota(\mathfrak{g}' \cap \mathfrak{z}) = \langle e_{r+1}, \ldots, e_{r-k+d} \rangle_{\mathfrak{o}}.$$

Such integers always exist as a consequence of the elementary divisor theorem.

Choose an $\mathfrak{o}$-basis $\mathbf{f} = (f_1, \ldots, f_d)$ for $\mathfrak{g}'$ such that

$$(\overline{f_1}, \ldots, \overline{f_k}) = (\overline{\pi^{b_1} e_{r-k+1}}, \ldots, \overline{\pi^{b_k} e_r}),$$
$$(f_{k+1}, \ldots, f_d) = (\pi^{b_{k+1}} e_{r+1}, \ldots, \pi^{b_d} e_{r-k+d}).$$

We define the structure constants $\lambda_{ij}^l \in \mathfrak{o}$ for $\mathfrak{g}$, with respect to these bases,

$$[e_i, e_j] = \sum_{l=1}^{d} \lambda_{ij}^l f_l,$$

where $i, j \in [r]$ and $l \in [d]$. These are then encoded in the commutator matrix,

$$\mathcal{R}(\mathbf{Y}) = \left(\sum_{l=1}^{d} \lambda_{ij}^l Y_l \right)_{ij} \in \mathrm{Mat}_{r \times r}(\mathfrak{o}[\mathbf{Y}]).$$

We define the submatrix,

$$\mathcal{S}(\mathbf{Y}) = \big(\mathcal{R}(\mathbf{Y})_{ij}\big)_{i\in[r],\, j\in\{r-k+1,\dots,r\}} \in \mathrm{Mat}_{r\times k}(\mathfrak{o}[\mathbf{Y}]),$$

comprising the last k columns of $\mathcal{R}(\mathbf{Y})$.

2.3. **Matrix valuations.** Let $y \in W(\mathfrak{o})$. We observe that $\mathcal{R}(y)$ is an antisymmetric $r \times r$ matrix with entries in $\mathfrak{o}$. Considering the Smith normal form of $\mathcal{R}(y)$, we see that that the elementary divisors are of the form $\mathfrak{p}^a$ where $a \in \mathbb{N}_0 \cup \{\infty\}$. Moreover, the elementary divisors come in pairs. If r is even, we have $r/2$ distinct elementary divisors and if r is odd, we have a remainder divisor which is equal to $\mathfrak{p}^\infty = \{0\}$. We define

$$\nu(\mathcal{R}(y)) := (a_1, \dots, a_{\lfloor r/2\rfloor}) \in (\mathbb{N}_0 \cup \{\infty\})^{\lfloor r/2\rfloor},$$

where $0 \le a_1 \le \cdots \le a_{\lfloor r/2\rfloor}$, [1].

Let $N \in \mathbb{N}_0$ and let $\overline{y}$ be the image of y in $W_N(\mathfrak{o})$. $\mathcal{R}(\overline{y})$ is an antisymmetric $r \times r$ matrix with entries in $\mathfrak{o}/\mathfrak{p}^N$ and the valuations of its elementary divisors are encoded by the tuple

$$\nu(\mathcal{R}(\overline{y})) := \nu_N(\mathcal{R}(y)) := (\min\{a_i, N\})_{i\in\{1,\dots,\lfloor r/2\rfloor\}} \in \{0,1,\dots,N\}^{\lfloor r/2\rfloor},$$

[1, 3.1]. We now specialise this definition for the $\mathcal{S}$ matrix.
Given a matrix $\mathcal{S} \in \mathrm{Mat}_{r\times k}(\mathfrak{o})$ we write $\tilde{\nu}(\mathcal{S}) = \mathbf{c}$ if it has elementary divisor type $\mathbf{c} = (c_1, \dots, c_k) \in (\mathbb{N}_0 \cup \{\infty\})^k$, which is to say, $\mathcal{S}$ is equivalent to

$$\begin{pmatrix} \pi^{c_1} & & \\ & \ddots & \\ & & \pi^{c_k} \\ & & \end{pmatrix} \in Mat_{r\times k}(\mathfrak{o}).$$

under elementary row and column operations, where $0 \le c_1 \le \cdots \le c_k$ [17].

2.4. **Poincaré series.** Given $N \in \mathbb{N}_0$, $\mathbf{a} \in \mathbb{N}_0^{\lfloor r/2\rfloor}$, $\mathbf{c} \in \mathbb{N}_0^k$, we let

$$\mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}} := \#\Big\{\mathbf{y} \in W_N(\mathfrak{o}) \,\Big|\, \nu(\mathcal{R}(\mathbf{y})) = \mathbf{a},\ \tilde{\nu}\big(\mathcal{S}(\mathbf{y})\cdot\mathrm{diag}(\pi^{b_1}, \dots, \pi^{b_k})\big) = \mathbf{c}\Big\}.$$

Given this count, we can compute the zeta function by [17, Proposition 2.9], which states,

$$\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) := \sum_{\substack{N\in\mathbb{N}_0,\\ \mathbf{a}\in\mathbb{N}_0^{\lfloor r/2\rfloor},\, \mathbf{c}\in\mathbb{N}_0^k}} \mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}}\, q^{-\sum_{i=1}^{\lfloor r/2\rfloor}(N-a_i)s - \sum_{i=1}^{k}(N-c_i)} = \zeta_{G(\mathfrak{o})}(s),$$

provided that p is odd or $p = 2$ and $c \ge 4$.

2.5. **$\mathfrak{p}$-adic integration.** Alternatively, instead of using the Kirillov orbit method, Voll explains in [19, Section 2.2] how the Poincaré series can be expressed in terms of the $\mathfrak{p}$-adic integral $\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau)$.

$$\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathfrak{o}}(-s/2, -1, us + v - d - 1) = \zeta_{G(\mathfrak{o})}(s),$$

[17, Corollary 2.11].
Where,

$$2u = \max\{\mathrm{rank}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{R}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\},$$

$$v = \max\{\mathrm{rank}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{S}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\},$$

$$F_j(\mathbf{Y}) = \{ f \mid f = f(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y}) \},$$

$$G_l(\mathbf{Y}) = \left\{ g \mid g = g(\mathbf{Y}) \text{ a } l \times l \text{ minor of } \mathcal{S}(\mathbf{Y}) \cdot \mathrm{diag}(\pi^{b_1}, \ldots, \pi^{b_k}) \right\},$$

$$\|H(X, \mathbf{Y})\|_{\mathfrak{p}} = \max\{ |h(X, \mathbf{Y})|_{\mathfrak{p}} \mid h \in H \} \quad \text{for a finite set } H \subseteq \mathfrak{o}[X, \mathbf{Y}].$$

$$\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau) :=$$

$$\int_{(x,\mathbf{Y}) \in \mathfrak{p} \times W(\mathfrak{o})} |x|_{\mathfrak{p}}^{\tau} \prod_{j=1}^{u} \frac{\|F_j(\mathbf{Y}) \cup F_{j-1}(\mathbf{Y}) x^2\|_{\mathfrak{p}}^{\rho}}{\|F_{j-1}(\mathbf{Y})\|_{\mathfrak{p}}^{\rho}} \prod_{l=1}^{v} \frac{\|G_l(\mathbf{Y}) \cup G_{l-1}(\mathbf{Y}) x\|_{\mathfrak{p}}^{\sigma}}{\|G_{l-1}(\mathbf{Y})\|_{\mathfrak{p}}^{\sigma}} \, d\mu(x, \mathbf{Y}).$$

The additive Haar measure μ on $\mathfrak{o}^{d+1}$ is normalised so that $\mu(\mathfrak{o}^{d+1}) = 1$.

3. Computation of twist zeta function

We now compute the twist representation zeta function via the Kirillov orbit method for the 3-nilpotent $\mathbb{Z}$-Lie lattice,

$$\mathfrak{f}_{3,2} = \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y, \ [x_1, y] = z_1 \ [x_2, y] = z_2 \rangle,$$

where all non-specified Lie brackets are trivial. This construction can be found in [2, Example 4.12].
We choose the bases $\mathbf{e} = (x_1, x_2, y)$ and $\mathbf{f} = (y, z_1, z_2)$. It is immediate from the presentation of the $\mathbb{Z}$-Lie lattice that $h = 5$, $d = 3$ and $r = 3$. We also see that $\mathfrak{z} = (z_1, z_2)$. From these bases we see that $\mathfrak{f}_{3,2}/\mathfrak{f}'_{3,2}$ has basis (x_1, x_2). It follows from the presentation that $\mathfrak{f}_{3,2}/\mathfrak{f}'_{3,2}$ is torsion-free and so we get that the isolator is $\iota(\mathfrak{f}'_{3,2}) = \mathfrak{f}'_{3,2}$. Therefore, we get $k = 1$.

This gives us the $\mathcal{R}$ commutator matrix,

$$\mathcal{R}((Y_1, Y_2, Y_3)) = \left(\sum_{l=1}^{3} \lambda_{ij}^{l} Y_l \right)_{ij} \in \mathrm{Mat}_{3\times3}(\mathfrak{o}[\mathbf{Y}]), i, j \in [3], l \in [3],$$

$$\mathcal{R}(Y_1, Y_2, Y_3) = \begin{pmatrix} 0 & Y_1 & Y_2 \\ -Y_1 & 0 & Y_3 \\ -Y_2 & -Y_3 & 0 \end{pmatrix}.$$

Since $k = 1$, the $\mathcal{S}$ commutator matrix is the final column of the $\mathcal{R}$ commutator matrix,

$$\mathcal{S}(Y_1, Y_2, Y_3) = \begin{pmatrix} Y_2 \\ Y_3 \\ 0 \end{pmatrix}.$$

We begin to compute $\mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}}$.
As $d = 3$, we have $W_N(\mathfrak{o}) = ((\mathfrak{o}/\mathfrak{p}^N)^3)^* = (\mathfrak{o}/\mathfrak{p}^N)^3 \setminus \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)^3$. Since the $\mathcal{R}$ matrix is of size 3×3, there is only one pair of elementary divisors and since the $\mathcal{S}$ matrix is of size 3×1 we only have one elementary divisor. This is easily spotted from our prior computation as $r = 3$ and $k = 1$, hence $\mathbf{a}, \mathbf{c} \in \mathbb{N}_0$.

This gives us $\mathcal{N}^{\mathfrak{o}}_{N,a,c} =$

$$\#\Big\{(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o}) \Bigm| \nu(\mathcal{R}(Y_1, Y_2, Y_3)) = a,\ \tilde{\nu}(\mathcal{S}(Y_1, Y_2, Y_3) \cdot \mathrm{diag}(\pi^{b_1})) = c\Big\}.$$

Observe from the definition of $W_N(\mathfrak{o})$ that we cannot have $Y_1, Y_2, Y_3 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$ simultaneously. This means that $\min\{\nu(Y_1), \nu(Y_2), \nu(Y_3)\} = 0$ hence,

$$\min\{\nu(Y_1), \nu(Y_2), \nu(Y_3)\} = 0 = \nu(\mathcal{R}(Y_1, Y_2, Y_3)),$$

for all $(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o})$.

As a result, $\mathcal{N}^{\mathfrak{o}}_{N,a,c} = 0$ for all $a \neq 0$. Therefore, we are left to compute
$\mathcal{N}^{\mathfrak{o}}_{N,0,c} =$

$$\#\Big\{(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o}) \Bigm| \nu(\mathcal{R}(Y_1, Y_2, Y_3)) = 0,\ \tilde{\nu}(\mathcal{S}(Y_1, Y_2, Y_3)) = c\Big\}.$$

Note that we drop the $\mathrm{diag}(\pi^{b_1})$ term. This is permissible since it merely adds b_1 to the valuation and we can absorb this into c. As well as this, note that by the definition of valuations of matrices, $\mathcal{N}^{\mathfrak{o}}_{N,a,c} = 0$ for all $c > N$.

We now approach the counting argument case by case.

1. **Case 1:** $Y_1, Y_2, Y_3 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$,
2. **Case 2:** $Y_2, Y_3 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_1 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.
3. **Case 3:** $Y_1, Y_3 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_2 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.
4. **Case 4:** $Y_1, Y_2 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_3 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.
5. **Case 5:** $Y_1 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_2, Y_3 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.
6. **Case 6:** $Y_2 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_1, Y_3 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.
7. **Case 7:** $Y_3 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, $Y_1, Y_2 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$.

These are all the possible cases for $(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o})$.

To proceed, we examine,

$$\tilde{\nu}(\mathcal{S}(Y_2, Y_3)) = \min\{\nu(Y_2), \nu(Y_3)\}.$$

Note that there is no Y_1 dependency as Y_1 does not appear in the $\mathcal{S}$ matrix. For $\tilde{\nu}(\mathcal{S}(Y_2, Y_3)) \neq 0$ we require $\min\{\nu(Y_2), \nu(Y_3)\} \geq 1$, therefore we must have $Y_2, Y_3 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$. Referring back to the case by case analysis we see this is only possible in case 5, therefore for all other cases we have both $a = 0$ and $c = 0$. Furthermore, in case 5 we have that $\tilde{\nu}(\mathcal{S}(Y_2, Y_3) > 0$. This means we can now split our count for $\mathcal{N}^{\mathfrak{o}}_{N,0,c}$ into two cases; when $a = 0, c = 0$ and when $a = 0, c > 0$. As shown there is no overlap between the two cases.

We first begin by counting how many such vectors exist for each case enumerated above when $c = 0$. Recall that we have defined q to be the cardinality of $\mathcal{O}/\mathfrak{p}$. It follows that given a variable $w_1 \in (\mathfrak{o}/\mathfrak{p}^N) \setminus \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, we have $q^N - q^{N-1} = q^{N-1}(q-1)$ choices for it and given another variable $w_2 \in \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$, we have q^{N-1} choices for it. This gives us,

(1) **Case 1:** $(q^N - q^{N-1})^3$ choices.
(2) **Case 2:** $q^{N-1}(q^N - q^{N-1})^2$ choices.
(3) **Case 3:** $q^{N-1}(q^N - q^{N-1})^2$ choices.
(4) **Case 4:** $q^{N-1}(q^N - q^{N-1})^2$ choices.
(5) **Case 6:** $(q^{N-1})^2(q^N - q^{N-1})$ choices.
(6) **Case 7:** $(q^{N-1})^2(q^N - q^{N-1})$ choices.

To find the total number of choices such that $c = 0$, (and $a = 0$), we take the sum of all the cases above and get $q^{3N-2}(q^2 - 1)$ total choices for the vector $(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o})$.

It remains to analyse case 5. We need to determine $\mathcal{N}^{\mathfrak{o}}_{N,0,c}$ for each $c \in [N-1]$. We proceed by splitting into subcases. Since $\tilde{\nu}(\mathcal{S}(Y_2, Y_3)) = \min\{\nu(Y_2), \nu(Y_3)\}$, we consider:

(1) **Case i:** $\nu(Y_2) < \nu(Y_3)$.
(2) **Case ii:** $\nu(Y_2) = \nu(Y_3)$.
(3) **Case iii:** $\nu(Y_2) > \nu(Y_3)$.

Case i:
Let $\nu(Y_2) = c$ then $Y_2 \in \mathfrak{p}^c(\mathfrak{o}/\mathfrak{p}^N)$. Therefore,

$$Y_2 = a_c\mathfrak{p}^c + a_{c+1}\mathfrak{p}^{c+1} \ldots + a_{N-1}\mathfrak{p}^{N-1}, a_i \in \mathfrak{o}/\mathfrak{p},$$

and

$$Y_3 = b_{c+1}\mathfrak{p}^{c+1} \ldots + b_{N-1}\mathfrak{p}^{N-1}, b_i \in \mathfrak{o}/\mathfrak{p}$$

Since $\nu(Y_2) = c$, this implies $a_c \neq 0 \in \mathfrak{o}/\mathfrak{p}$, hence we have $q-1$ choices for a_c. We get q choices for each of the $N - c - 1$ remaining coefficients as they lie in $\mathfrak{o}/\mathfrak{p}$ with no other restrictions. As a result, we have $(q-1)q^{N-c-1}$ choices for Y_2.
On the other hand, we have no restriction for any of the $N - c - 1$

coefficients found in the expansion for Y_3, so we get q^{N-c-1} choices for Y_3. This gives $(q-1)(q^{N-c-1})^2$ choices in total for case i.

Case ii:
We have $\nu(Y_2) = c = \nu(Y_3)$, hence

$$Y_2 = a_c\mathfrak{p}^c + a_{c+1}\mathfrak{p}^{c+1} \ldots + a_{N-1}\mathfrak{p}^{N-1}, a_i \in \mathfrak{o}/\mathfrak{p},$$

and

$$Y_3 = b_c\mathfrak{p}^c + b_{c+1}\mathfrak{p}^{c+1} \ldots + b_{N-1}\mathfrak{p}^{N-1}, b_i \in \mathfrak{o}/\mathfrak{p}.$$

Using the same counting argument as above we see that we have $(q-1)q^{N-c-1}$ choices for both Y_2 and Y_3, giving $(q-1)^2(q^{N-c-1})^2$ total choices.

Case iii:
This is symmetrical to case i therefore we also get $(q-1)(q^{N-c-1})^2$ choices.

Summing all three subcases together we see that for every $c \in [N-1]$ we have $(q^2-1)(q^{2N-2c-2})$ total vectors (Y_2, Y_3).

However, we still have a choice of Y_1 such that $Y_1 \notin \mathfrak{p}(\mathfrak{o}/\mathfrak{p}^N)$. As argued above this has $q^N - q^{N-1}$ possibilities. This means that overall case 5 gives us $(q-1)^2(q+1)q^{-3N-2c-3}$ choices for the vector $(Y_1, Y_2, Y_3) \in W_N(\mathfrak{o})$ for each possible value of c.

We are left with the sole case, $c = N$, which has only one possible vector, the zero vector. This is evident when referring back to the definition for the valuation of our $\mathcal{S}$ matrix,

$$\tilde{\nu}(\mathcal{S}(y)) = (\min\{c, N\}).$$

If $\min\{c, N\} = N$ then $N < \nu(Y_2), \nu(Y_3)$. This is only possible when $\nu(Y_2) = \nu(Y_3) = \infty = \nu(0)$. Hence we have that $Y_2 = 0 = Y_3$ and Y_1 must be a unit.

To summarise,

$$\mathcal{N}^{\mathfrak{o}}_{N,a,c} = \begin{cases} 0 \text{ if } a \neq 0 \\ \mathcal{N}^{\mathfrak{o}}_{N,0,c} = \begin{cases} 0 \text{ if } c \geq N, \\ \mathcal{N}^{\mathfrak{o}}_{N,0,c} = \begin{cases} 1, \ N = 0 \\ (q^2-1)q^{3N-2}, \ c = 0 \\ (q-1)^2(q+1)q^{3N-2c-3}, \ 1 \leq c \leq N-1 \\ (q-1)q^{N-1}, \ c = N \end{cases} \end{cases} \end{cases}$$

The Poincaré series simplifies as follows,

$$\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) = \sum_{\substack{N\in\mathbb{N}_0, \\ a\in\mathbb{N}_0}, \, c\in\mathbb{N}_0} \mathcal{N}^{\mathfrak{o}}_{N,a,c}\, q^{-(N-a)s-(N-c)} = \sum_{\substack{N\in\mathbb{N}_0, \\ c\in\mathbb{N}_0}} \mathcal{N}^{\mathfrak{o}}_{N,0,c}\, q^{-Ns-N+c}.$$

This gives,

$$\begin{aligned}\sum_{N=0}^{\infty}\sum_{c=0}^{N} \mathcal{N}^{\mathfrak{o}}_{N,0,c}\, q^{-Ns-N+c} = 1 &+ \sum_{N=1}^{\infty}(q^2-1)\, q^{3N-2}\, q^{-Ns-N} \\ &+ \sum_{N=1}^{\infty}(q-1)\, q^{-1}\, q^{(1-s)N} \\ &+ \sum_{N=1}^{\infty}\sum_{c=1}^{N-1}(q-1)^2(q+1)\, q^{3N-2c-3}\, q^{-Ns-N+c}.\end{aligned}$$

(1) **Term 1:**

$$\sum_{N=1}^{\infty}(q^2-1)\, q^{3N-2}\, q^{-Ns-N} = (q^2-1)q^{-2}\sum_{N=1}^{\infty} q^{(2-s)N} =$$

$$(q^2-1)q^{-2}\frac{q^{2-s}}{1-q^{2-s}} = \frac{(q^2-1)q^{-s}}{1-q^{2-s}}$$

(2) **Term 2:**

$$\sum_{N=1}^{\infty}(q-1)q^{-1}\, q^{(1-s)N} = (q-1)q^{-1}\frac{q^{1-s}}{1-q^{1-s}} = \frac{(q-1)q^{-s}}{1-q^{1-s}}.$$

(3) **Term 3:**

$$\sum_{N=1}^{\infty}\sum_{c=1}^{N-1}(q-1)^2(q+1)\, q^{3N-2c-3}\, q^{-Ns-N+c} =$$

$$(q-1)^2(q+1)q^{-3}\sum_{N=1}^{\infty} q^{(2-s)N}\sum_{c=1}^{N-1} q^{-c}.$$

First doing the sum over c we get,

$$(q-1)(q+1)q^{-3}\sum_{N=1}^{\infty} q^{(2-s)N}(1-q^{-(N+1)}) =$$

$$(q-1)(q+1)q^{-3}\Big[\sum_{N=1}^{\infty} q^{(2-s)N} + q\sum_{N=1}^{\infty} q^{(1-s)N}\Big].$$

Ultimately giving,

$$(q-1)(q+1)q^{-3}\Big[\frac{q^{2-s}}{1-q^{2-s}}-\frac{q^{2-s}}{1-q^{1-s}}\Big].$$

Putting this all together we get $\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) =$

$$1+\frac{(q^2-1)q^{-s}}{1-q^{2-s}}+\frac{(q-1)q^{-s}}{1-q^{1-s}}+(q-1)(q+1)q^{-3}\Big[\frac{q^{2-s}}{1-q^{2-s}}-\frac{q^{2-s}}{1-q^{1-s}}\Big].$$

Simplifying this expression gives $\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) = \zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}(s) =$

$$\frac{(q^s-1)^2}{(q^s-q)(q^s-q^2)}=\frac{(1-q^{-s})^2}{(1-q^{1-s})(1-q^{2-s})}.$$

4. Alternate computation

We compute the $\mathfrak{p}$-adic integral seen above, for our class 3 nilpotent example. As above,

$$\mathfrak{f}_{3,2}=\langle x_1,x_2,y,z_1,z_2 \mid [x_1,x_2]=y,\ [x_1,y]=z_1\ [x_2,y]=z_2\rangle.$$

From the $\mathcal{R}$, $\mathcal{S}$ matrices calculated in the previous section, we see that $2u = 2$ and $v = 1$. Therefore we have the integral,

$$\int_{(x,\mathbf{Y})\in\mathfrak{p}\times W(\mathfrak{o})}|x|_{\mathfrak{p}}^{\tau}\frac{\|F_1(\mathbf{Y})\cup F_0(\mathbf{Y})x^2\|_{\mathfrak{p}}^{\rho}}{\|F_0(\mathbf{Y})\|_{\mathfrak{p}}^{\rho}}\frac{\|G_1(\mathbf{Y})\cup G_0(\mathbf{Y})x\|_{\mathfrak{p}}^{\sigma}}{\|G_0(\mathbf{Y})\|_{\mathfrak{p}}^{\sigma}}\,d\mu(x,\mathbf{Y}).$$

Computing the 2×2 principal minors of $\mathcal{R}(Y_1,Y_2,Y_3)$ we get,

$$F_1(\mathbf{Y})=\{Y_1^2,Y_2^2,Y_3^2\}.$$

and computing the 1×1 minors of $\mathcal{S}(Y_1,Y_2,Y_3)$ we get,

$$G_1(\mathbf{Y})=\{Y_2,Y_3\}.$$

We note that both F_0 and G_0 are empty hence, $\|F_0(\mathbf{Y})\|_{\mathfrak{p}} = 1$ and $\|G_0(\mathbf{Y})\|_{\mathfrak{p}} = 1$. This simplifies our integral to,

$$\int_{(x,\mathbf{Y})\in\mathfrak{p}\times W(\mathfrak{o})}|x|_{\mathfrak{p}}^{\tau}\left\|\{Y_1^2,Y_2^2,Y_3^2\}\cup x^2\right\|_{\mathfrak{p}}^{\rho}\left\|\{Y_2,Y_3\}\cup x\right\|_{\mathfrak{p}}^{\sigma}d\mu(x,\mathbf{Y})=$$

$$\int_{(x,\mathbf{Y})\in\mathfrak{p}\times W(\mathfrak{o})}|x|_{\mathfrak{p}}^{\tau}\left\|\{Y_1,Y_2,Y_3\}\cup x\right\|_{\mathfrak{p}}^{2\rho}\left\|\{Y_2,Y_3\}\cup x\right\|_{\mathfrak{p}}^{\sigma}d\mu(x,\mathbf{Y}).$$

Where we have,

$$\|\{Y_1,Y_2,Y_3\}\cup x\|_{\mathfrak{p}}=max\{\|Y_1\|_{\mathfrak{p}},\|Y_2\|_{\mathfrak{p}},\|Y_3\|_{\mathfrak{p}},\|x\|_{\mathfrak{p}}\}=$$

$$p^{-min\{\nu(Y_1),\nu(Y_2),\nu(Y_3),\nu(x)\}}.$$

Notice that $x \in \mathfrak{p}$, so $\nu(x) \geq 1$ and by definition of $W(\mathfrak{o}) = \mathfrak{o}^3 \setminus \mathfrak{p}\mathfrak{o}^3$ we have $min\{\nu(Y_1), \nu(Y_2), \nu(Y_3)\} = 0$. This means that

$$min\{\nu(Y_1), \nu(Y_2), \nu(Y_3), \nu(x)\} = 0,$$

and so,

$$max\{\|Y_1\|_{\mathfrak{p}}, \|Y_2\|_{\mathfrak{p}}, \|Y_3\|_{\mathfrak{p}}, \|x\|_{\mathfrak{p}}\} = 1.$$

Substituting this back into our integral we get,

$$\mathcal{Z}_{\mathfrak{o}}(\sigma, \tau) = \int_{(x,\mathbf{Y}) \in \mathfrak{p} \times W(\mathfrak{o})} |x|_{\mathfrak{p}}^{\tau} \left\|\{Y_2, Y_3\} \cup x\right\|_{\mathfrak{p}}^{\sigma} d\mu(x, \mathbf{Y}).$$

We now split the domain of integration into two distinct regions,

$$A = \{Y_1 \in \mathfrak{p}, \text{and at least one of } Y_2, Y_3 \notin \mathfrak{p}\},$$
$$B = \{Y_1 \notin \mathfrak{p}, Y_2, Y_3 \in \mathfrak{p}\},$$

where $A \cap B = \emptyset$ and $A + B = \mathfrak{p} \times W(\mathfrak{o})$.

So we have $\mathcal{Z}_{\mathfrak{o}}(\sigma, \tau) =$

$$\int_{(x,\mathbf{Y}) \in A} |x|_{\mathfrak{p}}^{\tau} \left\|\{Y_2, Y_3\} \cup x\right\|_{\mathfrak{p}}^{\sigma} d\mu(x, \mathbf{Y}) + \int_{(x,\mathbf{Y}) \in B} |x|_{\mathfrak{p}}^{\tau} \left\|\{Y_2, Y_3\} \cup x\right\|_{\mathfrak{p}}^{\sigma} d\mu(x, \mathbf{Y}).$$

Observe that by the definition of A, $min\{\nu(Y_2), \nu(Y_3), \nu(x)\} = 0$, and so $\left\|\{Y_2, Y_3\} \cup x\right\|_{\mathfrak{p}} = 1$. This means that our integral over the region A simplifies to,

$$\int_{(x,\mathbf{Y}) \in A} |x|_{\mathfrak{p}}^{\tau} d\mu(x, \mathbf{Y}).$$

We compute the Haar measures to be,

$$\mu(\nu(x) = c) = q^{-c} - q^{-(c+1)} = (1 - q^{-1})q^{-c},$$
$$\mu(Y_1 \in \mathfrak{p}) = 1,$$
$$\mu(Y_2 \text{ or } Y_3 \notin \mathfrak{p}) = (1 - q^{-2}).$$

As a result, our integral becomes,

$$(1 - q^{-2}) \int_{x \in \mathfrak{p}} |x|_{\mathfrak{p}}^{\tau} d\mu(x) = (1 - q^{-2}) \sum_{c=1}^{\infty} q^{-c\tau} (1 - q^{-1})\, q^{-c},$$

and by simplifying we get,

$$(1 - q^{-2})(1 - q^{-1}) \sum_{c=1}^{\infty} q^{-c(\tau+1)} = (1 - q^{-2})(1 - q^{-1}) \frac{q^{-(\tau+1)}}{1 - q^{-(\tau+1)}}.$$

Now we compute the integral over the region B,

$$\int_{(x,\mathbf{Y}) \in B} |x|_{\mathfrak{p}}^{\tau} \left\|\{Y_2, Y_3\} \cup x\right\|_{\mathfrak{p}}^{\sigma} d\mu(x, \mathbf{Y}).$$

We have the Haar measures,

$$\mu(Y_1 \notin \mathfrak{p}) = 1 - q^{-1},$$
$$\mu(\nu(Y_2) = a) = (1 - q^{-1})q^{-a},$$
$$\mu(\nu(Y_3) = b) = (1 - q^{-1})q^{-b},$$
$$\mu(\nu(x) = c) = (1 - q^{-1})q^{-c}.$$

For ease of notation we define $m = min\{a, b, c\}$. Our integral becomes,

$$(1-q)^{-4} \sum_{a,b,c\geq 1} q^{-a-b-c} q^{-c\tau} q^{-\sigma m}.$$

This can be rewritten as,

$$\sum_{a,b,c\geq 1} q^{-a-b-c} q^{-c\tau} q^{-\sigma m} = \sum_{n=1}^{\infty} q^{-\sigma n} \sum_{m=n} q^{-a-b-c} q^{-c\tau},$$

and we notice that,

$$\sum_{m=n} q^{-a-b-c} q^{-c\tau} = \sum_{a,b,c\geq n} q^{-a-b-c} q^{-c\tau} - \sum_{a,b,c\geq n+1} q^{-a-b-c} q^{-c\tau}.$$

We let

$$S_n = \sum_{a,b,c\geq n} q^{-a-b-c} q^{-c\tau}$$

and so we have,

$$\sum_{m=n} q^{-a-b-c} q^{-c\tau} = S_n - S_{n+1}.$$

We now compute S_n explicitly,

$$S_n = \sum_{a\geq n} q^{-a} \sum_{b\geq n} q^{-b} \sum_{c\geq n} q^{-c} q^{-c\tau} = \frac{q^{-n}}{1-q^{-1}} \frac{q^{-n}}{1-q^{-1}} \frac{q^{-n(1+\tau)}}{1-q^{-(1+\tau)}} =$$
$$\frac{q^{-n(3+\tau)}}{(1-q^{-1})^2(1-q^{-(1+\tau)})}.$$

Using this general form, we simplify $S_n - S_{n+1}$ as follows,

$$\frac{q^{-n(3+\tau)}}{(1-q^{-1})^2(1-q^{-(1+\tau)})} - \frac{q^{-(n+1)(3+\tau)}}{(1-q^{-1})^2(1-q^{-(1+\tau)})} =$$
$$\frac{q^{-n(3+\tau)}(1-q^{-(3+\tau)})}{(1-q^{-1})^2(1-q^{-(1+\tau)})}.$$

Returning to the entire integral over the region B we obtain,

$$(1-q^{-1})^4 \sum_{n=1}^{\infty} q^{-\sigma n}(S_n - S_{n+1}) = (1-q^{-1})^2 \frac{1-q^{-(3+\tau)}}{1-q^{-(1+\tau)}} \sum_{n=1}^{\infty} q^{-n(3+\sigma+\tau)}.$$

This gives,

$$(1-q^{-1})^2\,\frac{1-q^{-(3+\tau)}}{1-q^{-(1+\tau)}}\frac{q^{-(\sigma+3+\tau)}}{1-q^{-(\sigma+3+\tau)}}.$$

Putting this all together, the original integral over the entire domain of integration is, $\mathcal{Z}_{\mathfrak{o}}(\sigma,\tau)=$

$$(1-q^{-2})(1-q^{-1})\frac{q^{-(\tau+1)}}{1-q^{-(\tau+1)}}+(1-q^{-1})^2\,\frac{1-q^{-(3+\tau)}}{1-q^{-(1+\tau)}}\frac{q^{-(\sigma+3+\tau)}}{1-q^{-(\sigma+3+\tau)}}.$$

To be able to compute the twist zeta function, we need the specialised form $\mathcal{Z}_{\mathfrak{o}}(-s/2,-1,us+v-d-1)$. This substitution gives us,

$$\mathcal{Z}_{\mathfrak{o}}(-1,s-3)=(1-q^{-2})(1-q^{-1})\frac{q^{2-s}}{1-q^{2-s}}+(1-q^{-1})^2\,\frac{1-q^{-s}}{1-q^{2-s}}\frac{q^{1-s}}{1-q^{1-s}}.$$

Finally, to compute our twist representation zeta function we simplify the expression,

$$1+(1-q^{-1})^{-1}\mathcal{Z}_{\mathfrak{o}}(-1,s-3),$$

which gives us,

$$\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}(s)=\frac{(q^s-1)^2}{(q^s-q)(q^s-q^2)}=\frac{(1-q^{-s})^2}{(1-q^{1-s})(1-q^{2-s})}.$$

5. Commutator Matrix Scaling

We aim to construct a class 2 nilpotent $\mathbb{Z}$-Lie lattice which has the $\mathcal{A}$ matrix equal to the $\mathcal{A}$ matrix of the Heisenberg group Kronecker product with the 2×2 identity matrix.

Let G_1 and G_2 be groups and let $\varphi : Z(G_1)\xrightarrow{\sim} Z(G_2)$ be an isomorphism between their centres.

Definition 5.1. The central product is defined as,

$$G_1\times_Z G_2 := G_1\times G_2/\langle (z,\varphi(z^{-1}))\rangle,\ \ z\in Z(G_1),$$

where $\times$ refers to the usual direct product.
Let $k\in\mathbb{N}$, the k-fold central product of a group G is defined to be,

$$\times_Z^k G := \underbrace{G\times_Z G\times_Z\cdots\times_Z G}_{k\text{ times}}.$$

Where,

$$G\times_Z G := G\times G/\langle (z,z^{-1})\rangle,\ \ z\in Z(G).$$

Note φ is taken to be the identity map in this case.

Definition 5.2. We define the following commutator matrices for the class 2 nilpotent specific case,

$$\mathcal{B}(\mathbf{Y}) = \left(\sum_{l=1}^{d} \lambda_{ij}^{l} Y_l\right)_{ij} \in \mathrm{Mat}_{r\times r}(\mathfrak{o}[\mathbf{Y}]),$$

$$\mathcal{A}(\mathbf{X}) = \left(\sum_{l=1}^{r} \lambda_{il}^{j} X_l\right)_{ij} \in \mathrm{Mat}_{r\times d}(\mathfrak{o}[\mathbf{X}]).$$

Recall from section two that $i, j \in [r]$ and also note that the $\mathcal{B}$ and $\mathcal{R}$ matrix are in fact the same matrix.

In [9, Proposition 2.5], Jones proves that for any $\mathcal{T}_2$-group, G, with k-fold central product $\times_Z^k G$, its $\mathcal{B}$ commutator matrix is given as follows:

$$\mathcal{B}_{\times_Z^k G}(\mathbf{Y}) \;=\; \bigoplus_{i=1}^{k} \mathcal{B}_G(\mathbf{Y}) \;=\; \begin{pmatrix} \mathcal{B}_G(\mathbf{Y}) & & \\ & \ddots & \\ & & \mathcal{B}_G(\mathbf{Y}) \end{pmatrix}.$$

By reordering the basis we get,

$$\mathcal{B}_{\times_Z^k G}(\mathbf{Y}) \;=\; \mathcal{B}_G(\mathbf{Y}) \otimes I_k,$$

where $\otimes$ is the Kronecker product.

5.1. **Example.** Let $\boldsymbol{H}(\mathcal{O})$ be the Heisenberg group over $\mathcal{O}$. It is well known that the unipotent group scheme $\boldsymbol{H}$ is obtained from the following $\mathbb{Z}$-Lie lattice,

$$\Lambda = \langle x_1, x_2, y \mid [x_1, x_2] = y\rangle, \text{[2, Example 4.10]}.$$

We have $\boldsymbol{e} = (x_1, x_2)$ and $\boldsymbol{f} = (y)$. With respect to these bases we obtain the commutator matrices,

$$\mathcal{B}(Y) \;=\; \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix},\; \mathcal{A}(X_1, X_2) \;=\; \begin{pmatrix} -X_2 \\ X_1 \end{pmatrix}.$$

The 2-fold central product of this $\mathbb{Z}$-Lie lattice is,

$$\Lambda \times_Z \Lambda \;=\; \big\langle x_1, a_1, x_2, a_2, y \;\big|\; [x_1, x_2] = y = [a_1, a_2]\big\rangle.$$

We take $\boldsymbol{e} = (x_1, a_1, x_2, a_2)$ and $\boldsymbol{f} = (y)$.
This gives the commutator matrices,

$$\mathcal{B}(Y) = \begin{pmatrix} 0 & 0 & Y & 0 \\ 0 & 0 & 0 & Y \\ -Y & 0 & 0 & 0 \\ 0 & -Y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0I_2 & YI_2 \\ -YI_2 & 0I_2 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} \otimes I_2,$$

$$\mathcal{A}(X_1, X_2, X_3, X_4) = \begin{pmatrix} X_3 \\ X_4 \\ -X_1 \\ -X_2 \end{pmatrix}.$$

5.2. **Non-existence of lattice.** A natural question that follows, is whether there exists a $\mathbb{Z}$-Lie lattice whose $\mathcal{A}$-matrix exhibits a similar scaling behaviour. In general, this is not the case, as the following counterexample demonstrates.

We return to our previous example, the Heisenberg group,

$$\Lambda = \langle x_1, x_2, y \mid [x_1, x_2] = y \rangle,$$

and we aim to find a class 2 nilpotent $\mathbb{Z}$-Lie lattice such that,

$$\mathcal{A}(X_1, X_2) \;=\; \begin{pmatrix} -X_2 I_2 \\ X_1 I_2 \end{pmatrix} = \begin{pmatrix} -X_2 & 0 \\ 0 & -X_2 \\ X_1 & 0 \\ 0 & X_1 \end{pmatrix}.$$

Given such a commutator matrix, it is easy to see $r = 4$ and $d = 2$, therefore the general form of the $\mathcal{A}$ matrix is,
$\mathcal{A}(X_1, X_2, X_3, X_4) =$

$$\begin{pmatrix} \lambda^1_{11}X_1 + \lambda^1_{12}X_2 + \lambda^1_{13}X_3 + \lambda^1_{14}X_4 & \lambda^2_{11}X_1 + \lambda^2_{12}X_2 + \lambda^2_{13}X_3 + \lambda^2_{14}X_4 \\ \lambda^1_{21}X_1 + \lambda^1_{22}X_2 + \lambda^1_{23}X_3 + \lambda^1_{24}X_4 & \lambda^2_{21}X_1 + \lambda^2_{22}X_2 + \lambda^2_{23}X_3 + \lambda^2_{24}X_4 \\ \lambda^1_{31}X_1 + \lambda^1_{32}X_2 + \lambda^1_{33}X_3 + \lambda^1_{34}X_4 & \lambda^2_{31}X_1 + \lambda^2_{32}X_2 + \lambda^2_{33}X_3 + \lambda^2_{34}X_4 \\ \lambda^1_{41}X_1 + \lambda^1_{42}X_2 + \lambda^1_{43}X_3 + \lambda^1_{44}X_4 & \lambda^2_{41}X_1 + \lambda^2_{42}X_2 + \lambda^2_{43}X_3 + \lambda^2_{44}X_4 \end{pmatrix}.$$

We now attempt to choose each structure constant of the Lie bracket, so as to give us the desired matrix. We require,

$$\lambda^1_{11}X_1 + \lambda^1_{12}X_2 + \lambda^1_{13}X_3 + \lambda^1_{14}X_4 = X_2.$$

This gives that $\lambda^1_{12} = 1$ and $\lambda^1_{11} = \lambda^1_{13} = \lambda^1_{14} = 0$.

$$\lambda^1_{21}X_1 + \lambda^1_{22}X_2 + \lambda^1_{23}X_3 + \lambda^1_{24}X_4 = 0.$$

This gives us that $\lambda^1_{21} = \lambda^1_{22} = \lambda^1_{23} = \lambda^1_{24} = 0$.

By anti-symmetry of the Lie bracket we should have $\lambda^1_{21} = -1$ but by requiring $\mathcal{A}_{21} = 0$ we get that $\lambda^1_{21} = 0$, thus it is clear that no such Lie lattice can exist.

In fact we see that if we want the $\mathcal{A}$ matrix to be two vertically stacked 2×2 diagonal blocks, X_1, X_2 cannot appear in the top 2×2 matrix. If

$$\lambda^2_{21}X_1 + \lambda^2_{22}X_2 + \lambda^2_{23}X_3 + \lambda^2_{24}X_4 = X_1,$$

then $\lambda^2_{21} = 1$ so $\lambda^2_{12} = -1$ so we do not get a diagonal entries.

Similarly, if

$$\lambda^1_{11}X_1 + \lambda^1_{12}X_2 + \lambda^1_{13}X_3 + \lambda^1_{14}X_4 = X_2,$$

then $\lambda^1_{12} = 1$ so $\lambda^1_{21} = -1$ and the matrix is not diagonal.

It is possible to get a diagonal block on top, but only in terms of X_3, X_4. First, consider,

$$\lambda^1_{11}X_1 + \lambda^1_{12}X_2 + \lambda^1_{13}X_3 + \lambda^1_{14}X_4 = X_4,$$
$$\lambda^2_{21}X_1 + \lambda^2_{22}X_2 + \lambda^2_{23}X_3 + \lambda^2_{24}X_4 = X_4,$$

then $\lambda^1_{14} = 1 = \lambda^2_{24}$. This determines the bottom submatrix, giving us,

$$\mathcal{A}(X_1, X_2, X_3, X_4) = \begin{pmatrix} X_4 & 0 \\ 0 & X_4 \\ 0 & -X_1 \\ -X_2 & 0 \end{pmatrix}.$$

This is does not have a diagonal 2×2 block on the bottom as we require. Now we consider,

$$\lambda^1_{11}X_1 + \lambda^1_{12}X_2 + \lambda^1_{13}X_3 + \lambda^1_{14}X_4 = X_3,$$
$$\lambda^2_{21}X_1 + \lambda^2_{22}X_2 + \lambda^2_{23}X_3 + \lambda^2_{24}X_4 = X_3,$$

then $\lambda^1_{13} = 1 = \lambda^2_{23}$. This gives,

$$\mathcal{A}(X_1, X_2, X_3, X_4) = \begin{pmatrix} X_3 & 0 \\ 0 & X_3 \\ -X_1 & 0 \\ 0 & -X_2 \end{pmatrix}.$$

We try to amend the bottom 2×2 matrix by identifying X_1 with X_2 as generators. In doing so we now have a new basis where $X_1 = X_2 = X'_1$, $X_3 = X'_2$ and $X_4 = X'_3$. This gives the respective $\mathcal{A}$ commutator of the form,

$$\mathcal{A}(X'_1, X'_2, X'_3) = \begin{pmatrix} \lambda^1_{11}X'_1 + \lambda^1_{13}X'_2 + \lambda^1_{14}X'_3 & \lambda^2_{11}X'_1 + \lambda^2_{13}X'_2 + \lambda^2_{14}X'_3 \\ \lambda^1_{21}X'_1 + \lambda^1_{23}X'_2 + \lambda^1_{24}X'_3 & \lambda^2_{21}X'_1 + \lambda^2_{23}X'_2 + \lambda^2_{24}X'_3 \\ \lambda^1_{31}X'_1 + \lambda^1_{33}X'_2 + \lambda^1_{34}X'_3 & \lambda^2_{31}X'_1 + \lambda^2_{33}X'_2 + \lambda^2_{34}X'_3 \\ \lambda^1_{41}X'_1 + \lambda^1_{43}X'_2 + \lambda^1_{44}X'_3 & \lambda^2_{41}X'_1 + \lambda^2_{43}X'_2 + \lambda^2_{44}X'_3 \end{pmatrix}.$$

As seen we can only consider a diagonal X'_3 block in this case if we want the top 2×2 block to be diagonal. This gives,

$$\lambda^1_{11}X'_1 + \lambda^1_{13}X'_2 + \lambda^1_{14}X'_3 = X'_3,$$

hence $\lambda^1_{14} = 1$ and so $\lambda^1_{41} = -1$. This means that the bottom 2×2 block can no longer be diagonal.

Consequently, we have shown that no class 2 nilpotent $\mathbb{Z}$-Lie lattice can give the desired $\mathcal{A}$ commutator matrix. In particular, we have shown that there is no direct analogue to [9, Proposition 2.5] for the $\mathcal{A}$ commutator matrix.

6. Construction and zeta function

In this section, we devise an identification between two copies of a class 3 nilpotent $\mathbb{Z}$-Lie lattice such that the $\mathcal{R}$ matrix exhibits a scaling effect identical to that of the $\mathcal{R}$ matrix for a class 2 nilpotent group under the 2-fold central product. We then analyse the presentation of our constructions to show their limitations.

6.1. **Construction of presentation.** Given that for any $\mathcal{T}_2$-group, the $\mathcal{R}$ matrix (equivalent to the $\mathcal{B}$ matrix), scales with the central product, it is natural to ask whether the same can also be said beyond nilpotency class 2. We attempt to find a construction with the required $\mathcal{R}$ matrix.

Note for all constructions below, we have that their centre lies within their derived subgroup. Furthermore, for every construction below the abelianisation is torsion-free. It is a well known theorem that if the quotient group is torsion-free, then the subgroup is isolated, hence it follows that their derived subgroup is isolated. This means that k is the $\mathfrak{o}$-rank of the derived subgroup quotient the centre. Also note that for all constructions below, we have the centre $\mathfrak{z} = (z_1, z_2)$.

We return to the 3-nilpotent Lie lattice considered earlier,

$$\mathfrak{f} := \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y,\ [y, x_1] = z_1, [y, x_2] = z_2 \rangle,$$

where all non-specified Lie brackets are trivial.

Let $\mathcal{R}_{\mathfrak{f}}(Y_1, Y_2, Y_3)$ denote the $\mathcal{R}$ matrix for $\mathfrak{f}$ and $\mathcal{S}_{\mathfrak{f}}(Y_1, Y_2, Y_3)$ the $\mathcal{S}$ matrix for $\mathfrak{f}$.

Given two copies of $\mathfrak{f}$,

$$\mathfrak{f}_1 := \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y,\ [x_1, y] = z_1, [x_2, y] = z_2 \rangle,$$

$$\mathfrak{f}_2 := \langle a_1, a_2, b, c_1, c_2 \mid [a_1, a_2] = b,\ [a_1, b] = c_1, [a_2, b] = c_2 \rangle.$$

We get the 2-fold central product with the presentation,
$\mathfrak{f}_1 \times_Z \mathfrak{f}_2 =$

$$\left\langle \begin{array}{c} x_1, x_2, y, z_1, z_2 \\ a_1, a_2, b, c_1, c_2 \end{array} \middle| \begin{array}{c} [x_1, x_2] = y, [a_1, a_2] = b, \\ [x_1, y] = z_1 = [a_1, b] = c_1, [x_2, y] = z_2 = [a_2, b] = c_2 \end{array} \right\rangle,$$

where we have the centres of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ identified.

Taking $\boldsymbol{e} = (x_1, a_1, x_2, a_2, y, b)$ and $\boldsymbol{f} = (y, b, z_1, z_2)$ and computing the $\mathcal{R}$-matrix gives,

$$\mathcal{R}(Y_1, Y_2, Y_3, Y_4) = \begin{pmatrix} 0 & 0 & Y_1 & 0 & Y_3 & 0 \\ 0 & 0 & 0 & Y_2 & 0 & Y_3 \\ -Y_1 & 0 & 0 & 0 & Y_4 & 0 \\ 0 & -Y_2 & 0 & 0 & 0 & Y_4 \\ -Y_3 & 0 & -Y_4 & 0 & 0 & 0 \\ 0 & -Y_3 & 0 & -Y_4 & 0 & 0 \end{pmatrix}.$$

We get that $k = 2$ so the $\mathcal{S}$ matrix is,

$$\mathcal{S}(Y_1, Y_2, Y_3, Y_4) = \begin{pmatrix} Y_3 & 0 \\ 0 & Y_3 \\ Y_4 & 0 \\ 0 & Y_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We observe that if we set $Y_1 = Y_2$ as generators this would then also shift Y_3 to Y_2 and Y_4 to Y_3 and we would get scaling in the $\mathcal{R}$-matrix analogous to that of the central product in the class 2 nilpotent case.

Naively, our first approach is to not only identify the centre of the two copies, but also the derived subgroup giving us the presentation,

$$\mathfrak{f}_3 := \left\langle \begin{matrix} x_1, x_2, y, z_1, z_2 \\ a_1, a_2, b, c_1, c_2 \end{matrix} \;\middle|\; \begin{matrix} [x_1, x_2] = y = [a_1, a_2] = b, \\ [x_1, y] = z_1 = [a_1, b] = c_1, [x_2, y] = z_2 = [a_2, b] = c_2 \end{matrix} \right\rangle.$$

Now we have the bases $\boldsymbol{e} = (x_1, a_1, x_2, a_2, y)$ and $\boldsymbol{f} = (y, z_1, z_2)$ which gives us the $\mathcal{R}$-matrix,

$$\mathcal{R}(Y_1, Y_2, Y_3) = \begin{pmatrix} 0 & 0 & Y_1 & 0 & Y_2 \\ 0 & 0 & 0 & Y_1 & Y_2 \\ -Y_1 & 0 & 0 & 0 & Y_3 \\ 0 & -Y_1 & 0 & 0 & Y_3 \\ -Y_2 & -Y_2 & -Y_3 & -Y_3 & 0 \end{pmatrix},$$

We compute $k = 1$ and so we get the $\mathcal{S}$-matrix,

$$\mathcal{S}(Y_1, Y_2, Y_3) = \begin{pmatrix} Y_2 \\ Y_2 \\ Y_3 \\ Y_3 \\ 0 \\ 0 \end{pmatrix}.$$

We see that as we have set $b = y$ as generators, we have inadvertently collapsed the last two columns into a single column. We amend this by considering the following construction,

$$\mathfrak{f}_4 :=$$
$$\left\langle \begin{matrix} x_1, x_2, y, z_1, z_2 \\ a_1, a_2, b, c_1, c_2 \end{matrix} \;\middle|\; \begin{matrix} [x_1, x_2] = y = [a_1, a_2], \\ [x_1, y] = z_1 = [a_1, b] = c_1, [x_2, y] = z_2 = [a_2, b] = c_2 \end{matrix} \right\rangle.$$

We have reintroduced b as a generator of the derived subgroup, but we have removed the identity $[a_1, a_2] = b$ as instead we have $[a_1, a_2] = y$. Note we do not have $b = y$.

The bases remain almost identical with $\boldsymbol{e} = (x_1, a_1, x_2, a_2, y, b)$ and $\boldsymbol{f} = (y, z_1, z_2, b)$ which finally yields the desired $\mathcal{R}$-matrix,

$$\mathcal{R}(Y_1, Y_2, Y_3, Y_4) = \begin{pmatrix} 0 & 0 & Y_1 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & Y_1 & 0 & Y_2 \\ -Y_1 & 0 & 0 & 0 & Y_3 & 0 \\ 0 & -Y_1 & 0 & 0 & 0 & Y_3 \\ -Y_2 & 0 & -Y_3 & 0 & 0 & 0 \\ 0 & -Y_2 & 0 & -Y_3 & 0 & 0 \end{pmatrix}$$

As we have no Y_4 dependency we can omit it giving,

$$\mathcal{R}(Y_1, Y_2, Y_3) = \begin{pmatrix} 0I_2 & Y_1 I_2 & Y_2 I_2 \\ -Y_1 I_2 & 0I_2 & Y_3 I_2 \\ -Y_2 I_2 & -Y_3 I_2 & 0I_2 \end{pmatrix} = \mathcal{R}_{\mathfrak{f}}(Y_1, Y_2, Y_3) \otimes I_2,$$

We compute $k = 2$, this means that we also get identical scaling in the $\mathcal{S}$ matrix,

$$\mathcal{S}(Y_1, Y_2, Y_3) = \begin{pmatrix} Y_2 & 0 \\ 0 & Y_2 \\ Y_3 & 0 \\ 0 & Y_3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y_2 I_2 \\ Y_3 I_2 \\ 0I_2 \end{pmatrix} = \mathcal{S}_{\mathfrak{f}}(Y_1, Y_2, Y_3) \otimes I_2.$$

6.2. **Analysis of constructions.** At first glance it seems we have succeeded in identifying two copies of our nilpotency class 3 $\mathbb{Z}$-Lie lattice in such a way as to give the desired scaled $\mathcal{R}$ matrix. However, it comes at a great cost, neither $\mathfrak{f}_3$ or $\mathfrak{f}_4$ are in fact Lie lattices. This is due to the violation of the Jacobi identity. In both instances we consider,

$$[x_1, [a_1, a_2]] + [a_1, [a_2, x_1]] + [a_2, [x_1, a_1]] = 0.$$

This gives us,

$$[x_1, y] + [a_1, 0] + [a_2, 0] = z_1 \neq 0.$$

This means that we cannot compute their twist representation zeta functions with the methods provided in this paper. In future work, it would be of great interest to compute the twist zeta functions for these constructions to determine whether they give a rescaled zeta function regardless. In this vein, [16] by Rossmann considers the zeta functions of other algebraic objects, beyond just $\mathcal{O}$-Lie lattices and explores how to compute them.

We remark that although the constructions are not Lie lattices, the $\mathcal{R}$ and $\mathcal{S}$ matrices are still well defined as the bracket is still bilinear and anti-symmetric. The commutator matrices encode the structure constants of any such bracket, regardless whether they satisfy the Jacobi identity.

Since a $\mathbb{Z}$-Lie lattice is uniquely determined by its structure constants, we see that in general, for nilpotency class greater than 2, it is not possible to construct a Lie lattice with a scaled $\mathcal{R}$ commutator matrix.

7. Central product scaling on higher nilpotency classes

For class 2 nilpotent groups, $\mathfrak{g}(\mathfrak{o})$, we have,

$$\zeta_{\mathfrak{g}(\mathfrak{o})}(ks) = \zeta_{\times_Z^k \mathfrak{g}(\mathfrak{o})}(s), \text{[9, Proposition 3.3]}.$$

In this section we show that this does not lift to higher nilpotency classes. In particular we show that it fails for our class 3 nilpotent example.

7.1. **Topological zeta function scaling.** Recall the definition of the local twist representation zeta function. We now specialise the $\mathbb{Z}$-Lie lattice case. As such we get,

$$\zeta_{G(\mathbb{Z}_p)}(s) \;=\; \sum_{i=0}^{\infty} \tilde{r}_{p^i}\big(G(\mathbb{Z}_p)\big)\, p^{-is}.$$

Denote by $V_1, \ldots, V_r$ separated $\mathbb{Z}$-schemes of finite type and by $W_1, \ldots, W_r \in \mathbb{Q}(X, Y)$ rational functions.
[15, Theorem 2.2] tells us that, for almost all primes p,

$$\zeta_{G(\mathbb{Z}_p)}(s) \;=\; \sum_{i=1}^{r} \#V_i(\mathbb{Z}_p/(p)) \cdot W_i\big(p, p^{-s}\big) = \sum_{i=1}^{r} \#V_i(\mathbb{F}_p) \cdot W_i\big(p, p^{-s}\big)\,,$$

since $\mathbb{Z}_p/(p) \cong \mathbb{F}_p$.

Definition 7.1. The topological representation zeta function of G is

$$\zeta_{G,\mathrm{top}}(s) \;:=\; \sum_{i=1}^{r} \chi\big(V_i(\mathbb{C})\big) \cdot \lfloor W_i \rfloor \;\in\; \mathbb{Q}(s),$$

where $\chi(V_i(\mathbb{C}))$ denotes the topological Euler characteristic and $\lfloor W_i \rfloor$ the constant term of $W_i(X, X^{-s})$ in its expansion at $X = 1$.

Lemma 7.2 (Scaling descends to the topological zeta function)**.** *Given two unipotent group schemes G, H over $\mathbb{Z}$, such that for almost all primes p,*

$$\zeta_{G(\mathbb{Z}_p)}(s) \;=\; \zeta_{H(\mathbb{Z}_p)}(ks),\;\; k \in \mathbb{N},$$

as meromorphic functions in s. Then,

$$\zeta_{G,\mathrm{top}}(s) \;=\; \zeta_{H,\mathrm{top}}(ks).$$

Proof. For almost all primes p, $\zeta_{G(\mathbb{Z}_p)}(s)$ and $\zeta_{H(\mathbb{Z}_p)}(s)$ are rational functions in p^{-s}, [8, Theorem 8.4].
As seen in [15], to compute $\zeta_{G,\mathrm{top}}(s)$ it suffices to expand $\zeta_{G(\mathbb{Z}_p)}$ as a formal Taylor series in $p-1$ and read off the constant term.

We have,

$$\zeta_{G(\mathbb{Z}_p)}(s) \;=\; \sum_{i=0}^{\infty} a_i(s)\,(p-1)^i, \qquad \zeta_{H(\mathbb{Z}_p)}(s) \;=\; \sum_{i=0}^{\infty} b_i(s)\,(p-1)^i,$$

with coefficients $a_i(s), b_i(s) \in \mathbb{C}(s)$.

By assumption we have,

$$\zeta_{G(\mathbb{Z}_p)}(s) \;=\; \zeta_{H(\mathbb{Z}_p)}(ks).$$

Comparing expansions in $p-1$ we get,

$$a_0(s) = b_0(ks).$$

By definition, we have,

$$\zeta_{G,\mathrm{top}}(s) := a_0(s), \qquad \zeta_{H,\mathrm{top}}(s) := b_0(s).$$

Hence,

$$\zeta_{G,\mathrm{top}}(s) = \zeta_{H,\mathrm{top}}(ks).$$

□

Corollary 7.3. *If for $k \in \mathbb{N}$ we have,*

$$\zeta_{G,\mathrm{top}}(s) \;\neq\; \zeta_{H,\mathrm{top}}(ks).$$

Then we must also have,

$$\zeta_{G(\mathbb{Z}_p)}(s) \;\neq\; \zeta_{H(\mathbb{Z}_p)}(ks).$$

Proof. Contrapositive statement of lemma. □

7.2. **Methods in Zeta.** Zeta is a Python package for SageMath developed by Rossmann for 64-bit Linux (x86_64) systems. Zeta can compute local and topological zeta functions arising from the enumeration of subalgebras, ideals, submodules, representations, and conjugacy classes for suitable algebraic structures.
In our case we will compute topological zeta functions using Zeta.

First we must rewrite our $\mathcal{R}$ matrix. Our $\mathcal{R}$ matrix has entries

$$\mathcal{R}(\mathbf{Y})_{ij} = \sum_{l=1}^{d} \lambda_{ij}^{l} Y_l.$$

To input this data into Zeta we replace each linear form in Y_l by the coordinate tuple of the corresponding commutator in the full basis $(e_1, \dots, e_n)$.

For our initial class 3 nilpotent $\mathbb{Z}$-Lie lattice,

$$\mathfrak{f} := \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] = y,\ [y, x_1] = z_1, [y, x_2] = z_2 \rangle,$$

and,

$$\mathcal{R}(Y_1, Y_2, Y_3) = \begin{pmatrix} 0 & Y_1 & Y_2 \\ -Y_1 & 0 & Y_3 \\ -Y_2 & -Y_3 & 0 \end{pmatrix}.$$

We input,

```
L = Zeta.Algebra([
[(0,0,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,0), (0,0,0,0,0)],
[(0,0,-1,0,0), (0,0,0,0,0), (0,0,0,0,1), (0,0,0,0,0), (0,0,0,0,0)],
[(0,0,0,-1,0), (0,0,0,0,-1), (0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0)],
[(0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0)],
[(0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0), (0,0,0,0,0)]
])
```

and then run,

```
Zeta.topological_zeta_function(L, 'reps')
```

This gives us the result,

$$\zeta_{G,\mathrm{top}}(s) = \frac{s^2}{(s-1)(s-2)}.$$

We now approach the 2-fold central product of our class 3 nilpotent $\mathbb{Z}$-Lie lattice,
$\mathfrak{f}_1 \times_Z \mathfrak{f}_2 =$

$$\left\langle \begin{matrix} x_1, x_2, y, z_1, z_2 \\ a_1, a_2, b, c_1, c_2 \end{matrix} \;\middle|\; \begin{matrix} [x_1, x_2] = y, [a_1, a_2] = b, \\ [x_1, y] = z_1 = [a_1, b] = c_1, [x_2, y] = z_2 = [a_2, b] = c_2 \end{matrix} \right\rangle,$$

via the same method. We input,

```
L = Zeta.Algebra([
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,1,0,0,0),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,1,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,1,0,0),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,1,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,-1,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,0,1),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,-1,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,1),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,-1,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,-1),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,-1,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,-1),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    [(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),
    (0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)],
    ])
```

and again run,

`Zeta.topological_zeta_function(L, 'reps')`

This gives us the topological zeta function,

$$\zeta_{G,\mathrm{top}}(s) = \frac{s^3}{(s-1)^3}.$$

It is clear that there exists no $k \in \mathbb{N}$ such that,

$$\frac{s^3}{(s-1)^3} = \frac{(ks)^2}{(ks-1)(ks-2)}, s \in \mathbb{C}.$$

Hence the central product does not provide a scaling in the twist representation for nilpotency class 3 by corollary 6.3.

Consequently, we have shown that for nilpotency class greater than 2, the k-fold central product does not, in general, scale the twist representation zeta function.

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The work contained in this thesis is my own work unless otherwise stated.